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The notion and role of "detection tests" in the Danish upper secondary "maths counsellor" programme

Niss, Mogens Allan; Jankvist, Uffe Thomas

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CERME 10

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Foreword: CERME10 in lovely Dublin

Susanne Prediger[1], Viviane Durand-Guerrier[2]

[1]Dortmund, Germany, prediger@math.uni-dortmund.de

[2]Montpellier, France, viviane.durand-guerrier@univ-montp2.fr

Since its beginnings in 1998, ERME, the European Society for Research in Mathematics Education was dedicated to supporting the so-called "three C's": communication, cooperation, and collaboration among researchers in Europe and beyond. The major occasion for the ERME spirit to come to life is the biannual congress, CERME. In this year, the 10th congress of ERME, CERME10, took place in Dublin from February 1st to February 5th, 2017.

The congress took place in Dublin in Croke Park, the stadium that is home of the Irish national sports of Gaelic football and hurling. Although the conference was bigger than ever, the 774 participants felt few compared to the 80 000 people who fit into the stadium on large sporting and other occasions. However, our local chair, Thérèse Dooley, her co-chair Maurice OReilly and all their colleagues did a fantastic job in making us feel at home and not lost in the huge venue. Their hospitality and engagement were praised by all participants.

The program of the congress was organized by the International Program Committee, chaired by Ghislaine Gueudet and the vice-chair Andreas Eichler in a very well structured, transparent and highly efficient way. Under their guidance, the IPC developed a substantial program with two very interesting plenaries, one presented by Elena Nardi (entitled "From Advanced mathematical thinking to university mathematics education: A story of emancipation and enrichment") and the other by Lieven Verschaffel (entitled "Towards a more comprehensive model of children's number sense"). In a panel on "Solid findings in mathematics education: What are they and what are they good for?" this 'hot' topic was discussed from different perspectives. Marianna Bosch, Tommy Dreyfus, Caterina Primi, and Gerry Shiel made up the panel. All of the plenary activities contributed substantially to the success of the conference.

However, the core and the heart of each CERME are the seven sessions in the Thematic Working Groups, which offer the main place for the spirit of inclusion realized in communication and cooperation. The 23 Thematic Working Groups were organized by 84 group leaders, an impressive number of people who invest their energy and time in the success of the congress. Several external conference organizers expressed their surprise that during the sessions, nobody was wandering around in the corridors. Of course not, we said, they are communicating and cooperating! And we become aware again that this intensity of work is specific, and perhaps even unique, to CERME.

Most of the CERME group leaders have taken this responsibility for several years and have established a long-term collaboration with substantial academic outcomes. This group of people engaged in the enormous effort of managing the process of quality development for 474 submitted papers and 94 posters, numbers much larger than ever before.

CERME is not only getter larger from congress to congress, but also increasingly international. The 774 participants came from 29 Europeans countries and 23 Non-European countries. The top ten

countries in terms of numbers of participants were Germany (127), United Kingdom (60), Norway (55), France (47), Italy (47), Ireland (41), Spain (39), Sweden (38), Israel (32), and the US (30). Austria, Belgium, Croatia, Cyprus, Czech Republic, Denmark, Faroe Islands, Finland, Greece, Hungary, Iceland, Kosovo, Malta, Netherlands, Poland, Portugal, Russia, Slovakia, Switzerland, Turkey, and Ukraine were included in the European countries. Among the non-European countries were Algeria, Argentina, Australia, Bangladesh, Brazil, Cameroon, Canada, Chile, Hong Kong, Kenya, Iran, Japan, Lebanon, Malawi, Mexico, New Zealand, Nigeria, Singapore, South Africa, Thailand, and Tunisia. It must be the specific style of the congress and the ERME spirit which attracts so many people from all over the world!

With the increasing numbers and diversity, the challenge of compiling proceedings is getting more and more complex. We thank the chairs who served as editors for this complex process and for finalizing it so quickly.

Such a huge and complex congress as CERME could not be conducted without the engagement of more than 15% of all participants (including TWG leaders, IPC members, LOC members and ERME board members). We thank everybody who has contributed to the ongoing work behind the scenes which allowed the congress to be a real success. Specific thanks go to Ghislaine Gueudet, Andreas Eichler, Thérèse Dooley and Maurice OReilly for their hard work with a wonderful outcome.

We encourage interested researchers to meet us at the next CERME that will take place from February 5th to February 10th 2019, in Utrecht (the Netherlands).

Susanne Prediger, ERME President since February 2017

Viviane Durand-Guerrier, ERME President until February 2017

Introduction to the Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10)

Thérèse Dooley[1], Ghislaine Gueudet[2]

[1] Dublin, Ireland, therese.dooley@dcu.ie

[2] Rennes, France, ghislaine.gueudet@espe-bretagne.fr

About CERME10

The Tenth Congress of European Research in Mathematics Education (CERME 10) took place in Dublin (Ireland) from 1st to 5th February 2017. Ghislaine Gueudet (France) was the chair of the International Programme Committee (IPC) which comprised Thérèse Dooley (Ireland, chair of the local Programme Committee), Andreas Eichler (Germany, co-chair), Marianna Bosch (Spain), Markku Hannula (Finland), Jeremy Hodgen (UK), Konrad Krainer (Austria), Despina Potari (Greece), Kirsti Rø (Norway), Cristina Sabena (Italy), Michiel Veldhuis (Netherland), Nad'a Vondrová (Czech Republic). Thérèse Dooley and Maurice OReilly were chair and co-chair respectively of the Local Organizing Committee (LOC).

CERME10 hosted 23 Thematic Working Groups, listed in the table below. The TWGs 21, 22, 23 and 24 were new groups, created following a call launched just after CERME9, and a selection process involving the CERME10 IPC and the ERME board. They all have been very successful, and will be part of CERME11 in February 2019. TWG7 (Mathematical potential, creativity and talent) has unfortunately been cancelled, due to a lack of contributions; while TWG14 has been split in two for the conference, because of the large number of papers received.

TWG	**Leader**
TWG1: Argumentation and proof	Gabriel Stylianides (UK)
TWG2: Arithmetic and number systems	Elisabeth Rathgeb-Schnierer (Germany)
TWG3: Algebraic thinking	Reinhard Oldenburg (Germany)
TWG4: Geometry	Joris Mithalal (France)
TWG5: Probability and statistics education	Corinne Hahn (France)
TWG6: Applications and modelling	Susana Carreira (Portugal)
TWG8: Affect and mathematical thinking	Pietro Di Martino (Italy)
TWG9: Mathematics and language	Núria Planas (Spain)
TWG10: Diversity and mathematics education: Social, cultural and political challenges	Lisa Björklund Boistrup (Sweden)
TWG11: Comparative studies in mathematics education	Paul Andrews (Sweden)
TWG12: History in mathematics education	Renaud Chorlay (France)

TWG13: Early years mathematics	Ingvald Erfjord (Norway)
TWG14a: University mathematics education	Alejandro Gonzalez Martin (Canada)
TWG 14b: University mathematics education	Irene Biza (United Kingdom)
TWG15: Teaching mathematics with resources and technology	Alison Clark-Wilson (UK)
TWG16: Students' learning mathematics with resources and technology	Hans-Georg Weigand (Germany)
TWG17: Theoretical perspectives and approaches in mathematics education research	Angelika Bikner Ahsbahs (Germany)
TWG18: Mathematics teacher education and professional development	Stefan Zehetmeier (Austria)
TWG19: Mathematics teacher and classroom practices	Charalampos Sakonidis (Greece)
TWG20: Mathematics teacher knowledge, beliefs and identity	Miguel Ribeiro
TWG21: Assessment in mathematics education	Paola Iannone (UK)
TWG22: Curricular resources and task design in mathematics education	Birgit Pepin (The Netherlands)
TWG23: Implementation of research findings in mathematics education	Uffe Thomas Jankvist (Denmark)
TWG24: Representations in mathematics teaching and learning	Elisabetta Robotti (Italy)

Editorial information

These proceedings are available as a complete volume online on the ERME website and each individual text is also available on the HAL open archive, where it can be found through keywords, title or author name. This has been the practice since CERME9, to increase the visibility of the huge work done in CERME conferences.

This volume begins with texts corresponding to the three plenary activities of CERME10: the presentation by Elena Nardi on University Mathematics Education; that by Lieven Verschaffel on Early Mathematics; and the panel on Solid Findings in Mathematics Education, chaired by Marianna Bosch and involving Tommy Dreyfus, Catarina Primi and Gerry Shiel.

After the plenaries, the reader will find 23 chapters corresponding to the work done in the TWGs of CERME10 (we remind the reader that TWG7 has been cancelled; moreover, TWG14 was split in two for the conference, but all the papers are in the same section in these proceedings).

These chapters follow a similar structure: they start with an introduction; then the long contributions (8-page papers) are presented – in alphabetical order by first author's name – and finally the short

contributions (2 pages). However, TWG17 has chosen a different order, corresponding to subthemes in the group.

There are two kinds of introductions to the TWGs, according to the team's choice: short introductions (4 pages) presenting the contributions; or long introductions (8 pages), which propose, in addition, an analysis of the current research on the theme of the TWG, and perspectives for the future. TWGs 6, 14, 15, 16, 17, 19 and 23 have chosen this form of long introduction.

The publication of these proceedings is the result of a collaborative work, involving CERME10 IPC, the TWG leaders and co-leaders, and the LOC co-chair. We warmly thank all these people for their involvement, and hope that this volume will contribute to the development of mathematics education research in Europe and beyond.

Statistical overview of CERME10

Maurice OReilly

Dublin, Ireland, maurice.oreilly@dcu.ie

CERME evolves over time, and so it is of interest to gather and present some quantitative data on the number of participants and on the scientific output at CERME10. The table opposite shows the number of submissions to these proceedings (excluding the TWG introductions and the plenary papers) along with other submissions that were made online by December 2016 but were not included here. The numbers of long and short contributions are noted. Of course, each TWG had more participants than the number of submissions, since (i) many papers had several authors and (ii) there were other participants who did not submit. The entries in the table are explained in its footnotes.

The data for the table comes from two sources: the submissions made online (to the CERME10 website) by mid-December 2016, and the registration and attendance database for the congress. The final column shows the 768 (distinct) participants at CERME10, allocating each to exactly one TWG. Although over 80 participants were active in more than one TWG, care was taken to ensure that no participant was included more than once (by fine-tuning the 'Additional authors' column appropriately). This was facilitated by taking into account the TWG explicitly chosen by each participant at registration. The foreword states that there were 774 participants; this figure includes the six presenters at the plenary sessions.

Of the 565 submissions made in advance of CERME – comprising 466 (long) papers and 99 posters (or short contributions) – the attrition by the time these proceedings have been edited was only 22 (about 3.9%). This low figure underscores the observation mentioned in the foreword that "nobody was wandering in the corridors". Contributions to the proceedings as a proportion of the total number of participants was 0.71 (= 543/768), this figure varying from 0.56 (for TWG2) to 0.82 (for TWG16). Another indicator of the intensity of the work at CERME is the low number of participants (84) who did not contribute papers – although they did contribute to the lively discussion! This was 10.9% of the total number of participants, with extremes ranging from 3.0% (for TWG10) to 20.0% (for TWG24). Yet another perspective on the hard work undertaken in the context of the congress is illustrated by the number (451) of 'long' papers as a proportion of all contributions: 83% overall, with a minimum of 71% (for TWG14) and an impressive maximum of 100% (for TWG2).

It has already been noted (in the foreword) that participants at CERME10 were drawn from 52 countries. It is part of the 'CERME spirit' to support academics who would normally have limited access to CERME (either from underrepresented or economically weak countries). This is made possible through the Graham Littler Fund which draws from those who can afford, in support of those who cannot. For CERME10, 46 participants were awarded grants totaling €21 100 (€12 300 for registration and €8 800 for travel and accommodation).

It is hoped that the data provided on these two pages helps quantify important aspects of CERME10, putting the scientific output in perspective.

TWG	Submissions		Contributions to the Proceedings		Additional	Additional	Total number of
	to Proceedings	Online only	Long	Short	authors	participants	participants in each TWG
1	28	0	27	1	12	6	46
2	14	2	14	0	5	4	25
3	19	2	16	3	2	5	28
4	20	0	15	5	4	2	26
5	25	1	21	4	3	2	31
6	22	2	18	4	6	5	35
8	25	1	23	2	3	3	32
9	26	1	22	4	6	2	35
10	26	1	19	7	5	1	33
11	13	0	10	3	4	1	18
12	18	0	16	2	5	2	25
13	20	0	15	5	7	3	30
14	58	5	41	17	13	11	87
15	25	0	19	6	6	4	35
16	27	0	20	7	1	5	33
17	16	2	12	4	2	3	23
18	31	0	28	3	4	5	40
19	23	0	22	1	5	5	33
20	26	4	23	3	8	3	41
21	25	1	23	2	7	3	36
22	23	0	17	6	7	2	32
23	15	0	14	1	2	2	19
24	18	0	16	2	2	5	25
Total	**543**	**22**	**451**	**92**	**119**	**84**	**768**

Submissions were made **online** by mid-December 2016, most of which are published in these **proceedings** (as either **long** or **short contributions**).
In each TWG, the number of submissions is augmented by '**Additional authors**' indicating the number of participants at CERME10 who contributed to (long or short) papers. The '**Additional participants**' attended CERME10 but were not authors of these papers.
The '**Total number of participants in each TWG**' is then the sum of all the submissions along with the additional authors and participants.

From advanced mathematical thinking to university mathematics education: A story of emancipation and enrichment

Elena Nardi

University of East Anglia, UK, e.nardi@uea.ac.uk

Between CERME1 and CERME9 there have been approximately two hundred and fifty papers with their focus directly, or a little less so, on the teaching and learning of mathematics at university level, starting from about a dozen in CERME1 and rising to several dozens in CERME9. ERME recognised the increasing significance of this emerging field with the launch of Working Group 14 (Advanced Mathematical Thinking) in CERME4 in 2005 which evolved into Thematic Working Group 14 (University Mathematics Education) in CERME7 in 2011. In this lecture, I draw on my experience as researcher in this field, and as participant in both groups (and inaugural leader of the latter), to identify epistemological – theoretical, substantive and methodological – trends in the transition from the one to the other. I aim that the story I tell is one of gradual emancipation from a relatively limited initial focus on cognitive aspects of the student learning experience in university mathematics to the grander vista of issues – also inclusive of pedagogical, institutional, affective and social issues – that studies presented at CERME nowadays address. I also aim that the story I tell is one of enrichment as the depth and diversity of said vista has been accomplished also through thoughtful appropriation of results from those earlier studies.

Keywords: University mathematics education, developmental / cognitive and sociocultural approaches to the teaching and learning of mathematics.

Introduction

In tandem with ERME, the area of research that is the focus of this plenary, *University Mathematics Education* research, has also been evolving rapidly in the last twenty years or so. Here I focus on some of the milestones of this evolutionary journey, with the particular emphasis that I promised in the above title and abstract. Before proceeding to these though, here is a bit of a pre-amble: Figure 1 presents a still from a scene in the film *A Serious Man* (2009) directed by Ethan and Joel Coen.

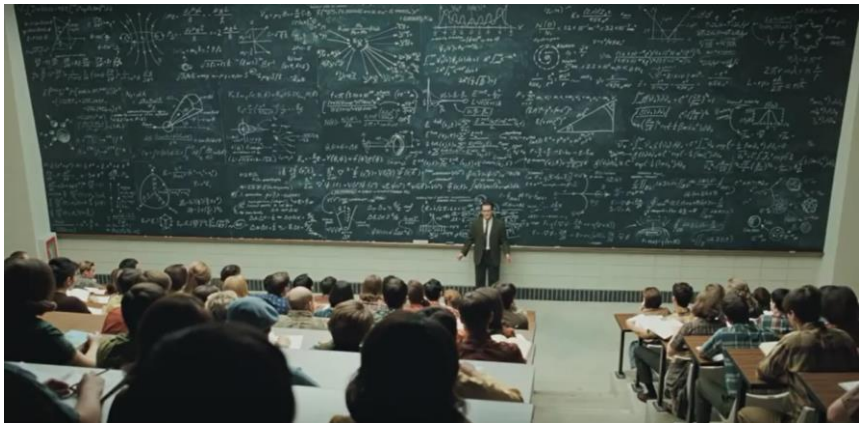

Fig.1. Still taken from *A Serious Man* (2009): https://www.youtube.com/watch?v=7iggyFPls4w

This is a typical imagining in popular culture of how mathematics teaching looks like at university. I will not go much further with a discourse analysis of what the still (or the scene, or the film itself) may convey. In what I see as some contrast, Figure 2 presents a sequence of images, taken from the publicity materials of my own institution's department of mathematics.

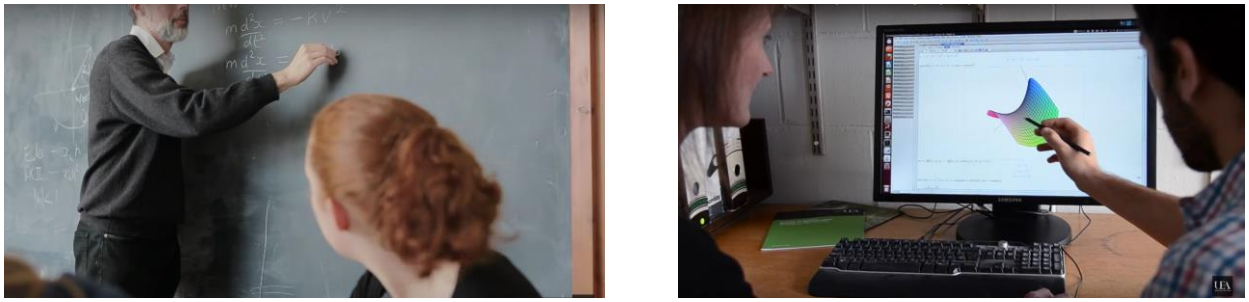

Fig. 2. Still taken from UEA promotional video: https://www.youtube.com/watch?v=gRzVX8c1be4

The students and the lecturer in these images work together, they are not physically too far from each other and there is a range of resources – from chalk to digital – present. The sequence illustrates how institutions may wish to present the kind of learning experience that potential incomers into a department of mathematics are likely to be offered.

To me, there is a clear contrast between the movie still from *A Serious Man* and these two images from the UEA promotional video. It is a contrast between a widespread perception of university mathematics lectures as the ultimate form of transmissive pedagogies – with all the repercussions of alienation and distancing these pedagogies may entail – and the aspiration (institutional but not only) for a more approachable, more inclusive and more engaging learning experience in university mathematics that is tailored to individual student needs.

As university lecturers today – in mathematics and in other disciplines – we lecture. But we also do much more: we coordinate seminars, we conduct individual or small group tutorials, we run workshops and drop-in clinics, we supervise dissertations, we advise students on academic and on pastoral matters and we assess students in a variety of ways (all the way from closed-book examinations to mini-projects and oral presentations). Our professional worlds are far from monotonous. In fact, they require us to be quite versatile.

I see as of little surprise, and rather pleasing, that the versatility of our jobs is being reflected in the diversity of University Mathematics Education research that is now presented at CERME. This diversity of focus – but also theoretical perspective and methodology – is to me a sign of richness. In fact, here I have taken the liberty of endorsing a metaphor, which originates in currently dominant theories of evolution and conservation (Figure 3). These theories equate species diversity with resilience. The story I tell here relies somewhat on whether this is a convincing metaphor.



Fig. 3. Image from: https://conservationbytes.com/2014/01/08/more-species-more-resilience/

I tell this story in five parts: *The "early years", CERME 1, 2, 3; The AMT years, CERME 4, 5, 6; The UME years, CERME 7, 8, 9; CERME10, the split*[1]*…*; and, *Taking stock / What next / Coming soon…* Before starting, I need to post a health warning though: that a lecture of this kind errs on the side of being impressionistic – and of course quite personal[2] too. I thank you in advance for your tolerance.

My own trajectory in CERME – and outside – mirrors some of the milestones and trends that this plenary aims to map out. I was present in 1999 at CERME1, in Osnabrueck, assisting with the coordination of Group 5, *Mathematical thinking and learning as cognitive processes*. To those more familiar with the increasingly sociocultural and discursive take that my work has been taking over the years, this commitment to Group 5 may sound a little surprising. It is not. I start Part I with an anecdote on exactly this.

Part I: The "early years", CERME 1, 2, 3; UME research evidenced in several TWG groups

My 1996 doctorate's title (Nardi, 1996) is *The novice mathematician's encounter with mathematical abstraction: Tensions in concept image construction and formalization.* The statement of intentions in this doctorate are clear:

> Mathematics is defined as an abstract way of thinking. Abstraction ranks among the least accessible mental activities. In [the UK educational context where the study took place], the encounter with mathematical abstraction is the crucial step of the transition from informal school mathematics to the formalism of university mathematics. This transition is characterised by cognitive tensions. This study aimed at the *identification and exploration of the tensions in the novice mathematician's encounter with mathematical abstraction*. (Nardi, 1996: Abstract)

However, the study's stated theoretical perspective is a little more perplexing. It is declared as "consisting of cognitive and sociocultural theories on learning". And, the two key parts of findings in the final chapter promise an account of the novice mathematician's encounter with mathematical abstraction "as a personal meaning-construction process and as an enculturation process" (ibid.).

It is quite easy, in hindsight, to be skeptical about the risky eclecticism of the approach – some may see this as standing on a fence, or, even, as pick-and-mix nonsense. But, I keep reminding myself that the study started in 1992 and was completed in 1996. It was therefore conducted at a time when

[1] Continuing with the biology inspired metaphors, I use the word "split" deliberately. Cell splitting is the process of subdividing a congested cell into smaller cells. Cell splitting or division is associated with reproduction and the creation of an entire new organism. This process is typically seen as increasing many of the capacities of a cellular system. In fact, in Parts III and IV, I aim to show the inevitability of cell splitting, emanating from the substantive, theoretical, and methodological diversity of UME research presented in CERME these days. It is in these parts that the main point of this lecture, signposted in the abstract by the words **emancipation** and **enrichment**, will, I hope, come through.

[2] I also need to thank at this juncture two overlapping groups of colleagues: my CERME 7, 8 and 9 TWG14 co-leaders and my co-authors of the 20-year anniversary ERME book in which UME research has been allocated a chapter (Winsløw et al., in press). Since 2010, when the UME TWG group was formed – for its first appearance in CERME7, in 2011 – these colleagues, have become what I like to call my academic family of friends.

the various shades of constructivism that form its theoretical backbones were then taking shape themselves. To signpost this a little more emphatically, allow me the gentle reminder that the seminal paper *Constructivist, emergent and sociocultural perspectives in the context of developmental research* (Cobb & Yackel, 1996) – a paper and a programme more broadly that impacted upon our debate around the co-determinants of mathematical learning in immense ways – appeared in *Educational Psychologist* in 1996, the year that my doctorate was completed. I often use this excuse when the slightly embarrassing thought comes to me that my study wanted to have its cake and eat it too!

So, here are some recollections from the early years, and, to start with, CERME1, that I see as pertinent for today: UME papers can be found in several groups but mostly in TWG1 (*Nature and content of mathematics and its relation to teaching and learning*) and TWG5 (*Mathematical thinking and learning as cognitive processes*). There is a pronounced epistemological focus on several papers – Grenier and Payan (1999) is one example – and there is a strong tendency in the few papers present to give a prominent position to the mathematical context and content of, for example, proposed course designs. Belousova and Byelyavtseva's (1999) paper on course design in Numerical Methods comes to mind; as do the Cabri designs for Linear Algebra put forward by Tommy Dreyfus, Joel Hillel and Anna Sierpinska (1999). There is also a tendency to consider this mathematical content regardless of whether this is present in school or university mathematics: there are, for example, propositions in this first CERME about using CAS (Computer Algebra Systems) for teaching functions; or, courseware for the teaching of Geometry from across school to university, and all the way to Differential Geometry.

There are two contributions to CERME1 though which, for me, stand out even more than those I sampled in my last comments. Both pre-empt the publication of two volumes that proved influential in the following years, in different, yet distinct ways. One is Leone Burton's (1999) preliminary analyses of interviewed mathematicians' epistemological perspectives which culminated in her monograph (Burton, 2004), *Mathematicians as Enquirers*. The other is Jean-Luc Dorier's paper (with Aline Robert, Jacqueline Robinet and Marc Rogalski, 1999) that sets the scene for the volume *On the teaching of linear algebra* (Dorier et al., 2000).

Both papers foreshadow – and I daresay contributed towards shaping – trends in UME research that became prominent in the years that followed. Burton's work signals a broadening of the UME church to include in its focus the university teacher (most other work at the time concerns the student or the mathematics alone). Dorier's work, and that of his colleagues, signals the still then not so imminent end of what I see as a shortcoming of UME research that is still present today, albeit to a lesser extent: the perception of research into university mathematics teaching practice as an a-theoretical aside of well-intended practitioners who are unaware of the epistemological and methodological underpinnings of mathematics education as an academic discipline. This work is distinct for its robust theoretical grounds and for its keen eye for intervention design, trial and evaluation – in a nutshell, for its systematic character. In this sense, of scope and ambition, it shares some common ground with another, powerful at the time – and still today –programme: that of APOS which originated in the USA and which was at the time also pushing the boundaries of work in UME beyond elementary Calculus and into Abstract Algebra.

Continuing with my observing trends that were to become influential in later years, within TWG5 (*Mathematical thinking and learning as cognitive processes*), which I mentioned earlier and which I assisted coordinating under the leadership of Inge Schwank, there are two themes that made an appearance – timidly and managing to occupy a small portion of the discussions only: the role of motivation in cognition (I see here inklings of evidence on the burgeoning importance of research on affect) and the emerging importance of theories of situated cognition.á An observation that stands out from these discussions was made in the paper by Pier Luigi Ferrari (1999): in advanced mathematical thinking, wrote Pier Luigi at the time, some learner behaviours cannot be accounted for simply in terms of semantics. His paper presented an argument that brings the role of language – ordinary and mathematical – and of communicational structures to the fore.

CERME2 and CERME3 are the two CERMEs that I missed. Nonetheless, returning to the proceedings after all these years, there are several papers presented in CERME2 and a couple of dozen papers in CERME3 that can be found across several Working Groups and contain implicit references to advanced mathematics, often as extensions of what is typically found in the school syllabus that each paper revolves around.

In CERME2 these papers are mostly found in Working Group 5 (*Mathematical thinking and learning as cognitive processes*) and Working Group 1 (*Creating experience for structural thinking*). Mathematical thinking (including a growing focus on proof and proving) is at the heart of these papers which are only implicitly and only occasionally concerned with the institutional, curricular and pedagogical context of university level Mathematics Education. There is concern in these papers with internal mental structures. Naďa Stehlíková and Darina Jirotková's paper (2001) is a good example: it focuses explicitly on processes of building an inner mathematical structure, which the authors abbreviate as IMS and which they acknowledge as hard to observe. They then resort to introspective, self-reporting accounts of mathematical thinking. John Mason's (1998) "researching from the inside" features largely as a theoretical influence on the paper. Naďa Stehlíková will carry on in this strand of work also in CERME3.

These works concern the learning of mathematics often at the cusp of the transition from school to (what is in many places) university mathematics. One example of this trend is Bettina Pedemonte's (2001) study of cognitive unity, or break, in the context of constructing mathematical arguments and proofs. Another is the paper by Baruch Schwarz, Rina Hershkowitz, and Tommy Dreyfus (2001) which presents a perspective on abstraction as always occurring in context and which focuses on three epistemic actions (Recognising, Building-With and Constructing, RBC). Its theoretical close relatives are an eclectic mix and include elements of Activity Theory (Alexei Nikolaevich Leontiev) and the construct of situated abstraction per Richard Noss and Celia Hoyles (1996).

In tandem with abstraction, there are two studies of mathematical intuition that I would like to close my reference to CERME2 with. One (Tsamir, 2001) regards infinite sets and another (Chartier, 2001) regards geometrical intuition as a stepping stone to the study of Linear Algebra. Both refer extensively – and in some sense stand on the solid shoulders of – the essential work on mathematical intuition by Ephraim Fischbein. The analysis in (Chartier, 2001) is also embedded in curricular and pedagogical aspects of the experiences of the post-graduate students who are its focus and draws out of the students' responses the kinds of geometrical intuition – helpful and less helpful

– they bring into their practice of Linear Algebra. Those links between mathematical encounters of the students in earlier and later phases of their studies will be a focus for Ghislaine Gueudet (then Chartier) also in CERME3.

Transitions, for example from Algebra to Analysis – as in the work also in CERME2 by Michela Maschietto (2001), even though technically concerning secondary school – is a theme that features strongly in CERME ever after. I note though that both Gueudet and Maschietto had their CERME2 work presented in Working Group 7 (*Metaphors and Images*) and that Maschietto's paper has an explicit focus on the concept of limit. This is a mathematical topic which, to this day, is a flagship topic for much UME research. In CERME3, for example, there are five papers with this focus, with three of the studies carried out in a computational environment. Again, UME research can be found interspersed in five (on my count) Working Groups: 1. Metaphors and images (including embodied cognition); 3. Building structures in mathematical knowledge; 4. Argumentation and proof; 6. Algebraic thinking; 7. Geometrical thinking. Colleagues such as Uri Leron, Ted Eisenberg, Cécile Ouvrier-Buffet contribute investigations that can be seen as closely relevant to those of us doing research in a university mathematics education context. However, these are works pitched beyond the context of the investigations at their heart. Participants are often called "subjects" and it is sometimes several pages into the papers that the reader learns whether these participants are school pupils, university undergraduates or pre-service teachers. This is a particularly evident tendency in the more explicitly psychologically-oriented works in Working Group 3 (*Building structures in mathematical knowledge*) and a little less pronounced in those in the rapidly growing Working Group 4 (*Argumentation and proof*) which had more than a dozen papers in it.

A clear exception to this rule is a paper that was not presented in any of the working groups I listed above: it was presented and discussed in Thematic Group 8 (*Social interactions in mathematical learning situations*) and, to me, it has an incredibly modern, up to date feel to it. It embodies several of the characteristics that were to become more salient in much later CERMEs. The paper is by Andreas Andersson (2003, later Ryve) and it involves observations of engineering students as they interact during mathematical activity. It also deploys the then just-emerging tools from the work of Anna Sfard and her colleagues (e.g. 2002). The tools are used to record patterns in participants' communication (*preoccupational analysis* for social aspects of the communication and *focal analysis* for patterns in the mathematical content of the communication). Both the explicit focus on a group of university students (and actually non-mathematics specialists) and the discursive tools deployed in the data analysis render the paper – retrospectively – a solid foreshadower of things to come, in CERME and elsewhere.

Part II: The AMT years, *CERME 4, 5, 6*

The quality and quantity of work I sampled so far from the first three CERMEs resulted in the recognition by ERME of the increasing significance of research in this area. Group 14 (*Advanced Mathematical Thinking*) was launched in CERME4 in 2005 with Joanna Mamona-Downs, Maria Meehan and John Monaghan as its inaugural leaders and attracted twelve papers.

There is a clear trend emerging from the bulk of these twelve papers: many of these works focus squarely on the students and their habits or preferences in mathematical thinking. The perspective is

largely developmental and dualist. Several papers explore perceived differences between the intuitive and the abstract, the procedural and the conceptual, processes and objects. The prevailing theoretical constructs are Richard Skemp's *instrumental and relational understanding* (1976), Shlomo Vinner and David Tall's *concept image – concept definition* (1981), Eddie Gray and David Tall's *procepts* (1994), APOS theory (Dubinsky, 1991) and Anna Sfard's *theory of reification* and *process – object duality* (1991).

These dualities prevail in the analysis in many of the papers – especially in studies that concern the mathematical topics of Calculus and Analysis, and proof and proving. Matthew Inglis and Adrian Simpson (2005) capture this well in their paper about dual process theory: intuition, formalism/abstraction. Students in these analyses – which have a strong developmental / cognitive flavour – appear frequently not at ease with the latter (formalism) and uncertain about the validity of the former (intuition). But, we are now well into the 2000s and the broader field is moving briskly towards what Steve Lerman (2000) had labelled a "social turn". (A note here: I find myself agreeing more though with the later labelling, by Eva Jablonka and Christer Bergsten (2010), of "social brand", and Lerman's own acknowledgment in the same volume that plurality is not a problem per se in mathematics education.) While attending CERME4, I was also preparing a review (Nardi, 2005) of Carolyn Kieran's, Ellice Forman's and Anna Sfard's 2002 volume *Learning Discourse: discursive approaches to research in mathematics education*. There was a palpable sense in the CERME4 sessions that this extended and accentuated tendency to use developmental/cognitive frameworks, rather than exploring connections between students' learning behaviours and the institutional, pedagogical and curricular context in which these behaviours manifest themselves, was leaving much more to desire from the presented analyses.

The paper by Erhan Bingolbali and John Monaghan (2005) on the impact of departmental settings for engineering and mathematics undergraduates' engagement with the notion of derivative, expressed this desire very well. The paper had a good go at exploring the dialectic between departmental setting, lecturers' teaching and student 'positioning'. Even better was the 2008 ESM paper by these authors, poignantly entitled *Concept image revisited*.

The paper that Paola Iannone and I presented at CERME4 (2005) also expresses, in a rudimentary form, this desire for more substantial exploration of the dialectic relationship between lecturers' and students' ways of communicating mathematically in writing and in speaking. We used the term "genre speech" (Bakhtin, 1986). The paper draws on the larger data pool that three years later became *Amongst Mathematicians* (Nardi, 2008) and has – a little over-ambitiously I admit – a multiple purpose. To explore the "genre speeches" of university mathematics is one. The other one is to bring to the fore an example of a "co-learning partnership" between university mathematics lecturers and mathematics education researchers. I note that "co-learning partnership" is a term that I had become familiar with from the work of my doctoral supervisor and research collaborator Barbara Jaworski (2003), who is also to be credited for introducing me to CERME in the first place! The rapprochement between the communities of university mathematicians and mathematics education researchers became a staple theme in much of the work that I became involved with in the years that followed – and it is one of the defining characteristics of the work that the UME group has showcased and also nurtured. More on this follows later.

Joanna Mamona-Downs continued to lead the AMT group in CERME5 too and the group grew bigger – about 50% bigger! But was it also healthier? I recall vividly the vibrancy of the sessions and also the fact that substantial findings were shared. Two strands made an impression on me at the time: the emerging strand of studies on students' generation of examples, non-examples and counter examples – for example by Maria Meehan (2007) – also emerging out of the then freshly published work in this area by Anne Watson and John Mason (2005). I also recall an emerging focus on studies that explore the easing of the transition from school to university – for example, in terms of the mathematical reasoning required. Matthew Inglis and Adrian Simpson (2007) at the time brought to our attention differences between 'vernacular logic' and 'mathematical logic' and belief biases in reasoning.

Closer to the focus that my work was gearing towards at the time, I also recall Winsløw and Møller Madsen's (2007) adaptation of ATD, the anthropological theory of the didactic, and their examination of the relationship between mathematicians' research activities and their teaching practices. Paola Iannone and I (2007) continued to report analyses from our interview study with university mathematicians: this time we chose to report a slice of our data that concerned the interplay between syntactic and semantic knowledge in proof production (Weber & Alcock, 2004).

With Lara Alcock, and also Matthew Inglis and Rina Zazkis, I was delighted to act as helper to Joanna Mamona-Downs and to observe the many elements of continuity from CERME4 – but also the elements of what I, to this day, see as evidence of healthy controversy. Mamona-Downs (2007), in her synopsis of the group's work captures this well. Here she lists the pertinent questions we were asked to engage with:

> (1) Is the perceived discontinuity between secondary and tertiary mathematics due to institutional and pedagogical practices, or is it caused by factors concerning the character of University Mathematics that demand new habits of behavior in reasoning? (2) What ways are there to ease the transition? (3) If AMT is taken as thinking skills needed for Advanced Mathematics, how are they beyond those required at school? (4) What commonalties or differences in mental processes are there in the two levels? (p.2228)

She then notes that our group discussion was:

> "rather diffused and mostly sidestepped the questions despite their fundamental significance. It was dominated by the view of some that the research field of AMT has largely changed its main focus from cognitive-based studies starting in the early nineteen eighties, to the tendencies found nowadays based more on societal and affect factors that make the long established work 'obsolete'. Others countered strongly this position on the basis of the existence of different scientific 'paradigms', in the sense of Kuhn, and on much of the actual output of recent educational research. Opinions were often put in a partisan spirit. […] A discussion was raised concerning the possibility that some tasks accessible to school students might pose the same kinds of problems in their resolution for undergraduates, and so it could be claimed that these tasks might be considered within the scope of AMT." (p.2228)

No consensus was found possible in the group at CERME5 as this quotation from Mamona-Downs suggests:

> "Several participants declared that the two interpretations are complementary and that there was no compelling reason not to retain the traditional name 'Advanced Mathematical Thinking' as an umbrella term [while there were] a few participants who felt that the themes stated in the program were mostly steered towards cognitive factors." (p.2228-9)

And, I recall, for example, the paper from Corine Castela (2007) offering evidence and taking a clear stance that this persistent focus on cognitive approaches may not be the most inclusive – or fertile – way forward for the group.

This tendency to question whether UME research was appropriately congregating under the AMT umbrella continued in CERME6. The AMT group maintained its size and also, as the group leaders (Roza Leikin, Claire Cazes, Joanna Mamona-Dawns, Paul Vanderlind) observe in their notes on the proceedings (2009), attracted papers firmly focused on the latter of the two ways of interpreting AMT (advanced thinking in mathematics, A-MT or thinking about advanced mathematics, AM-T). As I was reporting a study about prospective and practising teachers' perspectives on proof, I attended the proof group on that occasion. So I missed the wealth of findings in the CERME6 AMT papers on conceptual attainment, approaches to proof and proving, problem solving, instructional approaches and processes of abstraction. It is fair to say though that UME research was gaining even more critical mass with about twenty five papers across six groups!

One of these is Barbara Jaworski's (2009) paper which proposes the exploration of university mathematics teaching practice through a sociocultural perspective that embroiders elements of *Activity Theory* and the *Communities of Practice Theory*. There will be a stream of papers thereafter in CERME with a focus on the practices and perspectives of the university mathematics teacher.

My own work in this period, a part of it also with Barbara Jaworski, illustrates this focus rather emphatically. In a nutshell, I would describe my **r**esearch programme dating from 1990s to the mid-2000s as as shifting from studies of university **mathematics students' learning** of particular mathematical topics (as outlined earlier: Nardi, 1996; 2000) to a progressively growing focus on **university mathematics teachers' perspectives/practices** in mathematics and mathematics teaching (Nardi, Jaworski & Hegedus 2005; Nardi, 2008). These two sets of work illustrate the shift of my focus progressively towards university mathematics teachers' pedagogical and epistemological perspectives. UMTP (University Mathermatics Teaching Project) resulted in the 4-level Spectrum of Pedagogical Awareness (Nardi et al., 2005). *Amongst Mathematicians: Teaching and learning mathematics at University Level* (Nardi, 2008) was published in 2008, following a gestation period of several years that had started also in CERME with the presentations, with Paola Iannone, that I mentioned earlier.

Amongst Mathematicians (Nardi, 2008) tells the story of a co-learning partnership that illustrated research between mathematics educators and mathematicians with these five key characteristics: ***collaborative, mathematically focussed, context-specific, non-prescriptive and non-deficit*** as possible. In addition to reporting university mathematicians' pedagogical and epistemological perspectives, the book served a broader purpose too. It is written in the rather unconventional format of a dialogue between two fictional, yet data grounded characters – M, mathematician, and RME, researcher in mathematics education – and is intended as reflection on the perceived benefits,

obstacles and desires of the relationship between the two. Such conversations were of course not new. For example, Anna Sfard (1998) reported her discussion with Shimshon A. Amitsur, in the form of a dialogue and a range of authors from a variety of national and institutional contexts, including Michèle Artigue and Gerry Goldin, were writing at the time about this relationship. A common observation in these accounts was about its fragility. Research which consolidates and propels the rapprochement between the communities of mathematicians and mathematics educators remains a focus of my work today (e.g. Nardi, 2016) and it is fair to say that CERME, in the mid-2000s provided one of the first fora for kickstarting this work.

Let me conclude my reflections on what I labelled as "the AMT" years with a brief reference to a set of works that somehow foreshadow developments within the UME community in CERME: in the Modelling TWG, Berta Barquero, Marianna Bosch and Josep Gascón (2009) offered an ATD account of the institutional constraints hampering the teaching of mathematical modelling at university level. They coin the term "applicationism", an epistemological perspective which proposes a strict separation between mathematics and other disciplines (especially the natural sciences) and sees mathematical tools as built to be applied to solve problems in other disciplines – with this application not causing any change in the discipline of mathematics or for the discipline in which the application is made. As UME research is rapidly growing in the area of teaching mathematics to non-mathematicians, works such as this, in CERME6 and earlier, now acquire added significance.

Part III: The UME years, *CERME 7, 8, 9*

The proposal to the ERME board for the launch of *TWG14: University Mathematics Education* was born out of two main sources. First was my reading and writing at the time: While writing *Amongst Mathematicians,* my search across the literature was broad. In fact, as Michèle Artigue (2016) has noted in her INDRUM2016 plenary, there is a synthesis feel to the book. A more explicit, deliberate synthesis of hitherto developments in research into the teaching and learning of university mathematics that was the chapter that Artigue (Artigue, Batanero & Kent, 2007) co-authored with Carmen Batanero and Philip Kent for the second NCTM Handbook. Secondly, at PME, in Morelia (Nardi & Iannone, 2008) and in Thessaloniki (Nardi et al,, 2009) , two Working Sessions / Discussion Groups that I had co-ordinated with colleagues many of whom ended up co-leading the UME TWG in CERME, had attracted many colleagues and had generated vital, urgent discussions.

I recall that this sensation of vibrancy and urgency was not universally shared outside the bubble of researchers in this area. I recall that when we proposed the launch of the group, we were gently reminded by members of the board that we would need to attract at least eight papers to make the new group viable! I recall that we – the inaugural co-leaders of TWG14 – were nudging each other that, if each one of us submitted a paper, we would only need to find three more to be able to launch the group! We were of course wrong.

I need to make two brief notes at this juncture: first, that the account of the group's work since 2011 borrows heavily from the collectively authored texts in the CERME7, 8 and 9 proceedings (Nardi et al., 2011; 2013; 2015); second, that, given the volume of work presented at these conferences, I will

from now on stay largely away from extensive exemplification from specific papers. I will instead focus on the themes that mark the "emancipation" and "enrichment" themes promised in the title.

Our rationale for a UME TWG ((Nardi et al., 2011) was in a nutshell as follows.

Research on university level mathematics education is a relatively young field, which embraces an increasingly wider range of theoretical approaches (e.g. cognitive/developmental, socio-cultural, anthropological and discursive) and methods/methodologies (e.g. quantitative, qualitative and narrative). Variation also characterises research in this area with regard to at least two further issues:

- the *role of the participants*, students and university teachers, in the research – from 'just' subjects of the research to fully-fledged co-researchers; and,
- the *degree of intervention* involved in the research – from external, non-interventionist research, to developmental/action research in which researchers identify problems and devise, implement and evaluate reforms of practice (Artigue et al, 2007).

2011 marked the 20th anniversary of the publication of *Advanced Mathematical Thinking* edited by David Tall (1991). This is a volume that is often heralded as a first signal of the emergence of this new area of research. A few years later, a second signal was given by the 1998 ICMI study that resulted in *The teaching and learning of mathematics at university level*, edited by Derek Holton (2001). In the meantime, Advanced Mathematical Thinking (AMT) groups ran both in previous CERME and PME conferences; sessions exclusively on university mathematics education have been part of the EMF (*'Espace Mathématique Francophone*) conferences since 2006; the RUME, UMT and Delta conferences emerged in the USA, the UK and South Africa respectively; the International Conferences on the Teaching of Mathematics at University Level were launched in 1998; etc. The UME TWG emerged out of the above developments and out of the realisation that this is a distinct area of mathematics education research.

The distinctiveness of UME research can be attributed to several characteristics.

Firstly, the classic distinction between 'teacher' and 'researcher' does not always apply in UME as researchers in mathematics education in this area are often university-level teachers of mathematics themselves. In particular, there is a growing group of mathematicians specializing in research on mathematics education at university level, where expertise and experience in advanced mathematics is really an asset (if not a necessity). Secondly, mathematics education theories and research methods find new uses, and adaptations, at the university level. These adaptations are often quite radical as the post-compulsory educational context is different in many ways – including the voluntary presence of students, the important role of mathematics as a service subject, the predominance of lecturing to large numbers of students, the absence of national programmes for university education, the required shift to the distinctly different practices of university mathematics, to mention but a few. In this sense, UME is a distinct area of mathematics education research, not merely a mirror of mathematics education research at a more advanced educational level. Finally, in recent years, research in this area has been growing in different parts of the world. TWG14 is one forum where evidence of this growing research activity from Europe and beyond has been accumulating.

Across CERME7, 8 and 9, the WG14 Calls for Papers invited contributions from as wide a range of research topics as possible. Here is, for example, the list from CERME9: the teaching and learning of advanced topics; mathematical reasoning and proof; transition issues "at the entrance" to university mathematics, or beyond; challenges for, and novel approaches to, teaching (including the teaching of students in non-mathematics degrees); the role of ICT tools (e.g. CAS) and other resources (e.g. textbooks, books and other materials); assessment; the preparation and training of university mathematics teachers; collaborative research between university mathematics teachers and researchers in mathematics education; and, theoretical approaches to UME research.

We opted for widening participation as much as possible, both in terms of the substantive, methodological and theoretical takes of the proposed papers but also in terms of the disciplinary background and experience of the proposers. The 21, 29 and 45 (31 long 14 short) papers accepted for publication in the respective proceedings met those terms.

Across the WG14 discussions, certain themes and questions emerged as crucial. These included: exploring whether UME needs to generate new theories or adapt already existing ones; attending to issues of both theory and practice; acknowledging that research on teaching and learning in higher education develops also outside mathematics education, and benefiting from these developments; working towards the generation of new theories while valuing already accumulated knowledge in the field; etc. One oft-repeated observation was that, beyond staple references to classic constructs from the AMT years, several works presented in TWG14 employ (often in tandem with the above) approaches such as the *Anthropological Theory of the Didactic* (Chevallard, 1999) and discursive approaches, such as Anna Sfard's (2008) theory of *commognition*.

In CERME7 (Nardi, et al., 2011), we noted that an area of growth has certainly been studies that examine the different role of mathematics in courses towards a mathematics degree, courses for pre-service teachers, as a 'service' subject (physics, biology, economics etc.). While a substantial number of papers remains in the increasingly well-trodden area of students' perceptions of specific mathematical concepts (again calculus prevails in these), a focus on university teachers and teaching is also emerging, if often a little timidly, and diplomatically, resulting in descriptive, openly non-judgemental studies. In conjunction with those studies, a genre of collaborative studies, with mathematicians engaged as co-researchers, also seems to be on the rise. We signal the emerging trends in the CERME7 papers as: *Transitions*; *Affect*; *Teacher practices*; *Mathematical topics*.

In CERME8 (Nardi et al., 2013), we noted the appearance of new mathematical topics: infinite series and abstract algebra. We also noted that some of these papers are written by research mathematicians, using a mathematical, epistemological, or historical analysis, and drawing on their teaching experience. Others present research that makes use of different theoretical frameworks, and methodological tools, to analyse students' difficulties with these specific topics, to better understand the teaching of a specific topic and the consequences of this teaching, or to formulate propositions for the design of teaching to overcome these difficulties. The range of approaches vary from developmental ones (such as *concept image – concept definition*), to models for abstraction (such as

the *RBC model*), to analysis of discourse (*theory of commognition)* and the consideration of institutional matters (*anthropological theory of the didactic*)[3].

After CERME8, the team – in collaboration with TWG14 participants and others – worked towards a *Research in Mathematics Education* Special Issue on *Institutional, sociocultural and discursive approaches to research in university mathematics education* which focused on research that is conducted in the spirit of the following theoretical frameworks: *Anthropological Theory of the Didactic*, *Theory of Didactic Situations*, *Instrumental and Documentational Approaches*, *Communities of Practice and Inquiry* and *Theory of Commognition*. As we noted in the Editorial of the RME Special Issue (Nardi et al., 2014), there is a clear surge of sociocultural and discursive approaches – and the number of papers using ATD and TDS is also remarkable. An emerging focus seems to be also on systematic investigations of innovative course design and implementation and there is certainly a rise in the number of studies that examine the teaching and learning of mathematics in the context of disciplines other than mathematics, such as engineering and economics. Furthermore, this time we welcomed more colleagues from outside Europe and also noted the rise in the number of papers on assessment and examination[4].

In CERME9 (Nardi et al., 2015), there was a notable shift in terms of numbers of papers (two to one) in favour of the second of our two umbrella themes: *Teaching and learning of specific topics in university mathematics; Teachers' and students' practices at university level*. The breadth of topics covered especially in the latter is also telling: curriculum and assessment; innovative course design in UME; student approaches to study; relating research mathematicians' practices to student practices; views and practices of mathematics lecturers; and, methodological and theoretical contributions to UME research.

In CERME9 we also observed the further strengthening, maturity and increasingly more robust theorizing of studies into teaching practices. And, we also noticed in several papers the establishing of promising liaisons across different theoretical perspectives such as a discursive take on *mathematical knowledge for teaching* or an anthropological take on *documentational approaches*.

The critical – and growing – mass and quality of the work presented at CERME9 TWG14 led to the launch of an ERME Topic Conference, INDRUM2016, a conference of the newly established *International Network for Didactic Research in University Mathematics* (Montpellier, March 31 – April 2, 2016)[5]. The conference attracted more than 80 submissions and more than 100 participants. INDRUM2018 is currently in preparation.

[3] By the way, we closed our CERME8 text for the proceedings with a *Concluding note on rigour and quality* of UME research. While there is no space here to elaborate, I invite the reader to what I see as pertinent observations from the TWG14 team about these issues in CERME at large.

[4] In CERME10 there is a new TWG on assessment that spans across educational levels led by former TWG14 co-leader Paola Iannone.

[5] I chaired this conference with the tireless Carl Winsløw. Its launch and its 2016 success (Montpellier, France) relied heavily on the sterling work of ERME president Viviane Durand-Guerrier and the commitment of Thomas Hausberger.

Part IV: CERME10, the split…

There were 47 UME papers and 16 UME posters accepted for presentation and discussion in CERME10. Their presentation and discussion was in two isomorphic groups: TWG14A and TWG14B. From CERME11, it is expected that papers may be invited for two, also thematically distinct, groups – and the debate on possible configurations for this dominated some of the discussions at the conference. One way forward that I personally favour is for a grouping by the following distinction: studies that concern the transition to university studies of mathematics and the transition from university studies into the (various forms of) workplace; and, studies that concern the teaching and learning of mathematics while at university. The challenge of debating the numerous configurations of how the (new) group(s) can be (re)defined is certainly non-negligible. Isn't this a most wonderful place to find ourselves though, having to manage the now critical mass and quality of UME research present in CERME?

Part V: Taking stock / What next / Coming soon…

As I am drawing to a close, I would like to ask the question: what did we want to achieve with the establishment of TWG14? Have we achieved these objectives? Are we going to? For example: did we manage to encourage fledgling topics in UME research? Have we planted the seed for new ones?

In the sprawling vista of works that I aimed to sample in this lecture – and I am fully aware of the wafer thin way in which I have done so – I have aimed to identify trends in UME research (overall, in CERME, in my own work) that signify the benefits (the **richness!)** of opening up, of widening our substantive, theoretical and methodological horizons (the what, the how and the why of our research). Most of my examples have aimed to illustrate the benefits that **emancipation** from an individualistic, narrowly psychological, cognitive perspective has brought to UME research.

There are still though foci that have not yet merited our sufficient attention. One such research focus that seems to me to be not within the radar of current works is UME research is on more advanced topics in mathematics – and by that, I mean mathematics that is typically taught beyond the first two years of university studies.

On a less deficit tone, I am generally satisfied that we have come a long way but I also acknowledge that there is an even longer way to go. It is fair to say that, within the various UME communities around the world, we have gone (or are still going) through what I would like to label as a dismissive phase: that all so-called traditional pedagogies are "bad", lecturing in particular. I am observing – but I am also asking that we do so even better – that we become more nuanced and embracing of possibility. We are starting, for example, to recognise that lecturing can serve some purposes rather well; that it can be complemented by formats more tailored to the serving of students' individual needs; that there are interactive lecture formats that give participants the buzz of community belonging and building and prepare students for the less cocooned, less protected world of work where interaction, team work and communication are key. We are finding out that not all interaction and all the time is good per se and that there are particular types of communal engagement with mathematics that work better than others. TWG14 papers have been offering the evidence base for these claims, steadily and cumulatively. In a way, I find the choice made by the

mathematics department in my institution (see earlier snip in Figure 2) to include in its promotional materials images of lectures and also to close its promotional video (*https://www.youtube.com/watch?v=gRzVX8c1be4*) with a close-up of white chalk on a blackboard (Figure 4) somewhat refreshing. We are perhaps starting after all to embrace diversity in the ways that the students need to experience mathematics!

I believe the answers to the questions with which I started this section are reservedly optimistic and affirmative. In Part II, I showed an outline of my own research programme over the years and I am pleased to be able to say that most of the items there – and what followed these – have emerged out of collaborations with colleagues in CERME, including research plans for the immediate future.

CERME has indeed been a platform where I am trialling new topics for research. My CERME8 paper (Nardi, 2013) offers analyses of the challenges of teaching a graduate course on mathematics education to students with a variety of backgrounds, including bachelor degrees in pure mathematics, and native languages other than the language of instruction. The paper also outlines key didactic techniques and principles to cope with these challenges. It finally morphed into the more substantial analyses present in a paper included in the inaugural issue of IJRUME (Nardi, 2015) which examined ways to facilitating paradigm shifts in the supervision of mathematics graduates upon entry into mathematics education.

CERME has also been a platform where I have trialled new approaches to analysing data. In fact, I credit CERME for allowing me the creative space to have a go – and converse about – discursive, particularly commognitive, approaches to the analyses of my data. My CERME7 paper (Nardi, 2011) outlined interviewed mathematicians' perspectives on their newly arriving students' verbalisation skills; and, observed that discourse on verbalisation in mathematics tends to be risk-averse and not as explicit in teaching as necessary. At CERME9, Bill Barton and I (Nardi & Barton, 2015) presented a *commognitive* analysis of a "low lecture" episode (student-led inquiry oriented discussion on open-ended problems) to illustrate crucial steps of student enculturation into mathematical ways of acting and communicating, including a shift away from the lecturer's 'ultimate substantiator' role. Finally, both the papers I am involved in as co-author in CERME10 (Virman & Nardi, 2017; Thoma & Nardi, 2017) present commognitive analyses in contexts that said analyses are now just about starting to appear (teaching mathematics to non-mathematicians; analyses of closed-book examination tasks and student/lecturers' assessment discourses).

Returning to the anecdote that I started with, a somewhat self-deprecating recollection of the theoretical ambivalence of my doctoral work, I see my own research programme as an illustration of the **richness emanating from the emancipation**, from what I now see as a narrow, individualistic perspective in my earlier work. To me there is nothing vacantly rhetorical about the three Cs in the CERME spirit: COMMUNICATION, COOPERATION, COLLABORATION. The growth of my research programme through each one of these is to me unshakeable evidence of the pragmatic strength of these three words. In TWG14 these words have taken shape as specific actions. Here are two: (1) Certainly, we have assisted with the arrival of several new researchers in this field, some of whom are currently co-leaders; many have used the reviewing process as a stepping stone for their writing (from poster to conference paper then to completing theses and journal papers). (2) We have engaged practitioners of university mathematics teaching who now see themselves also as UME

researchers. To do so, we deploy the reviewing process and the discussions at the conference to convey the rigour that is required for UME research (in terms of engaging with theory, prior research and methodology) and to bridge the epistemological differences between the academic disciplines of mathematics and education.

I invite the reader to the collections of papers published in the TWG14 sections of the Proceedings, the 2014 *Research in Mathematics Education* Special Issue that followed CERME8, the proceedings of the 2016 *INDRUM* conference and the imminent (publication expected in 2018) *International Journal for Research in Undergraduate Mathematics Education* Special Issue that is following INDRUM2016 as testimonials of the growth and diversity I have tried to map here. And there is more to come: INDRUM2018 will be hosted by MatRIC at the University of Agder (Kristiansand, Norway) in April 2018 and its Scientific Committee aims to follow it up with a state-of-the-art volume soon after. And, of course, there is the UME chapter (Winsløw et al., in press) in the ERME 20th Anniversary Book that we aim to celebrate in CERME11, in 2019. The promise of UME research on the global scene is further corroborated by the healthy growth of the RUME and DELTA conferences, and the respective group within EMF. In closing, I return to the words of Michèle Artigue whose thoughtful INDRUM2016 plenary (Artigue, 2016) triggered the focus of the synthesis and analysis presented here:

> "The emergence of the [UME] field was […] characterized by the domination of cognitive and constructivist perspectives. I consider as a strength of our field the fact that we have succeeded in emancipating ourselves from these perspectives, whose limitations are evident, but also the fact evidenced by the consideration of most research publications, that this emancipation went along a reconstruction of their main outcomes, thus making possible some form of incorporation of these outcomes in the new paradigms."
>
> Michèle Artigue, from *Mathematics education research at university level: Achievements and challenges*, INDRUM2016 plenary lecture (p.19)

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Young children's early mathematical competencies: Analysis and stimulation

Lieven Verschaffel, Joke Torbeyns, and Bert De Smedt
University of Leuven, Belgium, lieven.verschaffel@kuleuven.be

In this paper we take a critical look at the state-of-the-art in the research domain of early mathematical development and education. We start with a brief review of the influential and successful (neuro)cognitive research in this domain - which is heavily focused on the development and teaching of children's (non-symbolic and symbolic) magnitude representation and strongly dominated by the theory of an approximate number system (ANS). We confront and complement this (neuro)cognitive approach with various other lines of research that may help to provide a more comprehensive picture of the development and stimulation of children's early mathematical competence and how it relates to their later mathematical proficiency at school.

Keywords: Early mathematics, approximate number system, number concepts, mathematical patterns and structures, preschool education.

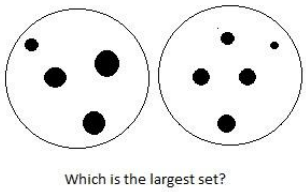
Introduction

The past 10-15 years have witnessed the emergence of a remarkably productive and highly influential line of research on children's early numerical magnitude representation, its development, its relation to school mathematics, and its assessment and stimulation (Torbeyns, Gilmore & Verschaffel,, 2015).

The starting point of this line of research - which has its origins in cognitive (neuro)psychology -, is the idea that young children, like many other species, are equipped with some foundational innate core systems to process quantities. This "starter's kit" is thought to involve (a) an "object tracking system" that has a limit of three or four objects and is thought to underlie "subitizing" (= to immediate and accurate estimate of one to four objects without serial enumeration), and (b) an "analogue number system" – for the internal representation of numerical magnitudes as Gaussian distributions of activation on a "mental number line" with increasingly imprecise representations for increasing magnitudes (Dehaene, 2011) - allowing them to compare non-symbolic quantities that are too numerous to enumerate exactly or to perform some very basic approximate arithmetic on these quantities (Andrews & Sayers, 2015; Butterworth, 2015).

With these foundational core number sense systems, these magnitudes are represented non-verbally and non-symbolically, but, over development and through early (mathematics) education, verbal and symbolic representations are gradually mapped onto these foundational representations, to evolve into a more elaborated system for number sense (Torbeyns et al., 2015).

People's numerical magnitude representations are commonly assessed via magnitude comparison and/or number line estimation tasks, of which there exist both non-symbolic and symbolic versions (Butterworth, 2015; Andrews & Sayers, 2015; Torbeyns et al., 2015). Examples are shown in Figure 1.



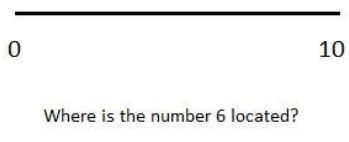


Figure 1: Example of a non-symbolic magnitude comparison and a symbolic number line estimation task

During the past decade, several research teams have set up correlational, cross-sectional and longitudinal studies to determine the contribution of children's numerical magnitude understanding - sometimes in combination with other specific early numerical competencies (such as subitizing, counting or numeral recognition) - to their concurrent and/or later overall mathematical achievement or to specific parts of it such as mental arithmetic or algebra (see, e.g., Bailey, Geary, & Siegler, 2014; De Smedt, Verschaffel, & Ghesquière, 2009; Jordan, Glutting, & Ramineni, 2010; Nguyen, Watts, Duncan, Clements, et al., 2016; Reeve, Reynolds, Humberstone, & Butterworth, 2012). These studies have demonstrated that children's numerical magnitude understanding is positively related to their concurrent and future mathematics achievement in general or in particular subdomains of mathematics.

Two recent meta-analyses have yielded a good overview of the outcomes of this research on the association between various measures of children's numerical magnitude understanding and their concurrent and future mathematics achievement. Schneider, Beeres, Coban, Merz, et al. (2017) performed a meta-analysis on the research about the association between performance on the magnitude comparison task and measures of mathematical competence. Their literature search yielded 45 articles reporting 284 effect sizes found with 17,201 participants. The results support the view that magnitude processing is reliably associated with mathematical competence as measured at least up to the end of the elementary-school years and by a wide range of mathematical tasks, measures and subdomains. Furthermore, the effect size was significantly higher for the symbolic than for the non-symbolic magnitude comparison task and decreased very slightly with age. So - the authors conclude - symbolic magnitude processing might be a more eligible candidate than non-symbolic magnitude processing to be targeted by diagnostic screening instruments and interventions for school-aged children and for adults. The association was also higher for mathematical competences that rely more heavily on the processing of magnitudes (i.e., early mathematical abilities and mental arithmetic) than for others (i.e., more general curriculum-based tests).

Schneider, Merz, Stricke, De Smedt, et al. (submitted) performed a similar meta-analysis for the association between people's score on the other main task to assess numerical magnitude processing skills, namely the number line estimation task, and mathematical competence. Using exactly the

same analytic procedure, and working with a set of 37 studies, they found that the correlations with mathematic competence - both in general and for particular parts of the curriculum - were significantly higher for number line estimation than for symbolic magnitude comparison or for non-symbolic magnitude comparison. Whereas the correlations did not substantially increase with age for comparison, an increase with age was found for number line estimation, which suggests that different underlying cognitive systems and processes are involved in magnitude comparison vs. number line estimation.

Furthermore, researchers working within this research tradition have tried to stimulate children's mathematical skills with (game-based) intervention programs that were (primarily or exclusively) aimed at enhancing their numerical magnitude understanding before or at the beginning of formal instruction in number and arithmetic in elementary school. While some intervention studies have resulted in positive effects (e.g., Kucian, Grond, Rotzer, Henzi, et al., 2011; Ramani & Siegler, 2011; Wilson, Dehaene, Dubois, & Fayol, 2009), the overall results are mixed (Torbeyns et al., 2015).

Being well aware of the prominence of this line of research in the international research of early mathematics education, the IPC of the 23rd ICMI study on "Whole number arithmetic" invited one of the leading scholars in that line of research, namely Brian Butterworth, as a plenary speaker of the conference, which took place in June 2015 in Macau, China. In his plenary lecture Butterworth (2015) presented a very informative overview of this cognitive (neuro)scientific line research, and strongly defended this research in the working groups and panels wherein he participated. However, at that conference, it also became clear that the dominant picture of early mathematical competences and education in current mainstream (neuro)cognitive research is dangerously narrow. In the present paper, we will try to broaden that picture in multiple ways. In doing so, we will partly rely on recent and current work done in our own research group, but also on the work of many colleagues who have been active in the field of early mathematics education during the past decade(s).

The ordinal and measurement aspect of number

A first important feature of the line of research summarized above is its focus on the cardinal aspect of number, or, to state it differently, its neglect of other constituent aspects of number, particularly its (1) ordinal and (2) measurement aspect. Hereafter we discuss these two neglected aspects.

The distinction between the *ordinal* and cardinal aspect of number knowledge is well known. Whereas cardinality refers to the capacity to link number symbols to collections, e.g., to know that four or 4 is the correct representation to denote a group of four objects, ordinality refers to the capacity to place number words and numerals in sequence; for example, to know that 4 comes before 5 and after 3 in the sequence of natural numbers. Given the wide recognition of the importance of ordinality for the constitution of number since Piaget (1952) developed his theory of children's concept of number, it is remarkable that, until recently, the ordinality aspect of number seems largely neglected in the above mainstream cognitive neuroscientific conceptualization, assessment and instruction of early numerical abilities.

Interestingly, recent neuroscientific evidence shows that accessing ordinal information from numerical symbols (e.g., decide whether three numbers are in order of size) relies on a different

network of brain regions and shows qualitatively different behavioral patterns when compared to the cardinal processing of magnitudes or numerical symbols or to the ordinal processing of perceptual magnitudes (Lyons & Beilock, 2011, 2013). And, how well a child is able to reason about ordinal relations between number symbols has been found to be one of the strongest predictors of mathematical skill such as mental arithmetic (Lyons, Price, Vaessen, Blomert, & Ansari, 2014) – much stronger, by the end of the first grade of elementary school, than non-symbolic or symbolic cardinal processing as measured by the numerical magnitude comparison task. So, the idea that emerges from this recent neuroscientific research is that children's sense of ordinality of number symbols may be distinct from their sense of cardinality and, in terms of developing skills needed for success in mathematics, that ordinality may even be the more significant one (Sinclair & Coles, 2015, see also Vogel, Remark, & Ansari, 2015).

This line of research pointing to the importance of ordinality also led math educators to criticize the mainstream neuroscientific view on how children's early number sense may be stimulated. The latter view suggests that working on linking symbols to sets of objects may reinforce the very way of thinking that young children need to overcome to become successful in school mathematics. But this is the current practice in many countries, where the emphasis in early mathematics education is firmly on linking number symbols to collections of objects - whether this is done through subitizing or counting. Based on the above theoretical and empirical arguments, Sinclair and Coles (2015, p. 253) asserted that this emphasis on cardinal awareness in learning number is misplaced and argued that what young children above all need is "support to work with symbols in their relationship to other symbols". This plea for paying more attention to the importance of ordinality has led these authors to the design of an innovative iPad app, *TouchCounts* (Sinclair & Jackiw, 2011) wherein the way numerosities are built, labeled and manipulated does not primarily require sense of cardinality but rather ordinality.

The cardinal emphasis on number knowledge has also been attacked from another, more radical, perspective. In his plenary address at the ICMI23 conference, Bass (2015) described an approach to developing concepts of number using the notion of quantity *measurement.* This approach is not new, of course, and is quite well-known among mathematics educators (see e.g., Brousseau, Brousseau, & Warfield, 2004), even though it has, to the best of our knowledge, hardly led to actual and wide-scale implementation in national curricula.

It has been articulated most prominently by Davydov (1990), a Soviet psychologist and educator, who developed, together with his colleagues, in the 1960s and 1970s, a curriculum for number and arithmetic based on this measurement approach. This curriculum delayed the introduction of number instruction until late in the first grade. Early lessons rather concentrated on "pre-numerical" material: properties of objects such as color, shape, and size, and then quantities such as length, volume, area, mass, and amount of discrete objects, but without yet using number to enumerate "how many". So, in this approach number is not intrinsically attached to a quantity; rather it arises from measuring one quantity by another, taken to be the "unit:" How "much" (or many) of the unit is needed to constitute the given quantity?

The discrete (counting) context in which whole numbers are typically developed in most approaches to early and elementary mathematics education is characterized by the use of the single-object set as

the unit, so that the very concept of the unit, and its possible variability, is rarely subject to conscious consideration. According to Bass (2015, p. 11), "this choice is so natural, and often taken for granted, that the concept of a chosen unit of measurement need not enter explicit discussion. If number is first developed exclusively in this discrete context, then fractions, introduced later, might appear to be, conceptually, a new and more complex species of number quite separate from whole numbers. This might make it difficult to see how the two kinds of numbers eventually coherently inhabit the same real number line. Indeed, this integration entails seeing the placement of whole numbers on the number line from the point of view (not of discrete counting, but) of continuous linear measure." (see also Behr, Harel, Post, & Lesh, 1992, for a similar argument coming from the research literature on rational numbers).

According to Bass (2015), this measurement approach has a lot of advantages over the counting based approach, especially if one takes a broader long-time mathematics educational perspective. First, it is a way of providing coherent connections in the development of rational numbers. A second advantage is that it makes the geometric number line continuum present from the start of the school curriculum as a useful mathematical object and concept. Third, the approach provides opportunities for some early algebraic thinking.

The above analysis suggests that it is important to balance cardinal, ordinal and measurement aspects of number in early mathematics education. This requires some serious reflection on the ingrained ways in which cardinality is now privileged in early mathematics education as well as further creative explorations of how the two other elements of ordinality and measurement can be mobilized to promote the development of a broad and balanced number concept.

Arithmetic reasoning skills

It is apparent that the mainstream analysis of early mathematics-related competences has capitalized on measures that emphasize children's *numerical* competences, i.e., their subitizing skills, counting skills, the ability to compare numerical magnitudes, and the ability to position numerical magnitudes on an empty number line. While such measures provided empirical evidence for the multi-componential nature and importance of young children's early numerical competences for future mathematical development, they reflect also in another way a restricted view on children's early mathematical competences.

Starting from Piaget's (1952) logical operations framework, there is a recent renewed research attention to children's early arithmetic reasoning skills, such as their understanding of the additive composition of number or their additive and multiplicative reasoning skills, as well as to their importance for later mathematical learning at school (e.g., Clements & Sarama, 2011; Nunes, Bryant, Barros, & Sylva, 2012; Robinson, 2016).

As documented in her extensive review of this research, Robinson (2016) points out that the research on children's conceptual understanding of these arithmetic concepts is heavily focused on additive concepts, that is, concepts involving the operations of addition and/or subtraction. Various principles including the additive composition of number but also the arithmetical properties such as the commutativity, the associativity, the addition-subtraction inverse, and the addition-subtraction complement principle have been intensively studied, sometimes also in relation to children's actual

use of these principles in their mental arithmetic (Baroody, Torbeyns, & Verschaffel, 2009; Verschaffel, Bryant, & Torbeyns, 2012). Quite a number of these studies already involve young children at or even before the age of 6-7 years old.

Similar multiplication and division principles have also been investigated, however, to a much lesser extent and with a more restricted developmental range, from late middle childhood to adulthood (Larsson, 2016; Robinson, 2016), which is not surprising given that, for most children, these operations are typically not yet formally introduced in the first grades of elementary school.

Only a few of these studies have explicitly addressed the question of how young children's emergent understanding of these additive and multiplicative principles is predictively related to their (later) achievement in school mathematics, in similar ways as has been done for the numerical aspects of early mathematical competence reviewed in the previous section. The limited available evidence from these few studies suggests that early mathematical reasoning of this sort makes a separate and specific contribution to achievement in school mathematics, even up to several years later (Nunes, Bryant, Evans, Bell, et al., 2007; Nunes et al., 2012).

As an illustration, we refer to the study of Nunes et al. (2012), which used data collected in the context of the Avon Longitudinal Study of Parents and Children (ALSPAC) involving about 4000 pupils, to assess whether arithmetic reasoning makes an independent contribution, besides calculation skills, to the longitudinal prediction of mathematical achievement over five years. Arithmetic reasoning was assessed at the start of children's elementary education (i.e., at 7 years) using a test that included three types of items: additive reasoning about quantities, additive reasoning about relations, and multiplicative reasoning items (see examples in Figure 2).

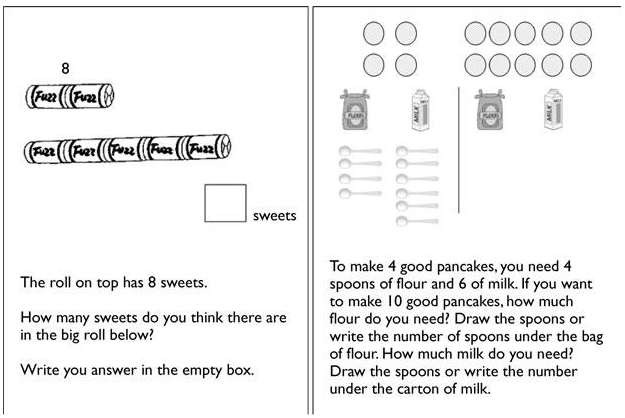


Figure 2: Examples of items from Nunes et al.'s (2012) arithmetic reasoning test

The outcome measures of mathematical achievement were standardized assessments designed to measure school standards by the end of elementary school. Hierarchical regression analyses were used to assess the independence and specificity of the contribution of arithmetic reasoning vs. arithmetic skill to the prediction of achievement in mathematics, science, and English at the end of elementary school, using age, intelligence, and working memory as controls in these analyses. Arithmetic reasoning and skill made independent contributions to the prediction of mathematical achievement, but arithmetic reasoning was by far the stronger predictor of the two. These

predictions were also specific, in so far that these measures were more strongly related to mathematics than to science or English.

In sum, according to Nunes et al. (2012), their findings provide a clear justification for making a distinction between arithmetic reasoning and numerical, counting and calculation skills. The implication for diagnosis and intervention in early mathematics education is that arithmetic reasoning should receive a greater emphasis from the early years in primary school on.

Understanding patterns and structures

In another attempt to identify and explain common underlying early bases of mathematical development and its stimulation, other researchers have looked at mathematical patterns and structures (Lüken, 2012; Mulligan & Mitchelmore, 2009; Mulligan, Mitchelmore, & Stephanou, 2015; Rittle-Johnson, Fyfe, Loehr, & Miller, 2015)

In what can be considered as one of the most enduring, systematic and influential research programs in this respect, based on a series of related studies with diverse samples of 4- to 8-year-olds, Mulligan and colleagues have identified and described a new construct, Awareness of Mathematical Pattern and Structure (AMPS) (Mulligan & Mitchelmore, 2009; Mulligan et al. 2015), that has been shown to be related to children's later mathematics achievement in school.

Mathematical pattern involves any predictable regularity involving number, space, or measure such as number sequences and geometrical patterns, whereas structure refers to the way in which the various elements are organized and related, such as iterating a single 'unit of repeat' (Mulligan & Mitchelmore, 2009). AMPS involves structural thinking based on recognizing similarities, differences and relationships, and also a deep awareness of how relationships and structures are connected.

An interview-based assessment instrument was developed and validated, the Pattern and Structure Assessment - Early Mathematics (PASA) (Mulligan et al., 2015). The PASA yields an overall AMPS score as well as scores on five individual structures: sequences, shape and alignment, equal spacing, structured counting, and partitioning. Some examples of tasks are sequences that have to be extended (e.g., a sequence of colored pearls on a string or a series of triangular dot configurations of increasing size) or structured counting tasks (e.g., counting by two's, counting the number of cells in a partly covered rectangular pattern). Based on the child's response, which may include drawn representations and verbal explanations of patterns and relationships, five broad levels of structural development were identified and described: pre-structural, emergent, partial, structural, and advanced structural (Mulligan & Mitchelmore, 2015). Validation studies indicated that high levels of AMPS were correlated with high performance on standardized achievement tests in mathematics with young students (Mulligan et al., 2015).

In alignment with the assessment of AMPS, an innovative, challenging alternative learning program, the Pattern and Structure Mathematics Awareness Program (PASMAP) was developed and evaluated longitudinally in the kindergarten (= the first year of formal schooling in Australia). This study first showed that kindergartners are capable of representing, symbolizing and generalizing mathematical patterns and relationships, albeit at an emergent level (Mulligan,

Mitchelmore, English, & Crevensten, 2013). The study also tracked and described children's individual profiles of mathematical development and these analyses showed that core underlying mathematical concepts are based on AMPS, and that some children develop these more readily and in more complex ways than others. Finally, this study also involved an attempt to provide an empirical evaluation part involving 316 kindergartners from two schools with and two schools without the PASMAP program. Highly significant differences on PASA scores were found for PASMAP children in comparison to children from the control schools, also for those children labeled as low ability, both at the posttest and the retention test, when children had already moved to Grade 1. On the other hand, there was no significant impact of PASMAP on improving children's mathematical achievement as measured by a general mathematics achievement test.

Other researchers have also performed analyses of (1) elementary school children's perceptions and understandings of patterns and structures, providing nice descriptions and accounts of young children's abilities and difficulties with respect to various mathematical patterns and structures tasks, (2) the predictive value of their mastery of pattern and structure for their later mathematical, i.e. algebraic proficiency, and (3) how instruction on patterns and structures can not only transfer to similar and other patterns and structures, but also to other mathematical domains such as ratios, and mathematics achievement in general (for an overview, see Rittle-Johnson, Fyfe, Loehr, & Miller, 2015).

Of course, the idea that patterns and structures play an important role in the learning of mathematics, and should play an important role in its teaching, is not new (Orton, 1999). After all, is the definition of "mathematics as a science of patterns" (Müller, Selter, & Wittmann, 2012) not one of our favorite definitions of what mathematics is all about? The critically new element in the research of the work of Mulligan and associates is that they give it such a prominent role in their diagnostic and teaching materials for early mathematics. In doing so, they contribute to broadening the picture of what (early) mathematics is all about – a picture that is largely undervalued in current early and elementary school mathematics with its strong focus on learning about numbers and arithmetic facts and procedures.

Spontaneous focusing tendencies

The studies and views on the early development of children's mathematical competence reviewed so far typically take a purely "ability" perspective. In doing so, they neglect other aspects of young children's early mathematical competence, such as their attention to or feeling for, numerical magnitudes, mathematical relations, or mathematical patterns and structures. During the past decade, researchers have started to explore children's spontaneous tendency to focus on numerosity (SFON), its development, its cultivation, and its predictive relation to children's later mathematical achievement (Hannula & Lehtinen, 2005). To a lesser extent, similar attempts have been done for quantitative relations (SFOR) and, even much less, mathematical patterns and structures (SFOPS).

These SFON, SFOR or SFOPS tendencies are not about what children think and do *when they are guided to* the mathematical elements, relations or patterns in the situation, but *what they spontaneously focus on* in informal everyday situations. SFON assessment instruments must therefore capture whether children spontaneously use their available number recognition or

quantitative or mathematical reasoning and patterning skills in situations where they are not explicitly guided or instructed to do so. So, the instruments used to assess these spontaneous focusing tendencies must meet several strict methodological criteria (Hannula & Lehtinen, 2005).

As far as SFON is concerned, the most frequently used task so far is the Elsi Bird Imitation task, wherein the child is instructed to imitate the experimenter's play behavior with toys, i.e., feeding berries into the beak of a toy parrot. A SFON score is given on an item as soon as the child is observed doing or saying something that shows that he or she has spontaneously attended to the quantitative aspect of the situation. Meanwhile several other SFON tasks have been developed, such as the Picture Description task, with cartoon pictures displaying both non-numerical and numerical information and the request to tell what is in the picture. If the child spontaneously refers to the exact numerosities - correct or not – in his or her verbal descriptions of the pictures, (s)he gets a SFON score (for an overview and critical discussion of SFON measures, see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016).

Observations of children's activities in SFON assessment indicated that already at the age of 3-4 years children can be spontaneously engaged in mathematically relevant practices in their everyday environments (Hannula & Lehtinen, 2005). This research also revealed great inter-individual differences in children's tendency to spontaneously focus on number. It further showed that children's SFON at the age of 5 or 6 is a unique and strong predictor of later development of mathematical skills even up to the end of elementary school. The hypothetical explanation for these findings is that children who spontaneously focus on the numerical aspects of their environment in everyday situations get much more practice of magnitude recognition, number comparison, combining of numbers, etc. than children who only do this when explicitly instructed by parents or teachers. SFON may support the development of numerical skills and more elaborated numerical skills may further strengthen the SFON tendency. However, convincing direct empirical evidence for this explanatory mechanism is still scarce (Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016).

In many everyday activities exact numerosity is not the only mathematically relevant aspect that can be focused on. In young children's daily life there are many opportunities to focus on more complex quantitative aspects, such as quantitative relations. Children can also recognize and use mathematical or quantitative relations without explicit guidance to do so. Based on a series of studies, McMullen, Hannula, and Lehtinen (2013, 2014) proposed that there is a similar tendency to focus on quantitative relations as SFON, which indicates that instead of mere numerosity children and school pupils can also focus spontaneously on quantitative relations (SFOR). McMullen and colleagues (McMullen, Hannula-Sormunen, Laakkonen & Lehtinen, 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2013; McMullen, Hannula-Sormunen, & Lehtinen, 2014). designed the Teleportation Task to measure SFOR. This task involves a cover story telling that a set of supplies containing three sets of objects was sent from earth through space with a teleportation machine. However, when doing so, the objects are transformed. Children are asked, first, to describe the transformation in their own words in as many ways as possible, and, second, to draw what they expect to happen with a different numerosity of the same objects. When describing or drawing the transformation, learners can pay attention to the various non-mathematical changes (e.g., in terms of

the colors or shapes of the objects), but also to the quantitative relation between the original and final numerosity of the three sets. The results of the longitudinal study of McMullen, Hannula-Sormunen, Laakkonen, and Lehtinen (2016) showed that there were substantial individual differences in students' SFOR tendencies,. It also revealed that SFOR tendency had a unique predictive relationship with rational number conceptual development in late primary school students during the 2-year follow-up period.

Interestingly, in their conceptualization of AMPS, Mulligan and Mitchelmore (2009) also tend to go beyond the pure ability aspect of early mathematical competence, by stating that AMPS may consist of "two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive, i.e., "spontaneous" (a tendency to seek and analyze patterns)" (p. 39). According to these authors, both are likely to be general features of how children perceive and react to their environment. However, neither in their assessment nor in their intervention materials, they have already tried to specifically and explicitly address this spontaneous focusing aspect.

Early mathematics and executive functions

In the previous sections, we have discussed various kinds of domain-specific competences that all have been claimed, and in many cases been shown, to be predictively related to general mathematical competence or to knowledge and/or skill in specific subdomains of the mathematics curriculum. However, it is a well-established research finding that formal mathematics achievement is also influenced by domain-general processes, such as sustained attention, inhibitory control, cognitive flexibility, working memory capacity, and - even more generally - intelligence (Bull & Scerif, 2001; De Smedt, Janssen, et al., 2009; Friso-van den Bos et al., 2013; LeFevre et al., 2010; Peng, Namkung, Barnes, & Sun, 2016).). While most of that research evidence comes from research with older participants, there is increasing evidence on the importance of executive functions in early mathematical thinking and learning too.

In one line of research, authors have analyzed the relative importance of general executive skills as compared to the role of domain-specific early numerical competences in predicting concurrent and later mathematical development. For instance, in a longitudinal study wherein we followed children during the first grades of elementary school, we were able to show that working memory at the start of primary education was predictively related to individual differences in mathematics achievement six months later in Grade 1 and one year later in Grade 2 (De Smedt, Janssen, et al., 2009). Interestingly, overviewing the research, Bailey et al. (2014), concluded that the contribution of domain-specific factors, such as children's early numerical competences to their later mathematical development is relatively small compared to these more stable domain-general factors, such as intelligence and working memory.

The relation between these executive functions and mathematical performance may also be more specific in nature. Research has revealed specific relations between certain executive functions, such as inhibition or working memory, on the one hand, and specific mathematical competences, such as mental arithmetic or word problem solving, on the other hand. Robinson and Dubé (2013), for instance, investigated the role of inhibition in children's use of the inversion and associativity shortcuts on mental addition and subtraction (e.g., $6 + 23 – 23 = ?$). Children who demonstrated the

highest use of conceptually-based shortcuts also scored highest on the Stop-Signal task, a standard measure of inhibitory abilities. This finding suggests that these children were able to inhibit their tendency to routinely solve problems from left-to-right and thereby process all of the presented numbers before executing the clever shortcut strategy.

So far, we have discussed the role of executive functions in children's performance on relatively complex mathematical tasks. However, to make the picture about the role of executive functions even more complicated, these executive functions are also assumed to play a critical function in the early mathematical tasks, such as the magnitude comparison task, the SFON tasks, the mathematical reasoning tasks, and the patterns and structures tasks discussed above. Take, for instance, the non-symbolic magnitude comparison task used to assess the approximate number system (ANS) and which lies at the basis of this whole line of research that has led to the pivotal role of the precision of children's early ANS representations in early mathematics diagnosis and intervention (see Section 1). In this task it is important to ensure that participants are basing their judgements on the numerosity of the visual arrays, rather than possible visual cues such as the size of the dots, or the area that the dot arrays cover. As Gilmore, et al. (2013) have argued, in an attempt to control for this possible confound, researchers introduce an inhibitory control aspect to the task, as for half of the items with which the child is confronted inconsistent visual cues must be inhibited to indicate the correct set. But if the non-symbolic comparison task is, in part at least, a measure of inhibitory control, then it is perhaps unsurprising that it is predictive of school-level mathematics achievement, but for other reasons than claimed by the advocates of this task.

Starting from the above research documenting in various ways the involvement of executive functions in mathematical thinking and learning, researchers have also asked the question about the possibility and efficacy of enhancing mathematical thinking and learning through training of these executive skills. At least for working memory, a recent meta-analysis by Schwaighofer, Fischer, and Bühner (2015) led to the general conclusion that attempts to improve working memory only improved performance on working memory tests but failed to improve mathematics achievement.

So, while there is increasing research evidence that, from a very young age on, an association between mathematics and executive functions exists, this complex and multi-aspectual association and its implications for early mathematics education and assessment is not well understood yet. Numerous questions remain (Robinson, 2016; Van Dooren & Inglis, 2015). As (early) mathematics educators we are traditionally not so much interested in these general executive functions. However, for various reasons related to theory, diagnosis and intervention, it may be unwise to neglect them.

The role of parents and early caregivers during the preschool years

As amply shown in the previous sections, before the start of formal mathematics education - typically at the age of 5-6, children already begin their initial explorations into everyday mathematics at home, progressively developing and refining their mathematical knowledge and skills as well as their mathematics-related orientations, beliefs, and affects. However, there is wide variation - linked in part to socio-economic status (SES) and culture - in the kinds of early mathematical learning experiences children have at home and the ways in which they are stimulated and helped by their parents. Further, in many cultures, the majority of young children spend

significant time in non-parental care, including family childcare and organized preschool education (DREME, 2016). Arguably, the quantity and quality of mathematics learning stimulation in these various settings also vary enormously, impacting children's mathematical development. For evident reasons, mainstream cognitive (neuro)psychological research on early numerical competences has paid little or no attention to these informal mathematical learning environments. But also within the mathematics education research community this topic is "under-studied". Indeed, we know relatively little about the role of parents and early caregivers during the preschool years when compared, on the one hand, to the development and stimulation of children's emergent literacy, and, on the other hand, to mathematics education in the higher educational levels. Fortunately, the last few years have witnessed an increased research interest.

First, several researchers have aimed for an understanding of children's preschool experiences at home and of how these experiences affect their early mathematical development. For instance, starting from the well-documented finding that children's early numerical competence before the start of formal schooling is highly predictive of their acquisition of mathematics in (the first grades of) elementary school, several authors have pleaded for a better understanding of children's preschool experiences at home. In a well-known study by Lefevre, Skwarchuk, et al. (2009), the mathematical skills of $\pm$ 150 Canadian children in Kindergarten, Grade 1, and Grade 2 were correlated with the frequency with which parents reported informal activities that have quantitative components such as board and card games, shopping, or cooking on a questionnaire. The results support claims about the importance of home experiences in children's acquisition of mathematics, given that effect sizes were consistent with those obtained in research relating home literacy experiences to children's vocabulary skills. In a more recent and more sophisticated study, Susperreguy and Davis-Kean (2016) analyzed the relation between the amount of mathematical input that preschool children hear from their mothers in their homes and their early mathematics ability one year later. Forty mother–child dyads recorded their naturalistic exchanges in their homes using an enhanced audio-recording device. Results from a sample of naturalistic interactions during mealtimes indicated that all mothers involved their children in a variety of mathematics exchanges, although there were differences in the amount of input children received. Moreover, being exposed to more instances of mathematics talk was positively related to children's early mathematical ability one year after the recordings, even after control for maternal education, self-regulation, and recorded minutes. Finally, starting from the well-documented finding that early numerical competences amongst children vary widely and from the belief that a better understanding of the sources of this variation may help to reduce SES-related differences in mathematics skills, Ramani, Rowe, Eason, and Leech (2015) examined two sources of this variation in low SES families: (1) caregiver reports of number-related experiences at home, and (2) caregivers' and children's talk related to math during a dyadic interaction elicited by the researchers. Frequency of engaging in number-related activities at home predicted children's foundational number skills, while caregivers' talk during the interaction about more advanced number concepts for preschoolers, such as cardinality and ordinal relations, predicted children's advanced number skills that build on these foundational concepts. So, these findings suggest that the quantity and quality of number-related experiences that occur at home contribute to the variability found in low-income preschoolers' numerical knowledge.

Complementary to these ascertaining studies, several intervention studies reported positive effects on children's early numerical and later mathematics performance at school. Again, we can give only a few examples. In a series of high-impact studies with children from low-income backgrounds, who were found to lag behind their peers from middle-income backgrounds already before the children enter school, Siegler and Ramani (2008) found that playing a research-based designed numerical board game for only a couple of hours already eliminated the differences in the two commonly used measures of understanding of numerical magnitudes, namely numerical magnitude comparison and number line estimation. Moreover, in a subsequent study (Siegler & Ramani, 2009), children who had played the number board game also performed better in a subsequent training on arithmetic problems. Thus, playing number board games was found to increase not only preschoolers' numerical knowledge but also to help them learn their school arithmetic. Van den Heuvel-Panhuizen, Elia, and Robitzsch (2016) report on a very recent field experiment with a pretest–posttest control group design, which investigated the potential of reading picture books to kindergarten children for supporting their mathematical understanding. During three months, the children from nine experimental classes were read picture books. Data analysis revealed that, when controlled for relevant covariates, the picture book reading programme had a positive effect on kindergartners' mathematics performance as measured by a test containing items on number, measurement and geometry. Finally, we refer to one of the best known research-based early mathematics programs, namely the Building Blocks (BB) program of Clements and Sarama (2011). This program, which is organized into five major strands: (numeric, geometric, measuring, patterning, and classifying and data analyzing), consists of daily lessons where children are encouraged to extend and mathematize their daily experiences through sequenced activities, games, and the use of technology. The daily lessons are organized in whole group activities, small group activities, free-choice learning centers, and reflection time. The program is complemented with a parallel in-service teacher training program. Studies on the effectiveness of the BB intervention program (Clements & Sarama, 2007, 2011) demonstrated that 3- and 4-year-olds who received the BB intervention program developed stronger mathematical abilities than children in the control group, with effects lasting up to the end of first grade. Bojorque (2017) recently successfully implemented the BB program in the Ecuadorian context, with significant effects on the quality of the kindergarten teachers' pedagogical actions as well as on children's progression both on a standard mathematics achievement test based on the national K3 curriculum and on their SFON.

The findings emerging from all these observational, correlational, and intervention studies are very informative for the design of educational environments and activities aimed at increasing young children's mathematics learning - far beyond the rather narrowly oriented (computer) games aimed at stimulating children numerical magnitude representations that have been derived from the cognitive neuroscientific line of research. But still a lot of work needs to be done to further advance knowledge on effective ways to increase parents' and professionals' engagement in preschoolers' mathematics learning, particularly in children growing up in poverty and/or in contexts of unfavorable immigration.

Preschool to elementary school transition

As explained in the previous sections, a large number of factors in the young child and in its home and caretaking environment have a strong impact on the ease with which (s)he will take the step to formal mathematics education at the age of 5-6 (depending on the country or culture) and profit from the elementary school mathematics curriculum. However, the child's mathematical development and achievement will evidently also be significantly affected by the quality of the transition from preschool to elementary school (see also Gueudet, Bosch, diSessa, Kwon, & Verschaffel, 2016).

Interestingly, researchers working on this theme typically take a much broader theoretical stance than the cognitive (neuro)scientific researchers who look for the elements in children's domain-specific and domain-general competences that are predictively related with success in school mathematics. Their inspiration comes from socio-cultural, sociological, anthropological, and critical mathematical theories (Dockett, Petriwskyj, & Perry, 2014; Perry, McDonald, & Gervasoni, 2015).

The transition from prior-to-school to school mathematics is primarily conceived by these researchers as a set of processes whereby individuals "cross borders" or undergo a "rite of passage" from one cultural c.q. educational context or community to another and, in doing so, also change their role in these contexts or communities. Dockett et al. (2014, p. 3) provide the following summation of this approach: "While there is no universally accepted definition of transition, there is acceptance that transition is a multifaceted phenomenon involving a range of interactions and processes over time, experienced in different ways by different people in different contexts. In very general terms, the outcome of a positive transition is a sense of belonging in the new setting." There is growing research evidence that developing practices that promote effective transitions, and that strive for giving agency of all involved and rely on the "Funds of Knowledge" available in children's home and local environments, results in positive effects - although most of this research is more qualitative and descriptive in nature and thus not primarily interested in following strict experimental designs and providing "hard" statistical data. A nice overview of this broader transition perspective is provided by Perry et al. (2015).

In an interesting newly funded project, Andrews and Sayers have begun to examine how two systems, England and Sweden, facilitate the early mathematical competences, and more specifically their foundational number sense (FoNS) (Andrews & Sayers, 2015), of children starting in Grade 1. Currently the project team is comparing the FoNS opportunities found in commonly used textbooks in the two countries (Löwenhielm, Marschall, Sayers, & Andrews, 2017a). Simultaneously the team has been interviewing first grade teachers in the two countries about their role as well as their perceptions of their pupils' parents' roles in the development of children's FoNS-related competence. Initial analyses (Löwenhielm, Marschall, Sayers, & Andrews, 2017b) have identified both similarities and considerable differences in the relationship between the school and home environment between the two countries.

It is a general complaint among stakeholders of early mathematics education that mathematics learning in preschool is often disconnected from the first grades of elementary school. This disconnect, which is particularly relevant for lower SES and immigrant children, can lead to

children experiencing uneven instructional practices, which can compromise their mathematical development in elementary school. So, policy makers, curriculum developers, teacher trainers, etc. should work toward creating greater alignment of and coherence between preschool and elementary school mathematics education, using research-based insights and recommendations. Unfortunately, there is still limited research on the impact of these policies and practices on the learning experiences and learning outcomes of children moving from preschool through the early elementary grades.

Professional development of caregivers and teachers

In the previous section, we emphasized the importance of a high-quality mathematical learning environment in the preschool years, the first years of elementary school, and the transition between the two. Evidently, this requires highly professional (mathematics) teachers, i.e., "teachers who know the content, who understand children's thinking, who know how to engage in pedagogical practices that support learning, and who see themselves as capable math teachers" (DREME, 2016, p. 4).

At the same time, many teachers and caregivers in the early care and education field may not be adequately equipped to provide appropriate math-related experiences and instruction to these young children. Research suggests that many practitioners working with preschool, kindergarten and early grade children (1) are themselves not competent in mathematics, (2) have important shortcomings in the pedagogical content knowledge, particularly with respect to the components of the early math curriculum beyond counting, number, and simple addition and subtraction, and/or (3) do not see themselves as competent in mathematics (see e.g., Lee, 2010). And, even if practitioners are mathematically capable and do view themselves as such, they may still hold pedagogical reservations against teaching mathematics to young children, believing that early childhood programs should focus primarily on social emotional and literacy goals (Platas, 2008).

While these problems have shown to be partly due to these professionals' restricted mathematical talents and negative earning histories in elementary and secondary education, research also indicates that the nature of the pre-service and in-service training they received does not greatly help to overcome these problems. As DREME (2016, p. 4) argues: "Professional teacher preparation programs rarely address how to identify the wide range of informal mathematical understandings that young children bring with them to the classroom, or how to translate these into intentional, individualized math experiences for children with diverse backgrounds and needs." Indeed, surveys of early childhood education degree programs (e.g., Maxwell, Lim, & Early, 2006) reveal that early education practitioners are exposed rarely to high-quality pre-service or in-service courses that address children's mathematical development, or the pedagogical content knowledge necessary for supporting it.

We emphasize that the above analysis is largely based on critical reflections upon the situation in the US. So, the situation may be better in other places in the world, although there are good reasons to restrain from being too optimistic, because the above observations about early math teachers' professional knowledge and beliefs and previous educational histories seem to hold, at least to some extent, for many other countries too.

To support the training of prospective and practicing early childhood teachers, there is a need of creating and implementing research-based modules for professional development that can be used in a variety of pre-service and in-service settings (DREME, 2016). The way forward for research is to attempt to figure out what are the key levers of professional development that might effect significant change in the quality of early math education and its learning outcomes. Given the above-mentioned depiction of the complex and multi-sided nature of caregivers' and early math teacher's professional knowledge base, it seems reasonable to expect the greatest effect from modules that do not focus on one single aspect of professionalism but work on the development of early math related knowledge, skills and beliefs, and that convey the idea that early mathematics is more than teaching young children some basic number knowledge and counting skills.

Conclusion

Inspired by developments in the field of neuroscience (e.g., Butterworth, 2015), the past two decades have witnessed the emergence of a very productive and highly influential line of (neuro)cognitive research on children's early number sense, its development, and its relation to school mathematics. Cross-sectional and longitudinal studies have demonstrated that various core elements of children's early mathematical ability - especially their numerical magnitude understanding, their subitizing and counting skills, and their ability to transcode a number from one representation to another - are positively related to concurrent and future mathematics achievement (Torbeyns et al., 2015).

However, other research, most of which is situated in other scientific circles and relying on other theoretical and methodological perspectives, has yielded increasing evidence for uniquely significant relations of mathematical achievement also with (1) young children's understanding of ordinal and measurement aspects of number, (2) their abilities related to mathematical relations, patterns and structures, and (3) their tendency to spontaneously attend to numerosities and to mathematical relations, patterns, and structures in their environment, and has confirmed the important role of domain general executive functions.

Moreover, researchers have started to explore and analyze the rich variety of early mathematical learning environments at home, in preschool and kindergarten settings, as well as the coherence between these informal learning settings and the first years of elementary school mathematics, with special attention to the professional quality of the early caregivers and teachers. Also, they started to set up various kinds of intervention studies aimed at the improvement of the quality of these environments and of the professionals operating in these environments. These studies have yielded evidence on the short- and long-term benefits of such attempts to provide high-quality early mathematics education in preschool settings and in the transition from preschool to elementary school.

While the small-scale, short-term and focused experimental intervention programs derived from the (neuro)cognitive research on early numeracy have their value in enhancing our theoretical insight into numerical cognition and learning, practitioners active in the field of early mathematics education may profit more from the studies describing the design, implementation, and evaluation of large-scale and more broadly conceived intervention programs that combine and balance several

of the elements that have been found to be foundational for future mathematics learning (see Sections 2-6) and that also integrate aspects of teacher development, working with parents, and community building (see Sections 7-9), with the *Building Blocks* program of Clements and Sarama (2007) and the *Pattern and Structure Mathematics Awareness Program* of Mulligan et al. (2013) as the most visible and successful examples. Still, as math educators, we should continue to follow, with an open but critical mind, the cognitive neuroscientific research on mathematical cognition and, equally important, also try to have an impact on their research agenda (De Smedt et al., 2011).

As a result of all this research, there is a lot of practically useful new knowledge, techniques and resources to promote young children's math learning. Still there remains much to learn about how to optimally enhance math learning at home and at school in the preschool years and about how to help teachers to be well prepared for delivering high-quality instruction to those young children, particularly the weaker ones. In this respect, we should applaud - and may-be also strive for an European counterpart – of the recent initiative called the DREME Network in the US, which is aimed at developing new researchers and enticing current elementary math education, child development, and policy researchers to expand their work to include young children's mathematical learning.

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Solid findings in mathematics education: What are they and what are they good for?

Marianna Bosch[1], Tommy Dreyfus[2], Caterina Primi[3], Gerry Shiel[4]

[1]Universitat Ramon Llull, Spain; marianna.bosch@iqs.edu

[2]Tel Aviv University, Israel; TommyD@post.tau.ac.il

[3]University of Florence, Italy; primi@unifi.it

[4]Educational Research Centre, Dublin; gerry.shiel@erc.ie

This paper presents the contributions of the participants at the CERME10 panel, as well as some of the questions raised during the discussion. Our main aim is to examine the notion of solid finding in mathematics education, the theoretical and methodological assumptions underlying their establishing and the degree of agreement (and disagreement) they provoke. We will consider their possible utilities and weaknesses, even jeopardies, taking into account two different standpoints: how solid findings are identified and what kind of common ground they rely upon; what are solid findings for, how can they be useful and what could be their risks or adverse effects. The panellists will adopt different perspectives on the topic, focusing on the specific selection of solid findings proposed by the Committee on Education of the European Mathematical Society, approaching the problem of the methodologies and use of psychometric models; questioning the use of evidence in policy development and curriculum evaluation.

Keywords: Mathematics education, solid findings, criteria, empirical proof, concept images, measurement, reliability, validity, invariance, math anxiety, statistics anxiety, PISA.

Solid findings in mathematics education: A necessary discussion

Marianna Bosch

Proposing to collectively discuss on 'solid findings' in mathematics education at CERME10 was certainly motivated by the initiative of the Education Committee of the European Mathematical Society (EMS) to start publishing a series of articles on 'Solid findings in mathematical education' in 2011 (http://www.euro-math-soc.eu/ems_education/education_homepage.html). This can be interpreted as an audacious attempt to establish a stable account of our young discipline, which otherwise might appear as made of too diverse principles, approaches and perspectives. It is thus supposed to facilitate the approach by people from the outside, especially mathematicians and teachers, by giving more visibility of the type of questions approached and the results obtained. Inside the discipline, it also appears as an effort to organise and elaborate a provisional common hard core (in the sense of Lakatos) of sound and relevant knowledge, without denying the provisional and dynamic nature of the considered findings.

Taking the EMS project and its products as initial motivation, the aim of the panel is to examine the notion of solid finding, the theoretical and methodological assumptions underlying studies on solid findings and the degree of agreement (and disagreement) they might provoke. We propose to consider the possible utilities and weaknesses, even jeopardies, of the reports on solid findings, taking into account two different standpoints: (1) how solid findings are identified and what kind of common ground they rely upon; (2) the purpose of solid findings, their potential utility, and also their possible risks or adverse effects.

The aim of the panel was thus to open a debate on controversial questions like:

(1) What is a solid finding in Mathematics Education? What criteria are used to select them? Who decides whether a finding is solid or not? Are solid findings linked to specific methodologies, theories or approaches? Can they be contested and how? What kind of evidence is required? Is it the same kind of evidence for the different 'findings'?

(2) What is the purpose of identifying solid findings? What are they for? How can they be useful? Are they necessary for teacher education? Could they help to give more visibility to our field and to negotiate with educational decision makers? Can there be a risk of disseminating false 'weak' solid results instead of disseminating the persistent questions addressed from research in mathematics education – which do not always coincide with those raised by the actors of the educational system (teachers, students, parents, decision makers, etc.)?

During the discussion among the participants at the panel session, the question of the diversity of theoretical perspectives was raised on various occasions. It is clear that solid findings are always anchored in a given research approach or paradigm (a set of close theories sharing the main theoretical principles or assumptions). Agreement on solid findings thus supposes agreement on these main assumptions too. This does not seem to be – at the moment – the historical situation of the research community in mathematics education, where a diversity of approaches coexists without a common shared ground. Not only the type of results provided by these approaches are different, but mainly the type of research questions asked, the methodologies used, and even the empirical units of analysis considered. If solid findings are presented without mentioning the approaches where they have been produced, we run the risk of interpreting solid findings as if they came from an a-theoretical perspective (or from a fully shared one), which is in fact a way of giving preponderance to the already dominant approaches in detriment of the less disseminated ones.

Other questions related the issue of solid findings to the problem of the dissemination of results. If solid findings should be closely contextualised within a given theoretical framework – or research perspective –, how to make them accessible to people not knowledgeable of the framework? To what extent, and under what conditions, could solid findings be extended to include frameworks? The question varies of course if we think about disseminating research outcomes outside the field, or about highlighting what are seen as important milestone in the evolution of the field, for instance to build the basis for productive debates.

Furthermore, participants also indicated that it is important to avoid not only taking the theoretical 'load' of solid findings for granted, but also to pay attention to the *values* they implicitly carry on, for instance, about the purpose of education, the purpose of research on mathematics education or

about the corresponding specific epistemology or conception of science. For instance, the choice of the term 'finding' seems related to a somewhat naturalistic perspective – the scientific discovery of a pre-existent reality –, while other options such as 'claims', 'proposals' or 'questionings' (in the double sense of raising questions and questioning the status quo) would entail other connotations. In this sense, maybe the dimension of problematizing can also be a possible direction to work with.

In fact, one of the questions from the audience addressed the issue of the relationships between solid findings, persistent phenomena and educational problems: Are solid findings restricted to phenomena that persist? Is it also possible to have a solid finding that eliminates a problem? In other terms, because advancing research also modifies our ways of problematizing reality, solid findings can also make some problems appear as simple difficulties that can be overcome, or as consequences of other factors to be approached. In the other sense, a solid finding can also consist in the awareness that a problem has not solution – at least in the framework where it is formulated.

The establishment of solid findings as such was also referred to by some participants. Some of them wondered if it is possible to identify some steps to help establish solid findings and build upon them more systematically. Others asked about efficient ways of guaranteeing cumulative research efforts, such as the replicability of the solid findings, which was proposed as a possible research avenue to pursue. One should not see naivety in this kind of demands – as if we were asking for 'recipes' –, but on the contrary, interpret them in terms of a reflection on the research methodologies followed (in terms of validity, truthfulness, reproducibility, etc.) and the level of exigence put on them. To enrich the debate, some participants provided related materials or counterexamples to this kind of reflexion, such as the U. S. webpage "What Works in Education" (http://ies.ed.gov/ncee/wwc/) or the special issue of the International Journal of Research & Method in Education (2016) *Is the Educational 'What Works' Agenda Working? Critical Methodological Developments*, including a paper on review procedures to optimise reviews' impact and uptake (Green, Taylor, Buckley, & Hean, 2016).

The three contributions that form the core of this paper address some of the issues raised from very diverse – and complementary – perspectives. Tommy Dreyfus, a member of the Education Committee of the EMS and co-author of some of the 'solid findings' articles, provides a very interesting account of two moments of reflection of our community around the issue of 'results' or 'findings' in mathematics education, and their related projects. He also presents two examples of 'solid findings', showing the criteria used to identify them and also some of the limitations of the efforts made. He argues for a collective effort toward the products of more systematic reviews on different topics or approaches, as a way to increase the impact of research outside the field but, also, to "establish and organize mathematics education as scientific discipline and to determine where we come from, where we are and where we might go as a research community". From a completely different side, Caterina Primi, an expert in the field of quantitative educational research, addresses the methodology problem – measurement tools to support rigorous research designs – for findings to be 'solid' or, in statistical terms, 'robust', 'reliable' and 'unbiased'. Even if the example taken and the questions raised are only related to quantitative methods – where statistical tools are more commonly applied –, the reader can do the mental exercise of transposing them to qualitative as well as theoretical studies to see how demanding the research work to make knowledge develop can be. Finally, Gerry Shiel, National (Ireland) Project Manager for the OECD PISA 2015 Study,

tackles what can be called the 'impact issue' of educational research, considering the PISA phenomenon, which is maybe the source of the most practical and political pressures nowadays in almost all countries. The relationships between 'solid finding' and *evidence-based decision making* provides a rich paradigmatic example and reminds us how intricate is the situation, especially when raw data is proposed without any protection from the procedure followed to generate it and the theoretical framework, including political ideologies, that underlies its generation.

To end this introduction, let me quote the British sociologist Martin Hammersley (2011) who, in his book on methodology, notes how extremely demanding it is to achieve the 'threshold of likely validity required by academic work' (p. 8). After presenting 'dedication', a 'heightened sense of methodological awareness' and 'objectivity' as important virtues for the researcher, the author recalls that, besides these individual virtues:

> [The] collective character of enquiry places additional obligations on researchers, as regards how they present their work, how they respond to criticism and how they treat the work of colleagues. In large part, what is required is that academic research takes place within an enclave that is protected from the practical considerations that are paramount elsewhere. […] In other words, academic discussion must be protected from political and practical demands, so that the consequentiality of proposing, challenging, or even just examining particular ideas or lines of investigation is minimised. […] [While] the 'findings' of particular studies should be made public within research communities, they should not be disseminated to lay audiences. What *should* be communicated to those audiences, via literature reviews and textbooks accounts, is the knowledge that has come to be more or less generally agreed to be sound within the relevant research community, through assessment of multiple studies. (Hammersley, 2011, p. 10)

I am not sure if the field of mathematics education has already reached a sufficient level of development to agree on what can be globally accepted as sound and relevant knowledge, and thus to identify, elaborate and disseminate 'solid findings' to lay audiences. However, I am certain that the community of research in mathematics education is mature enough to initiate a productive debate on this, as a way to make different research perspectives interact in a productive way. The effort of gathering, summarising, organising, and discussing the research produced about certain big questions or issues – as the one undertaken by the EMS Educational Committee – appears nowadays as an endeavour that cannot be postponed.

What are solid findings in mathematics education?

Tommy Dreyfus

Relying on earlier studies by an ICMI Study and the Education Committee of the EMS, the question what the term 'solid finding' might mean with respect to mathematics education is discussed and criteria are proposed. Examples are provided for solid findings that mathematics education research has produced.

Introduction

Mathematics education as a research community has grown over the past approximately 50 years: ERME, The European Society for Research in Mathematics Education is approaching its 20th anniversary in 2018 – CERME1, the first conference took place in Osnabrück, Germany, in August, 1998. PME, the International Group for the Psychology of Mathematics Education has held its 40th annual conference in 2016 - the first one took place in 1977 in Utrecht, The Netherlands. JRME, the *Journal for Research in Mathematics Education*, is now producing its 48th annual volume, and ESM, *Educational Studies in Mathematics* is currently in its 50th year of publication since Volume 1 appeared in 1968. One of the characteristics of research results in (mathematics) education is that they depend on the context in which the research has been designed and carried out. Nevertheless, after 50 years, one would expect the community to be able to make statements that go beyond "it depends on the context and the learning environment", which is often implicit in the results of even high quality research articles. Review articles could be expected to remedy this situation to some extent but few review articles are published in the domain.

What are the results of research in mathematics education – ICMI Study 8

The question whether we, as a research community, have obtained results with a certain scope, range or breadth of validity and what these results are, has been approached at least twice, once in the framework of the study conference of ICMI Study 8 in 1996 (ICMI stands for the International Commission on Mathematical Instruction), and a second time in the framework of the Education Committee of the European Mathematical Society (EMS) in 2011.

The task assigned by ICMI to the Study 8 program committee was to discuss *what is research in mathematics education and what are its results.* The title of the book published two years later as outcome of the study is *Mathematics Education as a Research Domain: A Search for Identity* (Sierpinska & Kilpatrick, 1998). Maybe significantly, the word 'results' has disappeared in the process. Nevertheless, one of the working groups at the study conference dealt with results (Dreyfus & Becker, 1998). One of the questions the working group dealt with was what counts as result; the term 'solid' did not appear. Rather, 'result' was interpreted as 'significant result'.

Working group members agreed that without a question, there can only be facts but no results. Results are more than data: They are based on data collected with questions in mind that have been asked within a theoretical framework, and consist of findings interpreted in that theoretical framework. Effects alone (e.g., statistical differences in achievements between different groups) are not results. In mathematics education, we need to *explain* the differences, not only show them. We

need to identify the variables of the didactic situation in order to combine the different facts into a coherent network of reasons, which informs the circular process of understanding the learning of mathematics and thus improving its teaching. Hence, results are often theoretical as well as experimental. Many of our theoretical frameworks are mathematics-specific (e.g., process, object, procept), and therefore our research questions and results are often domain-specific.

Context was seen as relevant with respect to theory as well as beyond: mathematical context (contents, concepts, symbols, representations and epistemological status), the community, the educational system, among others. Results might be tied not only to the theoretical framework but also to the institution that asked the question. It is not the result itself, but the conditions under which it was obtained, that make it significant.

The contextual nature of results implies that results are neither universal nor eternal, that their validity is situated in space and time, and that we have to be careful when trying to generalize. The validity of a result depends on the interpretation within a theory, and the theory might change with time and place, with mathematical content, learning environment, and so on. Hence results are permanent but their relevance might be ephemeral.

Characterising solid findings

It is on this background that the members of the Education Committee (EC) of the European Mathematical Society (EMS) asked themselves what solid findings mathematics education has produced. While the question was motivated by the intention of the committee to present mathematics education to mathematicians, in particular to EMS members, with an interest in mathematics education, committee members were well aware that the exercise of identifying solid findings contributes to establishing and organizing mathematics education as scientific discipline and to determining where we come from, where we are and where me might go as a research community.

Of course, the first, and possibly most difficult task of the EC was to discuss, agree and explain what they meant by ‘solid findings’. A major difficulty in defining what it means that a result it solid is the context dependence, mentioned above. A second and related difficulty is complexity. As we know well, things are more complex than one might think; we know, for example that the mathematics taught and learned in parallel classes with a similar population according to the same curriculum may be quite different (e.g., Even & Kvatinsky, 2010; Pinto, 2013). A third, and of course also related difficulty is that much of the research in mathematics education is qualitative. Since qualitative empirical research cannot be repeated in a strict sense, reproducibility is replaced by the question how close the results are that one obtains in similar contexts; and the answer to this question of course depends on the metric used to measure closeness. This lack of reproducibility may appear as a serious drawback of mathematics education’s claim to be a scientific discipline; however, reproducibility has recently been shown to be very low even in many hard sciences such as physics, chemistry and engineering (Baker, 2016).

Aware of these difficulties and with the ICMI 8 study characterization of (significant) results as background, the EC has observed that results with the potential of being considered solid usually do not stand alone but have emerged from a line of research consisting of a larger set of related studies.

Solid findings are typically yielded by such a line of studies. Next, the EC has built on three properties of research quality proposed by Schoenfeld (2007 – see there for a much more detailed discussion): trustworthiness, generality and importance. Each of these contributes to the solidity of research results. A characterisation adapted to the purposes of the EC was agreed upon and published in the Newsletter of the EMS (Education Committee of the European Mathematical Society, 2011a). I summarize this characterization here, adapting and supplementing it for the purposes of the present CERME panel.

Trustworthiness includes the explanatory power of research, its rigor and specificity, and whether it makes use of multiple sources of evidence. However, a study may be trustworthy but trivial, in terms of generality or importance.

Generality (or scope) refers to the question: What is the scope or generality of a research result? How widely does this finding, this idea, or this theory apply across content domains, learning contexts, cultures, etc.? For example, did researchers, in different countries and school systems obtain comparable or related empirical results? Do theoretical constructs turn out to be useful beyond the bounds of the individual studies in which they were developed?

Trustworthiness and generality together are expected to impart some predictive power to a result. A result that has no predictive power cannot be considered solid. On the other hand, the difficulties mentioned above, such as context dependence, will usually limit this predictive power. If a result is used to predict an outcome in a new context, and the prediction failed, a trustworthy explanation of the failure may in fact increase the solidity of the result.

Importance addresses the question: Does it matter? What is the (actual or potential) contribution of the research to theory and practice. Of course, importance is to a large extent a value judgment. As in any other field of study, beliefs about what is essential and what is peripheral are not static but change over decades, reflecting trends both within and beyond the discipline. Hence recognition of the significance of the result by experts contributes essentially to the solidity of a result.

The term ‘solid’ may remind the reader of the term ‘robust’ often used in related situations. ‘Robust’ often has a technical meaning that refers to a finding having been repeatedly observed or confirmed in many studies reporting the same or similar results leading to the same (general) conclusions (see Primi, below). The term solid has been chosen intentionally, to refer to results rather than findings, and imply that robustness in the technical sense is not possible, nor maybe desirable, in mathematics education.

While robustness can be defined and hence (dis-)proved, solidity cannot. The above is a characterization or description – not a definition. Hence, solidity cannot be proved but it can definitely be argued by on the basis of the above criteria of significance, trustworthiness, generality and adaptability to context.

Examples

The second major task of the EC with respect to solid findings was to provide a variety of examples of findings that are solid according to the EC’s characterization. While the selection of the examples to be presented was somewhat eclectic and partly determined by EC members who were willing to write about a topic, the EC as a whole discussed and approved the proposed topics; the EC also

revised every draft several times. As result a sequence of brief articles has been published presenting a rather representative selection including solid findings about cognition and about affect, about teaching and about learning, about elementary school and about university, about specific mathematical contents and about cross domain issues such as the use of technology, and maybe most importantly about theoretical and about empirical results. Most of the issues of the EMS newsletter from Issue 82 (December 2011) to Issue 95 (March 2015) present such a solid finding.

Here, I briefly present two of these, one reason for my choice again being personal preference and the other representativeness, at least in the empirical – theoretical dimension.

Do theorems admit exceptions?

Empirical studies on students' conceptions of proof have found that many students provide examples when asked to prove a universal statement. Universality refers to the fact that a mathematical claim is considered true only if it is true in all admissible cases without exception. A student who seeks to prove a universal claim by showing that it holds in some cases is said to have an *empirical proof scheme*. The same student is also likely to expect that a statement, even if it has been 'proved', may still admit exceptions. There is considerable evidence that many mathematics students, and some mathematics teachers, rely on validation by means of one or several examples to support general statements. The majority of students who begin studying mathematics in high school have empirical proof schemes, and many students continue to act according to empirical proof schemes for many years, sometimes into their college years.

While the issue of empirical proof schemes has already been mentioned by Polya (1945), Bell (1976) may have been the first to report an empirical study about students' proof schemes. Following Fischbein's (1982) seminal investigation on universality, the issue has been re-examined many times, usually with similar results. For example, findings by Sowder and Harel (2003) indicate the appearance of empirical proof schemes among university mathematics graduates.

The phenomenon of empirical proof schemes is general in the sense that it has been found in many cultures, countries, school systems, and age groups. It is persistent in the sense that many students continue to do so even after explicit instruction about the nature of mathematical proof. However, it is also complex. For example, the London proof study (Healy and Hoyles, 2000) showed that even for relatively simple and familiar questions, 14-15 years old high-attaining students' most popular approach was empirical verification but that many students correctly incorporated some deductive reasoning into their proofs and most valued general and explanatory arguments.

How can this pervasive phenomenon be explained? The notion of a "universally valid statement" is not as obvious as it might seem to mathematicians. Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences, as well as everyday experience. In everyday life, the "exception that confirms the rule" is pertinent. Students, in particular young children, have little experience with mathematics as a wonderful world with its own objects and rules. According to Fischbein (1982), the concept of formal proof is completely outside mainstream thinking, and we require students to acquire a new, non-natural basis of belief

when we ask them to prove. These explanations contribute to the trustworthiness of the findings on empirical proof schemes.

In summary, the studies on empirical proof schemes, only a few of which have been referred to here, firmly establish the solidity of the phenomenon of empirical proof schemes. (For a more detailed exposition, see Education Committee of the European Mathematical Society, 2011b.)

Concept images in students' mathematical reasoning

Vinner and Hershkowitz (1980) were the first ones to point out that students' geometrical thinking is frequently based on prototypes rather than on definitions. They have shown, for example, that junior high school students tend to think that the altitude has to reach the base (rather than its extension). Hence, they draw the altitude inside the triangle, even in a triangle with an obtuse base angle. Students' prototype altitude is one that is inside the triangle. This is so, even if the students know and can recite the (general) definition of altitude in a triangle.

Authors from many countries have reported, over the past 35 years, analogous patterns in students' reasoning in other areas of mathematics, even among talented students in elementary school, high school and college. For example, and in spite of 'knowing' the appropriate definitions, students tend to act according to rules such as multiplication makes bigger, inflection points have horizontal tangents, definite integrals must be positive, and sequences are monotonous.

A commonality in these and parallel studies is that students do not base their reasoning on the definition of the concepts under consideration (even though they are often aware of these definitions and can recite and explain them) but rather on what Tall and Vinner (1981) have called their *concept image:* "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). A student's concept image need not be globally coherent and may have aspects which are at variance with the formal concept definition.

The notion of a student's concept image is complex since it is influenced by all of this student's experiences associated with the concept. These include examples, problems the student has solved, prototypes the student may have met substantially more often than non-prototypical examples, and different representations of the concept including visual, algebraic and numerical ones. Images may deeply influence concept formation. As a consequence, the concept image is personal and continuously changing through the student's mathematical experiences.

How can this pervasive phenomenon be explained? While it is not possible to introduce a concept without giving examples, particular instances of the concept never suffice to fully determine the concept. As a consequence, specific elements of the examples, even if not pertinent to the mathematical definition of the concept, become for the student key elements characterizing the concept. And even if at the stage a concept is introduced a teacher might make an effort to present a rather varied set of examples, as the concept is being used over the coming months or years, some of these properties tend to be reinforced because they appear much more frequently than others that may recede. Examples abound, and the height of a triangle being vertical in the sense explained above is a typical one. Students may see many triangles in which the altitude is inside the triangle, and few in which it is not. They might consider these few cases as exceptions (Lakatos might say

monsters). This explanation contributes to the trustworthiness of the findings on empirical proof schemes.

In summary, a solid finding of mathematics education research, supported by dozens of studies in many difference contexts, is that students' mathematical reasoning is frequently based on their concept images rather than on a mathematical concept definition. A more detailed exposition of this solid finding has been published elsewhere (Dreyfus, on behalf of the EC of the EMS, 2014).

Conclusion

The list is of solid findings presented by the EC of the EMS is, of course, not exhaustive but limited by the time of service of the committee and the people who served on it. I would like to encourage CERME members (and other researchers) to write and publish articles about solid findings they are aware of and consider important. This might have the desirable effect of producing a type of article lacking almost completely from our literature – review articles. Let me make just one suggestion: Work to raise the awareness of issues and of research on teaching and learning among university lecturers and tutors is necessary; it usually improves students' attitudes but effects on learning are limited. Research in at least four countries (USA, Germany, England, Finland) has shown that work with students has more potential for large scale effects. It seems to me that a suitable review article might not only inform many mathematics educators of an important line of research but might have a considerable effect on university teaching centres, an effect that a single study report could (and should) never have.

In conclusion, the researchers and teams referred to above have shown that mathematics education has, over the past 50 years, produced theoretical and empirical results that are solid in the sense that they have explanatory and predictive power, that they can be applied in contexts beyond those involved previous studies, and that they are recognised as important contributions that have significantly influenced the research field, for example by providing a theoretical lens that allows to see an observed phenomenon differently from how it was seen before.

Acknowledgment

I am grateful to Günter Törner, chair of the EC of the EMS for initiating and sustaining the discussion about solid findings in mathematics education at and in between the six committee meetings from 2010-2014; I would also like to acknowledge the contribution of the committee members to my thinking about what makes findings solid and about what solid findings mathematics education has produced.

Solid findings in mathematics education:

A psychometric approach

Caterina Primi

The foundation of all rigorous research designs is the use of measurement tools that are psychometrically sound. The purpose of this paper is to present the scales' proprieties such as reliability, validity, and invariance that are fundamental prerequisite for assuring the integrity of study findings. Providing examples of how to assess the psychometric properties of tools used in mathematics research may be helpful for future researches in this topic.

In the document prepared by the Education Committee of the European Mathematics Society (2011), the description of "solid findings" includes an aspect of "robustness". That means that findings in the research on mathematics learning and teaching should be repeatedly observed or confirmed in many studies reporting the same or similar results leading to the same (general) conclusions. To achieve this goal rigorous research designs and measurement tools that are psychometrically sound are needed. Starting from these premises I will try to identify the contribution of psychometrics to solid findings in mathematics education.

Measurement

In many educational measurement situations, the variables of interest such as ability, beliefs, attitudes, and anxiety are not directly observable. As such, they are *latent variables* or *traits*. Indeed, they are easily described but they cannot be measured directly, as can height or weight for example, since these variables are concepts rather than physical dimensions.

To give an example of a measurement process, imagine that a researcher is interested in measuring mathematics anxiety (MA). Mathematical anxiety is commonly defined as an adverse emotional reaction to math or the prospect of doing math (Hembree, 1990). It is a state of nervousness and discomfort brought upon by the presentation of mathematical problems and may impede mathematics performance irrespective of true ability (Ashcraft, 2002; Ashcraft & Moore, 2009). The negative consequences of mathematical anxiety are well-documented (Morsanyi et al, 2017). Students with high levels of mathematical anxiety might underperform in important test situations, they tend to hold negative attitudes towards mathematics, and they are likely to opt out of elective mathematics courses, which also affects their career opportunities. Over the last decade there has been more interest in understanding how and when MA develops (Dowker, Sarkar, & Looiet, 2016; Harari, Vukovic, & Bailey, 2013; Jameson, 2013; Ramirez, Gunderson, Levine, & Beilock, 2013), investigating the incidence of MA, and its effects on primary school samples (e.g. Galla & Wood, 2012; Karasel, Ayda, & Tezer, 2010; Wu, Barth, Amin, Malcarne, & Menon, 2012), as well as its consequent influence on math achievement (Wu et al., 2012). Given the widespread prevalence of MA and its detrimental long-term impact on academic performance and professional development, it is important to measure this construct in a valid and reliable way.

From a measurement prospective it is not possible to directly observe MA. Following the latent trait theory (Lord & Novich, 1968), we can measure something that cannot be observed only by inference from what can be observed. Thus, while the trait itself is not observable, its interaction with the environment produces, at the surface level, observable indicators which can be used to infer the level or degree of the latent trait. Considering MA, although we cannot observe our latent variable, its existence may be inferred from behavioural manifestations or *manifest* variables (for example, as feeling tense, fearful and apprehensive about mathematics). These manifestations make it possible to measure MA asking, for example, a series of questions (the *items* of the instruments) that describe each manifestation (for example, "I feel nervous when I use numbers"). Indeed, a measurement instrument can be constructed using these items with the purpose of assessing the unobservable trait.

However, the primary goal of educational measurement is to determine the level of the latent trait that a person possesses. In general, *scaling* is the process of establishing the correspondence between the observations and the latent variable. Several mathematical approaches have been developed in order to define how to measure a latent trait through item responses, assuming that the latent trait is continuous. These approaches include Classic Test Theory (CTT) and the more recent Item Response Theory (IRT).

Traits, indicators, and their relationships can be represented graphically. Figure 1 represents the measurement structure of the *Abbreviated Math Anxiety Scale* (AMAS; Hopko, Mahadevan, Bare, & Hunt, 2003). This is a two-factor measure of MA that is considered a parsimonious, reliable, and valid scale. The two factors are *Learning Math Anxiety*, which relates to anxiety about the process of learning, and *Math Evaluation Anxiety*, which is more closely related to testing situations. The AMAS is one of the most commonly used measure of MA in college and high school students (for a review, see Eden et al., 2013). The scale has been translated into several languages, and it has been found to be a valid and reliable measure in a variety of populations (Polish version: Cipora, Szczygiel, Willmes, & Nuerk, 2015; Italian version: Primi, Busdraghi, Tomasetto, Morsanyi, & Chiesi, 2014; Persian version: Vahedi & Farrokhi, 2011). Recently, it has also been adapted for children between the ages of 8 to 11 (Italian version: Caviola et al. 2017), and 8 to 13 (English version: Carey et al. 2017).

Looking at the details of Figure 1, the ovals represent latent, unobserved variables, specifically, Learning Math Anxiety and Math Evaluation Anxiety. The squares represent the observed variables (items); five for *Learning Anxiety* (e.g., listening to a lecture in a math class), and four for *Math Evaluation Anxiety* (e.g., thinking about an upcoming math test one day before). The relations between items and the latent traits are represented with arrows that indicate that the traits cause the corresponding indicators. In Figure 1, the error components that we have to take into account in the measurement process are also represented. Any observed score has two parts: The true score part and the error part. Intuitively, we control for the error and we estimate the true score by taking several measures and averaging them. We assume that averaging several measures results in a better estimate of the true score. These ideas are formalized in the concept of *reliability*. We use multiple indicators or items to better measure the trait. This way, we can have more information and reduce the error components, that is, we can maximize the reliability or precision of the measurement tool.

Moreover, verifying the relationships among indicators and the corresponding traits, through a confirmative procedure, such as a confirmatory factor analysis (CFA), we can verify that the measurement tool truly captures the underlying trait, attesting the *validity* of the measurement tool (Zumbo, 2009). Indeed, obtaining evidence of validity is part of the measurement process.

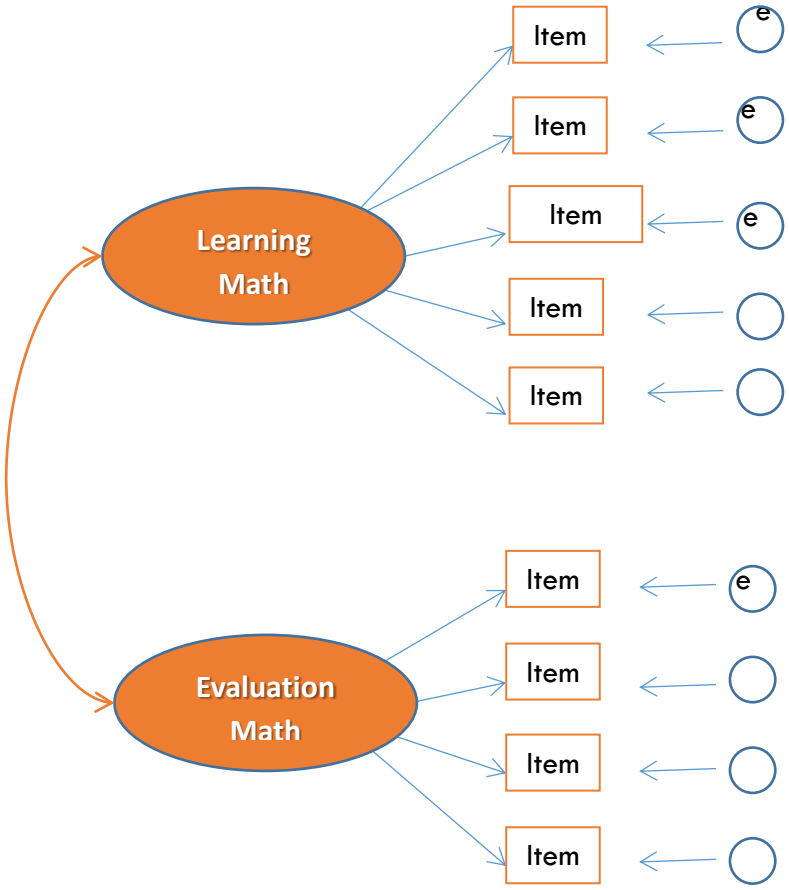

Figure 1: Model of the Abbreviated Math Anxiety Scale (AMAS)

Invariance

Measurement validity also implies that the meaning of the construct and its operationalization is the same in different social and cultural contexts. Testing the invariance of the test concerns the extent to which the psychometric properties of the test generalize across groups or conditions. Therefore, measurement invariance is a prerequisite of the evaluation of substantive hypotheses regarding differences between contexts and groups.

If the research question is, for example, about assessing gender differences in MA, and our test shows that female students have higher math anxiety scores than male students, we would be tempted to interpret test scores in terms of the trait that they are intended to reflect, i.e., that females have greater MA than males. However, it is possible that the test scores do not purely reflect the latent trait, i.e. MA in each group. That is, it is possible that the test is biased in some way.

Bias is used as a general term to represent the lack of correspondence between measures applied to different groups (Van de Vijver & Tanzer, 2004). There are different kinds of bias, for example

construct bias, when the meaning of the studied trait varies among groups; *item bias*, when the meaning of the item content is different in certain groups, or *method bias*, when the characteristics of instruments induce measurement errors for particular groups of respondents.

These biases violate the assumption of measurement invariance, which holds that measurement properties should not be affected by group membership (Zumbo, 2009). In other words, the observed scores should depend *only on the latent construct*, and not on group membership. An observed score is said to invariantly measure the construct if it is affected by the true level of the trait in a specific person, rather than by group membership or context (Meredith, 1993). This means that people belonging to different groups, but with the same level of a trait, are usually expected to display similar response patterns on items that measure the same construct. Thus, when studying test invariance, we determine whether the tool functions equivalently in different groups, that is, we test the absence of biases in the measurement process.

A well-known method to assess invariance is multiple group confirmatory factor analysis (MGCFA) in which the theoretical model is compared to the observed structure in two samples. Testing measurement invariance involves a step-by-step procedure in which nested models are organized in a hierarchal ordering. Specifically, the following invariance models are tested. The *configural* one, which refers to testing whether an instrument exhibits the same structure (Do the groups show the same general factor structure? Same number of factors? Same conceptual definition of latent constructs?). The next model, the *metric* one, tests whether the items function equally across groups. If this invariance is established, the groups can be said to have the same unit of measurement. The final model, the *residual* one, tests if measurement errors are the same across groups, which means that the scale is be equally reliable in both groups.

Applying this method, we tested the equivalence of the AMAS across male and female Italian students (Primi et al., 2015). With regard to the measurement issue, given that the assessment of MA relies on self-report measures, it is important to note that females are more willing to report their feelings of anxiety than males (e.g., Goetz, Bieg, Lüdtke, Pekrun, & Hall, 2013). This finding highlights the importance of employing measures of MA which are invariant across genders. That is, there is a need to test if the items measure the same construct when administered to male and female respondents, controlling for the differences in true group means. Indeed, to compare groups of individuals with regard to MA, one must be sure that the values that quantify the construct are on the same measurement scale.

The issue of measurement invariance has received considerable attention also in cross-cultural research because people from different cultures might have different understanding of the same questions included in an instrument (Milfont & Fischer, 2010). Indeed, testing invariance is of particular concern when using a translated version of a survey instrument, and it is a necessary prerequisite for the translated instrument to be used in cross-cultural research (e.g., Baumgartner & Steenkamp, 1998).

For this reason, we tested the invariance of the Italian version of the Statistical Anxiety Scale (SAS) developed by Vigil-Colet, Lorenzo-Seva, and Condon (2008). Learning statistics is often associated with statistics anxiety, defined as “extensive worry, intrusive thoughts, mental disorganization, tension, and physiological arousal [. . .] when exposed to statistics content, problems, instructional

situations, or evaluative contexts" (Zeidner, 1991, p. 319). In the original validation study, Vigil-Colet et al. (2008) analyzed the internal structure of the SAS using exploratory factor analysis. The results attested a three-factor structure: *Examination Anxiety* (referring to the anxiety involved when taking a statistics class or test), *Interpretation Anxiety* (referring to the anxiety experienced when students are making a decision about or interpreting statistical data), and *Fear for Asking for Help* (referring to the anxiety experienced when asking a fellow student or a teacher for help in understanding specific contents). The primary aim of our work was to confirm this factorial structure of the Italian version using CFA. As confirmation of the same base factor model was not a sufficient condition to establish the equivalence of the Spanish and Italian versions of the SAS, we tested the invariance of the factor model's parameters between the Italian sample and a comparison Spanish sample. Since the results indicated a substantial equivalence of the Italian and Spanish versions of the SAS, we can use the translated instrument in cross-cultural research, we can make meaningful comparisons between Italian and Spanish students' statistics anxiety, and we can develop intervention strategies to enhance students' achievement across Spanish and Italian educational frameworks.

To sum up, if measurement tools are not "invariant", instruments do not measure the same trait across the different groups or contexts, results are not comparable, and inferences about differences are misleading. As a consequence, methods for investigating biases should be implemented when new measures are created, when existing measures are adapted to new contexts or for different populations, or when existing measures are translated.

Conclusion

The foundation of all rigorous research designs is the use of measurement tools that are psychometrically sound. Confirmation of the validity and reliability of tools is a prerequisite for assuring the integrity of study findings.

In empirical research, comparisons are often made between distinct population groups, including groups from different cultures, genders, or that speak different languages. These analyses implicitly assume that the measurement of these outcome variables is equivalent across groups, although this assumption often remains untested. Measurement invariance can be tested and it is important to make sure that the variables used in the analysis are indeed comparable across groups.

In conclusion, testing the psychometric properties of tools, such as measurement invariance might help in increasing the robustness of findings across various groups and contexts**.**

Can the outcomes of PISA 2015 contribute to evidence-based decision making in mathematics education?

Gerry Shiel

Drawing on data from the OECD's Programme for International Assessment (PISA), which assesses mathematical literacy and other domains among 15-year olds in over 70 countries every three years, this paper explores the extent to which PISA outcomes in 2015 can be described as 'solid' and hence contribute to evidence-based decision making. It identifies aspects of PISA that render its findings 'solid', but also points to pitfalls that arise in interpreting PISA outcomes related to achievement. The paper concludes by examining how PISA can contribute to thinking about the nature of evidence-based findings in mathematics education.

Introduction

A key feature of the educational landscape since 2000 has been the Programme for International Student Assessment (PISA), a study sponsored by the Paris-based Organisation for Economic Cooperation and Development (OECD) that assesses performance in mathematics, reading literacy and science among 15-year olds in over 70 countries every three years. In addition to administering tests to students, PISA administers questionnaires to students, their parents and their school principals. The student questionnaire asks about students' socioeconomic status, their attitudes towards mathematics and other subjects, and their instructional experiences. This paper looks at performance outcomes in the two most recent PISA cycles – 2012, when mathematics was a major assessment domain, and 2015, when mathematics was a minor domain, and PISA moved from a paper-based to computer-based testing in most participating countries.

Interest in the extent to which PISA provides 'solid' or 'evidence-based' findings arises because of the strong impact that PISA has on policy making in many participating countries. In Ireland, for example, a significant drop in performance in mathematics and reading literacy in PISA 2009 led to the implementation of a *National Strategy to Improve Literacy and Numeracy 2011-2020* (DES, 2011). The strategy set out a series of measures designed to improve performance, including plans to enhance initial teacher education, curriculum and assessment. In parallel with the Strategy, revised curricula in mathematics at post-primary level have been rolled out in a phased basis since 2010 in an initiative known as 'Project Maths'. This involves a strong focus on developing students' conceptual understanding in mathematics, and on applying mathematical knowledge in solving problems in context using a range of methods. Ní Shuilleabháin (2013) described Project Maths as 'a philosophical shift in Irish post-primary classrooms from a highly didactic approach with relatively little emphasis on problem solving towards a dialogic, investigative problem-focused approach to teaching and learning mathematics' (p. 23).

A key feature of the National Strategy is the inclusion of national targets for performance in PISA mathematics. In an interim review of the Strategy (DES, 2017), there are targets of 10.5% of students achieving below proficiency level 2 by 2020, and 12.0% achieving levels 5-6. The first of these is quite an ambitious relative to current performance (15% performed below Level 2 in 2015), while the second is more modest (11% performed at Levels 5-6 in 2015).

Efforts to ensure that PISA findings are solid

The procedures around the development of PISA survey instruments, including the mathematics test, are designed to ensure that findings can be relied on and used by participating countries to enhance teaching and learning, and raise performance standards. The development of the PISA mathematics test and scale encompasses the following:

- An assessment framework is developed and published at the outset of each PISA cycle (e.g., OECD, 2013). The framework provides a definition of mathematical literacy in PISA, and outlines the content areas (mathematical content categories) and processes to be assessed, the contexts in which items are to be embedded and the item formats to be used. Items are then developed in a way that ensures that all elements of the framework are adequately addressed. The assessment framework is a key source of evidence to support the validity of the PISA tests.
- Items based on the framework are submitted by countries, or are developed by the consortium charged by the OECD with implementing PISA. Items are vetted by countries for cultural and linguistic appropriateness and suitable items are forwarded for field trialling.
- The PISA field trial is conducted on a sample of 15-year olds in each participating country, and the performance of items is assessed within and between countries. The outcomes of both classical item analysis and item response theory scaling are taken into account in determining the suitability of items. These items, along with any trend items not field-trialled, are then used to compile test forms for the main study.
- Considerable effort goes into ensuring that items are scored accurately, using scoring guides prepared by the PISA consortium. Many items are marked by two or four scorers, and real-time indices of inter-rater reliability are used to guide the quality of scoring.
- The PISA main study is implemented. Quality control is a key aspect of the Main Study, as countries are held accountable to quality standards (see below).
- Performance on PISA is scaled using Item Response Theory models and links with performance on earlier rounds are established.

A document, *PISA Technical Standards* (e.g., OECD, 2014), is issued in each cycle to guide countries in ensuring that their samples, response rates, security procedures, translation and coding practices are of a sufficiently high standard that their data warrants inclusion in international reports. For example, the 2015 Technical Standards indicate that response rates of 85% at school level and 80% at the student level are required. The achieved samples of countries failing to meet these criteria are examined in detail for potential bias. In some cases, countries have not been included in international reports because of low response rates (e.g., the Netherlands in 2000, and the UK in 2003).

At the end of each PISA cycle, a technical report is prepared by the PISA consortium and is issued by the OECD (e.g., OECD, 2017). It details the procedures used in each aspect of the implementation of PISA, including sample design, field operations, quality control, survey weighting, scaling, proficiency scale construction, and coding reliability.

The consortium charged with implementing PISA establishes expert groups for mathematics, science and reading literacy, and there is also a Technical Advisory Group, which advises the Consortium on its use of scaling and other procedures, and a Questionnaire Expert Group. These groups act as a further check on the quality of the PISA instruments and outcomes.

Hence, PISA has taken several precautions to ensure the quality and solidity of its findings. Notwithstanding the fact that PISA assesses the mathematics that students require for life after they leave school (or mathematical literacy) and for future study, rather than mathematics based on school curricula, the steps taken to ensure that findings are solid are extensive.

The introduction of computer-based assessment as a threat to the solidity of PISA findings

Prior to 2015, PISA implemented computer-based testing in subsets of countries on an optional basis. In 2012, for example, mathematics was assessed on paper in all 65 participating countries, and on computer on an experimental basis in a subsample of 32 countries. In 2015, however, there was a shift to computer-based assessment in most participating countries, with 56 of 73 countries, including all 34 OECD member countries, administering PISA in this format. The remaining countries administered PISA on paper.

The transition to computer-based testing in PISA presented some significant challenges for the OECD. A key component of PISA is the availability of trend data – that is, performance from one PISA cycle to the next must be placed on the same underlying scale so that average performance and performance across proficiency levels in each country and on average across OECD countries can be tracked from cycle to cycle. The task facing the OECD and its contractors[1] was to establish the feasibility of linking performance on the 2015 computer-based tests to scales based on performance on paper-based tests in earlier cycles. This was further complicated by a requirement to continue to provide trend data for countries that administered PISA in paper-based form in 2015.

There were several ways in which the transition to computer-based testing could have been managed, given the imperative to maintain trends. For example, all students (or equivalent samples of students) taking PISA 2015 could have been given paper-based and computer-based tests. Then trends could have been established with reference to performance on the paper-based measures and new computer-based scales could have been devised, based on the computer-based items, and used for trend analysis in the future. This would have eliminated any concerns about mode effects (an advantage or disadvantage arising from implementing PISA on computer).

The approach taken by the OECD and its contractors was to make adjustments in 2015 based on how the same items performed on paper and on computer in the PISA 2015 Field Trial, which took place in all participating countries in spring or autumn 2014. In the case of mathematics, which was a minor domain, items used in earlier PISA cycles (i.e., trend items) were transferred from paper to computer, and equivalent representative samples of students from each country took the paper- and computer-based tests. Hence, the purpose of the mode study was to ascertain whether tasks or items

[1] The lead contractor in PISA 2015 was the Educational Testing Service in the US. The lead contractor in all earlier cycles of PISA was the Australian Council for Educational Research.

presented in one mode (i.e., paper) functioned differently when presented in another mode (i.e., computer) and vice versa. For the purpose of analysis, items were pooled across countries, as individual countries did not have sufficiently large samples of students to allow for reliable comparisons of individual items across modes, or for an analysis of item-by-country interactions. Where item parameters were judged to be 'strongly invariant' (that is, similar on paper and computer), item parameters were constrained to be the same in the 2015 Main Study (OECD, 2017). In the course of the Field Trial analysis, a subset of items showed mode effects. To account for these effects in the Main Survey, different item parameters were estimated for paired paper- and computer-based items. According to the OECD (2017, Chp. 7, p. 53), 'this established an invariance model that assumes scalar or strong invariance for the majority of items and metric invariance for a minority of items for which difficulty differences were detected'. A correlation of .95 was found between paper-based and computer-based item parameters for mathematics in the Field Trial, further supporting a link between performance on computer-based tests in 2015 and paper-based tests in earlier cycles, as well as between computer- and paper-based tests administered in 2015.

The PISA 2015 Field Trial yielded other interesting findings that applied to mathematics as well as other domains. For example, across countries, students taking the Field Trial tests on computer had significantly fewer omitted responses than students taking the paper versions. Furthermore, there were fewer effects of cluster positon on performance when tests were administered on computer (that is, items administered by computer were more likely to perform in the same way regardless of whether they appeared early or late in the test). However, as Jerrim et al. (in press) note, while the Field Trial did not yield large differences across modes for male and female students, no analyses were conducted to examine potential interactions with variables such as ethnicity or socioeconomic status. They also questioned the representativeness of the samples used in the Field Trial, which, in some countries, could be described as convenience samples. They viewed this as weakening the external validity of the results, given the implications for the adjustments made within Main Study scaling to enhance cross-mode comparability.

Overall performance on PISA 2015 mathematics

The PISA main study took place in all participating countries in 2015. The OECD issued two volumes of findings in December 2016 that included country mean scores in mathematics, and comparisons with performance in earlier cycles. The mean score of students in Ireland in 2015 was 503.7 (OECD, 2016). This was significantly above the average across OECD countries (490.2), and was about the same as in 2012 (501.5), 2006 (501.5) and 2003 (502.8). Indeed, the only year in which average performance moved outside the 501-504 range was in 2009 (487.1).

While the mean mathematics score of students in Ireland was stable in the transition to computer-based assessment, a number of countries saw large declines in performance between 2012 and 2015. These included Korea (down 29.7 score points, though still well above Ireland at 517.4), Chinese Taipei (17.5), Hong Kong (13.3), Poland (13 points), and the Netherlands (10.7 points). On the other hand, a small number of countries experienced increases in achievement, including Sweden (15.7 points), Norway (12.4), the Russian Federation (11.9), and Denmark (11.1).

It is noteworthy, however, that Norway, Denmark and the Russian Federation were among the countries with the highest use of computers by students in mathematics classes in PISA 2012 for

purposes such as entering data on a spreadsheet, drawing a graph of a function, constructing geometric figures, re-writing algebraic expressions and solving equations (OECD, 2015). In contrast, Korea, Hong-Kong China and Ireland were among the countries with the lowest usage of ICTs by students in mathematics classes.

The fact that Ireland's overall performance on PISA 2015 is similar to 2012 can be interpreted in a number of ways:

- It suggests that students in Ireland are equally adept as solving mathematical problems in paper and computer-based formats; indeed, this would suggest that the mode of assessment does not matter, at least for students in Ireland.
- It suggests that students in Ireland improved in their mathematics between 2012 and 2015, but this improvement was largely hidden because of the transition to computer-based testing.

The second of these seems the most likely. PISA 2015 was the first PISA cycle in which all students in Ireland's sample had studied under the Project Maths syllabus. This interpretation is consistent with a finding that students in initial Project Maths schools (24 schools that had implemented Project Maths first) outperformed students in non-initial schools in PISA 2012 mathematics (see Merriman et al. 2013), though the difference was relatively small (4 score points) and not statistically significant.

A further relevant finding relates the optional computer-based assessment of mathematics administered as part of PISA 2012. In that assessment, students in Ireland had a mean score that was not significantly different from the corresponding OECD average score, despite achieving a mean score on paper-based mathematics that was significantly above the corresponding OECD average in the same year (Perkins et al., 2013). Hence, performance on PISA 2015 can be interpreted as being indicative of a possible improvement.

Interestingly, the OECD has continued to hold the positon that mode effects in PISA 2015 mathematics were small and did not impact on the performance of participating countries (OECD, 2016, 2017). Implicit in this is the view that performance on computer-based assessment in 2015 can be linked back to performance on paper-based assessment in earlier PISA cycles.

Other threats to the solidity of PISA 2015 findings

The transition to computer-based assessment in PISA is clearly one threat to the validity of scores reported by the OECD for PISA 2015 mathematics. However, there were several other changes to PISA 2015 which could also impact on the interpretation of outcomes, and hence the solidity of PISA findings. The changes – several of which occurred because a new consortium was contracted by the OECD to gather and analyse PISA data – include:

- Changes in the assessment design – the design of PISA 2015 was modified to reduce or eliminate differences in construct coverage for major and minor assessment domains for test takers. In practice, this meant that fewer students took mathematics in PISA 2015, compared with earlier cycles, but more mathematics items were included in the assessment, thereby allowing for broader construct coverage.

- Changes in the calibration sample – prior to 2015, item difficulty in PISA was estimated using the responses of students in the most recent cycle (e.g., in 2012, this comprised data from students who took PISA in 2009). Moreover, the calibration sample in earlier cycles comprised a random sample of 500 students per participating country. In 2015, item parameters were re-estimated using all students in all participating countries in the previous four PISA cycles. This change was implemented to reduce the uncertainty around estimates of the item parameters used in calibration.
- Changes to the scaling model – in earlier PISA cycles, a one-parameter Item Response Theory (IRT) model (with adjustment for partial credit) was used to scale performance. In 2015, item functions based on a two-parameter logistic IRT model for dichotomous data, and a generalized partial-credit model for polytomous data were used in scaling data in the case of new items, while functions based on a one-parameter model were used (as previously) with trend items. Unlike its predecessor, the new approach does not give equal weighting to all items when constructing a score, but assigns optimal weights to tasks based on their capacity to distinguish between high- and low-achieving students.
- Changes in the treatment of differential item functioning across countries – where items performed unexpectedly differently across countries, the calibration in 2015 allowed for a number of country-by-cycle-specific item parameters. In previous cycles, items that showed differential item functioning (e.g., because of differences across languages) were dropped from scaling. The change in 2015 was intended to reduce the dependency of country rankings on the selection of items included in the assessment (for a country) and hence improve fairness (OECD, 2016).
- Changes in the treatment of not-reached items – in PISA 2015, not-reached items (unanswered items at the end of a section, such as at the end of the first and second hour of testing) were treated as not administered when estimating proficiency (i.e., scoring student responses), whereas in previous PISA cycles they were treated as incorrect. A reason for this change was to eliminate the opportunity for countries and test takers to randomly guess answers to multiple-choice questions at the end of a section of the test. As in previous cycles, not-reached items were treated as not administered when computing item parameters (i.e., during scaling).

The OECD (2016) acknowledges that improvements to the PISA test design and to scaling in PISA 2015 can be expected to result in reductions in link error (the error associated with particular sets of items used in a particular cycle) between 2015 and future cycles. However, it also acknowledges that the changes described above may result in increased link error between PISA 2015 and earlier cycles, as past cycles used a different design (paper-based assessment) and used different scaling procedures. Furthermore, the OECD (2016) acknowledges that the change in the treatment of not-reached items could result in higher scores than would have been estimated in earlier PISA cycles for countries with many unanswered items.

Conclusion

The problem in terms of interpreting trend scores is that any of the changes implemented by the OECD and their contractors in relation to the design and scaling of PISA in 2015 could have impacted on the scale scores achieved by students. Interpretation becomes even more difficult when multiple changes are implemented, as these may interact with one another in complex ways. The OECD has sought to address this in a limited way by rescaling data from earlier PISA cycles using the methods implemented in 2015. Thus, in the case of Ireland, performance on PISA mathematics changed by +2 score points between 2012 and 2015 (see above), but, the change was 6.0 score points when newer scaling methods were applied to the 2012 mathematics data. On average across OECD countries, the impact of changes to scaling procedures was also reported to be small (a published drop of 3.7 score points between 2012 and 2015, and a drop of 2.5 score points following rescaling of the 2012 data) (OECD, 2016). For most countries, differences arising from re-scaling are within the error margins of the original difference scores reported by the OECD.

While the readjustment of scores from PISA 2012 using the new scaling procedures implemented in 2015 may go some way towards reassuring users that PISA outcomes are comparable over time, the sheer number of changes implemented in PISA 2015, including the change to computer-based testing, indicates that particular care should be exercised in interpreting PISA 2015 data.

Efforts to improve the design and scaling of PISA 2015 also contain some lessons for efforts to generate solid data in mathematics education. On the one hand, solid findings can be obtained by implementing the same testing procedures and methodologies on multiple occasions (e.g., pre- and post-intervention). In the words of Beaton (1990), 'when measuring change, do not change the measure' (p. 165). On the other hand, at least in the case of longitudinal, multi-year surveys such as PISA, there is an ongoing need to build innovation into all aspects of the project to maintain relevance and deliver more robust measures for the future. One clear danger is that, when mathematics becomes a major assessment domain in PISA 2012, the construct measured will also change, as new items specifically designed to take advantage of the affordances computers, will be introduced for the first time.

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TWG01: Argumentation and proof

Introduction to the papers of TWG01: Argumentation and proof

Gabriel J. Stylianides[1], Samuele Antonini[2], Orly Buchbinder[3], Kirsten Pfeiffer[4] and Kirsti Hemmi[5,6]

[1]Department of Education, University of Oxford, UK, gabriel.stylianides@education.ox.ac.uk
[2]Department of Mathematics, University of Pavia, Italy, [3]Department of Mathematics and Statistics University of New Hampshire, USA, [4]School of Mathematics, Statistics and Applied Mathematics, NUI Galway, Ireland, [5]Åbo Akademi University, Finland, [6] Uppsala University, Sweden.

Introduction

The role and importance assigned to argumentation and proof in the last decades has led to an enormous variety of approaches to research in this area. The 27 papers and 1 poster presented in the Thematic Working Group (TWG) "argumentation and proof" come from 18 countries across 4 different continents, and offer a wide spectrum of perspectives. These contributions intertwine educational issues with explicit references to mathematical, logical, historical, philosophical, epistemological, psychological, curricular, anthropological and sociological issues.

Taking into account this diversity, the contributions were presented and discussed in working sessions organized under the following 7 themes: (1) assessments issues of argumentation and proof; (2) theoretical and philosophical issues of argumentation and proof; (3) argumentation and proof in textbooks; (4) tools for analyzing argumentation and proof; (5) intervention studies on argumentation and proof; (6) argumentation and proof at the university mathematics level; and (7) task design in argumentation and proof. Since the themes are intertwined, each paper could be assigned to multiple themes. Therefore, the assignment of papers to themes was guided by a "best fit" approach as well as practical considerations. We will briefly discuss each theme separately.

Assessment issues of argumentation and proof

This theme included three papers, related to issues of assessment in the area of argumentation and proof: Kónya and Kovács' paper focused on development of inductive reasoning of prospective teachers by analyzing their problem-solving processes on a carefully selected problem. Hemmi, Julin and Pörn's paper investigated teachers' perspectives on the possibility of using students' common misconceptions, identified in prior research, as a starting point for activities that develop students' understanding and skills in proof. Demiray and Bostan's paper investigated pre-service middle school mathematics teachers' interpretations of statements regarding proof by contrapositive and the reasons for their incorrect interpretations. The discussion of the three papers in the TWG raised several important issues, such as:

The influence of sociocultural context should be considered in assessment findings.

Alternative variations of task design should be considered in the interpretation of students' performances.

Adopting a more positive model. Should researchers aim to identify students' competencies rather than misconceptions?

Theoretical and philosophical issues of argumentation and proof

The two papers in this theme addressed implications of Habermas' rationality theory. Conner's paper discussed how Habermas' rationality can be used to analyse how teachers support argumentation processes in their classrooms. Boero's paper showed the added value of analyzing individual student's thinking processes while attempting to prove a statement. The discussion of the papers raised several points, including the following:

What is the added value of applying Habermas' rationality to a particular kind of analysis, and what would be lost if it was not used?

The difficulty of applying the categories of Habermas' rationality to coding data and, in particular, the difficulty in distinguishing between teleological rationality and epistemological rationality.

Argumentation and proof in textbooks

The five papers presented in this theme were grouped based on their relation to argumentation and proof in textbooks. Žalská's paper described how different types of arguments enacted in one classroom were influenced by the textbook, the teacher beliefs, and the students. Wong's paper presented an examination of geometry chapters in a prominent Hong Kong textbook series and illustrated the limited opportunities for students to engage in the process of generalizing and providing proofs. Cousin and Miyakawa's paper described the evolution of proof in Japanese geometry textbooks and the role of the specificity of Japanese language on that process. Mesnil's paper described a reference for studying and teaching logic in France, while Bergwall engaged the TWG in analysis and discussion of reasoning-and-proving opportunities in Finish and Swedish textbooks on primitive functions. The discussion addressed several important topics, such as:

The role of language and linguistics in introducing, teaching, and writing proofs; and how the goal of teaching proof is articulated in a curriculum, represented in textbooks, and enacted in classrooms.

The role of mathematical logic in the teaching and learning of proving.

Definitions in research frameworks. Caution is required in the interpretation of the findings from different studies which operationalized certain terms in different ways.

The need for specialized analytical frameworks when examining argumentation and proof opportunities in textbook tasks versus textbook expositions.

Tools for analyzing argumentation and proof

The five papers in this theme concerned different tools for analyzing argumentation and proof. Ruwisch's paper concerned a one-dimensional model to rate reasoning competences at the primary level, considering both mathematical reasoning and its linguistic realization. With the same goal to better understand primary students' reasoning characteristics, Koleza, Metaxas and Poli used a simplified model of Toulmin's argumentation, drawing also on argumentation schemes described by Walton. The paper by Mata-Pereira and da Ponte aimed to understand how application of design principles regarding tasks and teacher actions can help provide students with opportunities to justify, and presented a framework that accounts for the level of complexity in students' justifications. In a longitudinal study, Fahse explored secondary school students' ways of argumentation on tasks concerning division by zero. He identified three different types of student argumentation and showed

how these relate to students' age. Focusing on teachers' competencies, Chua's paper presented a theoretical framework that classifies justification tasks by their nature, purpose and the expected element to be provided in the justifications. The discussion of the five papers raised some deep issues, including the following:

The validity versus utility of theoretical frameworks in argumentation and proof. The utility of a framework depends on how well it is designed to address a particular goal.

Multi-dimensional models of proof. Researchers should acknowledge the complexity of proof and specify the aspect(s) of proving that they are focusing on.

Language and argumentation. Investigating relations between language and argumentation requires clarifying what we mean by "mathematical language".

Classroom culture should be considered in interpretations of research findings.

Intervention studies on argumentation and proof

The five papers in this theme related to implementing proving activities in school mathematics classrooms. Reid and Vallejo Vargas' paper describes an intervention where 3rd graders learn division through "proof-based teaching" by developing a shared toolbox of justification principles. The study showed that 3rd graders are capable of reasoning deductively from premises when explaining their thinking. The paper by Soldano and Arzarello described students using game activities in Dynamic Geometry Environments (DGEs) to discover geometric properties of the mutual relationship between two circles. The authors found that games helped students to communicate their claims, formulate and check conjectures, and explain their thinking. Siopi and Koleza's paper focused on students' use of a specific tool, a pantograph, to explore geometrical properties of parallelograms. The paper by Pericleous and Pratt examined how a teacher helped students to foreground mathematical argumentation as they investigated geometrical properties within a DGE. Finally, Buchbinder reported on a study on professional development sessions where teachers became familiar with 'proof-task prototypes', applied them in their teaching, and reflected on this application. These activities helped teachers to involve proof-oriented activities in their ordinary mathematics classrooms. The discussion included the following issues:

What was the contribution of particular tools to students' learning?

Students' investigations within DGEs and the ambiguity of the expression 'play with the software'.

The need for structured support for teachers to implement proving activities.

Argumentation and proof at the university mathematics level

The five papers in this theme were concerned with teaching and learning of proof at the university level. Yan, Mason and Hanna's paper suggests an exploratory teaching style to promote the learning of proof, and describes specific pedagogical strategies. Selden and Selden's paper discusses theoretical perspectives for proof construction and its teaching. They suggest including psychological aspects of proving to these perspectives, and how these aspects should be considered in teaching and future research. Moutsios-Rentzos and Kalozoumi-Paizi's paper also considers psychological aspects by describing affective and cognitive experiences of a mathematics undergraduate student while producing a proof under exam-conditions. The innovative methodology of their study is to examine

students' facially expressed emotions during proving activities, as a way to study and influence students' attitudes towards proof. Gabel and Dreyfus' paper describes an attempt to analyze rhetorical aspects of proof presentation. They use Perelman's "New Rhetoric" as a framework to identify ways to analyze and increase the effectiveness of teachers' argumentation in mathematics classrooms. Azrou's paper suggests that students' lack of meta-knowledge about proof, such as features of mathematical proof and how a proof should be organized, influences their competence to write mathematical proofs. The discussions of these papers raised the following issues:

Phenomena (as behavioural issues) that have not been previously considered by psychologists or mathematics educators may play a role in students' difficulties to construct proofs.

The role of emotions and feelings in proof construction.

Task design in argumentation and proof

Although many papers touched on issues of task design, this was the main topic of the two papers in this theme. Komatsu and Jones's paper explores how task design can facilitate students' engagement with the mathematical activity of proofs and refutations in the context of a DGE. Hein and Prediger's paper explored the role of task design and scaffolding to foster students' learning of deductive reasoning, making explicit the logical structures and unpacking their verbal representations in geometry. The discussion included the following issues:

Proving something in a particular case: how can we help students see the generality of the proof?

The notion of scaffolding. How can we make explicit to students the logical structure of proving?

The special place of geometry in teaching, learning and researching of argumentation and proving.

Conclusions

We think that the TWG on argumentation and proof has offered the participants the richness of diversity in this research domain and the opportunity of fruitful discussions. At the last session of the TWG, the participants engaged in a discussion to identify areas in which they would like, and hope, to see more research in future CERMEs. The following areas were identified:

The teaching of proof and argumentation in both school and university settings, including in teacher education. The study of the classroom implementation of tasks rich in argumentation and proof opportunities, scaffolding and responding to unexpected student responses.

Issues of language in argumentation and proof. This also includes representations, structure, oral and written language, rhetoric and logic.

Aesthetics of proof and ways in which students of all levels of education can improve their attitudes, emotions, and beliefs about proof.

The identification of these areas is aimed at describing the state of the art of the field, without suggesting prioritizing certain areas of research. The TWG is committed to representing the diversity of perspectives and research areas on argumentation and proof in future CERMEs.

Meta-mathematical knowledge about proof

Nadia Azrou

University Yahia Fares, Medea, Algeria; nadiazrou@gmail.com

We deal in this paper with a particular difficulty with proof and proving at the undergraduate level, which concerns knowledge about proof at a meta-level. Some undergraduate students' difficulties or mistakes observed in their proof texts have been related to lack of that meta-knowledge. In order to test this hypothesis, interviews with a sample of students have been performed. Relationships with the didactic contract have been discussed.

Keywords: Proof, meta-knowledge, theorem, undergraduate students.

Introduction

Most mathematics teaching at all school levels is concentrated on teaching content; at the university level, students learn about functions, differential equations, matrices and integrals, by manipulating definitions and theorems. In order to assimilate the content, students are asked to solve problems and prove statements. The difficulties of students with proof have been largely investigated in research (Moore, 1994; Epp, 2003; Selden & Selden 2007; Harel & Sowder, 1998); some of these difficulties are related with the fact that students do not know mathematics at the meta-level, particularly as it concerns proof (Morselli, 2007; Hemmi, 2008). Several students do not see clearly the difference between a definition and a theorem, the difference between an example and a counter example. Knowledge about proof at the meta-level is neither presented in textbooks nor in courses of specific mathematical disciplines, but it makes one of the most important differences between mathematicians and students. In this paper, we will consider in particular the meta-knowledge about proof (MKP), such as the knowledge of the notion of proof and the rules related to how a proof must be organized. Many researchers acknowledge the fact that high school and university students do not understand what is meant by "proof" and "proving" (e.g. Schoenfeld 1989, Harel & Sowder 1998). "To most undergraduates, convincing their teacher (and thereby earning satisfactory grades) is typically the most important reason for constructing a proof" (Weber, 2004, p. 429) and "unfortunately, many students believe that they either know how to solve a problem (prove a theorem) or they don't, and thus, if they don't make progress within a few minutes, they give up" (Selden & Selden 2007, p. 96); students often believe that non-deductive arguments constitute a proof, or "an argument is a proof if it is presented by or approved by an established authority, such as a teacher or a famous mathematician" (Weber, 2003, p. 3); other different interpretations and conceptions of students regarding proof are described in Harel & Sowder (1998) and in Recio & Godino (2001). Meta-knowledge about proof is used implicitly by mathematicians when they construct proofs, "what may be assumed contextually and what needs to be explicitly proved, using logical deduction and previously established results, is highly non-trivial and, I would suggest, is implicit rather than explicit in the minds of most mathematicians" (Tall, 2002, p. 3). Our focus in this paper will be on the lack of MKP and what it might cause as difficulties to students when constructing proofs. The present study is developed using a past empirical study with undergraduates that consisted of investigating students' difficulties by analyzing their proof texts

responding to different tests (Azrou, 2015). We would like to examine if the following hypothesis is supported by an interview analysis: Is lack of MKP one of the reasons behind the messy proof texts? Moreover, we would like to answer the following question: Why students do not develop MKP?

Theoretical framework

We choose the definition of proof stated by Durand-Guerrier et al. in (Durand-Guerrier, Boero, Douek, Epp, Tanguay, 2012), inspired from the Vergnaud's conceptual fields (Vergnaud, 1990). According to Vergnaud, a concept (in our case proof) is learnt by acquiring three components: the set of different representations (oral, written, formal, etc), the situations of reference (proof in geometry, in algebra, in calculus, etc.) and the operatory invariants (related to the logical structure of proof: legitimated inference rules, status of hypotheses, thesis, axioms, etc). Mastering the MKP is mastering the concept of proof according to Vergnaud, as it was stated by Durand-Guerrier et al. (2012).

We are interested in comparing MKP that students acquire with how they have been presented proofs and how they have been taught MKP by their teachers. As we consider the relationship with the teaching regarding proof and proving, we will be referring to the didactic contract (Brousseau, 1988) that is defined as a set of rules framing the mathematical practices of teachers and students under the constraints of the teacher-students institutional relationships. Most of these rules regarding how, why and what teachers do mathematically (and students should learn to do) are implicit and thereby not declared by teachers, who often suppose that students would assimilate them over time and practice. Often, teachers use some particular intentions and rules with the proofs exposed to students, without being aware of and without feeling the importance of explaining them; consequently, sometimes students are misguided to make correct proofs. Let us take the example of proof writing: 'the processes used by mathematicians are often rough and informal, but students typically see proofs in their final forms, and rarely witness the process of creating a "rough draft", as a result, students often do not know where to begin when writing their own proofs' (Moore, R. C., 1994). We will examine what kind of MKP students learn from their teachers and how they manifest it.

Methods

We are more interested in examining in students' proof texts their MKP considering the three components of the definition of Durand-Guerrier et al. (2012); students' behaviors will be checked regarding definitions and other used mathematical statements (mathematical argument), regarding their modes of reasoning and argumentation and how they expressed them and presented them. Especially, how lack of MKP is manifested through students' proof texts and their interviews. A test composed of open questions (to which the proof cannot be procedural but rather syntactic or semantic (Weber, 2004)) have been submitted to 98 undergraduates during their third academic year, for a complex analysis course in a high level school of engineers in Algeria. The written language is French, but often, Arabic dialect is used, along with French, in the oral form. The analysis of students' proof texts indicates that one of the difficulties behind writing messy and disorganized proof texts to open questions was lack of MKP (Azrou, 2015). To receive further evidence for our findings, interviews were performed with a sample of students. We have chosen

fourteen students to interview whose proof texts contained well-organized, less organized and very disorganized proof texts. Our aim was to investigate, by analyzing students' words, whether they master the concept of proof at a conscious level, in other words if they have a mastery of its operatory invariants (Vergnaud, 1990).

A-priori analysis of the test

The test contained three questions, but in scope of this paper we can only include the first question.

1- Is it possible to find a holomorphic function that admits 0 as a simple pole such that Residue of *f* at *0* is *0 (Res (f, 0) =0)?*

By designing such questions, we aimed at ascertaining if students were able to construct the proof, based on known definitions and theorems, in a clear argumentation form by providing their own way of expressing the answering to the question. The question is about the possibility of having (P $\Rightarrow$ Q) and its negation (P and $\overline{Q}$) at the same time, which results in a contradiction and thus it is impossible. Logically speaking: the fact that the residue at a point is not 0 is a direct consequence of that point being a simple pole for the function. We have chosen to refer to the point 0 to simplify the formula. There was no doubt that students knew all these concepts because they had used them many times before, but always when performing direct calculations and procedures. However no request of identifying and exploring the links between concepts had been made, especially in a written form.

A preliminary analysis of students' proof texts

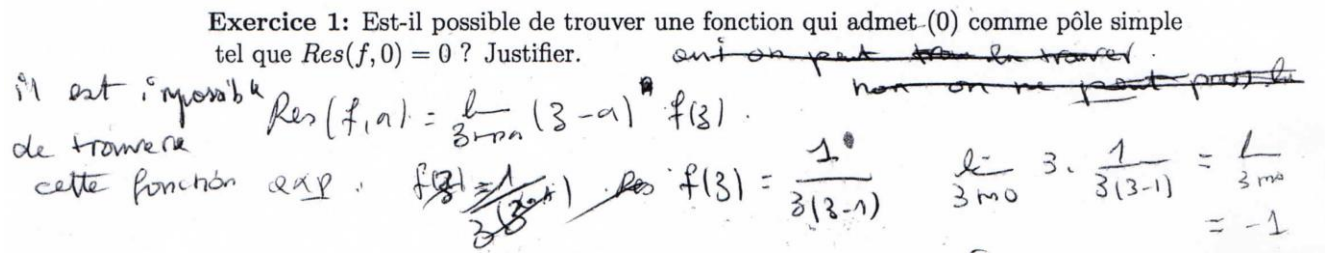
We have observed in students' proof texts, among others, the following behaviors related to the concept of proof:

- Lack of justifications: students do not know when the justification is necessary and when it is not; they might give a justification for an obvious fact and miss to justify a non-obvious statement.
- Students turn around confusing the hypothesis and the thesis (forwards and backwards between the premise and the result).
- An example is given instead of a justification to prove that the statement is true.
- Incomplete mathematical statements and/or formulas.
- Missing details that make holes in the proof.
- Lack of organization of proof steps.
- Disconnection between statements.
- Writing the proof text like a draft or a sketch.

Examples of students' proof texts

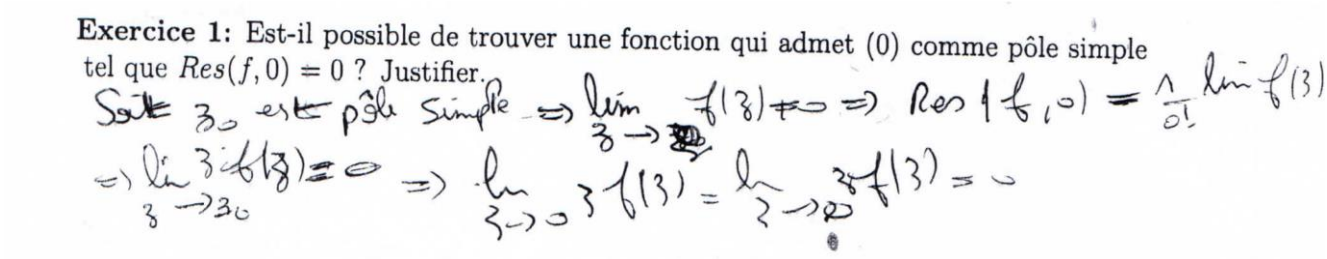
The following excerpts show some of the difficulties cited before; the language used is French.

Proof text 1



In this proof, the student responds by saying that such function does not exist and gives an example of a function that does not verify the conditions given by the statement. Clearly, the existence of a function that does not verify the conditions does not tell why these conditions cannot go together. This student considers that giving such example is the proof of the inexistence of functions verifying the two conditions.

Proof text 2



The student does not provide an answer to the question, the proof is a series of statements; each one derived from the previous one by an implication, but without any justification; moreover the last three ones are similar but incorrect, they present the simple pole definition, but the limit should be not 0.

Interviews

Based on the analysis of students' written productions, in the wider investigation this study belongs to, we have conducted interviews to address our previous questions, but also to receive more information about students' points of view. We will present only the questions of the interviews that deal with MKP about proof. The main interview contained three questions, each with three or four sub-questions.

Q1- If your answer would have been addressed to another teacher, would you have written it the same way?

a- What is important, to a teacher, to see in a student response to questions like this one?

b- Do you think another teacher, not familiar with the course, would have understood the answer?

c- How can the teacher know if the answer is right or not?

d- If the question has been proposed in homework, would you have presented it in a different way?

Q3- If a rigorous mathematician would have answered to this question, how would he presented his answer?

a- What is the difference from your answer text and those we find in mathematics books?

b- After this time, looking again to your answer, is there something you would add or change in yours answer or would you keep it as it is?

Results of interviews

Q1: Four students said 'yes', while the rest (ten) said 'no'. They intended their responses to be given especially to their own teacher, so they made their responses intentionally focusing on what is important to the teacher which is, according to them, their reasoning and showing that they got the idea of the process and understood enough the concept at stake in order to get the credit or a part of it. *'I know that my teacher will understand it even if it's not complete'*.

Q1.a: All students responded that the teacher would check in a proof whether a student got the whole idea of the solution or not: *'the teacher would see always the method'; 'the reasoning'; 'the process of the proof*'.

Q1.b: Half of the students said 'yes' and the others said 'no': *'no, because we are used to respond to get the credit, so we address the response to our teacher'.*

Q1.c: All of them responded mentioning the reasoning of the student (method, the logic in his response, whether it is convincing, if there is no contradiction): *'the teacher would follow the reasoning of the student to find out if his understanding is clear or not about the concept'*.

Q1.d: Twelve students answered 'no' and only two students said 'yes'. They would keep the idea or the method the same but make better the organization or the presentation: *'I would have changed the way I wrote, … the organization'*; *'I would have given more details'.*

Q3: Two students (good ones) said 'the same'; one didn't answer clearly, six said with more details and/or better organization; two said with better reasoning and three said with more symbols: *'a mathematician would have another goal, mine is to give the response and get the credit'; 'he would use only symbols till getting the final result, you see, I wrote a lot of comments'*.

Q3.a: Twelve students said that they would contain more symbols and less comments; with an academic rigorous style: *'it's different'; 'my answer is addressed to the teacher while mathematics books are addressed to all'*; '*with more symbols and less comments*'.

Q3.b: Four students among fourteen answered by keeping their text as they are. Five said they would improve the organization, three said they would add more details and two said they would make the explanation better: *'I might keep the idea, but I will give more details'; 'I would write it better'*.

Data analysis

Analysis of the written texts

Different students' weaknesses emerge from the analysis of students' written texts (difficulty of communication, lack of justification, using incomplete mathematical statements (or formulas) and lack of organization of the proof); the last three are of particular interest for MKP. Failing to give justification may be caused by the didactic contract supposing that the teacher would not mind it, by a lack of concept mastery or by a lack of meta-knowledge about proof. Mathematical statements are given incomplete because students might suppose that they are clear for the teacher, or because they are not well mastered by them or not important to be given complete in a proof text, which is related to MKP. The lack of organization of the statements displayed by students might be originated in didactic contract, in lack of concepts mastery, but also in lack of MKP.

Results of interviews analysis

The answers to Q1, Q1.b and Q1.c. confirm that students, when writing their proof texts, intend to address it particularly to their own teacher. The answers to Q1.d show that students are aware of their unclear text and possible missing details. According to them, they have to focus on two important points that have the same objective: how to get the most part of the credit and show to their teacher that they understood the concept at stake by presenting the main idea or the method of the proof; because they believe that the teacher will focus on that. This shapes their meta-knowledge about proof writing. Most answers to the third question and to Q3.a support more details would be given by a rigorous mathematician and mathematics textbooks, students mention that the organization would be better in both cases than theirs – but they reveal how their conception about proofs in mathematics concerns superficial aspects when they say that proofs in textbooks contain more symbols and less comments and words in comparison with their proofs and do not mention the structure of the proof. Answers to question Q3.b confirm that students are aware that their proof texts need improvement – but it must be related to previous consideration about superficial aspects.

Conclusion

Students' texts and interviews offered strong evidence for students' lack of MKP and its influence on proof writing. Students have many situations of reference for proofs at their disposal but do not master the operatory invariants of the proof concept and the form of the proof texts as conscious objects. Findings suggest particularly that the influence of the didactic contract is strong. Teachers generally write proofs in a direct linear way, making unfolding the steps till the conclusion. Students learn to do the same: when they first set some ideas about how to solve a problem, they write their first exploratory draft as a final text because they were never shown how to go further to the written proof text. Here, the didactic contract works against to the development of MKP because the contractual knowledge substitutes the knowledge about the concept of proof. An important element emerged in the interviews, which is the intention of the students to write the proof text only for their teachers, which supports our hypothesis of lack of MKP. Students acknowledge that their texts miss details, but do not see that these missing details would make the organization of the different parts of the proof clearer. This shows that the MKP and the didactical contract are strongly related. When students compare their texts with mathematicians' or textbooks' proofs, they only point out to symbols and comments, they do not see that in these perfect proofs, the statements are

linked through a deductive process from the hypothesis to the proof end, the proof text is organized, not only in its form, but also in the structure; avoiding holes, disconnections and missing justifications. This is evidence of students' superficial perception of proof texts, which indicates lack of mastery of proof structure and representation as a concept, which is related to lack at the operatory invariants level. As MKP is also built up through language, we hypothesize that students' weak mastery of French language, especially in the oral form, which should be translated to the written form, might have contributed to their unclear written texts.

Let us examine now why students are not able to develop their MKP; it seems that they are stuck in a constant perception of proof that does not help them to overcome their difficulties, if not causing some of them, and as long as there are not alternative ways of presenting proofs, they will hold on it. "Students need to understand that proofs are not generally conceived of in the order they are written" (Selden & Selden, 2007, p. 114) and that "successful reasoning can be carried out both by relying on the logic and formal structures of syntactic reasoning, and by relying on the informal representations of mathematical objects of semantic reasoning" (CadwalladderOlsker, 2011, p. 48). Changing or adjusting the didactic contract may favor students' autonomy to understand and make proofs; university teachers often mistakenly think that undergraduates understand what a proof is and how to make proofs by following the standard presented proofs. In fact, "while a traditional definition-theorem-proof style of lecture presentation may convey the content in the most efficient way, there are other ways of presenting proofs that may enable students to gain more insight" (Selden & Selden, 2010, p. 411). Teachers should provide samples of proof construction instead of final products, to be clear about what do they expect from students when they are asked to prove and to provide an opportunity to learn how to make proofs. "In general, professors should avoid "dumbing down" their assessments by asking routine questions that can be answered by mimicking. One needs to modify the "didactic contract" in order to achieve this; otherwise, questions requiring genuine problem solving and proving will be considered "unfair" " (Selden & Selden, 2010, p.414). We support that "university teachers should consider including a good deal of student-student and teacher-student interaction regarding students' proof attempts, as opposed to just presenting their own or textbook's proofs" (cf. Selden & Selden, 2007, p. 114). Finally, in order to gain control, students need to master meta-knowledge about proof; "the difficulty of students to reach a structural axiomatic proof scheme suggests that a capstone course including some attention to meta-mathematics as a topic might be of value to mathematics majors" (Harel and Sowder, 1998, p. 280).

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Conceptualizing reasoning-and-proving opportunities in textbook expositions: Cases from secondary calculus

Andreas Bergwall

Örebro University, Sweden; andreas.bergwall@oru.se

Several recent textbook studies focus on opportunities to learn reasoning-and-proving. They typically investigate the extent to which justifications are general proofs and what opportunities exist for learning important elements of mathematical reasoning. In this paper, I discuss how a particular analytical framework for this might be refined. Based on an in-depth analysis of certain textbook passages in upper secondary calculus textbooks, I make an account for analytical issues encountered during this process and identify aspects of reasoning-and-proving in textbooks that might be missed unless the framework is refined. Among them there are characterizations of generality, use of different representations, logical and mathematical structure, and ordering of material and student activities. Finally, implications beyond textbook research are discussed.

Keywords: Reasoning-and-proving, mathematics textbook, upper secondary calculus.

Introduction and background

Almost two decades ago, Hanna and de Bruyn (1999) pointed out that textbook research with specific focus on reasoning and proving was rare. Even though a number of papers with such a focus have been published in prominent journals since then, the field is still young. While the ultimate goal is to come up with well-founded prescriptions for textbook design, research is still striving to describe the current state of the art for reasoning-and-proving in textbooks (Stylianides, 2014).

Several studies have focused on (potential) opportunities to learn reasoning-and-proving (RP). Textbooks from different stages in mathematics education, from different educational contexts, and from different content areas have been studied (e.g., Davis, Smith, Roy, & Bilgic, 2014; Nordström & Löfwall, 2006; Otten, Gilbertson, Males, & Clark, 2014; Stacey & Vincent, 2009; Stylianides, 2009; Thompson, Senk, & Johnson, 2012). They typically include one or several of the following aspects of RP: generality (are statements justified with proofs or specific cases?), elements of proof-related reasoning (are students asked to make and investigate conjectures, find and correct errors, design counter examples?), proof methods (direct, indirect, by contradiction), purposes of proof (conviction, verification, discovery etc.), levels of formalism, and mathematical structure.

The variety of analytical frameworks developed for textbook studies can make it difficult to compare findings. However, some researchers have purposefully chosen to use frameworks and methods developed by others. For instance, the framework by Thompson et al. (2012) has been used with slight modification by Otten et al. (2014) and Bergwall and Hemmi (2017), and it was the basis for Bergwall (2015). Their framework is similar to the one developed by Stylianides (2009), which also has been used by Davis et al. (2014). While this simplifies comparison of findings, there is a risk that certain aspects of RP always are missed in the analysis. The purpose of this paper is to examine such potential aspects in relation to the framework by Thompson et al. (2012) and to contribute to a more refined conceptualization of opportunities to learn RP in mathematics textbooks.

Theory and analytical framework

Mathematics textbooks are widely used in classrooms around the world and are important links between national curricula and student learning (e.g., Stein, Remillard, & Smith, 2007). Tasks and expository sections, as they appear in a textbook, are potential sources for opportunities to learn RP. The concept of RP goes beyond formal proof and includes proving elements such as developing, outlining, or correcting an argument; deriving a formula; making or testing a conjecture; and providing a counterexample.

In this paper, I will focus on opportunities to learn RP through justifications in expository sections. I will use the framework and analytical procedure by Thompson et al. (2012). They employ a four item framework for justifications: A general proof is named a *general justification* (G); a deductive justification based on a generic case is named a *specific justification* (S); if the authors explicitly ask the student to provide a rationale it is referred to as *justification left to student* (L); and otherwise there is *no justification* (N). As in Bergwall and Hemmi (2017), I include all non-proof arguments in the S-category.

Stylianides (2009) uses a more refined framework with a separate category for specific justifications that are not generic. Otten et al. (2014) made modifications to the framework by Thompson et al. (2012) and distinguish between specific and general statements. They also have additional categories for justifications that only outline the general proof and for justifications that can be found in past or future lessons. We have adopted Thompson et al. (2012)'s methodology for the present and other studies (Bergwall, 2015; Bergwall & Hemmi, 2017). It has been put forward that mathematics education research needs more of cumulative research (Lesh & Sriraman, 2010) and we want to compare with – and build on – Thompson et al.'s extensive results on US upper secondary textbooks.

Textbook sample and analytical procedure

Cases for the present paper are chosen from the two most commonly used textbooks in Sweden and the only Finnish textbook available in Swedish (for Finland's Swedish speaking minority): Alfredsson, Bråting, Erixon, and Heikne (2012); Szabo, Larson, Viklund, Dufåker, and Marklund (2012); and Kontkanen, Lehtonen, Luosto, Savolainen, and Lillhonga (2008). I refer to them as SW1, SW2, and FI1 respectively.

In Bergwall and Hemmi (2017), we report our findings from an analysis of all expository sections and students' tasks on integral calculus in these textbooks (and others). In that study, we identified all mathematical statements presented as results and categorized their justifications using the framework described above. Like Thompson et al. (2012), we also checked if there were opportunities for the students to conjecture the result, how the statements were labeled, and what proving methods were used. Like researchers always do during such processes, we encountered a number of analytical difficulties. In the present paper, I will focus on these difficulties and on other issues that became apparent when the textbooks were compared to each other. I consider them a relevant base for discussing the development of frameworks for RP opportunities.

An upper secondary textbook cannot present a general theory for integral calculus. Thus its authors face the problem of what kind of justifications to include. This makes this topic relevant when examining frameworks for opportunities to learn RP. I will illustrate my findings with an analysis of the sections where students first encounter the definition of primitive function, the statement of the

representation formula $F(x) + C$ for all primitive functions to F', and the justification of this result. This particular choice was made since it includes a complete definition-theorem-proof chain for a central concept and a non-trivial result. Furthermore, the textbooks present this particular content quite differently.

Analysis and results

The analysis and results are presented as follows. I give a condensed description of how each textbook treats primitive functions, following the chronology of that textbook. This description will include all details needed to: (1) make an analysis according to the Thompson et al. (2012) framework, (2) describe analytical difficulties, and (3) make my points about the need to further develop the framework. Aside from the textbook's *definition*, *justification* and *statement*, I describe material placed immediately *before*, *after*, and *in between* them if such exists. This is followed by my *analysis* and description of *analytical difficulties* and *other issues*. Finally, I make a short *summary* of aspects of RP opportunities that could be better incorporated in the framework.

For easier reference, the descriptions of the justifications are presented as numbered lists. Note that the representation formula can be expressed as an equivalence. Therefor the (trivial) statement that $F(x) + C$ is a primitive function to $F'(x)$ will be referred to as 'the sufficiency', while the (non-trivial) statement that *all* primitive functions have this form is referred to as 'the necessity'.

SW1 (Alfredsson et al., 2012, pp. 173-174)

Before. There is one exercise where the student, based on graphical representations, shall identify which function has a certain derivative, and another where the student shall draw two different graphs with the same derivative. This is followed by a short note that it now is time to turn the problem of finding the derivative around.

Definition. The following text is framed and labelled 'Primitive function': "A function F is called a primitive function to f if $F'(x) = f(x)$."

In between. The authors write about three questions that need to be answered: How to find *one* primitive function, *all* primitive functions, and *the* primitive function satisfying a certain condition?

Justification.

1. x^2 and $x^2 + 5$ are presented as examples of functions with derivative $2x$ and the reader is told that "whatever constant C we add to x^2 we get a primitive function to $f(x) = 2x$".
2. There are plots of the graphs to $x^2 + 1$, x^2, $x^2 - 1$ and $x^2 - 2$, and the authors write: "Obviously, graphs to functions with the same derivative must for every x-value have the same slope. Hence the graphs have the same form, they are only translated in the y-direction".
3. The authors continue: "This means that if $f(x) = 2x$ then every function $F(x) = x^2 + C$, where C is a constant, is a primitive function to $f(x)$".
4. The authors ask if there are other functions with derivative $2x$ and immediately answer that it can be proven that there are no such functions.

Statement. The following text is framed and labelled 'Summary': "If $F(x)$ is a primitive function to $f(x)$ then $F(x) + C$, where C is a constant, denotes all primitive functions to $f(x)$".

After. There are two worked examples illustrating how primitive functions are determined, a table with some elementary primitive functions and then a student exercise set.

Analysis. (1) provides two specific cases for the sufficiency (x^2 and $x^2 + 5$), and it is said in words (without explanation) that any additive constant works. The necessity is touched upon in (2). This might be meant as an intuitive argument. But it is merely a formulation in words of the statement itself with no further warrants for the conclusion. The authors also chose to return to the sufficiency in (3) before they return to the necessity in (4), but once again without any argument. This means that in relation to the framework by Thompson et al. (2012) the sufficiency is justified with a specific case (S) and that there is no justification (N) for the necessity.

Analytical difficulties. The first difficulty was to decide if this justification should be counted as one or two. In Bergwall and Hemmi (2017), we chose the second alternative. However, if the unit of analysis is the justification of the statement as it is formulated in the textbook one could also choose the first. Then there are at least two alternatives: the justification receives the code N (since there are not justifications for both directions) or the code S (since there is a specific case justification for at least one direction).

The second difficulty was whether (2) should be counted as an intuitive justification of the necessity and receive the code S instead of N, since it seems to have a convincing purpose.

Other issues. Even a specific case such as $x^2 + C$ has some generality to it: the identity $(x^2 + C)' = 2x$ holds for *all* x. This indicates that when dealing with functions there is room for a more nuanced way of describing justifications than the categories G and S admit. Also, if the textbook statement had been that $x^2 + C$ denotes all primitive functions to $2x$, then the justification offered for the sufficiency is a general proof.

Summary. The analytical framework/method should be developed to better account for opportunities to learn: *the difference between an equivalence and an implication and how such are justified*; *the roles of different kinds of non-proof justifications, such as intuitive arguments based on visual impressions from a drawing of an "arbitrary" case*; and *that justifications can be specific in different ways when statements include several kinds of variables (dependent and independent), and that whether a justification is general or not also depends on how general the statement is.*

SW2 (Szabo et al., 2012, pp. 154-155)

Before. The authors demonstrate how velocity can be obtained by differentiating the distance function and then state that the opposite problem can be solved by asking which function has a certain derivative. In the margin there is a table with some elementary derivatives.

Definition. The following text is framed and labelled 'Primitive function': "A function *F* is a primitive function to f if $F'(x) = f(x)$."

Justification.

1. x^2, $x^2 + 5$, and $x^2 - 4$ are given as examples of functions with derivative $2x$ and in the margin it is emphasised that the derivative of a constant term is 0.
2. The authors write: "You can add and subtract any constant to a primitive function without altering its derivative. Thus a given function has an infinite number of primitive functions".

Statement. The following text is framed and labelled 'All primitive functions': "If $F'(x) = f(x)$ then $F(x) + C$, where C is a constant, gives all primitive functions to $f(x)$."

After. There are two worked examples illustrating how primitive functions are determined followed by a student exercise set.

Analysis. The sufficiency is justified with three specific functions in (1). That any constant C can be added/subtracted is explained in (2). However, it is not clear if the first sentence cited in (2) refers to a primitive function to *any* function or to a primitive function to $2x$. In the former case, the argument could have been expressed symbolically as $(F(x) + C)' = F'(x)$, which most teachers and mathematicians would have accepted as a proof. In the latter case, the sufficiency is only justified with a specific case. Concerning the necessity, there is neither a justification nor a remark that there is something more to prove. Summing up, this means that there is an ambivalence concerning the sufficiency ((S) or (G)) and that there is no justification (N) for the necessity.

Analytical difficulties. The question arises whether (2) is a general proof or not. There are two issues here: The use of words instead of algebraic symbols, and clarity in what the authors refer to.

Other issues. When comparing SW1 and SW2, we see at least three differences even though the classifications of the justifications are the same. First, SW1 discusses the necessity and states that it can be shown that there are no other primitive functions, which SW2 does not. But neither textbook clearly expresses the representation formula as an equivalence. Second, SW1 uses graphic representations and describes the meaning of the statement in terms of slope and form which SW2 does not. Third, SW2 is less vague in its labelling and formulations. While SW1 labels the statement "summary" and expresses that $F(x) + C$ "denotes" all primitive functions, SW2 uses the label "All primitive functions" and expresses that $F(x) + C$ "gives" all primitive functions.

Summary. The analytical framework/method should be developed to better account for opportunities to learn: *what needs to be justified, what has been left out of a certain justification, or if a justification is a proof or not*; *the role of different forms of representations*; and *the structure of mathematics, i.e. what part of a mathematics text that is a definition, a statement, and a proof, and what their different roles are.*

FI1 (Kontkanen et al., 2008, pp. 7-8)

Definition. The following text is framed and labelled 'Primitive function': "Assume that the functions f and F are defined in the open interval I. The function F is a primitive function to f for every $x \in I$, if $F'(x) = f(x)$."

In between. In worked examples, the authors demonstrate how one checks if a certain function is a primitive function to another given function. In one of these examples, it turns out that two different functions can be primitive functions to the same function. However, the algebraic descriptions of these functions are not such that it is obvious that they only differ by an additive constant.

Statement. The following text is framed and labelled "theorem": "Assume that F_0 is a primitive function to f. Then all functions of the type $F(x) = F_0\,(x) + C$ are primitive functions to f. The function f has no other primitive functions."

Justification. The justification is labelled "proof" and divided in two steps. First the sufficiency is justified by differentiation of $F(x) = F_0\,(x) + C$. Then the necessity is justified using the fact that if

a derivative is 0 everywhere the function is constant. For this fact, there is a reference to a theory section at the end of the book.

After. It is pointed out and illustrated in a diagram that the additive constant C corresponds to a vertical translation of the graph. The notation $\int f(x)dx$ is introduced. This is followed by three worked examples on calculation of primitive functions and a set of student exercises.

Analysis. The sufficiency and the necessity are both justified with general proofs (G).

Analytical difficulties: There are none that have not been mentioned so far.

Other issues: In FI1 it is clear that the statement contains two parts even though it is not formulated as an equivalence. The justification is labelled proof (SW1 and SW2 have no labels on their justifications). The justification comes after the statement (not before as in the Swedish books). There is a graphical interpretation of the statement but it is put after the proof (not before as in SW1) and it seems to have the purpose of illustrating the meaning of the statement (and not to justify it as in SW1). FI1 is the only textbook that emphasizes that being a primitive function actually is a global property (i.e. that $F'(x) = f(x)$ should hold for *all* x in an interval). However, as in SW1 and SW2 the definition is phrased using the word 'if' even though it should be interpreted as 'if and only if'.

SW1 and SW2 have activities and/or worked examples before the definition which together with their justifications give the student an opportunity to discover and conjecture the statement. In FI1 the section starts with the definition. The indefinite integral notation is used throughout FI1 but is completely avoided in SW1 and SW2.

Summary. The analytical framework/method should be developed to better account for opportunities to learn: *mathematical formalism, detail and notation*; *different purposes with different forms of representation*; *the conjecturing as well as the verifying nature of mathematical work*; and *the importance of clear definitions*.

Discussion

When opportunities to learn RP are studied in textbooks there are several aspects to take into account and there is always a risk that important aspects are left out. The examples mentioned above illustrate a number of such aspects identified when a specific analytical framework was applied to a few textbook passages on primitive functions. Here I chose to discuss the importance of four such aspects of RP and their relevance in a refined framework for RP.

The first aspect is *generality* and relates to opportunities to learn what makes a justification a proof. Students' difficulties with understanding the difference between a general proof and an example are well-established (e.g., Harel & Sowder, 2007). However, justifications can have different levels of generality, or 'scope of variation', which opens up for sub-categories of non-proof justifications (e.g., Bergwall, 2015). Also, a justification must be judged in relation to the statement's formulation and the level of detail in relevant definitions. Thus an analysis of textbook justifications should include an analysis of statements (which Otten et al. (2014) do) and definitions.

The second aspect concerns *forms of representation* and relates to opportunities to learn how proofs are communicated. Sometimes a justification is better expressed in words but often algebraic symbols bring more precision and detail to the argument. Graphical representations may be used to illustrate

meaning as well as the idea behind an argument. Frameworks should take the use of different forms of representation and their roles and purposes into account.

The third aspect is *structure* and relates to opportunities to learn the role of proof in mathematical theory. Here I include the logical structure of individual definitions, statements and justifications as well as the overall structure of the mathematical theory, with its definitions, theorems and proofs, and the connections between them. To some extent this is captured in an analysis of labeling (as in Thompson et al. (2012)) and references to other lessons (as in Otten et al. (2014)).

The fourth aspect is about *ordering* of the material, including student exercises and worked examples, and relates to opportunities to learn different purposes of proof, and to how justifications can serve different educational purposes. Student investigations, specific cases and intuitive arguments placed before a statements can emphasize the creative and conjecturing side of mathematical work, while formal general proofs placed after the statement can emphasize the verifying and organizing side.

All four aspects have one thing in common. They concern proofs and justifications as objects and not only as processes (e.g., Sfard, 1991). To analyze if textbooks offer opportunities to understand proofs and justifications as objects, the analytical frameworks and methods need to focus on opportunities to learn object properties of proofs and justifications. Generality, forms of representation, structure, and ordering are examples of such properties.

Finally, development of frameworks and methods that better capture important aspects of RP are of importance not only for textbook analysts and textbook authors. Similar frameworks can be used for analyzing lecture scripts and teaching episodes. Hence they can also aid teachers when they plan their lectures and teaching elements. A detailed framework risks being of limited analytical use but is an important contribution when conceptualizing opportunities to learn RP.

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Cognitive unity of theorems, theories and related rationalities

Paolo Boero

DISFOR, Genoa University, Italy; boero@dima.unige.it

The construct of cognitive unity of theorems was introduced twenty years ago to identify suitable conditions for students' "smooth" approach to proving. In this paper the Habermas' construct of rationality, adapted to mathematics education in previous research, is used to identify some factors in the activation of cognitive unity of theorems. In particular, I consider the dependence of cognitive unity on the specific rationality (e.g. analytic geometry rationality, or synthetic geometry rationality) according to which a conjecturing and proving problem is dealt with. The analysis of some examples will provide evidence for it, together with hints for further research.

Keywords: Theorems, conjecturing and proving, cognitive unity, proving as rational behavior.

Introduction

"Cognitive unity of theorems" (CUTHE) is a construct introduced in Garuti, Boero, Lemut & Mariotti (1996) to account for a phenomenon detected in a grade 8 (13-years-old students) classroom engaged in a conjecturing and proving activity, concerning a theorem of space geometry contextualized and verbally expressed in terms of Sun rays (instead of straight lines) and Sun shadows (instead of shapes projected on a plane according to parallel projection rules). The conjecturing task (see Garuti et al, 1996) may be shortly reported this way: "Is it possible that the Sun shadows of two non-parallel sticks are parallel on the ground? If yes, under which conditions?". After comparison and standard re-phrasing ("if… then…") of their conjectures, students were asked to validate their statements by "general reasoning". We observed that, while trying to validate their conjectures, several students resumed some pieces of personal reasoning (e.g. ways of looking at the Sun rays and the Sun shadows) developed during the production of the conjecture and the search for reasons for its validity, and arranged them in a deductive chain of statements. The ways of looking at the Sun rays and the space relationships had been different for different students; those ways corresponded to the different ways of proving the theorem by them. After having found other theorems (in geometry, and in elementary arithmetic) for which students behaved in a similar way, we defined "cognitive unity of theorem" (CUTHE) what happens for some theorems when:

> during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain (Garuti, Boero, & Lemut, 1998, p. 345).

The CUTHE construct was also extended to the case of the relationships between the exploratory phase of proving a theorem, and the subsequent construction of a proof for that theorem (Garuti et al., 1998): indeed, the exploratory phase of proving shares some common aspects with conjecturing (as re-construction of the meaning, and appropriation, of a statement; and identification of elements for its validity). The construct of cognitive unity resulted in various research developments. Pedemonte (2007, 2008) performed studies in which (given a theorem for which CUTHE is accessible to students) the mechanism of arranging arguments produced in the exploratory phase

does not result in a proof for some students, due to their difficulty of re-arranging inductive or abductive arguments into deductive arguments. These difficulties are not likely to emerge in algebraic conjecturing and proving (Pedemonte, 2008), while they frequently emerge in the case of plane geometry (Pedemonte, 2007). Leung and Lopez Real (2003) investigated CUTHE in the case of computer-based learning environments, which change the nature of students' exploration and make CUTHE difficult to activate, finding out ways of activating it in the new situation. Fujita, Jones and Kunimune (2010) studied conditions under which CUTHE may be activated in the field of synthetic geometry: they "analyze the circumstances when students *unite,* or not, their conjecture production and proof construction"; the potential of geometrical constructions for the activation of CUTHE was explored. The quoted studies suggest the opportunity of investigating the conditions for the activation of CUTHE for a given theorem. Boero et al. (1998) started a discussion on it, taking into account both the student (her skills, her knowledge and expertise in a given field of mathematics) and the field of mathematics in which a given statement is dealt with. Douek (1998) analyzed the individual variety of exploration strategies and their effects on conjecturing and proving; at present (personal communication) she is further deepening the idea of subject-relativity of CUTHE, together with the relationships between the quality of student's exploration (including its semiotic features) and the construction of the proof. In this paper I will try to identify some aspects of the relativity of CUTHE referred to a given system of discursive practices that concern the truth of statements, the ways of producing and validating them, and the ways of communicating with others - i.e. a "rationality", according to Habermas (1998).

Theoretical assumptions

Mathematical theory

It is possible to define a mathematical theory (shortly, a theory) by its characteristic components: primitive notions, and definitions related to them; postulates; inference rules to get true statements from the postulates and other statements proved as true. 'Characteristic components' depend on the historical period and, in a given historical period, on epistemological assumptions that may be different, according to different fields of mathematics. The case of Euclidean geometry before and after Hilbert's Grundlagen der geometrie is a well-known paradigmatic example of historical change in the ways of considering the requirements of a mathematical theory. In this paper, we will consider the following theories: Synthetic geometry (in particular, Euclidean geometry); Analytic geometry (including the algebraic treatment of conic sections); Elementary, verbal-semantic number theory (evidence for truth and inference rules rely on properties of the concept of number and its representations); Elementary, algebraic-formal number theory (evidence for truth of statements comes from the interpretation of an algebraic expression derived, through suitable syntactic transformations, from the algebraic expression which represents the problem situation).

Theorem

Mariotti (2001) defines a theorem as a statement and its proof with reference to a theory (and related inference rules). The definition results in the possibility of considering different theorems with the same statement (in particular, when different proofs referring to different theories are available). The definition encompasses theorems related to various kinds of theories and related inference rules: Euclid's as well as Hilbert's geometry; analytic geometry; graph theory, with its

reference to visual objects; 19th-century probability theory as well as Kolmogorov's axiomatic theory, etc.; and the different ways of considering proof since the Greeks, including verbal-semantic proofs (like in Euclid) and modern algebraic-formal proofs of arithmetic statements.

CUTHE and Habermas' rationality

In this paper we are interested in CUTHE, one possible aspect of the conjecturing and proving process, in order to identify for which theorems (in Mariotti's sense) it may be easily activated, thus we need a comprehensive frame to deal with the process of proving and its relationships with the product (proof) to be built up in a given theory. Habermas (1998, pp. 310–317) deals with the complexity of discursive practices according to three *interrelated* components, concerning: knowledge at play, and the answer to "why is it true" questions in a given cultural context (*epistemic rationality*); action and its goals, and strategies to achieve them, to be evaluated (*teleological rationality*); communication and related, intentional choices in a given social context on a given subject (*communicative rationality*). In Boero & Planas (2014) a detailed elaboration of the reasons for adapting Habermas' construct to mathematics education is presented, with references to how it has been used in different studies. In the case of proof and proving, according to Mariotti's definition of theorem, the adaptation of the Habermas' construct concerns:

- criteria for validity of inferences and truth of statements within a theory, and their dependence on historical periods, mathematical domains, and institutions and cultures. Inferences may rely on visual evidence, or conceptual meaning, or syntactic transformations, etc.;
- problem solving strategies that may be adopted to reach the goal of proving, along with their effectiveness; strategies may use analogies, abduction, and so on. Strategies and exploration are not constrained within the border of the reference theory;
- the choice and use of appropriate communication means for proof in a given context,

together with the relationships among them, taking into account the goal of the proving process — a proof, conforming to requirements specified for the first and the third components. The expression 'rationality frame' will be used to put into evidence the system of epistemic constraints, strategies and forms of communication, which works as reference for proving and proof in a given theory.

Examples

Moving to the school, the role of the following examples is to provide evidence for the hypothesis that CUTHE depends, for the same statement, on the specific rationality frame in which a conjecturing and proving problem is dealt with by the student; and also to provide elements for further investigation. The examples will include some excerpts from students' think aloud solving processes. *Italic* is for written texts. (…) is for omitted sentences. … is for a pause in oral speech.

Example 1

The same conjecturing and proving problem was proposed in grades VIII and IX: "Consider all the products of three consecutive natural numbers. What is their GCD? Prove that it is their GCD". S-A is a grade VIII (13-years-old) student not yet familiar with the use of letters to prove:

Student S-A: *1·2·3=6 2·3·4=24 3·4·5= 60 10·11·12=1320;* it is evident that 6 is the GCD of the first three products, because it is the greatest divisor of the first product and a divisor of the other products. Is it a divisor of 1320? … Yes, 1320 is

an even number divisible by 3 because the sum of its digits is a multiple of 3. Then 6 might be the divisor of all the other products too. But why? Probably, by looking at these four products, all the products are even… But why? OK, one factor is always even! Even numbers go two by two, thus among three numbers one number … one number at least is even, and they may be two, like in the case of 2·3·4. Look at, three is there! And a multiple of three is in the last product! Why? In the case of 2, multiples go two by two … In the case of 3, numbers go three by three. That is the reason! Now I try to write down the general reasoning: *The greatest common divisor is 6 because every product is divisible by 6 because every three consecutive numbers contain one even number (multiple of 2) and one multiple of 3, because multiples of 2 go two by two, and multiples of 3 go three by three* (The teacher writes the following question: *Why greatest*?) (after a while) *Because the first product is divisible by 6, and no greater divisor is there*.

S-A resumes the examples, which conjecturing was based on, to identify *general* reasons for the truth of the conjecture. The intention of proving is related to the emerging conjecture, through "But why?" self-posed questions of epistemic relevance. A narrow intertwining between epistemic, teleological and communicative components of rationality allows the student to move continuously from exploration to the production of the conjecture, to proof construction by exploiting relevant elements got during the exploration, and then to proof writing. We may consider S-A's solution as an example of CUTHE in the rationality frame of verbal-semantic elementary theory of numbers.

S-B is a grade IX student who tries to solve the problem after some classroom work (about 10 hours) on the use of letters to prove in an algebraic way. Note that he would be free (according to the didactic contract) to choose another way of solving the problem, as other schoolmates do:

Student S-B: $(n+1)(n+2)(n+3)=(n2+2n+n+2)(n+3)=(n^2+3n+2)(n+3)=n^3+3n^2+2n+3n2+9n+6=n^3+6n^2+11n+6$. I do not see anything. But if I consider, for instance, 2·3·4=24 3·4·5= 60 5·6·7=210 I see that… Yes, I see that 6 is always a divisor, because I see it as 2·3, as one half of 3·4, as 6 in the products. The same for 13·14·15. (…) 24 is also divisible by 12, and by 8, but 60 is not divisible by 8, but it is divisible by 12. Let us see 210: (…) not divisible by 12, thus 6 is the only remained candidate! With algebra: $n^3+6n^2+11n+6=6(n^2+1)+n(n^2+11)$. I do not see anything. Perhaps it is not true! 16·17·18 (…) not a good counter-example! Because 18 is divisible by 6. 21·22·23= (the student uses his cellular phone to make calculations; the product is divisible by 6). Perhaps it is easier by considering: $(n-1)n(n+1)=n(n^2-1)$.… I see nothing! I am not able to prove it!

S-B tries to solve the conjecturing and proving problem in the rationality frame of elementary algebraic theory of numbers; the difficulty to produce a conjecture in that frame is overcome by moving to the rationality frame of verbal-semantic theory, where afterwards he will also try to dispel a doubt on the truth of the conjecture by considering a further, more elaborated example. Differently from S-A, no effort is addressed to find general numerical regularities that might be exploited to build up a verbal-semantic proof. In terms of rational behavior, this is an example of lack of connection between two different strategies (teleological aspect): to produce the conjecture and afterwards to provide some empirical evidence for it; and to produce a general reasoning for

proving. As a consequence, CUTHE does not work in the rationality frame where it could have been activated (verbal-semantic theory). The same happened with the other students who tried to build up an algebraic - formal proof. Note that an algebraic - formal validation of the statement may be performed either in combinatorics, or in modular arithmetic. Some schoolmates get the conjecture in the rationality frame of verbal-semantic theory of numbers, then they consider the products $(n+1)(n+2)(n+3)$ or $(n-1)n(n+1)$ and realize that in these products one number is divisible by three and at least one number is divisible by two; thus proving still relies on semantic considerations related to the number line and the positions of multiples of 2 and 3 in it, like in the case of S-A. The algebraic expression of the product is only a device to favor the transition to a general reasoning. CUTHE works thanks to the intention of finding general regularities and a proof in the same rationality frame of verbal-semantic theory, where the conjecture had been produced.

Example 2

A conjecturing and proving problem was proposed by the same teacher in grade XI, in three parallel classes, as an individual task: "Among the triangles with a given side and the same perimeter, find the triangle with the greatest area". Those classes were familiar with conjecturing and proving in number theory (both in a verbal-semantic way and in an algebraic way), and in Euclidean geometry.

The first class at that moment was familiar only with proving in plane Euclidean geometry; according to the conjecturing style of Euclidean geometry, some students (one third of that class) got the conjecture (the solution of the problem is the isosceles triangle) by considering that, after drawing some triangles, an isosceles triangle looks as the "widest" one (students say: "the fattest") among the drawn triangles (but three students got the conjecture of a right-angled triangle with the same considerations); a few students got the conjecture through a "limit & symmetry" consideration related to the fact that, when the triangle becomes strongly asymmetric, the surface within it becomes very "small", if we want to keep the same perimeter. During the discussion on the produced conjectures, after disproving (through measures) the conjecture concerning the right angled triangle, some students proposed to consider another triangle with the same height of the isosceles triangle (thus with the same area), and to try to prove that its perimeter is longer than in the case of the isosceles triangle. But a rigorous proof is not easy to build up, and in fact no student built it up, in spite of a long time spent for it in the classroom, by working in small groups (and then at home as well!); a relatively easy proof needs an auxiliary construction and the use of related theorems. The exploration to get the conjecture only suggests a first step of a proving process, and does not provide the ingredients to build up the proof: CUTHE does not work.

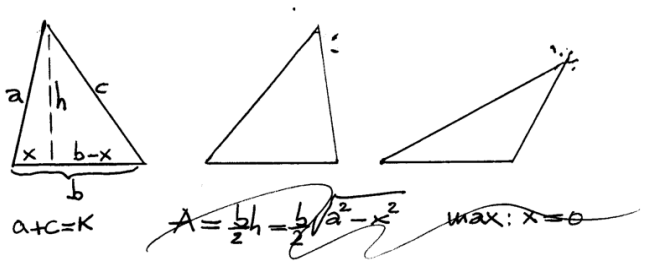
The second class had already met conic sections in synthetic geometry (they knew that an ellipse is the locus of points whose sum of distances from two given points is constant, and its basic properties concerning symmetry, axes, etc.). In this class, the conjecture was produced in a similar way as in the first class; but one fourth of students, thanks to the drawings of some triangles with approximately the same perimeter, arrived also to make a link with the ellipse in synthetic geometry. Students shared what had been discovered; then (by working in small groups) four groups out of six were able to solve the proving problem by considering the properties of an ellipse in synthetic geometry. The exploration provided students with a visual link with the ellipse in synthetic geometry, thus bridging conjecturing with proving – even if proving did not rely on the

considerations ("fatness" of triangles) that has generated the conjecture (and thus CUTHE did not work). Here is an excerpt from S-C's think aloud process:

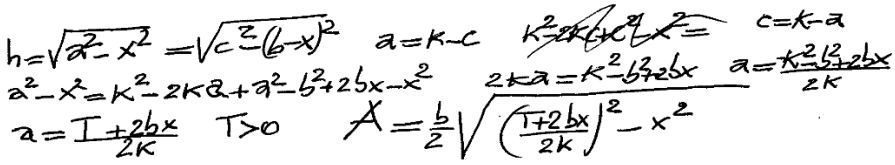
Student S-C: (…) Now I have a reasonable conjecture. How to prove it? (student C draws three more triangles, with the same side in common with the four previously drawn triangles, and approximately the same perimeter). It is even more evident that the isosceles triangle has the largest area. But it seems to me that all those triangles have something in common. Their free edges are …Yes! I understand: the same perimeter means that the free edges are on an ellipse. Thus I may try to see if I succeed to build the proof by using the ellipse. (…)

The third class had already constructed, under the teacher's guide, the equations of a circumference and a parabola by translating into algebraic equations the characteristic conditions of those geometric loci. They had not yet met the equation of an ellipse, or the notion of an ellipse in synthetic geometry. The teacher suggested to use algebra to solve the problem. Student S-D is a representative of those students (about one third of the class) who succeeded in finding the conjecture and proving it. S-D draws three triangles with (approximately) the same perimeter:

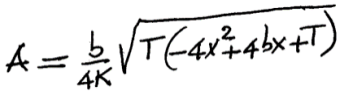
Student S-D: I must maximize an expression for the area of the triangle, when x changes:



The maximum is when x=0. Perhaps this is the solution! But I have not considered the condition a+c=K. And what I found is … it is obvious: x=0 means the rectangle triangle. Obvious: in that case the side of length a is vertical, namely, maximum height of the triangle. But that side has always the same length. But in this problem a is related to c. I should find how to take the condition a+c=K into account. Perhaps it should be good to compare two expressions for the height of the triangle, perhaps… in order to get the area depending only on b and K.



Good! Given K and b, the area depends only on x (algebraic calculations follow)



Now it works: I see the equation of a parabola; … if x=b/2 I get the the vertex of the parabola, it means the maximum… the maximum of the area. (S-D draws an isosceles triangle) OK, it looks fine: the isosceles triangle looks as the widest one!

We may observe how (as it usually happens in analytic geometry) conjecturing and proving are dealt with at the same time, thus CUTHE works. Exploration is driven by the goal to be attained through algebra, thus the initial figures are not exploited to get a conjecture. The first trial is abandoned after interpretation of the algebraic result, the second one develops and brings to the conclusion. Epistemic control works on formalization, choice of syntactic transformations, and interpretation of results (see Morselli & Boero, 2011, pp. 455–456).

Conclusion and discussion

The aim of this paper was to elaborate the idea of cognitive unity of theorems (CUTHE) by relating it to the rationality frame available to (or chosen by) students to solve a conjecturing and proving problem. Through the examples (particularly Example 1, S-A and also Example 2, S-D) we have seen how the same statement may be produced in a particular rationality frame and then proved in the same rationality frame by exploiting some elements produced during the conjecturing phase, in a continuous process where the intention to achieve the conjecture and ascertain why it is true drives the attention of the student to relevant aspects of the problem situation, useful to build up the proof. While the same statement of Example 1 resists S-B's effort of proving it in another rationality frame. The same for the statement of Example 2 in the frame of Euclidean geometry.

This paper brings some elements of novelty in the field of research, which deals with the relationships between the exploratory phase of conjecturing and proving (or of proving a given statement), and the phase of proof construction. Through the use of the rationality construct, the hypothesis of dependence of activation of CUTHE on the theory chosen as reference for conjecturing and proving, already briefly presented in Garuti et al (1998), is further elaborated, with a counterpart in some examples from classroom activities. The rationality perspective provides a lens to compare (and distinguish between) different rationalities in mathematics, with different opportunities to validate the same statement by activating CUTHE. The chosen examples (particularly in the case of S-B if compared with S-A and with some S-B's schoolmates) also suggest to move to a deeper consideration of the relationships between the student's intention (i.e. the teleological component of her rational behavior) and the production of those elements, which might be arranged in a deductive chain in order to get a proof. Another, possible research development (related to Douek's present work) concerns a connection with what is called "semantic proof production" in Weber (2005, p. 356–357): in his reported example the student produces a visual-graphic representation of the sequence (a_n)=(1,0,1,0,1, …) and a horizontal band, which 'shows' that the sequence is not convergent to a limit; that "informal representation" suggests and guides "the formal inferences that (she) would draw". CUTHE does not work: elements produced during the exploration are not resumed as steps of the construction of the proof in the rationality frame of formal Calculus. But those elements allow to bridge the exploration of the proving situation with the construction of a proof in terms of the teleological component of rationality, with some analogy with the case of S-C (in Example 2); both cases suggest to widen the idea of CUTHE by including that kind of productive relationships between exploration and proof construction.

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Supporting classroom implementation of proof-oriented tasks: Lessons from teacher researcher collaboration

Orly Buchbinder

University of New Hampshire, USA; orly.buchbinder@unh.edu

This paper reports on a professional development (PD) which aimed to support secondary teachers in incorporating argumentation and proof-oriented tasks in their classrooms. The teachers interacted with researcher-developed models of proving tasks in a variety of ways, including modifying the tasks to their classrooms contexts, implementing the tasks, sharing and reflecting on the experiences. In the process of modifying proof-oriented tasks by teachers some of the original researcher-intended goals were lost, while other unexpected affordances emerged. This raises important questions regarding modes of teacher-researcher collaborations around proof-oriented classroom interventions, and their potential effectiveness.

Keywords: Reasoning and proof, professional development, instructional activities, classroom interventions.

Introduction

As the body of knowledge on reasoning and proof grows, the focus of mathematics education research has shifted from examining individual students' conceptions of proof and theorizing about potential causes of students' difficulties with proof towards designing classroom interventions that aim to remediate these difficulties and provide instructional support for students and teachers in classrooms (Stylianides & Stylianides, 2016). In this process teachers play a critical role, as they are responsible for establishing learning environments in their classrooms. In line with the wide recognition of the importance of argumentation and proving to students' mathematical experiences (e.g., Reid & Kipping, 2010) teachers are expected to implement tasks that promote reasoning, and have students construct and critique mathematical arguments (CCSS, 2010).

While many teachers agree, in principle, with this vision of mathematics classrooms, they often find them challenging to implement and maintain over time (Brodie, 2010). Moreover, only a limited number of professional development (PD) settings explicitly focus on argumentation and proving in connection to classroom practices (Brodie, 2010). Hence there is a need to expand the theoretical and practical knowledge of successful strategies for supporting teachers in this area.

This paper reports on an experimental model of a PD intended to support teachers in incorporating argumentation and proving in their classrooms. The following sections describe theoretical grounds underlying a special feature of the PD: teachers modifying researcher-designed proof tasks for implication in their classrooms. I illustrate two such modified tasks and analyze them in terms of affordances for students' learning, and their (mis)alignment with the original designer's intentions. I close by discussing some implications for supporting teachers' implementation of proof-oriented classroom activities.

Theoretical framework

Supporting change in teacher practices: the emphasis on argumentation and proving

Research has identified key features of PD settings that have shown to be successful in supporting change in teachers' practices. Among them are: focus on content and pedagogical knowledge, active learning experiences, establishing strong connections to teachers' own classroom contexts, and providing ongoing support for teachers (Copur-Gencturk & Papakonstantinou, 2015). These general features can be adapted to provide targeted support for teaching argumentation and proving, for example, by emphasizing mathematical knowledge for teaching proof (MKT-P).

Building on Stylianides's (2011) notion of "comprehensive knowledge package for teaching proof", Buchbinder et al. (2016) suggest that MKT-P includes 4 types of knowledge. Two types are related to pedagogical content knowledge: (a) knowledge about students' conceptions of proof, and (b) knowledge of pedagogical practices for supporting students' development of correct conceptions of proof. The other two types of MKT-P involve subject matter knowledge: (c) robust knowledge of mathematical content involved in a given task, and (d) meta-mathematical knowledge of proof, such as argument validity, logical connections, types of proof, and the role of examples in proving. These four types of knowledge were addressed in the design of the PD reported in this study. In addition, the PD activities established strong connections to teachers' own classrooms by providing practical tools for teachers to develop and implement proof-oriented instructional tasks in their classrooms.

Task design

Choosing, adapting and designing mathematical tasks is one of the cornerstones of a teacher's work. With textbooks providing only limited opportunities for students to engage in argumentation and proving (Thompson et al., 2012) teachers have been encouraged to treat textbooks' tasks as a starting point for planning instruction: to modify tasks to increase their cognitive demand or develop their own tasks (Stein et al., 2000). Since PD efforts in this area have seldom specifically targeted argumentation and proving tasks, the knowledge on teachers developing and implementing such tasks has been limited. Adding to this concern, Stylianides and Stylianides (2016) argue that it is unrealistic to expect individual teachers to design their own instructional activities that successfully target persisting difficulties with proving. On the other hand, Kim (2016) has found that teachers regularly tend to omit, replace or substitute instructional activities, even when working with reform-based, research informed curricula, which often compromises the original designer intentions.

This dilemma can be addressed by fostering close collaboration between researcher-designer and the teacher (Cobb et al., 2003). While the researcher-designer brings in strong theoretical and empirical knowledge related to proving, the teacher has an intimate knowledge of specific instructional and institutional context. This partnership model was realized in this study by providing teachers with researcher-developed prototypes of proving tasks to modify and implement in their classrooms.

Proof-task prototypes

Six prototypes of proving tasks were developed by the author of this paper in a study of secondary students' conceptions of proof. The tasks, which can be used as diagnostic tools and as instructional activities (Buchbinder & Zaslavsky, 2013), were developed in generic form, so they could be adjusted for a variety of mathematical topics, while maintaining the original structure and goals,

such as recognizing the limitation of examples for proving general claims, or understanding the role of counterexamples. In the context of the PD reported herein, teachers received at least one content specific version of each type of task, and a generic template highlighting task structure. Figure 1 shows an algebraic version of the task *True-or-false*; and Figure 2 shows its generic version[1].

True or False? For each statement below decide whether it is true or false and justify your answer.	
1) Every three numbers *a, b, c* satisfy the equation: $\frac{a}{b+c}=\frac{a}{b}+c$.	4) There exist four numbers *a, b, c, d* that satisfy: $\frac{a+c}{b+d}=\frac{a}{b}+\frac{c}{d}$.
2) The (positive) difference between the squares of any two consecutive natural numbers is equal to their sum.	5) There exists a number $a\neq 1$ that satisfies the equation: $a+\left(1+\frac{1}{a-1}\right)=a\cdot\left(1+\frac{1}{a-1}\right)$
3) Every two numbers *n, m* satisfy the equation: $\frac{1}{m}+\frac{1}{n}=\frac{1}{n+m}$	6) There exist three distinct positive integers *a*, *b*, *c* that satisfy $\frac{a+c}{b+c}=\frac{a}{b}$

Figure 1: Algebraic version of the task *True-or-False*

The task *True-or-false* targets multiple aspects of proving and refuting. It requires distinguishing between universal and existential statements, and recognition that the type of statement affects the role of examples in proving or disproving it. To successfully complete the task, students need to construct general proofs, construct appropriate counterexamples to disprove false universal statements, and come up with supporting examples to prove existential statements.

Statement	Type of statement: U / E	Truth value: T / F	"Always-Sometimes-Never"			Type of justification required
			True for all values	Ture for some values	True for no values	
1)	U	F		✓		Refutation by a counterexample
2)	U	T	✓			General proof
3)	U	F			✓	Refutation by a counterexample
4)	E	T		✓		Proof by a supporting example
5)	E	T	✓			Proof by a supporting example
6)	E	F			✓	General refutation

Figure 2: The structure of the task *True-or-False*

The task *Always-Sometimes-Never*, builds on the task *True-or-false* by asking whether the *propositions* of the statements in the latter task are true for all, some, or no values of relevant variables. This often requires construction of additional arguments, e.g., although statement #3 in Figure 1 can be refuted by a single counterexample, one must construct a general argument to show that no values of variables satisfy the statement. Sequencing these tasks allows to contrast quantified

[1] For complete presentation of all 6 types of task prototypes see Buchbinder & Zaslavsky (2013).

statements, which are either true or false, with non-quantified propositions, which truth-value depends on the value of a particular variable. Creating a combination of statements to addresses all these aspects of proving is a complex undertaking, which could be supported by using a generic version of the task (Figure 2). The goal of this study was to explore the potential of using generic task prototypes to support the work of mathematics teachers with respect to incorporating argumentation and proving in their classrooms.

Methods

Participants. The study was conducted with 5 secondary teachers, all female, all from different schools in a Northeastern area in the United States. Their teaching experience varied greatly from 5 to over 30 years. Since the PD was advertised as explicitly devoted to classroom implementation of argumentation and proving tasks, all participating teachers were motivated to introduce such tasks in their teaching, but sought to gain practical skills in this area. Hence, the PD aimed to reinforce already existing teachers' motivation, provide ongoing professional support, and foster teachers' sense of self-efficacy as they transformed their practices.

The setting. The PD consisted of 9, two-hour long weekly meetings which took place on the campus of a state university in the Fall of 2015. During the sessions teachers interacted with the 6 types of researcher-developed proof tasks in several ways: they experienced the tasks as learners, examined samples of student work pertaining to these tasks, and analyzed opportunities to learn about argumentation and proving embedded in the tasks. This was done by comparing teachers' own experiences and student work with the generic task prototype to examine the extent that the designer-intended goals have been realized. Throughout the PD teachers were encouraged to try out at least two types of tasks in their classrooms and share their experiences with others.

Modes of Inquiry and Data Sources. All PD sessions were videotaped. Each teacher submitted the tasks they had created or modified for their classrooms, sample student work and a two-page report on the task implementation e.g., the mathematical topic, the number of students, and the modes of work: group, individual, whole class, or combined. Teachers were also asked to describe what kinds of learning opportunities they think their tasks afforded, and what challenges they encountered as they created and implemented the tasks. The teachers also completed a short survey assessing the perceived effect of the PD on their classroom practices.

Results and discussion

Perceived obstacles for classroom implementation

Although all participating teachers expressed their commitment and motivation to incorporate proof-oriented tasks in their teaching, they also frequently shared concerns about feasibility of such shifts in their practices. Their concerns included whether incorporating proof-oriented tasks would compromise curriculum "coverage", or would take out from the time originally allotted to test preparation; whether students would be willing to take social risks associated with sharing mathematical arguments in public, and to critique the arguments of others; and whether students be willing to engage in proof-oriented tasks that vary in form and content from what they are used to. These types of concerns reflect teachers' professional obligations towards the institution of schooling and towards individual students' social and emotional needs (Herbst & Chazan, 2011).

Of the total 11 proof tasks created by the teachers, 2 were of their own design and 9 were modifications of one of the researcher-designed task types: *Is this a coincidence?*, *True-or-false?* and *Always-Sometimes-Never*. The tasks addressed a variety of mathematical topics in algebra, geometry, number and operation, and logical reasoning. The modes of implementation involved: enrichment activities, practice, exam review, or introducing a new topic. In the following I focus on one teacher, Alison (a pseudonym), to illustrate how she had modified two tasks to fit her classroom context. These tasks were chosen because they stood out as one of the most creative modifications to the researcher-designed task prototypes that occurred within this group of teachers.

Alison's modification of the tasks *True-or-false?* and *Always-Sometimes-Never*

Alison has more than 20 years of teaching experience and is well-respected in her school. Similar to other teachers she joined the PD with mixed feelings: committed to provide students with proving experiences but sharing the abovementioned concerns. Alison was inspired to create two proof tasks when her students performed poorly on a particular item on an algebra test: a word problem about money invested and interest earned in two bank accounts. The students found it challenging to set up an equation to represent the total amount of money split between the two accounts, using a single variable. Alison used students' test responses to create a sequence of tasks: *Always-Sometimes-Never* (Figure 3) and a follow-up *True-or-false* task (Figure 4).

Always, Sometimes or Never?

Part 1: Tom invested $50,000 into two accounts. The first account earns 6% interest and the other account earns 11% interest. How much money did he invest at each rate if his total interest earned from both investments was $4,900.

The following cards are possible setups to describe the amount of money (x) invested for Tom's investments described above. Indicate whether the setups are:
true for all of values of x,
true for some values of x,
or whether there are no values of x which satisfy the set-up of the application.

Explain your reasoning.

A)

	Money invested
1st account	x
2nd account	x
Total	$50,000

B)

	Money invested
1st account	x
2nd account	x
Total	$4,900

C)

	Money invested
1st account	x
2nd account	$50,000 - x
Total	$50,000

D)

	Money invested
1st account	$50,000
2nd account	x
Total	x + $50,000

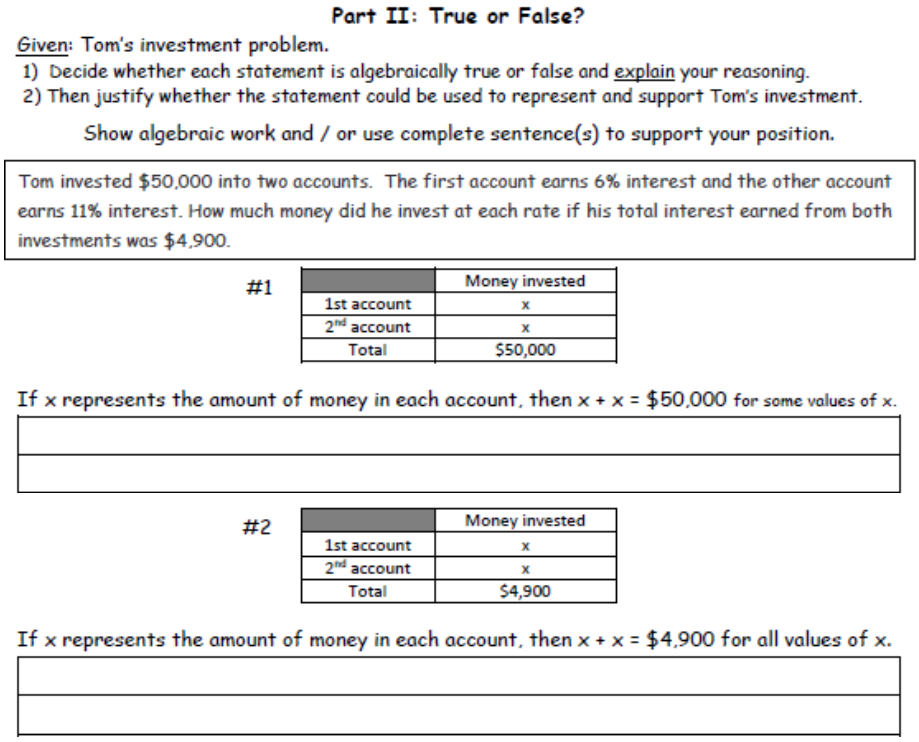
E)

	Money invested
1st account	x
2nd account	x
Total	X + $50,000

F)

	Money invested
1st account	x
2nd account	x + $50,000
Total	$50,000

Figure 2: Six out of 8 items from Alison's task *Always- Sometimes-Never*



Part II: True or False?

Given: Tom's investment problem.

1) Decide whether each statement is algebraically true or false and explain your reasoning.
2) Then justify whether the statement could be used to represent and support Tom's investment.

Show algebraic work and / or use complete sentence(s) to support your position.

Tom invested $50,000 into two accounts. The first account earns 6% interest and the other account earns 11% interest. How much money did he invest at each rate if his total interest earned from both investments was $4,900.

#1

	Money invested
1st account	x
2nd account	x
Total	$50,000

If x represents the amount of money in each account, then x + x = $50,000 for some values of x.

#2

	Money invested
1st account	x
2nd account	x
Total	$4,900

If x represents the amount of money in each account, then x + x = $4,900 for all values of x.

Figure 3: Two out of 6 items from Alison's task *True-or-false?*

Alison's goals in developing this sequence of tasks were to confront students with both correct and erroneous charts for setting an equation representing the money split between the two accounts, and have students analyze, validate or critique the equation setups. In the *Always-Sometimes-Never* task students were to determine whether the equation setups are true for all, some, or no values of x, where x is the amount of money in one account. In the *True-or-false* task the same setups were accompanied by conditional statements. Students were to determine whether each equation is algebraically correct, and whether it can be applied to the given word problem. The tasks were implemented with 74 students (4 classes). Students worked in groups of 3 or 4 on the *Always-Sometimes-Never* task in class, and then completed the *True-or-false* task at home.

Opportunities gained and lost through task modification

The design on Alison's tasks reflects the way she balanced her professional obligations. By using students' test responses as a content of the tasks Alison minimized social anxiety associated with presenting and critiquing mathematical arguments. She also addressed her curriculum goals while engaging students in proof-oriented tasks. The mathematical affordances of Alison's tasks encompass many of the original designer intentions. For example, the task *Always-Sometimes-Never* provided students with opportunities to reason through a variety of correct and incorrect equation setups, and evaluate whether they can be true for all, some or no values of the variable. The use of precise mathematical language echoes the goals of the original design. The two tasks build on each other, with *True-or-false* task emphasizing evaluation of conditional (if not quantified) statements. The tasks reflected additional learning goals Alison had for the students: to distinguish between equations that are mathematically correct but are inappropriate in the context of the problem. Further distinctions could be made between equations that do not account for an implicit problem requirement: the investment in either account cannot be $0 (equations D, E & F); and equations that do not account for the explicit requirement: the total interest earned must be $4900, meaning that

equal sums of money cannot be invested in two accounts[2] (equation A). These distinctions came up in students' written responses to the tasks. Alison was very satisfied with students' interactions with the tasks, and indicated that next year she plans to use them to *introduce* the topic of solving word problems, rather than a test review.

Despite the important affordances of Alison's task, many of the original proof-oriented goals of the tasks, such as the limitation of examples for proving general claims, the distinctions between quantified and non-quantified, universal and existential statements, were not realized in the tasks setup. Potentially, Alison's tasks could be used to highlight other issues related to proving, which although not intended by researcher design, arise naturally in the context of her tasks. Justifications for dismissing solutions A, D, E and F (Figure 1) bare resemblance to arguments by contradiction – a proposed equation, assumed as correct, is rejected because it contradicts one of the problem constraints. Such interpretation could pave a way to discuss proof by contradiction in algebra class.

Conclusions

This paper described an exploratory study that tested a PD model which aimed to support secondary teachers' implementation of argumentation and proving tasks in their classrooms. The researcher-designed tasks served as prototypes after which teachers could model their own tasks. The generic versions of the same tasks provided additional support for teachers by outlining the structure of the tasks and highlighting specific proof-related goals. By using researcher-designed tasks as a starting point for creating their own tasks, teachers became critical partners in designing classroom interventions to promote students' engagement with proving. As teachers collaboratively explored, modified and shared experiences of classroom implementation of their tasks, they negotiated a new understanding of what it means to engage students in argumentation and proving. In the post-PD survey, all participating teachers reported increased confidence in their ability to incorporate argumentation and proving tasks in their teaching. One teacher, called here Jenifer, wrote:

> The [PD] classes gave me great ideas to take back to my classroom, to look at proofs very differently than what I had always thought of as a proof. Proofs do not need to be the static, two column proofs from my school experience. They can take a couple of minutes or they could be something to wrestle with for a majority of the block. I liked that the activities were easily manipulated to fit a specific time frame or wanted outcome.

The study also revealed challenges associated with supporting teachers in developing proof-oriented tasks. Alison's tasks show that although she created powerful opportunities for students to engage with argumentation, some of the original researcher-intended goals, specifically related to proving, seem to have been lost. The available data sources do not provide sufficient information as to which aspects of proving were explicitly addressed in class, or whether they were completely overshadowed by discussions of the algebraic content. Hence, future studies should involve classroom observations. Finally, the results of this study concur with those suggesting that changing teacher practices is a gradual process which requires structured support (Brodie, 2010) to help teachers to develop a view of proof-oriented classroom activities as means to balance their professional obligations and enhance students' mathematical learning.

[2] Investing $25,000 at 6% in one account and $25,000 at 11% in another account would yield a total interest of $4,250.

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A framework for classifying mathematical justification tasks

Boon Liang Chua

National Institute of Education, Nanyang Technological University, Singapore;
boonliang.chua@nie.edu.sg

The large corpus of research on mathematical reasoning and justification in the mathematics education literature has yielded a wide range of tasks that require a mathematical argument to be established. This paper presents the DIVINE framework that classifies justification tasks by their nature and purpose as well as the expected element to be provided in the justifications. The framework is then used as a theoretical basis for appraising justifications produced by mathematics teachers.

Keywords: Mathematical justification, classification framework, teacher competency.

Introduction

Mathematical reasoning plays a crucial role in mathematics learning at all grade levels. It is a useful tool for exploring, discovering and understanding new mathematical concepts, for applying mathematical ideas and procedures flexibly to other situations, and for reconstructing previous knowledge in order to generate new arguments (Ball & Bass, 2003). To probe into the mathematical reasoning of students, another tool is needed to make such reasoning visible – justification. With the emphasis in schools worldwide on developing a broad set of competencies that are believed to be an imperative for success in the workplaces in the 21st century, greater demands are therefore being placed on students to reason and justify in the learning of mathematics.

Mathematical reasoning and communication are two key process skills in the framework of the Singapore school mathematics curriculum (Ministry of Education (Singapore), 2012) that have been advocated for a long time. The notion of communication refers to the ability of using mathematical language to articulate mathematical ideas and arguments precisely, concisely and logically (Ministry of Education (Singapore), 2012). In this sense, mathematical justification is considered part of communication. But very little is known about the justification ability of Singapore mathematics teachers and students at the secondary level. I am thus interested to find out more about it and commenced the investigation with a survey of the various justification tasks that secondary school students had been tested in the national examinations over the past ten years. The survey has found that the justification tasks are of varied nature and can be classified into different categories.

This paper seeks to address the following questions: What are the different types of justification tasks given to secondary school students? How might justifications for the different types of tasks qualify as acceptable? What elements should be present in an acceptable justification? It presents a theoretical framework for classifying mathematical justification tasks and discusses the expectation required in each type of tasks. The structure of this paper broadly follows these strands of work: (a) a perspective of what justification encompasses, (b) a view of justification tasks and the elements expected in the justifications, and (c) a discussion of justifications produced by Singapore mathematics teachers.

Theoretical framework

Justification According to Simon and Blume (1996), mathematical justification involves "establishing validity [and] developing an argument that builds from the community's taken-as-shared knowledge" (p. 28). The notion of justification as a means of determining and explaining the truth of a mathematical conjecture or assertion resonates strongly with many other researchers. For instance, it is consistent with Balacheff's (1988) perception of justification as "the basis of the validation of the conjecture" (p. 225) – a view also supported by Huang (2005) as well. To Harel and Sowder (2007), justification for validation serves two different roles: to *ascertain* the truth of a conjecture, and to *persuade* others that the conjecture is true. Even these two roles have slightly dissimilar intention. In Ellis' (2007) view, ascertaining the truth is meant to remove one's own doubts whereas persuading is one's attempt to remove others' doubts. As the discussion reveals, expressing justification for the purpose of ascertaining truth is a cognitive process whilst convincing others of the truth is a social process.

The notion of justification focuses traditionally on the notion of proof from the primary to the high school and university levels in the research literature (see e.g., Jones, 2010; Stylianides, 2007). Thus proof is viewed as a type of justification in this regard. So I think the definitions of proof available in the literature can help to deepen our understanding of mathematical justification. A prime example that stands out is Stylianides' (2007) definition of proof as a mathematical argument made up of a connected sequence of assertions for or against a mathematical claim. This definition echoes Hanna's (1989) definition of proof as "an argument needed to validate a statement" (p. 20) and is considered by far the most comprehensive meaning of proof.

Mathematical justification encompasses a broad range of arguments besides proof. The types of arguments that students are expected to produce depend on at least two factors: *the cognitive abilities of students* and *the nature of the task*. For primary and secondary school students, particularly those in the lower secondary grades, a justification does not need to measure up to a formal proof. This is because providing a theoretical argument for a mathematical result is sometimes not required in the light of their cognitive level until they reach higher level of study (Hoyles & Healy, 1999). This is illustrated by the justification task on algebra asking lower secondary school students to explain why $2n-1$ is an odd number for any positive integer n. This task presents a mathematical claim (i.e., $2n-1$ is an odd number for any positive integer n) and requires the students to provide supporting evidence to show why the claim is true. In short, the nature of such a task is to *validate* the claim. Therefore a reasoned argument within the conceptual reach of the students of this grade level could take the form as follows: with n being any positive integer, forming two groups of n, which can be expressed as $2n$ in notation, thus generates an even number, therefore subtracting one from it will result in an odd number. This justification simply uses everyday language rather than formal mathematical language, and does not draw on any theorems as in a typical theoretical argument.

Clearly not all justification tasks require a theoretical argument. Some lend themselves well to experiential justification, which is mainly supported by specific examples and illustrations. Consider asking students to justify why the rule $a^m \times a^n = a^{m+n}$ is true for any positive integers *a*, *m* and *n*. The students can rely on intuitive reasoning using several concrete numerical examples in the justification. This mode of argument may be rejected as an adequate and valid justification of

the rule because it does not cover all cases of the variables *a*, *m* and *n*. Although such an experiential justification does not involve any theorems and somewhat lacks mathematical sophistication, it does convey to some extent student understanding of why the mathematical claim is true, albeit a far less formal argument than a typical mathematical deductive proof (Becker & Rivera, 2009). But it is such justification that is valued because it "explains rather than simply convinces" (Lannin, 2005, p. 235).

Aside from presenting an explanation for or against a mathematical claim, a justification can also take the form of an *elaboration* of how a mathematical result is obtained, as pointed out by Becker and Rivera (2009). Consider, for instance, the topic of pattern generalisation. Becker and Rivera (2009) and Stylianides (2015) had asked students to justify how they established their general rules for figural patterns. The nature of this type of justification task expects the students to illuminate clearly the method used in rule construction. Like the *validation* task described previously, the justification for the *elaboration* task can also be articulated in two different modes: written as in paper-and-pencil tests and verbalised as in face-to-face interviews. Both modes were evident in Stylianides' (2015) study.

Justification tasks Different types of justification tasks are gleaned from the literature on mathematical reasoning, proof and argumentation. Justification tasks require individuals to make mathematical arguments, a process which is integral to mathematics learning in order for the individuals to make sense of the mathematical concepts and procedures, and learn mathematics with understanding. Additionally, these tasks provide insight into their thinking and reasoning as well. Justification tasks can be classified into what I call *elaboration*, *validation* and *making decision* tasks.

Elaboration justification tasks are very popular in the literature and have been widely used in research studies by many researchers, including Becker and Rivera (2009), Lannin (2005) and Stylianides (2015). Such tasks (for e.g., *Pizza Sharing* in Lannin (2005)) require individuals to elaborate the approach that was used to obtain a mathematical result. *Validation* justification tasks are questions that seek arguments to support or refute a mathematical claim. This kind of tasks (for e.g., *Mr. Right Triangle* in Chua (2016)) is used to gain insight into how individuals reason about a mathematical claim. *Making decision* justification tasks offer options for a mathematical situation and individuals have to exercise decision-making power to pick one of the options so as to answer the question. The geometry test item from the study by Küchemann and Hoyles (2006) is a case in point.

Apart from the three types of justification tasks discussed thus far, there is one more type which is seemingly less common in research studies but popular in the Singapore national examinations for secondary school students. Consider the algebra task in Figure 1 that requires individuals to make sense of the given context and then infer the significance of the positive solution of the quadratic equation from the context. Such a task exemplifies what I call an *inference* justification task. It is normally set in a real-world context and seeks an interpretation of a mathematical result.

A stone was thrown from the top of a vertical tower. Its position during the flight is represented by the equation $y = 50 + 21x - x^2$, where y metres is the height of the stone above the ground and x metres is its horizontal distance from the tower.

Explain what the positive solution of the equation $0 = 50 + 21x - x^2$ represents.

Figure 1: Inference task on algebra

In summary, this sub-section has highlighted four distinct types of justification tasks. All these tasks share a common objective, which is to elicit from someone a mathematical argument for a mathematical claim or result. As they vary in nature from one type to another, the essential elements to be expected in the argument for each type of task are therefore also not the same. In the next section, I introduce the *DIVINE* framework that classifies justification tasks by nature and purpose as well as the expected element to be provided in the justifications, and describe its usefulness. *DIVINE* is the acronym of the four types of justification tasks: making **D**ecision, **I**nference, **V**al**I**datio**N**, and **E**laboration.

The *DIVINE* framework

The conceptualisation and development of the *DIVINE* framework in Table 1 was informed by the literature on mathematical proof, reasoning and justification in the field of mathematics education, by analysis of justifications produced by students and mathematics teachers that I had encountered in the course of my teaching in recent years, and by my own disciplinary knowledge. It describes the *nature* and *purpose* of the justification tasks, and the expected *element* to be provided by individuals in their attempt to produce a correct justification.

Nature of justification tasks	Purpose of justification tasks	Expected element in the justification
Making Decision	Explain whether… Explain which…	a decision about the mathematical claim with evidence to support or refute the claim
Inference	Explain what…	the meaning of the mathematical result, with the key words in the task addressed
Validation	Explain why…	a reason or evidence to support or refute the mathematical claim
Elaboration	Explain how…	a clear description of the method or strategy used to obtain the mathematical result

Table 1: The *DIVINE* framework

The term *nature* can be described as the cognitive process that an individual undertakes when doing the justification task. The nature of the tasks places slightly different demands on thinking and reasoning. Making decision, inference, validation and elaboration are the four kinds of cognitive processes that have been identified in this paper. The *purpose* of a justification task refers to the reason for making the mathematical argument. Finally, the expected *element* is used to refer to the details that an individual is supposed to provide in order to give a correct justification.

It should be pointed out that although the expected element in a justification indicates what needs to be given for a particular type of justification task, the resulting justification may not necessarily be accepted as correct. For the justification to be judged as correct, I think it is imperative to also examine three other elements of a mathematical argument: the *mathematics* presented, the *clarity* in the argument and what Stylianides (2007) termed as the *modes of argumentation*. The mathematics presented refers to the mathematical concepts and procedures used in the justification, including the definitions and theorems that are used, the calculation that is shown and so on. The clarity in the argument means presenting the argument in a clear, easy-to-follow, and unambiguous way. The mode of argumentation concerns how a justification is developed. In other words, the form of the justification (such as a logical deduction, a proof by contradiction, exposition) has to be taken into consideration. A brief discussion of the potentiality of the *DIVINE* framework will now follow.

Usefulness of the framework Recognising whether a mathematical justification is correct is a vital task for teachers because they often have to evaluate the validity of students' justifications. But as Chua (2016) had noted, this task is fraught with difficulties as the teachers might not be clear about the rigour of justification. They may accept justifications as correct even when certain elements are missing. Teachers therefore need guidance in teaching justification. So the *DIVINE* framework shows them what essential elements to look out for so that they know whether certain details are still lacking in the justification. Teachers can also discuss the three components of the framework for the various types of justification tasks with the students to enrich their learning and appreciation of justification. In this way, students can develop a deeper understanding of constructing mathematical justification and become more confident in doing it. This pedagogical approach is particularly useful for those students who do not already have the justifying skill and struggle with justification. Additionally, for those who get stuck when attempting a justification task, the framework offers a structure for them to rely on and get unstuck instead of seeking immediate help from their mathematics teachers.

In the remaining sections, examples of justifications by both pre-service and in-service mathematics teachers will be discussed to demonstrate the rigour of the *DIVINE* framework as it currently stands. The pre-service teachers were Year 2 undergraduates undergoing their first course in mathematics pedagogy to prepare them to teach secondary school mathematics. The course content covers problem solving, learning theories and teaching strategies for a range of mathematics topics, including arithmetic, algebra, probability and statistics. The in-service teachers were from the same secondary school who attended my professional development workshop. A vast majority of them have taught mathematics for at least 5 years. The justifications were collected from the various classwork given to the teachers in my lessons. The names of the teachers are changed to protect their privacy. The discussion focuses specifically on *making decision*, *inference* and *validation* types of justification tasks. No *elaboration* task will be illustrated because the teachers were not given such tasks to do in my lessons.

Making Decision task: The justifications of Angel, Betty and Carl

The number pattern item in Figure 2 was given to the pre-service mathematics teachers. Before administering this item, the teachers had learnt the various generalising strategies for deriving the

general rule for both numerical and figural patterns, but not how to deal with justification tasks. This item was therefore given to see how they would handle and justify a *making decision* task.

> The first four terms of a sequence are 5, 9, 13 and 17.
>
> (a) Find an expression, in terms of *n*, for the *n*th term of the sequence.
>
> (b) Explain whether 207 is a term in the sequence.

Figure 2: *Making decision* task on number pattern

Part (a) was answered correctly by all the teachers. They established $4n + 1$ as the general rule of the sequence. However, the responses for part (b) were more varied, and the justifications produced by Angel, Betty and Carl are described below.

Angel began with the supposition $4n + 1 = 207$ and then solved the equation to obtain $n = 51.5$. He concluded: *Since n has to be a positive integer, then 207 is not a term*. Betty worked out the difference between 207 and the first term 5 to get 202. Then she wrote: *No. All terms in the sequence are divisible by 4 after being subtracted by 5. 202 is not divisible by 4*. For Carl, he started with the same supposition as Angel and found the value of *n*. He then stated: *n must be a whole number for the given number to be a term in the sequence*. The justifications of Angel and Betty, but not that of Carl, were considered fully correct. Their justifications contain all the vital elements for a *making decision* task: that is, a conclusion supported by evidence. Carl's justification is missing the conclusion, thus judged as partially correct. In all the three examples, the justifications are logical and easy to follow, and the mathematics is correct. Carl's case is a perfect example to illustrate the importance of the *DIVINE* framework. If he had known about the essential elements that he had to show in his justification, he would have constructed a complete and correct justification.

Inference task: The justifications of David and Eve

The algebra item in Figure 1 was administered to the in-service mathematics teachers. The item tested them on their understanding of the significance of the positive solution of the quadratic equation in the given context. I expected the teachers to explain what the following three parts mean in the context: (i) y = 0, which in this context means that the stone has hit the ground, (ii) positive, which represents the forward direction of the throw, and (iii) the numerical value of the solution, which refers to the horizontal distance from the tower. However, expecting all three parts was too demanding, so a reasonable justification should address at least (i) and (iii). The mathematics teachers were told to construct the justification that would get them the best mark because they were experienced in-service teachers. The justifications of David and Eve are illustrated below.

David: *x* metres is the distance of the stone from the tower, when *y* = 0 (at ground level).

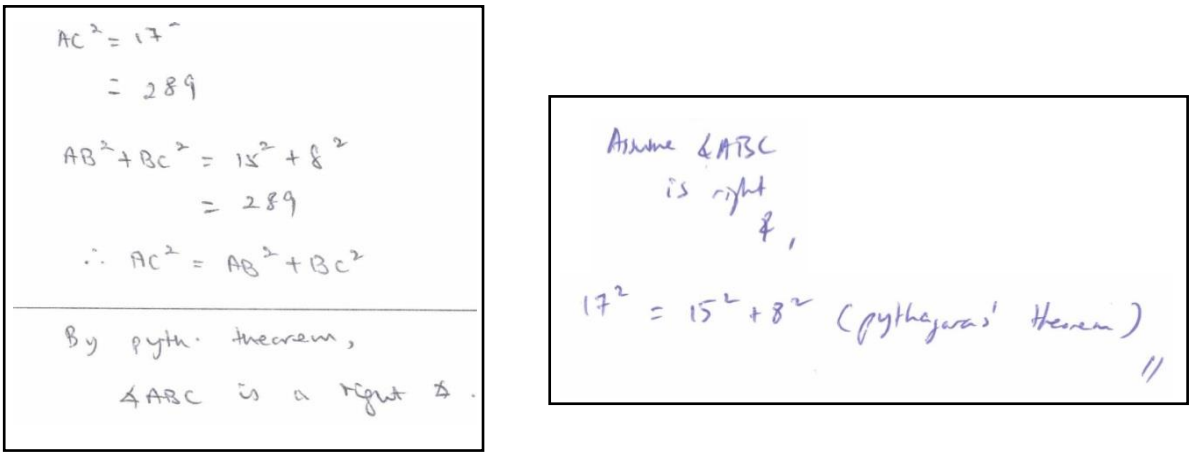
Eve: when *y* = 0, height above ground = 0, ∴ stone is lying on ground.

David and Eve showed evidence of their attempt to explain the meaning of the positive solution. David's argument was regarded as correct because he justified (i) and (iii) correctly. For Eve, her justification was not deemed correct since she justified only (i). Her case again underscores the importance of knowing the critical elements that are needed in the justification, thus manifesting the usefulness of the *DIVINE* framework.

Validation task: The justifications of Faith and George

A geometry item involving a triangle with all three sides provided (15 cm, 8 cm and 17 cm) was given to the same group of in-service mathematics teachers mentioned above. They had to justify why the angle opposite the 17-cm side is a right angle. Figure 3 presents the justifications of Faith and George.

Faith established the condition $AC^2 = AB^2 + BC^2$ by separately working out the values of AC^2 and $AB^2 + BC^2$, and noticing that both values were equal (see Figure 3a). Subsequently, she inferred that angle *ABC* is a right angle. The mode of argumentation is correct, the justification is logical and easy to understand, but there is a mathematical flaw. The correct warrant to use should be the *converse* of Pythagoras' theorem and *not* Pythagoras' theorem. On the other hand, the mode of argumentation of George's justification (see Figure 3b) was wrong because he began with the wrong supposition by assuming angle *ABC* is a right angle, which was what he had to prove. So Faith's justification was judged as partially correct whereas George's justification was wrong.



(a) Faith (b) George

Figure 3: Teachers' justifications for *Validation* task on geometry

What's next and conclusion

The *DIVINE* framework introduced in this paper is still emerging and will need further testing and refinement. For instance, it remains to be seen whether the framework can be put into use with student justifications and justification tasks in other mathematical topics. Furthermore, how do mathematics teachers judge what qualifies as a correct justification? What elements do they expect to see in the justifications? How would their judgement differ from peers and mathematics experts? Such evidence is needed to make the *DIVINE* framework more robust.

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An application of Habermas' rationality to the teacher's actions: Analysis of argumentation in two classrooms

AnnaMarie Conner

University of Georgia, United States of America; aconner@uga.edu

In this paper, I argue that Habermas' components of epistemic, teleologic, and communicative rationality provide insight into the differences in teachers' support for collective argumentation. I examine the teacher's supportive actions in two different classrooms. In their interactions with students, the teachers emphasize different components of rationality. I suggest that teachers may act in ways to support students' development of components of rationality by asking different kinds of questions and raise the question of whether it is useful to consider the components separately.

Keywords: Argumentation, proof, geometry, teaching.

Introduction

It is generally accepted that argumentation and proof are crucial to the study of mathematics. Argumentation has been shown to be particularly important to the learning of mathematics through social interaction. Numerous examples in the mathematics education literature have unpacked aspects of arguments in elementary and secondary classrooms (e.g., Krummheuer, 1995; Pedemonte, 2007); these cases have focused on the learning of mathematics through participation in argumentation, the similarity of argumentation to the structure of proof, the analysis of proof as argument, and the role of the teacher within argumentation. Recent research has examined "successful" argumentation within classroom discussions (Boero, 2011), argumentation that does not meet expectations (Cramer, 2015), and different aspects of rationality with respect to argumentation (Boero & Planas, 2014).

This paper explores the differences in collective argumentation that can be observed in classrooms. It addresses a temptation to characterize the argumentation in one classroom as productive and that in the other as problematic and suggests an explanation for the teacher's actions in each case can be found in Habermas' (1998) constructs of rationality as described by Boero (2006).

Background

In this paper, we explore the teacher's role in argumentation through the combined lenses of our interpretation of Toulmin's (1958/2003) description of arguments in multiple fields, our framework for teacher support of collective argumentation (Conner, Singletary, Smith, Wagner, & Francisco, 2014), and Boero's (2006) description of Habermas' (1998) components of rationality.

Habermas' (1998) components of rationality have been applied to argumentation in several ways. Boero (2006) analyzed a seventh grade student's argument (and the reactions of teachers to the argument) using three interrelated components introduced by Habermas: epistemic, teleologic, and communicative rationality. Boero gave the following explanation of these components.

> Epistemic rationality is related to the fact that we know something only when we know why the statements about it are true or false…the crucial requirement is that the person has elaborated an evaluation of propositions as true and is able to use them in a purposeful way and to account for their validity. The teleologic rationality is related to the intentional character of the activity, and to the awareness in choosing suitable tools to perform the activity…The communicative rationality

> is related to communication practices in a community whose members can establish communication amongst them…rational means that the subject has the intention of reaching the interlocutor in order that he/she can share the content of communication, with an adequate and conscious choice of tools to make it possible. (p. 189–190)

Boero concluded that the student acted in a rational way, using all three components of rationality, while the teachers' behavior did not meet these criteria for rationality.

Recently, multiple researchers have taken up Habermas' components of rationality to examine a range of issues with argumentation (see Boero & Planas, 2014). Within Boero and Planas' (2014) research forum report, Douek introduced the construct of rational questioning, suggesting that teachers can ask students questions in order to "organize the mathematical discussion according to the three components of rationality" (p. 1-210). The teacher plays an essential role in organizing and supporting argumentation in classrooms. In this, Habermas' construct of communicative rationality is key, but the teacher can also influence the teleologic and epistemic rationality of the classroom community.

We follow Krummheuer (1995) in adapting Toulmin's (1958/2003) description of argumentation to collective argumentation in mathematics classrooms. We define collective argumentation broadly as any instance in which students or teachers make a mathematical claim and support it with evidence. Our adaptation of Toulmin's diagrams (see Figure 1) includes the use of color (line style) to denote the contributor(s) of components of an argument and the addition of contributions and actions of the teacher that prompt or respond to parts of arguments (teacher support).

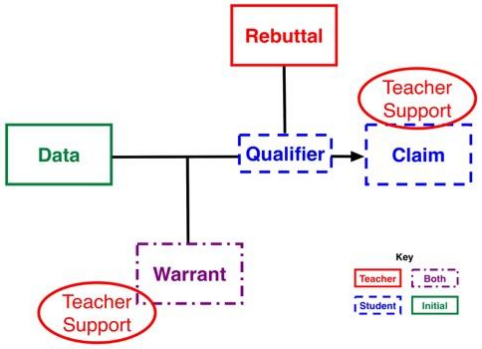

Figure 1: Adaptation of Toulmin's (1958/2003) Diagram for an Argument

Our framework for teacher support of collective argumentation includes three main kinds of supportive actions: direct contributions of argument components, questions, and other supportive actions such as gestures or diagrams (Conner et al., 2014). We defined a teacher's support for collective argumentation as any teacher move that prompted or responded to an argument component. We used Toulmin's (1958/2003) model to classify the direct contributions of argument components, and we used an inductive approach to develop categories of questions and other meaningful supportive actions the teacher used. More details about the development of the framework are available in Conner et al. (2014).

Methods

The analyses in this paper are based on data collected from a project that investigated the beliefs and argumentation practices of a cohort of secondary prospective teachers in the southeastern United States. In particular, the data for this paper include video recordings, field notes, and other artifacts

from two days selected from a unit of instruction in each of two student teachers' (Ms. Bell and Ms. Carr, all names are pseudonyms) classrooms. These days were selected as representative of the variety of instructional moves observed in each teacher's instruction. We diagrammed every episode of argumentation from these days of instruction and categorized every supportive action of the teacher that we identified. We used Habermas' (1998) components of rationality to examine the actions of the teacher within our framework categories.

Episodes from two classrooms

The episodes presented and diagrammed below capture essential qualities of the instruction in each teacher's classroom. For each teacher, we present an excerpt of an episode of argumentation, our interpretation as captured by a partial diagram, and a summary of the teacher's support for argumentation in the class. We then examine the teacher's (and students') actions using Habermas' (1998) components of rationality and argue that the teacher's actions with respect to argumentation reflect her teleologic rationality. That is, we examine the teacher's supportive actions as tools to infer her goals for students' learning and contributions to class and her classroom norms.

Ms. Carr's Class

This episode occurred when Ms. Carr and her students were at the beginning of a unit on congruence. The students had not yet learned any of the triangle congruence theorems. Thus they were proving figures congruent by their definition of congruence, which required all corresponding sides and all corresponding angles to be congruent. Ms. Carr posed the problem in Figure 2 to her class; the students and she worked together to mark relevant parts of the figure, and when we enter the discussion, they had modified the figure as shown. (They had extended segments BC, AB, and CD, and they marked angle ABE and angle DCE as angles of interest.)

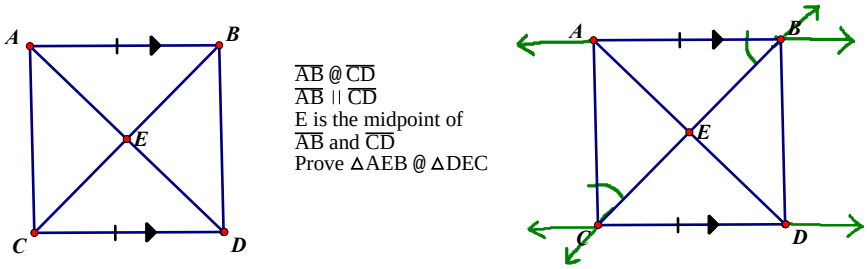

Figure 2: Initial Problem in Ms. Carr's Class and Modification

541	Ms. Carr:	Okay. So, what I have marked up here in green, we said are what? What is their special relationship?
543	Alice:	They are alternate interior angles.
544	Ms. Carr:	They are alternate interior. Ok. So, let's write that down. ABE, let's call it, and angle, what is it? Angle DCE…[writes ∠ABE ∠DCE on board, leaving space between the two angles] [unrelated conversation/interruption] Now, I left some space in there. What symbol needs to go, what do we know about these?
549	Students:	{congruent} {congruence}
550	Ms. Carr:	Awesome. They are congruent. Why do we know that?
551	Cameron:	Because they are alternate interior angles.

552 Ms. Carr: Alternate interior angles theorem [writes by alt. int. angles thm. on board]

Figure 3 shows the diagram of this excerpt of an argument. Notice that three parts of the argument were contributed by students with a significant amount of support from the teacher. Ms. Carr asked a question that prompted each of the argument components, pointed at or wrote something on the board for each of them, and restated or affirmed each part as well.

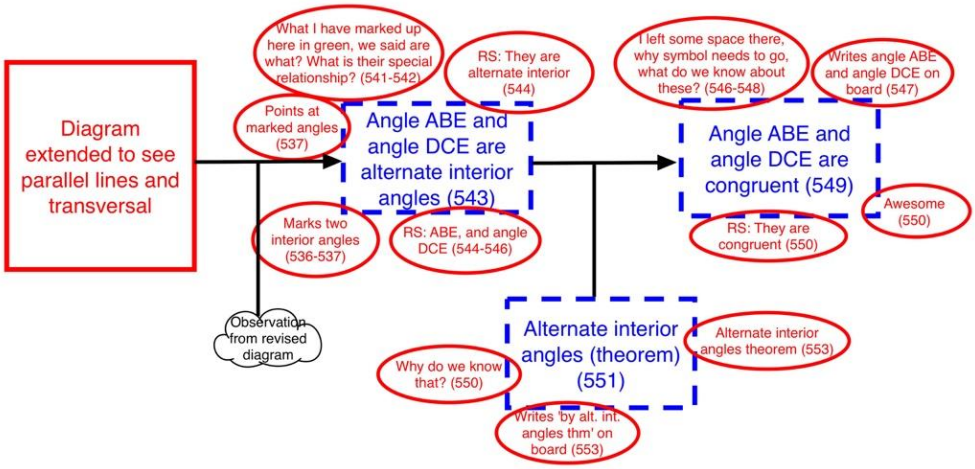

Figure 3: Diagram of First Excerpt of Argument in Ms. Carr's Class

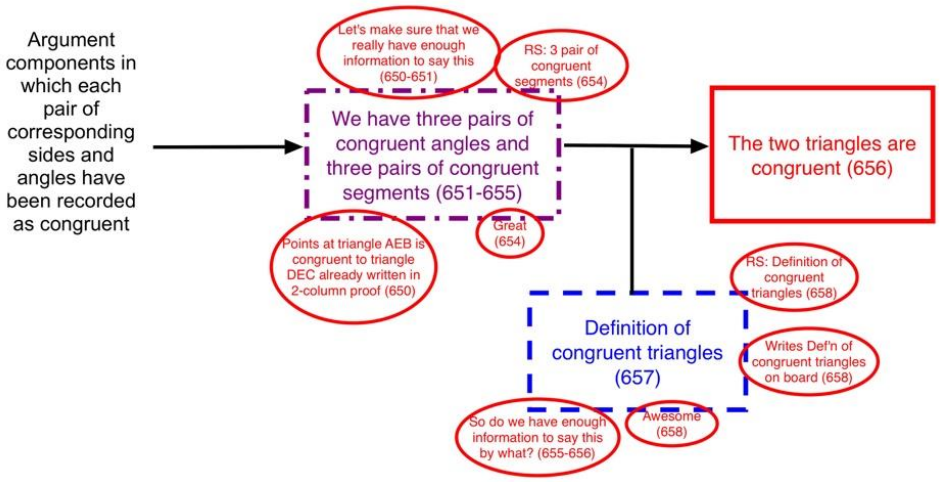

Figure 4: Diagram of Second Excerpt of Argument in Ms. Carr's Class

A little more than five minutes later, Ms. Carr and her students had compiled all of the information about the figure into congruence statements. They ended the proof construction by verifying that they had three pairs of congruent segments and three pairs of congruent angles, warranting the claim that the triangles were congruent with the definition of congruent triangles. In the diagram for this excerpt of argument (Figure 4) we see that the teacher contributed the final claim, the teacher and students jointly contributed the data, and a student contributed the warrant. Ms. Carr prompted both the data and warrant, and she supported each of these components with actions such as repeating, pointing, and writing on the board.

Ms. Carr supported her students in making arguments by contributing many argument components, including approximately one-half of the warrants in her class. In addition, she prompted most argument component by asking questions (primarily factual answer and elaboration questions,

Conner, et al., 2014), and she provided additional support for these argument components using other several kinds of supportive actions (including focusing, evaluating, informing, and repeating actions, Conner, et al., 2014). The importance of Ms. Carr's choices in supporting her students' arguments becomes clear as we reference Habermas' (1998) components of rationality. Ms. Carr asked several questions (line 541, lines 547–548) that requested a factual answer and then asked for elaboration by asking the students to justify that answer in line 550. In this interchange we see an assumption by Ms. Carr of her students' epistemic rationality. She invited them to participate in the argument and indicated by her questions that they should have reasons for their statements. This may be an instance of Douek's rational questioning, as described in Boero and Planas (2014), although Douek's rational questioning seems to presuppose all three aspects of rationality. Ms. Carr seemed to focus on epistemic rationality for her students, while Ms. Carr's own statements and actions indicate a focus on communicative rationality for herself. She repeated or restated (and often wrote the statement on the board) all of the student-contributed components of the arguments. If we consider the teleological rationality of Ms. Carr's actions, they appear to be very goal-directed. Her goal was student understanding of concepts and procedures. In search of that goal, her goal or focus for students was on epistemic rationality. She intended to make sure that they knew the reasons for the statements that were made. Across the class periods, this was evidenced by her many questions prompting argument components as well as her pervasive prompting and providing of warrants for arguments.

Ms. Bell's Class

In Ms. Bell's class, the excerpt exemplifying her instruction involved a task in which students had measured the interior angles of several polygons. Students were asked to find a formula for the sum of the interior angles of an *n*-sided polygon. The brief snippet of class we examine occurred when a student was presenting his group's work at the end of class. Prior to this excerpt, a student representing a different group presented a solution. Martin, the student in this episode, asked to present his solution because his group found the solution in a different way from the first student.

1444	Martin:	All right. I had the chart. This is the sides of the figure. That would be the sum of the interior angles.
…		[Martin talks as he constructs a chart containing numbers of sides and corresponding sums of interior angles for polygons with three to eight sides]
1456	Martin:	And then it changes by 180 degrees each time.
1458	Ms. Bell:	So Martin, the fact that it changes by the same number each time, when you're going up by one side, tells you what?
1460	Martin:	That it has--that that's the slope.
1461	Ms. Bell:	That's the slope. Which means it's? Karin, you said it earlier. What does that mean when it's?
1463	Martin:	Linear.
1464	Ms. Bell:	Linear. It's linear, right? If it changes the same amount each time, when you're going up by 1, it's going to be a linear function.
1466	Martin:	So I did $f(s) = 180s$
1467	Ms. Bell:	What is that 180?
1468	Martin:	It's the slope. But that doesn't work out right, because 180 times 3 is like
1470	Ms. Bell:	540

1471 Martin: [writes 180 x 3 = 540] 540. But then I just subtracted 180 from 540 and it equals 360. Yeah. So, subtract 360. [Writes f(s) = 180s - 360]

1475 Ms. Bell: So same thing; he got it a different way. When he got to the 540--so he got this 540 out when he plugging in 3 for his *s*, and he got 540. We wanted to get 180 when we plug in a 3. So he said, 'how am I going to get from 540 to 180?' So he found the difference between them and subtracted from this product. Do y'all see that?

1482 Martin: It works with all of them too.

Figure 5 shows the diagram of this argument. In this argument, Martin (the student) contributed all of the components except one warrant. Ms. Bell prompted three of the components with questions, and she supported five of the components by restating or rephrasing Martin's contributions. In general, Ms. Bell asked questions of multiple kinds to prompt argument components, and she contributed some components of arguments, but only about one-eighth of the contributed warrants. Students in Ms. Bell's class seemed to contribute more autonomously to arguments, as evidenced by components that were neither prompted by nor responded to by Ms. Bell.

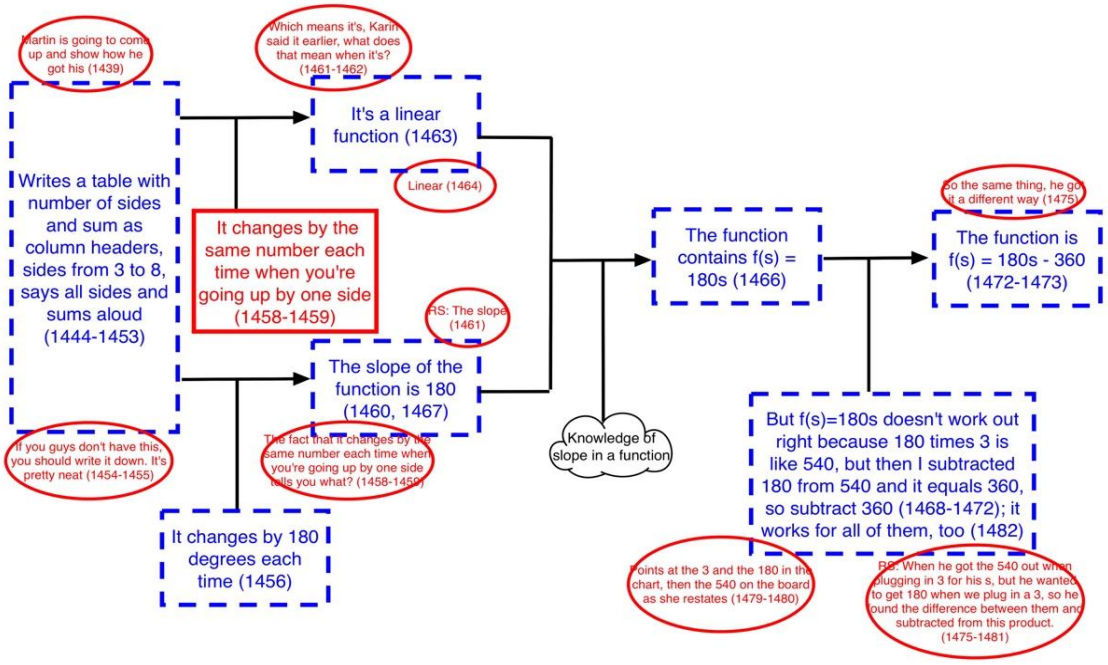

Figure 5: Diagram of Argument from Ms. Bell's Class

We see evidence of Ms. Bell's teleologic rationality (Habermas, 1998) in her actions and questions in support of her goal of engaging students in doing mathematics. Ms. Bell modeled actions related to all three components of rationality, and she seemed to encourage all three components of rationality in her students. Ms. Bell's actions show a strong emphasis on developing her students' communicative rationality, not only in her communication with her students (see line 1475 in which she restates the student's argument), but in her encouragement of her student to communicate his ideas more clearly (e.g., line 1458) and in the student's instinctive actions and statements (e.g., lines 1444 and following in which he explained the entries in the chart he drew on the board), which illustrate norms established in this class. Several times after the student gave a claim and warrant, Ms. Bell seemed to slow down the presentation to make sure it was clear to others, enhancing their understanding of communicative rationality. But instead of giving all the information herself, she

asked the student to do so (line 1467). She seemed to be balancing engaging in acts of communicative rationality herself and encouraging her student to do so. In addition, Ms. Bell assumed epistemic rationality in her student and encouraged him to express it (line 1458). The beginning and end of the episode evidence a classroom norm regarding goal-directed behavior and use of appropriate tools (lines 1444–1455; line 1482). The student indicated by his final statement that he had intentionally completed his goal of finding, expressing, and justifying the formula for the sum of the interior angles of a polygon, showing the teacher's encouragement of behavior exhibiting teleologic rationality. This episode illustrates a teacher's use of rational questioning, bringing the students' voices into the discussion and encouraging their implementation and understanding of all three components of rationality (Douek in Boero & Planas, 2014).

Discussion

The argumentation we observed in the two classes was very different. One classroom was characterized by a focus on students' epistemic rationality and the teacher's communicative rationality. The argumentation in this class seemed to be both somewhat shallow and more formal and proof-like. The other classroom was characterized by a more balanced focus on students' epistemic, communicative, and even teleologic rationality, and we saw the argumentation in this class as somewhat informal but characterized by student autonomy. The second classroom also illustrated some intentionality and awareness of components of rationality (although not with those words) in the interactions, as Douek suggested was necessary (Boero & Planas, 2014). The teachers also used different kinds of tasks in their classrooms. The choice of tasks in each classroom may also be related to the teachers' intentions with respect to the components of rationality; more research is necessary to examine this question.

Differences were observed in the kinds of questions each teacher asked. Ms. Bell asked a wide range of questions, while Ms. Carr asked primarily factual answer and elaboration questions. Perhaps the kinds of questions teachers ask may indicate their focus on a particular component of rationality. It is an open question as to the significance of these components of rationality in a mathematics class, but if we want to encourage students to view mathematics as rational and to act in rational ways when engaging in the study of mathematics, then it seems that it would be helpful for teachers to act in ways that encourage all components of rationality at appropriate points (as Douek suggested, to engage in rational questioning, Boero & Planas, 2014). As Boero (2006) suggested, teachers can model the components of rationality for their students at appropriate times. Perhaps introducing these components of rationality to teachers could provoke a wider focus. Examining the kinds of questions teachers ask in conjunction with their argumentation shows promise for revealing which components of rationality are privileged in their classes. And these components of rationality provide a useful explanatory mechanism for the differences in support for argumentation observed in classrooms. Future research will have to examine how important it is for a teacher to engender all components of rationality and whether it is possible or productive to address each component separately.

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Evolution of proof form in Japanese geometry textbooks

Marion Cousin[1] and Takeshi Miyakawa[2]

[1]Lyons Institute of East Asian Studies, Lyon, France; cousin_marion@yahoo.fr

[2]Joetsu University of Education, Joetsu, Japan; miyakawa@juen.ac.jp

This study discusses the evolution of mathematical proofs in Japanese junior high school geometry textbooks and the conditions and constraints that have shaped them. We analyse the evolution of these proofs from their inception in the Meiji era (1868–1912) to the present. The results imply that features of the Japanese language affected the evolution of proof form in Japan and shaped the use of proofs in Japan as written, but not oral, justification for mathematical statements.

Keywords: Secondary school mathematics, history of education, textbook analysis.

Introduction

Proving mathematical statements is a very important part of mathematics. However, there were no proofs in the texts of *wasan*, the traditional mathematics dominant until the mid-19th century in Japan. In *wasan*, following Chinese tradition, Japanese mathematicians concentrated on elaborating procedures to solve problems rather than proving statements. As one consequence of the educational reforms that accompanied the opening and modernization of the country in the Meiji era (1868–1912), axiomatic Euclidean geometry with mathematical proof was adopted in secondary school mathematics.

Today, Japanese students learn mathematical proof in junior high school, and often face difficulties doing so (MEXT, 2009; Kunimune et al., 2009), as do students in other countries (see Mariotti, 2006; Hanna & De Villiers, 2012). These difficulties vary by country, for two reasons linked to the cultural and social dimensions of teaching. The first involves what is taught; one recent study compared France and Japan and showed that proof to be taught, specifically what constitutes a proof and the functions of proofs, is different between the countries (Miyakawa, 2017). The second reason relates to how students employ and understand justification and argumentation in their daily life, which affect how they approach mathematical proof in the classroom and which differ across cultures (Sekiguchi & Miyazaki, 2000).

The Anthropological Theory of the Didactic (ATD) posits that knowledge taught/learnt in a given *institution* (here, the Japanese educational system and culture) is shaped by a process of 'didactic transposition' reflecting the conditions and constraints specific to that institution (Chevallard, 1991; Bosch & Gascón, 2006). In this paper, we study the didactic transposition of proofs in Japan and the effects of the cultural and social dimension. We expect that this will help us better understand the nature of these difficulties and will show the needs for studying this dimension of proof-and-proving in different countries to improve teaching and learning everywhere.

Methodology

We adopted ATD to frame our research question and determine what should be investigated so as to better understand the cultural and social dimension of proof. The research question we focused on is as follows: *What cultural and social conditions and constraints shape the nature of proof to be taught today in Japan?* To identify these conditions and constraints, we conducted a historical study of the

evolution of the proof in Japanese junior high school geometry textbooks from its first appearance during the Meiji era to the present.

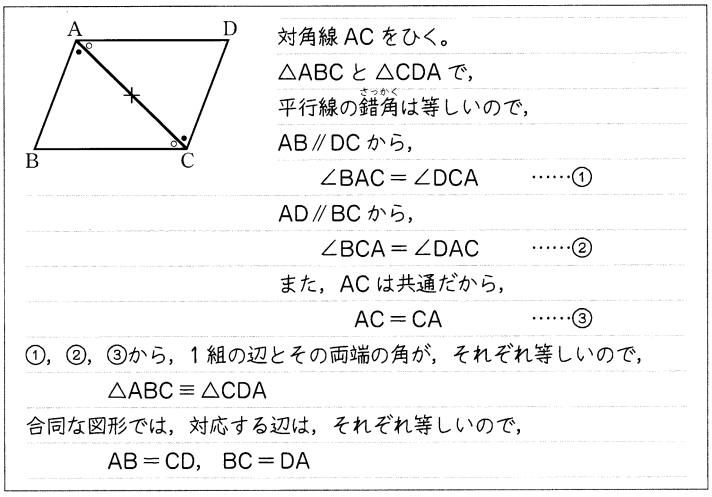
From out of the many textbooks published since the Meiji era in Japan we selected those that were widely used, to construct a representative corpus. Textbooks from the Meiji and Taishō (1912–1926) eras were more important than later ones, since proofs in geometry first appeared in Japan during these periods and since the way they were presented and taught changed more than in later periods. For the Meiji period, we identified major textbooks by consulting prior research (Neoi, 1997; Tanaka & Uegaki, 2015); however, for the Taishō era and up to the Second World War, we had no statistics on the use of textbooks, and so we selected textbooks that remain relatively well known today and that have been the topic of historical studies (Nagasaki, 1992). For the post-war era, we selected one or two textbooks that were widely used from the period following each successive reform of the national curriculum. The current system of selection of textbooks was firmly established by 1965 (Nakamura, 1997, p. 90) and the market share of each textbook series is known thereafter. From that point to the present, the most widely used textbooks have been those published by Keirinkan and by Tōkyō Shoseki.

The process of analysis we followed had three steps. First, we determined the role of the proofs in the geometry teaching approaches employed by the textbooks: Did the textbooks reflect a general strategy concerning proof learning? If yes, what was it? Were proofs important in geometry learning? Second, for each textbook, we analysed the forms (including intermediate steps) of sample proofs (worked examples) related to parallelograms, which were found in most of the textbooks, for overall formatting or organization, use of symbols, and formulation of properties (theorems, definitions, axioms, etc.) and statements. We use the terms *paragraph* and *semi-paragraph* to reflect the extent of sentences versus symbols in a proof, with paragraphs being all written language and semi-paragraphs a mix of words and symbols. Third, we looked at the authors' comments on the proof or on proof learning.

Below, we first describe the proofs one finds in Japanese mathematics textbooks today, and then show what they evolved from and how. However, as this work is currently only at a preliminary stage, our analysis remains general on the evolution of proof form in Japan.

Proof in Japanese mathematics textbooks today

Nowadays, the term 'proof' is introduced in Japanese junior high school mathematics, specifically in grade 8 geometry. Figure 1 shows a sample proof taken from a grade 8 textbook from Keirinkan, proving a property of parallelograms: 'Two pairs of opposite sides in a parallelogram are equal'. The figure provides an image of the proof with our own translation; the translation is quite literal, to maintain data integrity. One may first note the use of mathematical symbols for equality, parallelism, triangles, and angles. Statements (not properties) used as conditions or deduced as conclusions in a deductive step are written all in symbols (e.g. $\angle BAC = \angle DCA$). Deduced statements are given separately from other statements and properties, and some are numbered for use in later steps. In contrast, properties used in deductive steps, such as the condition for congruent triangles, are given as written Japanese phrases, without symbols—not in if-then form as in French mathematics textbooks (Miyakawa, 2017). The proof presented here thus represents the semi-paragraph type, with a mix of natural sentences and symbols; below, we consider the origin and history of such proofs.



(Our translation) Draw the diagonal AC.
In ΔABC and ΔCDA,
since the alternate-interior angles of parallel lines are equal,
from AB // DC,
∠BAC = ∠DCA … (1)
from AD // BC,
∠BCA = ∠DAC … (2)
And, since AC is common,
AC = CA … (3)
From (1), (2), and (3), a pair of sides and the angles of both sides are equal,
ΔABC ≡ ΔCDA
since corresponding sides of congruent figures are respectively equal,
AB = CD, BC = DA

Figure 1. A sample present-day proof from a Keirinkan textbook (Okamoto et al., 2016, p. 133)

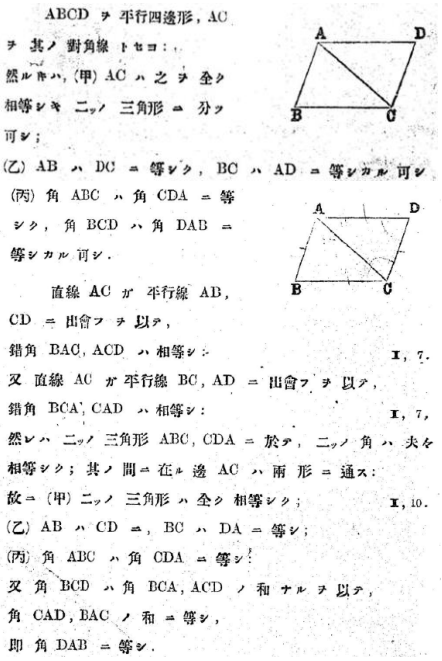
Proofs in geometry textbooks from the Meiji era to the present

Before the Meiji era—that is, before the modernization of Japan—geometry teaching was based on *wasan*, and centred on problem-solving: questions about the measurement of geometric figures were asked, and procedures (sometimes employing algebraic or analytic tools) were applied to acquire the correct answer. Although some *wasan* mathematicians questioned the accuracy of the results yielded by this method, proofs were not used in mathematical texts until the mid-19th century, at the beginning of the modernization movement began (for a general view of the evolution of Japanese mathematics and its teaching, see Ueno, 2012, and Baba et al., 2012).

With the *Decree on Education* (*Gakusei*, 1872), the Japanese government abandoned *wasan* teaching and imposed learning of Western-style knowledge and teaching methods (for example, one-on-one teaching was replaced with lecture-type classes in groups). Western textbooks were translated to provide teaching materials for schools of this new type, and the first geometry proofs in Japanese appeared in this context. Since proofs were new to Japan, no convention and no stipulation in the curriculum constrained how they were written or formatted, and the forms used by Western authors and their Japanese translators varied widely. The situation can be quite confusing. For example, in the Japanese translation of an American version of *Legendre*'s textbook (Nakamura, 1873), proofs were written in paragraph form only, whereas in translations of other American textbooks (Miyagawa, 1876; Shibata, 1879), symbolic expressions were also mobilized. This situation, and the fact that no author-translators provided any remarks on proofs or reasoning in geometry and sometimes even removed remarks on the nature of mathematical statements that had been present in the original textbooks (see Cousin, 2013) betrays the lack of importance attached by Meiji-era scholars and authorities to proof learning; it also may have occurred partly because of the need for rapid translation of textbooks to meet new requirements, which led translators to focus on developing a basic vocabulary for the new geometry in Japanese and producing textbooks understandable enough for use. We also encountered textbooks from this period in which some functions of proofs were obscured compared to the original source: for example, while the axiomatic systematization function of proofs is emphasized in Davies (1870), the abridged Japanese version of this textbook (Nakamura, 1873) does not preserve this emphasis (see Cousin, 2013).

During the 1880s, Tanaka Naonori (1853–?) compiled works by English, American, and French authors as well as Chinese and Jesuit translators to produce a series of textbooks that were adopted widely in Japanese junior high schools (see Cousin, 2013, pp. 277–282). Tanaka was better trained in Western mathematics than the 1870s author-translators and had teaching experience as well. His proofs used few formulas and provided exposition (the part of the proof where the hypothesis is expressed using specific names for the elements considered in the proposition) and determination (the conclusion expressed using these names) using only symbolic expressions. Moreover, unlike previous authors, Tanaka gave after each statement a reference number corresponding to the property he used to justify it, highlighting the need for systematic justification of every statement in a proof. He was also the first Japanese author to discuss the nature of proof per se, explain its role in geometry (see Cousin, 2013, pp. 305–310), describe inductive and deductive ways of proving, and emphasize that we 'prove the propositions thanks to the axioms, the postulates and the propositions that already have been proven' (Tanaka, 1882, p. 15).

In the late 1880s, the publication of textbooks by Kikuchi Dairoku (1855–1917) marked a new stage in Japanese geometry textbook production, and Kikuchi fixed a new Japanese mathematical language and proof form that would remain for decades, as his textbooks were used until the beginning of the Taishō era. In his view, it was important to create a Japanese mathematical language that unified oral and written expression so that geometry proofs could be written in paragraph form, without relying on symbols. Moreover, like Tanaka, he highlighted the systematic aspect of proof by putting on the right-hand side the number of properties used in each deductive step (Figure 2). Kikuchi was clearly influenced by his education in England, where the aim of geometry teaching was to cultivate young spirits to reasoning: 'Wherever Mathematics has formed a part of a Liberal Education, as a discipline of the Reason, Geometry has been the branch of mathematics principally employed for this purpose. […] For Geometry really consists entirely of manifest examples of perfect reasoning: the reasoning being expressed in words which convince the mind, in virtue of the special forms and relations to which they directly refer' (Whewell, 1845, p. 29). Kikuchi provided extensive explanation of



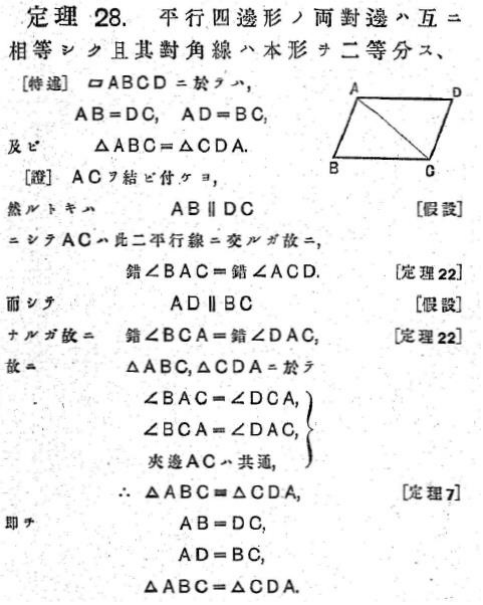
(Our translation)
Let ABCD be a parallelogram and AC be its diagonal;
Then (1) AC divides it into two completely equal triangles;
(2) AB is equal to DC, BC is equal to AD;
(3) The angle ABC is equal to the angle CDA, the angle BCD is equal to the angle DAB.

Because the line AC intersects with the parallel lines AB and CD, alternate-interior angles BAC and ACD are equal; I, 7.
And because the line AC intersects with the parallel lines BC and AD, the alternate-interior angles BCA and CAD are equal; I, 7.
Now, in the two triangles ABC and CDA, two pairs of angles are respectively equals, and the side AC between them is common to both figures.
So (1) the two triangles are completely equals; I, 10.
(2) AB is equal to CD, and BC is equal to DA;
(3) The angle ABC is equal to the angle CDA: and because the angle BCD is the sum of the angles BCA and ACD, it is equal to the sum of the angles CAD and BAD, which is the angle DAB.

Figure 2. A sample proof from Kikuchi's textbook (Kikuchi, 1889, pp. 53–54)

geometric reasoning, and paid particular attention to the language used and the organization of geometric properties; in doing so, he tried to highlight the importance of the systematization and justification functions of proofs.

However, the form of Kikuchi's proofs (Figure 2) soon came in for criticism by his contemporaries, for being difficult to teach. Nagasawa Kamenosuke (1861–1927), in his own textbook, criticized the paragraph form of Kikuchi's proofs in strong terms: 'Writing proofs of theorems with sentences in a complete and perfect manner is the vice of those who agree with the Euclid movement that came from England' (Nagasawa, 1896, pp. 3–4). Nagasawa instead wrote proofs in a semi-paragraph form very different from Kikuchi's, especially in terms of the use of symbols, as seen in Figure 3. In particular, Nagasawa put more importance on the proof as a written form, and in fact his proofs cannot be used for oral justification due to certain features of the Japanese language and the use of symbols. For example, the statement 'AB || DC' would usually be read or spoken aloud in Japanese as 'AB hēkō DC' ('AB parallel DC'). However, this is just a pronunciation of each symbol in succession and not a grammatically sound phrase; to be grammatical, it should instead be pronounced as 'AB wa DC ni hēkō' ('AB is parallel to DC'), whose shortened version would be 'AB DC ||', as an adjective with a *be*-verb should always be placed at the end of a phrase in Japanese. Beginning around the end of the Meiji era, proofs written in semi-paragraphs appeared in many Japanese geometry textbooks (e.g. Nagasawa, 1896; Kuroda, 1917), even Kikuchi's (Kikuchi, 1916), and Kikuchi's goal of a language that unified oral and written expression was abandoned.


定理 28. 平行四邊形ノ両對邊ハ互ニ相等シク且其對角線ハ本形ヲ二等分ス、

[特遣] □ABCD ニ於テハ、

AB＝DC, AD＝BC,

及ビ △ABC＝△CDA.

[證] AC ヲ結ビ付ケヨ、

然ルトキハ AB ‖ DC [假設]

ニシテ AC ハ此二平行線ニ交ルガ故ニ、

錯∠BAC＝錯∠ACD. [定理22]

而シテ AD ‖ BC [假設]

ナルガ故ニ 錯∠BCA＝錯∠DAC, [定理22]

故ニ △ABC, △CDA ニ於テ

∠BAC＝∠DCA,

∠BCA＝∠DAC,

夾邊 AC ハ共通、

∴ △ABC＝△CDA, [定理7]

即チ AB＝DC,

AD＝BC,

△ABC＝△CDA.

(Our translation)

Theorem 28. Two pairs of opposite sides of a parallelogram are equal to each other, and its diagonal divides it into two equal parts.

[Exposition] In □ABCD, AB = DC, AD = BC, and ΔABC = ΔCDA.

[Proof] Connect A and C,

in such a case,	AB ‖ DC	[Hypothesis]
and because AC intersects with these two parallel lines,		
	alt. int. ∠BAC = alt. int. ∠ACD.	[Theorem 22]
And because	AD ‖ BC	[Hypothesis]
	alt. int. ∠BCA = alt. int. ∠DAC,	[Theorem 22]
so	in ΔABC, ΔCDA,	
	∠BAC = ∠DCA,	
	∠BCA = ∠DAC,	
	the side AC is common,	
	∴ ΔABC ≡ ΔCDA,	[Theorem 7]
So,	AB = DC,	
	AD = BC,	
	ΔABC = ΔCDA.	

Figure 3. A sample proof from Nagasawa's textbook (Nagasawa, 1896, p. 53)

Moreover, until the end of the 19th century, although various ways of writing proofs were seen, all textbooks nevertheless followed a classic pattern in the teaching of geometry: theorems and problems were stated one after the other and, beginning in the 1880s, statements in proofs were justified with the reference number of the relevant property. Beginning in the Taishō era, however, the 'practical' approach, meaning one that tried to be more related to ordinary life, gained more and more success, influenced by the work of Treutlein (e.g. 1911), and Japanese authors distanced themselves from the classic pattern. For example, in the first quarter of Kuroda's textbook (1917), measuring instruments were presented and geometric matters were treated without theorems or proofs, while in the latter

part, several practical questions were asked. This evolution of geometry teaching also had an influence on proof form. In Kikuchi (1889), all the statements were expressed without using symbols and the justifications were expressed only by presenting reference numbers for properties (Figure 2), whereas in Yamamoto (1943), new statements were expressed with symbols and the justifications were expressed using literal expressions, without using numbers to refer to properties. Under this practical approach, the systematic aspect of justification in geometry came to be less emphasized.

With the 1942 curriculum reform, the national curricula explicitly adopted this practical approach. The general axiomatic system became less and less explicit in the textbooks, and more and more problems appeared that were related to everyday life. For instance, no proofs at all appeared in 1947's *Secondary Mathematics* (*Chūtō sūgaku*), published by the national Ministry of Education (Monbushō, 1947). Nevertheless, between 1949 and 1955, proofs gradually reappeared in geometry textbooks.

Since the 1960s, proofs have been introduced beginning in the 8th grade; however, although the concepts used in geometry teaching in Japan have not changed much in this period, proof form has continued to change, a little. For example, in Kodaira et al. (1974), in the New Math period, properties were always given on the right hand-side, in brackets, and symbols were frequently used (more than in any previous or later textbooks). Later, in Kodaira et al. (1986), the same authors returned to a strategy similar to that observed in the 1940s but also to that used today: symbols were used to express statements in the proofs, but natural language sentences were used to express the properties justifying these statements.

Discussion and conclusion

The proofs in Japanese mathematics textbooks take the forms they do as a result of the process of didactic transposition, which involves their exposure to different conditions and constraints that affect their nature as proofs. For instance, this study on the evolution of proofs in geometry education in Japan has shown that one factor that significantly affected proof form was certain features of the Japanese language. As mentioned above, Kikuchi tried to develop a Japanese mathematical language unifying oral and written expression, in order to help train students in rigorous logical thinking, adopting the approach of structuring proofs in paragraph form as part of this project; however, our study has shown that Kikuchi's paragraph-form proofs disappeared, as they were viewed as too hard to teach. It was replaced by the semi-paragraph form, which is still used for proofs in Japan today. One consequence is that the distance between the forms of the written proof and the oral justification is still bigger in Japanese education than in English or French, and statements written with symbols cannot be directly used in the oral justification. This leads us to think that Japanese students may experience a proof as a particular written object (like an algebraic equation), a formalism with little relationship to 'actual' oral justification or argumentation. As such a distinction implies, it will be useful to investigate the distance between written proofs and oral justifications across countries, which will help us benefit more fully from existing research results on argumentation and mathematical proofs.

Acknowledgment

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Pre-service middle school mathematics teachers' interpretation of logical equivalence in proof by contrapositive

Esra Demiray[1] and Mine Işıksal Bostan[2]

[1]Hacettepe University, Faculty of Education, Ankara, Turkey; esrademiray@hacettepe.edu.tr

[2]Middle East Technical University, Faculty of Education, Ankara, Turkey; misiksal@metu.edu.tr

The purposes of this study are to investigate pre-service middle school mathematics teachers' interpretations of logical equivalence in proof by contrapositive and the reasons for their incorrect interpretations. Data analysis indicated that pre-service middle school mathematics teachers were considerably unsuccessful in interpreting logical equivalence of statements. Lack of knowledge related to indirect proof methods, accepting a true statement as false, suggesting to apply direct proof instead of selecting given choices, and thinking contrapositive statements as unrelated could be regarded as the reasons for their incorrect interpretations.

Keywords: Contrapositive, logical equivalence, pre-service middle school mathematics teachers.

Introduction

Proof does not have simple roles in mathematics and mathematics education; it is a fundamental component and includes different forms and methods (Jones, 1997). A review of the literature indicated that there are limited number of studies focusing on particular proof methods (Antonini & Mariotti, 2008; Baccaglini-Frank, Antonini, Leung, & Mariotti, 2013; Bedros, 2003; Stylianides, Stylianides, & Philippou, 2004). According to Stylianides, Stylianides and Philippou (2004), the least attention has been given to proof by contrapositive compared to other proof methods such as mathematical induction, proof by contradiction, and direct proof. Thus, in this study the focus is given to proof by contrapositive. According to Bedros (2003), proof by contrapositive is a method of indirect reasoning. Since a conditional statement $p \Rightarrow q$ and its contrapositive $q' \Rightarrow p'$ are logically equivalent, in order to prove a given statement $p \Rightarrow q$, the statement $q' \Rightarrow p'$ can be proved by using direct proof (Bloch, 2000). In other words, when a statement is proved, its contrapositive is also proved (Antonini, 2004). This study focused on the logical equivalence of contrapositive statements, which is the key idea of proof by contrapositive method.

According to Baştürk (2010), students have difficulty in deciding which proof method to use and in applying the selected method. Moreover, students have many more difficulties in indirect proof methods rather than direct proof methods (Antonini & Mariotti, 2008). For example, Dickerson (2008) commented that undergraduate and graduate students have difficulty in understanding the language and logic of indirect proof methods. In the study by Stylianides, Stylianides and Philippou (2004), it was stated that some undergraduate students had difficulty in understanding logical equivalence in contrapositive and used incorrect equivalences such as $p \Rightarrow q \equiv p' \Rightarrow q'$ in their explanations. Similarly, many students could not distinguish proof by contradiction from proof by contrapositive (Goetting, 1995).

As seen, indirect proofs such as proof by contrapositive have the potential to reveal many difficulties that students possess in relation to proof (Bedros, 2003). Teachers' knowledge of proof plays an important role in developing students' understanding in proof. For instance, when mathematics

teachers present various proof methods in the class, it helps students to enhance their logical thinking and proof abilities (Altıparmak & Öziş, 2005). Therefore, mathematics teachers should have necessary knowledge and experience concerning different proof methods. Since pre-service middle school mathematics teachers are future teachers, their interpretations related to the logic of particular proof methods such as proof by contrapositive are important to investigate. Thus, to examine pre-service middle school mathematics teachers' interpretation of logical equivalence in proof by contrapositive and the reasons for their incorrect interpretations were determined as the purposes of the present study. Moreover, in the teacher education program, pre-service teachers take various mathematics courses and their ability in interpreting proof related concepts might depend on these mathematics courses since some of which place more importance on proof. In relation to this, how pre-service teachers' success levels differ by year level in the program was also investigated. By considering these purposes, the research questions were stated as follows:

1. To what extent are Turkish pre-service middle school mathematics teachers successful in interpreting logical equivalence in proof by contrapositive, and how does their success differ by year level?

2. What are the reasons for Turkish pre-service middle school mathematics teachers' incorrect interpretations?

Method

Since data were collected at just one point in time from a selected sample in order to describe certain characteristics of the population by asking questions (Fraenkel & Wallen, 2005), this study was designed as a cross-sectional survey. Using convenience sampling methods, the sample for this study was determined as 115 pre-service middle school mathematics teachers attending a state university in Ankara, Turkey. In terms of their year level, 19 were freshmen (16.5%), 25 were sophomores (21.7%), 39 were juniors (33.9%), and 32 were seniors (27.8%).

In Turkey, the middle school mathematics teacher education programs offer mathematics courses such as Calculus, Algebra; mathematics education courses involving Methods of Teaching Mathematics, Practicum; education courses such as Classroom Management; general courses involving Academic Oral Presentation Skills, and elective courses. The first two years of the program mainly consist of mathematics courses while the last two years put more emphasis on education, mathematics education, and elective courses.

This study was conducted as part of a larger study focusing on pre-service middle school mathematics teachers' interpretation of the logic behind proof methods. In this study, the answers given by pre-service teachers to three questions related to the logical equivalence of contrapositive statements were analyzed. These questions were prepared by reviewing the related literature (Knuth, 1999; Saeed, 1996). In more detail, Question 1 (Q1) and Question 2 (Q2) were prepared by the researchers by considering the format of the multiple choice questions in the study undertaken by Knuth (1999). The students were asked to select the correct statement that can be used to start to prove the given statement and explain their answers. The correct choice involves the proposition $q' \Rightarrow p'$ as the starting point to prove the proposition $p \Rightarrow q$ which is known as proof by contrapositive. The other choices were not appropriate to start any proof. The correct choices were identified as (d) for Q1 and (c) for Q2. Questions 1 and 2 are presented below.

Q1. Assume that m and n are positive integers. If mn=100, then m≤10 or n≤10.	**Q2.** Assume that a, b and c are real numbers and a>b. If ac≤bc, then c≤0.
To prove the given statement, which statement can you begin with?	To prove the given statement, which statement can you begin with?
a) Assume that m and n are positive integers. If m>10 or n>10, then mn=100.	**a)** Assume that a, b and c are real numbers and a>b. If c>0, then ac≤bc.
b) Assume that m and n are positive integers. If m≤10 or n≤10, then mn=100.	**b)** Assume that a, b and c are real numbers and a>b. If c≤0, then ac≤bc.
c) Assume that m and n are positive integers. If m≤10 and n≤10, then mn≠100.	**c)** Assume that a, b and c are real numbers and a>b. If c>0, then ac>bc.
d) Assume that m and n are positive integers. If m>10 and n>10, then mn≠100.	**d)** Assume that a, b and c are real numbers and a>b. If c≥0, then ac≥bc.
e) None of the above	**e)** None of the above
- Why? State your reasons.	- Why? State your reasons.

Figure 1: Question 1 and Question 2

Question 3 (Q3) was adapted from the study of Saeed (1996) and involves a discussion about the proofs of two contrapositive statements. In the question, the participants were asked to select the person they agreed with and explain the reasons for their choice. The students' answers were accepted as incorrect if they agreed with Pınar and correct if they agreed with Ahmet.

Q3. Statement A: If n^2 is an odd integer, then n is an odd integer.
Statement B: If n is an even integer, then n^2 is even integer.

Ahmet: I think statement A is true, Pınar.
Pınar: Let me see, if $n^2=9$, then $n=\pm 3$ is odd; if $n^2=25$, then $n\pm5$ is odd. So, statement A seems to be true Ahmet.
Ahmet: I also think that statement B is true, Pınar.
Pınar: Why?
Ahmet: Since n is even, then n=2k where k is some integer.
Therefore, $n^2 = 4k^2 = 2\ (2k^2)$ is also even.
Pınar: But Ahmet, this only show the statement B is true, but does not show that statement A is true.
Ahmet: This argument also shows that statement A is correct.

Questions;
- Considering the discussion above, who do you agree with?
Ahmet_____ Pınar_____

- Why? Explain your reasons.

Figure 2: Question 3

To investigate the research questions, descriptive statistics and item-based analysis were conducted. Firstly, pre-service middle school mathematics teachers' interpretations of logical equivalence in proof by contrapositive were analyzed based on the rubric given in Table 1. Then, the reasons for their incorrect interpretations were examined qualitatively by generating themes.

	Answer types in Q1 and Q2	**Answer types in Q3**
	No answer	No answer
Incorrect answer	Incorrect choice was marked, no explanation was stated	Agreed with no one or both of them
		Agreed with Pınar, no explanation was stated
	Incorrect choice was marked, explanation was stated	Agree with Pınar, explanation was stated
Correct answer	Correct choice was marked, no explanation was stated	Agreed with Ahmet, no explanation was stated
	Correct choice was marked, explanation was given but not referring to the logical equivalence	Agreed with Ahmet, explanation was given but not referring to the logical equivalence
	Correct choice was marked, explanation was given referring to the logical equivalence	Agreed with Ahmet, explanation was given referring to the logical equivalence

Table 1: Rubric for questions

Findings

In order to investigate the first research question, pre-service middle school mathematics teachers' answers to Q1 and Q2 were analyzed. The results of 115 pre-service middle school mathematics teachers' answers are presented in Table 2.

Answer types		**Question 1**		**Question 2**	
No answer		4 (3.5%)		4 (3.5%)	
Incorrect answer	Incorrect choice was marked, no explanation was stated	12 (10.4%)	50 (43.4%)	12 (10.4%)	55 (47.8%)
	Incorrect choice was marked, explanation was stated	38 (33.0%)		43 (37.4%)	
Correct answer	Correct choice was marked, no explanation was stated	43 (37.4%)	61 (53.1%)	33 (28.7%)	56 (48.7%)
	Correct choice was marked, explanation was given but not referring to the logical equivalence	7 (6.1%)		9 (7.8%)	
	Correct choice was marked, explanation was given referring to the logical equivalence	11 (9.6%)		14 (12.2%)	

Table 2: Frequencies of the answers to Q1 and Q2

Table 2 shows that 4 students (3.5%) did not answer to Q1 and Q2. When the answers of the students to Q1 were investigated, it was seen that 50 students (43.4%) answered incorrectly and 61 students (53.1%) selected the correct choice. In addition, 43 students (37.4%) marked the correct choice without stating their reasons and the answers of 7 students (6.1%) were correct but their explanations were not related to logical equivalence. The remaining 11 students (9.6%) answered correctly by providing an explanation based on logical equivalence of contrapositive statements. In terms of year level in the program, freshmen (73.7%) had the highest percentage of correct answers and seniors (40.6%) had the lowest percentage of correct answers in Q1. As an example of a correct answer with an explanation referring to logical equivalence, Participant 52 stated as follows:

p: mn=100 p':mn≠100 $p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p'$
q: $m \leq 10 \vee n \leq 10$ q': $m > 10 \wedge n > 10$
q'⇒p' (If m>10 and n>10, then mn≠100) (Participant 52, junior)

The analysis of the answers to Q2 showed that 55 students (47.8%) answered incorrectly whereas 56 students (48.7%) answered correctly. Thirty-three students (28.7%) marked the correct choice in the question but did not substantiate their ideas. Moreover, 9 students (7.8%) answered correctly without referring to contrapositive statements, and 14 students (12.2%) answered correctly by referring to the logical equivalence of contrapositive statements. While sophomores (64.0%) had the highest percentage of correct answers, freshmen (36.8%) and seniors (37.4%) had the lowest percentages of correct answers in Q2. To illustrate, Participant 97 answered correctly and explained by referring to logical equivalence in proof by contrapositive.

p: $ac \leq bc$ q: $c \leq 0$
Then, proof by contrapositive, $p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p'$ (Participant 97, senior)

Since Q3 has a different rubric from the multiple choice questions, pre-service middle school mathematics teachers' answers to Q3 are presented in Table 3.

Answer types		Question 3	
No answer		4 (3.5%)	
Agreed with no one or both of them		3 (2.6%)	
Incorrect answer	Agreed with Pınar, no explanation was stated	16 (13.9%)	75 (65.2%)
	Agree with Pınar, explanation was stated	59 (51.3%)	
Correct answer	Agreed with Ahmet, no explanation was stated	5 (4.3%)	33 (28.7%)
	Agreed with Ahmet, explanation was given but not referring to the logical equivalence	24 (20.9%)	
	Agreed with Ahmet, explanation was given referring to the logical equivalence	4 (3.5%)	

Table 3: Frequencies of the answers to Q3

According to Table 3, 4 students (3.5%) did not answer Q3. The answers of 3 students (2.6%) showed that they agreed with neither Pınar nor Ahmet but did not explain their rationale. Moreover, 75 students (65.2%) agreed with Pınar, which is accepted as incorrect answer and 33 students (28.7%) agreed with Ahmet, which is accepted as correct answer. Five students (4.3%) agreed with Ahmet without giving any explanation, 21 students (20.9%) agreed with Ahmet and explained without referring to logical equivalence, and 4 students (3.5%) explained their agreement with Ahmet by referring to logical equivalence of contrapositive statements. Moreover, juniors (38.4%) had the highest percentage of correct answers and sophomores (4.0%) had the lowest percentage of correct answers to Q3. An example of a correct answer, Participant 52 agreed with Ahmet and her explanation was related to logical equivalence used in proof by contrapositive.

p: n is even q: n^2 is even
$p \Rightarrow q$ was proved
$p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p'$
Thus, if n^2 is odd then n is odd. Therefore, Ahmet is right. (Participant 52, junior)

For the second research question, pre-service middle school mathematics teachers' explanations for their incorrect answers were analyzed. As presented in Tables 2 and 3, 50 students (43.4%) answered Q1 incorrectly and 38 of them (33.0%) gave explanations for their answers. Fifty-five students (47.8%) answered Q2 incorrectly, of whom 43 (37.4%) explained their answer. Lastly, 75 students (65.2%) answered Q3 incorrectly and 59 of them (51.3%) suggested explanations for their answers. Table 4 shows the reasons behind the students' incorrect interpretations grouped under four categories.

Reasons		Q1	Q2	Q3
R1	Lack of knowledge related to indirect proof methods	30 (26.1%)	35 (30.4%)	-
R2	Accepting a true statement as false	5 (4.3%)	-	-
R3	Suggesting to apply direct proof instead of selecting given choices	3 (2.6%)	8 (7.0%)	-
R4	Thinking that contrapositive statements are unrelated	-	-	59 (51.3%)
Total		38 (33.0%)	43 (37.4%)	59 (51.3%)

Table 4: Reasons for students' incorrect interpretations

The first reason for the incorrect interpretations is students' lack of knowledge related to indirect proof methods. As a result of this inadequacy, students thought that one of the choices in the question was related to contradiction or contrapositive; however, this choice was not related to these methods.

For example, in Q2, Participant 7 selected one of the incorrect choices and explained it as an assumption for contradiction.

> To prove by contradiction, we have to prove the converse situation. The choice b can be used in this situation. (Participant 7, freshman)

The second reason behind students' incorrect interpretations is that they accepted the given statement as false even though it was true and tried to find counterexamples to refute it. For instance, in Q1, Participant 114 could not see that the given statement was true.

> The given statement 'Assume that m and n are positive integers. If mn=100, then m≤10 or n≤10.' is not true.
> As counterexamples, m=12 and n=12 can be used.
> Then, mn=12.12=144≠100
> Therefore, 'if mn=100 then m≤10 and n≤10' is a true statement. (Participant 114, senior)

The third reason is that students mentioned using direct proof instead of selecting one of the given choices. For instance, the answer of Participant 106 to Q1 is given below:

> Firstly, we can assume that mn=100; we can try to deduce m≤10 or n≤10. We cannot start with the sentences given above. (Participant 106, senior)

The last reason for incorrect interpretations is that students thought that there was no relation between the given contrapositive statements A and B. For example, in Q3, Participant 30 cited that statements A and B were different.

> Because the statements are different, one of them starts with an even number and the other one starts with an odd number. The proof of statement A can't be the same with the proof of statement B. (Participant 30, sophomore)

Discussion

According to the results of pre-service middle school mathematics teachers' answers to questions, it was found that nearly half of the sample answered Q1 and Q2 correctly and almost one third answered Q3 correctly. In other words, students' achievement levels in interpreting logical equivalence in proof by contrapositive were found to be considerably low. The findings revealed that freshmen had the highest achievement level for Q1, sophomores had the highest achievement level for Q2, and juniors had the highest achievement level for Q3. Although seniors were expected to have been the most successful group by considering the number of mathematics courses they took in the program, they were not the most successful in terms of all questions. This result might stem from the fact that seniors did not take any mathematics course in their last year of the program. Therefore, seniors might not remember the details of the logical equivalence used in proof by contrapositive. To avoid this situation, teacher educators could offer elective courses related to logic and proof to enhance prospective teachers' reasoning skills.

Four reasons for the incorrect interpretations were detected from three questions. The first reason is preservice teachers' lack of knowledge related to indirect proof methods. This finding is consistent with the results of Atwood (2001), who stated that students had difficulty in using the words converse, contrapositive, contradiction, and counterexample, and that they might use them interchangeably, which is not correct. Moreover, in the case that where students generally memorize proof methods

instead of understanding the structure of the proof might cause them to have difficulty in related proof methods. Therefore, the participants in this study might use proof by contrapositive and proof by contradiction inaccurately and interchangeably. The second reason why students answer incorrectly is accepting a true statement as false and trying to find counterexamples based on this idea. Some of the terms and signs involved in the given statement in Q1 such as 'or' and '≤' might cause students to misunderstand the statement. Thus, students might have had trouble in deciding whether the given statement was true or false and evaluate it as false. The third reason is that students suggested proving the given statement with direct proof instead of selecting one of the given choices in the question. This situation may result from the fact that the majority of the proofs in the textbooks are given as direct proofs (Atwood, 2001). Therefore, students may have a tendency to use direct proofs since they are more familiar with this method. The last reason is that students thought that statements A and B given in Q3 were unrelated. In this study, students might fail to see the relation between proofs of given two contrapositive statements. Therefore, they might think that statement A which involves $p \Rightarrow q$ and statement B which involves $q' \Rightarrow p'$ should be proven separately.

In mathematics teacher education programs, proof should be considered as an important theme. Thus, the content or place of mathematics courses in teacher education programs might be revised and developed in order to enhance preservice teachers' understanding of reasoning, proof, and logical rules behind proof methods. For example, mathematics courses might be taught by paying attention to logical rules behind proof methods. This study pointed out the importance of having knowledge of logical rules in reading and interpreting a given proof statement or conducting proof by using particular proof methods. Moreover, similar findings related to the interpretation of logical equivalence used in proof by contrapositive might be achieved with pre-service mathematics teachers in different countries. Therefore, to compare and to gain a global perspective about pre-service mathematics teachers' understanding of logical rules behind proof methods, cross-cultural studies could be conducted. Based on the findings of such studies, teacher educators might develop strategies to overcome pre-service mathematics teachers' current difficulties in logic and proof by considering the characteristics of their teacher education programs.

The results of the study are limited to the data collected with three questions. For further studies, pre-service middle school mathematics teachers' interpretations of logical equivalence used in proof by contrapositive might be investigated by using alternative questions in various formats. An investigation of the effect of pre-service mathematics teachers' knowledge of logic on their ability to prove might also be undertaken. Moreover, to analyze the answers of the pre-service mathematics teachers and to determine the reasons for their incorrect interpretations regarding logic in-depth, follow-up interviews might be conducted in future studies.

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Issues of a quasi-longitudinal study on different types of argumentation in the context of division by zero

Christian Fahse

Institute of Mathematics, University of Koblenz-Landau, Campus Landau; fahse@uni-landau.de

In this study we explore students' ways of argumentation concerning division by zero. The answers of 365 students of four different grade levels in a German secondary school were analyzed on the basis of written texts of the students explaining their results of 7:0. Applying qualitative content analysis (Mayring, 2000), we were able to distinguish three different types of argumentation. The relative frequencies of these different types vary with the increasing age of the students: rich argumentations stagnate, apodictic references to an authority increase.

Keywords: Argumentation, reasoning, communication model, division by zero.

Motivation and interest of research

Argumentation, besides for instance modelling and problem-solving, is one of the main issues especially characterizing mathematical education. The aim to develop argumentative abilities in mathematics has been reinforced by German authorities since 2003. Consequential, there is a need to measure progress in this field on the level of learning groups and educational systems. Tests like PISA and other test series claim to perform this measuring (OECD 2015, p. 32), even though they are subject to partly strong criticism (e.g., Jahnke & Meyerhöfer, 2007). The tasks used in these tests, pretending to measure argumentative skills, necessarily are very restricted in regard to content and time on task, compared to a creative argumentation process performed in classroom. Furthermore, the dichotomous focus on right or wrong does not seem to be suited for an observation on argumentation.

The notion of "probe" in educational research

The idea of the main study is to use a so called "educational probe" (Ger. "Sonde"). This can be best described by an analogy: Car insurances ask if the car is parked in a garage. If so, the insurance fee is reduced. This is not done due to causal inference, but for statistical reasons (information from the insurance company HUK-Coburg, Coburg, by telephone, 2011): there is a robust (negative) correlation between parking in a garage and the probability of an accident. A "probe" for detecting abilities is a small bundle of easily carried out measurements, observing the patterns of reactions of learners to some standardized impulse together with an established correlation of those patterns to the intended ability of the learning group.

It is an open question if educational probes exist. As a first step, we try to find candidates for a "probe" on the ability of argumentation; in a second step, we have to validate these probes. A variety of measurements can be taken into account (e. g. videography), but here we concentrate simply on tasks given as impulses and evaluate written texts, thereby e. g. ignoring any social interaction. Several groups of tasks, all roughly of the form "Give your opinion and justify it", were given to the same students. In this article we consider only the task concerning division by zero (in short: "7:0=? Justify your opinion."). This special task was included in the study because of the

variety of possible justifications discussed in the literature (Knifong & Burton, 1980) and observed in classroom practice (author's experiences with eight classes of 6th graders).

Basic assumptions and research questions

Argumentative abilities are considered to be of general importance beyond mathematics. Therefore, this study does definitely not focus on proving, and argumentation is not considered as a preliminary step to establish a proof. This decision was a consequence of both, the observed argumentations of the students and a certain communication model which has shown to be compatible with the data.

Furthermore, argumentative and mathematical abilities are considered as different constructs. Therefore, the correctness of the solution of the problem cannot be a primary criterion to discern different types of argumentations. Mistakes have to be tolerated, "misconceptions" can be of a distinct rationality (Prediger, Gravemeijer, & Confrey, 2015, p. 881). This pedagogical view is supplemented by a historic mathematical fact: It is not true that division by zero is not possible or not to define. E. g. the inversion (holomorphic extension of 1/z) on the Riemann Sphere is a continuous function which imposes $1/0=\infty$.

On first sight, one can wonder if short written answers to tasks really make a difference compared to the testing in PISA. The described preponderant disregard of "correctness" of the given result and the completely different evaluation by a qualitative content analysis (QIA, see below) are characteristics of this study, distinguishing it from PISA. The resulting category system is developed by an approach which is in a first step "grounded", that is, constructed without reference to other theories. This methodical choice was taken because the interest of this study lies in the opportunity to compare our findings to other category systems found in the literature. However, in order to narrow the scope of this article we have to make two limitations: First, we will not report on a comparison between different category systems. Second, neither the analysis of misconceptions of division by zero nor suggestions for classroom practice are points of interest here, but presented in Fahse (2014).

Taking the hypothesis that the ability of deploying argumentation develops over time, an appropriate probe should provide different results for different ages of the learners. The research questions in this mainly descriptive study are:

- What types of mathematical argumentation can be found?
- Does the percentage of these types differ from grade level to grade level?

Theoretical background

In this section we concentrate on literature about types of argumentation and leave aside that on divisions by zero with the following exception: In congruence with our study, Tsamir and Sheffer (2000) analyze argumentation in regard to division by zero. They distinguish between concrete and formal (algebraic) arguments, and favor the formal ones (Tsamir & Sheffer, 2000, p. 94). In contrast Fahse's (2014, p. 24) empirical examples show that the use of concrete models of division can give insight into the problematics of division by zero, even if a "wrong" result is given (different from "division is impossible", caveat see above). Therefore, in our system, the distinctions of Tsamir & Sheffer can only be considered as subcategories, not as main categories.

Different classifications of argumentation schemes without reference to any special mathematical topic can be found in the literature. Argumentation can be set in contrast to proof (Duval, 1991), or these phenomena can be treated in regard to their mutual relationship. The latter can e.g. be done by analyzing the process that leads to a proof, wherein argumentation is conceived "as a process of producing a conjecture and constructing its proof" (Boero, Douek, Morselli, & Pedemonte, 2010, p. 183). Following Pedemonte (2007), argumentations are based on a system of conceptions and related to conjectures either as "structurant" or as "constructive" argumentation. Furthermore, an argumentation can be abductive, inductive, or deductive (Pedemonte, 2007). These characteristics could be applied to our data, but since our study does not focus on proof they do not adequately describe the variety of justifications found in our study.

Harel & Sowder (1998) use the term "proof schemes" which refers to "what the person offers to convince others" (p. 275). This fits well into the model of argumentation given below. Their way of classification scheme (externally conviction, empirical, and analytical proof schemes as superordinate categories; "analytical" is changed into "deductive" in Harel, 2008, p. 491) will be compared to the findings of our study in another article.

Communication models and specification of concepts

Since essentially different (Brunner 2014, p. 231) definitions exist of the notions argumentation, reasoning (regarded here as synonymous to justification, if referring to one fixed claim), proving and explaining we have to specify these terms. They are not conceived with regard to proof, but to the argumentations notated by the tested students.

Our study uses a model of argumentation that is based on communication theories (Bühler, 1934; Kopperschmidt, 1980, following Habermas (1984)). The sender and receiver refer to a knowledge (and communication) basis assumed to be shared. The objects of justifications are statements that have different grades of plausibility for the two interlocutors. The act of justification performed by the sender is an attempt to augment this grade, conceived as an ordinal structure, on the receiver's side. Therefore, this concept of argumentation is genuinely dialogic. Nevertheless, the receiver can also be an internal entity within the sender, or a universal audience.

In the following, short definitions of the principal terms used in this article are given. The discussion and the comparison of these definitions to those found in the literature go beyond the scope of this article. But for reasons of practicability we suggest to accept these definitions in the frame of this article despite a lacking consensus in the wider scientific community (Brunner 2014, p. 231).

Argumentation is conceived as a generic term (Bezold, 2009), including the process of finding hypotheses, and checking common bases of knowledge and communication. **Reasoning** or **justification** is a communicative reaction to a questioning of a statement. The aim of reasoning is to increase the degree of the receiver's acceptance (his attributed epistemic value) of the statement by relating the statement under discussion to the basis of knowledge and communication assumed to be shared (Kopperschmidt, 1980, p. 73). **Proof** is a sequence of argumentative steps relying on an accepted basis of statements approaching the ideal of a complete logical chain of deductive steps (Duval, 1991). A proof can be a justification, but does not necessarily be one. **Explanation** (of "why", not "how" or "how to do") is an addressee-oriented justification by the sender with the aim

of creating an "understanding", which in turn is conceived as a fitting to the (possibly accommodated or enlarged) factual knowledge of the addressee (Kiel, 1999, p. 72; Hanna, 2016, "pedagogical explanations", p. 2).

Justifications can attempt to explain, but also aim to refer to reliable sources. Furthermore, in this framework, a mathematical proof is only one method of justification and not necessarily effective, depending on the mathematical ability of the sender and receiver. E. g. an algebraic transformation $7:0 = x \mid \cdot 0 \rightarrow 7 = 0$ performed by a student does not really convince another student if their interpretation of the variable x is uncertain and the concept of equivalence transformation is known, but insufficiently familiar.

The study

The analysis of the data is not fully completed yet, but we can report first results on selected parts of the study. We asked the students to give the result of the division 7:0 and to "justify [their] opinion in a way that someone who doesn't know the answer is able to understand [the result]" (Unabbreviated original task, translated from German). The written answers of a group of N=365 students in grade 7, 9, 11, 13 were analyzed. In regard to the relative abundance of argumentation types we report data from a subgroup of N=300 pupils which did not take part in interventional courses. These were N=73, 86, 78, 63 students in the four grades resp.. In this convenience sample all students were of the same secondary school ("Gymnasium"), and all students of the four chosen grades were tested (absence of students < 5%, no denial).

Method of analysis

First we applied a qualitative content analysis (QCA, Mayring, 2000) with inductive category development. Therefore, we analyzed the student's written justifications in several steps. The first step was to classify the texts only by similarity without any recourse to theory. In the next steps we aggregated items with an increasing level of abstraction ("feedback loops") leading to the different types of argumentation described in our coding manual. To ensure reliability this manual was used to perform a separate "deductive category application" (Mayring, 2000, sec. 4.2) by a pair of university students in a second step (Interrater Reliability κ=.967, N=365).

Results - Three different types of argumentation

We found three types of argumentation: rich, pseudo-factual, and apodictic. A summary is given in Table 1. All examples of student justifications have been translated from German.

Rich justifications

In this category the content and the way of reasoning are essentially reasonable (see below) even if the results might be wrong, or the justification is partly false or incomplete. The statement of justification is connected to a domain which is relatively complex. Therefore, operations (e.g. changing the mode of representation, calculations) are more likely to be found.

"Essentially reasonable" means that with the same idea a correct argumentation is possible. "7 : 0 = 7. So, if you have 7 apples and you divide them among 0 persons, you still have 7 apples." The usage of a model for division in the warrant (Toulmin, 1958) can be regarded as an operation in

which the representation of division is changed from the "algebraic view" to "partitive distribution". This latter domain is sufficiently complex: it is simply not possible to distribute 7 apples to nobody. The mistake in the original quote, disregarded for the type of argumentation, can be interpreted in the following way: the result of the division does not show what is left, but how much each person gets. Beside partitive interpretation using concrete objects, measurement interpretations of the division: "0 fits infinitely often into 7", and algebraic calculations are typical examples of rich arguments.

	Rich	Pseudo-factual	Apodictical
Domain	Linked to a mathematically appropriate, well structured domain	Unapt or poorly structured domain, weakly linked	No mathematical domain or tautological
Warrants	Potentially useful/ basic concepts	Essentially wrong	No mathematical but social warrants (authorities)
Operations	Operations e. g. change of representation	Few operations	No operations
Need for ...	Mathematical justification		Reliable authority/ source
Value (see discussion)	Epistemic (Proof: Logical)	(Similarity)	Informational

Table 1: Characterization of justification types

Pseudo-factual justifications

Mathematical warrants (Toulmin, 1958) are quoted, but these are profoundly incorrect, e.g. if the link to a domain of the common knowledge basis is not reliable (one student used the analogy of 7^0=1 - not an appropriate domain (power calculations), and weakly linked by analogy). In others, the cited domain has no sufficient structure, e.g. when including "invented" calculation rules ("all calculations with 0 produce 0 as result"), or making statements about the nature of the task or objects ("there's nothing to calculate", "0 has no significance"). Because of the lack of rich structure in the used domains only few other acts or operations besides generalizations and analogies can be found. Other texts seem to imitate the logical and symbolic structure of a mathematical justification, or use invented terms.

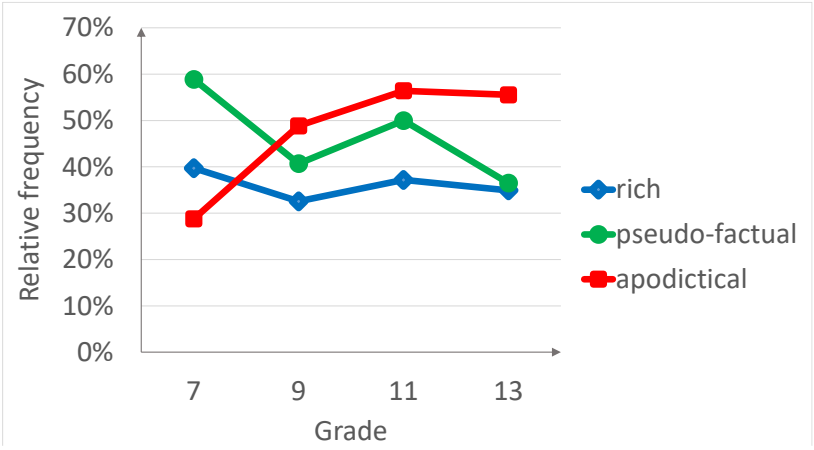
Apodictic justifications

Mathematical warrants are not used, but rather references to authorities like the teacher, the calculator, the textbook, or the world-wide-web given instead. A simple tautological repetition of a statement is interpreted as a reference to one's own ultimate knowledge and thus seen as an authority in the sense of: "That's how it is, I know it." Sometimes it is even stated that no justification is necessary. This shows a utilitarian understanding of mathematics, which can be convenient, e.g. for engineers in the course of their everyday work. Because there is no need for *mathematical* warrants, there are no domains and consequently no operations found. Typical examples: "[...] There is nothing to explain, that's the way it is", or "The rule says you cannot divide by zero. You just have to learn and remember it." This type seems to be very close to the authoritative type of Harel &

Sowder (1998). Precautionary it was given a different name, to be able to compare thoroughly the two types.

Quantitative results

Looking at the increasing grade level of the students, the relative abundances of the used argumentation types accordingly develop as follows: 1) Rich argumentations remain at slightly less than 40%. 2) Pseudo-factual argumentation is reduced to almost a half. 3) Apodictic argumentation almost doubles. There are very few justifications that can be seen as an algebraic proof: Only 4% use multiplication as the inverse of division. Note that the percentages in Figure 1 add up to more than 100% because each text can show more than one type of argumentation. The observed decrease, resp. increase was significant.



	Increase/decrease p.a.	CI (95%)	t(298)	p
Apodictical	4.5%	[.019;.071]	3.4	<.001***
Pseudo-factual	-2.8%	[-.054;-.001]	2.1	.039*
Rich	Not significantly ≠ 0	/	0.35	.724

Figure 1: Relative abundance of types with increasing grade and inference-statistical characteristics of the relative abundance of the different types when applying a linear model of increase

Discussion

The classification of justifications into different types presented in this study might be useful for the teacher's practice. They offer a framework for the moderation of class discussions. At first sight, there is no valuation concerning the different types. Proof increases the logical value, non-proof justifications try to augment the epistemic value of the statement (Harel, 2007, p. 497). In our model (Table 1), we add the *informational value*, conceived as the reliability of the source, which is increased by apodictic argumentation. This can be important for non-mathematicians, e. g. in the realm of engineering. Also in school, an algebraic argumentation (close to proof) by another student can be considered hard to trust. However, the aim of school education and the standards of mathematical education favor rich argumentations. For the practice of teaching it is recommended that apodictic justifications should *not* be discredited right away, but first discussed: They may be valuable for practical needs (unexamined statements as "black boxes"), but do not initiate an understanding. In contrast, pseudo-factual justifications should be criticized, even though at this point it is an open research question if this argumentation type can be considered a preliminary stage

of rich argumentation practice. As shown in an example in Fahse and Linnemann (2015), pseudo-factual justifications can be very appreciated by fellow students, because they seem to be particularly "mathematical" at first sight. In some cases, they do not increase any value of the statement, but try to augment the acceptance of the *justification* by similarity to genuine mathematical justifications. With the help of our classification, such misleading contributions could be more clearly discerned, both in discussions and written texts. Therefore, it might even be helpful to inform the students of these types (for first experiences, see Fahse & Linnemann, 2015). For teachers, these types could be useful for diagnostic purposes: to gain insight into the individual development of argumentation skills and to foster these abilities.

Taking the special task on division by zero as an educational probe seems to be promising: In all considered grades all three types occurred and were not marginalized. The different grades (and also the 16 different learning groups) showed significantly different distributions of types (not reported here). What is more, there are clear tendencies: The stagnation of the abundance of rich argumentations and the increasing of apodictic ones. One might think that this is caused by the increasing distance (mental and in time) of the students to this topic (taught in 5th grade). The following observations, though, question this interpretation: Even in 7th grade, the majority of the students does not remember the lessons on this topic, but refer to primary school. Similar questions are topics taught in grade 9 (square-root of negative numbers) and grade 10 (7^0, log(0)). Nevertheless, the use of algebra in the answers is rare. More likely, the increase of apodictic reasoning is caused by continuous repetition of the mere algebraic rule without any explanation, and perhaps by a neglect of argumentation as an educational objective. This last hypothesis will persist only if the used probe (division by zero) can be validated as a probe for argumentation ability in general. This will be a topic of future research in our investigations.

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The flow of a proof - Establishing a basis of agreement

Mika Gabel[1] and Tommy Dreyfus[2]

[1]Tel Aviv University, Afeka – Tel Aviv College of Engineering, Israel; mikagabe@post.tau.ac.il

[2]Tel Aviv University, Israel; tommyd@post.tau.ac.il

The notion of flow of a proof encapsulates mathematical, didactical and contextual aspects of proof presentation, related to the lecturer's choices regarding the presentation. We explore the relationship between mathematics teaching and rhetoric, suggesting Perelman's New Rhetoric (PNR) as theoretical framework to assess different rhetorical aspects of the flow of a proof. In this paper we relate particularly to the establishment of a shared basis of agreement between the lecturer and the students, and to potential fallacies in this basis. We present examples of analysis of the basis of agreement from a lesson in Number Theory, at the beginning undergraduate level.

Keywords: Proof teaching, flow of proof, Perelman's New Rhetoric, mathematical argumentation.

Theoretical background

Mathematics and Rhetoric - "Can two walk together, except they be agreed?"

Mathematics "possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature…" (Russell, 1917, p. 60) and is "independent of us and our thoughts" (ibid, p. 69). This perception of mathematics seems to stand in drastic contrast to rhetoric, the ancient art of persuasion, which over the centuries became mostly related to the study of the ostentatious and artificial aspects of discourse. Yet, over the last few decades, scholars have begun to discuss the mathematics- rhetoric separation and its consequences.

A pioneering effort of associating mathematics and rhetoric was made by Davis and Hersh (1987) who argued "that mathematics is not really the antithesis of rhetoric, but rather that rhetoric may sometimes be mathematical, and that mathematics may sometimes be rhetorical" (p. 54). Davis and Hersh challenged the opinion that mathematics establishes truth "by a unique mode of argumentation, which consists of passing from hypothesis to conclusion by…small logical steps…", and claimed that "mathematical proof has its rhetorical moments and its rhetorical elements" (ibid, pp. 59–60). They illustrated this by phrases that a college mathematics lecturer may use while presenting a proof (in addition to the expected logical transformations), such as: "It is easy to show …", "… simple computation, which I leave to the student, will verify that…"; they identified these phrases as rhetorical means in the service of proof. They acknowledged that the use of such phrases may be related to context, but rejected the myth that behind each theorem stands a flawless, logical proof. For them 'proof' is an amalgam of formality, of convincing arguments and of appeals to imagination and intuition.

Another example is the 'rhetoric of the sciences' movement, which studies the stylistic forms used by scientists in scientific texts (mathematics included), to persuade others that their claims are valid. So, as in the other sciences, the rhetoric of mathematics plays an essential role in maintaining its epistemological claims (Ernest, 1999). Ernest relates to another aspect of rhetoric in mathematics, namely the importance of persuasion for mathematics instruction.

Reyes (2014) asserts that it should be in the interest of rhetorical scholars to explore mathematics discourse, as it is the basis of techno-science. He analyses conceptual mathematical metaphors as an example for a mode of analysis of mathematics whose roots lie in rhetorical studies. Elsewhere, Reyes studies the rhetorical process during the invention of mathematics, and explores the introduction of infinitesimals by Newton and Leibniz as an example of the role of mathematical rhetoric in mathematical invention, in addition to its role in communicating the mathematics.

In conclusion, inquiry into relations between rhetoric and mathematics is growing in extent and richness. An increasing number of scholars explore the possibilities offered by the use of rhetorical concepts and ideas to gain better understanding of mathematics and mathematical education. Instead of viewing mathematics as a 'perfected, austere' product, they re-connect it to its 'human features', that in addition to formal logic utilizes persuasive argumentation and exploits rhetorical means.

Argumentation theory and 'The New Rhetoric'

Aberdein (2016) rejects the common thesis that mathematical reasoning is fundamentally different from everyday reasoning and that formal logic adequately models the practice of mathematical reasoning. Research in mathematical education uses argumentation theory to address aspects of mathematical argumentation other than formal logic, and for that purpose frequently uses Toulmin's model that permits schematic analysis of formal proofs as well as of arguments classified as deductively invalid. Toulmin's model has been shown to be an efficient framework to discuss local arguments as well as global argumentation structures (e.g. Knipping & Reid, 2013) and Inglis, Mejia-Ramos and Simpson (2007) claim that implementing Toulmin's full model (including rebuttals and qualifiers) should be used for this purpose. However, Toulmin's model has been criticized for not relating to the effect of the arguments on the audience, and for denigrating rhetoric in argumentation (Olbrechts-Tyteca, cited in Frank, 2004).

In 1958, Perelman and Olbrechts-Tyteca published 'The New Rhetoric' (PNR, translated in 1969), an argumentation theory based on the notion that argumentation aimed at justification of choices, decisions, and actions, is a rational activity complementing formal argument. PNR studies techniques used by an arguer to increase audience adherence to the arguer's theses and conditions that allow argumentation to begin and develop. PNR asserts that reducing an argument to its formal aspects undermines the rhetoric features that support its meaning; it recognizes the distance between dialectic and rhetoric but creates an alignment between them. This complex view at times produced an inherent ambiguity in definitions of some concepts. However, PNR adds meaningful layers of analysis beyond the analysis of argument structure and type achieved by using Toulmin's model.

PNR describes the 'threads that make the cloth of the argument': the starting points that establish a shared basis of agreement, the scope and organization of arguments, ways of creating presence to arguments, and different argumentation techniques. The audience plays a pivotal role in PNR since each 'thread', or aspect, is tied to what the arguer believes will deeply persuade the audience. This means that argumentation techniques should be adjusted to the audience's frame of reference, its previous knowledge, experiences, expectations, opinions and norms. So arguers construct arguments that they consider persuading for a particular audiences or convincing by a 'universal audience' (an arguer construct consisting of all reasonable humans) (van Eemeren et al., 2014).

In our study, we wish to analyze rhetorical aspects of proof presentation, in a scenario of a lecturer presenting a mathematical proof to a class of students. We use PNR as a theoretical framework as it incorporates aspects of rhetoric, argumentation and lecturer-classroom relations. Elsewhere (Gabel and Dreyfus, 2017), we demonstrate an analysis of other PNR aspects: scope and organization of the argumentation, and presence of proof elements. In this paper we address a different aspect: establishing a shared basis of agreement with the audience.

PNR's basis of agreement and its adaptation to proof teaching

According to PNR, argumentation can be successful if it advances from premises already accepted by the audience, i.e. the arguer established a shared basis of agreement with the audience. These premises are classified as follows: (1) *Premises relating to the real*: premises where the arguer claims recognition or acknowledgement of the universal audience. Those include: Facts, truths and presumptions. (2) *Premises relating to the preferable*: premises that have to do with the preferences of a particular audience. Those include: Values, value hierarchies and loci of preferable.

Facts and truths are statements already agreed to by the universal audience; they are considered to require no further justification. Truths stand for connections between facts. Presumptions are opinions or statements about what is the usual course of events which need not be proved, although adherence to them can be reinforced, and it is expected that at some point they will be confirmed. Values relate to the preference of one particular audience as opposed to another. They function as guidelines in making choices of the arguer (even though not all would accept them as good reasons). Values are normally arranged in value hierarchies, which are very important since different audiences may possess the same set of values arranged in different hierarchies. Values and value hierarchies generally remain implicit, but the arguer cannot simply ignore them. Loci of the preferable (aka commonplaces, Topoi) are premises used to justify values or hierarchies and express the preferences of a particular audience (e.g. quantity, quality, essence) (van Eemeren et al., 2014).

We have adapted PNR's classification of premises to the context of our study (analyzing proof presentation in class) as described in Table 1. We do not include in the table the loci of the preferable since they are highly abstract mental constructs which did not need adaptation.

		Adaptation to proof teaching
Premises relating to the real	Facts	Axioms, definitions, givens, previously consolidated symbols/results
	Truths	Lemmas, theorems, newly established symbols/results
	Presumptions	Statements or opinions about what previous knowledge to use, for example: mathematical techniques, proving methods, past theorems.
Premises relating to the preferable	Values	The preference or adaptability to a particular audience of a certain proving method or technique as opposed to another.
	Hierarchies of values	The hierarchies of values will affect audience preferences for choosing notation, proving method or mathematical technique.

Table 1: Adapting PNR types of premises to a proof teaching context

According to Perelman and Olbrechts-Tyteca, lack of agreement concerning the basis of agreement may occur at one or more of the following three levels:

a) *Status of premises*: e.g. if the arguer advances something as a fact but the audience wants to see it proven or if the arguer assumes a value hierarchy not accepted by the audience;
b) *Choice of premises*: e.g. if the arguer uses facts that the audience does not consider relevant to the argument or would have preferred not to see mentioned;
c) *Verbal presentation of premises*: e.g. if the arguer is presenting certain facts (acknowledged as relevant by the audience) in words which have connotations unacceptable to the audience.

The ability of creating a shared basis of agreement with the audience is crucial to the success of argumentation. Arguers should therefore carefully consider the status they ascribe to premises, the selection of premises, and the wording of explicit premises (van Eemeren et al., 2014). Examine, for example, two possible values related to proof teaching: (1) Certainty: every argument in the proof should be proved formally or at least justified; (2) Pedagogy: parts pf the proof should be left for the students. A lecturer may choose to leave parts of the proof as homework because her/his value hierarchy places (2) over (1). However, if the students have an opposite value hierarchy, this implies that the lecturer had a fallacy in the shared basis of agreement at the level of the status of his value hierarchy, which might consequently weaken students' persuasion.

The study – description and methods

Our research concerns the notion of 'The flow of a proof' (Gabel and Dreyfus, 2017) which encapsulates various aspects of the proof presentation. The flow is an outcome of the choices made by the lecturer regarding presentation of: (i) the logical structure of the proof (arranging the proof of the theorem into claims, which are proved in a specific order); (ii) informal features and considerations of the proof and proving process (e.g. examples, intuitions), and (iii) mathematical and instructional contextual factors. One aim of the research was to analyze global and local aspects the flow of the proof, in particular to examine rhetorical aspects of the proof presentation.

The research took place in a Number Theory course, given by the same lecturer to prospective mathematics teachers in two consecutive years. Each year, three lessons including the same suitable proofs (length, richness) were observed and audio-recorded. The three proofs were unrelated to each other. After each lesson students answered a questionnaire relating to cognitive and affective aspects; also, a reflective interview with the lecturer and individual interviews with students were conducted. The post-lesson interviews conducted with the lecturer in Year 1 were analyzed and interpreted, and a list of the lecturer's general considerations regarding proof teaching was produced. In this paper we relate to the second lesson in each year, in which the following theorem related to linear Diophantine equations $ax+by=d$, $x, y \in Z$ was formulated and proved:

Theorem: The greatest common divisor (gcd) of two integers a, b, *at least one of which is not 0, equals the smallest natural number of the form* $ma + nb$, *where* m, n *are integers:* $\gcd(a,b) = \min\{ma + nb > 0 : m, n \in Z\}$.

The full proof of this theorem requires the use of previously proven results. In the next section we present a partial analysis of the shared basis of agreement, demonstrating the different types of lecturer premises (in the PNR sense) reflected in the proof presentation, consider potential fallacies in these premises and demonstrate the lecturer's attempt to fix these fallacies.

Examples of analysis of the basis of agreement

All post lesson interviews conducted with the lecturer in Year 1 were analyzed and interpreted as two sets of lecturer considerations (Gabel and Dreyfus, 2017). One of the sets contains general considerations for proof teaching. In the current paper we relate to three of these general considerations: (a) A proof should be mathematically complete and exact; (b) Some of the proof elements should be left for students to prove by themselves; and (c) Proof structure should be clear. One aspect of the clarity of proof structure was exhibited when the lecturer referred to the myth about Ariadne's thread: "I use… Ariadne's thread many times since mathematical proofs are built in such a way that you need to find the tip of the thread and just follow it…" We relate to these lecturer considerations as values that affect the lecturer's choice and status of premises.

Our examples stem from the last part of the proof as presented by the lecturer, and we will address lecturer premises as reflected in his arguments. In Year 1, just after proving that d is a divisor of a, leaving the (almost identical) proof that d is a divisor of b to the students, the lecturer said:

Lecturer: The same way we showed that d is a divisor of a it follows that d is a divisor of b, so d is a common divisor of a,b. Now, it can't be smaller than the gcd, yes? Because once I write the equation $ax+by=d$ then…like we said in the beginning of the lesson, we said that this d must be divisible by $\gcd(a,b)$ … so it can't be smaller then $\gcd(a,b)$ and that means it is equal to $\gcd(a,b)$.

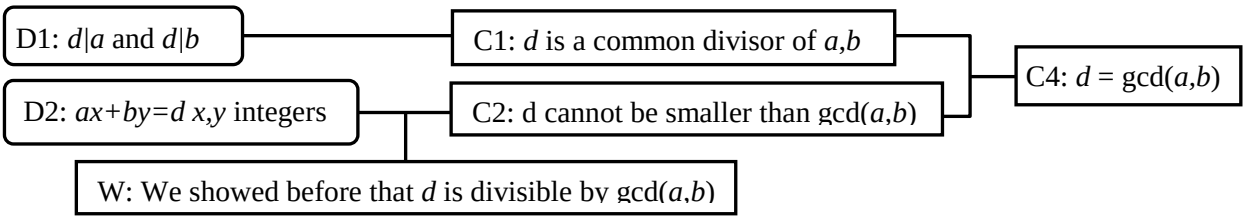

Figure 1: Toulmin's scheme representation of 1st explanation

The argumentation in this excerpt is represented by the Toulmin's scheme in Figure 1. We suggest a possible interpretation of the lecturer's explicit and implicit premises reflected in this explanation. For the lecturer this argumentation requires very little justification (if any) and he presents it as a chain of facts (D1, C1, D2 and possibly C2) that does not need to be discussed, and whose connection results in the conclusion (C4) in a self-evident way. The lecturer implicit presumption is that in order to prove that $d=\gcd(a,b)$ one needs to show two inequalities: $(d \le \gcd(a,b)$ and $d \ge \gcd(a,b)) \Rightarrow d=\gcd(a,b)$; he believes that this presumption does not need to be made explicit and that he and the students share this presumption. As for the values reflected in this explanation and their hierarchy, since the lecturer chose to leave some of the proof (that $d|b$) to the students, in this case the pedagogical value of leaving some of the proof elements for students was placed above the value regrading proof completeness. In addition, we recognize another implicit value: for the lecturer the 'tip of Ariadne's thread' here is to realize that d is a common divisor of a,b from which the rest of the proof just unfolds.

However, the students had difficulties following this first explanation and asked the lecturer to repeat it. A possible reason for this difficulty is that the premises that the lecturer considered as facts were not considered as facts by the students and required further justification. For example, the students probably needed an explicit justification for the argument "if d is a common divisor of a,b

then $d \le \gcd(a,b)$". Moreover, the lecturer's implicit presumption regarding the natural proving technique (the two inequalities) is not necessarily clear and natural to the students. In PNR language, there was a lack of agreement about the status and choice of the lecturer's premises, which caused a fallacy in establishing a shared basis of agreement. So, following the students' request, the lecturer instantly explained again the argument in Year 1 lesson as follows:

Lecturer: We said that d, as a minimal element of this set [$\{ma+nb>0 : m,n \in Z\}$], is of the form $a \cdot \text{integer} + b \cdot \text{integer}$. Now the first thing I have shown today is that in such a situation, actually this is a result of theorem 1 that we have used before,… it follows that d must be divisible by $\gcd(a,b)$, yes? Once I can write a number as a linear combination of two numbers a,b, with integer multipliers m,n, this d must be divisible by $\gcd(a,b)$. On one hand it must be divisible by $\gcd(a,b)$; on the other hand…it is a divisor of a,b. It can't be smaller then $\gcd(a,b)$ so it can only be equal to it. Because $\gcd(a,b)$ is the greatest common divisor, yes? And that finishes the proof… d is a common divisor of a,b that also has to be divisible by $\gcd(a,b)$ so we conclude that $d = \gcd(a,b)$…

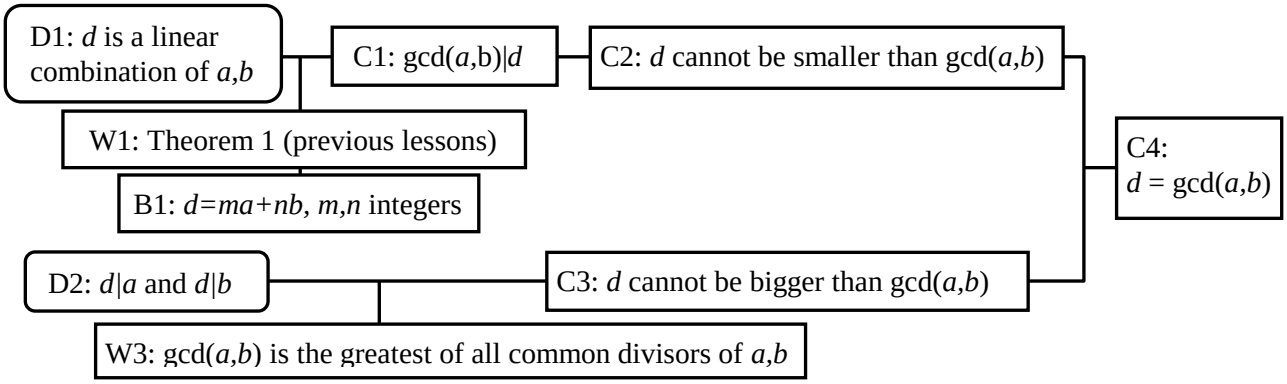

Figure 2: Toulmin's scheme representation of 2nd explanation

The argumentation in this excerpt is represented by Toulmin's scheme in Figure 2. In the second explanation the lecturer added some justification (W1, B1, W3) to the conclusions C1, C2 and C3; we interpret they were not presented as facts but rather as truths, i.e. the lecturer changed the status of the premises to establish a stronger basis of agreement with the students. However, his presumption still remained implicit – a point which we will revisit shortly.

Before the lesson in Year 2 the lecturer was informed by the researcher about some student difficulties that were found in the post lesson students' questionnaires of Year 1; in particular, the questionnaires reflected that the last part of the proof, where combining the inequalities $d \le \gcd(a,b)$ and $d \ge \gcd(a,b)$ leads to the equality $d = \gcd(a,b)$ was not trivial to the students. For lack of space we will concentrate on demonstrating the change in the lecturer's presumptions and his value hierarchy between Years 1 and 2. The lecturer took the reported students' difficulty into account and during the lesson in Year 2, just before the last part of the proof he explained:

Lecturer: Here we are doing something similar to what we already had in the past, when we wanted to prove that two numbers are equal…

Student: We assume that they are unequal…

Lecturer: No, we should prove two inequalities, right? Or refute two inequalities, right? I remind you, we already used it: when we wanted to show that two numbers are equal then we need to show that it is impossible that a is smaller than b… it is impossible that a is bigger than b, or in other words … to show that a is not smaller than b is actually showing that $a \geq b$, and instead of showing that a is not bigger than b we'll show that $a \leq b$. If I want to show that $a = b$, I need to show that $a \geq b$, i.e. not smaller than b, and that $a \leq b$, meaning that a is not bigger than b. Once I will show these two inequalities I am done, I've shown that $a = b$.

Here, the lecturer consciously makes his presumption explicit to the students, justifies the choice of this presumption and makes it relevant. By explicating his presumption the lecturer also caused a change of value hierarchies: he enhanced the clarity of the proof structure, making it more explicit before going into the details of the proof. Indeed, the lecturer also explicitly declared:

Lecturer: It remains to prove the other inequality: $\gcd(a,b) \geq d$. In fact, I will show you that this… minimum of the set, *d*, is a divisor of *a*,*b*… and this will end the story…

So before formally proving that *d* is a divisor of *a*,*b*, the lecturer spread out his proving plan, identifying "the tip of Ariadne's thread" and explained exactly why this "will end the story".

We conclude this short example by stressing that while Toulmin's model enables the presentation and analysis of the argumentation structure, PNR complements it by relating to other argumentation qualities, such as the adaptability to the intended audience. The fallacies that have been mentioned in the example were related to the status and choice of premises.

Concluding remarks

Weber and Mejia-Ramos (2014) demonstrated that mathematics students and mathematicians have different perceptions regrading students' responsibilities when reading a mathematical proof: the students believe that reading a good proof is quite a passive process, one in which they are not expected to construct justifications, diagrams or sub-proofs by themselves, and they may simply follow each and every step. Mathematicians believe the opposite. This tension between the students' and their teachers' beliefs supports our interpretation regarding the different value hierarchies that the lecturer and the students have. But beyond the identification of the difference, we argue that PNR has the potential to explain the consequences of that difference on the effectiveness of argumentation; in other words, PNR provides a suitable framework to identify ways to increase argumentation effectiveness, for instance by referring to the shared basis of agreement.

Moreover, PNR relates to other rhetorical and dialectical aspects of argumentation. Some of these aspects (scope and organization and presence) have been studied in Gabel and Dreyfus (2017); others, namely argumentation techniques and the manner by which PNR complements the use of Toulmin's scheme, need further study. One advantage of PNR is that because of its theoretical scope, all these aspects can be studied within a single unifying theoretical framework.

Although Perelman perceived PNR as a complement of formal logic and focused on disputes in which values play a part (van Eemeren et al., 2014), we argue that PNR can be adapted to be a productive theoretical framework in the context of proof teaching, particularly the flow of a proof: firstly, Perelman was much inspired by formal logic (mainly the work of Frege), and secondly, the

context of argumentation in the mathematics classroom resembles PNR's context of persuading an audience. Thus PNR is a comprehensive argumentation theory that can broaden and enrich researchers' perspectives regarding different aspects of mathematics classroom argumentation.

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Fostering and investigating students' pathways to formal reasoning: A design research project on structural scaffolding for 9th graders

Kerstin Hein and Susanne Prediger

TU Dortmund University, Germany
kerstin.hein@math.tu-dortmund.de, prediger@math.uni-dortmund.de

Major obstacles for students learning formal reasoning are the lack of transparency of the logical structure of formal deductions, their theoretical status, and their verbal representation. For fostering students' learning of formal reasoning, making explicit the logical structures and unpacking their verbal representations is therefore necessary. In the design research project presented, a teaching-learning arrangement of angle theorems was designed in which given if-then-statements were to be connected with formal deductions based on the design principle of structural scaffolding. A case study of a pair of 9th graders investigated students' pathways towards becoming aware of and using the logical structures and exemplifies the functioning of structural scaffolding.

Keywords: Formal proof, logical structure, verbal representation, structural scaffolding.

Introduction

Formal reasoning, the logical deduction of new theorems from other theorems, has been shown to be a huge challenge for many students at both secondary and tertiary level. Empirical research studies have identified different reasons for these difficulties (Harel & Sowder, 1998). The design research project presented here focuses on one major obstacle, namely understanding the logical structure of deductions and deductive theory development, which is rarely explicit in mathematics classrooms (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2011). For this obstacle to be overcome, researchers have suggested that it is important to make the logical structures of deductions and their verbal representations explicit (Durand-Guerrier et al., 2011).

This design research study follows this general suggestion. It draws upon the design principle of structural scaffolding (following general ideas of scaffolding, cf. Lajoie, 2005). It pursues two main research questions: *(1) How can a teaching and learning arrangement be developed to make logical structures of deductive reasoning explicit? (2) Which typical pathways towards formal reasoning can be initiated by such a teaching-learning arrangement, and which obstacles appear along the pathways?* The first two sections present the theoretical background and the methodological framework. The design outcome (a teaching-learning arrangement based on structural scaffolding) and a case study of two students in 9th grade on their pathway is presented afterwards.

Theoretical background: Approaching logical structures by structural scaffolding

Formal reasoning is crucial in mathematics, not only for *convincing one self and others of the truth* of theorems or for explaining connections, but mainly for *building* (at least locally) *deductive theories* (de Villiers, 1990). Even if, for example, students immediately are convinced of the truth of angle theorems, deducing them from each other encourages students to organize them in a logical and deductive sequence and give insights in mathematical evidence instead of empirical (cf. Fig. 1).

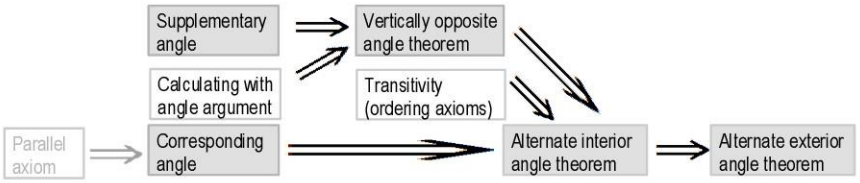


Figure 1: Locally deductive theory of angle theorems

Missing learning opportunities for formal deductions. In school mathematics, in contrast, most reasoning activities do not refer to formal deductive reasoning but to semantical reasoning where the epistemic value is prioritized over the validity of a statement (Duval, 1991). Formal deductions are presented mostly in a ready-made form (Harel & Sowder, 1998). This does not allow students access to awareness of how to compose an argumentation using logical structure. Although composing the deductions is only the last step of proving (Boero, 1999), it is still necessary to offer learning opportunities for this last step, e.g. by linearly ordering all elements and formulating their logical relations in written forms (Russek, 1998).

Logical structure of formal deduction and everyday argumentations. Everyday argumentations have often been described by the argumentative structure of data, warrant, and claim (Toulmin, 1958). While in everyday argumentation the warrant may be omitted and only made explicit when an opponent raises doubts (Rapanta, Garcia-Mila, & Gilabert, 2013), it is crucial in formal reasoning and has a theoretical rather than semantic status (Duval, 1991). That means that the existence of mathematical theorems and their statements, which are not characteristics of reality, are of relevance. Thus, the range of possible warrants must be made explicit in mathematics (Douek, 1999). Also, the status of preconditions in if-then-statements differ: in most everyday argumentations, if-then-statements are only formulated when the conditions are satisfied (Nunes, Schliemann, & Carraher, 1993, p. 130ff). In mathematics, in contrast, if-then-statements are hypothetical, so the validity of the preconditions always have to be checked before applying an if-then-statement as an argument.

Making explicit logical structures for students. Due to these differences, many researchers have suggested explicating the logical structures of formal reasoning in the learning process (Durand-Guerrier et al., 2011). Cho and Jonassen (2002) used Toulmin's (1958) argumentation scheme for this purpose for college students in non-mathematical contexts, and we will extend this approach for 9th graders in a geometrical context by including the check of preconditions.

Structural scaffolding as a design principle. Explicating alone is not enough. For students to become acquainted with the logical structure, and to produce it in their own deductions, this study draws on the design principle of scaffolding. Scaffolding is characterized as enabling learners to realize supported activities before they can conduct them independently (Wood, Bruner, & Ross, 1976). Initially only applied to one-to-one interaction for language learning, the idea of scaffolding has increasingly been elaborated into a design principle for materials and computer tools, whole-class contexts, for open geometrical proofs (Miyazaki, Fujita, & Jones, 2017), and other learning contents (e.g. Lajoie, 2005).

Methodology of the design research study

Design research as methodological framework. Because the study has the dual aim of designing a teaching-learning arrangement (here: on the logical structure of formal reasoning with the design principle of scaffolding) and developing an empirically grounded local theory of students' learning

pathways, we chose the methodological framework of design research with a focus on learning processes (Gravemeijer & Cobb, 2006). The concrete model of Topic-Specific Didactical Design Research (cf. Prediger & Zwetzschler, 2013) relies on the iterative and intertwined interplay of four working areas: (a) specifying and structuring learning contents; (b) developing the design of the teaching-learning arrangement, (c) conducting and analyzing design experiments; and (d) (further) developing local theories on teaching and learning processes.

Design experiments for data collection. Design experiments are the methodological core of design research studies (Gravemeijer & Cobb, 2006). For this project, 3 design experiment cycles were conducted with 20 ninth and tenth graders (age 14-16 years) in total. The case study reported here stems from Cycle 3 in which design experiments in laboratory settings were conducted and videotaped with 5 pairs of students, comprising two sessions of 60 minutes each (in total about 600 minutes video material). The students were familiar with the geometrical topic of angle theorems. The empirical part of this paper focuses on the case study of two female students, Katja and Emilia, from grade 9 and the first author as design experiment leader (in the following called tutor).

Methods for qualitative data analysis. The transcript of the video was analyzed with respect to students' development of explicating elements of the logical structure (using the analytic scheme of data, warrant, claim, cf. Krummheuer, 1995) and to how students articulate relations between these elements (linguistic analysis, not presented here). This makes it possible to investigate the functioning of the scaffolding tool and typical pathways and obstacles.

Design Outcome: Teaching-learning arrangement with structural scaffolding

Mathematical topic. Within the iterative design experiment cycles, a teaching-learning arrangement was developed for the mathematical topic of angle sets. This topic was chosen because the if-then-statements and the set of possible warrants are well limited in this field and locally organized (cf. Fig. 1).

Structural scaffolding. For structural scaffolding, we use *materialized argumentation structure forms* on paper as depicted in Fig. 2. In addition to Toulmin's (1958) argumentation structure, the materialized structure also makes explicit why the preconditions of the if-then-statements (named arguments) are satisfied. Every theorem that is already proven is offered as warrant for the next step of formal reasoning. Working with this materialized structural scaffold in each step allows the students to make explicit their often implicit ideas. In the following, the boxes (from above to below) are named data box, condition check box, argument box, and conclusion box.

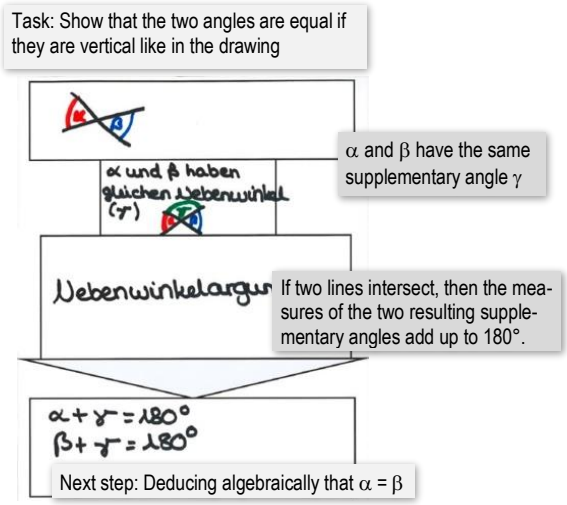


Figure 2: Materialized argumentation structure forms as structural scaffolds

Learning trajectory for introducing the structural scaffold. The intended learning trajectory starts by activating students' previous knowledge on angle sets in cases of determining angles for *concrete* constellations ("Find β if α = 120 °…"). When first asked to prove the *general* vertically opposite angle theorem, students' initial argumentative resources often include the critical feature, but are limited mostly by their semantic nature ("because

supplementary angle"). Starting from these initial reasoning resources, the tutor introduces the structural scaffold by explaining the new practice of formal reasoning as making explicit *all* aspects *implicitly* contained in the students' brief argumentation. The condition check box for checking if the precondition of the if-then-statement is satisfied had to be introduced after the first design experiment cycle in order to clarify the theoretical and hypothetical status of if-then-statements in mathematics. The structural scaffold serves different roles along the learning trajectory, (I) as a visualizer for the extended structure; (II) as a working tool for the students to check the completeness of their explicit reasoning; and (III) as a framework for writing down the proof. In our design experiment, after 120 minutes, the students write proofs with deductive chains of reasoning, even though they do not yet find deductive chains for more complex proofs on their own.

Empirical insights into Katja's and Emilia's pathways to formal reasoning

Katja and Emilia start their learning pathway in the way described above. Figure 2 shows the product of the phase of jointly introducing the structural scaffold ending with Sequence 1.

Sequence 1: Reasoning determined by empiricism instead of validity of statements

When asked to prove that α and β are equal, Emilia and Katja offer a typical initial, semantic three-word answer "vertically opposite angles" (unprinted Turn 339), assuming that classifying the type of relation between the two angles is enough. Becoming aware that they are supposed to prove the vertically opposite angle theorem by using arguments like the *argument of supplementary angles* (cf. Fig 2) and the *calculating argument* ("If there are angle measures, then it is possible to calculate with them like numbers."), they start by naming the angles γ and δ (Figure 3). Then they discuss the necessary conditions and conclusions.

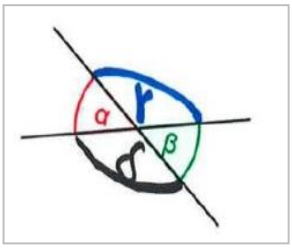


Figure 3: Marking angles as first step

362 Emilia: […] And now we could say actually that α plus γ results in 180 degrees.
363 Tutor: Mmm.
364 Emilia: Also like here [*points to conclusion box of the last task with α + 120° = 180°*]
365 Tutor: Yes.
366 Katja: Yes.
367 Emilia: And that, uh.
368 Katja: γ plus β
369 Emilia: Yes, so actually this can be – Yes, precisely – But we have no concrete numbers [*points to the conclusion box previous task*] – and then we can go on – so, I don't know, whether we can do this in such small steps, because we have no numbers at all, but then we could say, α plus γ equals 120, umm, 180 degrees. And β plus δ equals 120, umm, 180 degree
370 Tutor: Mmm.
371 Katja: And…
372 Emilia: And accordingly
373 Katja: γ plus β – plus δ and then
374 Emilia: yes, okay, but actually, actually we need only one, don't we? Then it is just unnecessary, this angle. [*points to the angle δ*] – So I would say…

375 Katja: … we have to – this with [*6 sec break*] yes, α plus γ 180 degree, then 180 degrees minus β
376 Emilia: No, so I would easily write
377 Katja: [“unintelligible”]
378 Emilia: α plus γ 180 degree and β plus γ 180 degree.
379 Tutor: Yes.
380 Emilia: And then, if, a system of equations could be created.

When asked to explain in more detail, the students offer details of steps of their calculation (“α plus γ equals 120, umm, 180 degrees”, Turn 369), but do not explicate the warrants for these relations (here the argument of supplementary angles). In this way, they find out that they do not need the angle δ. Interestingly, they formulate steps of action or calculation instead of general relations, and consequently, these steps are combined temporarily (“and then” in Turn 373) instead of logically.

Sequence 2: Filling the argumentation structure form without verbalizing the connections

When filling in the materialized argumentation structure form (Fig. 2), the students discuss whether they need δ and organize their process:

403 Emilia: Well then – eh, I would say – I know, I think, that here *[points to the argument box*], we first write that the supplementary-argument is our argument. Then we think which has to be there [*hints to the condition-check box*]
404 Katja: [*writes “supplementary angle” in the argument box,* 21 *sec break*] Yes, that here
405 Emilia: Ah, I wanted to write that
406 Katja: … that we γ here
407 Emilia: Yes, that α and β have the same supplementary angle.
408 Katja: [*3 sec break*] Where?
409 Emilia: Here [*hints a finger at the condition check box*]



410 Katja: [*writes in the condition check box: α and β have the same supplementary angle γ*]
[…] *[Discussion with the tutor, if the second angle δ is necessary]*
417 Emilia: Yes, okay. – Umm, then I would now write here, umm, - α plus γ equals 120 degrees and β plus γ equal 120, umm, 180 degrees. Why do I always say 120? Yes,
418 Katja: [*writes both equations in the conclusion box, cf. Fig .2*]

The students succeed in filling in the argumentation structure form mostly without help from the tutor. In particular, they correctly identify all elements of the logical structure, first choosing the argument and then checking whether its precondition is satisfied (Turn 403). After filling in the form, they condense the proven theorem as a new argument to be used for further proofs (in non-printed Turns 452-471): “Argument of vertically opposite angles: If two lines cross each other, then the opposite angles are equal. (They are called vertically opposite angles.)”. This illustrates how the scaffold supports them to produce a complete argumentation and to understand the logical structure. However, it is remarkable that they still do not use any logical connectors to relate the different elements to each other. The language is rather deictic (“here”, “there” in Turns 403, 404, 409, 417), but the logical relation between the elements is not verbalized by the students. To give an expert model of how the connections could be expressed, the tutor finally intervenes as follows:

474 Tutor: […] Also this condition of point of intersection was considered, so that we have two times two supplementary angles. Here as condition, and because we have supplementary angles, we could use the supplementary-argument that says that a pair of supplementary angles add up to 180 degrees. Therefore, it can be used for our supplementary angles and umm, here two times two were regarded, this means we have two times this equation with, umm, our angles. […]

Sequence 3: Mastering formal reasoning

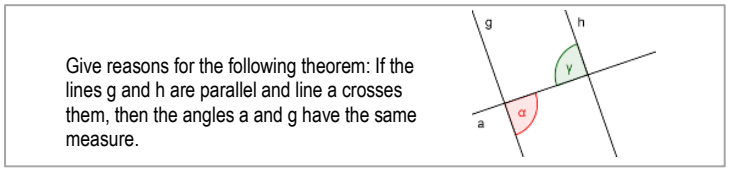


Figure 4: Alternate interior angle theorem

After determining a specific alternate interior angle, the next task for Emilia and Katja is to prove the general alternate interior angle theorem (Fig. 4). For constructing their formal argumentation structure, the students are given the equality argument (If $\delta = \mu$ and $\mu = \pi$, then $\delta = \pi$. (transitivity)) and the corresponding angle argument (which can only be derived from the parallel axiom and is hence left unproved for the students, cf. Fig. 1). Again, the students successfully construct a complete argumentation structure supported by the structural scaffold of the form. Based on an enriched sketch, they deduce the theorem in three steps (cf. Fig. 5): In Step 1, the use the vertically opposite angle for deriving that $\gamma = \beta$. In Step 2, they use the corresponding angle argument for deriving $\alpha = \beta$. For deriving that $\alpha = \gamma$, they use the equality argument and produce the last chain of reasoning in Step 3.

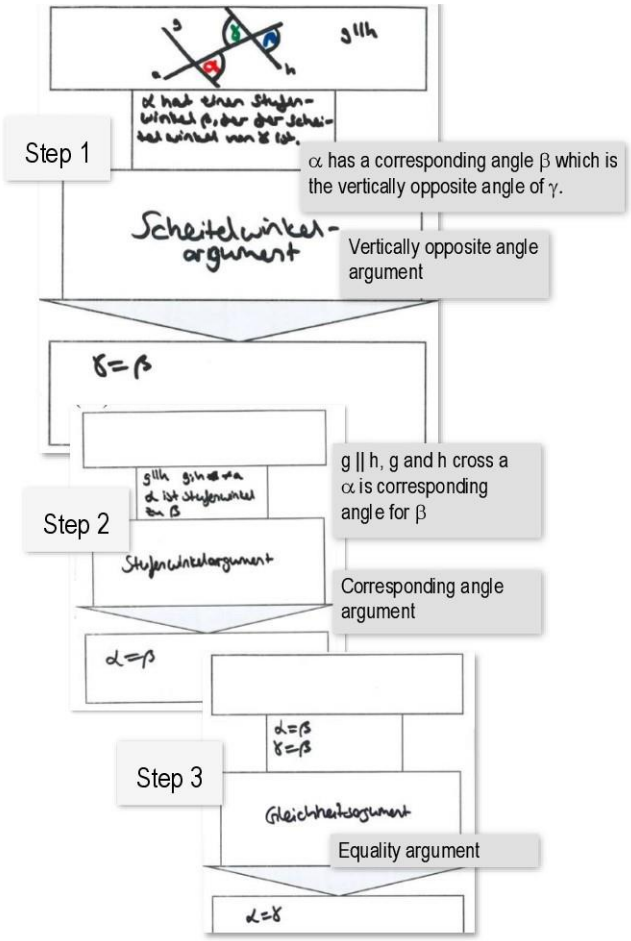
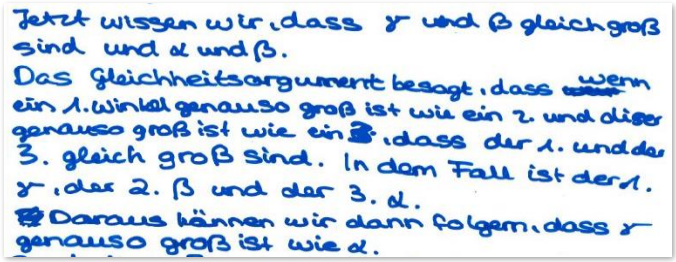


Figure 5: Katja's and Emilia's three steps of formal proof for the alternate angle theorem

The written text produced by Katja for this last step shows what she has learned (cf. Fig 6). Katja's text provides at least situational evidence that she has grasped the logical structure of formal reasoning and can express some of the logical connections. In contrast to the beginning of the students' learning pathway, she makes explicit the warrant ("the equality argument says that") and the conditions of its application ("Now, we know that γ and β have the same measure and α and β."). For expressing the logical connections, she adopts elements of a language offered by the tutor in Turn 747 ("the supplementary-argument that says"). She also expresses the deduction from the argument to the conclusion: "from this we can conclude". However, the order of aspects is still the order of discovery, not yet the strict order of formal reasoning as the conditions are again guaranteed after using the argument.



"Now we know, that γ and β have the same measure and α and β. The equality-argument says, that if the first angle has the same measure as a second and this has the same measure as a third, that the first and the third have the same measure. Here the first is γ, the second β and the third α. From this we can conclude that γ has the same measure as α."

Figure 6: Katja's verbalization of the third step of reasoning

Looking back to Sequence 1 – Sequence 3

In total, these three sequences from the students' learning process provide insights into the students' pathway from their everyday argumentative resources towards formal reasoning, their induction into mathematical proof as a cultural practice. The structural scaffold strongly supports comprehending of the logical structure, the designated Function (I). The students also capture the norm that the practice of formal reasoning is characterized by making explicit every element in the logical structure (every box must be filled, Function II). However, the scaffold alone does not sufficiently support the process of talking about the logical structures, as visible in Sequence 2. Hence, the structural scaffold had to be complemented by language scaffolds (in this case oral expert modelling offered by the tutor). The written product from Sequence 3, finally, shows that the students can adopt the language scaffolds for communicating about the formal deductions.

Discussion and outlook

The case study of Katja and Emilia gives a first indication for the potential efficacy of the structural scaffolding. Other pairs of the 10 students in Design Experiment Cycle 3 also succeeded in mastering formal reasoning, supported by the scaffold. Filling the boxes serves as prompts for identifying every single aspect of the logical structure (data, warrant, and conclusion) and the satisfaction of preconditions of argumentations. The specific strength of the materialized structure form is that it not only makes the logical structure visible, but also permits students to complete the form in non-linear order. Based on this structural scaffold, the students' written texts are mostly produced in linear, deductive order. As with any provided format, it can be done non-generatively, passively, locally filling each box but not attending to what role the boxes play in formatting the reasoning. However, not only Katja and Emilia but also other students we observed benefited from the scaffolding as they learnt to distinguish between preconditions, if-then-statements and conclusions. The scaffold supported the students to express the relations between the elements of the logical structure verbally and to reflect amongst other things about the generality of the statements or which characteristics of the sketches are important. These features are crucial to increase awareness of formal reasoning.

Of course, the study still has methodological limitations which have to be overcome in later cycles or future research. Limitations concern the sample size (2 students presented, 10 in total) which is not yet representative. So far, the teaching-learning arrangement is focused on one specific topic, the angle theorems, which need to be extended to other topics in future research in order to gain evidence of the overall claim of efficacy. The most important limitation in view for the next cycle of the presented project concerns the language: we intend to identify the phrases and syntactic structures which appear to be necessary for students to realize the need to articulate the logical connections

between the elements in the argumentation. This will provide support for the students on the linguistic level as well as on the logical-structural level.

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Misconceptions and developmental proof

Kirsti Hemmi[1,2], Erika Julin[1] and Ray Pörn[1]

[1]Åbo Akademi University, Finland; kirsti.hemmi@abo.fi; erijulin@gmail.com; rporn@abo.fi

[2] Uppsala University, Sweden; kirsti.hemmi@edu.uu.se

The issue of students' misconceptions in mathematics and how to prevent and deal with them in teaching has been a major concern of mathematics educators since at least four decades. At the same time our knowledge about the processes of developing understanding and skills in proof and argumentation from early school years has increased. We argue, that there are connections between these two areas of studies important to make explicit for teachers. In this paper, we first elaborate the relation between the research on students' misconceptions and the ideas of developmental proof. Then we present the relevant results of an empirical study about how mathematics teachers in the field interpret this relation. Our conclusion is that there are important connections between these two research fields that are not always visible for teachers.

Keywords: Misconceptions, developmental proof, teacher education, MKT.

Background

Teachers' knowledge about students' misconceptions in mathematics and how to deal with them has been pointed out as an important part of teachers' mathematical knowledge for teaching (MKT) (e.g., Hill, Rowan & Ball, 2005). Knowledge about them has also been raised as an important part of educative curriculum materials and many of these provide information about common student misconceptions and suggestions on how to address them (Cengiz, Kline & Grant, 2011). At the same time a growing number of research articles have raised the importance of enhancing students' skills and understanding of proof and proving and research has shown that it is possible and beneficial to start to develop students' proof-related competences during early school grades (e.g. Hanna & De Villiers, 2009; Hemmi, Lepik & Viholainen, 2013). This knowledge can also be considered a part of MKT, and in line with knowledge of students' misconceptions close to what Shulman calls pedagogical content knowledge. Research shows that teachers who have strong knowledge in different areas of MKT are more able to create opportunities for extending student thinking (e.g. Hill et al. 2008). This is especially important to consider in teacher education and teacher professional development. Yet, the relationship between specific aspects of MKT still remains unclear (cf. Cengiz et al., 2011). This study contributes to the field by investigating the relation between two particular research areas, namely the knowledge about students' misconceptions on the one hand and the development of understanding of and skills connected to proof and proving. More specifically, we do not consider students' misconceptions in a deficit (cf. Jaworski, 2001) manner but investigate how we could use them to enhance students' understanding of proof and vice versa. The following questions are in the focus of this study:

1) What kind of connections can be found between the research field concerning students' misconceptions and the development of students' knowledge and skills with respect of proof?
2) How do teachers relate to students' misconceptions, proof and the relation between them?

First, we offer an analysis about the connections related to the first research question. Then, we briefly describe the methodology of the empirical study focusing on upper secondary school teachers' conceptions about these areas. Finally, we report the results of the empirical study and discuss them in the light of the connections found in the theoretical section.

Misconceptions in mathematics related to developmental proof

Hanna and de Villiers (2008) introduced the idea of *developmental proof* as something that should grow in sophistication in action, perception and language as the learner matures towards more coherent conceptions. We have earlier concretized the idea of developmental proof by identifying in research literature and school curricula proof-related competences that could be developed through the school grades from 1 to 12 (Hemmi et al., 2013). These competences address besides the development of argumentation and proving, the meta-level knowledge about proof, such as the character of mathematical definitions, methods of proofs, logical and formal aspects that often remain invisible for students (cf. Hemmi, 2008) as well as investigations with validation of either students' own or others' reasoning and results. For a comprehensive description see Hemmi et al. (2013).

Research on misconceptions and how to prevent and deal with them has roots in cognitive research and constructivism. For example, some researchers claim that students find it difficult to give up their misconceptions as they have actively constructed them (e.g. Egodawatte, 2011). Further, research on misconceptions is often concentrated within a certain topic in mathematics or science. One of the earlier ideas about reasons for misconceptions is that of Fischbein (1994). He defines three basic components in mathematical practice: the formal, the algorithmic and the intuitive. According to Fischbein, the intuitive knowledge is often experienced as self-evident and may not be problematized or deeply justified in school mathematics and therefore may conflict with the mathematical, logically proved "truths". Scholars agree that the main function of proof in school is to offer explanations (e.g. Hanna & de Villiers, 2009). Yet, the explanations offered in early school years for operations of natural numbers do not always explain properties of operations when operating in other domains. For example, the concept of multiplication is often explained as repeated addition, in order to reveal connections between arithmetic operations. Yet, this explanation leads to an intuitive conception that multiplication always results to a bigger number than the one you multiply (cf. Fischbein, 1994). When students start to operate with rational numbers this intuitive belief could be made visible and be challenged by investigations, explanations and justifications developed by the classroom community. The transition from the domain of integers to the domain of rational numbers could offer a fruitful platform for developmental proof concerning some logical aspects of reasoning connected to universal statements' truth-values in different domains (see e.g. Durand-Guerrier, 2003). Hence, we argue that besides prevention of future misconceptions in mathematics, this kind of testing and challenging of intuitive rules could also develop students' understanding of proof in mathematics and the other way around.

Another example about misconceptions that can be connected to intuitive rules is "over generalizing", often involving improper use of analogical reasoning, for example in connection to ratio between area of a figure and volume of a figure (see, for example, Tirosh & Stavy, 1999;

Chick & Baker, 2005). These kinds of misconceptions also offer excellent possibilities for students' investigations and proofs where students could for example develop their understanding of differences between analogical and deductive reasoning. Indeed, new approaches to proof using students' investigations have been developed and tested in order to enhance students' skills and appreciation of proof as an important part of doing mathematics (e.g. Heinze & Reiss, 2004). These studies often advocate investigative approaches covering the whole process of proving, starting from the first experiments to generate an idea for a hypothesis up to the final step of writing down the complete proof. We think, that beside this, it is also beneficial to conduct continuously smaller investigations about truth-values of various statements, for example connected to algebraic laws. Scholars agree that several identified student misconceptions are due to students' difficulties in algebra. The idea of developmental proof has parallels with ideas about children's development from an understanding of arithmetic to algebra (cf. Hemmi et al., 2013). For example, the generality of reasoning is an important component in investigations and proving where the move from concrete and specific to general is needed when justifying the conjectures made on the basis of the observations of regularities.

Application of rules to situation where the rule is not valid is still another type of misconceptions found in the literature (e.g. Fischbein, 1994; Egodawatte, 2011). As an example consider the following typical misconception in simplifications of expressions (Egodawatte, 2011):

(1) $(2 + x)/x = 2$ (2) $(12 \cdot 2x)/2 = 6x$

The rule applied in the first example is valid for rational expressions with only multiplication in the numerator, but not with addition, while the rule applied in the second example is valid for rational expressions with addition in the numerator. Typical misconceptions also concern the use of the distributive law in situations where it is not valid (e.g. Fischbein, 1994). These kinds of misconception could be regularly used as an object for investigations in order to enhance students' understanding of treatment of algebraic expressions and derivation of rules. Explanation in mathematics often refers to making mathematical connections explicit. Kuchemann and Hoyles (2009) emphasize the importance of the mathematics instruction to move from a computational view of mathematics to a view that conceives mathematics as a field of intricately related structures in order to develop students' proving competences. Seeing connections and mathematical structures is also an important proof-related competence connected to developmental proof (Hemmi et al., 2013). Still another kind of misconception identified in the literature is connected to mathematical definitions. For example, several researchers present similar ideas about students who often operate as if all functions were linear (Tirosh & Stavy 1999). This is connected to development of understanding the role and character of definition in mathematics, also identified as an important aspect of developmental proof (Hemmi et al., 2013).

Scholars have also attempted to explain why some misconceptions are developed and how to deal with them to change them (e.g. Tirosh & Stavy, 1999). There are significant connections between the suggestions offered to deal with students' misconceptions and the developmental proof, for example, the understanding of counter example, critical thinking, and argumentation with peers. Several studies show that erroneous conceptions are so stable because they might be correct in some instances. Scholars state that teachers should encourage students to critically evaluate their solutions and develop a skeptical approach to their intuitive rules. Balacheff (2010) points out that proving is

the most visible part of validation and something that cannot be separated from the ongoing controlling activity involved in solving problems or achieving tasks. Scholars also advocate the use of common misunderstandings for planning of effective sequences of instruction by both using situations where the intuitive rule is valid and where it is not valid in order to create *cognitive conflict*. Creating cognitive conflict by using a counter example is not always fruitful if students do not understand the role and the logic of counter example in mathematics. Here the development of students' understanding of the role of counter example in mathematics is important and connected to developmental proof. Interestingly, the idea of creating cognitive conflict has also been used to change students' misconception concerning the use of specific examples in validation of mathematical statements and the promotion of students' feeling for the need of proof (Stylianides & Stylianides, 2009).

The empirical study

In Finland, proof was an important part of upper secondary school mathematics in the 1970s during a period of 'New Math' reforms but since then its importance has decreased significantly. Yet, the Finnish steering document for the compulsory school curriculum (2004) addresses a number of proof-related topics (Hemmi et al., 2013) and although the word proof is not mentioned in upper secondary curriculum, in textbooks for the advanced course, proof and deductive reasoning is an important part of the contents (Bergwall & Hemmi, 2017). There are two programs in upper secondary school mathematics in Finland. The *basic course* is for students who study humanities and social sciences while the *advanced course* is for those students who want to study mathematics, science and computer sciences at the university.

The empirical study was conducted with Swedish speaking[1] upper secondary school (about the age of 16-19) mathematics teachers in Finland (Julin, 2016). The aim of the entire study was to investigate teachers' knowledge, experiences and views of students' misconceptions and the role of proof in mathematics and in teaching. A questionnaire comprised mostly closed statements and questions that were developed from items in literature. For example were teachers asked to judge how often they had experienced seven common misconceptions and how they usually reacted to them when encountered them in their teaching (see Figure 1). Concerning their reactions teachers could choose from five methods applied from Chick and Baker's (2005) study: counter example, re-explain the procedure, re-explain the concept, cognitive conflict, and probe student thinking. These methods were shortly described in the questionnaire. The items in the questionnaire also addressed proof in mathematics and in teaching and finally the relation between misconceptions and proof. As a complement to the quantitative part we also posed an open question: "Explain shortly why you use/do not use proof in your teaching", and finally there was a possibility for the teachers to freely write their own thoughts about these issues.

The electronic questionnaire was sent to all mathematics teachers working in the Swedish speaking upper secondary schools in Finland, in all 90, and of them 36 teachers responded to the questionnaire. Both the gender and age distribution were representative for the whole group and the responding teachers' teaching experience varied from 1 to 40 years. All teachers responding to the

[1] About 5 % of the Finnish population has Swedish as a mother tongue.

questionnaire were certificated mathematics teachers. The responses to the quantitative part of the study were analyzed using descriptive statistics and the open questions were analyzed inductively.

Teachers' relation to misconceptions and proof

Most teachers (97 %) state they recognize the common misconceptions in their own teaching and all of them consider the knowledge about misconceptions relevant for their work. Almost 70 % of the teachers state that they know how to deal with these misconceptions.

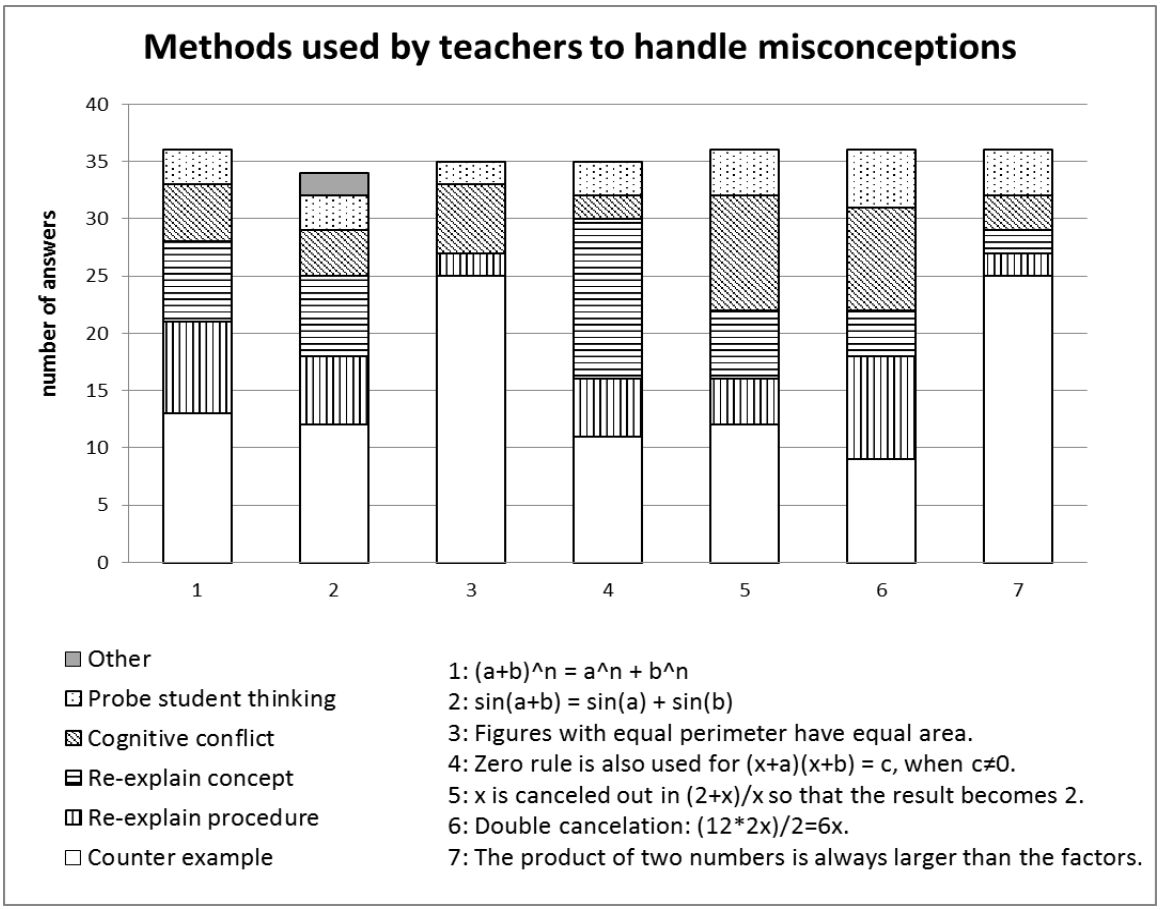

Figure 1: Methods used by teachers when encountering misconceptions

Further, over one half of the teachers who state that they do not know how to deal with students' misconceptions had less than 10 years of teaching experience and 36 % of them wanted to learn more. The methods teachers would choose to deal with students' misconceptions varied depending on the character of them. The use of a counter example and cognitive conflict are related and were dominating among the methods that the teachers preferred (Figure 1). The least popular method was the probing students thinking. Yet, it seems to be usual for the teachers to use the analysis of the steps in the reasoning when sorting out the situation and then utilize the other methods. Some teachers suggested that a teacher should focus on common misconceptions already when presenting the theory of a new topic and explain why that is not true in order to create a cognitive conflict from the beginning.

One can let students work pairwise and judge the correctness of different solutions and ask them to justify their judgments. Surprisingly, students are insecure and experience these tasks as

difficult. I have tested this with both students taking the advanced course and students studying the basic course in mathematics. This is really instructive for both a teacher and students. (All of my examples were authentic student solutions.)

Most of the teachers stated that they used their knowledge about the common misconceptions when they designed their teaching and chose tasks. Yet, peer instructions (students discuss and argue about the correctness of different solutions) was utilized (sometimes /more often) only by 25 % of the teachers (Figure 2).

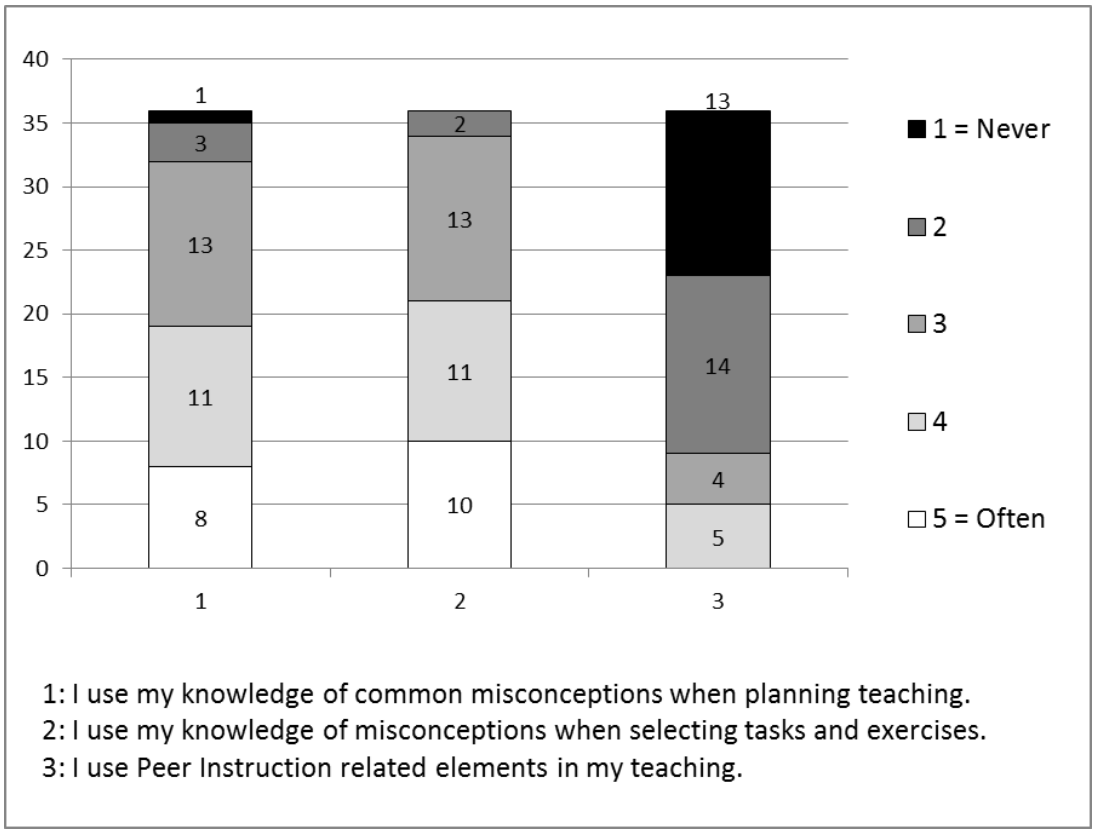

Figure 2: Teachers' use of misconceptions

Mathematics is a cumulative subject. Elementary school has a great responsibility. Unfortunately the textbooks I have seen are quite bad. Students cannot see the structure because of all details. Mathematics is not only using of Pythagorean theorem or calculation of percent that students do without understanding. It is a logical structure.

All the teachers consider proof as more or less important for mathematics as science and they present (sometimes or more often) proofs for students studying the advanced program in mathematics. They also agree that proof somehow contributes to the teaching of mathematics. Yet, only 2 teachers present proofs for students studying the basic program, one of them states that "proof gives often a greater broadness than 'learning by heart'" and if students learn to prove the formulas then they also can modify them so that they can solve a broader spectrum of tasks. Another view of the role of proof in school mathematics is shown by a teacher who states: "Proofs are good and beautiful but in the upper secondary school reality teaching is far away from building teaching around proving". About 30% of the teachers seldom let their students work with proof and proving by themselves. Concerning the teachers' views of the connections between students'

misconceptions and proof, most of them are not convinced that proof and proving would have a positive effect on students' misconceptions. Only 9 teachers agree with the statement "Proof and proving can help to change students' misconceptions" and 8 teachers agree with the statement "If students learned proving, their incorrect steps of reasoning would diminish."

Concluding remarks

The paper focuses on the relation between research on students' misconceptions and developmental proof. The elaboration of the research literature on students' misconception from the perspective of proof reveals several important connections between the ideas and results of the two research fields. However, the empirical study shows that these connections are not clear for teachers. For example, we find it significant that the teachers in our study most often use counter examples or cognitive conflict that are closely related to developmental proof as a method for changing students' misconceptions but at the same time only about 25 % of them consider proof and proving as beneficial to prevent and change students' misconceptions. We also recognize different views of proof in school mathematics among the teachers that may have crucial consequences for students' possibilities to develop their understanding and skills in proof and proving and therefore also using students' misconceptions as a starting point for this and vice versa. The idea of developmental proof is probably not in focus in mathematics teacher education. Furthermore, it is possible that teacher educators focus on both students' misconception and proof but because of the different research traditions the connections and the possibilities of these connections between these areas may not become clear for student teachers. More studies are needed to investigate teachers' views of proof in relation to their views of these connections in the teaching of mathematics.

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Primary and secondary students' argumentation competence: A case study

Eugenia Koleza[1], Nikolaos Metaxas[2] and Klio Poli[1]

University of Patras, Department of Primary Education, Greece

[1]ekoleza@upatras.gr, [2]nkm1012gr@yahoo.com

There is a significant corpus of studies indicating that children even from the age of primary school are capable of providing convenient arguments and that the cultivation of this ability fosters learning significantly. Based on these assumptions, the present paper examines the forms of the arguments that students of primary and middle school use to support their answers. In particular, this study was divided in seven independent activities, where students of a fifth and an eighth grade class had to complete mathematical tasks and support with arguments how they concluded to their answers. We used the simplified Toulmin's argumentation scheme and subsequently, enriched our findings with the argumentation scheme perspective, in order to gain a better understanding of student's reasoning characteristics.

Keywords: Toulmin, argumentation, scheme.

Introduction

Basic aim of the new teaching methods, starting from the new curriculum in mathematics, which was published by the National Council of Teachers of Mathematics in 1989, is to reinforce reflective thinking and shift from 'learn mathematics' to 'do mathematics'. According to Dewey (1903), children learn effectively through inquiry-based processes, which require from students not only to solve mathematical activities, but also to express their thinking, state their opinion and finally, compare their statements with their classmates'. Thereupon, it can be stated that reasoning organises students' ideas, builds strong conceptual connections and fosters mathematical thinking (Dewey, 1903). Following these assumptions, the cultivation of proper language skills plays a significant role in this thinking process, as it allows students to express accurately their thoughts by forming arguments. Halliday (1993) uses the phrase 'interpersonal gateway' to refer to the power of language as interactive tool in the communication between students and teachers. Regarding our study, we adopted Toulmin's argumentation model in order to explore the reasoning ability of primary and secondary students. Toulmin's model was first used by Krummheuer (1995) in the field of teaching mathematics. According to the latter, claims, data and warrants are not predetermined, but are constructed through the process of classroom discourse and interaction. Toulmin's model can describe the structure of an argument by specifying its components, but it cannot characterize the quality of the particular argument. Therefore, at a second analytical level, we enhanced our analysis by using argumentation scheme theory, used as in Metaxas, Potari, & Zachariades, (2009).

Theoretical background

According to Schwarz et al. (2003) constructing knowledge is the process of composing evidence in such a way, that the selected claim is supported by provided evidence and further supported related to other co-existing beliefs. Moreover, a plethora of studies have established that argumentation

plays a crucial role in the development of knowledge and scientific reasoning. The same holds in the field of education, either as a means to learn (argue to learn) or as a goal of instruction (learn to argue). Schwarz (2009) describes "learning to argue" as the acquisition of general skills such as justifying, challenging, counterchallenging, or conceding, whereas "arguing to learn" refers to the fulfillment of a certain goal through argumentation in a specific educational framework. In other words, the first path uses argumentation as a goal, while the second one as a tool that contributes to the learning process. In order to analyse students' argumentation we employed the classical Toulmin's model and subsequently, the recently developed, argumentation scheme approach.

Toulmin's theory

Argument structure has been used several times as a tool of analysing public discourse regarding mathematics and their teaching. A significant number of these analysis have been conducted by using Toulmin's theory. Toulmin (1958) has claimed that the traditional formal logical analysis of arguments is not rich enough to include parts of common arguments such as qualified conclusions, response to other arguments and inferences. He proposed a model for the layout of arguments that consists of six basic parts, each of which plays a different role in an argument (Metaxas, Potari and Zachariades, 2016). The first one to apply Toulmin's model in mathematics education was Krummheuer (1995). Since then, some researchers have focused on the analysis of mathematical arguments of students, including usage of proof in general (Yackel 2002), number skills (Evens & Houssart 2004), geometry (Pedemonte, 2007) and algebra (Pedemonte 2008). By studying the argument components a student is using when talking about a solution in mathematics, we could have some indication about his/her understanding and generally, his/her perception about mathematics.

Argumentation schemes

Toulmin's model describes the structure of an argument giving its components, but it does not reveal much about the quality of the particular argument. However, the content of the Warrant and the Backing in an argument should be considered in the evaluation process of an argument. For this reason, we combined Toulmin's model with the tool of argumentation schemes to analyse the quality of the Warrant and the Backing. For example, a Warrant that is based on the authority of a source (teacher has said so...) is fundamentally different to a Warrant that is based on a mathematical relation or to an intuitive remark. Standard accounts of argumentation schemes describe them as the representation of different types of plausible arguments that, when successfully deployed, create presumptions in favor of their conclusions (Metaxas, Potari and Zachariades, 2016). Argumentation schemes have been assigned a role in the analytical reconstruction of an argument, as well as its evaluation. In reconstruction, these schemes can be used to identify and categorise certain patterns of reasoning, contributing to the identification of implicit claims of the arguer. Moreover, a set of critical questions are associated with each argumentation scheme to be used in the evaluation of arguments and their correspondence with each category (Walton, Reed & Macagno 2008). Another significant aspect of argumentation schemes is that the evaluation of the argument is directly associated to the dialogue as a whole, rather than evaluating it independently and isolated from the context that is being constructed. Consequently, every argument will be evaluated via the critical questions, in the context of the dialogue of which it is a part of. Thus, critical questions are a kind of evaluative points, providing a list of individually necessary

conditions for the success of particular schematic arguments. For instance, an argument can be characterized as weak if it fails to answer appropriate critical questions that have been or might be asked in a dialogue (Walton, 2006). In addition, an argumentation scheme could inform us about the quality of a warrant or a backing as a form of an argument (Metaxas, Potari and Zachariades, 2016).

The structure of the course

The theoretical underpinnings for looking at the classroom discourse was the theory of symbolic interactionism. Individuals are seen to develop personal meanings as they participate in the ongoing negotiation of classroom norms (Cobb, 1999). The centrality given to the process of interpretation in interaction is one of its main principles (Blumer 1969). While individuals are interacting with each other, they have to interpret what the other one is doing or is about to do. Each person's actions are formed, in part, as he/she changes, abandons, retains, or revises his/her plans based on the others actions (Cobb, 1999). Moreover, the group discussions can provide participants with learning opportunities by turning their implicit supporting arguments into explicit. In addition, the objects of debate can result in a change of their status and engage them at a higher level of mathematical reasoning. The very act of argumentation could produce learning on the part of the arguer (Jermann and Dillenbourg, 2003). In our study, the materials used to trigger the discussion were tasks, which were based on topics that research and experience have highlighted as important.

Data analysis

In order to study the ability of elementary students to reason in mathematics, we implemented a series of activities, where the students of a fifth and an eighth grade classroom in Greece were asked to solve mathematical exercises and in addition, to provide with written arguments why they believe their answers were correct. Having analyzed all the written answers following the methodology of previous studies (McNeill, 2011), we drew the conclusion that students of that age have the ability to form arguments in order to support their solutions. More specifically, 66% of the students who provided some kind of argumentation, used to some extend the simplified Toulmin's argumentation scheme, which is consisted of three parts; claim, data and warrant. Although not all answers had all three essential parts, they could be adjusted to Toulmin's pattern arguments. Toulmin's model allows students to reason in a completed way, which presents the hypothesis, the explanation and the solving process. Subsequently, the arguments that followed Toulmin's model were analyzed according to their structure following the analysis of other relevant studies (Evagorou and Osborne, 2013) that have taken place in the past and adopted the modified version of Toulmin's Argumentation Pattern (Erduran et al., 2004). Out of the 66% that is mentioned above, nearly half of the students (Table 1) included all three essential parts, that is they were able to state their opinion (claim), provide all the necessary support (data) and finally, connect them in a sufficient way (warrant). This completed structure is followed by the students who managed to include the claim and the data to their answers, but weren't able to provide effective warrant (33.4%). Finally, a little less than 20% wrote only their opinion, without justifying or explaining how the concluded to this claim. There is a similar pattern in secondary school students, where there is only a slight differentiation around 2-3% in the first two columns.

Reasoning forms of Toulmin's Model			
	Claim – Data - Warrant	Claim – Data	Claim
Primary school	47.6%	33.4%	19%
Secondary school	45.2%	36.2%	18.6%

Table 1: Reasoning structure of Toulmin's Model

The following table presents an example of each category, taken from an activity that students had to form the biggest decimal number by throwing a dice and placing the digit in a suitable place.

Claim only	*I have to place the numbers with greater value in tenths etc.*
Claim-Data	*If I get 6, I'll place it in tenths because 6 is the biggest number I can get. If I get 1, I'll place it in thousands because is the smallest number I can get.*
Claim-Data-Warrant	*In order to win the game I have to make the biggest number. I need to place the bigger numbers in the integer part and the smaller ones in the decimal part. So, the best thing I can do is to place the numbers from the biggest to the smallest.*

Table 2: Excerpts from each category

Having completed the primary data analysis, we studied the produced arguments using the argumentation scheme theory, which helped us gain insight regarding the quality of the argumentation used.

Discussion

Having analyzed the data and in correlation with previous related studies, it can be clearly said that elementary students can form arguments in order to justify their mathematical thinking and that the most common way to state their reasoning is by using Toulmin's Argumentation Pattern (TAP). However, students of that age do not recognize the significant role of proof and therefore, they don't understand that justification of their thinking is essential. Even though they solved the exercises correctly and they presented the important data, they don't define clearly the connection between data and claim, which according to TAP is known as warrant. This deficiency must not be understood as lack of students' ability, as in many of their answers and especially when is asked by the teacher they expand their reasoning and include the semantic warrant. The obvious implication that follows the existence of a correct claim and a written data could be the reason the students don't regard as necessary to include a warrant in their answers. Consequently, this identification and evaluation of the missing premises or conclusions could be greatly enhanced by the employment of

the argumentation scheme theory (Walton, Reed & Macagno, 2008), where most of the arguments are considered forms of plausible reasoning that do not fit into the traditional deductive and inductive argument forms. In this case, a further analysis is needed in order to evaluate the content of the argument accurately. For example, the absence of the warrant or backing is due to people's belief that these are automatically entailed from the data given and there is no need for further justification. This deficient form of argumentation can be enriched and expanded in order for an argument to acquire the desirable structure. Likewise, short answers that were given by the students and were characterized by lack of structure, were in fact complete, if the essential parts that were considered obvious and were implied, are included so as to form a complete argument. Below there are given two examples of arguments that were at first deficient, but after expanding them, they transformed in complete arguments according to TAP. The first example is taken from an activity, where students had to form the biggest possible decimal number, using the digits that were given after rolling a dice six times.

Student: I will win by putting the number to the correct places. For example, if I get number 1 I will place it in the thousandths.

The above argument is considered short and deficient. However, it is clear that the student has understood the procedure in order to form the biggest number, but still prefers not to include all the essential information to his answer, as he believes that it is obvious. He argues by employing an argumentation scheme of illustration, which nevertheless remains without support. Nonetheless, after the teacher's claim, the student added the hidden parts in order to transform his deficient answer to a complete argument. We give a reconstruction of the argument:

Student: I will win by putting the number to the correct places [claim]. I have to place the small numbers in the decimals' places (tenths, hundredths, thousandths) and the bigger ones in the integer part of the number (tens, hundreds, thousands), because decimals have smaller value than integers [warrant]. For example, if I get number 1 I will place it in the thousandths [claim], because number 1 is the smallest I can get and thousandths have the lower possible value compared to the other places [warrant].

In analyzing student's elaboration of his argument, we can either consider the second argument as a continuation of the first one, in the sense of using the previous claim as the data of the second argument, or we could interpret the whole second syllogism as a backing of the first one. In any case, the scheme employed in the second argument is, again, a scheme from illustration but now connected to the previous scheme from established rule (I have to place …integers). As a result, regarding the quality of the schemes employed, the student actually elaborates his reasoning by using an established rule, which again is supported by a scheme from illustration.

The second example is taken from an activity, where students were asked to estimate the product of a decimals' multiplication without making the transaction, by simply observing the factors.

Student: I have to consider what the multiplication does; if it makes the number bigger or smaller.

The above claim contains the perception that multiplication can either grow or reduce the value of its factors. Even though he misses many essential parts, if the claim is expanded, we can take an efficient answer. A reconstruction of the above statement could be the following:

Student: Multiplication can either increase or decrease the value of its factors [claim], so I have to consider what this transaction will do. If one of the factors is smaller than zero, then the product will decrease. If the factors are integers, then it depends on their value [data]. So, when comparing two products, the bigger will be the one that contains the bigger factors [warrant].

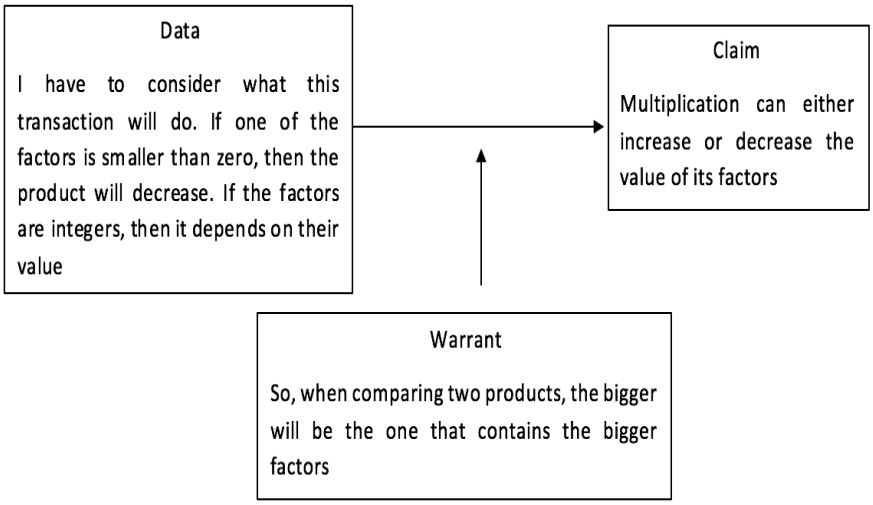

Table 3 - Analysis of student's extended argument

Again, taking into consideration the schemes employed, we could note the presence of a scheme from (positive) consequences (Walton, Reed & Macagno, 2008). The (implied) fact that in order to answer the posed question, we should consider the effect of the multiplication on the magnitude of the numbers, is a scheme from consequences. The explanation that follows is the elaboration of the scheme; the consequences in each case. The student explains in a more abstract (mathematical) way his reasoning, which is in a clear contrast to the previous excerpt (where the invocation of an illustration was employed).

Our thesis is that elementary students are capable of forming arguments and reasoning in mathematics, but one of the main characteristics of that age is the short way they express their arguments and therefore the absence of basic parts. The deficient character that defines most of the arguments can lead to the conclusion that all students reason according to TAP, but the structure is incomplete, as some parts are considered obvious and children believe are excessive. Additionally, another interesting point is the insignificant difference between the two grades, especially if considered that students from the seventh grade start using and structuring their first proofs. Nevertheless, by taking into account the types of the syllogisms employed, in the sense of argumentation schemes, we could shed a bit more light into the quality of arguments used. In the primary school case, students used mainly schemes from illustration and from consequences, which probably is due to the students' inadequate exposure to mathematical thinking or argumentation structuring in general. On the other hand, the eighth grade students employed more schemes from rules to cases, which accounts to their better understanding of the structure and function of a proof. As a result, although the Toulmin model is indicative of the structure of the arguments students use, it is not enough to discern the difference of the quality of their arguments. This could be easily overruled by using argumentation schemes. Finally, it should be noted that justification and correctness should be distinguished in the analysis of an argument. For example, a premise that is

based on an authoritative opinion or is justified by intuition or a meme could be turn out to be false. Consequently, representational tools as the argumentation schemes that could exhibit the implicit structures of arguments can enhance the reconstruction and comprehension of the syllogism. In further studies it would be interesting to examine ways that will cultivate the argumentative way of thinking and grow the ability to express completed arguments that contain all essential parts.

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Proofs and refutations in school mathematics: A task design in dynamic geometry environments

Kotaro Komatsu[1] and Keith Jones[2]

[1]Shinshu University, Japan; kkomatsu@shinshu-u.ac.jp

[2]University of Southampton, UK; d.k.jones@soton.ac.uk

Although the mathematical activity of proofs and refutations is widely recognised as significant in school mathematics, much remains under-explored about ways of facilitating such activity in the classroom. In this paper, we address this issue by focusing on task design in dynamic geometry environments. In particular, we formulate three principles for the task design and use these to develop classroom tasks. We analyse a task-based interview with a triad of upper secondary school students to show how the designed tasks stimulated their activity of proofs and refutations.

Keywords: Proof, refutation, counterexample, task design, dynamic geometry environment.

Introduction

The mathematical activity of proofs and refutations described by Lakatos (1976) is significant in school mathematics because it enables students to experience authentic mathematical practice (Lampert, 1992). Although several researchers have described student behaviour with Lakatos' terminology (Komatsu, 2016; Larsen & Zandieh, 2008), few studies have examined ways of purposefully introducing such activity into classrooms (Komatsu, Tsujiyama, Sakamaki, & Koike, 2014; Komatsu, 2017; Larsen & Zandieh, 2008). Given the importance of mathematical tasks for student learning (Kieran, Doorman, & Ohtani, 2015), this study aims at developing task design principles and actual tasks for realising proofs and refutations.

To achieve this purpose, we specifically focus on dynamic geometry environments (DGEs). Research has shown the capability of dynamic geometry software (DGS) for enhancing proof-related activities such as making conjectures and subsequently constructing proofs. In particular, some studies have shown how using DGS enabled students to discover the refutations of their conjectures and proofs and cope with these refutations (Healy & Hoyles, 2001; Olivero & Robutti, 2007). The successful use of DGS in previous research was accompanied by carefully-designed tasks (Hanna, 2000). Nevertheless, how the tasks were designed was often not clarified explicitly, and task design in DGEs remains understudied (Sinclair et al., 2016).

To address these issues, the study reported in this paper focuses on the following research question: What principles can underpin the design of DGE tasks that facilitate student activity of proofs and refutations?

The meaning of proofs and refutations

Based on Lakatos (1976), we conceptualise the meaning of proofs and refutations as depicted in Figure 1. Students make conjectures (or are provided with statements), and then attempt to prove them. In this, they are confronted with refutations of the conjectures/statements or proofs, and refine them by addressing the refutations (Komatsu, 2016).

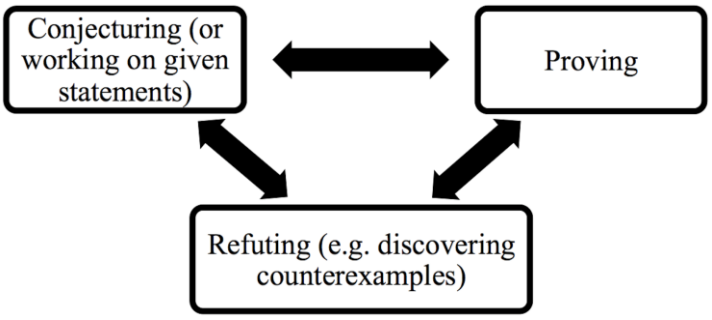


Figure 1: The conceptualisation of proofs and refutations

As there is insufficient space to explain Figure 1 fully, here we clarify only two points. First, we take the meaning of proof in a broad sense such that a deductive proof may be valid only for a subset of all cases considered in a conjecture and statement. Second, although the word *refutation* is sometimes used only for conjectures and statements, not for proofs, this study utilises *refutation* for conjectures/statements and for proofs. These two points are epistemologically consistent with Lakatos' view of mathematics. In *Proofs and Refutations* (Lakatos, 1976), he dealt with deductive proofs that were only partially valid. He also argued that proof was inextricably linked to refutations (Reid, Knipping, & Crosby, 2008) and coined the term *local counterexamples* to denote the refutations of proofs.

Task design principles

To design tasks for fostering the student activity of proofs and refutations, we develop principles for the task design from three aspects. First, Hanna (1995) pointed out that Lakatos' (1976) story rested on the topic of polyhedra, where it was relatively easy to suggest counterexamples. This confirms that it is necessary to create tasks intentionally where counterexamples can be produced. Because it is the ambiguous meaning of polyhedra that made counterexamples possible in Lakatos' research, it is essential to develop tasks whose conditions are purposefully ambiguous so that counterexamples can be proposed. In fact, we previously demonstrated that specific tasks that include hidden conditions, namely *proof problems with diagrams*, are useful for introducing proofs and refutations into secondary school geometry (Komatsu, et al., 2014; Komatsu, 2017).

Second, research indicates that students encounter difficulties in producing proper counterexamples (Hoyles & Küchemann, 2002). Thus, it is important to prepare tools that foster student production of counterexamples. DGS could play the role of such tools in geometry education because the main advantage of DGS is that students can easily transform diagrams by dragging (Arzarello, Olivero, Paola, & Robutti, 2002) and thus the students have access to various diagrams. From research on dragging modalities and measuring modalities in DGEs (Arzarello et al., 2002; Olivero & Robutti, 2007), the following are relevant to refutations of conjectures and proofs: dragging test, validation measuring, and proof measuring.

Third, several studies have reported that when students encounter counterexamples, some of them refuse to accept the counterexamples and do not try to revise their conjectures (e.g. Balacheff, 1991). For resolving this problem, we capitalise on the potential of contradictions, because if contradictions are appropriately induced, confusion generated by the contradictions can be beneficial for learning (D'Mello, Lehman, Pekrun, & Graesser, 2014). To trigger contradictions, it is likely helpful to combine tasks intentionally (rather than use a single task) where students can recognise contradictions

between their solutions to tasks and their thinking in subsequent tasks (Hadas, Hershkowitz, & Schwarz, 2000; Prusak, Hershkowitz, & Schwarz, 2012).

In summary, we formulate the following principles of task design for fostering the student activity of proofs and refutations: 1) Using tasks whose conditions are purposefully ambiguous and thus allow the occurrence of counterexamples; 2) Providing tools that enhance the production of counterexamples; and 3) Increasing students' recognition of contradictions that facilitates them to revise conjectures/statements and/or proofs.

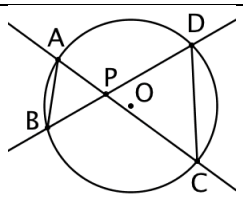
Methods

Participants

This paper analyses a task-based interview in which a triad of students, Kakeru, Sakura, and Yuka (pseudonyms), voluntarily participated. They were 11th graders (aged 16–17 years old) in an upper secondary school in Japan. According to their mathematics teacher, their mathematical capabilities were above average. The first author conducted the interview. The DGE was GeoGebra. Because the students had no experience with DGS, four hours was devoted, prior to the interview, to teaching the students the basic functions (e.g. basic construction, dragging, and measuring) of the DGE. The students had learnt geometric proofs using the conditions for congruent triangles and those for similar triangles. They were familiar with the inscribed angle theorem, the inscribed quadrilateral theorem, and the alternate segment theorem, all of which are related to tasks used in the interview.

Tasks

Q1. (1) As shown in the diagram given, there are four points A, B, C, and D on circle O. Draw lines AC and BD, and let point P be the intersection point of the lines. What relationship holds between ΔPAB and ΔPDC? Write your conjecture. (2) Prove your conjecture.



Q2. Construct the diagram shown in Q1 with DGS. Move points A, B, C, and D on circle O to examine the following questions. (1) Is your conjecture in Q1 always true? (2) Is your proof in Q1 always valid?

Figure 2: Tasks used in the interview

The tasks used in the interview are shown in Figure 2. We developed them according to the aforementioned design principles. Q1 is relevant to the first principle that involves ambiguous conditions. The condition of Q1 is vague because there is no reference to the locations of points A, B, C, and D in the problem sentences. If the locations are changed, refutations of the proof constructed in Q1 can be discovered, as shown below. The second principle corresponds to Q2, where students are invited to construct the given diagram with DGS and produce various diagrams by dragging. The third principle is related to the combination of Q1 and Q2. It is, of course, possible for Q1 to stipulate the use of DGS to produce various diagrams for making a conjecture before proving. However, we designed Q1 and Q2 in the way set out in Figure 2 because we expected that proof construction in Q1 could increase students' conviction in their conjecture and proof. This design could lead to students' recognition of a contradiction between their conviction and the subsequent refutations in Q2.

Data collection and analysis

The three students were asked to solve task Q1 collaboratively with paper and pencil and task Q2 with DGS on a desktop computer. The task-based interview lasted for approximately 35 minutes in total. It was video-recorded and the audio transcribed. We used two cameras for the recording, one placed to video the students and the other placed to record the screen of the computer. The worksheets the students completed, and the DGS file the students made, were collected. We analysed these data by focusing on what type of diagram the students produced and how they dealt with the diagrams.

Results

Conjecture, proof, and types of diagrams the students produced

Immediately after student Kakeru read the problem sentences in Q1, Sakura conjectured “similar?”. The students then wrote the following proof on their worksheet:

In ΔPAB and ΔPDC,

From the vertical angles, ∠APB = ∠DPC … (1)

From arc BC, since inscribed angles are equal, ∠PAB = ∠PDC … (2)

From (1) and (2), since two pairs of angles are equal, ΔPAB ~ ΔPDC

After that, the students worked on Q2. As they worked, they produced and examined the six types of diagrams shown in Figure 3. In Figure 3a, triangle PAB (or likewise triangle PDC) is not constructed, while both triangles are not constructed in Figures 3b and 3c. Point P is located outside circle O in Figures 3d and 3e. In the type of diagram shown in Figure 3f, the students regarded points A and C (or likewise with points B and D) to be coincident and considered line AC (or likewise BD) to be a tangent to circle O.

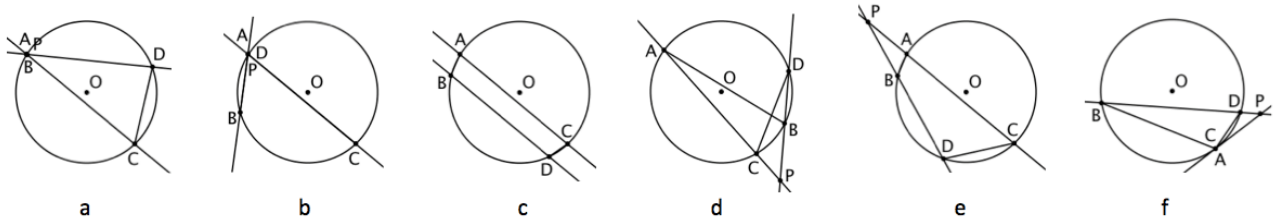


Figure 3: Types of diagrams the students produced

In the following, we report the cases regarding Figures 3e and 3f because the students devoted more efforts to these types than to the other types.

Case where point P is outside the circle

At the beginning of Q2, the students produced the type shown in Figure 3e:

116 Kakeru: Is the conjecture in Q1, similarity, always true? [Reading the problem sentence.]

117 Sakura: Not similar.

118 Yuka: In this case, … impossible.

119 Kakeru: The intersection point is outside the circle.

Here, Sakura and Yuna recognised a contradiction because although they proved their conjecture in Q1, they considered the type of Figure 3e to be a counterexample to their conjecture (lines 117 and 118). Kakeru then responded to their judgement:

132 Kakeru: We can say that they are similar.
133 Sakura: Why? We can't say that.
134 Kakeru: Because.
135 Sakura: Wait. Because.
138 Kakeru: PAB and PDC. These are similar. This and this [angle P] are common and equal. Then, because [quadrilateral ABDC is] a quadrilateral that is inscribed to the circle.
139 Sakura: That one.
140 Kakeru: This [angle PAB] and this [angle PDC] are equal.

A dispute between the students can be seen in this dialogue, where Kakeru argued that their conjecture was still true (line 132), whereas Sakura objected to his argument (lines 133 and 135). To respond to Sakura's objection, particularly for showing the congruence of angles PAB and PDC, Kakeru proposed using the inscribed quadrilateral theorem (lines 138 and 140): an interior angle is equivalent to the exterior angle of the opposite angle. Sakura agreed with his thinking (line 139), and, thus, they were able to resolve the dispute by proving the similarity of the triangles in the type of Figure 3e.

Case where a line is a tangent to the circle

After producing the diagram type shown in Figure 3e, the students examined the type shown in Figure 3f (note that, strictly speaking, this type is different from the original problem where line AC cannot be drawn if points A and C coincide.) When encountering this type, Kakeru was convinced that their conjecture would be still true, and proposed using the alternate segment theorem to prove the conjecture. Nevertheless, when he started explaining his idea to Sakura and Yuka, he had a doubt as to why line AP can be considered as a tangent. The students struggled to resolve this doubt. During their struggle, as the students mentioned only once that the type of Figure 3f might be a counterexample to their conjecture, they consistently anticipated that their conjecture would be true in this type. Eventually, they judged that line AP was the tangent by measuring the degree of angle PAO and finding that it was almost 90 degrees. The subsequent student interaction was as follows:

357 Kakeru: If we consider this as a tangent, we can use the theorem about the angle formed by a tangent and a chord.
358 Sakura: I see.
359 Kakeru: We can show the similarity.
360 Sakura: This (angle DCP) and this (angle PBA) and P.

This dialogue shows that the students were able to prove their conjecture in the type of Figure 3f with the alternate segment theorem.

Examination of the initial proof

The students concluded their activity without considering Q2(2), so the interviewer questioned them as follows: "Please read again the sentences in Q2 carefully. When you say it does not hold, do you mean your conjecture is false, or your proof is invalid?" When addressing this question, the students noticed that the reasons in their initial proof were not applicable to the diagrams that they produced.

In other words, they regarded these diagrams as local counterexamples to their proof in the sense of Lakatos' (1976) terminology. For example, the following is their discussion about Figure 3e:

456	Sakura:	We wrote, "From arc BC, since inscribed angles are equal, ∠PAB = ∠PDC".
457	Kakeru:	PAB and PDC. This [the initial proof] is for this case [shown in Figure 2].
458	Sakura:	This [the last line in the proof] is valid, but the sentences [the second and third lines in the proof] are not valid, right?
461	Kakeru:	This [the initial proof] is only for this [Figure 2].
462	Yuka:	If so, this proof …
463	Sakura:	Is not always valid, right?

After that, the students pointed out that it was sufficient to revise the reasons in their initial proof by replacing the equality of vertical angles and the inscribed angle theorem with the identity of the angles and the inscribed quadrilateral theorem, respectively. They also examined and revised the initial proof in the type of Figure 3f in a similar way, with the alternate segment theorem.

Discussion and conclusion

The students in the interview were able to engage in mathematical activity of proofs and refutations depicted in Figure 1. After making and proving a conjecture, they produced diagrams to scrutinise whether their conjecture was always true. Although they initially judged the type of diagram in Figure 3e to be a counterexample to their conjecture, they modified their judgement by proving that their conjecture was still true in this type. This proof was constructed without looking back at their initial proof and revising it. However, after the interviewer's intervention asking them to consider Q2(2), the students recognised that their initial proof was not applicable to the types of diagrams in Figures 3e and 3f, and revised the proof for these types.

The three design principles and the tasks developed based on the principles were generally helpful for fostering the students' activity. Based on the first principle, we used the proof problem whose condition regarding the locations of points A, B, C, and D is ambiguous (Figure 2). This task enabled the students to produce the six types of diagrams that had the potential to refute their conjecture and proof (Figure 3).

With regard to the second principle, DGS in general and its dragging function in particular (Arzarello et al., 2002), were highly useful for producing such a variety of diagrams. In our earlier research, many students in a lower secondary school encountered difficulties in drawing diagrams that refuted their proofs in paper-and-pencil environments (Komatsu, Ishikawa, & Narazaki, 2016). Although the tasks used in that study were more difficult than those in this study, without DGS it would likely be challenging for the three students in this study to produce various diagrams different from Figure 2.

The combination of Q1 and Q2 based on the third principle played a role in stimulating the subsequent students' activity. In the case where point P was outside circle O, Sakura and Yuka felt a contradiction between the truth of their conjecture that was proved in Q1 and the refutation in Q2 where they judged the type of Figure 3e to be a counterexample to their conjecture. This contradiction triggered the dispute with Kakeru, where Sakura and Yuka's judgement was revised through Kakeru's proof showing that their conjecture was still true. In the subsequent case where a line was a tangent to circle O (Figure 3f), the students did not seem to perceive such a contradiction. This was likely related to the students' earlier experience, where they could show that the type of Figure 3e, which was initially

regarded as a counterexample, did not refute their conjecture. This experience would constitute a source of their conviction in the truth of their conjecture as regards the type of Figure 3f. If the students encountered this type prior to the type of Figure 3e, they would think that it might refute their conjecture, and would perceive a contradiction between their conjecture and the refutation.

This study has limitations as it is based on a case with one set of tasks. It is necessary to develop other tasks based on the design principles of this study and conduct further empirical studies, including studies in real classroom settings, to inspect the values of the principles and tasks. Another interesting future issue is to examine whether the design principles of this study are applicable to content areas other than geometry (for example, number theory). The design principles are not conceptually restricted to geometry education; the 'tools' mentioned in the second design principle are not only DGS tools. This issue is worth addressing in order to extend the opportunity to introduce proofs and refutations from geometry into other topics in the mathematics curriculum.

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Can teacher trainees use inductive arguments during their problem solving activity?

Eszter Kónya[1] and Zoltán Kovács[2]

[1]University of Debrecen, Hungary; eszter.konya@science.unideb.hu

[2]University of Nyíregyháza and University of Debrecen, Hungary; kovacs.zoltan@nye.hu

The Hungarian curriculum for mathematics teachers' training specializes in a problem-solving Seminar aimed at teaching heuristic strategies. This fact motivated our research focusing on problem-solving competency of teacher trainees. In this study we deal with some aspects of inductive reasoning. We summarize the results of a diagnostic survey. We chose a closed problem which could be solved through inductive reasoning, and analyzed problem solving process of 94 students. Our primary interest was how students apply general phases of inductive reasoning, if they use it at all; that is, how they conclude general statements after pattern recognition, and whether they close it deductively or not.

Keywords: Problem solving, inductive reasoning, proof and proving, rational errors.

Introduction

In the midst of a long-term discussion on the role of metacognition and teaching heuristic strategies in order to enhance mathematical problem solving skills (Schoenfeld, 1985; Cai, 2010), the new Hungarian curriculum for mathematics teachers' training[1] explicitly specializes such a course. We think that there is no final rule concerning this polemics; moreover there is no comprehensive research focused on this student group in Hungary in this respect. One of the antecedent studies, the PhD dissertation (Pintér, 2012) focused solely on primary teacher trainees. Motivated by these facts we have begun a research to map the status quo in Hungary, with the aim to give didactical consequences and finding ways of teaching heuristic strategies, general problem solving skills effectively.

According to Polya (1954), heuristic reasoning is based on induction or analogy. In this study we focus on inductive reasoning process only. Csapo (1997) supports the proposition that inductive reasoning and skills of proof develop during broad age range (Grade 1-11). We therefore assume that problem solving skills, especially proper utilization of inductive reasoning strategy develop after entering higher education, and should be subject to development.

Inductive reasoning and inductive problem-solving strategy

The word "induction" means a scientific procedure starting from experience. In inductive reasoning, one makes a series of observations and infers a new claim based on them. The mathematics education offers the possibility of learning the way of inductive reasoning beside the deductive one. Within the process of inductive reasoning Polya (1954) distinguishes stages such as observation of particular cases, formulating a conjecture (generalization), testing the conjecture with other particular cases. Haverty, Koedinger, Klahr, & Alibali (2000) identify the "function finding task" as the "representative of inductive reasoning" and use this term in a narrower sense as we use it, thinking

[1] This curriculum was introduced in 2013.

only on open problems and determine three basic inductive activity such as data gathering, pattern finding and hypothesis generation. Yeo and Yeap (2009) make the difference between inductive observation and inductive reasoning clearer. If students observe a pattern when specialising, the pattern is only a conjecture and they call it 'inductive observation'. But if students use the underlying mathematical structure to argue that the observed pattern will always continue, this can be called 'inductive reasoning'. Motivated by these approaches we describe inductive reasoning with five phases. (1) Observation of particular cases including looking for possible pattern as well. (2) Following the observed pattern, i.e. applying it for other cases. It often happens without formulation of a general statement. (3) Formulating a general conjecture. (4) Testing it by other particular cases. The result of the inductive reasoning is a general statement, but the mathematical problem solving process requires its deductive closure (5). The form of deductive closure could be either a rigorous proof or justification using the underlying mathematical structure (Mason, Burton, & Stacey, 2010). Moreover, Rivera (2013) uses the term of *empirical structural argument*, as a type of justification. Empirical structural argument means that one uses the steps of a logical deductive proof with concrete numbers or objects instead of variables. This process is closely related to the phenomenon of *generic example* (Stylianides, 2009) and *transformational proof-scheme* (Harel & Sowder, 2007). Thus, we look not only for the clues of the formal proof but the clues of empirical structural argument too while investigating Phase 5.

Haverty et al. (2000) argue, in accordance with other studies, that the detection of patterns is crucial to inductive reasoning. In patterning activity there is a difference between near generalization, a description of a pattern allowing one to determine the next term in a sequence, and far generalization, the construction of a general rule or a far stage in the pattern (Rivera, 2013). Solving the problem we investigated in this research requires mainly the near generalization activity; the formulation of a general rule could be useful, but is not necessary.

We identified in many cases that a mental manipulation process led to the inductive observations. The same phenomenon was detected by Simon (1996) who defined the concept of transformational reasoning, which is rather a dynamic process. Transformational reasoning visualizes the transformation of a mathematical situation and the results of that transformation. The conjecture is drawn from the result of the mental manipulation.

Besides inductive strategy, some other strategies may work for many closed mathematical problems. As Ben-Zeev (1996) pointed out, the schema-based thinking could be a useful way for organizing mathematical experiences. Using a schema – the knowledge structure for a particular class of concepts – in a proper way can predict the solution of the problem. However, schematic reasoning often lead to rational errors when applied rigidly or without understanding the context. The term rational error "refers to process where student first induces an incorrect rule and then proceeds to follow it "correctly" in a logical consistent manner" (Ben-Zeev, 1996, p. 65).

In this current study our main focus is on the way of problem solving of our teacher trainees with special interest in inductive reasoning. Thus, we have formulated the following research questions.

Q1 Whether they use inductive arguments during the problem solving process or not?
Q2 If not, what are the frequent types of their reasoning?
Q3 What are the characteristics and the typical errors of their inductive reasoning process?

In order to answer these questions we constructed a problem which may be solved in different ways, among others using inductive strategy.

The problem

In Figure 1 $AC_1 = C_1B_1 = B_1C_2 = C_2B_2 = B_2C_3 = C_3B_3$.

1. If $\alpha = 15°$, find β.
2. How many isosceles triangles can be drawn following the algorithm presented in the figure?
3. For some α, we can draw exactly 9 isosceles triangles. Find α.

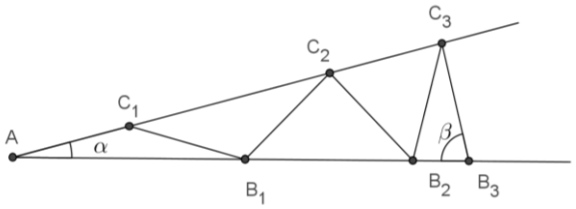

Figure 1: The initial problem

The first question requires only minimal geometrical knowledge; moreover, the completion is the only cognitive operation needed, where by completion we mean finishing arithmetic operations in this context. The second question tests whether the student could follow the algorithm given by the figure. Our hypothesis was that the third question should be a mathematical problem for our students. Since the solution is completely determined by the underlying geometrical structure, this problem is suitable for examination Phases 1-5 of inductive reasoning.

A possible strategy is based on two steps. (1) The nth isosceles triangle has angles with measure $n\alpha$ on its base. (2) If we can draw the 9th triangle, then $9\alpha < 90°$; moreover, we cannot draw the 10th triangle, thus $10\alpha \geq 90°$. It means $9° \leq \alpha < 10°$.

Our primary interest was in Step (1). If the student uses this general statement, how he or she concludes it. This "general approach", i.e. when we use a general n in the solution instead of a concrete number of triangles, may appear in all parts of the solution; however it is not necessary for this particular problem. The reason is that only near generalization is involved here, i.e. direct methods (drawing, counting, and determining all the angles) could be effective (Rivera, 2013).

We highlight here only one more question: how students deal with the last possible triangle? We briefly refer to this question as "condition for halt".

Dimensions and methodology of the research

In academic year 2015/16 we investigated the solution of the problem described above with involvement of 94 students, including 49 prospective primary school teachers and 45 prospective secondary and upper secondary school Mathematics teachers. Solving of the problem does not require advanced mathematical knowledge and skills. Thus, we do not distinguish between these two groups in our research. The base group consists of 83 students (S01-S83 in the transcripts). In this group we investigated students' written elaborations. The interview with 11 other students completed the frame of this research (S84-S94). During the interviews we followed students' activities and made sound records. Students were asked to say out loud what they are thinking of when solving the problem. We corrected numerical errors immediately; otherwise we did not put guiding questions.

Results: Students' activity during problem solving

Overview

Analysis of students' performance handling the third question represents the overall problem solving process well. We identified two classes of solutions (Figure 2). *Concrete solution class* means that student deals with 9 triangles only and sticks to the text verbatim. Because the third question is a near generalization of the previous one, this plan is acceptable. By *general solution class* we mean, that solution works for arbitrary number of triangles. Some students used more than one strategy. 18 students didn't show up any strategy, 5 of them ignored the problem, and 13 students could only compute the angle of the 5th triangle.

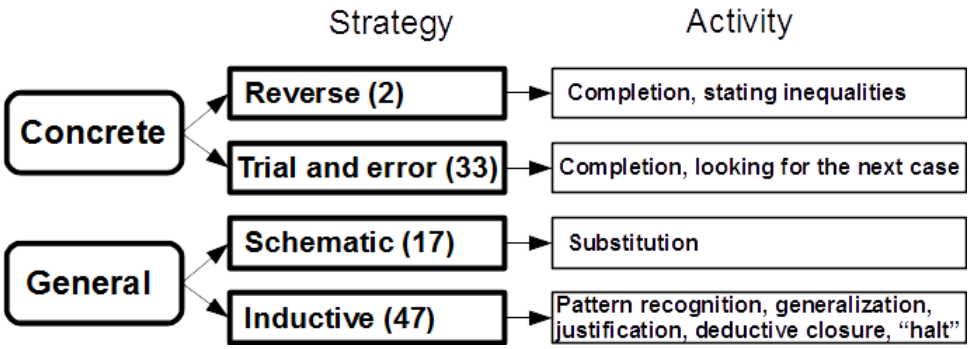

Figure 2: Strategies and activities with number of students following the particular strategy

Reverse strategy

By reverse strategy we mean here that student's starting point is the final configuration with 9 triangles. This is a successful approach, where the student investigates the figure with 9 and 10 isosceles triangles and computes all the necessary angles directly, with or without showing signs of pattern recognition. We encounter this approach in 2 interviews, but nobody completed the third question using this strategy in the base group.

Trial and error approach

Trial and error strategy is characterized by repeated, varied attempts which can be continued until success. Although this approach appeared 33 times, in most cases it played certain role in the inductive reasoning. 10 students applying this strategy did not show inductive or any other strategies; however, in 4 cases the activity was controlled with the (unproven) hypothesis that $n(\alpha)$ is a decreasing function, where n is the number of isosceles triangles. One student in the interviewed group followed this pattern. Her view demonstrates that trial and error could be a rational activity even for this problem. After reading the text, her first and immediate reaction was applying trial and error method. After two trials with angle measure 10 and 5 the interviewer interrupted her:

Interviewer: Do you think that the solution is an integer?

S86: Certainly.

Interviewer: Why?

S86: I don't know… It is a nice problem and the solution should be a 'nice' integer.

Interviewer: [He gave a hint that $\alpha \in \mathbb{R}$.] In what cases is trial and error effective?

S86: When we have small number of cases to check. [She gave up.]

The transcript points out the role of student's belief in the problem solving process (Schoenfeld, 1985). Theoretically she knows that her effort is hopeless, but her belief in 'nice' solution overwrites this knowledge. In this context the false trial and error strategy is a rational error here in the sense of Ben-Zeev (1996), because if α is an integer then we have finite number of integers to check. We detected 24 students with belief that the solution is an integer but in some cases with sign of uncertainty, e.g. "If we reject the condition that α is an integer, then we have infinite possibilities" (S37).

Schematic reasoning, false scheme

Schematic reasoning is the process of reasoning by which new information is interpreted according to a particular schema. In our problem the number of isosceles triangles is $n(\alpha) = \left\lceil \frac{90}{\alpha} \right\rceil - 1$, where α is the given angle. This function, to be more precise, some approximate idea of $n(\alpha)$ appeared in students' responses. First of all, $n(\alpha)$ is a decreasing function, and 15 students referred to this property properly or erroneously (i.e. in strict form) without proof or explanation. The following transcript demonstrates the typical usage of this observation. Previously this student settled that for $\alpha = 9°$ there are 9 triangles. "If $\alpha < 9°$, then the number of isosceles triangles is more than 9" (S10). We presume that the transformation reasoning (Simon, 1996) is behind this recognition. Some students showed explicit evidence of transformational reasoning. We encountered sentences like the following transcript 5 times in the base group. "If we decrease the angle, then we get more triangles" (S03). Students' observations are the result of the mental transformation of the angle.

In some cases it invoked the scheme of inverse proportionality or the misconception of *strictly* decreasing $n(\alpha)$ function. Two false solutions with inverse proportionality scheme appeared in our experiment and caused a rational error. Two other students referred to inverse proportionality, but later revised the idea.

Other false scheme was the direct proportionality scheme. Perhaps the following interpretation of the problem invokes it. "In case of 15° we have 5 triangles, how much is the angle if we have 9 triangles?" This is a common pattern in elementary word problems. Transcriptions of data demonstrated in Figure 3 strengthen this presumption. Data from the second question is not necessary to answer the third question, but students who applied the direct proportionality scheme connected data in this way. S13 misprinted the angle and used 30 instead of 15. S36 revised her outcome. S08 finds x by the 'proper' way: $5x = 9 \cdot 15$. He just began the division (the tick between digits 3 and 5 indicates this), but presumably rejects the result which he found too big and finishes the calculation "forcing" a more reasonable result. Direct proportionality appeared 5 times, but 1 student revised this solution.



Figure 3: Direct proportionality (left: S13, center: S36, right: S08)

Looking for a general solution using inductive strategy

We consider that the inductive strategy appears if a student reaches at least the first phase of the inductive reasoning process i.e. at least observes particular cases and looks for a possible pattern. Half of the students in this research (47 people from 94) used or tried to use this problem solving strategy. (Some of them used other strategies too.) Eight students stopped at the first stage because of the possible lack of near generalization ability. In the second phase (near generalization) the others determined the 9th and the 10th angle in the sequence in a way that they skipped some members and tried to transfer the "condition for halt" observed before. In our problem we didn't ask to formulate a general statement; however 11 students made it (far generalization, Phase 3). The statements were expressed either by symbols or by words, like S85 told "Thus the length of one step is equal to the opener." [The difference between the base angles in two consecutive triangles is equal to the given angle.][2]

We wondered whether the students feel the need of testing their conjecture by other particular cases or not (Phase 4). The following transcript represents this phenomenon well. After calculating $\alpha, 2\alpha$, and 3α S88 said: "I'm sure the result will be something similar. β equals probably 5α, but I will compute it." 13 solutions contained test of the near/far generalized conjecture.

Concerning Phase 5 (deductive closure) we confronted with the dilemma of "proving or not". The near generalization feature of the problem probably caused the fact that no one has felt the necessity of proving of the observed and applied conjecture. The following transcript represents a typical attitude during the interviews:

Interviewer: Why are you sure that the 9th angle equals 9α?"

S93: Because it was clearly visible, and I felt that it will work always in the same way.

The clue of empirical structural argument (Figure 4) appeared only in 5 works. Previously S88 determined the 5th angle without any skipping, after that she skipped to the 9th angle directly.

Here the recursive counting procedure confirms that the measure of the angle increases by α.

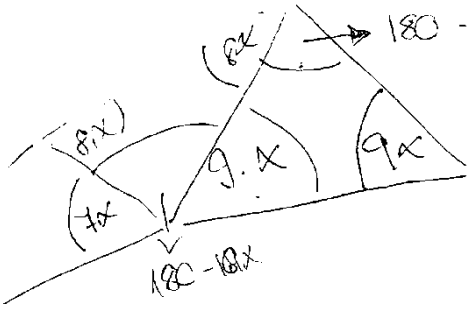

Figure 4: Empirical structural argument of S88

Four students were able to make a correct deductive closure of the inductive reasoning by mathematical induction proof after the interviewer asked them to prove their conjecture. One of them said "I can prove if you wish." (S93)

[2] Rephrased by the authors

Typical error during inductive reasoning: spurious abstraction from irrelevant feature

Solving the first and second problem, students have some previous experience in the third problem. In 18 solutions we found that they abstracted a false rule from a previous experience, what is more, from one particular case. We highlighted only a few examples here. In the third part of the problem 2 students used the same difference (i.e. 15) for the arithmetic sequence of base angles as in the first part of the problem. In 2 cases the starting point was that the measure of base angles of the last possible triangle always equals 75°. The most frequent spurious abstraction concerns the "condition for halt" (12 students). In Figure 1 $\measuredangle AC_3B_3 = 90°$ causes the halt. Generally this condition is $\measuredangle AC_nB_n \leq 90°$ (for the smallest n), but these students kept the equality instead of inequality. The following transcript is a typical answer to the third question: "$90/10 = 9$, because in this way the tenth triangle would have two right angles" (S16).

Findings and interpretation of results

The students involved in this research dealt with the presented problem in many different ways, and we detected many different solution strategies. Thus, we conclude that the chosen problem was an appropriate instrument to answer our research question in particular and to make some conclusions in general. We have summarized our findings for research questions as follows.

Q1 50% of the students used inductive arguments during their problem solving process.

Q2 In the other cases the most frequent type of their reasoning was trial and error strategy. Other strategies appeared, namely schematic, and reverse as well. Furthermore, we found that lot of students (19% in this research) did not go beyond the computational activity; they did not have any other idea. Yeo and Yeap (2009) describe the same phenomenon for weaker students.

Q3 We found an uncertainty in inductive reasoning: students formulated conjecture from a few particular cases; moreover, they did not confirm it and 95% of students did not make any form of deductive justification. They often abstracted a false rule from a previous experience, what is more, from one particular case. Students relied on their intuitions without doubt; and this behavior calls for rigid schemes. They often mixed or changed these strategies without any result.

Possible explanations of these findings are complex. First of all, our students are not familiar with heuristic strategies, especially with strategy for determining patterns. The recognized pattern which described the relation between the angles and the number of triangles was a plausible one in their mind instead of a definite pattern in situation with well-defined mathematical structure. Moreover, the common misconception appears in the interviews that particular examples prove a general statement. With respect to the function concept we conclude that it is not deep enough, students have difficulty with step function. In many cases our students had in mind natural numbers instead of real numbers, as possible values of an angle, which suggests that their number concept is very simple and/or their belief in "nice whole number" solution is very strong.

Acknowledgment

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Principles towards justification: A focus on teacher actions

Joana Mata-Pereira[1] and João Pedro da Ponte[2]

[1]Instituto de Educação, Universidade de Lisboa, Portugal; joanamatapereira@campus.ul.pt

[2]Instituto de Educação, Universidade de Lisboa, Portugal; jpponte@ie.ulisboa.pt

The aim of this paper is to understand how a proposed set of design principles regarding tasks and teacher actions provide students with opportunities to justify. We see justification as a reasoning process that relies on mathematical concepts, properties, procedures, ideas, and, in some situations, particular cases. The teaching intervention, part of a design-based research, is carried out in a grade 7 class of an experienced teacher in nine lessons about linear equations. Data is gathered by classroom observations (video and audio recorded) and a researcher logbook. Data analysis takes into account a set of design principles and a framework regarding students' justifications. The results show that paths of teacher's actions that rely on the design principles enable students to present rather complete justifications based on logical coherence and on mathematical aspects of the situation.

Keywords: Reasoning skills, teaching practices, teacher-student interaction.

Introduction

Developing students' mathematical reasoning is an important aim of teaching and learning mathematics. Students' engagement in reasoning processes allows them to move from using procedures with little or no understanding towards envisioning mathematics as a logical, interrelated, and coherent subject. We consider reasoning as making justified inferences (Brousseau & Gibel, 2005), using processes as formulating questions and solving strategies, formulating and testing generalizations and other conjectures, and justifying them. In this paper, we focus on justification as a central reasoning process. Enhancing students' mathematical reasoning in the classroom requires the set-up of challenging learning environments that go much beyond proposing students to solve exercises using well-known procedures. In order to better understand how teachers may foster students' mathematical reasoning, we conduct a design-based research (Cobb, Jackson, & Dunlap, 2016) that relies on whole class mathematical discussions triggered by exploratory tasks as privileged moments to promote students' mathematical reasoning. In this paper, we aim to understand how a set of design principles regarding tasks and teacher actions that focus on justification might promote students' justifications during whole class discussions.

Students' justifications

In the classroom, justifying, particularly justifying conjectures and generalizations, is a reasoning process that rarely emerges spontaneously. Often, students accept conclusions such as conjectures and generalizations without feeling the need to test or justify those (Henriques, 2010). In many situations, students focus mostly on what is familiar or on ideas that they superficially recall, paying little or no attention to the mathematical properties or concepts implicated (Lithner, 2000, 2008). However, justifying is a reasoning process central to mathematics learning, as it allows students to connect mathematical ideas, concepts, and objects, to present arguments to support statements and conjectures, to solve problems and to develop new mathematical ideas (Brodie, 2010). We consider

justifying as a reasoning process and as the way to prove statements by relying on concepts, properties, procedures, and mathematical ideas and, in some situations, on particular examples.

Justifications in the classroom can occur at different levels regarding formality and complexity. Brousseau and Gibel (2005) propose three different levels regarding the formality of a justification: Level A – Justification that is not formally presented, but that might be associated with the student's actions as a model of his/her action; Level B – A formal but incomplete justification with inferences based only implicitly in elements of the situation or on what is considered to be shared knowledge; Level C – A formal justification based on a sequence of related inferences, with explicit reference to the situation or to what is considered to be shared knowledge. The concept of formal justification referred in these three levels is not necessarily the same as a formal justification in mathematics related to a mathematical proof, but rather to what is considered to be formal in a specific situation, namely accordingly to the grade level and the knowledge of students. However, as students advance through their schooling, formal justifications should be increasingly more formal from a mathematical standpoint, being sometimes equivalent to proofs or to significant parts of proofs. Drawing upon the classifications of Lannin (2005) and Carraher, Martinez and Schliemann (2008), it is possible to consider six levels of complexity: Level 0 – no justification, when the answers do not include a justification; Level 1 – Appeal to external authority, when the justification refers to other individual or reference material; Level 2 – Empirical evidence, when the justification is based in particular examples; Level 3 – Logical coherence, when justification is based on logic; Level 4 – Generic example, when the justification is deductive, but stated in relation to a particular situation; Level 5 – Deductive justification, when the validity of the justification is based on a deductive argument that is independent from the particular cases or examples. At all these levels, a justification presented by a student may be correct, partially correct, or incorrect. Thus, it is important that the students understand what validates a justification and reject justifications based on authority, perception and common sense (Lannin, Ellis, & Elliot, 2011).

Tasks and teacher actions to enhance justification

Students learn to reason by "reasoning and by analyzing the achieved reasoning processes" (Ponte & Sousa, 2010, p. 32). Therefore, to enhance justification processes it is necessary to provide situations in which students have the possibility to justify their answers. As such, in mathematics teaching and learning, and particularly to enhance students' mathematical reasoning, a central aspect is the design of suitable tasks. It is important to understand the nature of those tasks, the ways students engage in them, and the interactions that may emerge (Brodie, 2010).

Several studies refer that problem solving and exploratory tasks have potential to develop students' mathematical reasoning (e.g., Francisco & Maher, 2011; Henriques, 2010). However, it is not necessary or even appropriate that all tasks involve questions at a high challenging level (Brodie, 2010). Such challenge may be infeasible due to time constraints and may lead to students' demotivation and loss of interest. Moreover, while designing a task, its structure and level of challenge should be considered according to the students to whom it is going to be proposed. In addition, we note that, just by themselves, exploratory tasks and problems, are not sufficient to foster students' mathematical reasoning. Teacher actions emerge as equally central to provide situations that promote students' mathematical reasoning. Regarding teacher actions to enhance

justifications, Bell (2011) highlights that the teacher should help students to make sense of justifications, ask for alternative justifications, emphasize what validates a justification, and focus on the explanation of "why". Also, it is important that the teacher encourages the students to share their ideas and various versions of their reasoning, seeking to consider students' incorrect or partial contributions and to broaden their valid contributions (Brodie, 2010).

Methodology: Design, participants, data analysis

This paper originates from a broader research study that aims to develop a local theory about enhancing students' mathematical reasoning in the classroom, following a design-based research (Cobb et al., 2016). In order to do so, we established several design principles (Cobb et al., 2016), i.e., heuristics that structure the intervention, based on the research literature and on a previous cycle of experimentation focusing on tasks and teacher actions to enhance students' mathematical reasoning. Four of these principles specifically focus on justification. One principle refers to task design and states that tasks must include questions that ask for a justification of answers or of solving processes. The other three principles concern teacher actions and indicate that the teacher must promote situations that prompt students to (a) justify and present alternative justifications; (b) identify valid and invalid justifications, indicating why; and (c) share ideas, namely by accepting and valuing incorrect or partial contributions, deconstructing, supplementing, or clarifying them.

This is the third cycle of design, after a first cycle that took place in lessons about sequences and a second cycle in lessons about linear equations. This third cycle took place in a public school in a grade 7 class with 27 students (12-13 years old), throughout nine lessons about linear equations. A detailed plan of each lesson was prepared considering tasks specifically designed to promote students' mathematical reasoning and considering possible teacher actions. Each lesson plan was proposed by the first author and discussed in detail with the teacher, who made all the amendments and adjustments that she felt necessary considering the class characteristics and the available resources. The participating teacher was selected because of her experience and her availability to consider changes on her practice. All participants in this study are volunteers, provided an informed consent, and their names are fictitious.

Data analysis is centered on the design principles regarding tasks and teacher's actions and also on students' justifications. The episode that we present is from lesson eight that aimed to lead students to be able to relate equations and functions. This lesson was directly observed and video and audio recorded and notes were made in a researcher's logbook.

An episode about equations

Task and context

The episode presented in this paper focuses on the first part of the task proposed in lesson eight of the nine lessons that constitute the linear equations unit. This segment of the task (Figure 1) aims to lead students to establish a procedure to figure out the intersection point of two functions. Earlier in the school year, the students learned about algebraic and geometric representations of linear functions, with no participation from the researchers.

Frances received a plant as a gift and she registered its growth. Simon thought it was a really nice idea and, on the same day, bought a plant and also registered its growth. The functions that follow represent the height of both plants on their first days with the students:

Frances' plant: $f(x)=0.4x$ Simon's plant: $s(x)=0.2x+2.2$

1. Represent graphically the functions *f* and *s*.
2. Based on the previous representations, indicate on which day the plants have the same height.
3. Consider the comment: "Graphs are not necessary to know on which day the plants have the same height. Knowing the functions that represent the height of each plant is enough to find out when they are equal". What would be the other approach to figure out the day when the plants have the same height? Justify your answer.

Figure 1: Proposed task about functions and equations

In the first two questions of this task, the students can support their answers by using GeoGebra app, as this particular school has iPads available by request. This was not the first time that the students used either the tablets or GeoGebra. Taking into account the design principle regarding task design, question 3 asks for a justification.

Justifying based on knowledge about functions

At the beginning of the lesson, the teacher asks a student to read the questions to the class and clarifies the aims of the task and the tools to use. Then, students work autonomously, in pairs, for a couple of minutes. After inserting the algebraic expressions of the functions in GeoGebra, some of them state that the plants have the same height on the 11th day. The teacher begins the whole class discussion by asking for a justification to that answer:

Teacher: How do you know that it is on the 11th day? (Several students raise their hands in order to answer.) Isa.

Isa: Because, if we check, both straight lines intersect in 11.

The teacher's invitation to justify (principle (a)) led Isa to justify her answer to question 2 based on her previous knowledge about functions. This justification is incomplete regarding referencing "in 11", however, it refers to elements of the situation, namely, the graph representations of both functions and the intersection point. Thus, Isa presents a generic justification regarding the available data (level 4 justification).

Aiming to complete Isa's answer (principle (c)), the teacher revoices this student's answer leading to a more accurate justification:

Teacher: In 11...

Isa: In point 11.

Teacher: In point 11?

Gabriel: Abscissa.

Teacher: In the point with abscissa 11.

By referring to parts of students' answers, the teacher implicitly identifies what is invalidating the justification (principle (b)), and based on the students' answers, the teacher highlights what completes the justification (principles (b) and (c)).

After validating Isa's answer, the teacher decides to go further on justifying, asking for another justification (principle (a)):

Teacher: Why am I looking at the intersection in *x*-axis . . . For the value in *x*-axis?

Isa: Because *x*[-axis] is the axis of objects...

Teacher: Yes… And how do I know if I am looking for an object or looking for an image?

Isa's justification relies on mathematical concepts (level 3 justification), however, her statement is not sufficient to provide a justification in this specific situation as it does not relate to the context of the problem. Once again, the teacher validates a partial contribution from Isa and encourages the students to complete that contribution (principles (b) and (c)). Another student tries to justify, but he does not add any information to what Isa has already said. Then, Gabriel participates in the discussion:

Gabriel: I think it is because the height is in… In... I just forgot the name.

Teacher: The axis…

Gabriel: The ordinate, the ordinate axis, and the days are in the abscissa [axis].

At this point of the discussion, Gabriel adds some relevant information to the justification, by relating objects and images of these functions to the context of the situation (level 4 justification). Despite this relevant relation, the required justification is still incomplete, and so the teacher continues on encouraging students to justify (principle (c)):

Teacher: And how do I know that days, in this particular case, are objects and heights are images?

Gabriel: Because there is… I forgot it…

Leonardo: Why it is that way, isn't it? Let me reason the other way around… If the height would be there [in *x*-axis]…

As the students struggle to address the teacher's question without being able to justify (level 0 justification), the teacher gives some more information in order to complete the justification (principle (c)):

Teacher: What do the functions *s* and *f* represent?

Several students: Height.

Teacher: Plant's height, right? Depending on what?

Several students: Time.

Teacher: The time that elapses, in days. OK, very well.

This information provided by the teacher leads the students to easily identify dependent and independent variables, thus completing the required justification (level 4 justification).

Both this and the previous justifications in this segment rely on students' prior knowledge about functions and emerge during the whole class discussion supported by the teacher's actions based on the defined design principles.

Justifying based on knowledge about equations

Right after discussing question 2, the teacher introduces question 3. At this point, a student immediately proposes a strategy to solve this question. This leads the students to engage in a new segment of whole class discussion, without having time to work autonomously on this question:

Teacher: Now, pay attention to question 3, because… (Santiago raises his hand). Tell me.

Santiago: So teacher, we have that thing that was G.C.D. (M.D.C. in Portuguese), I believe it was… Multiple (regarding the M in M.D.C)…

As the teacher allows Santiago to intervene, he brings to the discussion a strategy based on a mathematical concept that was not expected in this situation. Despite seeming a senseless idea, the teacher lets him go on with his explanation (principle (c)):

Teacher: Greatest common divisor?

Santiago: Yes, something like that. Can't we use it to answer to when do they intersect? . . . I can't recall it, but wasn't there something in common? Doing each number and then…

By allowing Santiago to justify it is possible to understand that, despite incorrect, the student's justification relies on an idea with some logical coherence (level 3 justification). Thus, both in G.C.D. and in intersecting functions one is trying to find "something in common", as he refers. At this point, the teacher poses more questions in order to deconstruct the misconceptions about G.C.D. which leads the other students to identify Santiago's strategy as not fitting to this situation.

After clarifying that, Clara presents her strategy:

Clara: We can use an equation (referring to $0.4x=0.2x+2.2$), and the number that we get is the day they have [the same height].

. . .

Teacher: What are you expecting as a solution of this equation?

Several students: 11.

Teacher: 11. So, confirm that.

Evoking the information obtained in the previous questions, the teacher supports Clara's strategy to solve this equation and, by asking to confirm the result, she prompts the students to justify (principle (a)) that 11 is the solution of the mentioned equation. Students do this in autonomous work, and then Daniel intervenes:

Daniel: Teacher, it isn't.

Teacher: It isn't? So, solve the equation over there (on the board).

As Daniel solves the equation on the board, the teacher realizes that he has just missed an x in one of the steps and, by following his solving process (principles (b) and (c)), the justification based on procedures is properly achieved (level 4 justification).

In this segment of the discussion, justifications, either valid or invalid, are based on knowledge about mathematical concepts. These justifications emerge when teacher's actions rely mostly on

encouraging students to share ideas and completing those ideas.

Conclusion

All the situations in the episode that we analyze were prompted by the proposed task. This task, focused on making sense of the relationships between equations and functions, provides an opportunity for students to develop a procedure to find where two functions intersect. This underscores the idea that collective activity in whole class discussions enable students to share, debate and clarify their reasoning and, in particular, their justifications (Galbrait, 1995).

This study shows that, if particular teacher action paths that rely on the design principles are followed, justifications are likely to emerge in whole class discussions. In this episode, when the principle regarding asking for a justification is enacted, the students present justifications. These justifications are based on previous knowledge about mathematical concepts or ideas or on known mathematical procedures. Thus, those are justifications based on logic or deductive justifications stated in relation to a particular situation. However, these justifications are often incomplete and sometimes incorrect, and, as it has been seen in previous research (Galbrait, 1995), the use of available information about a certain mathematical concept or idea is not always adequate given the definitions or assumptions of the task. When the justification is incomplete, the teacher tends to encourage the students to complete the justification, validating or invalidating their statements only implicitly. Depending on her appraisal of the support that the students need to mobilize their knowledge, the teacher provides them with more or less information. By relying on these principles, the complete justification emerges from the whole class discussion. When an invalid justification is at stake, and according to the defined principles, the teacher values students' contributions and keeps on encouraging them to present their ideas, leading them to present justifications based on logical coherence or on mathematical procedures. In these situations, where a student's justification is incorrect, teacher's actions strive either to abandon that justification and to focus on an alternative justification or, if possible, to adjust it to its correctness.

In this particular episode, students' justifications, despite sometimes incomplete or invalid, tend to be reasonably formal as they are based on mathematical aspects of the situation. Also, in the context of a whole class mathematical discussion based on the design principles, those justifications emerge often as justifications in a logical coherence level and, as students continue to add information, those justifications became generic example justifications. As this episode illustrates, in order to provide students with opportunities to move in-between levels of justification, it is not enough to ask students to justify and validate their justifications, but also to accept and value partial and incorrect justifications. Thus, the presented paths are likely to provide promising environments to develop students' justifying abilities, hence to be better prepared to deal with mathematical proof later in their schooling.

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A *reference* for studying the teaching of logic

Zoé Mesnil

Université Paris Est Créteil, Laboratoire de Didactique André Revuz, France; zoe.mesnil@u-pec.fr

This paper presents a work developed in my thesis on the teaching of logic in high school in France. The current official instructions specify that teachers don't have to make a mathematical logic course, but have to help their students develop a relevant use of some notions of logic as tools. Therefore, I point out in this paper that in order to take this constraint into account in the study of didactic transposition process, it is relevant to describe it from a reference knowledge for logic, but that such knowledge has never been established by the mathematics community. In order to offer such a reference, I base myself on a double epistemological and didactical study in which I favor the links between logic and language. I will also explain the choices made for this reference and its use as a methodological tool through the example of quantifiers.

Keywords: Logic, teaching, didactic transposition, quantifiers.

Introduction

In France, mathematics syllabuses for high school (students between 15 and 18) mention explicit goals concerning certain notions of logic. For example, they recommend that "students [be] trained on examples to wisely use the universal and existential quantifiers (the symbols ∀ , ∃ are not required) and to identify the implicit quantifications in some propositions, particularly in conditional propositions". These current recommendations come after twenty years during which the logic was excluded from syllabuses. This exclusion itself is a reaction to the too abstract and formal aspect of modern mathematics (taught in France between 1969 and 1981) for which mathematical logic and set theory were basic elements. They strongly highlight the logical notions as tools and even show distrust against their feature as objects. This distrust can be interpreted as a resistant mark of turning down modern mathematics. The syllabuses specify that "the concepts and methods of the mathematical logic should not be subject of specific courses but must naturally take place in all chapters of the syllabus". I described more precisely the characteristics of the conditions and constraints due to these official instructions in a contribution to CERME 8 (Mesnil, 2013): teaching goals are ill-defined and subjected to strong constraints. Moreover, mathematics teachers do not share a common reference for interpreting these syllabuses. Indeed, mathematical logic is not part of the domains necessarily studied by a mathematics student, bringing a diversity of teachers' knowledge, that is usually not filled during their training in which teaching of logic is often only superficially and quickly addressed.

In this contribution, I would like to pursue this issue of a reference for the teaching of logic. In my thesis, I studied the teaching of logic in high school as the result of a didactic transposition process, and I'll explain first why the nature of mathematicians' logical knowledge requires thinking this transposition not from a *mathematical knowledge* like mathematical logic, but from a *reference knowledge*, nonexistent for the moment in the mathematical community (Mesnil, 2014). Secondly, I will justify the choice I made then to construct a methodological tool to conduct the analysis of syllabuses and textbooks. This tool is a *reference* in which the notions of logic are presented from three points of view that root them deeply in mathematics, in mathematical activity, and in the

classroom. I will conclude by illustrating the choices I made for this *reference*, and its use through the example of quantifiers.

The need for a *reference*

During the didactic transposition process (Chevallard, 1985), a mathematical object is identified in a body of knowledge (in French, *savoir savant*), and a first succession of adaptations will make it able to become an object of teaching in a particular institution. It is then identified by mathematics teachers in the knowledge to be taught (in French, *savoir à enseigner*) and is subjected to a second succession of adaptations to become a taught object.

But except in some university courses, teaching logic does not mean teaching mathematical logic, but the logic at work in mathematical activity, which supports mathematicians' expression and reasoning. We can consider that mathematicians have a logical knowledge, which may be the subject of a didactic transposition process, but this knowledge is more visible in practices than in treaties. Mathematical logic can be perceived as a description of the principles of this logic. It was then explicitly taught during the time of modern mathematics, in France and elsewhere, but these experiments showed that this teaching, conducted formally and isolated, did not help pupils to express themselves and to reason (Adda, 1988). Several researchers in mathematics education now agree that such teaching should be explicitly linked together with mathematical activity in which the logic is omnipresent (see for example Epp, 2003; Durand-Guerrier, 2005). However, there are few research studies on the effect of teaching logic on reasoning and expression capacities, which is still an open issue.

Thus, to take into account this particular nature of the logical knowledge and its connection to mathematics, and to appropriately study the didactic transposition process of this knowledge, I propose to describe this transposition not from a mathematical knowledge, but from a reference knowledge. I take this notion from Rogalski and Samurçay (1994) who thus characterize a knowledge produced by practices, but decontextualized from situations where knowledge is apparent into action. These authors state that it is necessary that this reference knowledge can "be expressed with its concepts, its methods, its systems of representation and its language" (*ibid*, p. 46). However, concerning notions of logic, such a reference knowledge does not appear in that there is no corpus collecting the logical knowledge necessary to mathematical activity and about which everyone agree the choices of concepts which are featured and of their representation.

I then conducted a study to answer the following research question: what kind of reference knowledge would be epistemologically and didactically relevant to the logic teaching? This study allowed me to construct a *reference*, which I then used to analyze the knowledge to be taught in high school in France. I call this analysis tool *the reference*, and not reference knowledge, because the production of a knowledge falls under a long and collective process.

The importance of language for reasoning, and its links with logic

The study of various logical systems across the ages allowed me to identify invariants and differences in the role assigned to logic and in the ways that are given to it to fulfil it. The study of didactic studies shows how issues relating to the teaching of logic meet the concerns and choices of these logicians.

All these logical systems are built from a work on language. The concept of proposition is primordial in them all. Aristotle describes it in terms of subject-copula-predicate, and it was not until Frege at the end of the XIXe century that this analysis was to be replaced by an analysis in terms of function and argument allowing two things essential for mathematical language: on one hand to consider predicates with several arguments, on the other hand to pull out the act of quantification from the proposition by making it expressed by quantifiers which act on variables. Predicate logic that is then born is able to model mathematical propositions. From a didactic perspective, several research studies have shown many examples of situations in which the predicate logic is a relevant reference for didactic analysis which allows to highlight the importance of issues on quantification (Durand-Guerrier, 2005).

The current language of mathematicians is inspired by Frege's formalism, but it isn't a strict use of a formal language. Focusing particularly on problems of language in mathematics teaching, Laborde (1982) showed that there is a particular use of language in mathematics, due to the interaction of the two codes of symbolic writing and natural language. This interaction allows mathematicians to use reformulations useful for conceptualization. Teachers are familiar with the particular features of mathematicians' language, but they can cause difficulties for students who "discover together the concepts and the way we talk about them" (Hache, 2015, p.28).

The common goal of the studied logical systems is to ensure the validity of reasoning, with a preliminary work on language. But for the authors of *The Logic of Port Royal*[1], logic above all needs to be trained and the formalization of reasoning is seen as an obstacle to the use of intuition, whereas for Leibniz and Frege on the contrary, logic must provide a system of signs in which reasoning can be expressed, and this formal expression guaranteeing its infallibility. Gandit (2004) denounces the excessive place taken by the formal aspect in the beginning of proof learning. But being careful about formalization at the time of the discovery of deductive reasoning does not mean that it cannot subsequently help those who begins to have a good practice of it. Thus, in higher education, Selden and Selden (1995) suggest presenting theorems and definitions in an informal formulation, which allows intuitive understanding, and in a formal language, which allows linking structure of the statements and structure of its proof.

Organization of *the reference*

In the *reference* I proposed, I decided to give a broad place to language. Entering into logic by the language is consistent with the epistemological study, granting it an important place is consistent with didactic studies. Of course, in the same way as it is done in the studied logic systems, especially in a didactic perspective, the study of notions of logic as components of mathematical language has to be articulated with their use in reasoning.

Finally, these epistemological and didactic studies led me to propose a *reference* in which the presentation of logical concepts combines three approaches:

[1] Name of a famous french treatise, which original title is « La logique ou l'art de penser », written in 1662 by A. Arnauld and P. Nicole, who were very influenced by Descarte's method.

- The mathematical logic. It is a recent branch of mathematics that can be considered as a result of what has been sought by different logical systems creators since Greek antiquity. Mathematical logic seems so particularly suitable as a formal reference to describe the logic at work in mathematics.
- The study of the language practices of mathematicians. In this way, the presentation of the logical concepts is rooted in mathematical activity taking into account how they are expressed in mathematical discourse, using predicate logic to uncover some complex and sometimes ambiguous formulations that are yet a part of the language practices of mathematicians, widely imported in the mathematics classroom.
- The research in mathematics education. In this way, the presentation of the logical concepts is rooted in mathematics classroom, taking into account the difficulties that the complexity of these notions can bring for students.

In this *reference*, the components of mathematical language are shown, beginning with the primordial notions of proposition and variable. Then for the connectives AND and OR, implication, negation, quantifiers, I have consistently adopted the three approaches mentioned[2]. Although the focus is on the language, reasoning is of course not absent from the *reference*. A difficulty for pupils and students is to distinguish, in a text of a proof, mathematical propositions concerning mathematical objects, and parts of the text which allow to follow the progression of reasoning, such as variable introductions, or justification of an inference. The confusion between implication and deduction falls under this type of difficulty.

The example of quantifiers in the *reference*

As announced, the *reference* contains first a presentation of logical concepts from mathematical logic. Predicate logic uses two quantifiers: applied to a variable x, and from a proposition P, the universal quantifier allows to obtain the proposition $\forall x$ P, and the existential quantifier allows to obtain the proposition $\exists x$ P (description of the syntactic aspect of the quantifiers: they operate on a variable and a proposition to build a new proposition).

Let E be a set in which the variable x can take values. The proposition $\forall x$ P[x] is true[3] when for any element a of E, the proposition P[a] is true. The proposition $\exists x$ P[x] is true when there is at least one element a of E such that P[a] is true (description of the semantic aspect of quantifiers: truth conditions of a quantified proposition). Quantifiers have an important effect on the variables: a variable that is in the scope of a quantifier is a dummy variable in the quantified proposition, and this proposition does not give information on the object designated by the variable, but on the set in which it can take its values.

Some important results on quantified propositions may be established by a semantic way, using the sense, as well known equivalence between NOT($\forall x$ P[x]) and $\exists x$ NOT(P[x]), or the fact that if $\exists y$

[2] For each of these themes I have summarized the mathematical content, the language related issues, and research findings on student difficulties.

[3] This semantic characterization can be described as "naive" because I do not strictly define what "being true" means. But of course, this characterization may be more rigorous with the notion of satisfaction of a formula in a model introduced by A. Tarski.

$\forall x$ P[x,y] is true, then $\forall x \ \exists y$ P[x,y] is true. These results are then used in syntactic manipulations, independent of the sense, in the same way we manipulate algebraic equalities.

In mathematical language, the quantifiers are a way to express the quantification, but there are many others. We can see this through some examples of mathematical propositions[4]:

1) Le carré d'un nombre réel est positif (The square of a real number is positive)
2) Le carré d'un nombre réel est toujours positif (The square of a real number is always positive)
3) Tous les réels ont un carré positif (The square of any real number is positive
4) Tout réel x est tel que x^2 est positif (Any real number x is so that x^2 is positive)
5) Pour tout réel x, x^2 est un réel positif (For all real number x, x^2 is a positive real number)
6) $\forall x \in \mathbb{R}$, $x^2 \geq 0$

They are several formulations of the same property, but universal quantification is expressed very differently. In proposition (1), quantification is implicit, implied by the word *un* (translated with *a*). We frequently use the indefinite article *un* to mark a universal quantification, in everyday language as in mathematics. But *un* is also sometimes used to mark an existential quantification, which is obviously confusing! Sometimes, the two usages coexist in the same proposition, such as in "un réel positif possède une racine carrée" ("any positive real number has a square root", in English, the first *un* is rather translated with *any* and the second *une* is rather translated with *a*, and there is no confusion). In proposition (2), the adverb *toujours* (always) is used to explicitly mark this universal quantification, as the word *tous* (any) in proposition (3). Propositions (4) to (6) are distinct from the first by the use of a variable. Furthermore, one can identify in each of these propositions an expression which express the quantification (here universal quantification) and which has the property that it can be separated from the proposition "x^2 is positive" (or equivalent formulation). Such expression works as quantifiers of mathematical logic, and I therefore also calls them *quantifier*.

Finally, we saw that in the language practices of mathematicians, quantification can be implicit or explicit, and in the second case, possibly marked by a quantifier which is an expression observing syntactic rules of use. Propositions (4) and (5) may seem closer to propositions (1) to (3) as they are formulated "with words" contrary to the proposition (6) which only uses mathematical symbols, and that may seems much more formal. I would like to stress that such a vision hide formalization still existing in these propositions, in the sense of a shaping according to certain rules, even if that formalization is not accompanied by a symbolization.

I will conclude by mentioning some of the difficulties of high school students or senior students in related to the use of quantifiers. First, the implicit quantifications are not always perceived by students. The case of the universal quantification associated with the implications and the formulation *if ... then ...* is highlighted for a long time (Durand-Guerrier, 1999). Quantification is often encapsulated in stiffened structures (for example, "u_n is as big as we want by taking n big enough") that the expert mathematician knows how to reformulate by explaining the quantifications, but these reformulations in more formal language tend to disappear from the language used in high school, and

[4] I give the examples in French first because I will explain some difficulties linked to the word un which is used in this language in different meanings, and which is translated in English with *one*, with *a*, with *the*… Commentaries following the examples refer to the French expression.

are a source of difficulty when students meet them in higher education. Another difficulty concerns the failure to take the order of quantifiers into account when there is an alternation. We know that students have rather an interpretation *for all... there exists...* even though they are facing a proposition *there exists... for all...* (Dubinsky & Yiparaki, 2000). Furthermore, Chellougui (2004) showed student difficulties with the use of an existential proposition. In a proof text, there is generally a confusion between the affirmation of the existence of an element checking a property, and the act which consist to consider it and give it a name. Likewise, mathematicians do not get trapped by the "dependence rule" in the statements *for all... there exists...* and identify easily this error in a student production. However, they do not necessarily explain this error to the students by linking it to formal rules of manipulation of variables and quantifiers (Durand-Guerrier & Arsac, 2005).

Examples of the treatment of quantifiers in school textbooks

The *reference* I have developed allows an analysis of resources available to teachers highlighting sensitive issues that need to be paid attention to. I will then conclude with an analysis of two extracts from school textbooks.

The reintroduction of logical concepts in the syllabuses had an effect on the textbooks: those published in 2010 for the first class of high-school (15 years old students) all contained passages identified as speaking of these concepts. Nine textbooks out of ten have chosen to dedicate a few pages (between one page and nine pages) to notions of logic, usually located at the beginning or at the end of the textbook (only one textbook does it in a disseminated way). Moreover, they all contain exercises with a stamp "logic" (from ten to fifty-four exercises in the studied textbooks).

In the eight textbooks that deal with quantifiers, the letters are introduced by examples. Seven of the eight textbooks give only examples of true quantified proposition, and this choice eliminate the syntactic aspect of quantifiers: quantifiers are used only to affirm something, there isn't the idea of a proposition built with a quantifier that one could wonder whether it is true or false.

The textbook *Indice* gives the example of the proposition "le carré d'un réel est positif" ("the square of a real number is positive") and states that "cette proposition est vraie quel que soit le nombre réel" ("this statement is true for all real numbers"). The authors of this textbook probably want to emphasize on the various possible meanings of the word *un*, but do not offer at the same time an example of the meaning as existential quantification. Moreover, to know that the meaning of *un* in this proposition is a universal quantification, it is necessary… to know that the universal proposition is true! Mathematical knowledge is therefore needed to decide between the two possible meanings of the word *un*, which calls for caution when using this word in a context where student knowledge is potentially fragile. Let's go back now on the comment "this statement is true for all real numbers". It makes no sense to say that the proposition "the square of a real number is positive", which is equivalent to "for all real x, the square of x is positive" is true for all real numbers, since the variable x is dummy in this proposition. The proposition referred to in this commentary is not the quantified one, but the not quantified proposition "the square of x is positive". Finally, there is a confusion between the use of "quel que soit" ("for all") to simply mark the universal quantification, and its use to assure that this universal proposition is true.

Now let's look at an example of exercise, taken from the textbook *Repères*, but it is an exercise that is found in many textbooks. Students must "complete the sentences (for example "… real number

x… *f*(*x*)>0") using either *for all… we have…* or *there exists… such that…*" from the graphical representation of the function *f*. Note first that the application is not explicitly to complete so that sentences are true! Furthermore, the instruction "complete using either… or…" suggests that each time only one of the both quantifiers is correct. Yet, when the proposition "for all *x* P[*x*]" is true, the proposition "there exists *x* such that P[*x*]" is also true, so when it is possible to complete with the universal quantifier, it is also possible to complete with the existential quantifier. In everyday language, we respect the principle of maximum information, according to which we give to our interlocutor all information in our possession. So, if I say "on my holidays, it rained some days", I say in the same time that it did not rain every day. The practice of this principle leads us in this exercise to complete naturally with the universal quantifier when possible. However, the notion of truth of a proposition will be contradicted by saying that using the existential quantifier is a mistake, because in mathematics, when the proposition "for all *x* P[*x*]" is true, it is not "more true" than the proposition "there exists *x* such that P[*x*]". Some students, however, adopt this position, and we can doubt position of the authors of the teacher's textbook who offers as a correction only the universal quantifier when it is possible.

Conclusion

I presented in this paper a methodological tool, a *reference* to study the teaching of logic. It is of course to be completed, to be improved, both from an epistemological and from a didactic point of view. Important work remains in particular on the concept of proposition, generally not made explicit in teaching, and on the notion of variable (didactic of algebra is very concerned about the status of the letters, but it seems to me that a logical point of view on the concept of variable, such as taking in Epp, 2011, or as I suggested in Mesnil, 2014, is more unusual), especially to identify students difficulties with these concepts that can be related to their epistemological complexity, or their use in the classroom.

Moreover, I have used for the moment this *reference* to analyze the syllabuses and textbooks, and a training for teachers who offers a similar approach of notions of logic. But it could also be used to study the practices of teachers, students' activity and conceptions. It would be particularly interesting to compare the effect of knowledge in mathematical logic that teachers have or have not.

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Revisiting Odysseus' proving journeys to proof: The 'convergent-bounded' question

Andreas Moutsios-Rentzos[1] and Faidra Kalozoumi-Paizi[2]

[1] University of the Aegean, Greece; amoutsiosrentzos@aegean.gr

[2]KJPTh, Kinder- und Jugendwohnheim Leppermühle, Germany; faidra.kp@gmail.com

In this paper, we investigate the cognitive and affective task-specific experiences of Odysseus, a mathematics undergraduate, as he attempts to answer to an exam-type proving question: the convergent-bounded question. The concurrent investigation of Odysseus proving strategies and his basic emotions appears to help in gaining deeper understanding about his proving experience.

Keywords: Proof, proving strategies, emotions, examinations, thinking styles.

Cognitive and affective aspects of proving

The notion of proof lies at the heart of modern mathematics (Thurston, 1994) and of mathematics education research (Furinghetti & Morselli, 2009; Reid & Knipping, 2010). In this paper, we focus on the cognitive and affective task-specific proving experiences (drawing upon Moutsios-Rentzos, 2015). Researchers have identified different proving strategies that the students employ when facing with a proving task (Weber, 2005), while others have investigated the type of the argument utilised in a proof (Inglis & Mejia-Ramos, 2008). Considering the affective aspects of proving, famous mathematicians stress the pleasure that a proof brings (for example, G. H. Hardy; Hoffman, 1998), which is in contrast with the reality as pictured by mathematics undergraduates (Rodd, 2002) and with the gloomy in-class mathematics experience, with 16-year old students reporting that "I hate mathematics and I would rather die" (Brown, Brown & Bibby, 2008, p. 10).

Emotions "give information about progress, or ability to progress, relative to goal states and anti-goal states" (Skemp, 1979, p. 18) set by an individual. The pleasure that derives from our dealing with a task is linked with our concentrating our cognitive efforts to solve it (Changeux & Connes, 1998). It is argued that research should attempt to co-consider cognitive and affective aspects of a proving experience (Furinghetti & Morselli, 2009). Moreover, we draw upon the idea that a theory may act as a meaningful attractor (Moutsios-Rentzos, 2015) of the different methodological-theoretical perspectives investigating a phenomenon. Furthermore, since the assessment process is strongly linked with the learning outcome of any educational system (Boud & Falchikov, 2007), we focused on the exam-type proving questions that all mathematics undergraduates undertake. Considering that being successful in exams is a highly goal-oriented activity, we adopt a theory developed for such experiences which also addresses both cognitive and affective aspects: Skemp's (1979) theory of social survival and internal consistency. Consequently, we address the question: *What are the affective and cognitive task-specific experiences of a mathematics undergraduate as he attempts to produce an exam-acceptable answers in an exam-type proving question?*

Theoretical – methodological approach

Skemp (1979) theorised that the learners survive both socially and internally. They survive socially by meeting the socially accepted, (usually externally) set criteria of a task (for example, exams), whilst they survive internally in the sense of achieving consistency within their internal reality (for

example, by satisfying their inner need for being creative or for identifying and following the rules), which crucially includes both cognitive and affective aspects. Hence, considering that producing exam-acceptable answers is essentially a goal-oriented activity, Skemp's theory is employed to give meaning to both aspects of the investigated phenomenon: proving strategies and basic emotions.

The students' proving strategies refer to the students' answering a proving question, rather than reflecting upon an answer. The A-B-Δ proving strategy classification scheme (Moutsios-Rentzos, 2009) was utilised to identify the students' qualitatively different proving strategies when they deal with exam-type questions. At the crux of the scheme lies the potential tension between proving to oneself and proving to others (respectively, ascertaining and persuading; Harel & Sowder, 1998). The scheme has been developed explicitly for exam-type proving questions (see Moutsios-Rentzos & Simpson, 2011), corresponding to well-known classifications, such as Weber's (2005) syntactic–semantic–procedural proof constructions, or the deep–surface–achieving/strategic approaches (Zhang, Sternberg & Rayner, 2012). Five strategies are identified organised in three types. In the α-type strategies (A & Δ_A), the students demonstrate a need to first investigate whether the given statement makes sense. Once an ascertaining argument has been chosen, a persuading argument is employed, thus potentially separating ascertaining from persuading. In an A (alpha) strategy, the ascertaining argument is appropriately 'mathematised' to serve as a persuading argument, whereas in a Δ_A (delta-alpha) strategy persuading appears to constitute a completely new process. In the β-type (B & Δ_B), the students immediately commence the persuading process, without pondering whether the given statement is meaningful to them or not. In a B (beta) strategy, the students attempt to recall either the proof of the statement or a proof that may serve as a template for proving the given statement, whilst in a Δ_B (delta-beta) strategy, the students concentrate their efforts on producing symbolic mathematical expressions to construct an exam acceptable proof. Finally, in a δ-type (Δ_Δ; delta-delta), the focus is on producing a proof that would get the maximum grade in exams, through symbolic mathematical expressions based on a variety of means (including, theorems, images and examples). The students may investigate whether the given statement makes sense, but only for their facilitating their mathematical expressions producing pursuit.

In this study, emotions refer to a state of alertness that mobilises the human body with respect to a stimulus, including psychological and neurophysiological effects (Oatley & Jenkins, 1996). These emotions are clearly differentiated from the mentally processed, socially situated, affective reactions towards a proving situation (Hannula, 2012). Ekman identifies seven evolutionally derived basic emotions that are universally manifested in the humans' facial expressions (Ekman & Friesen, 1978): sadness, anger, contempt, fear, happiness, disgust, surprise. Thus, we attempt to map the reflexive affective exam-type proving experiences. Certain combinations of micro-movements of the facial muscles are linked with specific basic emotions as described in the 'Emotional Facial Action Coding System' (EMFACS; Ekman, Irwin & Rosenberg, 1994). Considering emotions and conviction, a positive affective state is linked with more superficial and/or authority-based judgements, whilst a negative/neutral affective state is linked with more thorough judgements, reducing the effect of authority (Oatley & Jenkins, 1996). Nevertheless, these studies mainly refer to judgements, rather than to multifaceted mental productions such as proof.

Overall, in this study, we discuss the proving cognitive and affective experiences of a mathematics undergraduate, Odysseus, as he deals with the exam-type proving question "Let a sequence $(a_n) \in \mathbb{R}$,

$n \in \mathbb{N}$. Prove that if (a_n) is convergent, then (a_n) is bounded" (*'convergent-bounded'*). In Moutsios-Rentzos (2009), it was posited that the students' general thinking preferences reveal aspects of their inner realities, thus affecting their initial strategy choices. Their back-up strategy choices indicate that the ineffectiveness of the initial attack lead them to re-evaluate the given situation and to choose a strategy that more appropriately fits with this new experience of the situation. In Moutsios-Rentzos and Kalozoumi-Paizi (2014), a small part of those data (of Odysseus) was subjected to additional analyses to illustrate the advantages of the synchronous mapping of cognitive and affective experiences as he dealt with six proving questions. In this study, we concentrate on only one task that Odysseus dealt with to elaborate on his affective-cognitive task-specific experiences.

Odysseus proving experience of the 'convergent-bounded' question

Odysseus' proving strategies were identified through video-recorded clinical interviews (in the sense of Ginsburg, 1981), in which he was asked to produce an exam-appropriate proof and to think aloud during that process. Since the focus was on the choice of means, Odysseus would be provided with any mathematical information (including definitions, figures) he would need (in line with Weber, 2001). During the think aloud process, his emotions were identified through the video-taped proof productions by an EMFACS trained and certified researcher. Following Ekman, all the emotions and emotional blends (more than one emotion in a single instance) were interpreted within the context they occurred. Finally, the Odysseus' perceived internal and external reality is reported (by identifying his mathematics attainment, thinking dispositions and understanding of exam-acceptable answer) to gain deeper understanding of the findings.

Odysseus' experienced realities: thinking styles and exam views

Odysseus was an above average attaining, 2nd-year student, attending a 4-year BSc-equivalent degree in Mathematics in a Greek University. Considering Odysseus' broader experienced internal or social realities, his thinking styles profile (i.e. his broad thinking dispositions; Sternberg, 1999) was identified as 'ground breaking' (expected to prefer creative, original and non-prioritised thinking; Moutsios-Rentzos, 2015). Considering his views about exams and exam-acceptable answers, Odysseus concentrated mainly on the peripheral aspects of their answer: the *amount of information*, the *language* used, the *structure* of the solution, and the *aesthetics* of the presented proof. Considering 'amount of information', he wondered: "*Hmm ... this is one of my greatest problems when I write down a solution ... should I ... Do I have to prove this? […] and when I know something and it doesn't have a name whether I should describe it ...*". Considering 'language' and 'structure', Odysseus noted that an exam-type proof should be axiomatically based, written symbolically in a linear form, since a proof presented this way was considered to affect positively his grade. Furthermore, he was particularly concerned about the 'aesthetics' of the presented proof, stressing: "*Presentation is very important ... that is why I use draft first [...] If I had more time, I would spend 10 or 15 minutes on figuring out how exactly I would present it* ".

Odysseus' Alpha (A) proving strategy to the convergent-bounded question

In the following excerpt, Odysseus employs an Alpha strategy to deal with the 'convergent-bounded' question. He reads the question and then he tries to produce a 'draft' solution. Odysseus tries to 'reconstruct' the definition, 'giving meaning' (Pinto, 1998) to his concept image.

Odysseus: I'll 'create' it ... I usually don't remember the formulas ... I 'create' them ... but ...

Researcher: Do you want me to tell you the definition?

Odysseus: Err ... in exams, if this [the interview] is a simulation [of exams] ... I would not remember it [the definition] ... I would try to 'create' it ...

Moreover, Odysseus draws upon his concept image to generate hypotheses and to validate these hypotheses. He conceptualises convergence as something 'constraining', evident both in his verbal and non-verbal communication, which suggests the meaningful interplay between concept image and concept definition (typical of an Alpha strategy).

Odysseus: ... well these ε and n_0 must have a relationship ... for every ε I should be able to find a n_0 ... not the way I have put it ... [many gestures].

Researcher: Do you say that based on your memory? Or ...?

Odysseus: No! I don't say that based on my memory, I say it 'logically' ... I mean ... I say for every $n>n_0$ there exists $\varepsilon>0$ so that $|a_n-a_{n0}|<\varepsilon$ … this is what I have written ... [He makes gestures as he talks that 'show' what he talks about.] … But this should be true for everything ... the ε ... there is a an infinite number of ε that are suitable ...this is true ... therefore I need something more 'constraining' ... therefore this [the writings] does not describe convergence, because convergence is something that is constraining ... it converges [gestures] to a specific number...

Furthermore, Odysseus' images are not pictorial, but 'fuzzy' and he likes to call them 'thoughts'.

Researcher: Do you have a specific 'image' in your mind?

Odysseus: No, I don't have it as a picture. I have it as ... I would call it 'thought'...

Once Odysseus is satisfied with the definitions of the mathematical notions included in the statement he is asked to prove, he focuses on proving it (see Figure 1, definition). For Odysseus, it is crucial that the statement that he wants to prove is what he terms as 'logical'; that it *makes sense*. He *needs* to be convinced that the statement makes sense, before he tries "to solve it".

Odysseus: Convergent ... belongs to $\mathbb{R}$ ... ok ... it begins from an a_0 and it goes to something else [Gestures] ... therefore ... logically ... if it is let's say in a straight line ... it would be from here ... here there would be something that 'blocks' it ... unless it goes up and down ... but since it converges somewhere it will reach somewhere that ... it might follow a different route that might go like this or like that ... I don't mind ... it will reach here ... the route has an end ... and therefore ... it is 'logical' that it is bounded ... and so we will try to solve it.

In this process, Odysseus draws upon his concept image, which is evident from his gestures and figures: the straight line ('a', Figure 1) that denotes the real numbers and the boundaries he draws (on the left and right of this line; 'b', Figure 1); the curved lines ('c', Figure 1) denote the potential 'routes' the sequence might follow from 'a_o' (the first term of a_n) to 'a' (the limit of a_n).

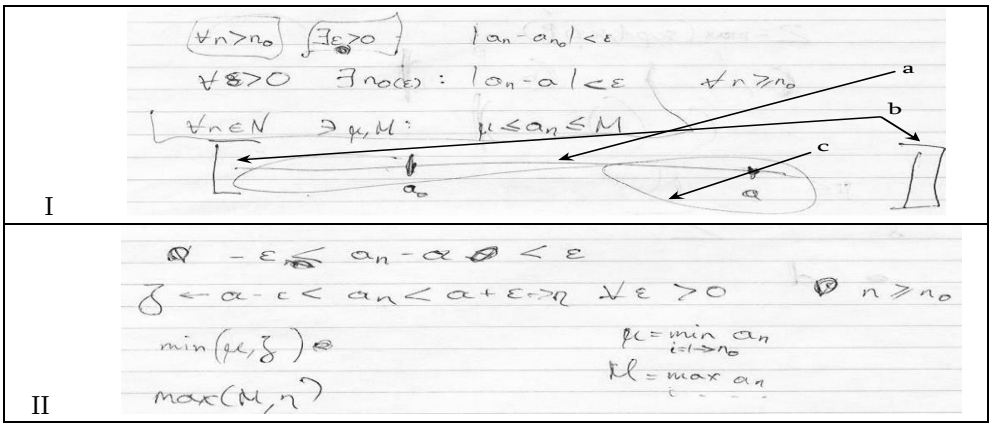

Figure 1: Odysseus' 'draft' definition (I) and answer (II)

Odysseus builds on the above to 'solve' the question. He looks at the definition of the convergent sequence, expands the inequality using the property of the absolute value, reaches a double inequality and stops to explain his rationale:

Odysseus: […] Until n_0 it is a finite set ... so it would have to be a finite segment [He shows it on the curved lines] ... from λ to κ [gestures] ... and after that [n_0] we have shown that they are all under a+ε for every ε I have chosen, right? ... so we have a big boundary that reaches to n_0 and after that all will be bounded somewhere else ... they could be here [he shows segments with his hands] ... or there ... or inside .... so let this be η and this be ζ [Figure 1] and we will have that the minimum ... it's not still a proof... we will have that the minimum of μ and ζ … no value will be below this [minimum] ... where μ is the minimum of a_i, i from 1 to n_0 ... and M is the maximum of a_i ... from these two [μ and ζ] it will be the minimum value ... it cannot be less than this... similarly for the maximum ... now what I have to do is to write this mathematically, but *for me it is already finished*...

Odysseus' argument convinced him of the 'truth' of the statement he is asked to prove. As he presents his argument, Odysseus draws upon his concept image using gestures to generate and validate his argument. For example, for validation, Odysseus is certain of the validity of this 'proof': for him "it's already finished". His certainty appears to derive from his image manipulation and his gestures: at first, he notes that his argument "is not still a proof", but subsequently he claims that "it cannot be less than this". At the same time, he acknowledges that this argument cannot be *presented* as a proof and that he needs "*to write this mathematically*". His mathematised argument is close to the original argument and though the ascertaining argument draws upon his concept image, his mathematised argument is a 'translation' to a mathematically 'acceptable' language.

Finally, it is noted that, Odysseus' 'formal' proof was carefully structured like a textbook proof based in axioms and definitions (unlike the less linear, based on image manipulation 'draft' proof).

Emotions in proving according to EMFACS

Time	Emotions (EMFACS; Ekman et al, 1994)	Excerpt	Answering phase
'Draft' answer			
12:59:84	Sadness	Researcher: So would you like to give it a couple of tries first and then ... Odysseus: Yes	Definition construction
14:27:00	Sadness-Anger	Odysseus: But this should be true for everything ... the ε ...	
14:27:60	Sadness-Contempt		
14:56:64	Fear	Researcher: Do you have a specific 'image' in your mind?	
14:56:80	Happy -Fear	Odysseus: No, I don't have it as a picture. I have it as ... I would call it 'thought'...	
14:57:16	Happy		
17:11:03	Happy	Odysseus: and therefore ... it is 'logical' that it is bounded ... and so we will try to solve it	'Truth' investigation
21:54:24	Sadness-Anger	Odysseus: more or less I am done […] How much time do I have left?	
'Formal' answer			
36:50:80	Sadness-Anger	Odysseus: How much time do I have left?	Beginning
37:04:88	Contempt	Odysseus: …and in exams there are many similar problems [such as time constraints]	
39:22:72	Contempt-Anger	Odysseus: I'll write it in a different way … it is not essentially different	Writing-up
44:03:24	Contempt-Anger	Odysseus: I'll write it down differently [instead of writing down two more lemmas]	
44:29:20	Happy-Contempt	Odysseus: In mathematics if you can avoid too many variables it is better, because … in the end you get lost…	
45:09:96	Contempt-Sadness	Odysseus: Because I consider it [a theorem] as given… it might be silly of me, but ..	

Figure 2: Odysseus, emotional journey to proving the 'convergent-bounded' task

The results of the EMFACS analysis are outlined in Figure 2 along with the corresponding excerpt and answering phase. Odysseus' positive emotions are few, mainly linked with his mathematical ideas: when he describes them as 'thoughts' or when they make sense. His negative emotions or emotional blends are predominantly linked with his attempting to meet the requirements of an exam situation: time constraints, appearance, amount of information included in the formal answer. In line with the rationale of differentiating amongst different proving strategies, Odysseus' emotions can be differentiated between *internally referenced* (linked with his inner reality; Skemp, 1979) or *externally referenced* (linked with the perceived by Odysseus social reality of the given situation, including the exam-status of the given questions). For example, Odysseus in his 'draft answer' manifested an internally referenced 'happiness' emotion (17 min) when convinced of the truth of the statement (ascertaining): "It makes sense to me that it is bounded and so I'll try [to prove] it". In contrast, in the end of his 'draft answer', when he completed the persuading process, he expressed an externally referenced sadness-anger blend (21 min), because the moment he realised that "more or less I am done", he almost immediately wondered "How much time do I have left?". His emotional clash is in line with his cognitive clash due to his tendency for choosing more α-type strategies (potentially differentiating ascertaining from persuading), linked with his ground breaking thinking styles profile (Moutsios-Rentzos, 2009).

Concluding remarks

In this study, we investigated the proving strategy and the emotions of a mathematics undergraduate, Odysseus, as he dealt with an exam-type question. Skemp's theory of internal consistency and social survival helped in gaining deeper understanding of the concurrent phenomena. A complex proving reality was revealed, diversely affecting Odysseus' experiencing a need for constructing a proof (Zaslavsky, Nickerson, Stylianides, Kidron & Winicki-Landman, 2012). His negative emotions were linked with the externally experienced communication of the answer, whereas his positive emotions were linked with the internally referenced success in finding a proving argument. Emotions are *non-verbal, facially expressed reflexes*, indicating Odysseus' emotionally interiorising of his previous proving experiences. The presented approach complements existing studies based on language and/or introspection (Furinghetti & Morselli, 2009), by revealing the students' real-time emotional states. It is stressed that the identified emotions are affected by the thinking aloud protocol and, thus, a current project is focussed on identifying the students' emotions as they prove without thinking aloud and on their evaluating written proofs. Overall, the proposed line of research may help in designing pedagogies reinforcing the positive affective aspects of proving, thus promoting the students' deeper engagement with proving, which is expected to facilitate their developing a fully-fledged internal need for proof.

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Play and pre-proving in the primary classroom

Maria Pericleous[1] and David Pratt[2]

[1]UCL Institute of Education, United Kingdom; mariapericleous19@gmail.com

This report focuses on a research study the aim of which is to investigate the activity of proving as constituted in a Cypriot classroom for 12-year-old students. By drawing on Cultural Historical Activity Theory, this study explores the way the teacher is working with the students to foreground mathematical argumentation. Analyses of video-recorded whole class discussions show how explaining and exploring provide a key pathway for the development of proving. We refer to these developments as pre-proving. However, inherent contradictions within explaining and exploring hinder the constitution of proving in the classroom.

Keywords: Proof, exploration, explanation, play, CHAT

Introduction

It is now acknowledged that proof and proving should become part of students' experiences throughout their schooling (Hanna, 2000, Yackel and Hanna, 2003, Stylianides, 2007). However, secondary school students as well as undergraduate students face difficulties when giving formal mathematical arguments. At the same time, research that shows how upper primary school students approach and construct proofs is still limited (Stylianou et al, 2009). It is also argued that argumentation, explanation and justification provide a foundation for further work on developing deductive reasoning and the transition to a more formal mathematical study in which proof and proving are central (Yackel and Hanna, 2003). But what is meant by proof and proving? Mathematical argumentation is a discursive activity based on reasoning that supports or disproves an assertion and includes the exploration process, the formulation of hypotheses and conjectures, explaining and justifying the steps towards the outcome and the proof of the statement. Thus, proof is at the core of mathematical argumentation, as a justification, an explanation and a valid argument.

Research has responded to the need to conceptualize proof and proving in such a way that it can be applied not only to older students but also to those in elementary school (Stylianides, 2007). The challenge remains however to understand how proof and proving is shaped by the practices in the mathematics classroom. This is in accordance with Herbst and Balacheff (2009), who argue that the focus should not only be on proof as the culminating stage of mathematical activity, but also on the proving process and how this is shaped by the classroom environment. Thus, in understanding how proving is constituted in the classroom, a wider network of ideas is required as these ideas no doubt have an impact on how proof in the narrow sense is constituted.

To address this issue, we refer to pre-proving, that aspect of mathematical reasoning that might nurture proving. What are the roots of proving? Given that proof is both a justification and an explanation, it can be argued that emphasis should be placed in these two aspects of mathematical reasoning. In considering those functions of proof that are considered important for school mathematics (Hanna, 2000), evidence has been reported that the establishment of sociomathematical norms (Yackel and Cobb, 1996) for explanation and justification (Yackel and Hanna, 2003) might foster deductive reasoning in the classroom. That is, describing, conveying and exchanging ideas through the act of communication, explaining and justifying statements influences

the appearance of proof and the transition from unsophisticated empirical arguments to the level of sophistication that might be expected at the tertiary level. It is through exploration and investigation that all these elements surface and develop in the process of proving. Thus, when discussing the roots of proving, exploration, which activates intuition and encourages thinking, constitutes another notion that should be taken into consideration. Thus, pre-proving refers to those elements that direct mathematical reasoning towards the ultimate goal of formal proving; that is exploration, explanation, justification and communication. In the social environment of the classroom, where hypothesizing, explaining and justifying conjectures is encouraged, the tools and tasks used, the rules of the classroom, the way the students work together, the way the teacher negotiates meanings and other external factors all interact, interrelate and influence each other in forming classroom activity. The purpose of this study is to explore pre-proving and proving in the elementary mathematics classroom and the way the structuring resources of the classroom's setting shape this process.

CHAT based theoretical constructs

As this study is exploring the various forces that impact on the activity of proving, Cultural Historical Activity Theory (CHAT) is being employed as a descriptive and analytical tool alongside collaborative task design (a means of gaining access to the teacher's objectives), to capture the interaction of different levels, such as the actions of teachers, students and the wider field as evidenced in curricula and research documentation. The analysis and discussion in this paper draws upon the following CHAT perspectives: (i) the object of the activity and (ii) the notion of contradictions. Initially, the unit of analysis in CHAT is an activity, a "coherent, stable, relatively long term endeavor directed to an articulated or identifiable goal or object" (Rochelle, 1998, pp.84). The object of a collective activity is something that is constantly in transition and under construction, has both a material entity and is socially constructed and its formation and transformation depends on the motivation and actions of the subject indicating that it proves challenging to define it. Among the basic principles of CHAT is the notion of contradictions. Contradictions are imbalances, ruptures and problems that occur within and between components of the activity system, between different developmental phases of a single activity, or between different activities. These systemic tensions lead to four levels of contradictions (Engeström, 1987). This conceptualization, should be differentiated from mere problems or disorienting dilemmas from the subject-only perspective as they are more deeply rooted in a sociohistorical context (Engeström, 2001). Contradictions are important because they may lead to transformations and expansions of the system and thus become tools for supporting motivation and learning. This paper focuses on a primary contradiction on the teacher's object. The primary contradiction can be identified by focusing on any of the elements of the activity system (subject, tools, object, rules, community, division of labor). For instance, within the mathematics classroom, the clash between the teacher's goal of teaching a specific content of the mathematics curriculum and her need to continually manage student behavior and maintain focus, leads to a primary contradiction within the system's subject (the teacher).

Data collection and analysis

This study was conducted in a year 6 classroom in a primary school in Cyprus. Apart from the researcher, the participants were the teacher, a Deputy Principal at the school who endorses the integration of technology in teaching mathematics, and 22 students (11-12 years old) of mixed abilities. Even though using computers was part of the classroom's routine, the students were not familiar with dynamic geometry environments, DGEs. The data collection process as relevant to this paper included video data from the classroom observations and field notes. The content of the curriculum covered during the classroom observations was the area of triangles, and the circumference and the area of circle. The overall process of analysis of the collected data was one of progressive focusing. According to Stake (1981, pp.1), progressive focusing is "accomplished in multiple stages: first observation of the site, then further inquiry, beginning to focus on relevant issues, and then seeking to explain". The systematization of the data led to the evolution of two broad activities: (i) the activity of exploration including the exploration of mathematical situations, exploration for supporting mathematical connections and exploration of the DGE and (ii) the activity of explanation which focuses on clarifying aspects of one's mathematical thinking to others, and sometimes justifying for them the validity of a statement. These activities were then interpreted through the lens of CHAT, by generating the activity systems of both exploration and explanation. Achieving this also made possible the identification of tensions.

There is insufficient scope in this short paper to consider in detail these various levels and so this specific study focuses on illustrative episodes, which were generated during classroom discussion, to show one aspect of how the teacher was working with the students to foreground mathematical argumentation. To elaborate more, while the teacher was endeavoring to provide opportunities for exploration and investigation, it was observed that the teacher would sometimes interrupt this exploration. This interruption was often followed by the teacher either translating students' exploration as playing and/or was providing the step that needed to be followed. This paper focuses on the teacher using the word 'play' as part of the activity of exploration. The relevance of this emphasis of the paper lies in the connection that exists between exploration and play. That is, analyzing 'play' provides important information in portraying the activity of exploration and identifying the way this might influence the activity of explanation, and thus, shed light on how proving is constituted in the classroom.

Results

This section provides a chronological overview of the protocols that illustrate the teacher intervening in the classroom by using the word 'play'.

Protocol 1

On the first lesson related with the area of triangles, the students are expected to say the area of rectangles presented on the interactive whiteboard and make explicit the way they worked towards the answer.

Teacher: 12 again … but why are your playing? We are not doing something on the computers now. Stop.

In this protocol, the teacher is relating exploration of DGE with ‘play’. That is, exploration was interpreted by the teacher as ‘playing’ instead of learning.

Protocol 2

After demonstrating on the interactive whiteboard how to construct a rectangle in which the triangle is inscribed, the students worked in pairs and constructed rectangles on a DGE. When they finished, the teacher asked:

Teacher: Now that you constructed the rectangles, can they help you to find the area of the triangles?

Students: Yes.

Teacher: What is the area of triangle PRS?

Student1: 3.

At this point, the teacher interrupted the classroom discussion as she was concerned with a student ‘playing’ with the computers.

Teacher: Student2 you are still talking. You are playing all the time and I will move you from the computers.

In this protocol, the teacher is relating exploration of DGE with ‘play’. That is, exploration was interpreted by the teacher as ‘playing’ instead of learning.

Protocol 3

On the third lesson, after revising the mathematical formula for the area of triangles, the students would construct triangles with specific areas on the DGE. Before the teacher demonstrated to the class the steps followed so as to construct a triangle on the DGE, she said:

Teacher: Now we will go back to the DGE to play with its features. I will give you instructions and you will construct … to see how it operates and then we will move on to a game where you will play in pairs on the computers online.

In this protocol, the teacher relates both exploration of the DGE, and playing a game on the computer with ‘play’, something encouraging and constructive.

Protocol 4

On the first lesson related with circle, after defining circle, the class moved to the computers. Before engaging in tasks so as to explore mathematical relationships related with circle, the teacher introduced the class to a new DGE. Eventually, the teacher, referring to a circle presented on the DGE made the following comment:

Teacher: I can make it bigger or smaller. Look what we will do next. We will play later. Construct a circle and move it. Click on the center. Did you all do it? Nice. Stop.

This protocol focuses on the teacher relating exploration of DGE to ‘play’, something that has a positive value.

Protocol 5

On the second lesson, after revising the definition of a circle, the teacher asked the students to tell her the mathematical relationships explored the day before. At this point, several students could not give an answer. The following comment comprises the teacher's interpretation of the hesitation these students had in participating in the classroom discussion:

Teacher: You shouldn't only play but concentrate and listen in the classroom.

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration was interpreted by the teacher as 'playing' instead of learning.

Protocol 6

In the following part of the lesson, the class engaged in discussing ways in calculating the area of circle. Among the students' ideas was to count the squares inside a circle. However, it was concluded that this could prove difficult to achieve. Other students hypothesized that the area might be equal to circumference times radius. Others said that the area could be equal to circumference times diameter. The teacher encouraged them to investigate and test these hypotheses while exploring a task on a DGE, by making the following comment:

Teacher: I will leave you for a while to play.

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration is translated as something encouraging and constructive.

Protocol 7

During the third lesson that followed the exploration of the mathematical formulas of the circumference and area of circle, the teacher asked the students to find the radius and area of a circle with a given circumference.

Student1: But how?

Student2: I do not understand.

At this point, the teacher interpreted the queries the students had as a result of 'playing' with the computers.

Teacher: We came up to some conclusions. We have been working on the computers for two days now. We should not only play but also find …

Through classroom discussion, the students were able to use the mathematical formula, separate the variables, use division and find the radius and area of a circle with a given circumference.

In this protocol, the teacher is relating exploration of DGE with 'play'. That is, exploration was interpreted by the teacher as 'playing' instead of learning.

Discussion

Analysis of the above protocols indicates that the word 'play' as used by the teacher has differing connotations. This leads to the emergence of two contrasting values, play/learn.

Initially, one value the word ‘play’ entailed was related with the teacher interpreting exploration as ‘playing’ instead of learning. That is, while the teacher would encourage the students to explore an activity in order to reach some conclusions, she would also make a negative comment about this exploration as something that had no didactical value. In protocols 1 and 2, the teacher is relating exploration of DGE that preceded the classroom discussion with ‘play’. That is, exploration was interpreted by the teacher as ‘playing’ *instead* of learning, though, in protocol 2, the student was still exploring the DGE, so as to construct the triangle. In protocol 5, the teacher translated the fact that some students could not really summarize the work which was done previously as ‘play’ instead of learning. Furthermore, in protocol 7, the teacher is relating the exploration of DGE (as in protocol 6 which is described subsequently) with ‘play’, something that has no didactical value. What is striking is the fact that the teacher is referring to the DGE tasks that were designed in collaboration with the researcher in such a way that could initiate the formation of hypotheses and mathematical argumentation. This intervention was followed by the teacher guiding the classroom discussion.

The word ‘play’ had an opposite value when used by the teacher to refer in a general way to the exploration of the activity. In protocols 3 and 4, exploring the features of the DGE and working in pairs for the construction of triangles, and investigation of mathematical relationships accordingly, is translated as something encouraging and constructive. What should be noted though for protocol 4, is that it appears that exploring the environment by following the teacher’s instructions has more value than the students exploring the environment themselves, which is called ‘play’. In protocol 3, the teacher announces that the students will have the opportunity to play a game on the computer. In this protocol, the word play is used with its authentic meaning, even though, from an educational and didactical perspective, it can be considered as a form of reflection, evaluation and further understanding. In protocol 6, the teacher is encouraging investigation and exploration of a mathematical situation that would lead to explanation and justification.

The ‘play’ dichotomy relates to the notion of the play paradox (Hoyles and Noss, 1992) and the notion of the planning paradox (Ainley et al, 2006). Hoyles and Noss (1992) introduce the notion of the play paradox to describe the multiplicity of paths that are available to students when using a tool in an exploration related with a mathematical task. That is, the students, through their exploration, might not encounter the mathematical ideas that were perceived as the objectives set by the teacher or the curriculum materials. Thus, the teacher may decide to close down an exploration opportunity as she may interpret students’ exploration as shifting away from her own objectives. In a similar way, Ainley et al. (2006) call the conflict that may occur in the daily mathematical classrooms, due to contextualize tasks as the planning paradox. This tension may also be related to the notion of ownership as perceived by Papert (1993) in his formulation of Constructionism. That is, while the students are provided with the necessary tools to participate and to take ownership of the learning process, the teacher is at the same time attempting to avoid facing these paradoxes.

Considering the dichotomy related with the word ‘play’ through the CHAT constructs, this tension is a manifestation of a primary contradiction. The primary contradiction that emerges is inherent in the component related with the object of the activity system. In the activity of exploring as part of pre-proving, the object for the teacher is related with exploring triangles and circles. At a first glance, this object seems to be clear and distinct. However, this object is multifaceted. To be more precise, the object for the teacher is related with the investigation of situations that lead to

conclusions related with the aforementioned parts of the mathematics curriculum. The teacher on one hand understands the importance of providing enjoyable exploring opportunities that keep students' motivation and interest to engage with the problem. As a result, the teacher provides opportunities that can be approached by the students in their own way. On the other hand, students, through the exploration of these opportunities are expected to reach those conclusions regarding triangles and circles as pre-determined by the teacher. The two poles of the object lead to a constant struggle in the teacher's everyday practice. The teacher, due to this multifaceted object, is faced with the play/learn dichotomy and thus the play and the planning paradoxes. That is, students' free exploration may lead to paths other than those expected by the teacher. This initially shows that the students share the teacher's object. Thus, the object related with exploring is being reached. However, if the exploration moves away from the teacher's motive, the teacher will inevitably close down the exploration opportunity and guide the students towards the exploration that leads to the conclusions that satisfy her. Time management and the pressure of the coverage of the curriculum further highlight this tension. Inevitably, even though closing down the exploration is necessary, the object will not be met because of this contradiction.

Manifestation of this contradiction leads to a clash between the activity of exploration and explanation and, subsequently, with the way pre-proving activity occurs in the classroom. It has been illustrated that pre-proving activity is closely connected with exploration and explanation. That is, those aspects of reasoning that appear to have the qualities of proving, even though they may not be proving in themselves, entail exploration and explanation that provide a point of reference for proof production. Correspondingly, the object of developing proving in the classroom is related with these notions. The object of the central system of pre-proving activity is related with exploration that leads to explaining and justifying for a specific part of the mathematics curriculum. However, closing down the exploration has an impact on how explanation and justification are established in the classroom. Furthermore, closing down an exploration opportunity may have a negative impact on the students' ability to approach the construction of a proof. Referring to exploration as 'play' may also have a negative impact on students' confidence in relying on their intuitions when exploring a situation.

Concluding remarks

The aim of this paper was to shed some light on the area related with the activity of proving as constituted in the naturalistic setting of the mathematics primary school classroom. The elements that drive pre-proving activity and influence the way proving may be established in the classroom have been identified. That is, in mathematical argumentation, pre-proving is coming out of reasoning through exploring, explaining and justifying and can lead to proving. This paper reports on a teacher whose object is related with exploration that leads to explaining and justifying. However, this object is being conflicted as while a play-like exploration can facilitate learning, this can prove quite challenging for the teacher, as she wishes to maintain focus and is worried that exploring detracts from that focus. The contradiction between emphasizing exploring and maintaining focus is one of the tensions which make the constitution of pre-proving in the classroom inherently complex. However, this does not tell the whole story. Exploring opportunities that were closed down were exploited so as to negotiate and establish socio-mathematical norms in the classroom. As these norms are related with the very nature, functions and characteristics of

proof and proving, they can lead to explaining and justifying. Consequently, their establishment strengthens the activity of explanation and thus, the activity of proving.

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Proof-based teaching as a basis for understanding why

David Reid and Estela Vallejo Vargas

University of Bremen, Germany; dreid@uni-bremen.de, vallejo@uni-bremen.de

The importance of proofs as a way to gain understanding has been observed many times. In this paper we show the result of two different experiences with division of natural numbers. The first comes from children in grade 3 who have learned about division and divisibility through what we call proof-based teaching (PBT), and the second comes from students who just finished their school studies and intend to become preservice primary teachers. Our main aim is to point out how different school experiences might lead to different (divergent) ways of gaining insight into the relationship between the divisor and the remainder. We particularly focus on describing some elements we identified in the third graders' instruction that might have allowed them to articulate their own understandings.

Keywords: Arithmetic, elementary school mathematics, number concepts, proof.

Proofs as a way to gain understanding

The goal of proof-based teaching is that students gain understanding through proving. Hence, it is based on past work on the role of proof as a means to understand or explain.

> In mathematics education, explanation and understanding go together. The goal is understanding, so any explanation offered is aimed at having someone understand *why* a mathematical claim is true. This implies that a proof, to be useful in the classroom, should embody explanation. It should show not only that a result is true, but also *why* it is true. It should be concerned not only with its conclusion, but also with its main ideas, its overall structure, and its relationship to other mathematical fields and concepts (De Villiers, 2004; Hanna, 1990, 2000) (Hanna, 2016, p. 2).

Hanna (2016) discusses a number of different views of what makes a proof explanatory, but for our purposes, one aspect of these views is important. Explanatory proofs make reference to what we call a key notion, but which is also called a "characterizing property" or "salient feature". "An explanatory proof makes reference to a characterizing property of an entity or structure mentioned" (Steiner, 1978, p. 143). "A proof can be explanatory only if 'some feature of the result is salient' and the proof builds upon that salient feature (Lange, 2014, p. 489, cited in Hanna, 2016, p. 4).

Elsewhere (Vallejo & Ordoñez, 2015; Reid, 2011) we have suggested that proof-based teaching (PBT), in which students learn mathematics through explanatory proving that builds on a shared body of knowledge, offers an opportunity for the development of relational understanding.

In the following we first elaborate on the elements of proof-based teaching based on our experience of a 3-year design research with third graders. We then show examples of the understandings of division of the third graders after a short unit of proof-based teaching instruction. Finally, we contrast these understandings with those of students who have completed secondary school and are about to begin university studies and draw some conclusions for teacher education.

Proof-based teaching

Reid (2011) proposed proof-based teaching as "a way to develop understanding of mathematical concepts" (p. 28), and Vallejo has elaborated this idea in a 3-year design research intervention with third graders, the main goal of which was constructing division and divisibility knowledge.

The first intervention took place in 2013 in a Peruvian public school. This intervention was framed in the context of a master thesis (Ordoñez 2014) for which Vallejo was the supervisor. The second intervention took place in 2014 with a different group of third graders in the same public school. It addressed weaknesses identified in the first intervention through observing the difficulties students encountered in the lessons. We will report on the third intervention in the next section.

In the three interventions Vallejo taught all the sessions as a guest teacher in the classroom of another teacher. Written classwork assignments and quizzes were collected which helped the researchers to assess the students' progress in their knowledge construction. All the sessions were videotaped, and significant parts of the first and third interventions were transcribed. In all three cases, the students had no prior knowledge of these topics at the time the interventions began as the goal was to see knowledge being constructed.

Elements of proof based teaching

Through this research several elements of proof-based teaching have been identified as important: a 'toolbox' of shared knowledge, an expectation for explanation, and deductive explaining.

The toolbox

In order to prove students must share a common set of accepted principles. A 'toolbox' of such principles is an essential feature of PBT and this also reflects the practice of professional mathematicians. We adopt the term "toolbox" from Netz (1999) who uses the term to describe the set of theorems and assumptions that are used in classical Greek proofs without explicitly referring to them. Thurston, (1995) describes the same phenomenon in contemporary mathematical practice:

> Within any field, there are certain theorems and certain techniques that are generally known and generally accepted. When you write a paper, you refer to these without proof. … Many of the things that are generally known are things for which there may be no known written source. As long as people in the field are comfortable that the idea works, it doesn't need to have a formal written source. (p. 33)

In the interventions, Vallejo assessed prior knowledge through an individual diagnostic test, but more importantly, she established through a class discussion three "key notions" related to division and divisibility. These provided "a framework of established knowledge from which to prove" (Vallejo & Ordoñez, 2015, p. 231). The three key notions are:

> Fair distribution: Distributions must give the same number of objects to each person.
>
> Maximum distribution: The maximum number of objects possible must be distributed.
>
> Whole distribution: Each person must receive a whole number of objects.

These key notions were the basis for the proof-based teaching of division and divisibility employing a mixture of written (individual and groups) tasks and class discussion.

An expectation for explanation

From the very beginning, Vallejo's students were accustomed to being asked 'why?' for every conclusion they made or in general for every answer they gave based on the "key notions". "In the course of the sessions students also gave incorrect answers. Occasions of this type were exploited to promote discussion and justification by students since they were the ones who corrected the answers" (Ordoñez 2014, p. 334). She established in this way "an expectation that answers should be justified within this framework" (Vallejo & Ordoñez 2015, p. 231). It became part of the didactic contract (Brousseau, 1997) established in the classroom. In the context of proof-based teaching this is what we call an expectation for explanation.

Deductive explanations

As part of the common 'toolbox' the whole class also shared an understanding of conjecture and justification, explained and modelled by the teacher, which was in tune with the meaning of proof given in A. Stylianides (2007). As part of the didactic contract the students knew that they could make as many conjectures as they wanted. The teacher wrote the students' conjectures at the blackboard to be analyzed by the whole class. But they were constantly reminded that in order for their conjectures to be upgraded to 'mathematical truths', they should provide strong support in the form of deductive arguments that were evaluated by the teacher.

Third graders' understandings of division

We report here some results from the third cycle of the research design we discussed above. This intervention took place in a public school in Peru, in 2015, with a group of 21 third graders (7-8 years old). The intervention consisted of 23 sessions, each of them made of around 90 minutes. It was session 3 when these third graders discovered the relation between remainders and divisors and explained the relation using the key notions through a whole class discussion.

At the end of the intervention (session 23), the third graders were given a final test, including two items related to remainders:

> Is it true that in a division by **4** we can have a remainder of **6**? ☐ Yes ☐ No Justify your answer.
>
> In a whole, fair and maximum distribution among 5 people, how many objects may be left over at most? Why can no more objects be left over? Justify your answer.

These two items were number 8 and 9 on a test with 11 items. We report here on the children's responses to these two items, which are summarized in Table 1.

	First Item	Second Item
Correct answer with explanation	9 (43%)	12 (57%)
Correct answer with unclear explanation	2 (10%)	2 (10%)
Correct answer with no explanation	1 (5%)	1 (5%)
Incorrect answer	1 (5%)	3 (14%)
No response or question misunderstood.	8 (38%)	3 (14%)

Table 1: Summary of results from the third graders' test

Of the 21 children, 12 (57%) answered the first question correctly and 15 (71%) answered the second question correctly. Most of those who answered the second question correctly were also able to give an explanation. Their answers are based on a relational understanding of division, bringing together knowledge of the key notions they learned and experience with explanatory proving in this context.

For example, Bruno answered the first question "No. Because if we divide by 4 the remainder is at most 3, and 6 is more than 3". This shows that he understands why the answer is no, and can explain by making reference to specific knowledge about division by 4, and implicitly to a general rule concerning the maximum remainder possible. Some children who answered the first item correctly (2 of the 12) provided a similar explanation, although their knowledge of the possible remainders when dividing by 4 was faulty. For example, Eduardo wrote "No, because in a division by 4 the only remainders are 1, 2, 3". Eduardo omits one possible remainder, but his explanation is still appropriate, as he points out that 6 is not among the possible remainders in a division by 4.

On the second item, Max answered "Question 1: 4 can be left; Question 2: because I can keep distributing (objects)". His answer shows his understanding of why the remainder cannot be more than 4 when distributing objects among 5 people. Although he does not refer to the condition by name (maximum distribution), he uses a condition that makes reference to it ("because I can keep distributing objects") as the question makes reference to cases in which the maximum number of objects has not yet been distributed. Max's answer is an example of the kinds of arguments they were able to produce.

Similarly, Renato's answer "There can be at most 1, 2, 3 and 4 left over. More objects can't be left over because it wouldn't be (a) maximum (distribution)", shows he understands why the maximum value for the remainder in a division by 5 is 4. He is actually the only student who makes explicit reference to this condition by its name in his written work. Even though Renato's answer is incomplete (he doesn't consider the remainder zero) his explanation is correct. The use of this common toolbox was consistent in this intervention.

From the very beginning Vallejo invited the students to share their ideas orally, and they seemed to feel comfortable to communicate in this way. However, some students had troubles with their writing skills while communicating their ideas individually, though they could still share well-thought ideas orally. Hence, after the final test Vallejo decided to interview some of the students who had performed well in whole class discussions, but not so well on the written tasks. These semi-structured interviews revealed that some students who had not given explanations had not understood the questions being asked. For example, Piero had answered the first test item by giving an example of a division by 4 that does not result in a remainder of 6. He did not understand that the question refers to dividing by 4 in general. When Vallejo asked the same question in the interview, he answered "No, because if there would be 6 left over, it would be 6 divided by 4, and I must continue distributing (objects)". Like Piero, most of the students who gave an answer classified as "Question misunderstood" showed in the interview that they had not understood the question in the first item. However, when the question was clarified and they were given time to reflect, most were able to provide reasons.

We feel that the explanations given by the third graders demonstrate a relational understanding of divisibility, which arose through the proof-based teaching they experienced. We have not (for practical and ethical reasons) attempted to make a comparison with a matched group of third graders taught about divisibility in another way. Instead, in the next section we compare their understandings with those of students at the end of secondary schooling, who have had many other opportunities to develop their understandings of division.

University students' understandings of division

We analyze here the answers given on a diagnostic test given at the beginning of university studies to 148 students enrolled in primary level teacher education. These students were enrolled at a private university and had received a government scholarship to support their teacher education. Hence, they can be assumed to be among the best students enrolling in primary level teacher education. Around 65% of these students came from the capital city, Lima, where the university is located, and the other 35% came from the other parts of Peru. The test was given prior to any instruction at the university, which means that it assessed only the understanding the students retained from their school experience.

The students were asked the following question: "In a division of natural numbers with the divisor equal to 3, what are all the possible values the remainder can take? Why?" To ensure that the terminology used in the question was understood, the question was accompanied by the diagram shown (which is a translation of the real one) in Figure 1.

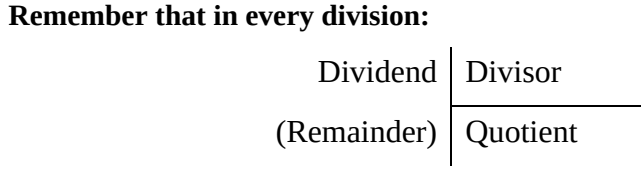

Figure 1: Reminder included with the question

The task presented to these students is not exactly the same as presented to the third graders. This reflects the background knowledge of these two different groups. In the case of the prospective teachers, they were not familiar with the language fair, whole and maximum distributions and the third graders were not introduced to the terms dividend, divisor, or quotient. Despite that, one can see that both tasks ask for the same knowledge about the divisor and remainder relationship.

Table 2 summarizes the results from the pre-service teachers.

	Correct answer	Partially correct	Incorrect answer	No response
With explanation	9 (6%)	1 (1%)		
Reference to a general rule	25 (17%)	11 (7%)		
Without explanation	21 (14%)	13 (9%)	38 (26%)	30 (20%)

Table 2: Summary of results from the pre-service teachers' diagnostic test

A correct answer was given by 55 (37%) of the pre-service teachers. But of these only 9 gave explanations that show they understood the reason why the remainder must be 0, 1 or 2. For

example, Elizabeth wrote “It can only take values less than 3, in this case they would be (0, 1, 2) x<3, because a number multiplied by 3 cannot be less than this. (x: Remainder)]”.

Among those giving a correct answer the most common way to answer the question “Why?” was by reference to a rule such as “the maximum remainder is one less than the divisor’s value” or “the remainder is always less than the divisor”. These answers may reflect understanding, but the rule may have been memorized without understanding. The remaining 21 responses include no explanation, an unclear response, or empirical evidence as “explanation”. Figure 2 shows a response of this last kind. Note that the divisions are of small numbers, but were done using a standard algorithm. In the first two cases, the dividends 1 and 2 were treated as if they were 10 and 20 for the purpose of determining the remainder, although the first decimal place of the quotient is worked out as if 1 and 2 are being divided. The pre-service teacher writes “Por lo tanto:” [Therefore] suggesting she feels that her six examples are sufficient to explain her answer. She also wrote “¿Por qué?” [Why?] with an arrow pointed to her examples, which is consistent if she believes these examples answer the question. It seems she was not able to provide a mathematical explanation.

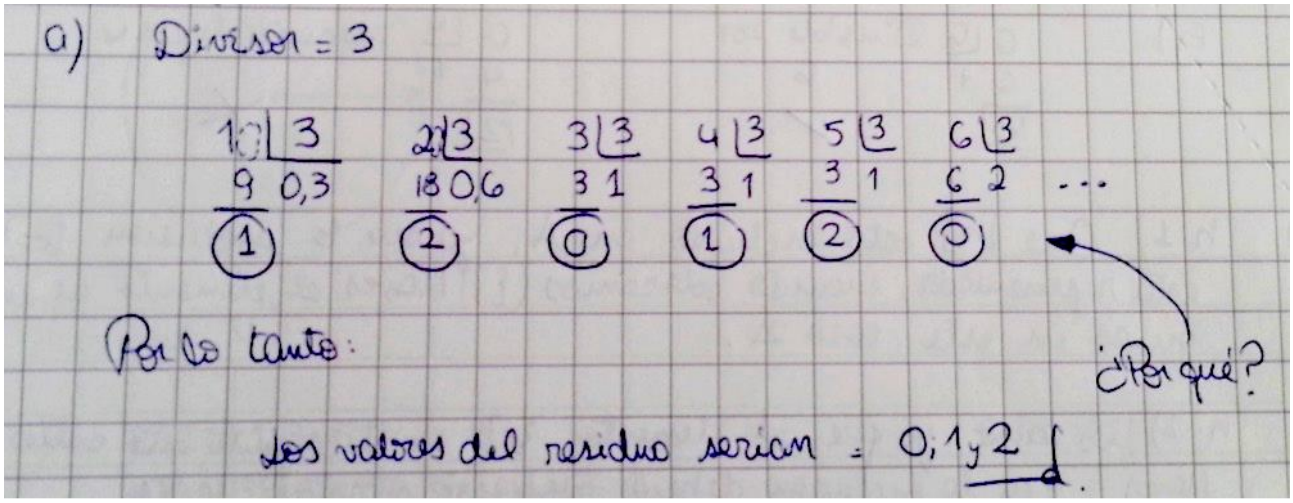

Figure 2: A pre-service teacher’s response, showing a correct answer without an explanation

Another 25 (17%) pre-service teachers gave partially correct answers (listing two of the three remainders, or listing 0 and 3 as distinct possibilities resulting in four remainders), 38 (26%) gave incorrect answers and 30 (20%) gave no answer. Overall, the responses of the pre-service teachers show an instrumental understanding (in the sense of Skemp, 1987) of division and limited number sense. Only 46 could give an explanation or cite a general rule and most used procedural approaches to determine the possible remainders in spite of the small numbers involved.

Conclusions

We do not claim that this comparison replaces an experimental design with a control group, but this was not a goal of our design based research in any case. Nevertheless, it does offer some food for thought. One might expect that adults at the conclusion of more than a decade of schooling would have had many opportunities to develop concepts related to division, a basic operation in arithmetic and one that is basic to understanding of rational numbers and algebra. Why compare them with children who have had only twenty-three lessons on the topic? What we wish to compare are the

two different school experiences with division of natural numbers these two groups have had. Most of the preservice teachers can be assumed to have had a typical school experience in mathematics. That the third graders have a better understanding of division we feel reflects the non-traditional learning context they experienced, that allowed them to make sense of division. We strongly believe that proof-based teaching was important in their achievement of this understanding, but further research is needed to confirm this.

However, this comparison also raises an important question for teacher education. If, at the conclusion of secondary school, future primary school teachers do not understand basic concepts related to division, will they be able to guide children in the development of these concepts? If they are to develop these concepts as part of their teacher education, how can this best be done? Clearly the approaches taken in their schooling were unsuccessful. Our current research focusses on such pre-service teachers, and explore whether a proof-based teaching intervention at the university level can allow adults with instrumental understandings to develop relational understandings.

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Mathematical reasoning in the written argumentation of primary students

Silke Ruwisch

Leuphana University Lueneburg, Germany
ruwisch@uni.leuphana.de

In Germany, there is increasing interest in language competences in mathematics lessons. Based on national curriculum standards, argumentation should also be strengthened in primary school mathematics classes (KMK, 2005). The reported interdisciplinary (linguistics and mathematics education) study on reasoning presents a model to rate arithmetic reasoning competences at primary level, in which mathematical reasoning and its linguistic realization are separately coded. In a pilot study, 243 third, fourth, and sixth grade students solved a number of arithmetic reasoning tasks. The results support a one-dimensional scale for the model of reasoning; its components identify differentiated requirements, which are formulated concretely in the coding guidelines and may point to didactical potential for language support in mathematical reasoning itself, as well as in mathematics lessons at primary level.

Keywords: Reasoning, written argumentation, primary school.

Reasoning in early mathematics learning

Early mathematical argumentation can be divided into four steps: detecting mathematical regularities, describing them, asking questions about them and giving reasons for their validity (Bezold & Ladel, 2014; Meyer, 2010; Bezold, 2009; Wittmann & Müller, 1990). The content base for argumentation is achieved by description of the detected structures or by reference to common knowledge (Krummheuer, 2000); reasoning is then needed to verify the described regularities as true (Toulmin, 2003/1958; Schwarzkopf, 1999).

The didactical value of reasoning in mathematics learning lies in gaining deeper insights into mathematical structures, so developing one's mathematical knowledge. In this sense, reasoning leads to questions about mathematical statements to ensure their correctness and to develop new mathematical connections (Steinbring, 2005). Two intertwined processes can be distinguished: one's own understanding and the process of sharing this understanding with others. In most cases, these processes don't occur separately, but are the response to cognitive-social needs (Harel & Sowder, 2007; Hersh, 1993). It follows that, in its epistemic function, mathematical reasoning may be monologic in leading to deeper individual understanding; in its communicative function, where mathematical structures are explained and justified, it is dialogic and dependent on other people (Neumann, Beier & Ruwisch, 2014; Ruwisch & Neumann, 2014).

In primary classrooms, mathematical reasoning usually occurs in the form of oral communication between pupils and in interactions with the teacher. These communicative processes have been widely studied. From an epistemological perspective, the emergence of shared knowledge and its structures has been described (e.g. Steinbring, 2005) while a more interactionist perspective traces the type and structure of argumentation in classroom interactions (e.g. Krummheuer, 2015).

Mathematical reasoning in this sense must be distinguished from reasoning in language classes, especially at primary level. While both are seen as concepts that develop out of situated everyday ("vernacular") speech (Elbow, 2012), reasoning in language learning focuses much more on self-evident facts and personal meanings than on provable structures in special content areas. It follows that argumentation in language learning leads to more addressee-oriented cognitivization (Krelle, 2007), as reasoning of this kind is much more about persuasion than proving. Nevertheless, typical linguistic forms of reasoning are learned in these everyday situations, and students must learn how to use these in different content areas (e.g. Wellington & Osborne, 2001; Lemke, 1990). So, in combining mathematical and linguistic views of early reasoning, we can hope to gain a broader and deeper understanding of early reasoning.

While most age-related studies of primary students focus on oral communication, experts in language learning emphasise writing as an important instrument for deepening individual understanding (Becker-Mrotzek & Schindler, 2007; Pugalee, 2005; Galbraith, 1999; see also Wellington & Osborne, 2001; Morgan, 1998; Miller, 1991). Although primary school children are not yet expert in writing, fourth-graders are capable of constructing expository texts with a relevant number of causes in elaborating a topic (Hayes, 2012; Krelle, 2007). It may therefore be fruitful to look at their written argumentations, and especially at how they offer reasons for mathematical regularities (Ruwisch & Neumann, 2014; Fetzer, 2007).

Modelling written mathematical reasoning

To investigate children's written reasoning, we developed a theoretical model that combines mathematical and linguistic aspects of reasoning (Ruwisch & Neumann, 2014; Neumann, Beier & Ruwisch, 2014).

Arithmetic reasoning tasks

Following the four steps of argumentation in primary mathematics (see above), we decided to give the children an already structured situation (see Figure 1), which explicitly requires detection, transfer and description before offering reasons for the validity of their suggestion.

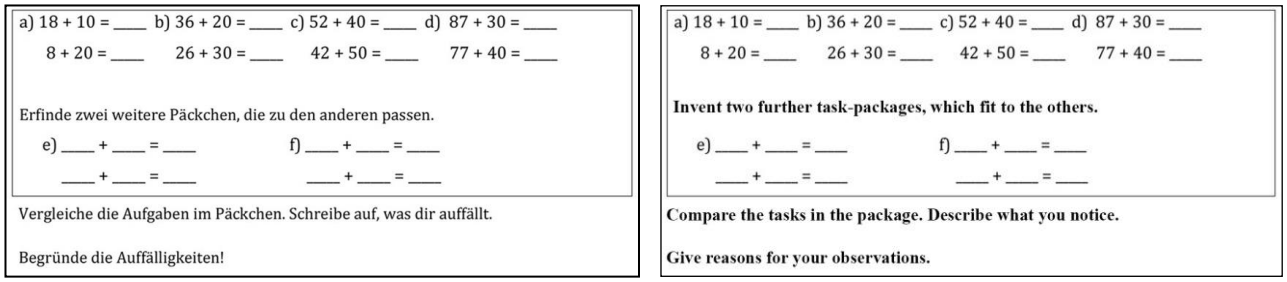

Figure 1: Complex addition task (CA) as a sample item (left: original version; right: English translation)

For the purposes of this study, four different arithmetic tasks were designed. Although differing in complexity of regularities, all of these tasks focused on detection and reasoning and were easy to compute. Format ZF involved three number sequences to be continued: +9, +7, and +2n. Format EA asked the children to continue a given additive structure by increasing all three summands by one so that the sum increases by three. In solving formats CA and CM, the children had to identify two

structures at the same time. To answer the complex addition task in Figure 1, the children had to find two tasks with the same sum. At the same time, they had to take into account that the summands must be changed by 10 in opposite directions. The multiplication tasks (CM) showed a constant difference in the product caused by the difference between the multipliers while the multiplicands remained constant.

Sample

The data include 477 justifications written by 243 students. In total, 41 third-graders (♀21; ♂20), 96 fourth-graders (♀43; ♂53) and 106 sixth-graders (♀52; ♂54) worked out two of the four arithmetic reasoning tasks.

Data analysis: Theoretical model of rating scales

The separate evaluation of mathematical and linguistic aspects of reasoning is fundamental to our model, which we assume allows differentiated exploration of the sub-skills of reasoning. We also wish to check whether experts in either domain (mathematics teachers and German language teachers) differ in their evaluations. As our tasks demanded both computing and continuation of a given structure (see Figure 1), competencies involving detection of a mathematical structure are distinguished from ability to offer reasons for its validity. Students' writings are rated by one detection scale and two reasoning scales (see Table 1).

Mathematical detections: Children were required to compute the arithmetic tasks on the sheet to identify the underlying structure and transfer it to two further packages of tasks. This process might be realised fully or only partly; sometimes, only irrelevant aspects were used to create new tasks. If the structure is transferred fully, the results of the given tasks are also correct, and three levels of this rating scale therefore seemed sufficient. This scale will not be discussed in the following application of the model, as it provides little information about reasoning skills.

Mathematical detections	Mathematical aspects of reasoning	Linguistic aspects of reasoning
irrelevant aspects as regularities	regularities (partially) described	indicators without reason-effect structure
	rudimentary reasoning	reason-effect structure
regularities partly transferred	reasoning through examples	explicit linguistic reference to the task
	partially generalized reasoning	completeness and consistency
regularities totally transferred	generalization/ formal reasoning	use of math. terminology/ decontextualization

Table 1: Rating-scales to evaluate written mathematical reasoning

Mathematical aspects of reasoning: Mathematical reasoning must be based on a description of mathematical elements. If only some regularities are described without giving reasons, this is coded as level 1. If rudimentary reasoning is given in addition to a description, the work is coded as level 2. To be rated as level 3 to 5, all relevant aspects must feature in the argumentation. If this is done

by use of examples, the work is rated as level 3; if already partly generalized, it is rated as level 4; and if it is totally general or constitutes a formal proof, it is rated as level 5.

Linguistic aspects of reasoning: Realisation of a mathematical argument by written language is also rated in terms of five levels, defined in terms of use of connectors and identifiable coherence of the text. If explicit linguistic indicators are used without any structured reasoning, the text is classified as level 1. If the text shows a reason-effect structure, it is coded as at least level 2. If explicit linguistic reference to the tasks is also included, the text is classified as level 3. A level 4 text shows consistent and complete argumentation. To achieve level 5, there must also be use of mathematical terminology for identifiable decontextualization.

Process of coding

Each written argumentation was assessed by at least four raters; while preservice mathematics teachers concentrated on the mathematical scales, German language teachers rated the linguistic aspects. There was 62% absolute agreement in the judgments across all tasks and scales. Deviations of more than one stage occurred in 8% of cases—mainly for linguistic ratings, which were reported as more difficult. Coding quality could be seen to increase during the course of the project. Although there were acceptable internal consistencies across all tasks (Cronbach's $\alpha = .80$), these values increase if only ZF ($\alpha = .82$) and EA ($\alpha = .84$) (which were used later in the project) are considered.

By excluding the multiplication task for the following overall scaling, an acceptable average internal consistency of individual scales was achieved for the remaining tasks ($\alpha = .86$ for mathematical detections, $\alpha = .81$ for mathematical aspects of reasoning and $\alpha = .71$ for the linguistic aspects of reasoning).

First results

Given the number of raters and in the interests of acceptable inter-rater-consistency ($\alpha > .70$), the following results are based on the means of ratings.

Overall scale

The IRT scale for the three tasks and all texts shows a common scale across all components (see Table 2). As items also conform to the model (WMNSQ .85-1.09), early mathematical reasoning in arithmetic as measured by the three tasks and ratings on our scales can probably be described as a one-dimensional construct. Looking at the three scales, it becomes clear that, as expected, it is easier to detect and transfer mathematical structures than to give reasons for their validity (negative deviation from zero). Comparing the two reasoning scales, it seems easier to realise mathematical aspects of reasoning than to find an appropriate linguistic structure. At the same time, the most stable dimension is *mathematical detections*, with a maximum difference of .783 as compared to 1.446 for the *linguistic aspects of reasoning* and 1.516 for the *mathematical aspects of reasoning*.

Comparing the three tasks, it seems that complex addition is the most difficult to transfer; simple addition and number sequences show almost no difference. The justifications show that it was easiest to realise both mathematical and linguistic aspects of reasoning in the number sequence tasks, followed by complex addition and then simple addition. Granted these differences, all tasks can be characterised as suitable for capturing mathematical reasoning in arithmetic.

	Mathematical detections		Mathematical aspects of reasoning		Linguistic aspects of reasoning	
Item	Estimate	WMNSQ	Estimate	WMNSQ	Estimate	WMNSQ
(ZF) number sequences	-1.556	1.02	-0.459	1.06	0.124	0.85
(EA) simple addition	-1.628	1.09	1.057	1.09	1.570	0.93
(CA) complex addition	-0.845	0.98	0.506	0.92	1.230	0.97

Table 2: Item parameters (estimated) for IRT scaling

Student performance

Performance of the total sample is distributed normally to slightly right-shifted. On the raw scores level, 21.2% are one standard deviation above the mean; 9.6% are one standard deviation below; 6.2% are two standard deviations above; and 4.2% are two standard deviations below.

To facilitate comparison of the three groups of students, all scores were transposed to a scale with mean 100 and standard deviation 20. Figure 2 shows almost the same mean performance across third-graders (M = 102, SD = 29), fourth-graders (M = 98, SD = 19) and sixth-graders (M = 101, SD = 17).

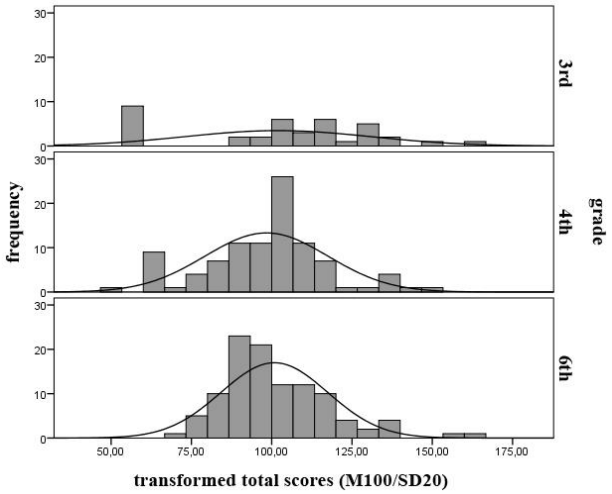

Figure 2: Student performance by grade

Unexpectedly, reasoning competences seem not to increase over time. Interpreting the differences in standard deviations across the three groups, it seems that third-graders differ more within their group than fourth-graders, and both differ more than sixth-graders, suggesting homogenization during schooling. However, the relatively limited data and lack of comparative data means that any general conclusions remain speculative.

Discussion of the model by application to examples

Five examples are presented here for deeper discussion of the model's adequacy (see Table 2). As it might prove difficult to discuss the argumentations only in the translated version, the original sentences are included on the left.

Mathematical aspects of reasoning: The child in example 1 has recognized that "something" is the same and "something" has changed. However, as he/she does not refer to any connection between the tasks or mention that the results are the same, this answer was coded as level 1. Examples 2 and 4 were coded as level 2; in example 2, it is clear that the child focused on only one of the two

relevant aspects. It is arguable whether child 4's argumentation is complete; as it is confined to one example in the task, it might be evaluated as level 3. In our opinion, the change of the summands in opposite directions is only implicit in "that's always 10 less". Answer 5 shows both connections clearly. Furthermore, the child is able to conclude (using an example) that the results must be the same, and so it is clearly to be coded as level 3. As child 3 is doing almost the same but also exhibits some generalization in using "always", we coded this as level 4.

1)	es sind immer die gleichen Aufgaben nur umgedreht weil wenn man es rechnet merkt man das.	The tasks are all the same but vice versa because if you calculate, you'll realize it.
2)	Es sind immer 10 mehr und 10 weniger.	It's always 10 more and 10 less.
3)	Dass es die gleichen Ergebnisse sind, kommt davon, weil bei der einen Aufgabe immer 10 weniger sind als bei der anderen. Aber bei der Aufgabe wo 10 weniger sind, ist die Zahl die noch dazu gerechnet wird wiederum 10 größer als die über ihr.	The results are the same because in one task it's always 10 less than in the other one. But in the task that has 10 less, the number to be added is 10 bigger than the one above.
4)	Das es immer 10 weniger sind. Zum beispiel 18+10=28 aber wenn man 10 weg nimmt und in der mitte 10 dazu nimmt z.b. 8+20=28 und dann kommt das gleiche ergebnis wie bei der 1. Aufgabe	That's always 10 less. For example, 18+10 = 28. But if you take away 10 and put 10 in the middle—for example, 8+20 = 28—then you'll get the same result as in the first task.
5)	mir fällt auf das immer die Ersten 2 Ergebnisse gleich sind. Die Ersten zwei Ergebnisse sind gleich weil die bei zum beispiel a) 18+10=28 und dann haben die bei 8+20 einfach 18, 10 weniger 8, und bei 10 zehn mehr, 10-10 ist 0, also bleibt das so	I notice that the first two results are always the same. The first two results are the same, because, for example a) 18+10 = 28, and then at 8+20 it's simply 18, 10 less 8, and at 10 ten more, 10-10 is zero, so it remains the same.

Table 3: Examples of written argumentation for the arithmetic sample item (left: original version; right: English translation)

Linguistic aspects of reasoning: Example 1 is coded as level 2 because as well as the comparative connector "because", a link between the sentences is also given. As example 2 includes only the indicator "always", without any link, it is coded as level 1. Examples 3 and 4 are coded as level 4 because there is a clear reasoning structure as well as a link to the tasks. As the argumentation in example 5 is ambiguous, and the language used is imprecise, it is coded as level 3.

Conclusion

The model was again presented for discussion here to improve its didactical value in evaluating the written reasoning of fourth-graders. Although our descriptions in the coding book have continued to improve over time, there are still deviations of more than one level between raters. While we wish of course to develop the model for its psychometric interest, the levels should also help teachers to evaluate written reasoning.

Although these tasks provide a good deal of information about children's written reasoning, we have to be aware that because they focus on products collected in a test situation, the argumentation was necessarily ad hoc. As requests of this type are not part of students' normal mathematics lessons, and they do not have time to restructure their texts, neither the requisite procedural knowledge for writing nor situated mathematical argumentation can be grasped in this way. It follows that competence in mathematical reasoning—even in written form—may be higher than is indicated by the results to date.

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An expanded theoretical perspective for proof construction and its teaching

John Selden[1] and Annie Selden[1]

[1]New Mexico State University, Department of Mathematical Sciences, New Mexico, Las Cruces, U.S.A.; jselden@nmsu.edu, aselden@nmsu.edu

This paper presents a theoretical perspective for understanding and teaching university students' proof construction. It includes features of proof texts with which students may be unfamiliar. It considers psychological aspects of proving such as behavioral schemas, automaticity, working memory, consciousness, cognitive feelings, and local memory. We discuss proving actions, such as the construction of proof frameworks that could be automated, thereby reducing the burden on working memory and enabling university students to devote more resources to the truly hard parts of proofs.

Keywords: Proof construction, behavioral schemas, automaticity, consciousness, feelings.

Introduction

We report an expanded theoretical perspective to better notice, understand, and alleviate difficulties of university students' proof construction including: features of proof texts, types of proofs, situation-action links, behavioral schemas, automaticity, non-emotional cognitive feelings, and local memory. Most difficulties were observed and documented during a 10-year teaching experiment—a proof construction course. Our explanations call on the psychological as well as the education literatures. Local memory (Section 4.10) arose from observing our own proving experiences.

Features of proof texts

The proving process involves many more actions (both physical and mental) than appear in the final proof text (e.g., Mamona-Downs & Downs, 2009; Selden & Selden, 2016). Indeed, researchers have distinguished argumentation from proof, noting that an informal line of reasoning may "justify" a theorem to the prover's satisfaction, but this often differs from the corresponding final proof text written to the standards of the mathematical community (e.g., Pedemonte, 2007).

The genre of proofs

Students sometimes find the manner in which proofs are written perplexing, as it is often at variance with other genres of writing. We have identified some significant features that generally occur in proof texts: (1) Proofs are not reports of the proving process. (2) Proofs contain little redundancy. (3) Symbols are (generally) introduced in one-to-one correspondence with mathematical objects. (4) Proofs contain only minimal explanations of inferences, that is, warrants are often left implicit. (5) Proofs contain only very short overviews or advance organizers. (6) Entire definitions, available outside the proof, are not quoted in proofs. (7) Proofs are "logically concrete" in the sense that quantifiers, especially universal quantifiers, are avoided where possible. (Selden & Selden, 2013a).

Structures of proofs

A proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part that depends only on unpacking the logical structure of the statement of

the theorem, associated definitions, and earlier results. In general, this part *does not* depend on a deep understanding of the concepts or genuine problem solving in the sense of Schoenfeld (1985, p. 74). We call the remaining part of a proof the *problem-centered* part. It *does* depend on problem solving, intuition, heuristics, and understanding the concepts involved (Selden & Selden, 2011).

Proof frameworks

A feature that can help write the formal-rhetorical part of a proof is what we call a *proof framework*, of which there are several kinds, and in most cases, both a *first-* and a *second-level* framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a first-level proof framework would be "Let x be a real number. Suppose $P(x)$. … Therefore, $Q(x)$," with the remainder of the proof ultimately replacing the ellipsis. A second-level framework can often be obtained by "unpacking" the meaning of $Q(x)$ and putting the second-level framework between the lines already written for the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down.

Operable interpretations

Another feature that can help write the formal-rhetorical part of a proof is converting definitions and previously proved results into *operable interpretations*. These interpretations are similar to Bills and Tall's (1998) idea of operable definitions. For example, given a function f: $X \to Y$ and $A \subseteq Y$, one defines $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa." One might think translation into an operable form would be unnecessary or easy especially because the symbols in $\{x \in X \mid f(x) \in A\}$ can be translated into words in a one-to-one way. But for some students this requires practice.

Dimensions of potential proof construction difficulty

The need for previous results—proof types: 0, 1, 2, 3

We have classified theorems of increasing difficulty to refine our inquiry-based "proof course" notes (Selden & Selden, 2013b). *Type* 0 often follows immediately from definitions. *Type 1* may need a result in the notes. *Type 2* needs a lemma, not in the notes, but relatively easily to discern, formulate, and prove. For *Type 3*, at least one of discern, formulate and prove should be difficult. A sample Type 3 theorem is: *A commutative semigroup S with no proper ideals is a group*, when provided only the definitions of semigroup and ideal. One needs to observe that, for $a \in S$, aS is an ideal, so $aS=S$. This implies equations of the form $ax=b$ are solvable for any $b \in S$. Using some clever instantiations of this equation, one can obtain an identity and inverses, and conclude S is a group.

The need for unguided exploration

In constructing some proofs, one may reach a point where there is no "natural" way forward. In what we call *unguided exploration*, one may need to find, or define, an object and prove something about it, with no idea of its usefulness, that is, one may need to "explore" the situation. For example, in proving the above semigroup theorem, this can happen three times. First, one notes aS is an ideal and thus $aS=S$. Then one sees equations of the form $ax=b$ are solvable for any $b \in S$. Such exploration may require self-efficacy (Bandura, 1994), which can be encouraged by arranging early student successes.

The need to unpack the logical structure of a theorem statement

An *informal statement* is one that departs from the usual use of predicate and propositional calculus or fails to specify variables. For example, *Differentiable functions are continuous*, is informal because a universal quantifier and a variable are omitted, and because it departs from the usual "if-then" form of the conditional. Such statements are commonplace in everyday mathematics. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians and less reliably by students. An informally stated theorem can be memorable and easily brought to mind, but it may be difficult to unpack and prove (Selden & Selden, 1995).

Psychological aspects of the proving process

We view proof construction as a sequence of actions that can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one's focus from the hypothesis to the conclusion or trying to recall a theorem). The sequence of actions that eventually leads to a proof is usually considerably longer than the final proof text and is often not constructed from the top down.

Situations and actions

When considering proving, we use the term, *action*, broadly as a response to a situation in a partly completed proof. We include not only physical actions, but also mental actions. The latter can include trying to recall something or bringing up a feeling, such as a feeling of caution or of self-efficacy (Selden & Selden, 2014). In addition, we include "meta-actions" meant to alter one's own thinking, such as changing focus to another part of a developing proof construction.

Situation-action links, automaticity, and behavioral schemas

If, during several proof constructions in the past, similar situations have corresponded to similar reasoning leading to similar actions, then, just as in classical associative learning (Machamer, 2009), a *link* may be learned between them, so that another similar situation evokes the corresponding action in future proof constructions without the need for the earlier intermediate reasoning. Using such *situation-action links* strengthens them, and after sufficient experience/practice, they can become overlearned and automated. We call automated situation-action links *behavioral schemas*.

Features of automaticity

In general, it is known that a person executing an automated action tends to: (1) be unaware of any needed mental process; (2) be unaware of intentionally initiating the action; (3) execute the action while putting little load on working memory; and (4) find it difficult to stop or alter the action (Bargh, 1994). However, not necessarily all four of these tendencies occur in every situation.

Forming behavioral schemas converts S2 cognition, which is slow, conscious, effortful, evolutionarily recent, and calls on considerable working memory, into S1 cognition, which is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory (Stanovich & West, 2000). This conversion into S1 cognition conserves working memory resources.

Behavioral schemas as a kind of knowledge

We view behavioral schemas as belonging to a person's knowledge base. They can be considered as partly conceptual knowledge (recognizing and interpreting the situation) and partly procedural knowledge (doing the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". We suggest that, in using a situation-action link, or a behavioral schema, almost always both the situation and the action (or its result) will be at least partly conscious.

Here is an example of a behavioral schema that can conserve resources. One might be starting to prove a statement having a conclusion of the form *p* or *q*. This would be the situation. If one had encountered this situation a number of times before, one might readily write into the proof "Assume *not p*" and prove *q* or vice versa. While this action can be warranted by logic (if *not p* then *q*, is logically to, *p* or *q*), there would no longer be a need to bring the warrant to mind.

The genesis and enactment of behavioral schemas

The action produced by the enactment of a behavioral schema might be simple. It might also be compound, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioral schema that was "triggered" by the action of the preceding schema in the procedure. We have developed a six-point theoretical sketch of the genesis and enactment of behavioral schemas (Selden, McKee, & Selden, 2010, pp. 205-206). Very briefly, here are the six points: 1) Within very broad contextual considerations, behavioral schemas are immediately available. 2) Simple behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action – one just does it. 3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs. 4) Behavioral schemas were once actions arising from situations through warrants that no longer need to be brought to mind. Behavioral schemas cannot be "chained together" and act outside of consciousness, as if they were one schema. 5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person's actions, but such influence is normally not precise enough for doing mathematics. 6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing a detrimental behavioral schema requires similar, perhaps longer, practice.

Implicit learning of behavioral schemas

It appears that the process of learning a behavioral schema can be implicit, although the situation and the action are in part conscious. That is, a person can acquire a behavioral schema without being aware that it is happening. Indeed, such unintentional, or implicit, learning happens frequently and has been studied by psychologists and neuroscientists (e.g., Cleeremans, 1993). In the case of proof construction, we suggest that with the experience of proving a considerable number of theorems in which similar situations occur, an individual might implicitly acquire a number of relevant beneficial behavioral schemas. As a result, he or she might simply not have to think quite so deeply as before about certain portions of the proving process, and might, as a consequence of having more working memory available, take fewer "wrong turns".

Detrimental behavioral schema

Many teachers can recall having a student write $\sqrt{(a^2 + b^2)} = a + b$, giving a counterexample, and then having the student make the same error somewhat later, perhaps in a different context. Rather than being a misconception (i.e., believing something that is false), this may well be the result of an implicitly learned detrimental behavioral schema. If so, the student would not have been thinking very deeply about this calculation when writing it. Furthermore, having previously understood the counterexample would also have little effect in the moment. It seems that to weaken/remove this particular detrimental schema, the triggering situation of the form $\sqrt{(a^2 + b^2)}$ should occur a number of times when the student can be prevented from automatically writing "$= a + b$" in response.

Feelings and proof construction

The word "feeling" is used in a variety of ways in the literature so we first indicate how we use it. Often feelings and emotions are used more or less interchangeably, perhaps because both appear to be conscious reports of unconscious mental states, and each can, but need not, engender the other. We will follow Damasio (2003) in separating feelings from emotions because emotions are expressed by observable physical characteristics, such as temperature, facial expression, blood pressure, pulse rate, perspiration, and so forth, while feelings are not.

Feelings, such as a feeling of knowing, can play a considerable role in proof construction (Selden, McKee, & Selden, 2010). For example, one might experience a feeling of knowing that one has seen a theorem useful for constructing a proof, but not be able to bring it to mind at the moment. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it (Clore, 1992, p. 151). We call such feelings *non-emotional cognitive feelings*.

For the nature of feelings, we will follow Mangan (2001), who has drawn somewhat on William James (1890). Feelings seem to be summative in nature and to pervade one's whole field of consciousness at any particular moment. Non-emotional cognitive feelings, different from a feeling of knowing, are: a feeling of familiarity and a feeling of rightness. Rightness is "the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge". (Mangan, 2001). About such feelings, Mangan (2001) has written that "people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction." Finally, we conjecture that feelings may eventually be found to play a larger role in proof construction than indicated above, because they provide a direct link between the conscious mind and the structures and possible actions of the unconscious mind.

The roles of affect and self-efficacy

In order to prove harder theorems--ones with a substantial problem-centered part--students need to persist in their efforts, and such persistence is facilitated by a sense of self-efficacy. According to Bandura (1995), self-efficacy is "a person's belief in his or her ability to succeed in a particular situation". Of developing a sense of self-efficacy, Bandura (1994) stated that "The most effective way of developing a strong sense of self-efficacy is through mastery experiences," that performing a task successfully strengthens one's sense of self-efficacy. Also, according to Bandura, "Seeing people similar to oneself succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities to succeed."

Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the Moore Method (Mahavier, 1999). Such courses are taught from a brief set of notes consisting of definitions, requests for examples, and statements of major results, together with lesser results needed to prove them, but no proofs. The students provide the proofs and present them in class.

The development and uses of local memory

Some may think that proof construction consists mainly of conscious thought (i.e., as communication with oneself or others using speech, vision, etc., or their inner versions, as suggested by Sfard, 2010). However, we take a somewhat different view. In constructing a proof of some complexity, often much more relevant information can be activated than can be held in one's short-term working memory (ST-WM). When such information is lost from consciousness, it may not return to its original state, but rather to a state of partial activation. Nonetheless, conscious thought can sometimes influence the activation of related information in long-term memory (LTM), that is, help bring something to mind. Ericsson and Kintsch (1995) stated that "reliance on acquired memory skills will enable individuals [experts] to use LTM as an efficient extension of ST-WM in particular domains and activities after sufficient practice and training." We speculate that mathematicians can do this when conducting their own research. We have observed of ourselves, when attempting an intricate complex proof, that a considerable amount of information is generated, but cannot all be kept in mind; however, it is easily recalled. We refer to such partially activated information as *local memory* -- it is available as long as we are seriously engaged with the proof. It seems analogous to Ericsson and Kintsch's (1995) idea of long term working memory (LT-WM).

Teaching and future research considerations

We believe this perspective on proving, using situation-action links and behavioral schemas, together with information from psychology, is mostly new to the field. Thus, it is likely to lead to additional insights and teaching interventions, which brings up the question of priorities. Which proving actions of the kinds discussed above are most useful for mid-level university mathematics students to automate when they are learning how to construct proofs? Since such students are often asked to prove relatively easy theorems—ones that follow directly from definitions and recently proved theorems—it would seem that noting the kinds of structures that occur most often might be a place to start. Indeed, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start. Furthermore, we have observed that some students do not write a second-level proof framework, perhaps because they have difficulty unpacking the meaning of the conclusion. This may be because a relevant definition needs to be converted into an operable interpretation in order to construct the second-level proof framework. Thus, helping students interpret formal mathematical definitions so that these become operable might be another place to start, even though students should eventually learn to make such operable interpretations themselves.

Finally, this theoretical perspective is likely to allow one to see parts of the teaching of proof construction in unusual ways and lead to new questions. For example, unguided exploration can be helpful for some proofs, but a student could easily feel the time required for exploration might reduce (timed) test grades. A feeling of self-efficacy might overcome that, but how are feelings "taught"? Early successes with proofs can help, but arranging for these might require detailed

planning of the course before it starts. Such planning could perhaps be aided by following a textbook, but most advanced mathematics textbooks prove the most important and useful theorems themselves, thereby taking away from students the opportunity to experience the proving of even parts of such theorems.

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Tool-based argumentation

Kalliopi Siopi[1] and Eugenia Koleza [2]

[1,2] University of Patras, Department of Primary Education, Greece;

kalsiopi@gmail.com and ekoleza@upatras.gr

The study presented in this report is part of a research project concerning the mediation of artifacts in teaching and learning geometry. In this paper we analyze the first step of our research which concerns the student-pantograph interaction and the identification of the math laws incorporated in the machine. During this interaction we are specifically interested in arguments that students produce for supporting their claims. Tools, especially mathematical machines, may support argumentation processes focusing either on the structure of the machine, or to the embodied math concepts that emerge from the machine's movement. Our research has shown that these arguments hold mainly on the topological conception of geometric figures.

Keywords: Argumentation, instrumentation, figural concept, topology, pantograph.

Introduction

Mathematics is in close relation with material and non- material artifacts. Artigue (2002, p. 245) points out that *"*the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computations*"*. Teaching and learning geometry may be mediated by visual or design artifacts. Research on the use of artifacts in Geometry teaching as means to facilitate understanding and learning, has tended to focus mainly on technology integration into curriculum- such as computer software packages focusing on how learning takes place when students use such artifacts. In addition to the use of information technology in schools, the MMLab researchers (e.g., Bartolini Bussi, 2010; Mariotti et al., 1997) have recommended and investigated from an epistemological and pedagogical aspect the use of mechanical artifacts -mathematical machines- as a way to generate mathematical ideas or concepts in the classroom. The geometrical machine *"*is a tool that forces a point to follow a trajectory or to be transformed according to a given law*"* (Bartolini Bussi & Maschietto, 2008). These machines (for example pantographs) are linkages that allow the implementation of geometrical transformations, such as symmetry, reflection, translation, and homothety.

The study presented in this report is part of a research project concerning the teaching of geometry at an upper secondary school in Greece (early 2016). Our research project was conducted in the framework of an attempt to incorporate artifacts that bear geometrical machine characteristics, in the instruction of Euclidean geometry.

Theoretical framework

The theoretical framework of the instrumental approach was used for analyzing our observations (Verillon & Rabardel, 1995). According to this approach, the artifact is the material or symbolic object, while the instrument is defined as a mixed entity made up of both artifact and utilization schemes. In order for an artifact to lead to the development of an instrument, "the user has to develop mental schemes, which involve skills to use the artifact […] the birth of an instrument requires a process of appropriation, which allows the artifact to mediate the activity. This complex process is

called the instrumental genesis" (Drijvers & Trouche, 2008, p. 370). "The instrumental genesis, is a two sided process. On the one side, the construction of schemes is oriented toward the use of the artefact: the instrumentalisation. On the other side, the construction of schemes is oriented toward the task to be achieved: the instrumentation." (Goos et al., 2009, p.313). In our study we investigate the instrumentalisation process, i.e. the discovery of the elements and qualities of the artifact by the user. For our purpose, teaching homothety, the pantograph was the most convenient tool. Following Drijvers and Trouche (2008, p.369) we consider that the utilization schemes students construct during the instrumentalisation process, contain operational invariants that consist of – explicit or implicit – knowledge in the form of concepts-in-action or theorems-in-action.

Martignone and Antonini (2009) analyze more specifically the pantograph utilization schemes. They identified the "utilization schemes linked to the components of the articulated system (as the constraints, the measure of rods, the geometric figures representing a configuration of rods, ect.) and the utilization schemes linked to the machine movements" (p. 1253). In the second case, they distinguished two main sub-families: (1) the utilization schemes aimed at finding particular configurations obtained stopping the action in specific moments (limit zones; generic or particular or limit configurations) and (2) the utilization schemes aimed at analysing invariants or changes during continuous movements (wandering, bounded, guided; of particular configuration; between limit configuration; of dependence or in the action zones) (p. 1254). They have also conducted research on the argumentations produced in activities employing the pantograph (Antonini & Martignone, 2011). They distinguish between arguments about a) the drawings traced by the machine, b) the movement of the machine (as some dynamic properties of the articulated system), and c) the structure of the machine. Arguments about the structure are distinguished between referring to the figural aspect of the machine and the conceptual component of the geometric figure, which they discern in the structure.

In this paper we analyze the kind of argumentation produced by 16-17 year old students during the phase of investigating (a) the structure of a pantograph and (b) the configurations and (con) formations produced by the movements of its structural components.

Our research hypothesis was that the argumentation produced by the students, is in close relation with the machine's characteristics: students' explanations in the form of concepts or theorems in action are the exteriorization of precise utilization schemes developed by investigating the structure and the movement of the pantograph.

Methodology

The first step of our research, that is the subject of this paper, concerns the student-pantograph interaction. 26 students of an 11th grade class (16-17 years old), of different learning abilities and interests, took part voluntarily in the experiment. Participating students worked in 6 groups (4 groups of 4 people, and 2 groups of 5 people). All the participating students had no prior experience with any artifact, except for compasses and rulers. Two meetings were carried out with the groups, of four hours in totals, and members of two groups (A and B) were interviewed. The working sessions and interviews were audio recorded, and afterwards transcribed. The transcripts, visual material (photographs), and written reports of the groups constitute the data for the analysis.

The artifact, with which the students were asked to work, is a geometrical machine (linkage) with the characteristics of a pantograph, specifically a version of Scheiner's pantograph (Figures 1, 2). The building blocks of the pantograph model were two equally-sized wooden rods 30 cm long [OD=AE=AC=BD], held together by the links/pivots [A, L, D, and K] in the middle, thus forming a parallelogram [ALDK]. The rods had notches allowing reassembly of the linkage while maintaining its properties, provided that the links were placed in such a way that the ratios of the lengths of rod's parts were equal in each rod. The pantograph's linkage was mounted on a wooden platform (60cmx60cm).



Figure 1: The linkage

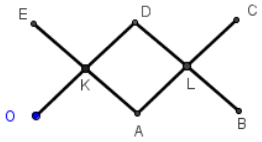

Figure 2: Schematic representation of the linkage

The tasks given to be treated by students concerned exploring the pantograph's structure and the investigation of special configurations and formations produced by the movements of the structural components of the linkage. The choice of the tasks was made following the distinction by Martignone and Antonini (2009) of the utilization schemes during pantograph exploration: the utilization schemes linked to the components of the articulated systems and the machine movements. The students invited to observe carefully the articulated system and to describe elements and characteristics of its structure such as length relationships and the mode of the rods' connection; to try to detect schematic shapes, properties and relationships that comprise its form and to create schematic representations of the articulated system (forms of system).

The analysis of the transcript was done following the classification of Martignone and Antonini (2009) about mathematical machine utilisation schemes and the kinds of arguments students use during the exploration of a geometrical machine, as a pantograph (Antonini & Martignone, 2011). In our research we examine utilization schemes linked to the structure of the machine in a static and in a dynamic status (: movement of the machine). For the first case (: static structure), hereafter, we use the symbols SF and SC, for the figural aspect of the machine and the conceptual component of the geometric figure which students discern in the structure, respectively. For the second case (: dynamic structure) we use the symbols MF and MC. In fact, in this second case we investigate the way students justify the embodied mathematics in the structure of the machine.

In this paper, we present and analyze the arguments produced by Group A -four girls hereby referred to as S1, S2, S3 and S4- as they investigate the configurations produced by the movements of the pantograph's structural components. Apart from space constraints, the omitted group (B) was already familiar with the abstract math concepts involved, as opposed to group A whose gradual discovery of the tool yielded high resolution into the thought process addressed by our research hypotheses.

Analysis of a transcript

The students of Group A took advantage of the capability of the linkage pivots to alter its form by opening and closing its parts, identified the property of the midpoint for the position of the pivots (as K, L in Figure 2) and inferred the equality of their lengths (equal rods).

S1: We identify a rhombus configuration (Figure 4), because the sides are equal as half of equal segments (SF, MC).

The student S1 perceives the components of the machine's structure as geometric objects and identifies in them geometrical relations. The equality of segments arose as an ascertainment while opening and closing the linkage. She uses the definition of rhombus (: because the sides are equal*)* and together with the figural aspect of the linkage's structure (: half of equal segments) to argue that the quadrilateral ALDK is a rhombus (Figure 2).

S2: Can we mention implied properties too?

Interviewer: Describe what you consider important.

This encouragement led the students to operate the artifact more dynamically, not only by opening and closing the rods, but also exploring characteristics and properties of specific conformations and support their claims, taking advantage of the capabilities provided by the pivots.

S2: Isosceles triangles are formed (*with her finger traces on the artifact the triangles OKA and ALB in* Figure 2) … if we assume that the articulated system can close its ends *(she points at the end of the rods and moves them until the linkage is closed,* Figure 3*)* and if we assume that it has a base because those (she *points at the pivots K and L,* Figure 2) are midpoints of equal sides (MF, MC).

The student executes the motion mentally ("if we assume … that it closes … and if it possesses a base"). They envision triangles in the linkage structure, though triangles do not exist. For these students the triangles they refer to are figural concepts (Fischbein, 1993). The students imagined the triangle and the reason that it is isosceles by moving the rods so that one rod meets the other (MF), while they deduce the equality of sides as halves of equal segments (MC).

S3: Maybe they're not triangles because they don't close?

The dimensions (thickness) of the rods don't allow them to coincide. The limitations of the artifact create a conflict between the figural aspects of the structure and the conceptual aspects of the geometrical figures of the articulated system. The students doubt whether they can actually consider it as a triangle.

S1: If we move the linkage, in a special position, we have a square (MC).



Figure 3: Closed linkage



Figure 4: The rhombus



Figure 5: The square

The student mentions the word "square" without justification. They have been taught that square is a special case of the rhombus. By moving the artifact, they predict that for a specific position of the rods, a square will be formed (Figure 5). Their square is of a conceptual nature. At the same time, it has an intrinsic figural nature: only while referring to the artifact one may consider operations like "if we move….we have a square". As a matter of fact, the square to which they refer cannot be considered as either a pure concept or a mere concrete representation.

Interviewer: How do you know it's a square?

S1: We have a right angle.

Interviewer: How do you know the angle is right?

S1: Since we have the capability of opening and closing an angle, then it can take all values between 0 and 180 … then definitely one of these values will be 90 … so one (angle) will be 90 degrees (MC).

The students imagine through the rod motion a continuous creation of angles between 0°-180° and that obviously 90° will exist as mean of the interval, and consequently will correspond to an angle.

S3: It is not just one particular (right) angle … but anyone… for a specific position of the rods (MC).

Here is shown clearly the conceptual aspect of the argument. To the students the angle is not a characteristic of a static position of the artifact, but is dynamically created independently of the nature and position of the artifact on the planar surface (generalization). The right angle has been disconnected from the particular tool and is being described dynamically through its measure as a specific value in the interval 0°-180°.

S4: (*She opens and closes the rods and observes where the ends move and where the joint*) … the ends of the rods (the points O, A and B in Figure 2) … remain always on the same line as O (MF).

Interviewer: How do you know that those are on the same line?

S2: From the triangles […]

Interviewer: You assume that the base of one isosceles is an extension of the other, how do you know?

The student assumes that the bases of two isosceles triangles are on the same line. The researcher's intervention is critical. She is suspecting that parallelism will allow her to transfer angles so as to justify the conjecture of the points' co-linearity.

S1: Those (*points at pairs of opposite rods*) … are parallel … they are always parallel (MF)… from construction … because their distance remains always equal.

Parallelism is suggested to the students by the artifact's structure, and is reinforced during the artifact's movement. Substantiation of equal distance bears is theoretically unfounded.

The fact that the distance between the rods is always the same does not arise from a mathematical justification, but from analysis of the tool's structure. The student considers the tool as embodiment of some geometrical properties. From the moment they regarded the square as a special case of rhombus, they pointed out the constant distance (: opposite sides of the square) between the lines containing opposite rods, a fact that leads them to parallelism. In fact the student overgeneralizes (Gärdenfors, 2004, p.151) the equal distances in the case of the square, to any other position of the rods. She implies that in any configuration (even in the case of rhombus) the distances are equal.

Interviewer: How do you know this distance is always equal?

S1: The distance from here (*the student opening two fingers represents the supposed distance between two opposite rods*) is always equal to this one (DH=DZ, in Figure

6) and are equal in all positions (Figure 7) … same with this one (Figure 8-a square) … they can also be unequal, of course they're not always the same but they're equal in every position (MF) and the maximum distance is when it (the tool) forms a square (MC).

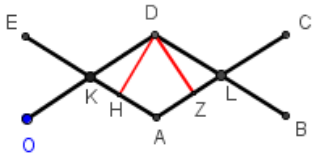

Figure 6

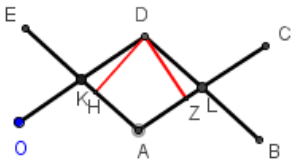

Figure 7

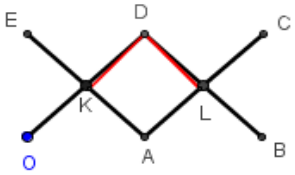

Figure 8

Figures 6,7,8: Schematically represent the conformations of the artifact that the student trace on the drawing paper.

The student's spatial conception is topological in nature. This conception appears first as an overgeneralization, and following the interviewer's persistence, it is expressed clearly through the movement of the artifact. For S1 rhombus and square are topologically equivalent (homeomorphic), leading to the conservation of equal distances (Figures 6, 7, 8). Piaget and Inhelder (1967) consider the structures of topology, to be the origin of the ontogenesis of spatial thinking.

Interviewer: Why is it the maximum distance?

S1: Because if I go over here (*she refers to her equivalent to Figure 6 drawing on the board*) from a right triangle (*shows on the drawing the triangle DZL to which she's referring*) I have a smaller distance, because this distance (DZ) is the smallest, as it's a leg … so these form this angle equal to this angle (*referring to the right angles of the right triangles DHK and DZL in her drawing*).

The argument the student formulates in her attempt to justify why the distance of opposite rods becomes maximum, in the case of a square, has two components. One concerns a geometrical property (: the perpendicular segment is shorter than any oblique) (MC), while the second component is based on figural characteristics related to the conformations of the tool (rhombus –square) (MF). In line with Radford (2003), we could consider the student's argument as a factual generalization: "A factual generalization is a generalization of actions in the form of an operational scheme that remains bound to the concrete level." (Radford, 2003, p.65).

Interviewer: But why are they parallel?

S3: Since the general shape is a rhombus (MC) and the distances between them (*meaning the opposite sides*) are always equal (MC).

The students identify the characteristics of a geometrical problem they have tackled in the past (distances of opposite sides in a rhombus are equal), but being unable to give a geometric proof, remain on a topological approach.

Discussion and conclusions

Our research hypothesis was that the argumentation produced by the students, is in close relation with the machine's characteristics. In fact, in agreement with the findings of Antonini and Martignone

(2011) students produce arguments both on a figural and a conceptual level even for the same investigation. The arguments used by the students are supported by the linkage's continuous movements in the action zones or between limit configurations.

Students produce arguments based on the figural characteristics of the tool mainly in two cases: when the structure of the tool renders their observation probable (3 collinear points), or when the tool produces the "proof" mainly through the linkage's continuous movements (: the triangle is isosceles, because the rods overlap). The figural characteristics make some facts "obvious" for the students to the point to give to the observer the impression of lack of substantial comprehension of geometrical definitions. For example, the students, in spite of recognizing the rhombus, do not readily deduce the parallelism of its side, and seek more complex proof, mainly via the artifact's attributes.

The instrument's structure may favor the theorems-in-action formulation (Fischbein, 1993), providing convincing argumentation: an angle is right, as it can be constructed by the artifact on a specific moment of its movement. Although the students were familiar with elementary proof in the framework of Euclidean geometry, their way of thinking was mainly topological. It seems that the mathematical tool, through its capacity for motion, favors such approaches. This was demonstrated not only in the justification that an angle is right, but also in the justification of two lines' parallelism. This fact implies the benefit of artifact use in Geometry teaching, before a formal introduction to the concept of proof.

Nevertheless, the tool's restrictions (e.g. rods not fully overlapping), create a conflict between the figural aspects of the structure and the conceptual aspects of the geometrical figures of the articulated system. For example, when trying to superimpose the wooden sides of the triangle in order to check if they are equal, students face the tool's restriction. Nevertheless, they can imagine it as an isosceles triangle. Those restrictions have two outcomes: a positive one being that students are forced to think in a more abstract manner and the negative one that the restrictions may lead them to false conclusions, giving the occasion of fruitful discussions.

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Learning with the logic of inquiry: game-activities inside Dynamic Geometry Environments

Carlotta Soldano[1] and Ferdinando Arzarello[2]

[1]University of Torino, Italy; carlotta.soldano@gmail.com

[2]University of Torino, Italy; ferdinando.arzarello@unito.it

The paper describes a game-activity proposed to 7th grade students with the goal to make them discover the geometric property concerning the mutual relationship between two circles. The activity, called "the game of the two circles", is composed of a strategic game that students play in GeoGebra and an investigative task which requires conjecturing and generalization. The aim of the activity is to trigger an approach to mathematics based on the logic of inquiry. We analyse students' dialogues and actions paying particular attention to the additional values the game confers to the more traditional exploratory activities with dynamic geometry software.

Keyword: DGE game-activity, played-game, reflected-game, inquiring and justifying processes.

Introduction

Many studies in mathematics education have documented the importance of making students explore mathematical situations before asking them to construct proofs (Boero et al. 1996, Pedemonte 2007). The exploration triggers the formulation and the checking of conjectures, introducing students into logical ways of reasoning. As pointed out by Dewey, all forms of logics, included the deductive logic, are consequence of inquiry processes:

> all logical forms (with their characteristic properties) arise within the operation of inquiry and are concerned with control of inquiry so that it may yield warranted assertions. This conception implies much more than that logical forms are disclosed or come to light when we reflect upon processes of inquiry that are in use. Of course it means that; but it also means that the forms originate in operations of inquiry. (Dewey, 1938, p.3,4)

Boero et al. (1996) have observed the possibility of a *cognitive continuity* between the processes of discovering and justifying. It occurs when the students, in the construction of the proof, exploit the argumentations employed for producing the conjecture. Pedemonte (2007) has distinguished between a cognitive unity in the *referencial system* and in its *structure*. The first occurs if some expressions, drawings, or theorems used in the proof have already been used in the argumentation for supporting the conjecture. The second occurs if the inferences produced in the argumentation and in the proof are connected through the same structure (abduction, induction, deduction). Hintikka (1999), an eminent Finnish logician, analyzing Sherlock Holmes way of reasoning in inquiry processes, showed that the clever deductions he made are obtained by reversing the abductions (Peirce, 1960) produced while investigating. His works demonstrates the existence of an *epistemic unity* between inquiring and justifying processes.

In this study, we favor and emphasize the possibility of connections between the discovery and justifying processes by introducing *strategic games* within Dynamic Geometry Environments (DGEs): these are games in which players have to make strategic choices meant for setting up and coordinating actions aimed at the achievement of a goal. As it is known from the literature (Arzarello

et al. 2002, Baccaglini-Frank & Mariotti, 2010), DGEs are particularly apt for triggering inquiring processes. Our conjecture is that in virtue of the game, the conjectures and abductions produced inside the DGE are not left isolated but can be connected together and reorganized in logical chains. In fact, for making a strategic choice within a game situation, players reflect backward on the moves made and forward on the possible moves to make. These reflections can support the construction of logical links through which reorganize the geometrical invariants observed during the players' moves. In fact, since the moves are made on dynamic figures and involve geometric elements, we wish that the strategic reflections made on the moves could affect also the geometric elements involved in the moves. For this reason, *DGE game-activities* can promote a kind of thinking which is different from the one triggered by more traditional explorative activities with DGS.

Theoretical framework

The interrogative logic or logic of inquiry, introduced by Jaako Hintikka (1999), proposes a back to the origin consideration of the discipline. According to Hintikka, the modern logic switched from the study of excellence in reasoning to the study of infallibility in reasoning: "preserving one's logical virtue becomes a more important concern than developing virtuosity in drawing logical inferences" (Hintikka, 1999, p.28). The rules of inference are definitory rules, which inform us about the possible inferences, but do not say anything about which inferences are appropriate in the current moment, which are not so and which ones are better than others. These types of considerations are the concern of strategic principles.

Hintikka conceives the process of seeking new knowledge as an *interrogative game*, which is a two-player game between an *inquirer*, who asks the questions and an *oracle*, who answers him. Observations can be thought of as answers put to an environment, a controlled experiments, a database stored in the memory of a computer, a diagnostic handbook, a witness in a court of law, or one's own tacit knowledge partly based on one's memory can be considered as questions asked to nature. "Strategies of questioning play a central role in interrogative games, these include strategies of information seeking by means of different choices of questions to be asked and of the order in which they are asked." (Hintikka, 1999, p.34).

Hintikka models the processes of verification and falsifications through a *semantical game* (Hintikka, 1998), which is a two-player game between a verifier, whose goal is to show the truth of a mathematical formula or statement and a falsifier, whose goal is to confute it. In order to establish the truth of the mathematical formula $\forall x\, \exists y \mid S[x, y]$ it is possible to imagine a game in which the falsifier choses a value x_0 "in the most unfavorable way as far as the interests of the verifier are concerned" and the verifier should find a value y_0 for y such that $S[x_0, y_0]$ is true. The formula is true if there exists a winning strategy for the verifier of the game, while it is false if there exists a winning strategy for the falsifier of the game.

In our study, taking inspiration from Hintikka's semantical game, we designed DGE game-activities in order to aid students in their discovery of geometric properties, through the game-play and the guiding questions. Analysing students' actions, we distinguish between two ways of using the game: the played-game and the reflected-game (Soldano & Arzarello, 2016). In the *played-game*, the students' aim is to win against their opponent. To reach this goal they activate strategic principles which help them to select the best move to make in a given situation. In the *reflected-game* students

play the game in order to answer the questionnaire and to communicate with each other. They play the game in a fictitious way: the game helps students to formulate the correct answer. In the reflected game we distinguish between the two main cognitive processes that characterizes dragging practices (Saada-Robert, 1989; Arzarello, 2002; Olivero, 1999): ascending and descending processes. We recognize ascending processes when students use the game in order to explore the situation and formulate a conjecture and descending processes when they use the game to check it. We have integrated this analytical tool with a new cognitive modality: the detached modality, in which students refer with words to the dynamic observed in the game, but they do not use it concretely.

The game of the two circles

The game-activity presented in this paper is based on the relationship between the distance between the centres of tangent circles and the sum/difference between their radii. Students play the game on the GeoGebra file shown in Figure 1. The GeoGebra window is divided in two parts: on the right there is the numerical window with sliders and variables, in the left the graphic window here is a graphic representation of the geometric objects.

Sliders *a, b,* and *c* control respectively the distance between the centres, the radius of the circle with centre O and the radius of the circle with centre O'. The variables *d*, *e*, *f* are respectively the absolute value of the difference between the radii (d=|b-c|), the distance between the centres and the sum of the radii (f=b+c). When students drag sliders *b* or *c*, they can observe the synchronic variation of the values of *d* and *f* and of the length of one circumference.

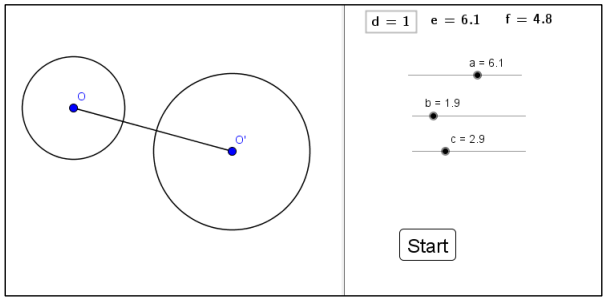


Figure 1: Game-activity

The game develops as follows: player B, the verifier, controls slider *b*, player C, the falsifier, controls slider *c* while player A, the referee, controls slider *a* and the hourglass. The goal of player B is to make e=d or e=f, the goal of player C is to make e≠d and e≠f. At the beginning of each match, the referee chooses the value of *a* and turns the hourglass over. Each time a player reaches his goal, the referee turns the hourglass over and the turn moves to the opponent. If the player cannot reach the aim within the time on his/her hands, he/she loses. The dynamic described is that of a semantical game played on the following statement: for every value of *c* there exists a value of *b* such that the circles are internally or externally tangent.

Each time that player B reaches his goal he produces an example of internally or externally tangent circles (look at Figure 2 a, b). Contrastingly, each time player C reaches his goal he produces an example of non-tangent circles (look at Figure 2 c, d, e). Since the interval of the sliders can take values from 0 to 10, players can produce also degenerate configurations (look at Figure 2 f, g). Player B can win also in this situation (look at Figure 2 h).

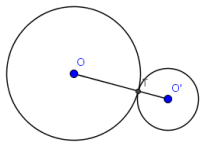
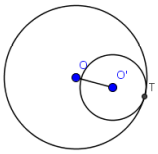
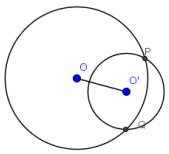
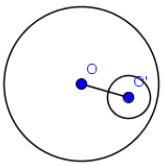
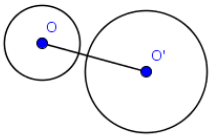
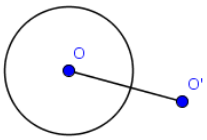
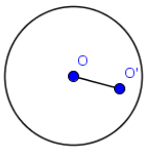
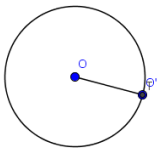
a) 	b) 	c) 	d) 
Externally tangent circles e=f	Internally tangent circles e=d	Non-tangent circles e≠d ∧ e≠f	Non-tangent circles e≠d ∧ e≠f
e) 	f) 	g) 	h) 
Non-tangent circles e≠d ∧ e≠f	Degenerate non-tangent circles e≠d ∧ e≠f ∧c=0	Degenerate non-tangent circles e≠d ∧ e≠f ∧c=0	Degenerate tangent circles d=e=f ∧c=0

Figure 2: Example space associate with "the game of the two circles"

Theoretically it is always possible for players to reach their goals. Therefor the outcome of the game is determined by the time limit.

After playing the game, students are required to answer to the following questions:

1. Which are the mutual positions between the two circles each time player B reaches his aim?
2. Which are the mutual positions between the two circles each time player C reaches his aim?
3. What do the sliders a, b and c represent?
4. What do the value of d, e and f represent?

The questionnaire is intended to help students shift their frame of reference from the game to the geometric theory. In particular, the first two questions are intended to change the focus of attention from the numerical values of variables *d*, *e* and *f* to the mutual positions between circles. In this way students discover the geometric invariants which characterizes player B's moves: each time the verifier reaches the goal the circles are tangent. Question number three is intended to link the values of the sliders to the length of the radii and the distance between the centres. Finally, question number four is intended to link the values of the parameters to the sum and difference of the radii. In this way students can discover another invariant which characterized player B's moves: each time the verifier reaches the goal the distance between the centres is equal to the sum or difference between the radii

Methodology and data collection

The study reported in this paper involves one classroom of 7th grade Italian students. The game of the two circles is the first of a group of four game-activities related to the geometry topic of circles. Note that the properties on which the game are designed are not part of the classroom knowledge: the goal of the activity is to guide students in their discovery. Each activity lasts almost two hours: in the first hour and half students are divided into groups of three students and they play the game and answer a questionnaire using a computer or a tablet. In the last half an hour the teacher revisits students discoveries and systematizes the mathematical knowledge. The data for the analysis includes the transcript of students' dialogue and the GeoGebra diagrams explored during the game and the

questionnaire. We videotaped two groups and the final class discussion but, for space reason, we will represent only one group's work.

Analysis

The videotaped group is composed by three students: Gu and Al are males, Bia is a female. They play the game on the computer. In the first match Bia is the referee, she chooses the value of a and she turns the hourglass. Gu is player C, the falsifier. He has to move slider c so that e≠d and e≠f. Al is player B, the verifier. He has to move slider b so that e=d or e=f. Figure 3 contains, in the first row, the diagrams produced during the first match. Below each diagram are the reported values of sliders and variables which appear in the numerical window. Finally the last row contains students' role (Falsifier (F), Verifier (V)) who produces the diagram, the type of example created and the time spent producing it. Remember that slider a controls the distance between the centres, slider b the radius of the circle with centre O and the slider c the radius of the circle with centre O'. The values of d, e and f are, instead, the respective absolute values of the difference between the radii, the distance between the centres and the sum of the radii.

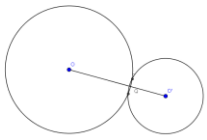
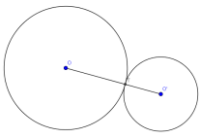
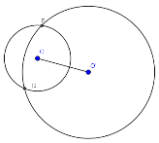
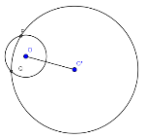
a)	b)	c)	d)
			
$a = 8 \;\; b = 5.1 \;\; c = 3$	$a = 8 \;\; b = 5 \;\; c = 3$	$a = 8 \;\; b = 5 \;\; c = 10$	$a = 8 \;\; b = 3.3 \;\; c = 10$
$d = 2.1 \;\; e = 8 \;\; f = 8.1$	$d = 2 \;\; e = 8 \;\; f = 8$	$d = 5 \;\; e = 8 \;\; f = 15$	$d = 7.7 \;\; e = 8 \;\; f = 13.3$
F, secant circle,7 sec	V, externally tangent, 6 sec	F, secant circle, 4 sec	V, secant circle, Time's up

Figure 3: First match

The match lasts short length time and it ends with the winning of the falsifier (Figure 3). In the last move the time ends before Al reaches his goal, hence Al loses even if, theoretically, he could have won. Al knows that he could have won if he had had more time, in fact he says "it should have been like this", making internally tangent circles. After Al demonstrates the winning configuration, Bia says "So B should always win". This sentence reveals the activation of the anticipatory thinking (Harel, 2001). After playing another match, students move to the first question.

Gu: In order to reach the goal they have to touch each other in only one point.

Al: On the other hand the answer to the question: 'In which mutual positions are the two circles when C reaches his/her goal?' is any position. They can touch each other in two points or nowhere.

Bia: No, they always touch each other in exactly two points. (looking at an example of secant circles).

Gu: They can touch each other in two points, but they can also not touch each other.

Bia: Ah… (moving *c* so that the circles do not intersect each other. Then she moves c back and forth for 30 seconds) Yes, that's right!

Al and Gu approach the question in a different way from Bia. They are in detached modality, they rethink what has happened in the played-game and then they answer the question. Bia, instead, uses

the game in order to investigate the situation, she is in descending modality: she is using the reflected-game in order to check her schoolmates' claims. The group repeats this approach (detached versus descending) in answering the subsequent questions. When they get to the last one, the students do not agree with each other: according to Al and Bia, *d* and *f* are the radii of the circles, while Gu does not agree with them. The disagreement invokes the need of a justification.

Al: Let me prove that it's the radius (taking the mouse)! They have to coincide perfectly… (moving the centre O' on the other circumference, see Figure 4e)

Gu: It's not the radius… Because if you change the radius of a circle, you don't automatically change the radius of the other one! Both values [d and f] change! You should change just one [d or f] by moving it [b] by changing the length of one radius… you are not changing the other one!

Al: Don't you notice? Don't you see? (he moved c to 0, obtaining that e is equal to d and f, see Figure 4f). Point T just appeared and, putting this on zero. Do you notice? It's 2.9 2.9 2.9.

In order to refute Al's conjecture, Gu tries to explain the contradiction he noticed between the graphic and the numerical window. In detached modality, Gu explains why Al's claims creates a dynamic contradiction in the conversion (Duval, 2006) from the numerical to the graphic register. Al, instead, tries to provide evidence for explaining that what he claims is true. In this attempt, he uses the value of the parameter *e* (distance between the centres) in order to measure the length of the radius of one circle. His goal is to show that this value is equal to the value of *d* or *f*. If this process were to be applied on a generic example, it would have led to a contradiction, but since Al moves the value of the radius of the circle O' to 0, he produces a supporting example.

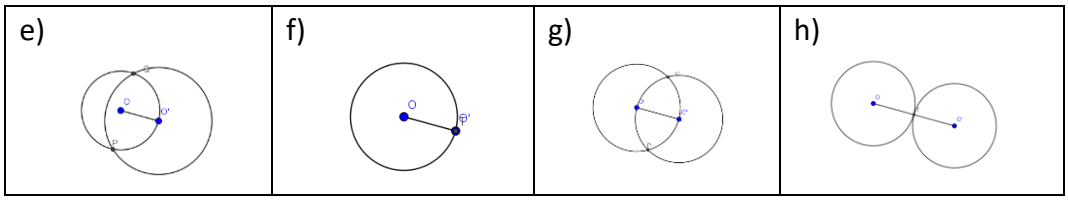


Figure 4: Reflected-game

After exploring silently the situation, Bia who at first supports Al's claim, changes her mind.

Bia: Anyway he's right… If you put them like this (each centre belongs to the other circle, see Figure 4g), the radius is the same thing, isn't it? I mean, it's the same here and here (pointing at the two circles), but here they are different (pointing at the values of sliders d, f)… Then it must be another thing, do you get it?

Adopting Al's strategy, she uses the distance between the centres to create two circles with the same radii in the graphic window. She notices that the two parameters which are supposed to be the radii are not equal in the numerical window. In contrast to Gu, who creates a dynamic counterexample, Bia exhibits a static counterexample, but this one also fails to convince Al that he is mistaken. The discussion continues with Gu who repeats his dynamic example, Bia who produces other static counterexample and Al who moves back to his supporting special example and cannot understand why he is wrong. Finally, observing the value d=0 when circles are externally tangent (Figure 4h)

and with the same radii, Al formulate a new conjecture: d is the difference between the radii of the circles. This discovery allows him to unlock the situation and to explain his special example.

Discussion and conclusion

Game-activities can operationalize a functional approach to geometry. Within these activities, students deal with *soft* tangent circles, namely dynamic circles in which some constructive steps that make the circles *robustly* tangent (the tangency is preserved by dragging) are voluntarily not performed (Healy, 2000). A constructive step creates a functional dependence between the geometric elements, which is hidden in the robust construction of the figure. In DGE game-activities, this functional dependence is made explicit through the verifier/falsifier's dialectic. More precisely, when the verifier has observed the invariant tangent configuration produced by his/her moves, he/she can create a cause-effect link between his/her goal and the invariant produced.

We describe the verifier's dragging as follows: $b \xrightarrow{values\ of\ the\ sliders} tangent\ circles$ which indicates that the verifier, by moving the slider b, can observe the invariant tangent configuration as the effect of making sliders values coincide. Once discovered the invariant, the verifier can accomplish the move with the goal of building tangent circles. In this case, he observes the coincidence of the values of the slider as the effect of making tangent circles. This time the verifier's dragging is described in this way: $b \xrightarrow{tangent\ circles} values\ of\ the\ slider$. By switching the focus of attention of the move, the DGE game-activities, create a sort of frame, which helps students to appreciate the "if and only if" relationship between tangent circles and the fact that the distance between the centres is equal to the sum or absolute values of the difference of the radii.

DGE game-activities enrich the exploration supporting the in-depth investigation of situations: the presence of the falsifier, who tries to create trouble to the verifier, exposes the verifier to different initial situations triggering the exploration of both standard and non-prototypical examples of tangent circles. In this way, the game-activities enlarge students' *accessible example space* (Goldenberg & Mason, 2008) associated with tangent circle configuration. This is a very important aspect for the construction of mathematical concepts: proposing students only standard configurations can be source of mathematical misconceptions.

Finally, the game tool enriches and supports students' arguing abilities and the coordination of numerical and graphic information. In order to communicate their claim, students activate a versatile use of the game: not only for formulating and checking conjectures but also for supporting their claim, confuting different opinions and explaining ones' point of view. The game assumes a fundamental role in promoting mathematical ways of reasoning. Al, for example, uses the game to show evidences of the truth of his claim, hence uses the game for constructing a supporting example, while Bia and Gu use it to show that he is wrong, hence for constructing counterexamples to what has been claimed by Al. In producing these arguments, students make conversion between numerical and graphic registers. Concluding, the game instruments help students not to assume the absolute truth of external opinions, but to establish a dialectic approach to them.

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Reasoning-and-proving in geometry in school mathematics textbooks in Hong Kong

Kwong-Cheong Wong

The Hong Kong Polytechnic University, Hong Kong SAR, China; wongkwongcheong@gmail.com

To promote learning mathematics with understanding, mathematics educators in many countries recommend that proof play a central role in school mathematics. In response to this recommendation, this study examines the opportunities for students to learn reasoning-and-proving from the geometry strand of a popular school mathematics textbook in Hong Kong. The study adopts the methodology of Stylianides (2009). Results show that Hong Kong takes the traditional but problematic approach in which proof is taught mainly in geometry and in which two-column proof is emphasized. Overall, results suggest that proof plays a marginal role in school mathematics in Hong Kong.

Keywords: Reasoning-and-proving, school mathematics textbooks, geometry, Hong Kong.

Introduction

In addition to verifying the truth of a mathematical statement, proof can have many other important functions in mathematics, including explanation, which can promote sense making and understanding in mathematics (de Villiers, 1990). As a consequence, many mathematics educators around the world, especially those in the US, recommend that proof (and proof-related reasoning) permeate school mathematics at all levels and in all content areas (e.g. Ball et al., 2002; NCTM, 2000). Furthermore, since textbooks can have an influence on what students learn, many studies have been conducted in different national curricula (e.g. US, Israel, Australia) to examine the opportunities for students to learn reasoning and proof from school mathematics textbooks. These studies were conducted at various grade levels (e.g. middle school, high school) and content areas (e.g. algebra, geometry); for example, see Stylianides (2009) and the articles devoted to this topic in Stylianides (2014). However, almost all of these studies were conducted in Western countries whereas only few studies have been conducted in East Asian countries (e.g. Singapore, South Korea, Hong Kong) where students have consistently performed very well in international studies of mathematics achievement such as TIMSS (e.g. Mullis et al., 2012). The present study is part of an on-going project aimed to complement the international research knowledge by examining the opportunities for students to learn reasoning and proof when they are using a popular secondary school mathematics textbook from Hong Kong. It is expected that the results obtained will shed light on how proof is being treated in school mathematics in one of those high-achieving countries (or regions) and provide insights into the influences that Chinese culture may have on issues concerning understanding in school mathematics. This paper reports our findings in geometry; for our findings in algebra, see Wong & Sutherland (2016).

The context: Hong Kong SAR

Being a special administrative region (SAR) of China, Hong Kong enjoys curriculum independence, in the sense that Hong Kong designs her own school curriculum, which is different from that in China. In 2009, Hong Kong launched her new academic structure, under which the number of years for senior secondary school changed from four years to just three years (Secondary 4, 5 and 6). Accordingly, at the same time Hong Kong initiated her New Senior Secondary Mathematics

Curriculum (Education Bureau HKSARG, 2007). This new curriculum consists of two parts: the Compulsory Part and the Elective Part (also called the Extended Part). It should be pointed out that teaching proof is not one of the stated goals (or processes) of the curriculum, which mentions proof only locally in the learning targets of geometry, namely, to "formulate and write geometric proofs involving 2-dimensional shapes with appropriate symbols, terminology and reasons" (ibid, p.15). The textbook series chosen for this study is the popular *New Century Mathematics* (2nd edition, Leung, Frederick K. S. et al., 2014–16). There were three reasons for choosing this textbook series: (a) this textbook series was one of the most popular ones in Hong Kong (if not the most popular one), (b) it was on the recommended booklist by the Educational Bureau, which means that it was guaranteed to be fully aligned with the new mathematics curriculum, and (c) it was coauthored by a prominent mathematics educator. Within this textbook series, there were two books (Books 4A and 4B) for Secondary 4, two books (Books 5A and 5B) for Secondary 5 and one book (Book 6) for Secondary 6. All topics in these five books were categorized into three strands: Number and Algebra, Data Handling, and Geometry (in the curriculum document (ibid.) the name "Measures, Shape and Space" was used instead).

Analytic framework and method

We followed the methodology of Stylianides (2009) in his investigation into reasoning and proof in school mathematics textbooks in the US. The framework he used was based on his conceptualization of *reasoning-and-proving* (RP), a term describing the overarching activity encompassing all of the four major proof-related mathematical activities: (a) identifying patterns, (b) making conjectures, (c) providing proofs, and (d) providing non-proof arguments. As shown in Table 1 below, the first two activities were grouped into the category of making mathematical generalizations and the latter two into the category of providing support to mathematical claims. The idea behind this conceptualization was that making mathematical generalizations (*identifying a pattern* and *conjecturing*) and providing support to mathematical claims (*proving*) are two fundamental and interrelated aspects of doing mathematics (Boero et al., 2007). Further, there were two kinds of pattern: plausible and definite; two kinds of proof: generic example and demonstration; and two kinds of non-proof argument: empirical argument and rationale. For the exact definitions of these terms, see Stylianides (2009).

Reasoning-and-Proving			
I. Making Mathematical Generalizations		II. Providing Support to Mathematical Claims	
(a) Identifying a Pattern	(b) Making a Conjecture	(c) Providing a Proof	(d) Providing a Non-proof Argument
1. Plausible Pattern 2. Definite Pattern	3. Conjecture	4. Generic Example 5. Demonstration	6. Empirical Argument 7. Rationale

Table 1: The analytic framework (Stylianides, 2009, p. 262)

In this study, we focused on the Compulsory Part of the curriculum. We examined all of the eight chapters comprising the Geometry strand (see Table 4 below for the names of these chapters). Following Stylianides (2009), we focused on the exercises in these chapters and examined all of them. In each of these chapters, exercises were categorized under various headings: Q&A, Review Exercise, Instant Drill, Instant Drill Corner, Exercise, Supplementary Exercise, Class Activity, Inquiry & Investigation, and Unit Test. Within each category of these exercises, there were many tasks. A *task*

here means any problem in the exercies or parts thereof that have a separate marker (Stylianides, 2009, p. 270). *Task* served as unit of analysis in this study and there were totally 2,929 tasks to be analyzed and categorized into the seven subcategories of the constituent activities of reasoning-and-proving set out in Table 1 above. Additionally, we extended Stylianides' framework by further dividing the subcategory "Demonstration" into seven (sub)subcategories that correspond specifically to the different proof methods that were used in the exercises of the Geometry strand of our chosen textbook series; these included (i) Proof by Definition, (ii) Proof by Calculation, (iii) Proof by Calculation and Definition, (iv) Paragraph Proof, (v) 2-Column Proof, (vi) Proof by Contradiction, and (vii) Existence Proof (see Table 3 below). To decide if a task was an RP task, we considered how it appeared in the textbook (e.g. key phrases "Prove that", "Explain your answer"). In cases where the requirements were not clear, we consulted the Teacher's Manual (which contained suggested solutions, but only suggested solutions) in order to infer what types of response was expected for students.

Examples of analysis

Although there was a considerable amount of tasks, the forms of expression of tasks providing RP opportunities were very limited. Tasks phrased with the obvious "Prove that" or "Show that" were treated as RP activities (see Examples 2, 4 and 5 below). Those tasks ending in "Explain your answer." were also treated as RP activities, because they were asking for some kind of justification (see Examples 1 and 3 below). However, in some cases there was no explicit request to explain the answer, but judging from the solutions in the Teacher's Manuals, justifications were actually expected and hence these tasks were also treated as RP activities (see Task 3 of Example 5 below). Some tasks, usually in Class Activity or Inquiry & Investigation, were special in that they were part of a template for illustrating reasoning-and-proving. Such tasks were *dually* coded: on the one hand as a unit of analysis on its own, and on the other hand as part of the constituent activity (or activities) of reasoning-and-proving being illustrated (see Example 5 below).

Example 1

4. Q(−1, 3) is rotated anticlockwise about the origin O through $90°$ to Q_1.
(a) Write down the coordinates of Q_1.
(b) If Q_1 is reflected in the x-axis to Q_2, are Q and Q_2 the same point? Explain your answer.

Solution (from Teacher's Manual 4B, p.228):

(a) Coordinates of $Q_1 = (-3, -1)$
(b) Coordinates of $Q_2 = (-3, 1)$
Coordinates of Q ≠ Coordinates of Q_2
$\therefore$ Q and Q_2 are not the same point.

Figure 1: Task 4(b) of Supplementary Exercise of Ch. 12 of Book 4B

Here Task 4(a) was not coded as any RP activity. Task 4(b) was coded as "Demonstration – Proof by Definition," by which we mean *one-step* deductive reasoning which can be derived directly from some definition (or property or theorem). This type of tasks does not involve substantive reasoning – its aim is simply to check students' understanding of the definition (or property or theorem).

Example 2

58. $A(0, \sqrt{2}+1)$, $B(-\sqrt{2}-1, 0)$ and $C(\sqrt{2}+1, 0)$ are the three vertices of ΔABC.
(a) Show that $AC = 2 + \sqrt{2}$.

Solution (from Teacher's Manual 5B, p.96):

(a) $$AC = \sqrt{\left[0-(\sqrt{2}+1)\right]^2 + (\sqrt{2}+1-0)^2}$$
$$= \sqrt{2(\sqrt{2}+1)^2} = \sqrt{2}(\sqrt{2}+1) = 2+\sqrt{2}$$
$\therefore AC = 2 + \sqrt{2}$

Figure 2: Task 58(a) of Supplementary Exercise of Ch. 7 of Book 5B

As shown, the solution involved substitution of values into the distance formula and algebraic manipulations to calculate *AC*. This task was coded as "Demonstration – Proof by Calculation". This proof method is also called "Mechanical Deduction" in the literature (e.g. Reid & Knipping, 2010, p. 124). Though involving mechanical algebraic manipulations and little reasoning, logically it should be regarded as a proof (for more on this point, see Slomson, 1996, p.11, "Proofs as Calculations").

Example 3

5. In the figure, *BM* = *CM* = 6 cm, *AM* = 8 cm and *AB* = 10 cm. *AMD* is a straight line. Is *AD* a diameter of the circle? Explain your answer.

Solution (from Teacher's Manual 4B, p.135):

$AM^2 + BM^2 = (8^2 + 6^2)\ cm^2 = 100\ cm^2$
$AB^2 = 10^2 = 100\ cm^2$
$\because\ AM^2 + BM^2 = AB^2$
$\therefore\ \angle AMB = 90°$
$\therefore$ AM $\perp$ CB
$\because$ AD is the perpendicular bisector of BC.
$\therefore$ AD passes through the centre of the circle.
$\therefore$ AD is a diameter of the circle.

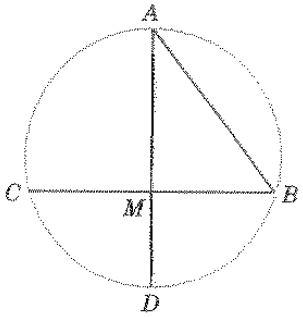


Figure 3: Task 5 of Supplementary Exercise of Ch. 10 of Book 4B

This task was coded as "Demonstration – Paragraph Proof", because it involved not just algebraic manipulations, and was written in the paragraph (or narrative) form – a less formal form in which it is not required to provide justification for every step. Paragraph proofs in geometry correspond to level 2 (informal deduction) of van Hiele Levels (Usiskin, 1982).

Example 4

11. In the figure, *PCQ* is a straight line. Chord *AB* is parallel to *PQ*. If $\overset{\frown}{AC} = \overset{\frown}{BC}$, prove that *PQ* touches the circle at *C*.

Solution (from Teacher's Manual 4B, p.182):

$\because\ \overset{\frown}{AC} = \overset{\frown}{BC}$	given
$\therefore\ \angle ABC = \angle BAC$	equal arcs, equal angles
$\angle BCQ = \angle ABC$	alt. $\angle$s, AB // PQ
$\therefore\ \angle BAC = \angle BCQ$	
$\therefore$ PQ touches the circle at C.	converse of $\angle$ in alt. segment

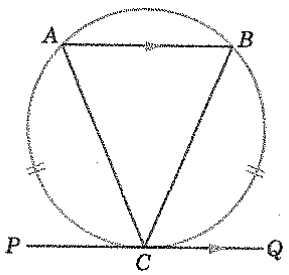


Figure 4: Task 11 of Exercise 11F of Ch. 11 of Book 4B

This task was coded as "Demonstration – 2-Column Proof", because it involved not just algebraic manipulations, and was written in the traditional two-column form – a more formal form in which every step is required to be justified with a reason and to be presented in the rigid two-column format as shown in Figure 4 above. Two-column proofs in geometry correspond to level 3 (formal deduction) of van Hiele Levels (Usiskin, 1982).

Example 5

Inquiry & Investigation 9.1: Alternative proof for the sine formula

Inquiry

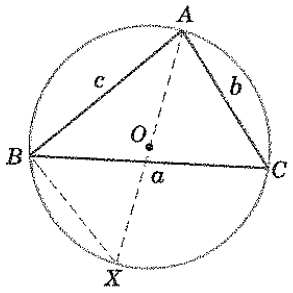
In ΔABC, what is the relationship among the radius r of the circumcircle, $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$?

Investigation Steps

In the figure, O is the centre of the circumcircle of ΔABC. The radius of the circumcircle is r. Produce AO to meet the circle at X. Join BX.

1. Find $\angle ABX$.
2. Consider ΔABX. Express $\sin X$ in terms of c and r.
3. What is the relationship between angles C and X?
4. (a) Use the results of Questions 2-3 to express $\frac{c}{\sin C}$ in terms of r.

   (b) Use similar method to express $\frac{a}{\sin A}$ and $\frac{b}{\sin B}$ in terms of r.



Conclusion

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} =$ _____

Figure 5: Task of Inquiry & Investigation 9.1 of Ch. 9 of Book 5B

This exercise consisted of six tasks (1, 2, 3, 4(a), 4(b) and Conclusion). It was a template for illustrating a direct proof. So these tasks were dually coded. First, each task was coded as a unit of analysis on its own. In this example, Task 3, and only Task 3, could be interpreted as an RP activity (Demonstration – Paragraph Proof or 2-Column Proof) in case the solution given in the Teacher's Manual would include a justification (e.g. "Angles in the same segment"). However, the solution given was just "$C = X$", so it was not regarded as a reasoning-and-proving task. Neither were the tasks 1, 2, 4(a), 4(b) and Conclusion. Then, each task was coded as part of the template illustrating RP activities. In this case, all of them were coded as "Demonstration – Paragraph Proof". For more examples, see the full version of this paper.

Results and Discussion

We have three major findings. Firstly, as shown in Table 2 below there were relatively limited opportunities (444 out of 2,929 tasks, i.e., 15.2%) for students to learn RP from the exercises of the Geometry strand of the chosen textbook series (Secondary 4 – 6). The majority of these exercises were to drill procedural skills. Secondly, there was a large difference between Making Mathematical Generalizations (24 tasks) and Providing Support to Mathematical Claims (420 tasks). This suggests that these two categories of activities were treated, in large part, in isolation from each other. This is problematic as they are two fundamental and interrelated aspects of doing mathematics (Boero et al., 2007; Cañadas et al., 2007; Hsieh et al., 2012). Thirdly, the majority of the RP opportunities were Demonstration (364 out of 444, i.e., 82%). However, as shown in Table 3 below, out of these 364 demonstrations, 32.1% were Proof by Definition or Proof by Calculation or Proof by Calculation and

Definition, all of which involve little reasoning. If we excluded them from Demonstration, the total RP opportunities would reduce to 11.2% (= 444 – 83 – 26 – 8 out of 2,929 tasks). A consequence that might be attributed to this lack of sufficient emphasis on proof even in geometry is that, as informed by TIMSS 2011 (Mullis et al., 2012, p.148 & p.150), "Hong Kong students in general do well in Knowing items, and relatively badly in Reasoning items" (Leung, 2015, p. 3).

Reasoning-and-proving subcategory	*Frequency*	*(Percent)*
I. Making Mathematical Generalizations:	24	(5.4%)
(a) Identifying a Pattern:		
1. Plausible Pattern	0	(0.0%)
2. Definite Pattern	12	(2.7%)
(b) Making a Conjecture:		
3. Conjecture	12	(2.7%)
II. Providing Support to Mathematical Claims:	420	(94.6%)
(c) Providing a Proof:		
4. Generic Example	18	(4.1%)
5. Demonstration	364	(82.0%)
(d) Providing a Non-proof Argument:		
6. Empirical Argument	38	(8.6%)
7. Rationale	0	(0.0%)
Total:	444	(100%)

Table 2: Frequency and Distribution of RP Tasks across RP Subcategories

On the other hand, as shown in Table 4, 36% of the total RP tasks were concentrated in one chapter, namely, Book 4B Ch. 11 *More about Basic Properties of Circles* – more specifically, in Section 11.5 *Geometric Proofs on Circles*, which began with "We learnt many theorems relating to properties of circles in Book 4B Chapter 10 and this chapter. In this section we will learn how to use these theorems to prove more geometric properties." In the exercises of this section, almost every task asked for a 2-column proof, suggesting that the curriculum took a traditional approach in which proof is taught mainly in geometry and in which 2-column proof is emphasized. However, this approach to proof is problematic as it gives a misrepresentation of the nature of proof in mathematics (Wu, 1996) and its emphasis on form over meaning can lead to a shallow, syntactic kind of knowledge, rather than a connected understanding of the mathematics involved (Schoenfeld, 1988).

Given that Hong Kong teachers rely heavily on textbooks in their teaching (Tam et al., 2014), the above results not only confirm that in secondary school classrooms in Hong Kong students' activities mainly focus on practicing and memorizing mathematical concepts and procedures (Leung, 2001), but also suggest that proof plays a marginal role in school mathematics in Hong Kong. The fact that school mathematics textbooks in Hong Kong stress drilling on procedural (or calculation) skills far more than reasoning and proof may be due to influences from Chinese culture (or, more specifically, the Confucian heritage culture or CHC). According to Leung (2006, p. 43), CHC believes that "the

process of learning often starts with gaining competence in the procedure, and then through repeated practice, students gain understanding." Additionally, CHC is an examination-oriented culture. In fact, the curriculum in Hong Kong is highly examination-driven. The fact that Hong Kong school mathematics textbooks stress practicing procedural (or calculation) skills far more than reasoning and proof may be a reflection of the strong influence of public examinations on textbook design. For more on how Chinese learn mathematics, see for example Fan et al. (2004).

Proof method	*Frequency*	*(Percent)*
Paragraph Proof	174	(47.8%)
Proof by Calculation and Definition	83	(22.8%)
2-Column Proof	71	(19.5%)
Proof by Definition	26	(7.1%)
Proof by Calculation	8	(2.2%)
Proof by Contradiction	1	(0.3%)
Existence Proof	1	(0.3%)

Table 3: Frequency and Distribution of Proof Methods used in Demonstration

Topic	*Frequency*	*(Percent)*
Book 4A Ch. 2 Equations of Straight Lines	56	(12.6%)
Book 4B Ch. 10 Basic Properties of Circles	46	(10.4%)
Book 4B Ch. 11 More about Basic Properties of Circles	160	(36.0%)
Book 4B Ch. 12 Basic Trigonometry	47	(10.6%)
Book 5B Ch. 7 Equations of Circles	67	(15.1%)
Book 5B Ch. 8 Locus	26	(5.9%)
Book 5B Ch. 9 Solving Triangles	21	(4.7%)
Book 5B Ch. 10 Applications in Trigonometry	21	(4.7%)

Table 4: Frequency and Distribution of RP Tasks across Topics

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An exploratory teaching style in promoting the learning of proof

Xiaoheng Yan[1], John Mason[2] and Gila Hanna[1]

[1]OISE, University of Toronto, Canada; xh.yan@mail.utoronto.ca; gila.hanna@utoronto.ca

[2]University of Oxford, United Kingdom; John.Mason@open.ac.uk

Our overall concern is with helping students learn to construct and re-construct proofs. Here we investigate an exploratory style which invites learners to think for themselves, with the instructor circulating among them while listening, probing, and suggesting. The objectives of this investigation are, to understand how the actions of teachers can contribute to the development of their students' thinking, and to provide explicit pedagogic strategies that teachers can use to promote their students' appreciation and understanding of mathematical proof.

Keywords: Explanation, exploration, proof, proof construction.

Theoretical underpinnings

Our paper adopts the notion of acts of teaching (Mason, 2009; 2004): something that initiates and directs; something which is acted upon; and something which mediates between these, enabling the action to take place. We proceed from this stance to analyse an exploratory style as it was observed in an advanced undergraduate geometry class, with the aim of identifying what specific contributions this style brings to the learning process.

When teachers introduce a proof task, they are likely to have a complex set of expectations of what learners will get from engaging with the task. They are (we hope) aware of, or have access to connections with pervasive mathematical themes, with other contexts in which similar ideas arise, and with the specific powers that learners have. Teachers will have views on how these powers might be developed through working on the task, and on opportunities to interact with learners during which both mathematical thinking and appreciation and comprehension of some particular mathematical topic will be deepened and enriched. Tasks can vary from following an exposition, through exploring relationships, exercising new-found procedures, and making use of newly encountered technical terms as part of their personal and collective developing narrative.

Following the *Systematics* of Bennett (1956-1966; 1993), we distinguish six modes of interaction which arise from the teacher, the learner and the content playing the roles of the three impulses comprising any action: initiating, responding and mediating, and conveniently labelled by six 'exs': *expounding, explaining, exploring, examining, expressing* and *exercising* (Mason, 2004). In *expounding*, the initiative is with the teacher who uses the presence of learners (actual or virtual) as the mediator to make contact with the mathematical content in a significant way. The teacher draws the learners into the teacher's world and ways of perceiving and acting. By contrast to expounding, and also in contrast to its every-day sense, *explaining* in this framework involves the teacher using the content as a mediator in order to make contact with the student, through listening, watching and probing. The teacher tries to enter the world of the learner. As soon as the teacher thinks they 'know' where the learner's difficulty lies, the action usually reverts to exposition. *Exploring* involves students taking initiative, mediated by the teacher who may suggest a starting point for exploration, and may

make suggestions based on what students are saying and doing. *Examining* in this framework involves the student seeking to validate their own criteria against those of the teacher'. *Expressing* is what the student does when they feel the urge to articulate insights or make conjectures. This can occur for example in response to a teacher asking probing questions. *Exercising* is what a student does when they feel the desire to practice in order to gain fluency. Our interest here is on the interactions labelled as *exploring*, *expressing*, and *explaining* in the technical sense used in this framework.

Our investigation builds upon previous research. Grenier (2013) has shown that an experimental teaching approach focusing on various "research situations" can succeed in helping students master mathematical reasoning and proving. Selden and Selden (2013) considered a division of proofs into a *formal-rhetorical* part and a *problem-centered* part. In their view, the formal-rhetorical part of a proof depends only on unpacking and using the logical structure of a theorem, while the problem-centered part depends on exploration and understanding, which is essential to the learning of proof.

Several researchers have investigated the use of explorations in various learning contexts. In the use of dynamic geometry, for example, Mariotti (2000), and others have shown the value of teaching proof through exploration. In the context of argumentation and proof and of the axiomatic organization of mathematics, several other researchers have examined students' use of empirical explorations (Durand-Guerrier et al., 2012; Hanna, 2010; Hemmi, 2010; Jahnke & Wambach, 2013; Reid & Knipping, 2010; Stylianides & Stylianides, 2009). Additionally, Garuti et al. (1998) and Pedemonte (2007) highlighted what they termed *cognitive unity*, the continuity between the process of conjecturing achieved through exploration and the production of an acceptable proof.

The exploratory style in this study

We investigate a particular exploratory style in which university students already familiar with linear algebra are taking an advanced geometry course in which they are exposed to proofs by means of explorations that invite them to think for themselves, with the instructor circulating among the students, listening to their discussions, asking probing questions intended to help the students think more deeply about the issue at hand, and at times offering suggestions.

Participants and classroom setting

The 24 participants were undergraduate mathematics students in a mixed-year class (2nd to 4th year) at a large urban university in central Canada. The advanced geometry course covered plane geometry, spherical geometry, and briefly, some hyperbolic geometry. It addressed the critical role of transformations (symmetries and isometries) in all of these geometries, the use of dynamic geometry software (The Geometer's Sketchpad), and proof. The instructor of the course was a geometer. The class met once a week for 3-hour sessions, 20 classroom meetings in total. The course was intended to keep equal proportion of instruction time and exploration time to facilitate the exploratory style of teaching and learning. Three classroom sequences were assigned to the investigations on conic sections. This paper focuses on one of the sessions – the exploration of ellipse.

To facilitate investigations and communication, the classroom setting of the geometry course was unconventional: 7 large round tables with chairs around each filled up the classroom, with 5 large blackboards mounted on three walls. In addition, manipulatives and visual aids associated with geometry were kept in the closet at the back of the classroom with free access for the students.

Data collection and analysis

The exploratory style of learning proofs was documented through (1) classroom observations in the form of audio recordings and field notes; (2) follow-up questions for students; (3) students' course reflections on the explorations; and (4) the researcher's research journal. In particular, the follow-up questions consist of 4 open-ended questions about students' explorations in geometry throughout the course. 17 out of 24 students completed the follow-up question sheets. All students submitted the course reflections.

The data drawn through classroom observations was organized and analyzed by the framework of 6 'exs'. The data from students' written reflections was analyzed using NVivo 10 software to explore themes and patterns of responses. The unit of analysis was a statement. Each participant's work was divided into statements and grouped in categories.

Investigations of conic sections: Findings and analysis

This paper focuses on how proving was promoted through the initiative taken by the instructor and the students, while the content connects the instructor and the students. It does not measure the students' achievement because it is concerned with perceptions of their own understanding of proof.

Initiatives of the instructor – Expounding and Explaining

The class made use of a hands-on investigation involving flashlights to explore conic sections. The instructor held a flashlight aimed at a wall at different angles. "The flashlight bulb and reflector make a "cone" of light. The wall cuts the cone with a plane, making a conic section. So moving the light changes which section we have," the instructor explained. As the beam was forming a circle, ellipse or parabola, he asked students to identify the particular shape made on the wall and to pay attention to the critical points where there was a change from one conic section to another, as the angle changed. Then the instructor raised a question about hyperbolae with a suggestion, "Now, what features will confirm the shadow is a hyperbola? You may look for asymptotes - lines which the light approaches as it goes up the wall."

Initiatives of the students - Exploring and Examining

After the demonstration, students worked in groups to create all four conic sections by using the light source of their smartphones or the flashlights provided. Students quickly discovered that when the beam was perpendicular to the wall, it gave a circle; when it was tilted a bit, it gave an ellipse; when it was tilted more, with the ellipse vanishing, a parabola emerged. Group discussions mainly focused on the creation of a hyperbola and the difference between hyperbolae and parabolae.

S8: How do you know it is a parabola or hyperbola?

S22: It depends on the angle you hold it at.

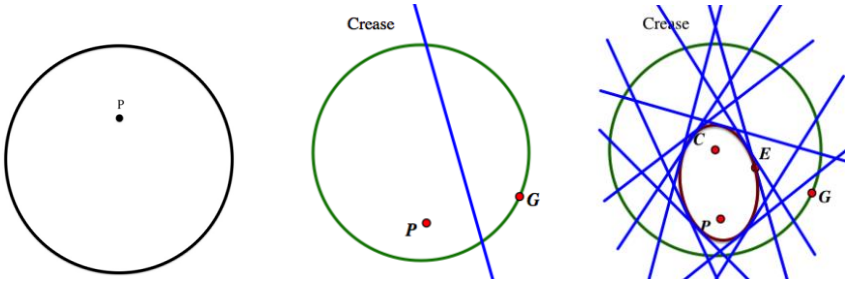
S8: Right, but how does an angle tell us whether it is a parabola or hyperbola?

S12: Well… if you look at this graph I found online, the parabola's axis is parallel to the cone's side. If it were not parallel, it would become a hyperbola.

S8: I see. So how is this related to what we are doing? The wall is the cutting plane and the light source is the cone. When the wall is not parallel to the borderline of the beam, it is a hyperbola. I cannot make them parallel precisely, but I get it.

This group discussion shows that in the course of their exploration, students did not limit their exploration to the flashlight demonstration but went on to research the problem by retrieving information online so as to better understand the features that confirm that the shadow was a hyperbola, the difference between hyperbolae and parabolae, and also to explain it to their peers.

Following the flashlight investigation, a series of paper folding activities was carried out. Taking the ellipse as an example, each student was given a clean sheet of paper with a circle and a point P inside the circle but away from the center (Figure 1a). Students were first asked to pick a point on the circle, say G, and fold the paper until P was lying directly on top of G (Figure 1b), and then to make a neat crease. Then students were asked to repeat the fold and crease action for a few dozen relatively evenly spaced points on the circle and to observe what shape emerged (Figure 1c).



a b c

Figure 1: The sheet of paper for folding to create an ellipse

Some students struggled to work out which point was being folded and to where – mistakenly folding a chosen point onto some other point on the circle, while others struggled to create a precise fold, due to the nontransparent nature of paper sheets. Although students worked on the folding at their own pace, within groups, they were talking to and helping each other as they proceeded, which allowed the ones who were struggling to listen, watch and move forward. With a number of creases created, conjectures were emerging in groups. For instance:

S6: I know that it is not going to be a circle. It is not circular. It would only be a circle if you can fold it onto the middle point. If you can fold it onto other point, it will be off-site the shape.

S17: I think it is going to be a parabola and P is going to be the focal point of the parabola. It makes sense.

Instructor: You need more folds. You can select a few more points on this side of circle (pointing at the sheet that S7 was holding).

S7: Oh, wait. It is an ellipse! I have a lot of lines. You can envision other points are going to be there. It is very clear it is an ellipse. You can really see it!

S17: (looking at S7's paper) Yeah, it is an ellipse… P looks like one of the focal point.

S6: Where is the other focal point?

Here we see that the students did not always do more than offer a conjecture about the shape of the conic created by the creases. The instructor felt it necessary to intervene and re-direct the students' attention to the core idea (the content) of the session.

Initiatives of the content - Exercising and Expressing

With more questions raised, the instructor asked each group to focus on the following questions:

(1) Now you have this ellipse, you know how to paper fold it. How do you prove it is an ellipse?
(2) If you pick one of the folds, how does this fold help us prove it is an ellipse?
(3) What can you say about this ellipse and the circle? How and why are they related?

S17: If you have this fold, and you have this distance from point P, then this distance (PE+EC) is going to equal to that distance (GC) because this is a reflection.

S6: Yes, but what does that have to do with the ellipse?

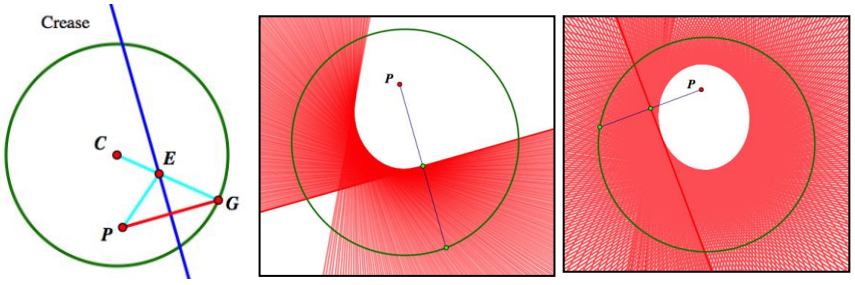
Instructor: Note: GC is the radius. These two distances (PE and EC)…

S6: No, I didn't get it.

Though the students had all the information they needed, they still had difficulty reaching the final step of the proof. By posing a prompt and question, the instructor tried to direct students' attention to the sum of the two distances (PE+EC) and the fact that it is equal to the radius. Then, a GSP graph, similar to the folding sheet in Figure 1 above, was shown to have students focus on the relationship between the two (see Figure 2a).

S17: OK. We are looking at P to E and E to C. The distance is equal to GC, which is the radius of the circle. Oh… That makes an ellipse because this distance (PE) plus this distance (EC) is fixed, the radius of the circle.

S6: And the ellipse must have something to do with the center as a focal point. If P is moving around according to the center point, the ellipse will just move as you move the P. So the center is another focal point. You can take any circle and a point off the center of the circle, and it will always be the case.



a b c

Figure 2: A GSP graph (a) and animation of the formation of an ellipse (b and c)

A Geometer's Sketchpad (GSP) animation was shown at the very end of the class. Figures 2(b) and 2(c) were two snapshots of the GSP animation indicating how an ellipse was constructed. This process taught them that, "A hypothesis is evaluated by deductively drawing consequences and by investigating whether these consequences agree with experience or should be accepted for other reasons" (Jahnke & Wambach, 2013, p. 469).

In a subsequent class, students were shown a demonstration in which sand was poured onto a circular disk with an off-center hole in it corresponding to the point in Figure 1a. A number of the students voluntarily poured the sand through a plastic strainer. As more sand poured onto the board, more a "ridge" clearly emerged. Looking down on it from above, it appeared to be an ellipse.

Students' reflections

Paper folding and sand pouring

More than half of the students admitted that they found paper folding complicated. While struggling with the first step, they missed the instructions for the second step. Following what was being said and doing folds correctly was an obvious challenge. Despite the confusion and the errors they made in folding, students appreciated the geometry embedded in the experiments. As S3 put it: "The reason for why the specific folds result in the shape never bothered me until I had this experience. I felt that I had a teacher explained the mathematical relevance." S4 observed that seeing the ellipse emerge during the sand pouring, was completely unexpected: "You can actually see an ellipse due to gravity pulling the excess parts down, forming a 'hill'." When asked about the definitions of conic sections, S14 stated that, "it is much easier, at least for me, to recall a process or property that I have physically manipulated or seen carried out visually than to recall a written definition."

The GSP demonstrations

The students were asked whether the GSP graph and demonstration directly or indirectly helped them with the proof of ellipse. The majority of the students claimed that the animation directly helped with the proof. The responses that support this claim can be categorized as follows:

- *Have attention focused.* "We had all the information but we couldn't prove it until we saw this picture (Figure 2a) which has all the lines with the colors" (S6).
- *Accuracy*. "The animation is more visual and accurate than the paper folding" (S7); "The animation showed an infinite number of straight lines without making mistakes" (S12).
- *Legitimate process*. "The animation presented a 3D visual to experiment with" (S6); and "It provided an extended version of the folds we did in class and allowed me to continue my experience in a less time consuming and more efficient way (S17).
- *Exposure of the final product*. "It showed the final product of paper folds when you fold 100 or 1000 times" (S4).

However, 2 students claimed that the GSP indirectly helped with the proof. One student explained that the animation "helped more with my understanding as opposed to helping me write down the formal proof" (S16), whereas the other student believed that the role of the GSP was to show how the ellipse was constructed and to show how the proof connects to the demonstration (S5). Compared to the hands-on investigations, S7 and S20 claimed that the GSP animation presented in class did help with their understanding, but they had difficulty interacting with it. This is so because they did not know how to use the geometry software. As S11 said, "If I knew how to use the software, I would definitely use it more".

Discussion

Adopting an exploratory style of teaching is metaphorically a bit like heading off into unknown territory; perhaps a city or forest not previously visited, and coming across blaze marks. On the surface, the 'exploratory style' of teaching involved initial stimuli provided by the instructor, then the instructor circulating listening, probing and suggesting. Questions raised by group members were fundamental and critical for the trajectory of the sessions. Beneath the surface lie the subtleties in how much time students were given to think for themselves, to discuss with each other, to try to

resolve questions that arose, and to seek assistance from the instructor. Three different contexts in which the same shapes emerge can be seen as a form of variation (Marton, 2015) with both conceptual and procedural aspects. The instructor's commitment to experiential style of engaging students with the mathematical content provided opportunity for different forms of interaction: students were stimulated to explore, to express, to seek explanation when they felt they needed it, and even to exercise their developing ideas. This prepared them to be able to make sense of what little exposition was provided during the sessions. Being fully engaged, with their hands, their own thoughts, and discussing with their peers, enabled them to produce a proof in which they had a high degree of confidence. Because the conjectures came mostly from them, they had an interest in proving, and a desire to find a proof. The physical and virtual phenomena directed student attention to the dynamic changes and alterations of the objects that they were creating, particularly when the instructor noticed students' struggles during the exploration. Student attention was directed by their peers through formulating conjectures, raising questions, and communicating their thoughts. These allowed learners to be immersed in an environment which engaged them to make conjectures, to try to express their vague thoughts, to modify their own conjectures and to challenge the conjectures of their peers, which is in line with the observations of Grenier (2013) and Hemmi (2010).

Pedagogical implications

One of the features that distinguish mathematics from other disciplines is that mathematical conjectures ultimately require proof. One importance of exposure to mathematical reasoning and proof is that it provides learners with an opportunity to "know that they know", not because someone has asserted something but because they can justify it on the basis of previously agreed properties. In the case of the paper folding and sand pouring, it is a means to provoke students' curiosity of why it works and invite them to discover a geometric proof of ellipse on their own. The exposure to the necessity of 'why' could have great impact on promoting students' learning of proof when the activity is carefully designed and chosen.

The Geometer's Sketchpad was used in the classroom throughout the course. Introducing and using geometry software in the classroom at a regular basis can gradually change the way that students approach geometry. However, as a teacher, exploiting manipulatives and geometry software effectively requires familiarity with the materials. Each context has its own trajectory, in terms of time required to make sense of the actions and to interpret the effects. Perhaps the most important pedagogic implication is the need to stimulate students to make connections, to develop their own personal narrative concerning the connections between different manifestations of the same mathematical object.

For the student teachers in the class, we believe that the exploratory teaching style allowed them to grow as a student and as a teacher. At first they looked to be told what they must do and how they will be assessed. By engaging them in mathematical activity, they had a chance to experience themselves as mathematician-learners, and to exercise and develop their own powers to imagine and express, specialize and generalize, conjecture and convince (Mason, 2004). As S3 put it, "My focus throughout the course remained on learning rather than passing, as it should be."

Thinking in terms of modes of interaction has enabled us to add a little bit of detail to the notion of an 'exploratory style' of teaching. However, in order to provide specific advice for teachers, it will be necessary to discern even finer details, which deserves further studies.

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Looking for the roots of an argument: Textbook, teacher, and student influence on arguments in a traditional Czech classroom

Jana Žalská[1]

[1]Charles University, Faculty of Education, Prague, Czech Republic; zalska@hotmail.com

As part of a larger investigation aimed at getting a deeper insight into how particular teacher beliefs influence the role of the teacher, the students, and textbook materials in arguments that take place in one classroom, this study shows specific teacher beliefs that determine the role of each of the two other factors: the students' contributions and the textbook influence. This paper presents findings observed in a case study of a teacher who holds more traditional beliefs about teaching and learning of mathematics, in a 7th grade classroom. Namely, I present cases of conflict in preferences for particular warrant forms between: a) the teacher and the textbook authors b) the teacher's own beliefs, and c) the teacher and the students. I then interpret these in terms of the teacher's particular beliefs and show how they affect the theoretical model.

Keywords: Mathematical argument, teacher mathematical beliefs, textbook curriculum.

Introduction

Whether providing a mathematical proof of a theorem, explaining a formula, or a solution to a word problem, arguments are an inseparable part of mathematics teaching and learning. A lot has been said about argumentation practices and norms that guide those practices in various contexts. Literature has focused on the role of argumentation in textbooks, uncovering differences among textbooks in terms of arguments presented (Stacey & Vincent, 2009; Thompson, Sharon, & Johnson, 2012; Zalska, 2012), opportunities for arguments (Stylianides, 2009), and in providing teachers with support in argument-based tasks (Stylianides, 2008). The differences become even more complex when researchers describe mathematics teachers' beliefs concerning argumentation in their classrooms (e.g., Staples & Bartlo, 2012), the acceptability of certain types of arguments (Biza, Nardi, & Zachariades, 2009), and students' perceptions and preferences of arguments (e.g., Levenson, 2013; Levenson et al., 2006).

But what specific arguments actually take place when the teacher's beliefs meet the textbook authors' and the students' in a classroom? What is the role of the textbook and what role do students' contributions play in the class conducted by a teacher with a specific set of beliefs?

Theoretical framework

According to Remillard's (2005) review, the textbook curriculum's role is important but the levels of participation in the intended curriculum vary greatly. The general model holds that a teacher selects tasks from the text, designs their implementation, supplements it with other tasks, and, finally, improvises based on the student contributions (Remillard, 2005). Sherin and Drake (2006) further find that teachers approach these activities in different ways and link these to teachers' experiences as mathematical learners. Schoenfeld (2010) argues that the actions an individual takes can be explained by their enacted beliefs, goals, and resources (including knowledge).

The individual students' mathematical knowledge and their perceptions of the expectations put on arguments they produce, as well as their own preferences and beliefs, can differ from their teacher's

expectations (Planas & Gorgorió, 2004; Levenson et al., 2006). The students' contributions, requests or choices of arguments are the result of their own knowledge, beliefs, and goals; they have their weight in the negotiation of socio-mathematical norms regarding mathematical arguments and explanations.

Based on the above literature, I adapt Remillard's (2005) model to propose a framework for studying the potential influence of the three main participants on the arguments, or the "enacted" arguments. Namely, the model theorizes that: 1) the curriculum may: provide examples of, requests and opportunities for arguments (tasks) to be enacted; it may also provide guidance for the teacher, based on the textbook authors set of beliefs, resources and goals; 2) the teacher may, based on their own set of beliefs, resources and goals: evaluate (select), design and provide the examples, requests and opportunities for particular arguments and may need to make immediate decisions about arguments prompted by students, and 3) the students' actual requests for arguments, for clarifications of arguments, as well as their own arguments or claims, which in turn are given by their own beliefs, goals and resources.

In my research, I focus on the similarities and differences between the interactions of the three participants in this model, in classrooms with teachers with different general beliefs about mathematics and its teaching and learning. Further, I investigate what specific teacher beliefs underlie the particular interactions when it comes to specific types of arguments or warrants. In this paper, I present some findings in the case of a teacher who holds a set of beliefs that tend to be associated with traditional views.

Participants and data

Karen is an experienced mathematics teacher who was identified, within a broader investigation (see Zalska & Tumova, 2012), as a teacher with strong utilitarian beliefs about mathematics education. Two extensive interviews as well as short post-lesson interviews were conducted with Karen to infer her professional beliefs as well as her own intentions and interpretations of events in the lesson. She had been working with her class for almost two years prior to the data collection, to ensure that social norms in the classroom have been established. The number of students ranged from 15 to 18, with about an equal distribution of male and female students. The class use a main-stream textbook series, one of the most popular ones in the country. Karen was among the teachers who approved the choice of the textbook in her school, and her students each have a copy of it.

The data consists of interview and lesson audio-recordings, fully transcribed, and photographs from series of five lessons that Karen taught on the topic of percent. The analyzed textbook text included the unit on percent and the corresponding text in the teacher's book where authors provide teachers with commentaries for particular parts of the text and the tasks.

Data analysis

In order to be able to establish differences between mathematical arguments, I will adopt the following terms from the widely used Toulmin's model in the following way: a (mathematical) *argument* denotes a sequence of statements (including visual statements) that is provided with the intention to show that a mathematical *claim* (specific or general) is true (or not). In this study, arguments include explaining of an answer to a problem, as well as the working out of the answer to a problem. A *warrant* is one such statement that directly supports the claim. In the context of a

classroom, it is a statement that does not require further explanation, i.e. is accepted as true. I will consider two arguments to be different if they contain different forms of warrants (e.g. representations) or a different sequence of warrants.

The textbook data was analyzed in accordance with the theoretical framework: the arguments that were provided were analyzed in terms of warrant forms and sequences. The tasks were analyzed as requests and opportunities for arguments (towards a claim that contains a problem's solution). There were no specific requests for arguments.

The transcript of the lessons and the text was first analyzed for episodes of argumentation to establish specific context for argumentation and social norms in the classroom. Next, the identified episodes of argumentation were broken down to individual arguments and warrants and warrant forms were identified in order to investigate where differences between arguments were present. The kind of student and teacher participation on the argument was also taken into account, in order to separate the cases of arguments provided by the teacher (i.e. when Karen elicited an argument step by step and students only provided the final part of a requested warrant) from those suggested by students themselves.

The arguments observed were then compared to the examples of arguments in the textbook, comparing warrants and warrant forms. Further, the relevant part of teacher's manual was analyzed for commentaries and any additional rationale given a particular argument in order to get insight into the text author's beliefs. Karen's own comments about particular arguments and warrant forms, in class and during interviews, were also analyzed to gain insight into the beliefs behind her decisions.

In this paper, I present the instances when an argument chosen by Karen did not correspond with a) the textbook, b) her own belief about mathematics, and c) her students' contributions. I selected them to illustrate the choices made by Karen, to pinpoint her specific beliefs, linking them with the students' and textbook influence.

Efficiency and insight: Karen and the textbook

The arguments that Karen exemplifies in her classes when she teaches her students to solve problems involving percent differ from those in the textbook in two aspects. The textbook introduces the rectangular representation (see Figure 1) as part of problem-solving, a form of warrant(s); the authors sketch out the known and unknown quantities.

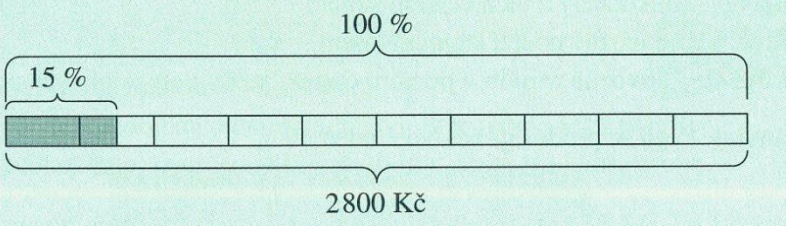

Figure 1: A rectangular representation of a 15% percent discount

Similarly, the textbook introduces one method for solving word problems with percent. The authors base the arguments on the concept of direct proportion, in particular, on the fact that the percent part changes in the same ratio as the percent. This idea is then used as a warrant in the method of the ratio-based rule of three (see Figure 2), which is explained and practiced in an earlier chapter in the book, the unit on ratios.

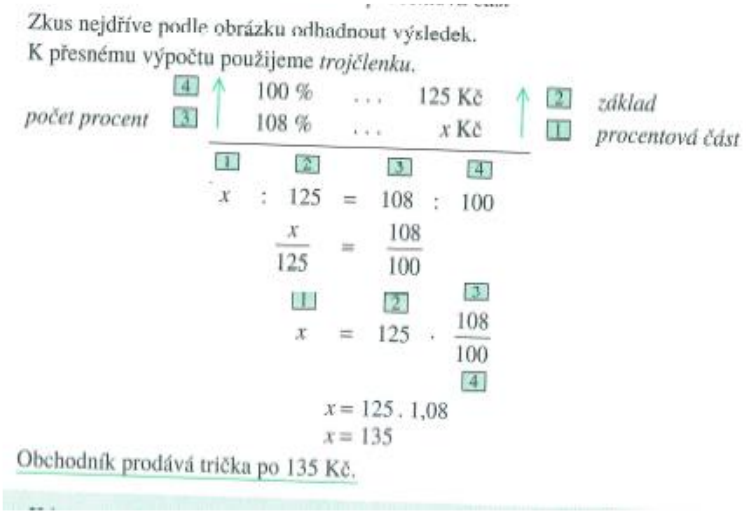

Figure 2: The rule-of-three method

In contrast, Karen does not use the rectangular representation at any moment in her classes. The arguments that she does show students are given names ("one percent", "with a decimal", and "ratio") and referred to as "methods". The majority of warrants for methods are based on the multiplicative relationship of percent part and the base, and on the definition of one percent, as corresponding to one hundredth, either as a fraction or decimal.

In the authors view, in the teacher's book, the geometrical representation helps students to get a better insight into the problem. Similarly, the authors assign the use of the ratio warrant the prominent role of helping students to get an insight into the problem.

This belief about a need to understand the problem through the use of a particular method or warrant seems to collide with Karen's beliefs about what is important for her students. Rather, she values efficiency and straightforwardness in problem-solving. Hence, she introduces neither the rectangular representation nor the rule-of-three arguments when solving word problems in her teaching. In fact, she discourages her students from using it (albeit acknowledging its existence and its effectiveness):

Teacher: Someone mentioned a third method, in case you study from your textbook, [I don't recommend it, only if someone gets] really lost and needs a crutch […] but in the time you write it all out (referring to the method), you might as well have finished other three problems [using the other methods].

Choosing not to justify – Karen's beliefs in conflict

The below example of a dialogue gives us a sense of how Karen's beliefs about the need to provide mathematical arguments for methods and general mathematical statements manifest themselves when the class discuss the percent – decimal relationship.

Teacher: So, if we have 18% (writing on the board), how do we get a decimal?

Students: Eighteen divided by 100.

Teacher: We divide by 100. Why? Because 18% is 18 hundredths (writing 18% = 18/100 = 0.18 on the board), to divide by a 100 means 18 hundredths.

Karen expressed her belief in having the responsibility to provide students with justification of mathematical statements. This responsibility is felt even in the one moment in the observed lessons when Karen acknowledges that she doesn't know how to provide a mathematical argument for the procedure, and states that students just "have to remember". The problem Karen posed to class is: "From a class of 22 students, six participated in a math competition. What percent of the class was that?" Karen goes on to exemplify two methods for solving the argument.

Teacher: The first one is the 1% method. Again, I think that this method is more convenient and easier… ok, what's the base in this problem?

Students: [suggest ideas]

Teacher: Yes, base or 100% is 22 pupils. There are 22 pupils [She writes a record of the solution on the board, writes "1% =".]. Now, we'll calculate, Ada?

Ada: 1% will be 0.22. [Karen writes this on the board.]

Teacher: Now you just have to remember that the percent, […] I don't know how to help you remember … you need to remember. You can calculate the percent this way […] we divide the percent part we want to express in percent by one percent.

The argument that she is reluctant to share with her students is in fact the ratio argument used in the rule-of-three method: firstly, that the *percent part : percent* ratio is a constant, and for all non-zero real numbers *a, b, c,* and *d*, if $a : b = c : d$ then $a = c \cdot b / d$. Clearly, this presents a conflict of beliefs for her, and she chooses not to present the argument, because this, in her mind, is too complicated and not possible to grasp with their current knowledge, especially for some students.

In the textbook, authors let the reader observe the first warrant through a series of examples, and then simply refer to the rule-of-three as practice established in the previous unit (on proportion). However, in the teacher's book they also admit that the equivalence of the two equations is, as yet in the curriculum, inaccessible to students and has not been established with students at this particular stage.

The stronger and the weaker: students' and Karen's preferences

The following passages will show examples when different arguments are provided by students. The exchanges take place at the beginning of the second lesson, students were converting a series of fractions into percent. They had just converted 4/5 by expanding to tenths and then hundredths. Now Sam tries to convert 3/8 in the same way:

Sam: I'll multiply the fraction by twelve and a half.

Teacher: Why twelve and a half?

Sam: Because if I multiplied 8 times 125 [unintelligible]

Teacher: So by 125, right?

Sam: But that will be a thousand, so …

Teacher: Doesn't matter. But (writing on the board) 8 times 125 is 1000. What is 3 times 125?

Student: 375.

Sam is trying to expand the fraction to hundredths (realizing that expanding by 125 and simplifying to hundredths is the same as expanding by 12.5) but the teacher feels that this is not straightforward and accessible to all pupils, so she takes over and breaks the argument down. After a few more simple problems, where students don't need to calculate, they are asked to convert the fraction 9/40. At first, a student (Will) suggests to reduce by two and expand by five. Then he adds:

Will:	Or multiply (sic) by two and a half.
Teacher:	Excellent, two and a half. Do you [all] agree?
Kim:	And couldn't you expand to thousands?
Teacher:	Also. And if you were to do that, by what number would you expand?
Kim:	So, that would be times … (thinking) … two hundr …
Teacher:	Twenty five. Either, as Will said, we expand by two and a half, which is not very common, (she turns to the board and writes) if we want hundredths in the denominator we expand by two and a half (she writes this on the board), do you agree? Forty times two and a half is one hundred, right? And the numerator … 18 and 4 and a half […] 22 and a half. So what percent is 9/40?
Students:	Twenty two and a half.
Teacher:	Or, as Kim said, expand by 25 (she writes on the board), the numerator (sic) is 1000, do you agree? And the denominator (sic) is …
Students:	225.
Teacher:	And we got the same thing, 22.5 %.

At this point, Karen allows a student (Will) to carry out an argument that is (like Sam's) based on expanding by decimals, but this time the student breaks it down into two warrants first, and Karen praises it. Will feels encouraged to suggest expanding by a decimal. Finally, another student supplies an argument based on the expansion to thousands (which had been shown by Karen before, see the transcript above). Both methods are now endorsed by the teacher, publicly, as valid arguments, and demonstrated on the board. When Karen summarizes these approaches, however, she qualifies Will's solution as "not very common".

Conclusions

The above examples illustrate how the enacted arguments were influenced by the three participants, the teacher, the textbook, and the students. Even though Karen was the most influential provider of mathematical arguments, arguments that were made in the classroom included students' own warrants, and became accepted as correct and valid by the teacher. At the same time, even as Karen acted as the decision-maker when it comes to choosing what representations are useful in warrants, i.e. efficient, for her class, what was her choice not to include the textbook's geometrical representation warrants based on? Clearly, the textbook does not give it a utilitarian value, i.e. it does not provide opportunities for its direct use, and makes the representation void of value, outside the possible provision of better insight, as the authors claim, but Karen did not find the claim convincing enough. In that sense, her decision was very much determined by two factors: a) by her pedagogical

content belief about the efficiency of a certain type of arguments and b) by the problems (opportunities for arguments rather than argument forms themselves) presented by the textbook authors in the unit. The second factor, in turn, is given weight by Karen's utilitarian view of the goals of mathematics education, i.e. being able to correctly solve problems provided by the curriculum.

The case of the rule-of-three method is perhaps even more interesting, especially as the ratio warrant that underlies it is also at the heart of a method Karen presents when she shows the procedure for finding the percent in a word problem, but decides that the justification is not straightforward enough for her class, and backs the procedure up with her own authority. What made her do that? When asked about the need to mathematically justify mathematical statements, Karen conceded that not all arguments are accessible to students (or not all of them). As I showed above, the textbook authors also use a warrant that they acknowledge is out of the students' immediate reach. Again, we observe similar tendencies, and at the same time it appears that in this case Karen's perception of her students' abilities accounted for her decision not to justify.

In her classes, Karen also allowed students to provide arguments that she had not intended to take place, and accepted them as long as they were mathematically correct. At the same time, she manipulated such publicly expressed arguments according to her perception of accessibility to all students and made frequent evaluative comments about the methods and arguments, labeling them as efficient, common practice, convenient, easier, or universal. This qualitative evaluation springs from her beliefs about her students' mathematical ability and what it means to be good in mathematics: in her view, some students are better at understanding the problem, and innately capable of finding and choosing the most efficient, original, or convenient method, an attribute she also gives mathematicians in general. For the others, she needs to show simply which method to use, and they need to learn it by solving many similar problems, i.e. for some students drilling is the only way to succeeding in mathematics. The episodes seemed to confirm that this belief corresponds with the students' contributions: the weaker students would rely on arguments promoted by Karen, while students who feel confident in their own warrants, could keep using their own.

In terms of the teacher's influence, it appears that the teacher is independently imposing her own beliefs that are very local, e.g. the choice of method, but the choice of representation is also clearly determined by the curriculum (and its tasks) and beliefs that are much more global. Further, the teacher's choice of not justifying mathematically can be caused by her own belief but also reinforced by similar examples in the textbook. Finally, the students' arguments are evaluated by the teacher in terms of their mathematical correctness, their efficiency, and their accessibility to all other students (as perceived by the teacher). They are then often re-formulated by the teacher, which potentially reinforces the dependency of the weaker students on the teacher's choice of argument.

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Identifying phases and activities in the proving process of first-year undergraduates

Katharina Kirsten

University of Münster, Germany; k.kirsten@uni-muenster.de

Keywords: Proof, process-oriented perspective, proving cycle, undergraduate.

It is well known that undergraduates commonly have to deal with great difficulties in constructing proofs, especially at the beginning of their mathematical studies (e.g. Weber, 2001). In the presented study, students' approaches to proving are analysed from a process-oriented perspective. The primary aim is to empirically confirm a process-oriented model for the proving competence of undergraduates. This model may be used to analyse proving processes at an individual level and, therefore, to gain more detailed information about proving process and its phases in general.

Theoretical framework

The study is based on a theoretical model of the proving process that describes different phases and activities in proof construction and which is mainly following the considerations of Boero (1999). Boero (1999) describes a proving process that starts with conjecturing and exploration activities and ends up in selecting arguments and linking them to a deductive chain. However, proving tasks at university level often consist of a statement estimated to be true, especially during the first year of studies. The construction of a proof is in this case rather aimed at justification instead of conjecturing and problem exploration. According to this, we suggest the following variation of Boero's model.

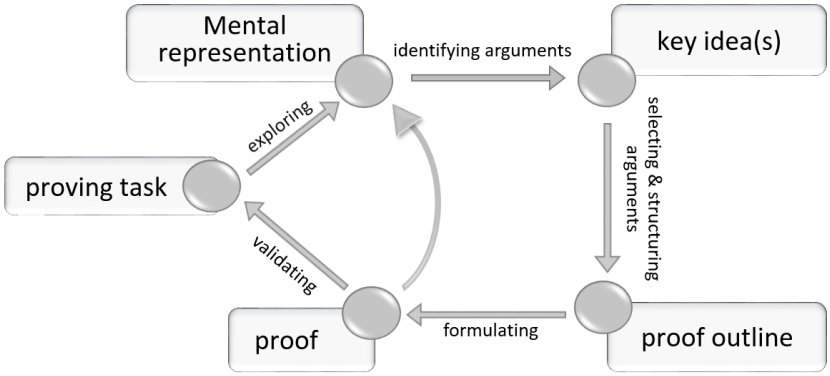

Figure 1: Proving cycle

This model differs from the existing model in three ways:

1. The model focuses on proving activities concerned with the justification of a given statement. In particular conjecturing activities are excluded. However, reducing the process described above does not mean to exclude all exploration activities, but locating them at the beginning of the proof construction. Leaning on Reusser's (1997) approach of a situation model, a mental representation of the given statement is estimated to be developed by exploring the proving task. This representation could affect the proving process in a meaningful way.
2. The model includes validating activities at the end of the process. In this phase, which is already implicitly considered by Boero (1999), the final proof is reviewed regarding content, structure and linguistics. Besides, further (shorter or more elegant) proofs can be considered.

3. The underlying structure of the model is a cycle. This kind of structure provides the assumption that proving processes are not supposed to be linear. In fact, the proving process is shaped by interruptions, revisions and turns.

Research question and method

The modifications lead to the following research question: Is the proving cycle an appropriate tool for analysing proving processes? That means, is it possible to reconstruct the different phases and activities stated in the proving cycle empirically? Is there in particular evidence for the existence of an exploration phase? In accordance with the research question, a qualitative study has been designed with the purpose to provide evidence of the proving cycle and to gain more detailed information about the different phases. Therefore, first year undergraduates and first year pre-service mathematics teachers (grammar school) are asked to work on proving tasks in the field of real analysis. To encourage the participants to talk about their ideas and approaches, the working processes are organized in pairs. The proving process of each pair is videotaped, transcribed and finally encoded according to Mayring's (2007) structuring content analysis. The coding is based on a system of categories, which consists of the five theoretical stated phases in the proving cycle.

Results

The analysis of data from six cases shows that the system of categories seems to be well suited to describe proving approaches of undergraduates. Each of the suggested phases could be empirically confirmed in nearly all cases. However, formulating a precise and clear proof is an activity, which is sometimes omitted. An exploration phase could be reconstructed in all cases, although it varies in quality and quantity. The structure of the analysed proving processes is linear insofar as many phases could be reconstructed in the suggested order and turns mainly concern consecutive phases. Only in those cases, where the identified key ideas turn out to be inadequate, the proving process starts with repeated exploration cyclically. As there is no need for further categories the proving cycle can be used as a tool for analysing proving processes. Additionally, this tool can serve as a basis for deeper inductive investigations of the proving competence from a process-oriented perspective.

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TWG02: Arithmetic and number systems

Introduction to the papers of TWG02: Arithmetic and number systems

Elisabeth Rathgeb-Schnierer[1], Renata Cavalho[2], Judy Sayers[3], Darina Jirotkova[4]

[1]University of Kassel, Germany: rathgeb-schnierer@mathematik.uni-kassel.de

[2] UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal

[3] Stockholm University, Sweden

[4] Charles University in Prague, Czech Republic

Introduction

Working Group 2 was formed in 2011 in Rzeszów, Poland (CERME 7) as a forum for presenting and discussing theoretical and empirical research on the teaching and learning of arithmetic and number systems. The scope of the working group comprises grades 1-12 and emphasizes research-based specifications of domain-specific goals, analysis of learning processes and learning outcomes in domain-specific learning environments and classroom cultures, new approaches to the design of meaningful and rich learning environments and assessments as well as research on teachers' professional development.

According to the great variety in the field of learning arithmetic and number systems, the group intensively discussed fifteen papers and one poster addressing research for different ages and different approaches[1]. The key themes were number sense and structure sense, estimation and estimation tasks, flexibility in mental calculation, derived fact based strategies of multiplication in low-achieving students, understanding of rational numbers and ratio, didactical models as scaffolds for the evolution of mathematical knowledge as well as teachers' knowledge about rational numbers, ratio and place value according to big numbers.

In comparison to former working groups, we had an even greater variety of themes, and therefore the challenges and opportunities to identify common ideas and approaches in the papers presented. Two papers put the emphasis on teaching arithmetic, five on learning arithmetic, five followed a design based research approach with combining teaching and learning, and two papers focused on teachers' professional development.

Teaching arithmetic and number systems

The topics of the two papers focusing teaching were diverse. One paper reflected the role and function of didactical models, the other analyzed and compared textbooks.

Marita Barabash and Raisa Guberman presented a theoretical study on didactical models (DMs) to support students' development of mathematical concepts and ideas. The authors discuss didactical models as a mathematical model (supporting the development of appropriate and consistent

[1] Fourteen papers were resubmitted after the conference and are published in the proceedings.

concepts) as well as a learning tool (providing an experience of using models). In this sense, DMs may help to cope some discontinuity in mathematics teaching across different levels.

With the focus on a foundational number sense, *Anna Löwenhielm Gosia Marschall, Judy Sayers and Paul Andrews* analyzed and compared how English and Swedish textbook tasks facilitate children's learning of those number-related competences that require instruction. Analyses identified both similarities and differences. Swedish pre-school curriculum seems to have prompted a conceptually focused textbook, while the strongly framed English pre-school curriculum seems to have precipitated a procedurally focused textbook.

Learning arithmetic and number systems

The subject area "learning arithmetic and number systems" includes papers on number sense, structure sense, and understanding of rational numbers and ratio.

Students' development of structure sense in arithmetic is the focus of *Andrea Maffia and Maria Alessandra Mariotti.* Structure sense can be mobilized by students to compare and to transform arithmetical expressions, however sometimes it can lead to mathematical inconsistency that pupils might not be aware of. Their paper provides evidence of this type of phenomenon through syntactical transformations.

The assessment of students' ability in number estimation is the topic of a study conducted with Brazilian second and third graders. *Beatriz Vargas Dorneles, Mariana Lima Duro, Nohemy Marcela Bedoya Rios, Camila Peres Nogues and Clarissa dos Santos Pereira* compare a Number Line Estimation Task and a Numeroisty Task. Results show that the Number Line Estimation Task is more accurate in assessing students' performance in estimation.

Luciana Corso and Beatriz Vargaz Dorneles put the emphasis on number sense and investigate the relation between three domains: working memory (especially the central executive and the phonological component), number sense and arithmetical performance. Based on different valid instruments, data was collected regarding each single component. The analyses reveal a significant correlation between the central executive component of working memory and number sense.

Ayşenur Yılmaz and Mine Işıksal-Bostan examine to what extent middle-grade students agree on statements about the ordering of two negative integers given within a real-life context, and what kind of procedural and conceptual strategies do middle-grade students generate to order those numbers. The results reveal that students did not explain the concept of ordering in daily life considering their conceptual meanings, and have problems in their procedural knowledge repertoire.

Ozur Soyak examines students' difficulties in proportional reasoning in rate and ratio problems. Students' difficulties are caused by the confusion of unit rate identification and by algorithmically based mistakes. A lack of understanding in the unit of measurement, the difference between additive and multiplicative reasoning, and some errors in the computation process seems to underlie such difficulties. The study suggests emphasizing proportional reasoning in the learning process.

Teaching and learning arithmetic and number systems

This time we had a considerable amount of papers following a design based research approach. All studies were characterized by designing a teaching and learning arrangement and investigating students learning and understanding according to this specific context.

With the focus on students' development, *Laura Korton* explores a teaching-learning arrangement for the inclusive mathematics classroom to foster flexible mental calculation. The approach considered both the level of design (consideration for use) and the level of research (quest for fundamental understanding). Initial findings appear to suggest positive outcomes when Mutual Learning processes are integrated.

Michael Gaidoschik, Kora Maria Deweis and Silvia Guggenbichler exhibit results of an ongoing design research project. Based on a specific instructional design that emphasizes conceptual understanding and derived fact strategies, the study investigates the exhibited strategies in basic multiplication of lower- achieving students. Within the analyses three different strategy types could be developed. Results also show the influence of the instructional context.

Cristina Morais and Lurdes Serrazinha describe an approach to develop conceptual understanding of decimal numbers by using and adapting the hundred square model. Within a teaching experiment 3rd and 4th grade students worked with three different models focusing on part-whole meaning. Results indicate that models promote students understanding of rational numbers, and suggest the decimat as an important part-whole model.

Helena Gil Guerreiro and Lurdes Serrazina discuss an approach to focus students' conceptual understanding of rational numbers based on teaching percentages in elementary school. Within a teaching experiment, the authors collected data by logbook, audio and video-recording, and analyzed qualitatively. Results suggest that this specific approach supports to understand multiplicative relations and rational numbers.

The pilot study of *Carlos Valenzuela García, Olimpia Figueras, David Arnau and Juan Gutierrez-Soto* contributes to the development of better mental objects for fractions. Using a Theoretical Model for fractions, they designed and developed a seven stage teaching sequence based on the use of applets created with GeoGebra and the number line as a conceptual and didactical resource. Results of the first two stages suggest students' preferences to represent fractions as proper fractions (unit segment). The majority of the participants paid more attention to the graphical aspects of the applet.

Teachers' professional development

The two papers in this field focused on teachers' knowledge and its impact on students' learning.

Frédérick Tempier and Christine Chambris paper aims to reveal teachers understanding of place value related to larger numbers, and its impact on teaching and students understanding. Based on the "Theory of Didactic Transposition" the authors investigate the relations between all three aspects. The qualitative analyses of a teachers' lesson compared with the results of a questionnaire of students' knowledge provide interesting insights in teaching large numbers.

Gulseren Karagoz Akar conducted a single case study on teachers reasoning about ratio. Using a theoretical framework based on the concepts *within (state) ratios* and *between (state) ratios*, the teachers' conceptions of ratio are discussed. The results suggest different levels of reasoning in between-state ratios (as an operator and a combination of two extensive quantities), but some lack understanding about in between-state ratios as a single intensive quantity.

Summary and outcomes

The great variety of papers provided a fruitful base for interesting and often animated discussions in a supportive quizzical environment. We agreed that we were able to broaden our own perspectives in terms of new perspectives on different mathematics educational research domains, and different uses of terms, concepts and theoretical frameworks. In our discussions, we went far beyond the specific themes of the single papers and covered general aspects that influence our work as international and interdisciplinary group of researchers. Those were

- the synergy of cognitive science and mathematics education,
- the similarity and differences in research paradigms and approaches,
- the intercultural differences,
- the different terminologies and the relevance of creating common terminologies and
- the necessity to clarify concepts and theoretical frameworks.

This spectrum of research focus in the papers reflect the number of open questions that still persist in the teaching and learning of arithmetic and number systems, and in need of further research. The intense group discussions raised further awareness amongst the group and questioned some positions that were taken for granted in both the field and in specific contexts. With the commonly perused aim improving mathematics education in the field of arithmetic and number systems in all academic levels, we perceived common ideas as a base for further research and discussion: (1) The importance of number sense regarding different age levels, number systems and perspectives form different domains. (2) The necessity to develop models for fostering flexible relational thinking about numbers. (3) The requirement to enhance our notion of number literacy as interaction of various components.

For our further work, the group agreed on enhancing the exchange of research on arithmetic and number systems from cognitive psychologists and mathematics education researchers, and work on specifying terms for both fields to agree on.

A conception of between-state ratio in fraction form

Gulseren Karagoz Akar

Boğaziçi University, Turkey; gulserenkaragoz@yahoo.com

This study investigated one prospective secondary mathematics teacher's (Jana's) reasoning on between-state ratios in missing value problems and comparison problems. In two one-and-one-half hour written problem solving sessions followed by one hour-long clinical interview, Jana's use of informal and formal strategies and justifications behind those strategies in the context of ratio were examined. Extending previous research, results of this study showed that someone could quantify between-state ratios acting as an operator in fraction form once she has understood ratio as an association of amounts of quantities in within-state ratios. Results indicated a dichotomy within the boundaries of identical groups conception in terms of within-state ratios and between-state ratios prior to an understanding of between-state ratios as a single intensive quantity.

Keywords: Between-state ratio, within-state ratio, division, multiplication, extensive and intensive quantity.

Theoretical framework

Students might approach to a proportion such as a/b=c/d by comparing the first set of ratios a/b or c/d or the second set of ratios a/c or b/d (Noelting, 1980). In the first case, the ratios a/b or c/d are called *within (state) ratios*, where the ratio represents the original quantities within one state. In the second case, the ratios a/c or b/d are called *between (state) ratios*, where the ratio represents quantities between two situations (Noelting, 1980). For instance, envision the Recipe 1 Problem in this study. The original quantities of 9 tablespoons of oil and 4 tablespoons of vinegar could be represented by the within-state ratio, 9/4; and, the 4 tablespoons of vinegar and the 7 tablespoons of vinegar from two situations could be represented by the between-state ratio, 7/4.

Researchers investigating prospective teachers' conceptions of ratio have revealed how teachers interpreted the relationships between quantities in ratio situations to quantify some attribute of interest such as lemon-lime flavor (Heinz, 2000; Karagoz Akar, 2007; Simon & Blume, 1994; Simon & Placa, 2012; Thompson, 1994). For instance, envision the Mixture Problem in this study. For this problem, research has shown that one might interpret the relationship between the quantities of 36 grams of pure lemon juice and 32 grams of pure lime juice in the following three different ways: First, someone having a robust conception of ratio conceptualizes that ratio is a *single intensive quantity* that expresses the size of one quantity (i.e., amount of lemon) relative to the size of the other quantity (i.e., amount of lime) represented by within-state ratios (Simon & Placa, 2012). In this conception one can utilize both partitive and quotitive division of the quantities in within-state ratios to quantify the attribute (i.e., lemon-lime flavor) in the situation. That is, either engaging in partitive or quotitive division, one might interpret the quotient (i.e., 1.125) of the original quantities represented in the within-state ratios, 36/32 , as a single intensive quantity representing the invariant multiplicative relationship between the quantities (Simon & Placa, 2012). This concept of ratio is also called as ratio as measure conception (Simon & Blume, 1994).

Secondly, in order to quantify the lemon-lime flavor, one might think of the within-state ratio, 36/32, as representing an association of amounts of two quantities (Johnson, 2015). In this regard, s/he thinks of the quantities making up a particular combination that quantifies the taste of the mixture. This understanding

aligns with the identical groups conception (Heinz, 2000) and ratio as a composed unit (e.g., Lobato & Ellis, 2010). Within the boundaries of such conception, one might find equivalent ratios by dividing for instance, 36 and 32, simultaneously with 4 and come up with 9/8 ratio (i.e., as a composed unit, Lobato & Ellis, 2010), representing the same lemon-lime flavor (Beckmann, 2011). Third, to quantify the attribute, one might engage in partitive division of quantities in within-state ratios, 36/32 (Heinz, 2000; Karagoz Akar, 2007; Johnson, 2015). The quotient 1.125 then represents an association between the quantities of 1.125 grams of pure lemon juice per one gram of pure lime juice. Therefore, ratio as identical groups conception (Heinz, 2000) and ratio as per-one conception (Simon & Placa, 2012) involves one's interpreting within-state ratios as an extensive quantity rather than a single intensive quantity (Heinz, 2000; Karagoz Akar, 2007; Johnson, 2015). The study reported in this paper attempted at extending the previous research results in the following way: As the previous research has shown, students interpreting within-state ratios as representing an association between quantities (i.e., identical groups conception) could utilize equivalent fractions to handle missing value problems and /or comparison problems (Heinz, 2000; Lobato & Ellis, 2010). However, they cannot reason in missing value and/or comparison problems with quantities non-integer multiples of each other (Heinz, 2000). In this study, data from one prospective teacher documented that given that she interpreted within-state ratios as representing an association of quantities, she could reason with between-state ratios for situations involving quantities non-integer multiples of each other. In particular, Jana quantified the relationship between the quantities in between-state ratios as representing a particular combination and acted it on the within-state ratios as an operator. This is important because earlier research focused *only* on students' reasoning on the relationship between the quantities in within-state ratios. However, there is also need to focus on how someone reasons with between-state ratios; because, the conceptions of between-state ratios and within-state ratios have cognitively different underpinnings and that the understanding of proportion integrates both of these conceptions (Noelting, 1980). Also, the results from Karagoz Akar (2007) study showed that an understanding of between-state ratios as an intensive quantity (as percent-increase/ decrease) does not necessarily depend on an understanding of within-state ratios as per-one. Together with the results of Karagoz Akar (2007) study, the results of this study indicated a dichotomy within the boundaries of identical groups conception without having within-state ratios as per-one. Also, knowing about different levels of sophistication in the conception of ratio might shed light on determining and detecting students' reasoning along the way to advanced understandings of ratio, such as ratio as measure. In this regard, this study scrutinized the following research question: How might a prospective secondary mathematics teacher quantify the relationship between the quantities in between-state ratios and within-state ratios in missing value and comparison problems?

Methodology

The voluntary participant was a prospective secondary mathematics teacher, Jana, at one of the universities in the United States. In this study data was collected through the structured task-based clinical interviewing method (Clement, 2000) following two one-and-one-half-hour long written problem solving sessions. During the written sessions, Jana, was asked to provide solutions with explanations and justifications to the tasks. The reason for doing written sessions was to determine Jana's solution processes prior to the clinical interviewing so that her reasoning, justifications of her solution processes, and the connections she made

among her interpretations of multiplication, division and part-part-whole relationships in missing value problems and comparison problems could be further elucidated. The interview was videotaped. The transcript of the interview and artifacts from written problem solving sessions and the interview were all used as data sources in the analysis.

In analyzing clinical interviews, the researcher "…is constructing a model of hidden mental structures and processes that are grounded in detailed observations from protocols" (Clement, 2000, p. 549). In this regard, the unit of analysis was Jana's strategies, solution processes and justifications she provided in externally written or uttered arguments (the observations from the point of view of the researcher). The goal was to determine what underlying conceptions of ratio Jana might be revealing. Thus, the analysis was interpretive (Clement, 2000). In this respect, reading the whole transcript line-by-line having in mind previous research, I determined chunks of relevant data that would allow generate the descriptions of Jana's mental structures such as her thinking of ratio as extensive or intensive quantities. Then, to further validate interpretations I went back to how she reasoned during the written sessions and how she reasoned on different tasks. Then I wrote a narrative. Following, another researcher was consulted to challenge the conjectures and/or to affirm their reasonableness to further validate the plausibleness of the interpretations.

Tasks (used in the study)

For the study, I wrote the Hair Color 1 and 2 problems and adopted the others from the existing literature (see Table-1). The rationale for the choice of problems was the following: Heinz (2000) study showed that prospective teachers had quantified ratio at different levels. For instance, within the identical groups conception, some teachers engaged in partitive division of the quantities in within-state ratios and quantified within-state ratios as an association of amount of one quantity per one unit of another quantity. To the contrary, some teachers engaged in quotitive division to quantify the within-state ratios as a single intensive quantity. Thus, I wrote The Mixture Problem in reference to the distinctions in partitioning and measuring. Also, Heinz (2000) stated that within the identical groups conception someone might have used either their part-whole understanding to make sense of the problems, or have gone back to additive thinking. Thus, I wrote The Hair Color-1 Problem. Further, within the identical groups conception teachers were not able to deal with the quantities non integer multiples of each other (in between-state ratios). So I hypothesized that there might have been teachers who could do so by using adjustment strategies (e.g., Kaput & West, 1994). Thus, I adopted and modified the Recipe-1 Problem from Kaput and West (1994) since also they ranked it among the highest levels of difficulty (13th out of 15th difficulty). I also wrote The Hair Color-2 Problem based on the research results on rational number as operator (Marshall, 1993).

Hair Color- 2 Problem: Kelly likes to color her hair with a mixture of red and brown dyes. When her hair was shorter, she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture of 15 grams of red and 17 grams of brown will not be enough to color all of her hair. So she thinks that if she adds 7 grams of red dye to the original amount of 15 grams of red dye, some amount of brown dye to the original amount of 17 grams of brown dye, she can color her hair. How much of brown dye do you think she needs to use to make sure that she has the original color? In a group of 7th graders, following two solutions were given: a) The amount of brown dye Kelly needs to add to the original amount is 17*(22/15). b)The amount of brown dye Kelly needs to add to the original amount is 22* (17/15). Explain how each answer fits or does not fit the story? Depending of your choice of answer, explain what (22/15) represents and what (17/15) represents. You can use diagrams, shapes, graphs etc in explaining your answer (Karagoz Akar, 2007)
Recipe-1 Problem: A recipe for salad dressing asks for 9 tablespoons of oil for every 4 tablespoons of vinegar. If we want to make a batch that includes 7 tablespoons of vinegar, how much oil we need to be consistent with the recipe? How do you know that the result makes sense? [adopted from Kaput & West, 1994] You can use diagrams, shapes, graph etc in explaining your answer.
Mixture Problem: A groups of 5th graders are given a project. In the project, they will create two mixtures and will put them in two jars to keep in the refrigerator in the kitchen so that during the break they can serve it to kindergartens. In the first jar, there is a mixture made up of 36 grams of pure lemon juice and 32 grams of pure lime juice. In the second jar, there is a mixture made up of 20 grams of pure lemon juice and 16 grams of pure lime juice. To label the jars, 5th graders want to use one number representing the lemon-lime flavor in the mixture. As a solution to their problem, three ways these students provided are as follows: a) There is 4 more lemons than limes in each mixture. So, put 4 on the labels. b) The ratio of 36/32 and the ratio of 20/16. c) 1.125 , by dividing 36 with 32 and 1.25 by dividing 20 with 16. Which one(s) would indicate the lemon-lime flavor? Why? Please, in explaining your reasons for why some of these options would indicate the mixture's lemon-lime flavor, feel free to provide any kind of diagram, figure, or graphs. (Karagoz Akar, 2007)
Hair Color-1 Problem: Kelly likes to color her hair with a mixture of red and brown dyes. When her hair was shorter, she used 4 grams of red and 5 grams of brown dye. Now, her hair is longer. She knows that the original mixture of 4 grams of red and 5 grams of brown will not be enough to color all of her hair. So she thinks that if she adds 1 gram each of red and brown dyes to the original 4 and 5 grams, respectively, she will preserve the color. Will the new mixture be the same color as the original? Explain your reasoning. You can use diagrams, shapes, graphs etc in explaining your answer. (Karagoz Akar, 2007)

Table-1: Tasks used in the study

Results

Jana's understanding of between-state ratios

Data from the "b" option of The Hair Color-2 Problem and the Recipe-1 Problem showed that Jana left the between-state ratio in the fraction form, contrary to the previous research results (e.g., Karagoz Akar, 2007; Heinz, 2000). She did not think of finding the quotient in between-state ratios once the problem required her to think of it as quantifying percent decrease/increase. On the other hand, data from the interview showed that once *her goal* was to find out *how many times the quantities were incremented*, she was able to divide the quantities in between-state ratios. Jana had solved the Hair Color-2 Problem using the cross and multiply rule during the written sessions. So, during the interview, the first question I asked Jana was The Hair Color-2 Problem "a" and "b" options.

> R: All right, okay, without solving the problem. What does that 22 divided by 15 represent in the problem?
>
> J: It doesn't represent. Umm, 22 over 15, it kind of just says that she is adding 7 grams to the new amount over and it is over the old amount… well, 17, She put it in a fraction that new amount over the old amount, 22 over 15, she multiplied it by 17 because that was the old amount of brown, so that is what she was doing…she already know what she wants to change the red one to, so, she has to make one of the numbers and she has to make sure that the other color is the same ratio as before.

It is interesting that, although I told her "22 divided by 15" Jana thought that 22/15 represented the change in color, as in a fraction of the new amount of red to the old amount of red. She knew that the other color needed to be kept in the same ratio, and she knew that she could do it by multiplying the other quantity in

the original ratio with the same number. Yet, whether she thought of the 22/15 as the "change factor" was not clear. In fact, further data clarified this. Jana's reasoning about the between-state ratios, once given in the simplest, reduced form, was the same on The Recipe-1 Problem, too. During the written sessions, Jana had written the following (see Figure-1):

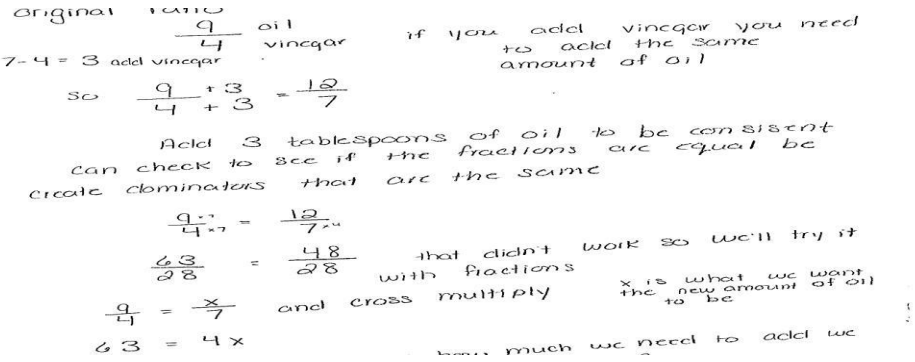

Figure-1: Jana's reasoning on the Recipe-1 Problem in problem solving sessions

Two interesting points need to considered: Jana thought of adding 3 grams to both ingredients, which was a characteristic of the identical groups conception. Kaput & West (1994) also stated that students revert back to additive reasoning once the numbers used in the original ratio are very close to each other. Thus, first, if Jana had the conception of ratio as a single intensive quantity, she would not have thought of subtracting the quantities magnitudes of which are close to each other; rather, she would have thought of dividing (e.g., Heinz, 2000; Karagoz Akar, 2007). Secondly, Jana used equivalent fractions, after her addition strategy, to check her solution, leading her to the conclusion that her solution was not correct. Her use of equivalent fractions indicated that she did not have any other way of verifying whether the proportion held. This claim will be further supported by her reasoning on the Hair-Color-1 Problem. To figure out the extent of her knowledge, I asked Jana during the interview to account for a solution for The Recipe-1 Problem provided by another student as 9x(7/4). Jana said, "Because you are trying to get the same combination, so this is like the new combination of the vinegar where it is changes from 4 to 7 so it is like a new ratio and you want the ratio of oil to be the same as it was before so you are allowed to multiply the old oil times its new ratio in order to get the new oil". Her reasoning on this problem was similar to the Hair-Color-2 Problem option "b," a between-state ratio represented a particular combination in fraction form. Taken together, data indicated a deviation from the identical groups conception: She did not solely go back to additive reasoning when the numbers in the original ratio were very close to each other. Also, she interpreted the fraction form of the between-state ratio as an extensive quantity, creating a particular combination, representing so many of the old quantity (from the first situation) for so many of new quantity (from the second situation).

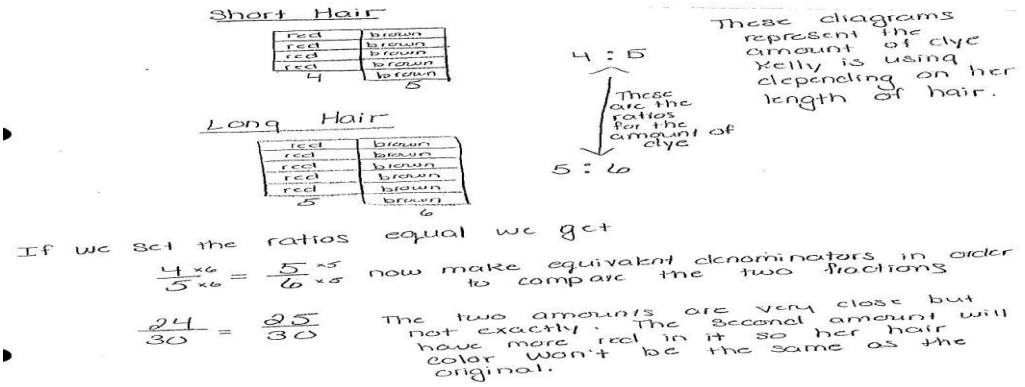

Figure-2: Jana's reasoning on The Hair Color-1 Problem in problem solving

Data above (see Figure-2) together with her statement in the interview below, once again indicated why Jana"s stage of knowing was within the scope of the identical groups conception, albeit with deviations from it. During the interview, Jana stated " this[referring to equivalent fractions] helps us to compare because you need to make one of them the same in order to compare actually compare ". The excerpt and her solution above (see Figure-2) are important in two ways: first, it shows that Jana used the equivalence of fractions as a way to *compare* whether two different dyes are the same color. Second, it shows how she related the equivalence of fractions and the common denominator algorithm. Jana thought that the within-state ratio represented an association between two extensive quantities, representing so many for so many other parts. This was evident when she said she could *change the order* (brown to red) of the ratios. So she did not need the second quantity to compare the ratios once she equaled them out. Here, she again deviated from identical groups conception since she was able to deal with quantities non-integer multiples of each other.

Limitations of Jana's understanding in within-state ratio context

Jana's understanding in the within-state ratio context showed some limitations and deviations from the identical groups conception. During the interview, for the "c" option of The Mixture Problem, Jana claimed the following:

> [First Part] J: Yeah, fractions even though when you actually these fractions, when you divide the fractions you get this number 1.125 but you when you look at that number you don't know how much lemon juice there is and how much actual lime juice.
>
> [Second Part] R: Does this tell like tell you anything like the lemon and lime about the juice or does this represent anything [referring to 1.125]
>
> J: Well, if you have different number which I don't, I can't calculate numbers, where you have a different amount of like I don't know if you have like x and y this is lemon over lime and when you divide it you get 1.125 then you know this combination [referring to b option] equals this one [referring to the ratio of x to y], that they will taste the same… because they are in the same ratio, so that kind of.
>
> [Third Part] R: other than that this is going to help you?
>
> J: No, actually, you can't, you can't, it is not going to help because you can't create more juice like this from just this number, you have to, because you don't know how much lemon juice is in there compared to actually how much lime.

The first part shows that Jana understood that, given the fractions of 36/32 and 20/16, when she divided those numbers she got 1.125 and 1.25 respectively. However, although she realized that when she divided 36 by 32 she would get 1.125, she *had not abstracted* the fact that the quotient was the invariant multiplicative relationship that quantifies the taste. In the Second Part, data also suggested that Jana could tell that two fractions are equal if they equal the same decimal, but she did not think of the quotient as indicating something about the situation modeled by the ratios (Simon & Blume, 1994). Data from the Third Part suggest that Jana did not realize 1.125 lemons per lime as *at least the representation of a particular mixture*: for 1.125 grams of lemon there is 1 gram of lime. This indicates that Jana *did not anticipate* the quotient as per-one. If she had, she would have been able to add the quantities of 1.125

lemons and 1 lime until she reached the targeted quantities. To the contrary, she claimed "you can't create more juice like this from just this number [referring to 1.125]", deviating from the identical groups conception.

Discussion

Results showed that, regardless of the type of tasks, Jana interpreted the relationship between quantities in within-state ratios as association of amounts of quantities. This is similar to the previous research results (Heinz, 2000; Johnson, 2015). Yet, she deviated from such level of reasoning by interpreting the between-state ratios as an operator acting on the quantities in the original ratio situation (i.e., within-state ratios) since she was able to deal with the non-integer multiples of quantities. Also data from the Hair-Color 1 and 2 and the Recipe-1 Problems indicated that Jana understood between-states ratios as a particular combination of two extensive quantities. For instance, for her, the 7/4 ratio from one situation to the other in the Recipe-1 Problem was a new combination of vinegar, a new ratio, acting as an operator (Noelting, 1980). Also, deviating from the identical groups conception, when the numbers in the original ratio were very close to each other, she did not go back to additive reasoning, though attempting at it. Such attempt indicated that she did not have an understanding of between-state ratios as an intensive quantity, quantifying the change from one situation to the other in percent-increase decrease Karagoz Akar (2007). She also deviated from the identical groups conception (Heinz, 2000) and ratio as per-one (Johnson, 2015; Simon & Placa, 2012), such that she was not able to anticipate the quotient in The Mixture Problem as how much of one quantity associates with one unit of another quantity even when she divided it. These results suggested a different level of reasoning in between-state ratios and also a dichotomy within the continuum of identical groups conception in term of the conceptions of within-state and between state ratios prior to interpreting between-state ratios as a single intensive quantity. These results also have some implications for teaching ratio to both students and prospective teachers: The tasks in the study might be used to introduce prospective teachers with different strategies students might engage in while solving missing value and comparison problems. Secondly, Jana's reasoning seem to be at a higher stage than an understanding of ratio as an association of quantities, as reported in the field (e.g., Johnson, 2015). Teachers and teacher educators might expect to observe these different kinds of reasoning while developing an understanding of ratio on the part of their students. Also, they might refer to these kinds of reasoning while assessing their students' understanding of ratio at different levels.

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The double nature of didactic models in conceptualizing the evolution of number systems: A mathematical model and a learning tool

Marita Barabash and Raisa Guberman

Mathematics Department, Achva Academic College, Israel; maritab@013.net

This paper is a small part of an on-going theoretical study on didactic models as a form of didactic transformations of mathematical notions, concepts and ideas, i.e. of adjusting mathematics for teaching. In what we propose here we argue that regarding this adjustment as mathematical modeling should be inherent to the mathematics teaching: they may enhance the concept development as the on-going result of students' learning; foster embedding the "big ideas" approach to mathematics learning, and lead to more self-consistently evolving mathematical knowledge. The "big idea" is that numbers are to be studied in the context of number structures, i.e. together with operations defined on them and properties of these operations, and that a familiar number system may serve a model for studying a new one. We illustrate the didactical model's approach at the initial stage of learning fractions.

Keywords: Didactic models, acquiring notion of number systems, evolution of mathematical knowledge, arithmetic at elementary school.

Introduction

This paper is a small part of an on-going theoretical study on didactic models as a form of didactic transformations of mathematical notions, concepts and ideas, i.e. of adjusting mathematics for teaching in a way that would preserve to a maximal possible extent its structure and spirit. "We want the students to be exposed as early as possible to the idea that beyond the nuts and bolts of mathematics, there are unifying undercurrents that connect disparate pieces" (Wu, 2009). The theoretical framework for the research is inspired by ideas such as Wu's idea cited above, and sprouts from the works by Freudenthal (in particular, Freudenthal, 1975), Kirsch (2000) and from the discussion on applied-mathematic nature of didactic transformations (Borovik, 2012). These and other sources reflect the need for the merged input of deep mathematical, psychological and didactic considerations in constructing the mathematics feasible and meaningful for students of various ages and levels of mathematics learning.

Any form of teaching mathematics involves adjusting it for the students. In what we propose here we argue that regarding this adjustment as mathematical modeling should be inherent to the mathematics teaching. The philosophy behind mathematic models is applying a user-attainable mathematical apparatus to study an unknown subject or phenomenon. No mathematical model fully represents the subject being studied. One should always be aware of limitations of a model being used along with its purpose and benefits, and also of what Freudenthal (1975) presents as the dual character of mathematical models: "*Models of something* are after-images of a piece of given reality; *models for something* are pre-images for a piece of to be created reality" (p. 6).

An educated usage of models is supposed to shed light on phenomena and subjects being studied and seems to be indispensable as a tool at any level of mathematic studies. This implies that the mathematical models usage should become an integral part of the teaching/learning procedure in

mathematics lessons. In particular, concerning the *models for something* – the "to be created reality" in school mathematics is first and foremost the new mathematical knowledge hopefully to become in due time reality for the pupils. Thus, when the unknown subject to be studied belongs to mathematics, the model illustrating it serves didactic purposes; in this case, we are referring to didactic models (DMs) (see Figure 1):

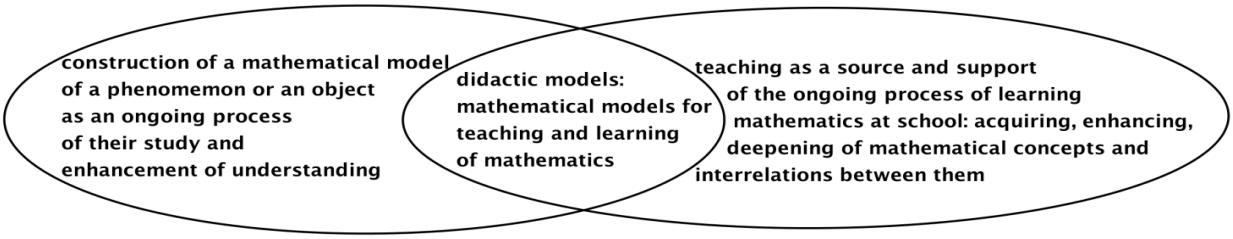

Figure 1. Didactic models as mathematical models and as a learning tool

Examples of well-known and widely used DMs are the Dienes model demonstrating the principles of the decimal system, and the rectangle-area model used to impart some properties of multiplication and division. In this paper, we would like to look closer at didactic models as mathematical models of the *to be created reality*, in order to appreciate their educational value provided they are used knowingly and systematically. We suggest that mindfully and systematically applied to teaching, they may enhance the concept development as the on-going result of students' learning; foster embedding the "big ideas" approach to mathematics learning, and lead to more self-consistently evolving mathematical knowledge.

A more-or less usual applied-mathematics scheme for the mathematical model usage is (Figure 2):



Figure 2. Applied- mathematics scheme for the mathematical model

An initial model is the result of a simplification rendering the phenomenon being studied mathematically feasible, solvable, analyzable. The mathematical model is applied to obtain results, which are supposed to reflect at least to a certain extent the "real thing" – the phenomenon or object being studied. The analysis of the results of the model application usually indicates situations at which the model fails to reflect adequately and fully the "real thing", and should therefore be improved to better reflect it. The improved model leads to more understandings concerning the object, provided it is mathematically feasible for the user. This looks like a never-ending story, and in mathematics it usually is.

Applying this concept of mathematical model *as is* in the didactic context, i.e., as a didactic model, does not differ conceptually from application of mathematical models to any other field. It is just that the "real thing" being studied is of mathematical nature. In what follows, we illustrate the approach, emphasizing the need for the expertise in using it, which means knowledgeably following the main steps represented in the scheme above, accounting for the students' level, so that the model being used is feasible to them; otherwise it cannot serve a basis for the further learning.

Didactic models

Didactic models are the result of didactic transformation, being a form of applied mathematical activity aimed at teaching: „We have to accept that, in mathematics, didactic transformation is indeed a form of mathematical practice. Moreover, it is in a sense applied research since it is aimed at a specific application of mathematics teaching." (Borovik, 2012, p.99). Didactically transforming a mathematical concept is no trivial matter, since it is supposed to cater to both the mathematical and the didactic aspects of the concept: to simplify without distorting the mathematical concept and to present it to pupils in an accessible form.

Shternberg & Yerushalmy consider didactic models to be a means for learning mathematics on the basis of mathematics already familiar to the students and rigorous mapping of learned operations onto the formal mathematical operations (2004). In line with this, we argue that a properly and consistently applied DM approach is a way to enhance the concept-development aspect of mathematical learning.

Prior to presenting the examples illustrating this assertion, we will sum up the main principles on which we propose to base the DMs approach in school. First of all, following Kirsch who claims that activating the existing knowledge is the way to attain accessibility of the new knowledge; DMs should be based on the existing mathematical knowledge, skills and understandings of a student (Kirsch, 2000). Second, the mathematical idea in the basis of the notion should not be distorted as a result of simplifications leading to a didactic model. The definition, operations and properties of a mathematical object should be lucid to those who construct a DM for its learning and to those who use it in its teaching (teachers, textbook writers, curriculum designers etc.)[1] Kirsch (2000) asserts that simplification is a way of making mathematics accessible, but explicitly refers to the "dividing line between legitimate simplification and falsification that does not get past critical pupils" (p. 267). "Not getting past critical pupils" does not ensure that less critical and mathematically aware students do not acquire the falsification as a true image of the mathematical concept. Third, properly used DM approach is a link between the student's existing mathematical knowledge, the knowledge being currently acquired and the future study of the subject, exactly as the mathematical model is the main tool of the on-going upgrading of the mathematics-based understanding of a phenomenon or an object. Thus, properly used DM approach is a tool for inherently mathematical way of studying it.

In addition, no DM is unique in presentation of a mathematical object (as any mathematical model is not the unique mathematical presentation of any object, for that matter). No contradiction should exist between various DMs; they are supposed to complete each other in the representation of the mathematical object. A student may be exposed only to some DMs representing the concept, appropriate to the didactic circumstances (such as the stage of acquaintance with the subject; level of mathematical development of the students; aims of the specific lesson etc.); the properties represented by a DM must be coherent with mathematics, even if it is not explicitly presented to a student.

The efficient usage of DMs involves two equally important components of the DM-based approach: regarding DM as a mathematical model and as a learning tool. As a mathematical model, the proposed approach enables gradual building-up of an appropriate and consistent concept using the mathematical phenomena, objects and skills familiar to a student. As a learning tool, it provides a

[1] This is obviously the matter of Specialized Mathematics Knowledge for Teaching (see Hill et al., 2004).

precious experience of utilizing models in the process of acquiring a new piece of mathematical knowledge, which necessitates critical and mindful insight into the existing knowledge being used. In what follows, we consider two possible appearances of DMs at school. The outline of the first one appearance is presented in Examples 1-4. The reference to the second one we found appropriate to include as a part of the Discussion.

The beginning of fractions

To illustrate what we consider to be a consistent and educated mode of DMs usage in elementary school, we will apply it to the initial stage of fractions learning. This is an example that we believe to be especially valuable at the elementary school level, when the young pupils do not yet have experience in the process of developing a mathematical concept, while they gradually accumulate some mathematical knowledge. Beliefs, skills and concepts they have acquired are supposed to serve them for the further study. The properly planned and applied model usage for learning may be one of the most important experiences in learning mathematics (Van Den Heuvel-Panhuizen, 2003).

Much too often the term "fraction" is used as a synonym to "a number smaller than 1", which is the more problematic since in the very beginning of fractions learning the pupils really meet mostly fractions smaller than 1. Moreover, dominating approaches to the beginning of the fractions teaching are based on the "part-of-the-whole" concept and on geometric-visual representations (Hurst & Hurrel, 2014). Important and intuitively supportive as they are, they are detached from the only arithmetic and the only number system the students have come to know to a certain extent at this stage, which is the system of natural numbers. Hurst & Hurrel (ibid.) suggest that it might be plausible to present fractions already at the early stages of learning in a way that will not inhibit, but rather support the future acquiring of the fraction concept without having to significantly change it. Their approach is that of "big ideas", which we interpret as constructing coherent DMs consistent with the future evolution of fractions into (final and infinite) decimal fractions, notion of ratio, algebraic fractions, the slope of a line and the derivative, and other advanced mathematical appearances of fractions. We suggest that the big idea behind the notion of fraction is the division operation (Mamede & Vasconcelos, 2016). In mathematics, a fraction is either the division operation itself or its result (quotient) (not necessarily a number). If the numerator and the denominator are both natural numbers, the fraction represents a rational number. Fraction is also an operator acting on other mathematical objects, and this is also directly related to its being the division operation. Hence, "the big idea" we propose as the mathematical background, is fraction as division: operation or result. Needless to mention that the idea itself is not intended for elementary school pupils, but the teachers should be cognizant of it.

We illustrate the DMs approach at the initial stage of learning fractions, the model being the arithmetic of natural numbers. We are fully aware of the risk of inhibition effect of this approach. Davis (1989) includes whole number schemes among inhibitors on the way to the rational numbers. Nevertheless, we assert that there is no other mathematical knowledge to build upon for the simple reason that the natural number arithmetic is more or less everything the pupils know before their first encounter with fractions, but for their possible acquaintance with ½ (also justly included by Davis among inhibitors), some primary geometric intuition and some idea of a number line.

Following Shternberg & Yerushalmy (2004), we provide here examples of "mapping" ideas familiar to pupils from the natural numbers arithmetic onto the new mathematical object - fractions, applying the usual scheme for a mathematical model use presented above.

We use it in the first example to impart a meaning of fractions needed for the understanding of addition of fractions; in the second example - to impart conventions of fractions presentations; in the third example – to adjust to fractions a handy geometric model used for integers. In all three examples we refer briefly both to advantages and to limitations of the chosen model, and propose an improved model. Last but not the least is the fourth example of a meaning of natural numbers inapplicable to fractions.

Example 1: Addition of fractions.

The model: a natural number as a cardinal number of a finite set of objects. In a fraction whose numerator and denominator are natural numbers, the numerator serves as a *cardinal number* of a set of equal parts - unit fractions, into which the whole is divided. The denominator indicates the number of parts and their magnitude. Different unit fractions are different objects and cannot be added, unless they are united into one set, just as apples and pears are to be united into the set of fruit to be counted together. For unit fractions, this means representing them with a common denominator. *Limitations:* applicable only to rational numbers. Any other fraction, for example, $\frac{\sqrt{2}}{1+a}$, has to be understood otherwise, namely, as the division operation $\sqrt{2}:(1+a)$ written in another form.

Example 2: **Conventions concerning representation of fractions.**

The model: the decimal representation of natural numbers. The decimal representation is an equally important appearance of two ideas: of a *representation* of numbers *per se*, and of *conventions* in mathematics. As a decimal representation, it is the *model* applied almost as is to decimal fractions, when the pupils are prepared to deal with them. As an example of a *representation convention*, it may pave the way to the understanding that in mathematics there may be different forms of presenting commonly used objects; these forms should be familiar to everybody; this is the part of the mathematical language. $\frac{7}{5}$ is just another form of writing 7:5, meaning either the operation or the number resulting from it. *Limitations:* the final decimal representation is inapplicable for some numbers; it has to undergo adjustments to infinite (periodic or non-periodic) decimal representations, and provide meanings for their truncations of various kinds.

Example 3: The area model

To adjust the useful *area model* from a rectangle whose sides' length are integers to the rectangle whose sides are rational numbers, it suffices to count "unit rectangles" whose sides are unit fractions corresponding to the factors' denominators, instead of unit squares. *Limitations:* the area model "as is" is hardly applicable, for example, to infinite decimal fractions[2], to fractions with irrational nominators or denominators, and would demand serious amendment to apply it to negative rational

[2] Some ideas as presented e.g. in Nelsen (1993, pp.118-122) are based on this type of visual reasoning linking the area notion to numerical reasoning and convergence ideas.

numbers. Nevertheless, speaking of irrational numbers - the segments division in an arbitrary ratio is defined for incommensurable segments as well, for example, by Thales similarity theorems in geometry, on the basis of segments measurement directly related to the number line. Having recognized that the segments ratios is attainable for irrational lengths as well, one can happily keep using the rectangle model for distribution properties of multiplication and division provided it is transfigured so that a subdivision neither into unit squares nor into small "unit" rectangles is needed anymore to apply it. Moreover, the basic fact that the whole segment of length a may be represented as the sum of the two parts, for example, $\frac{a}{(1+\sqrt{2})}$ and $\frac{a\sqrt{2}}{(1+\sqrt{2})}$, is consistent with a similar idea for rational ratio, which again is beneficial for the further goal of regarding the system of real numbers as a whole.

Example 4: Addition of natural numbers as continued counting.

Consider the addition of natural numbers as continued counting: m+n as n times the addition of 1 to m, or m times the addition of 1 to n. Here the *limitations* of the model render it inapplicable as a model for fractions. Obviously, these examples are not meant to be used simultaneously and immediately and not necessarily explicitly in the beginning of acquaintance with the notion of fraction. We do assert though that the ideas represented in these examples must be intertwined in appropriate detail in the course of primary school arithmetic as a general approach to mathematics teaching and learning (DM being a learning tool) and as a groundwork to further encounter with irrational numbers (DM being a mathematical model).

Discussion

The examples above include instances of appearance of new features when the object evolves from an existing one, of transforming the existing feature to adjust to the evolving object, and instances when some features disappear in the new object. Systematically focusing on such occurrences as a teaching norm may foster the concept development as an integral part of learning, provided the notion being taught is regarded as a concept to be permanently developed as a result of teaching and not merely as a topic in a curriculum. One important observation should be made here: should this approach be adopted for fractions or for real numbers, it has to be kept in mind already in the natural numbers teaching. More generally, it will hardly be useful if applied sporadically instead of being a systematic mindful approach. The more so in view of constraints of educational systems: in Israel, for example, and in many other countries, the primary schools are separated from the secondary and the mode of mathematics teaching at different levels is not always coordinated. This transition between the levels is therefore intrinsically discontinuous. We believe that systematic adoption in the primary school of DMs may help to cope with this discontinuity. We regard this to be an issue worth theoretical and empiric study.

Speaking of the encounter with the real number system, we refer here again to the double-sided role of DMs. *DMs as a mathematical model*: similarly to the initial encounter with fractions which is based on natural numbers as a model, in the case of irrational and in general, real numbers, the initial models to build upon are those originating from the system of rational numbers more or less familiar to the students. *DMs as a learning tool*: should the students have acquired appropriate mathematical concepts and learning skills prior to the encounter with real numbers system, these will determine their ability to take-in this new, rather advanced concept, and the extent to which they may take it in. They should have experienced testing the properties of new numbers and operations on them vs. the

familiar ones and the appearance of a new system that includes the previous one not only as a set of numbers, but also as a number system.

Examples of the challenges anticipating the students in their encounter with irrational and in general with real numbers in which the DM approach seems to be promising and worth a close empiric study, are appearance of the root operation and sign; operations on roots (arithmetic and later - algebraic), on expressions of like $a+\sqrt{b}$, and rules of these operations; decimal representations, in particular, decimal approximations of irrational numbers and the necessity to decide when, whether and how to approximate; inclusion of rational and irrational numbers in the same number system, etc. One of the key problems with the notion of irrational numbers is based first and foremost on *impossibility* of writing an irrational number as a fraction of two integers. Thus, their mere existence seems to claim for a new model because of the impossibility of using the previous one. On the other hand, *any* number *a* may be written as a fraction at least in a trivial way as $\frac{a}{1}$ meaning nothing more than *a:1*. A representation of an irrational number by a fraction means just that at least one of the two parts of the fraction: its nominator or denominator or both, are not rational[3]. This does not prevent one from using operations on these fractions the way they were used on fractions as rational numbers; sometimes this representation calls for formulation of new rules. For example, to avoid fractions with irrational denominator, the students are sometimes taught to expand them following the rule familiar from rational fractions and based on division properties, for example, $\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$. On the other hand, an equality like $\frac{\sqrt{3}}{\sqrt{5}}=\sqrt{\frac{3}{5}}$ represents actually a new rule, to be both understood and adopted into the set of mathematical skills. Thus, the notion of fraction as division operation and its properties retains its usefulness. Comprehension of this can be the result of recurrent examination of the notion of fraction and operations defined on it and by it for "new" numbers, based on the DM approach.

Not less important, we suggest that a process of learning that systematically involves DMs is intrinsic to mathematics. **N**o DM adequately represents "the real thing", in our case eventually the system of real numbers. Various facets of the same complicated mathematical object awaiting the students in their forthcoming studies based, at least to some extent, on the analytic abilities acquired with the help of DMs, is a didactic challenge not less that it was a mathematical challenge, and mathematical and didactic tools should be combined in their teaching and learning. We suggest that the didactic models should be seriously regarded as a tool for this type of learning and closely studied in various theoretical and empiric aspects.

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[3] This seems to be a vicious cycle but it is not: we are referring not to a definition but to a possibility of using fractions to write a division operation.

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Dealing with large numbers: What is important for students and teachers to know?

Christine Chambris[1] and Frédérick Tempier[2]

[1]LDAR, Université de Cergy-Pontoise, France; christine.chambris@u-cergy.fr

[2]LDAR, Université de Cergy-Pontoise, France; frederick.tempier@u-cergy.fr

National French assessment shows difficulties with writing large numbers at the beginning of the 6th grade. But, what do students need to learn and teachers need to teach? What do they actually learn or teach? We investigate these questions at different levels of the didactical transposition: students' knowledge, teaching practices and reference knowledge. We show a lack of mathematical understanding of large numbers and make a proposal for teaching knowledge which provides justifications for the use of large numbers which could foster a 'number sense' understanding of such numbers.

Keywords: Large numbers, numeration, units, teaching, knowledge.

Introduction

The topic of place value related to whole numbers is a major one in the primary grades, especially because it is the fundament of basic arithmetic. There is an abundant research literature on the issue. Most often it focuses on "small numbers": from two to four digits. Several authors (Wagner and Davis, 2010; Howe, 2015) consider larger whole numbers, and specific issues on this topic. In this literature, order of magnitude – even more relative magnitudes - appears as a key one. For instance, it is needed for a thorough understanding of economic, political and scientific issues. National French assessment shows difficulties in writing large numbers at the beginning of the 6th grade. The latter can be seen as an anecdotal subject compared to number sense understanding of this kind of numbers. Yet, is there any link between both? How writing and reading large number tasks can be connected to quantity sense or number sense? What do students need to learn and teachers need to teach? What do they actually learn or teach?

Theoretical framework, previous works, and methodology

Theoretical framework

The Theory of Didactic Transposition (TDT) (Chevallard, 1985) (figure 1) considers school mathematics as a reconstruction by the educational institutions from the mathematical knowledge produced by academic scholars. The TDT has been often used for secondary school, more rarely for primary school where scholarly knowledge as a reference is not always taken for granted.

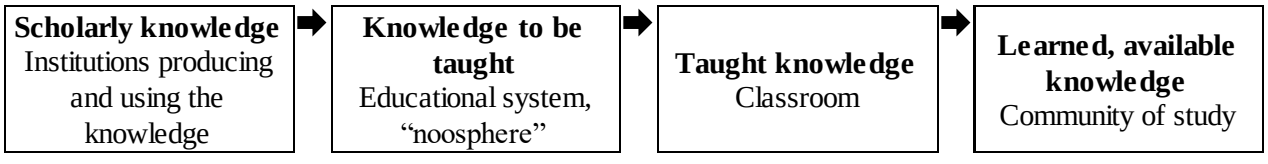


Figure 1: The didactic transposition process (Bosch and Gascon, 2006, p. 56)

The Anthropological Theory of Didactics (ATD) (*ibid.*) extends the TDT. It postulates that practicing math, as any human practice, can be described with the model of *praxeology*. It is constituted by four

pieces: a *type of tasks* -a set of similar problems-, a *technique* -a "way of doing" for all the tasks of the type-, a *technology* justifies the technique and is justified by a *theory*.

Previous works

Chambris (2008, 2015), Tempier (2016) have analyzed teaching and learning of decimal numeration in French context, notably in second and third grades. *Classical* mathematical theory in numeration which embeds units (tens, hundreds, etc.) and relations between them was the reference knowledge up to the New Math. Chambris (2008) proposed the wording "numeration unit" for the units used in numeration (ones, tens, hundreds, etc.). Beginning in the 1980s, classical "scholarly knowledge" has been replaced by transposition of *academic* theory (polynomial decomposition with the exponential notation) within which there is no unit. This might explain why relations between units (e.g. 10 tens = 1 hundred) are little mastered –sometimes not taught at all- in present French context. In turn, Houdement & Chambris (2013), Tempier (2016) designed interventions for reintroducing units for "small numbers" in teaching practices, especially the relations between units, as well as explicit properties of positional notation: 1) The position of each digit in a written number corresponds to a unit (for example hundreds stand in the third place) ("positional principle"); 2) Each unit is equal to ten units of the immediately lower order (e.g. one hundred = ten tens) ("decimal principle"). Ten digits are enough to write any whole number thanks to an iterative process. The names of small numbers present many irregularities: numeration units provide a way to bridge the gap between irregular number names and positional notation. In short: thirty ↔three tens ↔3 tens ↔ 3 in the second place (Houdement & Chambris 2013).

In a broader francophone context (France and Switzerland), there is a range of literature (e.g. Mercier 1997, Ligozat & Leutenegger 2004) on another issue in numeration: students' and teachers' difficulties in the topic "how to write large numbers". Here we present some of their findings. The teachers being observed seem to consider relations between number names and positional notation as a linguistic issue which does not require mathematical knowledge. This generally leads them to teach two rules for writing numbers: 1) replace the words thousand, million by a space[1] (sometimes a dot), 2) put three digits between two spaces. These rules appear to be little powerful to solve the most complex problems with "mute zeros". Mercier (1997) (related to French context) argues this reflects an institutional problem: the lack of mathematical knowledge on the topic, in the teaching system for several decades. Moreover, only one of the five teachers observed attempts to teach general base ten property of positional notation for large numbers. In all these contexts, it is finally social knowledge which leads to validate (or not) an answer! About "13180" "St.: One-hundred-thirty-one-hundreds and eighty. (…) T.: This would be one of the ways to name this number; but: will everybody understand immediately?" (Ligozat & Leutenegger 2004 p. 15).

These scholars present mathematical knowledge to fill the vacuum: Mercier (1997) indicated the general rule for positional notation using exponential notation algebraically, as well as a brief history of number names. Ligozat and Leutenegger (2004) proposed to distinguish and link two pieces of knowledge: for positional notation (base ten), for number names (base 1000). They formulate this using "powers of ten written with figures" (Chambris 2015, p. 57) notation:

[1] In France (among other countries), a space is used between the periods for writing large numbers: 34 020 (unlike 34,020 in some other countries). This space is sometimes replaced by a dot (34.020) but never by a comma. The latter is dedicated to decimal numbers: 34,020 is thirty four ones and twenty thousandths. This paper uses "French" notation.

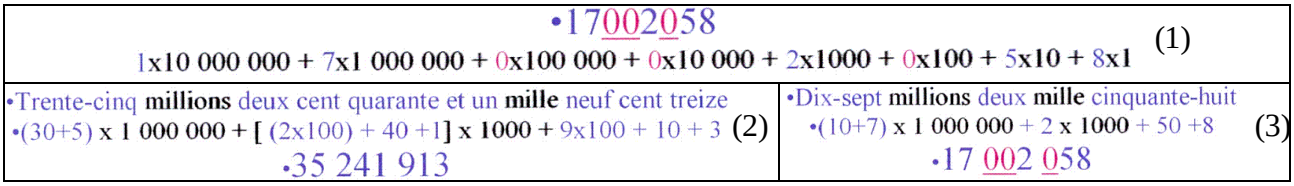


Figure 2: Relations between positional notation and powers of ten (1), between number name and powers of thousand (2 & 3). (Ligozat & Leutenegger 2004, p. 3)

They state: "the point is an institutional "foregone knowledge" phenomenon about how to name the numbers and how this (foregone) knowledge is linked with positional notation" (p. 17, our translation). They suggest tasks like 13180 = 131 hundreds 80 ones. Years before, Fuson (1990) already indicated this in term of knowledge, the systems of multiunits that are intertwined (named base-ten numeration units, and base-1000 numeration units in the present paper). For large numbers, ten "thousands" make a new unit, a "ten of thousands" which is written in the 5th place; ten "tens of thousands" make a "hundred of thousands" which is written in the 6th place, etc. This reveals that the issue of large number names is mathematically connected with relations between units: in base-ten and in base-1000. That is clearly a first step to "quantity sense".

Research questions: Finding praxeologies

What are the relations between knowledge learned (by students), knowledge taught (by teachers) and scholarly knowledge? How do they contribute to give sense to large numbers?

Method and data

Within praxeological analysis, exercises to be performed generally indicate the tasks, explanations related to students' mistakes as well as introduction of new type of tasks often bring the teacher to make explicit the aimed technique and/or technology in classroom episodes, and definitions show technologies. Data will be analyzed in term of praxeologies. A mathematical analysis about reference knowledge is deepened. The data were designed and collected by the second author as follows. A teacher (Soline) was trained on 3-to-4-digit numbers teaching in a collaborative research project aiming at designing a resource for teachers paying specifically attention to the use of base-ten units in relation with written numbers (same vein as Tempier 2016). This grade-4 teacher was later observed during a lesson on another subject: numbers larger than 9999. The lesson was audio-recorded, transcribed, and notes were taken. During an interview (with note-taking by interviewer) just before the lesson, Soline was asked to explain her plan for this lesson. The different tasks of the lesson have been identified then three episodes corresponding to three tasks have been selected as follows. The two first ones are related to students' mistakes (1- in relation with the introduction of the first 5-digit number, 2- in relation with the first mute zero in a 5-digit number). The third one is the introduction of one million. Finally, a written questionnaire was designed to better identify students' knowledge and difficulties (n=159, end of grade 6).

Results

A teacher's mathematical praxeology about large numbers (taught praxeology)

Planning the lesson

Reading and writing numbers (in digits) is the only explicit task ("to be taught") in the French syllabus. The preparation plan shows that Soline chose this writing task with increasingly large numbers from 4 to 8 digits, with various places for "mute zeroes". She was also planning to introduce the definition of the million

as one thousand thousands, but no base-ten relation with the new units (despite the previous study). The teacher wonders whether it is enough to teach how to write large numbers, and that a million is one thousand groups of one thousand.

Implementing the lesson

The first number greater than 9999 to write in digits is "twelve thousand five hundred". A student, Anaïs, is not able to write it. Perhaps she refrains from writing a two digits number in the thousands place in accordance with the technique learned before. The teacher does not identify this cognitive conflict. She shows some confusion by calling out to the researcher; then she tries to help Anaïs.

> You see […] I already have Anaïs who has troubles. She is able to write three thousand but does not know how to write twelve thousand. Does it change something Anaïs? Think about it. Twellllllve thousand. Twellllllve thousand five hundred. Twelve thousand it is twelve groups of one thousand.

While the students learn for the first time to write a five-digit number, the teacher seems to consider there is nothing new to know about the *new* 5th place in the written number and about the *old* word (thousand) in the number name. She tries to help the student by emphasizing "twelve". The technique aimed is to write the number heard before "thousand" (here 12) and then the next number (here 500) with eventually a dot between them.

Later, the teacher asks to write "thirty four thousand and twenty". It is the first time with a "mute zero". Axel writes "34.20": it is what he hears: 34 and 20 with a point instead of the word "thousand".

Soline (T): Thirty four **thousand**. Twenty. It doesn't look strange? After the word one thousand how many digits are there?

Axel: Three

Soline (T): And here you have only two digits. How could you make to have three?

Soline writes on the black board "34.20" and underline 20.

Axel: To put a zero?

Soline (T): Where? (*Axel's answer is inaudible. But Soline writes "34.200" on the board).*

Soline (T): Look Axel (a*nd Soline writes on the board: "34.020").*

Once again, the teacher seems powerless in front of a student's mistake. She gives then the answer without any explanation. With this case the technique to write a number in digits incorporates a new element: after "thousand" there must be three digits. Thus, when there is a two digits number after the thousands it is necessary to write a zero.

Later, after the writing of the number 999 000, Soline intends to introduce the millions. She asks the students how many groups of one thousand there are just after 999 000 and introduces the million thereby:

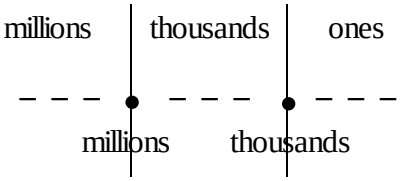
Soline (T): One thousand times one thousand is called otherwise. How it is called?

A pupil: One million.

Soline (T): One million, it is a new word. For the moment we said the word one thousand, now we say too one million.

A pupil: How we are going to write it?

Soline (T): This is what we are trying to discover together. It is necessary to make a kind of small chart (*Soline draws then this place value chart on the black board).*



The million is defined as one thousand thousands, what confirms Soline's plan. She tries to give meaning to large numbers relying on relations between base-thousand units. According to this point of view, the million doesn't appear as ten groups of hundred thousands. In the above table there are the names and places of periods but no reference to the name of base-ten places like the hundred thousands place for example. These technological elements rely only on a period viewpoint of the written number. The corresponding technique for writing numbers is: "write the number heard and points for the words *millions* and *thousand*". This is confirmed during the rest of the lesson. For example for the writing of two million, the teacher tells: "I write my *two* with my little point which means *million*". With this technique, teacher is not able to help students with mute zeros as a whole period or within a period. For example, at the end of the lesson, to write twelve million fifty some students write 12.50.000, 12.000.50 or 12.050.000 (with points or spaces). The teacher continues to explain the three digits in a period without a "base ten/place" viewpoint in addition to this "base 1000/period" viewpoint.

A questionnaire to inform about 6th grade students' knowledge (learned praxeologies)

The first part of our questionnaire concerns the writing of large numbers in digits (table 1).

Numbers to be written in digits	Beginning of gr. 6 (2008)	End of gr. 6 (2016)
Four hundred and seventy-five (475)	94%	
Three thousand and three (3 003)	96%	
Six hundred and twenty seven thousand (627 000)	76%	87 %
One million six hundred thousand (1 600 000)	76%	89 %
Three million fifty thousand three hundred and twenty (3 050 320)		79 %
Seventeen million two thousand and fifty-eight (17 002 058)		69 %
Five hundred and three million thirty-seven (503 000 037)		82 %

Table 1: Results of national assessment (2008) and our assessment (2016) of 6th grade students

To complement the data we also proposed conversions in order to examine the relations between large units and determine which of the relations between base-ten units and base-thousand units are better known. Such tasks are inspired from our prior research, and Ligozat & Leutenegger (2004)'s analysis. Both tasks raise as much difficulties. Approximately half of the students succeed in converting 4 millions into hundreds of thousands (48%) and 3 millions into thousands (50%). Many students did not write any answer. This was not the case for the "writing number" tasks. Perhaps they never performed conversion tasks before.

Complements for reference knowledge for teaching large numbers

We have already recalled some considerations about the specificity of the written and spoken numeration systems and their imbrications in a double system. The above lesson analysis and our previous studies

about "small" numbers show that there is a need for an intermediate system between these numeration systems in order to articulate this double system: the numeration units systems for bases ten and 1000. These systems of units enable various decompositions of numbers related to base 10 and base 1000. Up to 9999, number names can be linked with base ten numeration units, whereas beginning of 1000 they are linked with base 1000 numeration units. The transformation from a base-ten decomposition into a base-thousand decomposition is made by conversions.

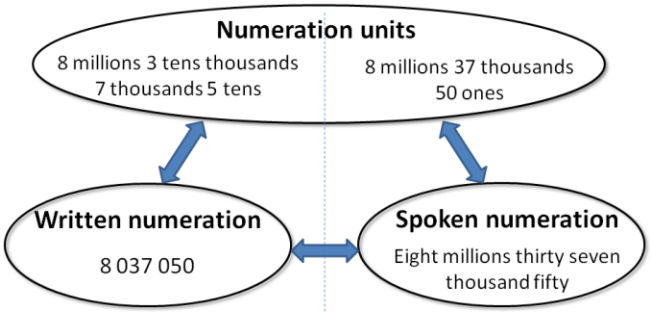


Figure 2: Links between numeration units, written numeration and spoken numeration

The numeration units system supplies a way to justify the writing (in digits) of a number name. It enables to justify the mute zeros. See, for example, the task "writing eight millions thirty seven thousand fifty" (figure 2) in digits. The number can be written or spoken in numeration units "8 millions 37 thousands 50 units", which can be converted in this register "8 millions 3 tens of thousands 7 thousands 5 tens". It relies on conversion of 37 thousands in 3 tens of thousands and 7 thousands. The digit of millions is written in the 7th place, that of hundreds of thousands in the 6th place, etc. It is necessary to write a 0 to mark the lack of missing units. The obtained number can then spell 8, 0, 3, 7, 0, 5 and 0 by the positional principle of the written numeration. This example shows how this designation is relevant for reference knowledge for teaching. It enables to make links between the written and spoken system and to explicit the knowledge at stake.

Discussion

The analysis of Soline's lesson shows that her knowledge of the spoken numeration of large numbers, for one side, and her previous teaching of small numbers with an important concern about relations between base-ten units, on the other side, brought her to a beginning of a teaching of relations between base-thousand units. The latter seems not to be the case in the observations reported by Mercier (1997) or Ligozat and Leutenegger (2004) where only one teacher focused on the relation between base-ten units, and none on those between base-thousand units. Yet it is not enough to provide explanations for writing large numbers. Indeed, the teacher seems sometimes deprived to help her students during this lesson. In addition, our observation at the end of the lesson shows that it is insufficient to enable the students to avoid some mistakes, particularly those which are related to the mute zeros. The teacher explanations and her place value chart let us think that she assigns a base-thousand system to the written system and ignored its base ten operating. The two first episodes are coherent with our literature review considering large number names as an "institutional foregone knowledge", and a linguistic issue.

By introducing numeration units in the reference knowledge to make links between base-ten and base-thousand units, we aim to provide a more explicit knowledge for teachers and students. It enables to take into account the double system units for large numbers (base ten and base 1000). It can, on one side, enrich the understanding of the written system by realizing that the system always works in the same way

according to the base ten and, on another side, learn the names of large numbers by putting them in connection with the written code. For example, to justify the writing of twelve thousand five hundred, the teacher has to clarify the link between ten thousands, the ten thousands unit and the corresponding places in the written number. In the writing of mute zeros, as in thirty four thousand twenty, the link between the values of 3 and 4 and the corresponding places in the written number can justify the writing of a zero in the hundreds place. Questionnaire results seem to confirm a more effective technique is needed for mute left-hand zeros.

This dual system of units can favor recognition of relative magnitudes of large numbers. For example, understanding the million involves the relation with smaller numbers. For example to understand the million involves the relation with smaller numbers: a million it is "ten times one hundred thousand" as well as "one thousand times one thousand". Tasks of mental computation on numbers with only one non-zero digit can strengthen these relations. For example, "ten times two hundred thousand" could aim at leaning on the relation between one hundred thousand and one million, relation which does not explicitly appear in our way to speak these numbers. Other tasks aim at the extension of the written numeration. For example it is possible to recover and adapt small numbers tasks as the situations of collection counting and ordering (Tempier 2016). Counting a collection, with representations of large groups, can then be used to introduce new base-ten units. In addition, ordering a collection can be used to produce various decomposition of number under base ten and base 1000 with numeration units. Under this approach the spoken numeration could be secondly brought, in connection with these decompositions.

Conclusion

Obviously students can succeed in writing numbers without knowing relative order of magnitude of numeration units in base ten and in base 1000. Yet, relations between units can provide justification for even writing numbers. This knowledge can be expressed with numeration units. It is missing in French institutional system. It might also contribute to the understanding of quantity sense. In this context, the work on small numbers is not enough to train teachers (even it is only for one case); surely, it helps the teacher to question her usual practices on large numbers. However, it does not provide tools for justification, neither the specific stakes of large numbers. Further research is needed in order to provide powerful tasks for teaching and learning large numbers.

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Working memory, number sense and arithmetical performance: Relations between these domains.

Luciana Vellinho Corso[1] and Beatriz Vargas Dorneles[1]

[1]School of Education - Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil
l.corso@terra.com.br; beatriz.dorneles@ufrgs.br.

Studies correlating working memory, number sense and arithmetical performance show controversial results which create the need for further investigation. This research aims to verify the relationship between two working memory components (central executive and phonological loop) and numerical competence assessed through two different tasks: the Number Knowledge Test and the School Achievement Test. It involved 79 Brazilian students from 4th to 7th year of elementary school. The results suggest a significant correlation between the central executive and number sense. The same relationship was observed for the arithmetical performance. The phonological component showed no significant correlation with number sense nor arithmetical performance. The educational implications of the study are pointed out.

Keywords: Working memory, number sense, arithmetical performance.

Background

Both number sense and working memory are fundamental skills for arithmetical learning (Geary, 2011; Jordan et al, 2013). Studies investigating the relationship between working memory and arithmetical performance have been widely discussed in the literature (Anderson & Lyxell, 2007; Geary et al., 2007; Passolunghi & Siegel, 2004). However, the research that deals with the relationship between working memory and number sense is recent. Number sense is considered the basis of arithmetical learning and, consequently, it is assumed to be associated with working memory (Friso-Van Den Bos, Van Der Ven, Kroesbergen & Van Luit, 2013).

Working memory is a cognitive system that supports the development of various learning processes. It is a limited capacity system which allows the temporary storage and manipulation of verbal or visual information required for dealing with complex tasks. During learning the student constantly uses the resources of working memory to perform a series of activities, from the simplest tasks, such as remembering instructions, to the more complex ones, such as solving problems, that require the storage and processing of information and the control of learning progress. In the case of arithmetic, for example, a multidigit calculation (23 + 48) requires several subprocesses (retrieval of arithmetic rules and arithmetic facts from long-term memory, calculation and storage of intermediate results, arithmetic procedures that involve carrying and borrowing operations) that must be coordinated and executed by the working memory system. Students with deficits in this ability would face problems. Thus, a difficulty especially related to the coordination of simultaneous operations of processing and storage can interfere in the execution of arithmetic tasks, resulting in slower performance and more errors in computation (Andersson & Lyxell, 2007).

Considering the tripartite model proposed by Baddeley and Hitch (1974), the working memory system is formed by three components: two storage systems (visuospatial component and the phonological loop) and the central executive, the nuclear component of working memory, responsible for the processing of cognitive tasks, coordinating the information stored within the other two components. It is generally agreed

that arithmetical achievement is associated with working memory performance (Alloway & Alloway, 2010; Andersson & Lyxell, 2007; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). However, there is a lack of consensus regarding the relative importance of the central executive (Andersson, 2008; Geary et al., 2007; Passolunghi & Siegel, 2004), the phonological loop (Andersson & Lyxell, 2007; Passolunghi, Mammarella & Altoè, 2008) and the visuospatial component (Geary, Hamson & Hoard, 2000; Mclean & Hitch, 1999) in relation to arithmetical performance.

Some studies found that the central executive is the most affected working memory component (Corso & Dorneles, 2012; Geary, Hamson & Hoard, 2000; Geary, Hoard & Hamson, 1999). The central executive has three main functions: inhibition (prevent irrelevant information from entering or remaining in working memory), shifting (shifting between pieces of information and response sets) and updating (active processing and refreshing of information in working memory). There is no consensus regarding the role that each specific executive function plays in number sense. Some studies point out that inhibition is central to number sense development (Kroesbergen,Van Luit, Van Lieshout, Van Loosbroek & Van De Rijt, 2009), but such a result was not found by others (Lee et al., 2012; Navarro et al., 2011). Updating is usually seen as the most important predictor of number sense (Kroesbergen et al., 2009; Lee et al., 2012), but, it is important to emphasize that research focusing on the relation between number sense and executive function is still limited (Friso-Van Den Bos et al., 2013).

Concerning number sense, the literature shows consensus related to the important role that this construct plays to mathematical development, but there is a lack of consensus regarding the best way to define, assess and intervene in number sense (Gersten, Jordan & Flojo, 2005). Considering studies in the areas of mathematical education and cognitive development, Berch (2005) compiled a list of 30 characteristics presumed to compose the number sense concept. According to the author, number sense means awareness, intuition, recognition, knowledge, ability, desire, feeling, expectation, process, conceptual structure or mental number line. When defining number sense, some authors point out the conceptual, abstract aspect of numerical processing.
For example, Dehaene (2001) emphasizes that number sense refers to the ability to mentally represent and manipulate numbers and quantities. Gersten and Chard (1999) define number sense as the flexibility with numbers and the understanding of the meaning of numbers and ideas related to them. Other researchers use definitions emphasizing the performance that is facilitated by that conceptual understanding of number, such as counting ability, number identification, number awareness, estimation, measurement, mental operations with numbers (Jordan, Glutting & Ramineni, 2010). We believe that both definitions of number sense are complementary: In order to succeed in the comprehension and execution of tasks involving numbers, relations and quantity, an abstract understanding of numerical processing is necessary. Therefore, the conception of number sense that characterizes this paper is that it is a general construct, which encompasses a very broad set of concepts, which the student develops gradually from his interactions with the social environment. Number sense is a way of interacting with numbers with its various uses and interpretations, enabling the individual to deal with daily situations that include quantification and the development of efficient strategies (including mental calculation and estimation) to deal with numerical problems (Corso & Dorneles, 2010).

Recently, researchers are interested in the association between the different components of working memory system, especially the central executive and the number sense. Children are expected to employ working memory capacity while experiencing number sense tasks such as counting, understanding magnitude, doing basic arithmetic calculation, using mental number line (Gersten, Jordan & Flojo, 2005).

However, there is a small body of literature targeting the association between working memory and number sense.

The current study

The literature presented above suggests that working memory plays an important role for the development of numerical competence. However, the importance of each working memory component is not well defined. In order to contribute to this discussion, the present study aims to verify the relationship between two working memory components (central executive and phonological loop) and numerical competence assessed through two different tasks: the Number Knowledge Test and the School Achievement Test.

Method

This is a cross-sectional study involving 79 (10- to 14- year-old) Brazilian students (36 girls and 43 boys) from the 4th to the 7th year of five public elementary schools. Mean age was 11.9 years. Students were indicated by their teachers considering the students' performance (average and low average) in the math curriculum according to each school year.

Instruments

1) Working Memory

1.1 The central executive component of working memory was measured using two different tasks: a non numerical - processing of verbal information (adapted from Hecht et al., 2001) and a numerical task - processing of numerical information (Yuill, Oakhill & Parkin, 1989). In the first task the students were required to answer yes or no to sets of two to four questions and then say the last word in each of the sentences, for example, in the two-question set, "Do tables walk?" and "Do lamps run?", a correct response would be "no" to each question and then "walk" and "run". For the numerical task, the students read aloud a growing sequence of three-digit sets and, at the end of each set, should remember, in order, the last digit of each set. For example, for the sets (2 5 7) and (1 8 6), the digits "7" and "8" must be remembered.

1.2 The phonological component of working memory was assessed through the Memory of Digits, Sentences and Short Stories task (Golbert, 1998). It consists of a growing sequence of digits, sentences and short stories to be repeated by the student.

2) Numerical Competence

2.1 Number Knowledge Test (Okamoto & Case, 1996): This task is designed to assess the students' knowledge and comprehension about counting, numerical concepts and basic arithmetic calculation. The instrument is divided into four levels of complexity, being presented from the simplest (level 1) to the most complex (level 4). Although this instrument was designed for assessing students up to 10 years of age, we decided to use it, even having few students in the sample older than this age group. The students who were 13 and 14 years old were repeating students who were facing difficulties in some foundational components of numerical proficiency. The sample of this study was formed by average and low average arithmetic learners, but no ceiling effect in this task was observed.

2.2 School Achievement Test (Stein, 1994): This is a Brazilian standardized psychometric instrument designed to verify the students' arithmetical achievement. It presents 38 items (3 word problems and 35 written calculations). The word problems involve magnitude comparison and simple addition and

subtraction calculation. The written computations involve basic operations, operations with decimals, fractions, operations with integers and potentiation.

Data were analyzed using the correlation analysis between number sense and arithmetical achievement measures with the working memory tasks (central executive and phonological tasks), using the analysis of Pearson correlation at the significance level of $p<.05$.

Results

Considering the central executive component of working memory, a significant correlation was found between the two working memory tasks and the tests that measured both number sense (WM1 r = 0,449, p = 0,000; WM2 r = 0,316, p = 0,005) and arithmetical achievement (WM1 r = 0,303, p = 0,007; WM2 r = 0,344, p = 0,002). The phonological component of working memory, though, did not reveal a significant correlation between the three tasks designed to assess the phonological loop neither with the number sense task nor with the arithmetical measure. Only a weak correlation was found between number sense and the task that assessed the recalling of short stories (r = 0,226, p = 0,045). Correlations among the measures are reported in Table 1.

Table 1 - Pearson correlation coefficient (r) and the significance level (p) between the number sense test and the mathematical subtest of the TDE with the different components of working memory (central executive and phonological loop)

	WM1		WM2		MD		MS		MSS	
	r	p	r	p	r	p	r	p	r	p
NKT	0,44	0,000*	0,316	0,005*	0,153	0,177	0,169	0,137	0,226	0,45*
SAT	0,303	0,007*	0,344	0,002*	0,189	0,096	0,12	0,915	0,069	0,547

NKT = Number Knowledge Test; SAT= School Achievement Test; WM1 = Working Memory 1 (non numerical task);WM2 = Working Memory 2 (numerical task); MD = Memory of Digits; MS = Memory of Sentences; MSS = Memory of Short Stories.
* p-value < 0,05

Discussion

The results of the study presented a significant correlation between the two central executive tasks and the number sense test. This point emphasizes the fact that dealing with number sense activities requires working memory involvement, in this case, specially through the central executive system, since it was not found a positive relation between the phonological component of working memory and the number sense measure. This outcome reinforces what research has shown in relation to the strong association between the central executive (updating function) and number sense in children (Lee et al., 2012). Results in the same line are presented by Friso-van den Bos et al. (2013) who found that updating has the highest correlation with number sense, when compared to the shifting and inhibition functions of the central executive. As pointed out earlier, a small number of investigation has targeted the association between number sense and the central executive component of working memory and, therefore, more investigation is needed considering that research of this kind will bring contributions to preventing and remediating arithmetical difficulties.

The results of this research are in line with those that emphasize the positive association between working memory and arithmetical achievement (Geary, Hoard, Byrd-Craven & Desoto, 2004; Passolunghi, Mammarella & Altoè, 2008) reinforcing that the working memory is critically involved in a variety of numerical and arithmetical skills. In this study, this positive association refers to the executive component of working memory, but not to the phonological one. As mentioned before, there is a controversy in the literature regarding the role of each component of the working memory system in arithmetical achievement (Meyer, Salimpoor, Wu, Geary & Menon, 2010). Studies that include students with difficulties in mathematics in its sample indicate problems with the three components of the working memory system, but the central executive seems to be specially affected.

The non-conclusive results related to the contribution of each working memory component to numerical competence can be related to the following aspects: the large variability in the tasks used to assess the different components of working memory, the different kind of arithmetical tests being used and the ages of the subjects being assessed. We know that different cognitive demands require distinct working memory resources and these resources, in turn, can vary according to the age of the subject (Andersson & Lyxell, 2007). Therefore, although there have been advances in this area of study, more investigation is needed.

Conclusion

The study showed a significant correlation between the central executive component of working memory (updating) and number sense. It contributed with more investigation with regard to the association between working memory and number sense, as studies linking these two domains are still scarce and present controversial results. The next steps for future investigation will involve more detailed analysis aiming to identify how the different tasks that compose the Number Knowledge Test (counting, numerical magnitude, mental number line, estimation, arithmetic calculation) are associated with the central executive, including in this analysis not only the updating component of the central executive, but also shifting and inhibition. This sort of analysis can give us a better view of the intensity of the involvement of the central executive function in different number sense tasks.

The outcomes of this investigation are in agreement with previous studies highlighting the significant correlation between working memory and arithmetical skills. The educational implication of such a finding deserves our attention. It is crucial to know the cognitive abilities that are impaired in the learner since the way the teaching process is conducted, will directly influence the effect that the cognitive deficit has on learning. For instance, students who are very slow to calculate, need the teaching of more efficient counting strategies and procedures in order to avoid being based overmuch on their working memory (when the counting all procedure is being used, for example), overloading it and increasing the chance of error in the calculation. Problems in working memory end up affecting the set of everyday situations in which mathematical tasks are involved. Those difficulties lead the students to present some characteristics that make the learning of mathematics more difficult, for example: counting on the fingers for a longer time, not performing mental calculation, forgetting the result of calculation they just made, not remembering the sequence of steps of an operation (Geary et al., 2007).

Studies are being designed focusing on the development of interventions to improve the working memory capacity by asking children to engage in tasks that require simultaneous processing and manipulation of information (Klingberg, 2010). However, the results of these studies are still controversial (Melby-Lervag & Hulme, 2013). Recent intervention research emphasizes the importance of combining working memory interventions with interventions that target the specific mathematical areas in which the student is showing

delay (Sperafico, 2016). This type of work has brought promising results to help students with working memory difficulties to learn mathematics.

Finally, this study contributed to the field of arithmetical learning by bringing some evidence of the positive associations between working memory and numerical competence (number sense and arithmetical achievement). Knowing the cognitive abilities underling arithmetical learning is fundamental to guide curriculum planning considering the working memory demands of the tasks, their level of difficulty and the characteristics of the learners. Further studies in this area will offer advances in the processes of preventing and remediating learning difficulties in mathematics. By identifying which components of working memory are weak, it is possible to avoid that at-risk students develop future problems. In the same way, research of this kind will support our understanding of possible cognitive obstacles that interfere in learning mathematics, so that we can face them trough the selection of adequate teaching resources and content as well as good teaching strategies. Maybe this is the most important contribution that the constructs of working memory and number sense can make to mathematical education.

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Number estimation in children: An assessment study with number line estimation and numerosity tasks

Beatriz Vargas Dorneles[1], Mariana Lima Duro[1,2]; Nohemy Marcela Bedoya Rios[1]; Camila Peres Nogues and Clarissa dos Santos Pereira[1]

[1]Universidade Federal do Rio Grande do Sul/UFRGS, Brazil; beatriz.dorneles@ufrgs.br

[2]Instituto Federal do Rio Grande do Sul, Brazil

Number estimation is an important skill for both everyday and school situations that involve a group of cognitive abilities. The ability to estimate may provide a feedback tool to check performance in different mathematics learning contents. The most widely used task to assess number estimation is the Number Line Estimation Task (Siegler & Booth, 2004), and some researchers used a kind of Numerosity Task (Luwel, Verschaffel, Onghena & De Corte, 2003). This research compares the students' accuracy in two tasks that assess the ability of Brazilian children (N = 60), attending the 2nd and 3rd grades of a public school, to estimate. The children's success in the Number Line Estimation Task suggests it is more accurate than the Numerosity Estimation Task in assessing children's performance in estimation maybe because of the different cognitive functions required by the two tasks. The study´s educational implications are discussed.

Key words: Number line estimation, numerical development, spatial representation of number magnitude.

Introduction

The decimal number system is used to establish exact quantifications, in contrast to quantity estimation processes. In everyday situations, we often use either exact quantification or number estimation (Feigenson, Libertus, & Halberd, 2013), and sometimes estimation can be easier than exact quantification (Siegler & Booth, 2004). Number estimation is a cognitive process used for quick or approximate answers or, for example, to calculate the duration of a movie or the distance between two places. It can be used as a feedback tool to check performance in different areas of mathematics including those requiring exact quantification. From our point of view, two complementary ideas define number estimation: a non-counting based quantitative answer to represent a set of objects; and a translation (Siegler and Booth, 2004) between two different ways of representing a number. We know that mathematical competence involves a group of abilities and cognitive processes. Number estimation has been considered one such process (Levine, 1982) despite the fact it has been less studied than exact quantification (Piazza, Mechelli, Price & Butterworth, 2006) although its importance has been highlighted (Rousselle & Noel, 2008). This can be explained by the variability of the tasks used to assess estimation skills in children, adolescents and adults as well as the different situations in which we use estimation (Siegler & Booth, 2004). The recognition of small amounts may be related to the ability to represent quantities in a mental number line and this ability would assist in comparing the magnitude between two numbers (Schneider, Grabner, & Paetsch, 2009).The estimation performance can be necessary in solving some mathematical tasks and the development of estimation is also considered a good predictor of later symbolic math skills (Park & Brannon, 2013). Despite the importance of its use, number estimation is not part of the school curriculum in many countries, including Brazil. Changes designed to improve mathematical achievement, including the introduction of number estimation in the curriculum, are currently being introduced in Brazil. For teachers, assessing the ability to estimate using different tasks might be a good starting point to analyze the

importance of the ability as well as to highlight the topic´s importance in mathematics education. One of the most consistent conclusions reached by studies about the development of estimation is that children are not very able estimators, even when estimation is used in various daily applications. However, some researchers have hypothesized that children's estimations reflect their internal representation of numbers (Siegler & Opfer, 2003).

Moreover, there is some evidence to suggest estimation is related to mathematical competence in general and arithmetical performance in particular (Siegler & Booth, 2004; Booth & Siegler, 2006; Schneider et al., 2009; Mazzocco, Feigeson, & Halberda, 2011; Laski & Siegler, 2007) and can be improved (Park and Brannon, 2013). Recent research has highlighted the importance of estimation for mathematical development (Link, Nuerk, & Moeller, 2014; Laski & Yu, 2014). This research indicated that the better the students' accuracy in mental number line is, the better their performance in other numerical and arithmetic tasks (Link, Nuerk, & Moeller, 2014). Hence, it is important to understand the estimation process, the abilities involved, how to assess them and its role in mathematical performance, especially regarding the proposed changes to the curriculum in Brazil. It can be said that even though the estimation processes have been studied for the past twenty years, there is no consensus regarding the assumption that the estimations assessed by different tasks reflect a pure mental representation of numbers as proposed by Siegler and Booth (2004) and some new data indicate that it is affected by the limited knowledge of numbers (Ebersbach, Luwel & Verschaffel, 2015), as well as by visuospatial abilities (Crollen & Noël, 2014).

In the current scientific debate, among the explanations for the development of number estimation in children, two stand out. The first, the multiple representations of numbers model (Siegler & Booth, 2004), assumes that children initially represent numbers in a less accurate algorithmic way and develop a more accurate linear representation with age and experience. That is, in children the mental number line is compressed and they tend to maximize the distance between the magnitudes of numbers at the low end of the range and minimize the distance between the magnitudes of numbers in the middle and upper ends of the range. This tendency was named logarithmic representation. Gradually, children develop a linear representation, which maintains the same distance between the numerical magnitudes. Empirical evidence to support this logarithmic-to-linear shift model has largely come from the Number Line Estimation Task (NLET) proposed by Siegler and Booth (2004) in which children or adults must estimate the magnitude of a number by marking its proper position on a number line. However, the logarithmic-to-linear-shift hypothesis has been questioned by researchers studying number-line estimation (Barth & Palladino, 2011; Ebersbach, Luwel & Verschaffel, 2015). Some of the issues raised led to the development of a second explanation, the proportion-judgement strategies model (Barth & Palladino, 2011), which suggests that in the NLET children estimate the size of a part, the numerical magnitude of a specific number, relative to the size of the whole, thus making a judgement about the proportion of the size of the former. Hence, according to the numerical range, the more reference points (landmarks) made available, the more accurate the estimation will be, especially close to the landmarks. In other words, estimation performance reflects the strategies chosen to solve the tasks. Recently, a third model, the Two-Linear Account has been proposed as a plausible alternative (Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008) and explains the developmental changes in number estimation as a result of children's familiarity with numbers. In this model, the mental numerical representation can be alternatively described as a combination of two linear patterns with different slopes, depending on number familiarity. In other words, this linear representation of numbers changes according to the age and numerical range known by children; the unknown numbers have

a slower linear representation than the known numbers. A recent paper (Dackermann, Huber, Bahnmueller, Nuerk & Moeller, 2015) proposes the integration of these accounts, which is a line of reasoning that we support as they introduce the idea that aspects of all three accounts may complement each other and facilitate a more comprehensive understanding of children´s development of number line estimation.

Among the tasks most widely used to assess the ability to estimate are the NLET, described above, and the Numerosity Estimation Task (NET), which requires the subject to estimate the quantity of objects in a set (Luwel, Verschaffel, Onghena & De Corte, 2003; Barth, Starr & Sulivan, 2009). To the best of our knowledge, there is no research that indicates which of these tasks best assesses students' accuracy in estimation and therefore which would be best for application in research and in schools. Thus, the purpose of the study was to compare the accuracy in the numerical estimation of 60 children from the 2nd and 3rd grades of a public school in the city of Porto Alegre (Brazil) in order to determine which task (NLET or NET) best assesses the students' accuracy in number estimation. As the children in both grades were used to manipulating objects and completing tasks similar to the NET and admittedly had no contact with the number line before, we assumed the students would perform better in the NET than in the NLET. We chose the two tasks because the NLET is the most frequently used in estimation research and is widely used to examine how the human mind represents numbers (Barth & Palladino, 2011; Ebersbach, Luwel & Verschaffel, 2015), while the NET is similar to another task often used by some research groups that have a slightly different theoretical viewpoint regarding estimation, for example, Barth, Starr & Sulivan (2009). Both tasks are assumed to assess the same numerical estimation ability.

Method

Using two different tasks, we compared performance in number estimation within a group of sixty children (mean age = 8.4, SD = .69, age range from 7.4 to 11.2 years) who were recruited from one public school, 37 boys and 23 girls: 28 from 2nd grade, (M = 7.8, SD = .29) and 32 from 3rd grade (M = 8.9, SD =.50). Two tasks were used: the NLET (Siegler & Booth, 2004) and the NET, adapted from Luwel, Verschaffel, Onghena & De Corte (2003). We used only one criterion to determine the students' accuracy in the presented tasks: the measure proposed by Siegler and Booth (2004), described below. The task was applied collectively in the classroom, the workplace and the school activities were affected as little as possible. The time to perform the activities in each class was about 40 minutes per task. The solution to both tasks involved the use of pen and paper.

The NLET requires the subjects to mark points corresponding to specific numbers along a number line bounded by 0 and 100. Children were asked to mark the place they considered most suitable for the position of the number to be estimated. Before each item, the experimenter said, "This number line goes from 0 at this end to 100 at this end. If this is 0 and this is 100, where would you put n?" (n being the number specified in the trial). 29 number estimations were required, one at a time. Each number was presented twice. Each child received a booklet with a number line drawn on each sheet to mark their answers. The difference between the two estimations of the same number provided a measure of the variability of the estimations. The 29 numbers comprised the 24 proposed by Siegler and Booth (2004), plus another 5 numbers that were also used in the second task. The 29 numbers presented were 3, 4, 6, 7, 8, 9, 12, 17, 21, 23, 25, 29, 33, 39, 43, 48, 49, 52, 57, 61, 64, 72, 78, 79, 81, 84, 90, 95, 96. They were presented in random order and then repeated in the same random order. Children had no pre-determined time to finish the task, each one could complete the task in the time they wanted.

The NET requires the children to estimate the amount of dots distributed in a checkered 10x10 grid. Before starting the task, the students were told the empty grid contained 0 dots and the full grid contained 100 dots. To reduce the possibility of verbal counting, the stimuli were presented quickly (1 second for each group of ten dots presented) and immediately followed by a white screen. Students were asked to perform a numerical estimation of the amount observed, writing them down in a notebook. They could not use any additional tool to solve the task. In the task, eight numbers from 0 to100 were randomly matched (4, 7, 9, 17, 25, 49, 78, 95) in two different ways. In the first, the dots were presented in clusters and, in the second, they were presented dispersed. Both tasks were carried out collectively on different days. There was no feedback for correct or wrong answers.

Results

To calculate the accuracy of the estimations given, the calculation of absolute percentage error of each child was used, adapted from Siegler and Booth (2004), and represented by the formula:

$$\frac{\text{Mean Estimation} - \text{Estimated Quantity}}{\text{Scale of Estimations (100)}}$$

To illustrate how this measure works, if a child was asked to estimate the location corresponding to the number 60 (or quantities of dots) in a number line from 0 to 100 (or 10X10 grid) and his/her answers were 65 in the first estimation and 75 in the second, we calculated the mean between the estimated values in the two attempts (in this example, (65+75)/2=70). The absolute percentage error would be 10%, corresponding to the result of (70 - 60)/100, according to the above formula. We used this calculation because previous analysis showed that the difference between two answers for each number in both tasks was not significant (p=.24) for NLET or (p=.06) for NET.

After that, a descriptive analysis of the accuracy of each child was conducted to identify the general standard of performance for each task. These analyzes show the students' accuracy is higher in the NLET. The reported performance tends to be more cohesive in the NLET (Table 1).

	Mean	SD	Significance
Number Line Estimation Task	.089	(.050)	p =.018*
Numerosity Estimation Task	.158	(.250)	

*p<.05

Table 1: Comparison of the Mean of Percentage Error in each task

To determine the correlation between the two tasks, Pearson´s correlation coefficient was carried out (r =.67, p<.01), and indicated a positive correlation between both tasks, suggesting that either demand similar cognitive functions or both are related to other skills that were not measured.

To test for differences in the children's estimation accuracy when required to estimate smaller and larger amounts, we considered the same numbers estimated in both tasks. A paired-samples t-test only indicated difference when estimating the same number in each task in larger quantities (Table 2).

	Number Line Estimation Task		Numerosity Estimation Task		
Number to be estimated	Mean	SD	Mean	SD	Sig.
4	.034	(.043)	.086	(.333)	*ns*
7	.063	(.086)	.084	(.397)	*ns*
9	.081	(.101)	.083	(.229)	*ns*
17	.109	(.112)	.188	(.495)	*ns*
25	.117	(.087)	.151	(.184)	*ns*
49	.085	(.062)	.204	(.145)	.000*
78	.118	(.064)	.249	(.224)	.000*
95	.091	(.080)	.219	(.441)	.022*

*p<0.5

Table 2: Comparison of estimations between the same numbers

This analysis showed that the students' estimations were all more accurate in the NLET, however, these differences were only statistically significant with the numbers 49, 78 and 95 (Table 2), while there is greater variability in the children's estimation in the NET. This may suggest important differences between the tasks performed. Maybe the three numbers (49, 78 and 95) are closer to "quarters" (50, 75 and 100), which would help children to identify the position of the numbers in the number line. Hypotheses to explain these variations will be discussed later. In the NLET, one of the most common strategies used by students was to fill the number line with marks that represent the numbers before or after the number proposed, as illustrated in Figure 1. In the NET, some students tried counting groups of dots and imagining how many similar groups could be in the whole. Progressions related to the speed with which the children performed the estimations were not tested in this analysis, considering that the time for execution was the same for all participants.

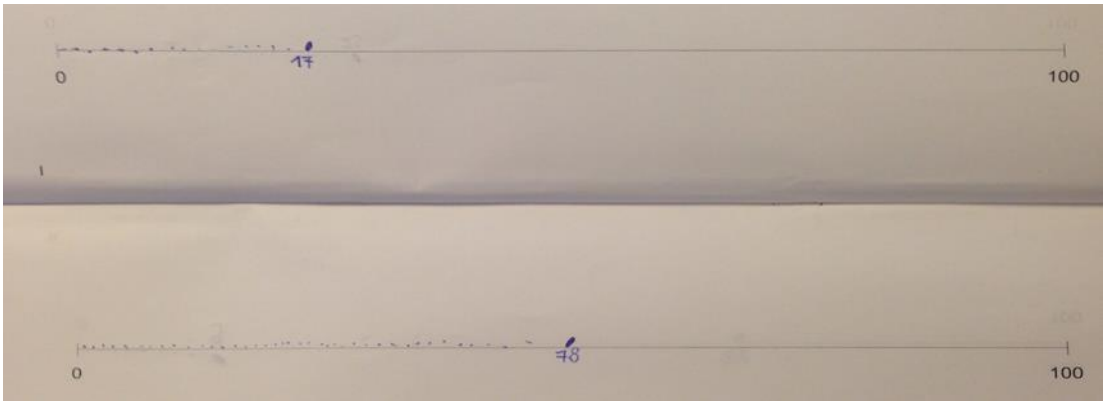

Figure 1. Example of marking strategy in two numbers of NLET

Discussion and conclusions

As we described above, number line estimation is related to both basic and complex arithmetical abilities. Moreover, there is evidence to suggest number estimation is related to mathematical achievement. Despite this, number estimation is rarely taught in Brazil and other countries. In our research, we observed children more accurately estimate numbers on a number line than dots on a grid. Although the two tasks measure the ability to estimate, as indicated by the general correlation between the students' accuracy in the two tasks, they may be linked to different cognitive functions, as suggested by the differential performance when the tasks involve numbers over 25. Although speculative in nature, some ideas help us to understand the results. The NLET requires transposition from numerical knowledge to a position on a line, whereas the NET requires transposition from a perceptual estimation (linked to quantities) to numerical knowledge. Both demand translation between different representations. However, unlike the NET, the NLET allows children to try a discrete quantity representation, sometimes marking the number line with lines or dots from the beginning to the point that could represent the required number. This marking strategy was the most widely used by the children and helped them identify an almost correct answer. Children used different strategies or representations when estimating. The linear distance of numbers along the number line seems to be an important support for the estimations, as Siegler and Booth (2004) have described. Additionally, children used decision-based strategies considering the proximity of the extremities, for example, to represent number 78, they made marks from 100 to 78 in descending order. Alternatively, some decided to begin from 50 and made marks from 50 to 78 in an ascending sequence. Another strategy was to mark the quarters (e.g. 25, 50, 75) as landmarks. All these strategies have been identified as means to improve the way children estimate (Siegler & Opfer, 2003; Siegler & Booth, 2004). The use of these strategies suggests the children did not estimate the numbers by chance, but instead coordinated mathematical knowledge and spatial skills to assess the place to mark. This tactic fits very well with the proportion-judgement strategies model (Barth & Palladino, 2011). The central number (50) was understood by some students as a reference mark used to estimate the other numbers.

The difference found in the tasks with the numbers 49, 78 and 95 can be explained by the fact the three numbers are close to "quarters", which would help children to identify the position of the numbers on the number line. Also, we must remember that, generally, younger children tend to overestimate small numbers and compress large numbers toward the end of the scale, whereas older children, using the same number range, tend to estimate more accurately (Siegler & Booth, 2004). Moreover, in our research, it may be the

case that due to lack of familiarity with the number line task, the older children continued to overestimate small numbers, as described in the logarithmic model. Another possible factor influencing the results was the time. In the NLET, children have time to think about the relation between the number and the place on the line, while in the NET they have only a few seconds to observe the quantities and more time to estimate the quantity represented. The NLET requires the ability to coordinate the knowledge of number systems with a kind of spatial graphic representation on a line. Maybe the effort made by children to coordinate these two cognitive demands helps them to estimate more accurately, despite having little experience with number line tasks. Moreover, the NET may not help students to access their knowledge of the number system, since it provides no clues that would allow students to adopt some proportion-judgement strategies coordinated with their number system knowledge. We suggest the NET requires more "guessing" than the NLET because there is no opportunity to mark "quarters" or anything like that to help them to estimate. One limitation of the research is that only eight numbers were repeated in both tasks. For future research, we suggest amplifying the analysis of the cognitive processes involved in both tasks. We did not control the number system knowledge of the subjects. We do not discuss whether or not the number line task reflects an internal mental number line, since this was not the subject of the paper. However, considering the results of our research, we can say that the NLET is a relevant measure even if it does not reflect an internal mental number line. We support this idea because, although the children were not familiar with the number line estimation task and were more used to NET-like activities, they performed better in the NLET. This surprising result provides the opportunity to introduce the discussion about estimation in the Brazilian curriculum. Furthermore, considering that the ability to estimate is correlated to many aspects of mathematics, for example, number comparison, addition, and subtraction, it is important to take in consideration the assessment of this ability in mathematics education as well as the ways to improve it.

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Do lower-achieving children profit from derived facts-based teaching of basic multiplication? Findings from a design research study

Michael Gaidoschik[1], Kora Maria Deweis[2] and Silvia Guggenbichler[3]

[1]Free University of Bolzano, South Tyrol, Italy; Michael.Gaidoschik@unibz.it

[2]Alpen-Adria University Klagenfurt, Austria; KoraMaria.Deweis@aau.at

[3]Univ. College of Teacher Education Carinthia, Austria; silvia.guggenbichler@ph-kaernten.ac.at

We present findings made during the first cycle of an ongoing design research study on the working out of basic multiplication in 8 Austrian classes. Their teachers had tried to implement an instructional design that put conceptual understanding and derived facts strategies centre stage. Focusing on the degree of fact mastery reached at the end of grade 3, we present a typology of strategy use within a sample of 48 students. We take a closer look at lower-achieving students, in particular those 8 students who had little if any success in mastering basic multiplication. While 6 of them used derived facts strategies quite often, their deficiencies either in adding and subtracting or with regard to the conceptual basis of derived facts strategies seem to have hindered them from mastering more facts. We discuss implications for the planned second cycle of the study.

Keywords: Basic multiplication, derived facts strategies, lower-achieving students, design research.

Introduction

Sherin and Fuson (2005), in an overview of prior work on teaching and learning basic multiplication, refer to 4 different though interlinked threads of research. Like them, in this paper we focus on only one of these, namely the development of computational strategies. We agree with Sherin and Fuson's assertion that strategy development must be examined with close reference to the ways multiplication is taught. Hence, after looking over different ways learners solve multiplication tasks, we summarize contemporary approaches as to how to work out basic multiplication in primary grades. Against this backdrop, we focus on the multiplication learning of lower-achieving students. We contribute to that issue some findings of a design research study on 8 classes whose teachers had tried to base the learning of multiplication on the targeted working out and practicing of derived facts strategies.

Empirical framework and research questions

A taxonomy of strategies used for basic multiplication

Sherin and Fuson propose that the "most important changes" in the development of strategies for basic multiplication are primarily "driven by relatively incremental changes to number-specific computational resources" (Sherin & Fuson, 2005, pp. 353–354). So a child might solve, e.g., 3x4 initially by drawing 3 groups of 4 circles each and "counting all" of them; later by "rhythmic counting" ("one, two, three, *four*, five, six, seven, *eight*, nine, ten, eleven, *twelve*"); then by "repeated addition" of 4+4+4 or maybe by saying the "count-by sequence" ("four, eight, *twelve*"). Finally, the child may "retrieve from memory" that "3x4=12". These strategies as well as "hybrids" such as applying a "derived facts strategy" ("2x3=6, then 4x3 must be twice as much") form part of the taxonomy of strategies the authors devise drawing on prior research. But even if the strategies as listed follow a progression, this is not to be seen as the consequence of an increasingly sophisticated

understanding of multiplication, but mainly driven by a child's growing abilities to, e.g., add, subtract, double and halve, and thereby compute products with increasing ease (Sherin & Fuson, 2005).

This is why, for a long time during the learning process or even permanently, a single individual will presumably use a variety of strategies for solving different basic multiplication tasks, depending on the value of the operands and his or her computational resources. What is more, "because the learning of number-specific resources is very sensitive to instructional emphasis" (Sherin & Fuson, 2005, p. 379), strategy development may differ significantly across classrooms. Such differences are sufficiently documented. For instance, Sherin and Fuson (2005) report that rhythmic counting, while being ascribed an important role by Anghileri (1989), was not observed at all during the interviews they carried out with students whose teachers had not promoted that strategy. Steel and Funnell found no evidence for the use of repeated addition within a sample of 241 children in grades 3 to 7, but a large amount of children using count-by sequences. Their teachers stated they had not encouraged repeated addition, whereas writing down sequences had been used as a method (Steel & Funnell, 2001). This leads to the question of how basic multiplication *should* be taught at the beginning.

The need of further design research on the teaching and learning of basic multiplication

There seems to be an international consensus nowadays that students should *both* acquire a sound conceptual understanding of multiplication *and* eventually solve all basic tasks accurately and effortlessly (cf., e.g., Padberg & Benz, 2011; CCSSI, 2016). It is also widely agreed that fact mastery should not be pursued by rote learning of multiplication tables. As an alternative, children should first learn how to solve harder problems by deriving them from those that are fairly easy to remember, i.e. the problems with 2, 5 and 10 as the multiplier. Only later should they move on to retrieving more and more facts directly from memory (cf. Gasteiger & Paluka-Grahm, 2013; Van de Walle, 2007).

However, when it comes to "details of instruction" that might be relevant for teaching success (Sherin & Fuson, 2005), there remain a lot of unresolved issues. One of them is whether or not to make children work within separate multiplication tables (e.g., the "table of 6" from 1x6 to 10x6). A specific answer quite commonly given in Austrian textbooks is "short tables": Children are supposed to learn the whole body of basic facts by consecutively learning the facts of one table at a time before moving to the next table. Within each table, they are supposed first to automate 1 times, 2 times, 5 times, and 10 times the respective number and to derive from these core tasks the other tasks of that table. In the second step, they should practice all the tasks of the table with the objective of automation.

In contrast, Gaidoschik (2014) advocates what he calls a "consistent" approach to derived facts strategies. According to this, strategies should be worked out in a targeted manner without any consideration of separate tables. For example, as soon as children have learned that they can easily derive a 9-times fact from a 10-times fact, they should be encouraged to do so with *any* 9-times fact. The same applies to any other strategy like halving 10-times to derive 5-times facts or doubling 2-times to solve 4-times facts. If the commutative property is emphasized from the very beginning, there is little need for strategies that demand computations that are more difficult. For instance, 7x9 can be done more easily by thinking of 9x7, which is 10x7–7, than by adding 2x9+5x9. Therefore, to abandon activities that are restricted to single tables is supposed to reduce the overall workload. Secondly, it should help get a better understanding of any single strategy since it is applied to a wide range of numbers as soon as it has been established, and not just within a single table. Last but not

least, this approach emphasizes the wide reaching power of these strategies, which is supposed to contribute to the children's willingness to acquire them (Gaidoschik, 2014).

Evidence as to whether and how such differing approaches to the teaching of basic multiplication indeed have an impact on children's learning is rather scarce and fragmentary. Cook and Dossey (1982), comparing teaching the tables with a derived facts strategy approach, find empirical support for the latter, but remain vague about the specifics of either approach. Woodward (2006) gives more details about an integrated approach combining explicit teaching of derived facts strategies with timed practice drill which yielded significantly better results than drill only. However, he reports not on regular classroom activities, but on a remedial programme applied when multiplication had already been worked out. So do most other studies in this field (e.g., Kroesbergen, Van Luit, & Maas, 2004).

One of the few studies known to the authors that deal with the initial workout of basic multiplication in a regular classroom and try to deliver a "rich description of the way the design works" (Swan, 2014, p. 151) is Selter's (1994) report on a teaching experiment with one grade-2 class in Germany. The teaching in this class favoured derived facts strategies throughout the second half of the school year, while it deliberately downsized drill. The study indicates that this concept was quite successful with regard to its conceptual targets. However, the account of whether it worked out equally well regarding the development of fact mastery is less satisfying. The study is rather sketchy in that respect. Selter (1994, p. 106) rates it as the "preliminary ending to a research project" to be followed by more detailed inquiries into single issues, such as "learning processes of underachieving pupils" (p. 281).

Questions addressed in the study presented in this paper

Teaching basic multiplication with a focus on derived facts strategies is still not at all common in Austria, where there is a long tradition of basically drilling tables with little, if any, consideration of derived facts strategies. Teachers particularly tend to be sceptical as to whether lower-achieving students would a) understand derived-facts strategies at all and b) reach fact mastery without rote-learning of the tables (Gaidoschik, 2014).

Against that backdrop and in consideration of the empirical framework outlined above, we started a design research project on the teaching and learning of multiplication in grades 2 to 3. In this paper, we present some findings collected during the first cycle of the project that lasted from September 2014 to June 2016. Out of the numerous issues we address in that project, in this paper we have to restrict ourselves to the following:

1) If multiplication is taught with a clear focus on derived facts strategies in the "consistent" way suggested by Gaidoschik (2014) and shortly outlined above, what types of strategy performance can be identified at the end of grade 3 with respect to the target of fact mastery?
2) To what extent, when taught like this, do children who have been identified by their teachers as mathematically lower achieving actually use derived facts strategies and reach fact mastery?

Method

We report on 48 students from 8 classes in Carinthia, Austria. Their teachers had volunteered to participate in a design research study aiming at evaluating and refining the concept of teaching multiplication in grades 2 to 3 as formulated in Gaidoschik (2014), with the following main ideas:

1) In the first half of the second school year, arithmetic lessons should have a clear focus on what children need to understand and be able to compute fluently in order to get comfortable with the strategies that are useful to derive multiplication facts. That is, they should be able to double and halve two-digit numbers effortlessly and to add and subtract fluently up to 100.
2) Subsequently, instruction should concentrate on the conceptual understanding of multiplication and its properties, particularly commutativity and distributivity. At the end of this stage, children should be able to translate smoothly terms such as 3x4 into actions, visual representations (identical groups as well as arrays), word problems, and vice versa.
3) In the next step, teachers should secure that all children know how to double and decuple any number at least up to 10 with ease, then learn how to derive the 5-times facts from the 10-times facts by halving and do so more and more effortlessly.
4) On that basis, a *guided discovery-learning* approach should be complemented with *direct instruction when needed* by single children to convey derived fact strategies as a convenient way to solve multiplication tasks. To this end, single lessons should be devoted to groups of facts as defined through the multiplier, for instance 9-times facts, 6-times facts, and so on. Children should be encouraged to find an easy way for themselves to solve tasks of such a group by deriving the solution from facts they already know, using representations such as arrays of dots or equal groups of interlocking cubes. Strategies found by the children should be discussed and compared in the classroom. Different strategies for the same task are welcome as long as they are mathematically correct. However, children who constantly fall into cumbersome ways to derive a task (such as, e.g., computing 6x9 as 10x9–9–9–9–9 instead of 10x6–6 or 5x9+9) or even resort to repeated addition or counting strategies should receive direct instruction to develop understanding for one derived facts strategy after the other, including the knowledge for which tasks that strategy fits well.
5) Subsequent practice should comprise substantial tasks such as explorations of mathematical patterns as well as timely restricted "strategy drill" (Van de Walle, 2007), e.g. using flash cards with the objective of performing a certain strategy with growing ease and speed.

To convey and discuss this concept in detail, the researchers met the participating teachers for 8 working sessions (3 hours each) once a month during the second and 4 follow-up sessions once every 2 months during the third school year of their classes. During each session, the researchers would give theoretical inputs and make concrete proposals for daily classroom activities. Each teacher was visited in the classroom 3 times by one of the researchers to receive feedback on his or her teaching practices. The teachers were interviewed individually 4 times during the cycle to cover as comprehensively as possible if and how closely they had followed the researchers' recommendations.

To assess the children's development, we selected 6 children out of each class to be interviewed a total of 7 times, from October 2014, at the beginning of their second school year, till the end of April 2016, before the classes started to move on to multi-digit multiplication. Always 2 of the 6 children had been rated as being above average, average, and below average with respect to their arithmetical performance as perceived by their teacher at the beginning of grade 2. The students were interviewed by the researchers during school time in some quiet extra rooms of the school. The first interviews were centred on addition and subtraction up to 20, which had been the main contents of arithmetic instruction till then. The semi-structured qualitative interviews to follow each reflected what had been the major classroom topics since the previous interview, from the base-10-system in January 2015 to

a focus on multiplication in the later interviews. In accordance with the instructional design, the interviews were restricted to the conceptual understanding of multiplication in March 2015, but starting with May 2015 encompassed both understanding and computation of multiplication tasks.

In the computation part of the interviews the children were presented always with the same 15 tasks, each of them written on a DIN A7 card, 7 being core tasks (10x7, 2x8, 4x10, 9x2, 5x7, 8x5, 5x5), and 8 harder tasks (in the order of the interview: 9x4, 7x7, 6x4, 6x9, 7x8, 6x7, 8x8, 4x7). The children were requested to solve each task mentally the way they usually would, and to state the result verbally as soon as they knew it. Immediately thereafter they were invited to explain or show how they had arrived at the solution. Strategies were evaluated on the basis of video recordings. "Fact mastery" was assigned to any solution that was accurately produced within 3 seconds (cf. Van de Walle, 2007).

Apart from computing, we invited the children inter alia to explain verbally to a fictitious first-grader the meaning of a task such as 3x5 and demonstrate that meaning with different materials (wooden cubes, arrays of dots). Moreover, we asked them to clarify whether and how an easier multiplication task could be of help to solve the not so easy tasks 9x7 and 4x8, respectively.

Findings

Types of strategy use in solving basic multiplication tasks

Based on the multiplication strategies exhibited by the children in April 2016, we performed an empirically grounded construction of types and distinguished 3 main-types of strategy performance at the end of grade 3:

A) "Masters": These students solved all tasks accurately either by retrieval or effortless derivation within about 3 seconds or, in most cases, instantly. Only in single cases, if at all, they would give a wrong answer or take slightly longer to produce a correct one. 19 out of the 48 children go smoothly with this type, and 5 children fall somewhere in between Type A and B (see below).

B) "Experienced users of derived facts strategies with limited fact mastery": These children, while exhibiting mastery of the core tasks, relied on derived facts for at least 3 of the 8 harder tasks. As a rule, these tasks were solved with rather little effort in 6 seconds or less. However, up to 2 of the 8 harder tasks still posed quite a challenge to these children, either taking more than 9 seconds or being answered incorrectly. 15 to 20 (see above) of the 48 children rank among this type.

C) "Users of derived facts with limited mastery and some trouble in deriving": These children typically solved core tasks by retrieval and harder ones by using derived facts strategies. At least 3 tasks caused them perceivable trouble, with solution times in excess of 9 seconds and/or resulting in incorrect solutions. 6 of the 48 children quite clearly fulfil this description.

These 3 types cover 45 of the sample's 48 children fairly adequately. Note that fact retrieval and derived facts strategies were the only strategies to be found within these types at the end of grade 3, with no single child relying on strategies like counting all, counting by or repeated addition.

3 children do not fit into this typology. One of them, whom we refer to by the fake name Leo, was the only child in the sample that relied on count-by sequences. He did so when solving 9x4, 6x4, and 4x7, all of them within 3 or 4 seconds. All the other facts he solved by retrieval, except 6x7, which

he derived from 6x6. Leo's teacher reported that Leo had consistently been getting "a great deal of support" by his mother. The teacher had "not been able to convince her of not drilling the tables".

Whereas Leo had reached a high degree of fact mastery, this clearly is not true for 2 other children. One of them, we call her Mia, had only mastered 4x10, 10x7, 2x8, 5x7, and 5x5. Whereas she retrieved these facts accurately, all the other tasks she either refused to try at all, or admittedly guessed upon. Only 4x7 did she solve correctly by a rhythmic count-all supported by her fingers. In no case did she use a derived facts strategy. The other child, whom we name Lara, solved all the core tasks as well as 7x7 and 8x8 by retrieval. On 7x8, she used a hybrid strategy by counting down 8 and another 8 from 9x8=72, which she had retrieved from memory. She tried to derive 6x4 as well, this time incorrectly by computing 5x4+5. 9x4 she rated as her "battleground task"; she eventually solved it by drawing 4 rows of 9 circles each and counting them all, which took her about 90 seconds.

Out of the 16 children who had initially been selected as below average, 3 were assigned to Type A at the end of their third year, 6 more to Type B, and one child between A and B. Out of the other 6 students who started as below average, 4 finally belonged to Type C, whereas the remaining 2 of this group have been introduced as Mia and Lara. There were 2 other children who had been rated as average by their teachers at the beginning of the second year but were finally assigned to Type C.

Interrelations with the performances on other tasks

As set forth, 8 children within our sample demonstrated only very moderate, if any, success in having mastered basic fact multiplication by the end of their third school year. To help better understand this, we refer to some of the findings we made aside from the computation part of the interviews. First, the tasks that were conceived to test the conceptual understanding of multiplication unsurprisingly revealed considerable differences, in particular with regard to the verbal competences of the children. However, all 48 children, including the 8 lowest achieving, could without any exception give a comprehensible and adequate verbal explication of what meaning could be ascribed to a term like 3x5, and support this by laying equal groups or arrays of wooden cubes.

What is more, all these 8 children seemed to have at least some clue of how to derive facts from other facts. This even applies for Mia, albeit in a very restricted way. She was the only child of this subgroup who had not used one single derived facts strategy during the computation part of the interview. Subsequently, when questioned whether there was an easier task that could help her solve 9x7, she spontaneously answered: "No idea". However, when directly asked whether 10x7 could be helpful, she said without any hesitation: "Yes, you just have to take away 7." Significantly, though, when asked whether knowing 5x8 could help a child solve 4x8, she stated: "Yes, you have to take away 4." The same mistake was made by Lara. Lara also erred in computing 6x4 as 5x4+5 in the computation part (see above), but she, too, stated that 9x7 could be derived as "10 times 7, then take away 7".

Only 2 other of the 8 lowest achieving children made such mistakes with regard to the *logic* of a derived facts strategy (as distinguished from *calculation errors* in executing a mathematically correct strategy; such errors were made by other children, too). For instance, one child said that to get 6x9 you have to compute 60–9, and the other that 10x8–2x7 equals 7x8.

The other 4 children of Type C not only used a variety of derived facts strategies in a mathematically correct way during the computation part, but also gave comprehensible explanations of how to derive 9x7 and 4x8. It seems to be noteworthy that these 4 children as well as Mia and Lara and 2 children

of Type B were the only ones in the whole sample who had considerable trouble with the addition and even more with the subtraction tasks they had to solve in a separate section of the interviews.

Discussion and final remarks

We are well aware of the limitations of our paper. As stated by Swan (2014, p. 151), "writing up design research is problematic". Due to space restrictions, the design is presented rather sketchily, as is the account of its implementation and the ways different children performed. Having conceded this, we still hope that some of our findings are of use and interest for other researchers in this field.

First, our results seem to corroborate the view of Sherin and Fuson (2005) that the development of strategies for solving multiplication tasks is to a high degree dependent on instruction. We recorded the tedious use of count-by sequences known to be rather typical of low-achievers in higher grades for only one out of 48 students at the end of grade 3. In that case, we have clear indications of parental drill. We may add that this strategy had been equally rare in the preceding interviews. On the other hand, all but one student used derived facts strategies autonomously for those facts that they had not yet automatized. Both occurrences correspond with the applied instructional approach that, as outlined, deliberately neglects working within single multiplication tables, thus aiming to prevent children from using count-by sequences as a solution strategy. As far as can be judged from teacher interviews, classroom visits and the examination of working sheets used in the classrooms, all teachers basically adhered to that concept.

This leads to a second finding: Teaching multiplication with a clear focus on derived facts strategies yielded what might be seen as quite satisfying results even for lower-achieving students. Of course, the study was not conceived to prove the superiority of one design over another, but to examine qualitatively whether and how certain measures may contribute to children's learning. With regard to lower-achieving students, it can be stated that the chosen combination of discovery-learning and direct instruction of single strategies followed by strategy drill seems to have helped almost all of them *use* these strategies *correctly* (as can be judged from the computation tasks) on the basis of a sufficient conceptual *understanding* (as can be judged from the additional tasks described above).

Thirdly, 8 students in the sample show severe problems with basic multiplication at the end of grade 3. 6 of them, constituting the Type C, still use derivations and do so quite frequently. But they do it in a way which indicates what may contribute to their struggling with multiplication: 4 of them, while apparently knowing how to use known facts as a basis for deriving unknown ones, are not able to add and subtract efficiently. That is why they often need considerable time to solve a task and sometimes miscalculate. We assume that out of the same reason their repeated use of derived facts strategies on harder tasks has not resulted in the mental linking of these tasks and the respective solutions (cf. Van de Walle, 2007) and therefore not contributed to the automating of basic facts to the same extent as it has done for their more successful peers. In this regard, their learning difficulties differ *qualitatively* from those of the other 2 children within Type C who are indeed quite proficient at adding and subtracting, but then again seem to have a limited conceptual basis of their strategy use.

All in all, these 6 children and so much the more Lara and Mia, who were least successful within this sample, have severe problems not only in multiplication but also in areas of elementary arithmetic that form a prerequisite for learning multiplication on a conceptual basis (Gaidoschik, 2014). That is why in a second cycle of this design research project we will focus on how to better foster children

who lag behind in adding and subtracting at the start of grade 2 before and while working on multiplication. From where we stand in our analysis, we assume such measures will have to include individual support for some children additional to what a single teacher can manage in the classroom.

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Fostering rational numbers understanding through percentage

Helena Gil Guerreiro[1] and Lurdes Serrazina[2]

[1]Agrupamento de Escolas Braamcamp Freire, UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; hg@campus.ul.pt

[2]Escola Superior de Educação de Lisboa, UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; lurdess@eselx.ipl.pt

This paper discusses an approach that fosters students' conceptual understandings of rational numbers with an initial focus on percentage, in elementary school years. This approach enables students to work with multiple representations associated with percentage which are taken as models of contextualized situation and reconstructed as models for reasoning through an emergent modeling process. A classroom teaching experiment was developed following the methodological procedures of a Design Research. Data were collected through participant observation, supported in a logbook, audio- and video-recorded lessons and students' productions in the classroom. The analysis of data reported in this paper seems to highlight that through this approach, percentage - if privileged in the introductory steps of its learning - strengthens the interpretation of multiplicative relations and fosters understanding of rational numbers through an emergent modeling process.

Keywords: Elementary school mathematics, rational numbers, models, percentage.

Introduction

The development of rational number understanding is considered a very important, but also a complex mathematical topic (Behr, Lesh, Post & Silver, 1983). It involves conceptual understanding that is the entwined comprehension of concepts, operations, and relations (NCTM, 2014). Students are called to construct new knowledge supported by multiplicative relations, through active sense making and extending previous knowledge and experiences with whole numbers. This conceptual understanding is closely related to number sense, which is an essential competence that students should gradually develop from an early age. Percentage, with its changing nature and multiple interpretations, can be more than a representation of rational numbers (Parker & Leinhardt, 1995). Percentage is itself a useful topic to learn rational numbers with understanding (Moss & Case, 1999).

The purpose of this paper is to provide useful insights into how percentage can be powerful, when privileged in elementary years, to foster rational numbers' learning process with understanding. This should be addressed with an active involvement of students through an emergent modeling process supported by multiple representations.

Percentage for a meaningful learning of rational numbers

Percentage is part of student's everyday contexts. Although, as a mathematical content, compared to other areas of arithmetic, there are few recent researches that discuss the issues surrounding its learning in the classroom (Pöhler, Prediger & Weinert, 2015). Since early years, students become familiarized with percentage in real life situations, for instance, on food labels, clothes tags, in discounts. Its practical use develops children's intuitive sense in dealing with it before being at school (Moss & Case, 1999). This very common use of percentage points to its introduction in elementary education, within rational numbers domain (Hunter & Anthony, 2003; Moss & Case, 1999). Percentage is "a language of privileged

proportion" (Parker & Leinhardt, 1995, pp. 472) that is based on multiplicative comparison to 100. Exploring percentage can be an opportunity to begin thinking relatively and to deal with multiplicative situations, enhanced on students' early intuitive understandings of proportional relations (Lamon, 2007). Being a language, percentage has different interpretations as it assumes properties of number, part-whole, ratio, function or statistic (Parker & Leinhardt, 1995). Grounded in real-world experience percentage can be a way to start the work of developing a solid understanding of those meanings as rational numbers subconstructs, which highlight essential characteristics of rational numbers (Lamon, 2007).

Learning percentage should begin with understanding its relational language using elementary strategies, including benchmarks, proportional reasoning, and additive building-up strategies, rather than learning formal calculation procedures (Moss & Case, 1999; Parker & Leinhardt, 1995). This process might start in elementary grades, but the concept of percentage at its very rich sense will be reached later. Thus, understanding percentage requires the development of appropriate models to grasp its various meanings and its relational features in order to support students' attempts to make sense of the numbers and the relationships that connect them (Dole, Shelley, Cooper, Baturo, Conoplia, 1997; Parker & Leinhardt, 1995). In problem solving, models can arise from the use of multiple representations – enactive, iconic, symbolic (Bruner, 1962) and oral and written language (Ponte & Serrazina, 2000), when associated with percentage can provide support to develop its conceptual understanding. Models should display the relationship between quantities and allow to describe comparisons in multiple ways, encouraging proportional reasoning (Parker & Leinhardt, 1995). Gravemeijer (2002) refers to emergent modeling process to explain both the "process by which models emerge" (p.3), as representations are used and progressively become models and the "process by which these models support the emergence of more formal mathematical knowledge" (p.3). Gradually models of contextualized situation are reconstructed and evolve to models for reasoning through this emergent modeling process.

The above ideas are the key components of the conceptual framework that supports this research. They seek to associate conceptual understanding, number sense and emergent modeling process to support percentage as an entry point for developing students' understanding of the rational numbers. The interrelationship between these components sets a developing framework for analysis, which will provide the lens to describe and analyze how rational numbers' conceptual understanding takes place in a specific classroom learning ecology (Gravemeijer & Cobb, 2006).

Methodology

Data reported in this paper was collected as part of a classroom teaching experiment in a design research approach (Gravemeijer & Cobb, 2006), within a broader study that aims to deepen how students construct their understandings of rational numbers through a learning trajectory with an initial focus on percentage. The cyclic process of the design research involved a first phase where a conjectured local instruction theory was defined supported by design principles, used to guide the design and development of the classroom teaching experiment. In the second phase, the teaching experiment took place and, through micro-cycles of design and analysis, the process of the students' participation and learning was analyzed. The last phase involved a retrospective analysis, which is still running. The conjectured local instruction theory, which is about a possible learning process together with theories about possible means of

supporting that learning process, is refined and improved all along, supporting a revised local instruction theory (Gravemeijer & Cobb, 2006).

The classroom teaching experiment was designed considering a mathematical content dimension and a pedagogical one. The first one is based on an hypothetical learning trajectory, inspired by Moss and Case (1999) experimental curriculum for teaching rational numbers. The first stage of this trajectory begins with the understanding of percentage in a linear-measurement context. Then two-place decimals are introduced. Finally, fractions are the focus concerning the use of different interchangeable representations. The pedagogical dimension attempts to account for the means of supporting the co-participated learning process as it occurs in the social context of the classroom. The classroom experiment involved a total of 20 lessons spread across two three-month periods (Grade 3 and Grade 4) in the same classroom, where the first author was also the teacher. It took place in a public elementary school in Lisboa with students aged between 8 and 10 years. In this paper, we analyze some episodes that took place in the first period when the students were in third grade.

The dual role as teacher and researcher raised significant ethical challenges. To avoid potential conflicts and assure students' and families' protection, an informed and voluntary consent was taken. Anonymity and confidentiality were guaranteed to be maintained. To establish credibility and allow convergence, multiple sources of evidence were chosen (Confrey & Lachance, 2000). For data collection, we used the transcript of video and audio recorded moments of all classroom sessions, students' written work, and teacher´s research journal.

A preliminary analysis was developed during the classroom teaching experiment, which supported the process of redesigning and testing instructional activities and other aspects of the design. (Gravemeijer & Cobb, 2006). This analysis involved an analytic induction strategy where we identified significant episodes from students' activity while exploring tasks. Then, we scanned those episodes for evidence of students' conceptual understandings related to rational numbers and for relationships among them. Thereafter, a retrospective analysis was made through content analysis. In this analysis, all data was revisited and divided into content categories generated from the interrelationship between the conceptual framework components through a typological analysis (Goetz & LeCompte, 1984). This analysis process creates a cross-coding system that evidence relationships among the various categories, emerged from data and anchored in the conceptual framework, which is still in progress. The analysis discussed in the next section of this paper focuses on students' activity with meaningful representations during four lessons of the classroom teaching experiment. It was carried out using three interrelated categories as indicators of students' conceptual understandings of rational numbers through percentage learning. Each of these categories involves working subcategories, handling percentage to: (1) support reasoning strategies (decomposition/composition; half/double; 10%; multiples of; unitizing/reuniting); (2) foster numerical relations (benchmarks; magnitude of numbers; orderliness and comparison; equivalence) (3) encourage a modeling process (mobilize familiar representations; interpret subconstruct situations; emerge of symbolic representations).

Learning rational numbers by focusing on understanding percentage

A mobile phone battery was one of the first iconic representations related to percentage chosen to be used in a problem-solving context. In this task, students were asked to estimate the percentage represented in batteries in Figure 1.

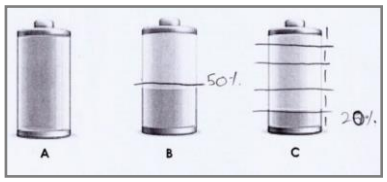

Figure 1: Ana's group resolution using a battery representation

Considering battery C, some students stated that the shaded part would represent 25% of charge, and others claimed that it would be 20%. During the discussion, students' arguments were shared in order to justify their reasoning strategy.

Simão: That is not 25 percent.

Teacher: […] So, what do you think it is?

Students: 20 percent.

Teacher: Why? Marco.

Marco: Because it fits 5 times.

Students who claimed 20% reasoned that if the shaded part was iterated it would fit the unit five times, so, it should be 20% and not 25%. This idea expresses a reasoning strategy drawn on division and laid out on numerical relations. The measure subconstruct allowed students to see the unit represented by the full battery as a distance, and the percentage as a relative quantity of that distance. Familiar battery representations were used to encourage a modeling process. They are used as models to think about the task as a measurement situation, allowing percentage to be conceived as an iteration of a unit part, rather than representing a part out of a whole.

Status bar (Figure 2) was another iconic representation regarding percentage that was privileged. Students interpreted status bar in an easy way by analyzing its fullness, as they are used to do it when downloading a file.

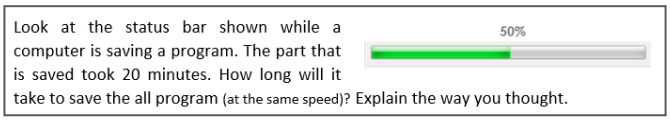

Figure 2: A task that privileged status bar representation

They perceived the comparison of quantities using proportional relationships. Percentage allowed students to see the multiplicative relation between the minutes taken to save the complete program and the amount of program saved shown by the status bar. "The whole program took forty minutes to save because if a half is twenty, the double is forty" as Mafalda's group explained. This use of splitting/doubling procedure seemed to make sense to the others as it expressed the multiplicative relation between half and double, keeping the ratio constant.

The students were able to see that saving half of the program would took as long as the remaining half, although not all students have yet clearly perceived it as a ratio. Status bar seems to encourage the modeling process during the teaching experiment while supporting intuitive ratio understanding, involving quantities of a standard unit.

Thereafter, the proportional judgments established with the status bar were mobilized to work percentage using another representation – the ratio table (Figure 3).

Percentagem	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
Tempo (min)	4	8	12	16	20	24	28	32	36	40

Figure 3: Clara's group resolution using a ratio table

This representation was used to make comparisons between entities in multiplicative terms, applying to multiples of 10%, as reasoning strategies. Percentage benchmarks allowed to foster numerical relations highlighting the relationship between time and the amount of program saved. Students could realize multiple numerical strategies through collaborative engagement.

Heitor: The process was always four in four minutes.

Simão: 10% are 4, is always times 4.

Mafalda: The all process took ten times four minutes.

Teacher: The whole process of saving the program took ten times four minutes…

Hélio: We could also look at 36 minutes and see which would be the number that could give 40.

The ratio table seemed to be powerful in the modeling process as it supported different reasoning strategies suggested by percentage, such as halving, doubling, multiplying by 4, showing why the relationship between quantities is multiplicative.

Some less stereotyped representations came up in classroom from other contexts. For example, during a specific group project work about dogs, the representation of a dog food bag was used to encourage the modeling process as it becomes a model of acting in a real meaningful situation. In this task, the representation of the dog food bag was tailored to consider its fullness (100%) that is 20kg as the top of the bag. To this iconic representation, two vertical number lines were added to foster numerical relations allowing to relate weight with the percentage of the amount of food in the bag. In this way, the dual set of numbers involved in percentage comparisons become explicit and can be used for calculations.



Figure 4: Carolina's group resolution using the bag representation

All groups found out immediately that 50% of the weight of the bag was 10 kilograms, as shown in the example on figure 4. When students had to compute other amounts of food, they consider percentage as a relative quantity according to the attained unit and applied to multiplicative reasoning strategies. Some divided using halving strategy, and reconstructed the unit, considering the unit the bag weight in each moment (Figure 5).

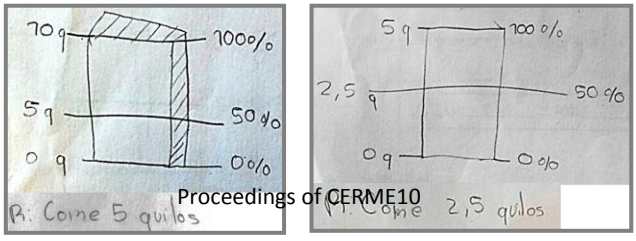

Figure 5: Mafalda's group resolution using the bag representation

Students understood that the relationship between the quantities in both number lines kept constant and perceived that they vary together, even though they have different magnitudes, interpreting ratio subconstruct. In this modeling process, the food bag representation became a model that supported reasoning strategies and proportional relationships.

Later on, this modeling process was extended to the double number line. Double number line fostered the emergence of numerical relations, as it allowed ordering and establishing equivalences, invoking benchmark percentage values in a measure context (Figure 6).

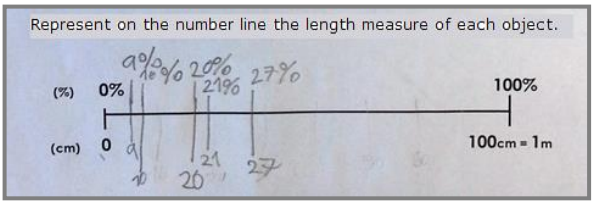

Figure 6: Dina's group resolution using the double number line

The double number line became a model of a measurement situation in which students supported their reasoning strategies (Figure 7). They established comparative relations between equivalent representations, as percentage and measure of an amount (e.g. 9% and 9/100). For this, a key aspect is the unit identification. Considering one as the unit, the decimal numbers representation emerged when a number was located. Together with percentage, this allowed students to invoke whole number's knowledge, establishing numerical relations between each decimal number and its position.



Figure 7: Whole-class resolution using the double number line

Two-place decimals were used as a symbolic representation by students to express an equivalent percentage of "fullness" of the unit, when the unit is 1, applying to establish numerical relations.

Simão – It's not enough to make up one meter…

Clara – …You have zero comma ninety-one meters.

Teacher – Then, this is Clara's suggestion [writes 0,91 at the number line]

Clara – Because it's not enough to make up one meter, so it´s zero meters and then we write ninety-one centimeters.

In this situation, the decimal representation was convened to identify a quantity less than one in a measurement context, fostering a modeling process as the number line became a model for a more formal reasoning.

Final remarks

As data analysis highlight, percentage enables an approach to multiplicative situations, describing comparisons in multiple ways and exploring relationships, enhancing an introductory step to rational numbers conceptual understanding (Lamon, 2007; NCTM, 2014; Parker & Leinhardt, 1995). The context within percentage appears in these tasks and suggests it can be linked with ratio and measure subconstructs of rational numbers, which are important although less common in these years of elementary school (Parker & Leinhardt, 1995). The meaning of percentage, supported by the use of number line in the dog food bag representation allowed to show the relation between the four numbers that make up the percent proportion, in an intuitive way. The focus on measurement revealed a growing flexibility of the reasoning strategies and provided opportunities for students to build a meaningful connection between percentage and decimal numbers notation. Ratio situations allow to recognize the importance of 100 as a privileged base and to develop sensitivity for the comparative uses of percentage, fostering numerical relations. Representations associated with percentage, drawn from students' real-life experience, like the mobile phone battery or the status bar, revealed to be essential. Students used them to make sense of problem situations and to solve them. Gradually, those representations became models as they encouraged students' reasoning strategies and promote rational numbers' conceptual understanding. In this emergent modeling process, as called by Gravemeijer (2002), percentage had a key role. It contributed to disclosure the multiplicative nature of rational numbers, connecting prior whole numbers' knowledge with intuitive understandings regarding relative proportion (Moss & Case, 1999).

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The fostering of flexible mental calculation in an inclusive mathematics classroom during *Mutual Learning*

Laura Korten

TU Dortmund University, Germany; laura.korten@math.tu-dortmund.de

The topic of teaching mathematics in an inclusive classroom provides – by the increased heterogeneity range – a big challenge between individualizing and mutual learning. (How) Can we make sure that all children work and progress on their individual level but at the same time learn with and from each other? Based on this question the aims of this project are the development of, and the research on a teaching-learning arrangement for the inclusive mathematics classroom to foster flexible mental calculation. The approach is a Design Research approach to face research interests on the level of design (consideration for use) and on the level of research (quest for fundamental understanding). This contribution focuses on the research level: first insights about mutual learning processes of elementary students with and without cognitive learning disabilities concerning flexible mental calculation will be presented.

Keywords: Inclusive education, heterogeneity, cooperative learning, flexible mental calculation.

Theoretical framework

Developing flexible mental calculation has been considered as a 'central goal' for more than a decade, not only for middle and high achievers, but also for less advanced children. However, empirical insights about teaching and learning processes of flexible mental calculation in inclusive classrooms do not exist. Although, *inclusive education* is a current international discussion, based on the UN-Convention on the 'Rights of Persons with Disabilities'. *Supporting everyone's learning process* and at the same time *encourage cooperative learning with and from each other* are the two central matters of inclusive education, which imply more than sharing a room. But, in school practice, teachers emphasize the difficulty of learning with and from each other in arithmetic (Korff, 2015). Building on the two matters of inclusive education, this study focuses on the *goal-differentiated fostering of flexible mental calculation* in an inclusive classroom during *Mutual Learning* takes place.

The *goal-differentiated* fostering of flexible mental calculation in an inclusive classroom

Developing flexible mental calculation is not only a central goal but also a 'critical point' in everyone's learning process (Heinze, Star, & Verschaffel, 2009), especially for students with cognitive learning disabilities. In this process, error-prone counting strategies should be replaced with more beneficial calculating strategies. Current literature offers different definitions, which commonly include the two aspects of 'flexibility' and 'adaptivity'. In most of the cases flexibility is understood as the ability to switch between different solution tools (Rathgeb-Schnierer & Green, 2013), while adaptivity is more emphasizing on the selection of the most appropriate strategy. In this project 'adaptivity' is related to the recognision of problem characteristics, number patterns and numerical relations. Consequently, *flexible mental calculating* is a situation-dependent and individual response to specific number and task characteristics and the corresponding construction of a solution process using strategic tools (ibid.).

The fostering of flexible mental calculation competences is influenced by the outlined general assumptions. If flexible calculation is related to number and task characteristics and relations, activities have to be chosen, which support children to focus on these. Thus, the crucial aim is to develop the competence to recognize problem characteristics, number patterns and numerical relations, and to use them for solving problems. Rechtsteiner-Merz and Rathgeb-Schnierer (2016) call this "Zahlenblick" and found out that it is a good vehicle for developing flexible calculation.

Today it is proven that also less advanced students can develop flexible mental calculation (Verschaffel, Luwel, Torbeyns, & van Dooren, 2009) and that the focus on developing "Zahlenblick" especially supports less advanced students (Rechtsteiner-Merz & Rathgeb-Schnierer, 2016). Schröder (2007) points out their problems in the usage of flexible strategies: Even if they know strategies, they very often cannot adapt and use those. Reflecting on characteristics and relations is especially essential for children with cognitive learning disabilities (ibid.) and at the same time supportive and preventive for everyone's learning process, because generally all children show little task-adequate action (Selter, 2000). Further, the content provides opportunities for high-performing students to establish mathematical structures and to generalize. Consequently, flexible mental calculating meets the requirements for a *common content* for an inclusive classroom to encounter the diversity of abilities and skills and to make goal-differentiated learning possible.

Mutual Learning in an inclusive mathematics classroom

The expression of *Mutual Learning* as it is used here combines the two central matters of inclusive education, which were mentioned above: individualizing as well as interacting and cooperating. Mutual Learning means to consciously induce learning situations *as often as possible* in which all children work and learn at a common content, in cooperation with each other, on their individual level, and by use of their current individual skills (Feuser, 1997). This definition is based on a wide sense of inclusion, acknowledging the diversity of all children and counteracting all forms of discrimination and special learning needs. Nonetheless this research project focuses on learning and interaction processes of children with and without *cognitive learning disabilities*. (This distinction is not used to label deficits, but rather with regard to make research and communication possible.)

In consideration of the two central matters of individualized learning and at the same time learning with and from each other, first supportive principles for successful Mutual Learning can be derived: '*content variability'*, *'goal-differentiated learning process -'*, and '*interaction orientation'*. As already outlined, the content of flexible mental calculation meets the requirements of the first two. Having regard to the principle of '*interaction orientation'*, the construction of mathematical knowledge is understood as an active, social and explorative process; also for children with cognitive learning disabilities as today several studies show. Gaidoschik (2009) points out, children with problems need more time and more support to learn arithmetic but they don't need something different. The exploration, understanding and use of arithmetical patterns and particularly the communication about number and task characteristics and strategic tools is especially important for children with cognitive learning disabilities (ibid.; Schröder, 2007). In social-communicative processes, individual mathematical learning develops through 'irritations', 'contradictions' and 're-interpretations' on the basis of individual interpretation processes (e.g. Steinbring, 2005). Therefore, this very *individual processes of learning flexible mental calculation in the context of interaction processes,* needs to be fostered on different cognitive levels for successful Mutual Learning.

Research question and methodological design

For this purpose, it seems important to research when and how successful Mutual Learning occurs and which support means can be reconstructed. Therefore, it is necessary to reconstruct the *individual learning processes* concerning flexible mental calculation as well as the *interaction processes* to be able to evaluate weather Mutual Learning with its two central matters took place.

To meet the aspiration of designing a teaching-learning-arrangement to foster flexible mental calculation on the one side, and to investigate learning and interaction processes on the other, a *Design Research* approach is used (Prediger & Zwetzschler, 2013). This requires research questions on the level of design and research. Nevertheless, this contribution focuses only on the level of research and the investigation of individual learning processes. The following question will be addressed: *How do individual learning processes of elementary students with and without cognitive learning disabilities concerning flexible mental calculation develop during the cooperative-interactive phase of Mutual Learning?*

Iterative design research cycles as an approach for answering the question

To investigate this, a teaching-learning arrangement was designed, tested and refined by conducting *design experiments* in three iterative cycles. Within each cycle the *individual learning processes* as well as the *interaction processes* were reconstructed to be able to evaluate weather Mutual Learning took place. To, in a next step, reconstruct support means for successful Mutual Learning.

Theoretical sample: The design experiments were conducted in classes two and three (7-9 years old), at three different German primary schools. Laboratory situations with couples of learners allowed to learn more about their thinking, their individual learning and interaction processes. Each design experiment consisted of three phases (Figure 1) and took place in a pair setting with one child tested and "termed" with and one child without learning disabilities. The participants were selected with the help of the class teacher and the special needs teacher in order to find pairs of children who like each other to have a positive basis of communication. In the design experiments, the learners processed the learning activities largely by themselves. The researcher, on the one hand, acts as a teacher, in order to give the learner a stimulus or help, and on the other hand as a researcher, who wishes to learn more about the thinking processes and the ways of proceeding by means of observation and targeted inquiry.

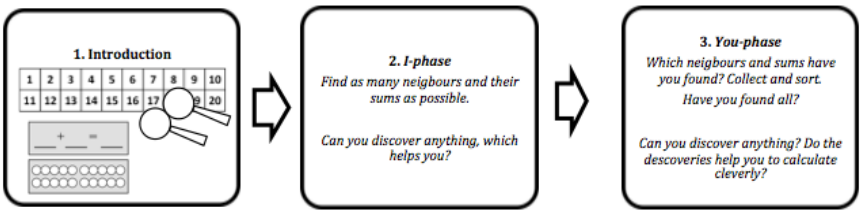


Figure 1: The structure of a design experiment

The teaching-learning arrangement - "We explore neighboring sums": After a mutual introduction (Figure 1), the children individually explore neighboring numbers on a 20frame (*I-/individual-phase*). The focus on neighbors – which are *next to*, *under* or *crosswise* to each other – and their sums enables them to discover number and problem characteristics and relations, as well as to develop mental calculating strategies based on individual abilities, arithmetic-, and context-characteristics. In the following, two children – one with and one without learning disabilities –

work together (*You-/cooperative-interactive-phase*), which enables them to communicate, use, reflect, refine, and/or improve their discoveries and strategic tools. In this way, singular accesses and comprehensions can evolve, through communicative exchange, to new comprehension and understanding. Due to the focus on neighboring sums the arithmetical patterns stay the same even in higher number ranges. This makes communication possible, even though some children already transfer their discoveries to neighbors on the 100frame or generalize the mathematical structures.

In order to reconstruct and categorize the development of goal-differentiate heterogeneous learning processes concerning flexible mental calculation, a model (Figure 2) has been drafted on the basis of previous research (Rathgeb-Schnierer & Green, 2013; Rechtsteiner-Merz, 2013). As mentioned before, *a 'process of solution'* is a situation-dependent and individual response to specific '*cognitive elements'* (Rathgeb-Schnierer & Green, 354) (e.g. 'characteristics and relations of numbers and problems' or 'automatized procedures') and the corresponding construction of the actual solution process using '*tools for solution'* (Rathgeb-Schnierer & Green, 2013, 355). In this sense, 'cognitive elements' are individual experiences, which comprise background-knowledge and -expertise for the individual process of solution. Referring to the theoretical background, the grey fields in Figure 2 are predictors of flexible calculation. Those will be fostered on different cognitive levels as an aim of the designed teaching-learning arrangement.

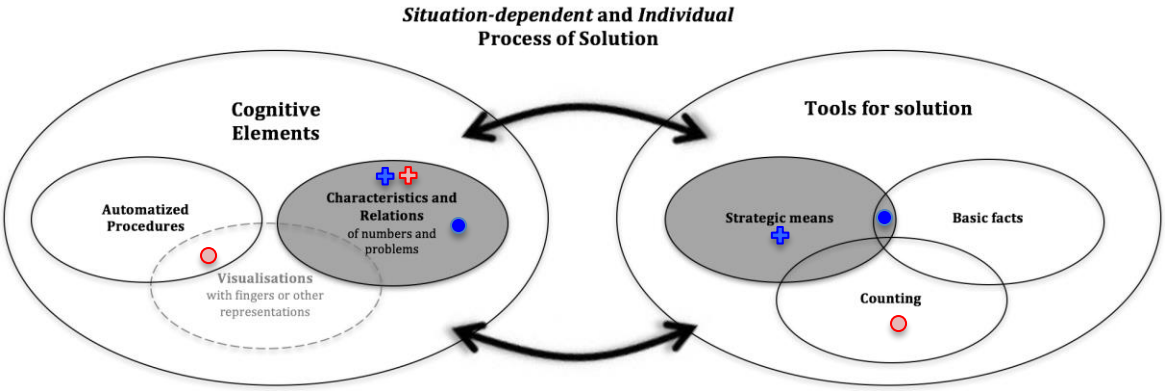


Figure 2: A model to reconstruct children's learning paths (cf. section 'Selected Results')

Each field of this model is described and defined by certain characteristics in order to group children's learning paths (for more information see Korten, i.V.).

Two analytical perspectives for answering the question and for developing local theories

The process of generating local theories gets content-specific theoretically and empirically justified. The data was collected in form of transcribed videos and gets analysed from two perspectives: 1) An *epistemological perspective*, to learn about individual learning processes on different cognitive levels in terms of the common content. 2) An *interactionist perspective*, to learn about the interactive structures during the cooperative-interactive phase of student with and without learning disabilities, and how these interaction processes influence the learning processes.

In order to address the two central matters of successful Mutual Learning, both perspectives are essential to evaluate if the children progress on their individual level and at the same time learn with and from each other. The interpretation of statements and actions, reconstructs interactive knowledge construction. Accordingly, an Interpretatively Epistemological Analysis Approach of Interactive Knowledge Construction (Krummheuer & Naujok, 1999; Steinbring, 2005) gets used.

At the same time this reveals information for the analysis of the teaching-learning arrangement and gives answers weather Mutual Learning in the sense of inclusion is supported or not. Thus, the empirical findings allow elaborating and enhancing the teaching-learning arrangement, as well as local theory building about mutual learning processes. Here the focus will be on the latter.

Selected results

In this section, the described analysis approach is illustrated with a short exemplary cooperative-interactive phase. Afterwards, selected general results concerning the research question addressed in this contribution will be presented. In the exemplary situation, a child with learning disabilities (S1) and a child with average mathematical skills (S2) work together. They explore crosswise neighboring numbers and their sums. Figure 3 shows an example.

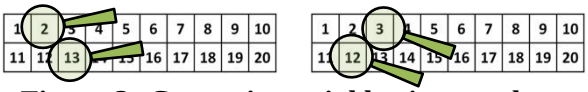
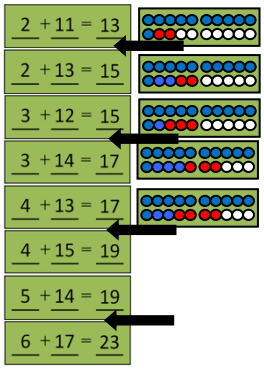


Figure 3: Crosswise neighboring numbers

S1: We need [the sum] 24 in between… 23 (points on 7+16=23) #, 24 (points between 7+16=23 and 7+18=25), 25 (points on 7+18=25)

S2: # No, this is… No... Here is the same. (points on 4+13=17) Also always one. (points on 3+12=15) See, there is 16 missing… here 14 is missing. (points between 2+11=13 and 2+13=15) Here 18. (points between 4+13=17 und 4+15=19) Oh here even (points between 5+14=19 und 6+17=23) two … no, 3…



S1: What? Now I am confused.

S2: Why? Ah! See, …

Interactionist perspective: S1 assumes that the sum 24 is missing and questions the completeness of the sums. This '*incorrect assumption'* (key impulse) leads S2 to exemplify relations between the sums. Her empirical argumentation leads to the hypothesis that the sum 24 does not exist. Both participants communicate with each other about the common content, according to individual assets. A '*balanced cooperation'*, in which both are involved and utterances are linked can be observed. Regarding to Naujok (2000) they are 'collaborating' with the focus on the same topic. Additionally, both develop on their individual level as the reconstruction of learning processes reveals:

Epistemological perspective: The children respond to the same 'incorrect assumption' (Figure 4, sign/symbol) in different ways by referring to *number relations* on the basis of their individual cognitive abilities. Figure 4 and 5 show the progress of the scene from an epistemological

perspective: S1 argues with counting and refers to the number word series (ordinal). S2 uses empirical arguments to prove that the sum of 24 does not exist by referring to arithmetical patterns (relational). Due to S1´s incorrect assumption, S2 discovers, exemplifies and later even generalizes number relations between the addition problems. S1, like this situation shows, is able to see and to question number patterns. This focus of attention on number characteristics and relations only started due to the interaction with S2. In the following, this situation leads S2 even to explore, explain and generalize the *constancy of two sums* (a+b)=(a-10)+(b+10) (Figure 5). From this point on, as a reaction on the interactive situation, she is not only referring to numbers characteristics and relations anymore but to *problem relations*, which she is using later to solve new problems.



Figure 4 & 5: Analysis of interactive learning processes (Steinbring, 2005)

The example shows how the individual learning processes developed during the cooperative-interactive phase of Mutual Learning. Both progressed according to their individual levels, triggered by a key impulse in the interaction, in this case an 'incorrect assumption'. These key impulses – here called '*productive moments*' – seem to be opportunities for fruitful Mutual Learning (Korten, 2017). Previously to the cooperative-interactive phase, **S1** exclusively refers to '*automatized procedures'* and used '*counting'* as her 'tool for solution' with the help of counters as '*visualisations'* (cf. Figure 2, red dots). She only relied on the procedure of counting, which seems to be like a "dead-end road" for developing flexibility in calculating (Rechtsteiner-Merz & Rathgeb-Schnierer, 2016, p. 359). But due to the impulses resulting from the interactive-cooperative phase she starts to look at *number characteristics and relations*. She is able to sort the addition problems according to characteristics and puts them into relation (cf. Figure 2, red cross). This recognition of number patterns and numerical relations is after Rechtsteiner-Merz and Rathgeb-Schnierer (2016) an important skill to overcome counting on the way to flexible calculation. **S2** was also stimulated by the exchange with S1: She was a flexible calculator from the start on. She used '*basic facts'* and adequate '*strategic means'* (e.g. decomposing and composing, using decade analogies, deriving solutions from similar problems such as 2 more and 20 more) by recognizing *relations* (cf. Figure 2, blue dots). Due to the interaction with S1 she was forced to explain and argue, which led her to discover new relations (*constancy of two sums*), which she used to refine her 'strategic means' later (cf. Figure 2, blue cross). It can be concluded that it was rewarding for both children to work in heterogeneous pairs in an arithmetic classroom. A study of Häsel-Weide (2016) about replacing persistent counting strategies with cooperative learning, supports this finding.

These were typical *learning* and *interaction processes*, which took place in all design experiments. In regard to the research question, which is addressed here, five types of individual learning paths

and their development during the cooperative-interactive phase of Mutual Learning could be reconstructed: 1) Children, who did not progress during the cooperative-interactive phase and used none or pre-existing individual insights into number/task characteristics and relations. 2) Children, who gained *new insights* into number/task characteristics and relations due to the task´s context. 3) Children, who gained *new insights* into number/task characteristics and relations due to impulses in the cooperative-interactive phase (e.g. S1). 4) Children, who *refined* their strategic means by taking advantage of new insights into number/task characteristics and relations due to the task´s context. 5) Children, who *refined* their strategic means by reflecting, inquiring and evolving insights into number/task characteristics and relations due to impulses in the cooperative-interactive phase (e.g. S2). In the cases 2)–5) changes could be identified because *new insides* were gained or strategic means were *refined*. Concluding, a development of learning process concerning flexible mental calculation was reconstructed for these situations. With respect to the title of this article and the definition given at the beginning, successful Mutual Learning took place and flexible mental calculation competencies were fostered on different cognitive levels in this inclusive situation.

Outlook

All research cycles demonstrated regularity in the appearance of the 'productive moments' in the interaction, which trigger individual learning as shown in the example. These moments mainly appeared during a ‘balanced cooperation’. Generally, a distinction can be made between ‘*direct-didactical*, *indirect-didactical* and *interactive* productive moments’ (Korten, 2017). In the future, research questions on the level of design will be addressed in order to reconstruct support means for successful Mutual Learning. It will be investigated in more detail how the developed teaching-learning-arrangement can specifically foster these ‘balanced cooperation’ and the 'productive moments'. First analyses show that beneficial and meaningful interaction must be specifically encouraged by an emotional benefit for all participants, which must be created from the outside. This, for example, can be a goal, which they can only reach together and functions as an ‘*extrinsic positive dependence’* (Korten, i.V.). This idea takes up the principle of ‘positive dependency’ from the concept of ‘cooperative learning’ (e.g. Johnson, Johnson, & Holubec, 1994) and advances it for the conditions of an inclusive classroom. Without this ‘extrinsic positive dependence’ a ‘balanced cooperation’ with ‘productive moments’ seems to be impossible in an inclusive setting. With this Topic-specific Design Research Approach the two interests of the goal-differentiated fostering of flexible mental calculation in an inclusive classroom and the general understanding and supporting of Mutual Learning processes could be integrated and first local findings were presented.

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Opportunities to acquire foundational number sense: A quantitative comparison of popular English and Swedish textbooks

Anna Löwenhielm[1], Gosia Marschall[1], Judy Sayers[1] and Paul Andrews[1]

[1]Stockholm University, Sweden; anna.lowenhielm@mnd.su.se, gosia.marschall@mnd.su.se, judy.sayers@mnd.su.se and paul.andrews@mnd.su.se

In this paper, we present analyses of popular grade one textbooks, one from each of England and Sweden. Focused on Foundational Number Sense, we examine how each book's tasks facilitate children's learning of those number-related competences that require instruction and which underpin later mathematical learning. Analyses identified both similarities and differences. Similarities lay in books' extensive opportunities for children to recognise and write numbers and undertake simple arithmetical operations. However, neither offered more than a few tasks related to estimation or simple number patterns. Differences lay in the Swedish book's greater emphases on different representations of number, quantity discrimination and relating numbers to quantity, highlighting conceptual emphases on number. The English book offers substantially more opportunity for students to count systematically, highlighting procedural emphases.

Keywords: Foundational number sense, mathematics textbooks, England, Sweden, grade one.

Introduction

In this paper we offer a comparative analysis of how commonly used textbooks, one from each of England and Sweden, enable year one pupils' acquisition of foundational number sense (FoNS). FoNS, which has been discussed in earlier CERME papers (Back, Sayers & Andrews, 2013; Andrews, Sayers & Marschall, 2015; Sayers & Andrews, 2015), comprises those number-related competences that underpin later mathematical learning, both in the short and the long term, and require instruction. Derived from a systematic review of the literature (Andrews & Sayers, 2015), FoNS comprises the eight broad categories shown in Table 1. Focused on the FoNS-related opportunities initiated during whole class teaching, the framework has structured analyses of grade one lessons in various European countries (Back et al., 2013; Andrews et al., 2015; Sayers, Andrews & Björklund Boistrup, 2016) and identified didactical emphases commensurate with earlier research undertaken in the same countries.

Until now, we have not examined the framework's effectiveness with respect to identifying FoNS-related opportunities in textbooks. This is a significant omission, particularly as both textbook production and deployment are unregulated in England and Sweden. This significance is heightened by uncertainty with respect to pre-school students' likely FoNS-related experiences. On the one hand, the English pre-school curriculum specifies that children should "count reliably with numbers from 1 to 20, place them in order and say which number is one more or one less than a given number. Using quantities and objects, they add and subtract two single-digit numbers and count on or back to find the answer" (Department for Education, 2014, p.11). On the other hand, the Swedish pre-school curriculum, which specifies no such detail, expects children to develop an understanding of the basic properties of quantity, number and number concepts (Skolverket, 2016). Thus, while there are no explicit FoNS-related expectations in the Swedish pre-school curriculum, a number, but not all, are addressed in the English.

FoNS Characteristic	Learners are encouraged to
Number recognition	Identify, name and write particular number symbol
Systematic counting	Count systematically, forwards and backwards, from arbitrary starting points
Number and quantity	Understand the one-to-one correspondence between number and quantity
Quantity discrimination	Compare magnitudes and deploy language like 'bigger than' or 'smaller than'
Different representations	Recognise and make connections between different representations of number
Estimation	Estimate, whether it be the size of a set or an object
Simple arithmetic	Perform simple addition and subtraction operations
Number patterns	Recognise and extend number patterns, identify a missing number

Table 1: Summaries of the eight FoNS categories

Of particular interest to this paper is Bierhoff's (1996) comparison of the number-related opportunities offered in commonly used English, German and Swiss textbooks. Focused on the transition from "working with numbers up to 20… to working with two-digit numbers" (p. 143), she found that English textbooks were the least coherently structured. Also, students were expected to calculate with large numbers before consolidating their understanding of the integers up to 20, a situation made problematic by the English overemphasis on place value. Turning more explicitly to studies focused solely on English textbooks, Newton and Newton (2007), in an evaluation of the professional support school textbooks might afford primary teachers, examined eighteen textbooks written for use with English 7-11 students. They found few tasks that would facilitate mathematical reasoning, being primarily focused on skills acquisition.

With respect to Sweden, as in England, the production of textbooks has been unregulated since 1991 (Ahl, 2016) and several recent studies have examined Swedish mathematics textbooks against various criteria. For example, at the university level, Lithner (2004) found tasks typically promoting low levels of imitative reasoning. At the upper secondary, or post-compulsory, level Nordström and Löfwall (2005) analysed the extent to which students were offered opportunities to engage with proof in two commonly used sets of textbooks. They found little evidence of proof in any of their examined topics, although there were many implicit opportunities in many of the tasks analysed. In similar vein, Lundberg (2011) compared three of the most commonly used textbooks from the perspective of proportional reasoning and found not only that direct proportion dominated but also that while both dynamic and static notions of proportion were present in all three textbooks, justifications were rare. With respect to the final years of compulsory school, Ahl (2016) examined the proportional reasoning in two popular textbooks. She found that "the impact of research findings on the representation of proportional reasoning is scant" in both (Ahl, 2016, p. 198) and that the books failed to encourage learners to understand the distinction between additive and multiplicative situations. In short, the limited available evidence indicates that textbooks written for older Swedish students present few opportunities for them to make mathematical connections or engage in mathematical reasoning.

However, little is yet known about the ways in which textbooks written for young children present mathematical ideas.

This study is a first attempt to evaluate the FoNS framework as a tool for analysing grade one textbooks. Thus, while it is not an explicit attempt to evaluate the content of the books themselves, it is an important first comparison of textbooks from the two countries. In making this comparison, we acknowledge Rezat's (2006, p. 482) position that the mathematics textbook "can be regarded as an artefact in the broad sense of the term. It is historically developed, culturally formed, produced for certain ends and used with particular intentions". In other words, comparative analyses of this nature highlight well cultural differences in expected learning outcomes.

Methods

Two popular textbooks, one from England and one from Sweden, were identified for analytical purposes. In focusing on popular textbooks, we believed we would gain insight into not only how a reasonably high proportion of children in both countries experiences FoNS but also what teachers and schools value in their choice of textbooks. Before formal analyses were undertaken, all four authors met for two days to discuss and evaluate a range of textbook tasks in order to operationalise the FoNS categories. Drawing on the studies of Li (2000) and others, only those tasks explicitly addressed to the student were analysed. For example, both of the examined textbooks included instructions or suggested activities that teachers might use. However, these were not analysed as they did not explicitly address the learner and typically included too little detail to show how they might have been used with children. For similar reasons, since tasks included in teacher guides were not focused directly on students, teacher guides were not included in the analyses. After this first pass, each of the first two authors took responsibility for analyses of the Swedish and English textbooks respectively. In these roles, each was supported by the third and fourth authors with respect to ambiguous or difficult to interpret tasks. In addition, random exercises from each textbook were also coded by both the third and fourth authors as part of a moderation process.

Operationalising the codes

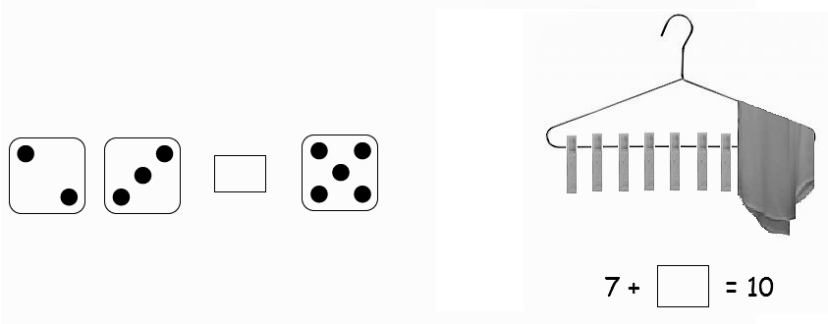

Figure 1: Additive tasks from the Swedish and English textbooks respectively

Figure 1 shows one example from each of the textbooks, Swedish on the left and English on the right. In one of several similar tasks in one exercise, Swedish students were asked to "compare the number of dots" and then "write either = or ≠" in the box. This particular task, which occurred before the introduction of addition, was thought to encourage completion by counting and coded for *systematic counting*. The expectation that students would address issues of equality or inequality led to its also being coded for *quantity discrimination*. In addition, the dot patterns not only offered *different*

representations of number but allowed for subitising and an *awareness of the relationship between number and quantity*. The goal of the English task, based on a coat hanger with ten pegs of which some of which had been covered with a cloth, was to identify the number of hidden pegs. The way in which the task was presented explicitly involved *number recognition*, while its focus was on *simple arithmetic*. In addition, its allusion to cardinality led to its being coded for *awareness of the relationship between number and quantity*. In short, many tasks attracted multiple codes.

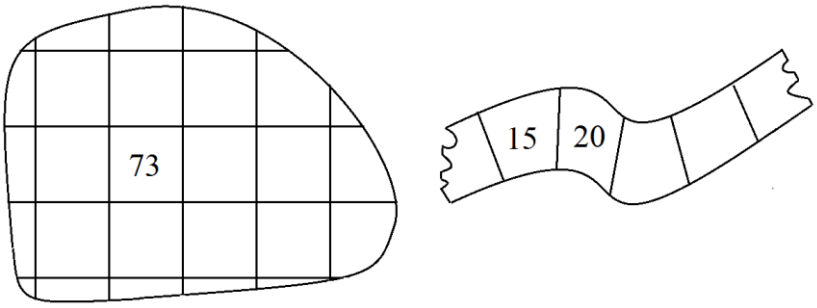

Figure 2: Number patterns tasks from the Swedish and English textbooks respectively

Some FoNS categories, as shown in Table 3, were rare in both textbooks. In this respect, Figure 2 shows tasks, one from each textbook, with explicit foci on *number patterns*. The Swedish task on the left was based on a section of a hundred square, with students being expected to complete the missing values. In addition to being coded for *number patterns*, the explicit focus of the task, it was also coded for *systematic counting*, *number recognition* and, implicitly, *simple arithmetical operations*. These decisions drew on the facts that the task required students to count on, recognise numbers and, in moving from one row to another, add or subtract ten. The English task on the right was one of several based around a section of a multiplication table torn from a longer strip of paper that invited students to count on in fives and enter the missing numbers. In addition to being coded for *number patterns*, these tasks were also coded for *number recognition*, *systematic counting* and *simple arithmetical operations*.

Results

Below we present two analyses offering similar but importantly different perspectives on the data. The first is based on frequencies and the second on proportions.

A frequency analysis

The figures of Table 2 show the distribution of the eight FoNS categories across the two textbooks, one from England and one from Sweden. The first thing to notice, acknowledging that both books are intended to provide the complete learning experience for year one students, is that the Swedish book offered 444 tasks appropriate for FoNS coding, while the English only 257. That is, while both figures represented similar proportions of the totality of tasks within their respective books, the Swedish textbook comprised 187 (73%) more FoNS-related tasks than the English. Table 2 also shows that of the eight FoNS categories, number recognition was the most frequently observed, with 532 out of 691 tasks providing opportunities for learners to recognise, write and say numbers. In similar vein, *simple arithmetical operations* were common occurrences throughout both books. Neither of these results, we suggest, is surprising as arithmetical competence is an unequivocal curricular goal, which

relies extensively on number recognition. The least commonly observed FoNS category was estimation, with just 18 occurrences.

Category present in task (444 Swedish tasks and 257 English tasks)

	No	Yes	No	Yes	No	Yes
	Number recognition		Systematic counting		Number and quantity	
England	29	228	145	112	194	63
Sweden	130	304	354	80	259	175
	159	532	499	192	453	238
	χ^2 = 31.8 (A)		χ^2 = 50.9 (A)		χ^2 = 17.9 (A)	
	Quantity discrimination		Different representations		Estimation	
England	237	20	202	55	250	7
Sweden	370	64	181	253	423	11
	607	84	383	308	672	18
	χ^2 = 7.33 (B)		χ^2 = 88.9 (A)		χ^2 = 0.03 (C)	
	Simple arithmetic		Number patterns			
England	154	103	232	25		
Sweden	232	202	406	28		
	386	305	638	53		
	χ^2 = 2.74 (C)		χ^2 = 2.45 (C)			

Chi squares marked A yielded $p<0.0005$, B yielded $p<0.01$ and C were not significant

Table 2: Frequencies and chi square tests for each category for each country.

When data are compared, some interesting results emerge. On the one hand the English books comprised significantly higher proportions of tasks involving *number recognition* (χ^2=31.8, $p<0.0005$) and *systematic counting* (χ^2=50.9, $p<0.0005$) than the Swedish. On the other hand, the Swedish books offered significantly higher proportions of tasks involving opportunities for students to *relate numbers to quantity* (χ^2=17.9, $p<0.0005$), engage in *quantity discrimination* (χ^2=7.33, $p=0.007$) and experience *different representations of number* (χ^2=88.9, $p<0.0005$). Proportionally, the figures of Table 2 show no significant differences with respect to *estimation*, *simple arithmetical operations* or *number patterns*. These results take us to the second step of the analysis.

A proportional analysis

A second perspective on the data can be seen in Table 3. Firstly, several FoNS categories were found in similar proportions in both textbooks. These included relatively high occurrences of *simple arithmetical operations*, implicated in just under half of all tasks in both textbooks. In smaller proportions, around a quarter of all tasks in both books, were opportunities for students to *relate number to quantities*. In very small proportions in both books, were found *number patterns* and *estimation*. Secondly, several categories distinguished the expectations found in one book from the other. On the one hand, the English textbook comprised a significantly higher percentage of number recognition tasks (89%) than the Swedish (70%) ($t=6.31$, $p<0.0005$). Also, almost half of all English tasks involved systematic counting in comparison with less than a fifth in the Swedish ($t=6.95$, $p<0.0005$). Alternatively, the Swedish textbook comprised nearly three times as many tasks involving different representations of number as the English ($t=10.57$, $p<0.0005$), twice as many tasks focused

on quantity discrimination (t=2.92, p=0.004) and almost twice as many tasks relating numbers to quantity (t=4.42, p<0.0005). Finally, Table 3 shows that the percentage of tasks coded for *estimation*, *simple arithmetical operations* and *number patterns* were comparable in both books, confirming that the two analyses, one essentially parametric and the other non-parametric, yielded equivalent results.

	E%	S%	t	p
Number recognition	89	70	6.31	0.000
Systematic counting	44	18	6.95	0.000
Relating number to quantity	25	40	-4.42	0.000
Quantity discrimination	8	15	-2.92	0.004
Different representations	21	58	-10.57	0.000
Estimation	3	3	0.16	0.874
Simple arithmetical operations	40	47	-1.66	0.097
Number patterns	10	6	1.49	0.136

Table 3: Percentage of all tasks coded for each FoNS category along with t-tests

Discussion

In this paper our objective was to examine the efficacy of the FoNS framework as tool for evaluating the learning opportunities embedded in commonly used textbooks and to undertake a comparative analysis to determine the framework's sensitivity to different cultural expectations. In both cases, we believe the study to have been successful. For example, with respect to the identification of the different FoNS categories, very few tasks were identified with an emphasis on *estimation*, a finding resonating closely with earlier classroom observations showing no evidence of teachers in England, Hungary, Poland, Russia or Sweden emphasising it in their teaching (Back et al., 2013; Andrews et al., 2015; Sayers et al., 2016). This, it seems to us, is an issue of some concern and the basis of further systematic inquiry. Indeed, acknowledging that estimation skills are important indicators of later mathematical competence (Booth & Sigler, 2006), that both older students (Sowder & Wheeler, 1989) and many otherwise competent adults (Hanson & Hogan, 2000) are uncomfortable with estimation tasks, it seems sensible to ask; why does estimation play such a lowly role in the classroom practice and textbooks of these two countries? This, we argue, is particularly pertinent in light of evidence from other countries that teachers see little relevance in teaching estimation (Alajmi, 2009). Furthermore, the similar frequencies of other FoNS categories are unsurprising. For example, it is reasonable to assume that the relative lack, in both textbooks, of tasks focused on *number patterns* may be explained by the fact that most year one curricular goals emphasise learners' *number recognition, relating number to quantity* and the beginnings of *arithmetic*. In other words, while number patterns are important in preparing students for later mathematical learning (Lembke & Foegen, 2009), they may be subordinated in children's early number experiences to more pressing developmental needs.

With respect to cultural sensitivity the data yielded several hitherto uncovered insights. For example, on the one hand, the higher proportions of Swedish tasks coded for *different representations of number*, *relating number to quantity* and *quantity discrimination* allude to a book focused on conceptual understanding. On the other hand, the apparent lack of a conceptual emphasis in the English book finds further support in the high proportions of tasks coded for *systematic counting* and extremely high proportions of tasks addressing *number recognition*, which tend to suggest a book

focused on the development of procedural knowledge commensurate with the low levels of mathematical challenge found in earlier studies of English textbooks (Bierhoff, 1999; Haggarty & Pepin, 2002; Newton & Newton, 2007). However, the conceptual emphasis found in the Swedish textbooks seemed not to match the generally negative findings of earlier Swedish studies (Ahl, 2016; Lundberg, 2011; Nordström & Löfwall, 2005). In this respect, it is not improbable that these differences may be because these earlier studies addressed textbooks for students in grades 7 and upward rather than on those for young children. Finally, drawing on Bernstein's (1990) notion of curricular framing, it is interesting to note that the weakly framed Swedish pre-school curriculum seems to have prompted a conceptually focused textbook, while the strongly framed English pre-school curriculum seems to have precipitated a procedurally focused textbook. Such matters allude to research beyond the scope of this paper but which will form a key aspect of any further analyses we make.

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Seeking symmetry in distributive property: Children developing structure sense in arithmetic

Andrea Maffia[1] and Maria Alessandra Mariotti[2]

[1]University of Modena and Reggio Emilia, Italy; andrea.maffia@unimore.it

[2]University of Siena, Italy; mariotti21@unisi.it

Structure sense can be mobilized by pupils to compare and to transform arithmetical expressions, however sometimes it can lead to mathematical inconsistency that pupils might be not aware of. This paper provides evidence of this type of phenomenon. Through the analysis of an interview with a third grader, it is shown that the development of structure sense can result in transformations as $a\times b+a\times c\rightarrow(a+a)\times(b+c)$*. It is concluded that a development of structure sense requires a dialectical control between the syntactic and semantic interpretations of symbolic sentences.*

Keywords: Structure sense, distributive property, arithmetic, syntactical transformations.

Learning distributivity

Distributive property appears to be less accessible to young students if compared to other multiplication's properties (Larsson, 2016). This phenomenon could depend on the fact that it is not a property *of* one operation but it states a relation *between* two operations. Lo and colleagues (2008) found that many prospective primary teachers show difficulties in applying the distributive property: a frequent erroneous transformation is $18\times26=10\times20+8\times6$ in which tens are multiplied just by tens and units are multiplied just by units (*ibidem*).

According to Carpenter et al. (2005) "an implicit understanding of the distributive property can provide students a framework for learning multiplication number facts by relating unknown facts to known facts" (Carpenter et al., 2005, p.55). For this reason they sustain that it is important to foster the use of fundamental properties of operations to transform mathematical expressions rather than simply calculating. An awareness of the structure of arithmetical expressions appears as fundamental to recognize the equivalence of two arithmetical sentences without carrying out calculations. Mason et al. (2009) use the expression '*structural thinking*' to refer to such awareness.

We wonder which difficulties might students face when they are introduced to structural thinking, and specifically when they elaborate expressions through operations' properties – distributive property in particular.

Structure sense in arithmetic

Apparently, before Mason et al. (2009) introduced the construct of structural thinking, different words have been used to express similar ideas. Linchevski and Livneh (1999) found that many of the typical difficulties faced by students while interpreting algebraic expressions can be found also in the arithmetical context. In particular, they notice students' difficulties in determining the order in which additions and subtractions have to be performed both in the arithmetical and in the algebraic context. These authors conclude that "difficulties revealed in children's understanding of structural properties of the algebraic system originate in their understanding of the number system" (Linchevski & Livneh, 1999, p.192).

> Undoubtedly, students must be exposed to the structure of algebraic expressions. However, it must be done in a way that enables them to develop *structure sense*. This means that they will be able to use equivalent structures of an expression flexibly and creatively. (*ibidem*, p. 191)

Similarly, Caspi and Sfard remarked how "structures of algebraic formulas are not unlike those of arithmetic expressions" (Caspi & Sfard, 2012, p. 64) and they interpreted such similarity as based on the fact that school algebra can be conceived as a gradual formalization of meta-arithmetic (*ibidem*); thus the development of an effective algebraic calculation competence has been referred to as the development of structure sense in arithmetic.

Hoch and Dreyfus (2004) characterize structure sense in the context of high school algebra. They define it in terms of a collection of abilities. According to the cited literature, these abilities can be related to similar abilities in arithmetic. So we propose to modify Hoch and Dreyfus' (*ibidem*) definition to adapt it to the context of primary arithmetic. Thus, structure sense in arithmetic can be described as a set of competences:

1. recognising an arithmetical expression or sentence as an entity, for instance comparing two arithmetical sentences without calculating partial results.
2. Recognising an arithmetical expression or sentence as a previously met structure, for example noticing that $3\times4+5$ is less than $10+3\times4+5$ because the second one is a sum which includes the first one.
3. Recognising sub-expressions in which an arithmetical expression can be divided, as in the case of a student who can describe $5\times7+8\times7$ as composed by two multiplications.
4. Recognising mutual connections between sub-expressions, that means being able to identify which are the operations connecting the terms of an arithmetical sentence, even when such terms are not just single numbers but shorter expressions.
5. Recognising which manipulations are possible to perform. For instance, on an arithmetical sentence like $7+8\times7+3+4$, many transformations could be done ($9\times7+7$ or $14+8\times7$) but there are also transformations that are not executable (as $15\times7+3+4$).
6. Recognising which manipulations are useful to perform. According to the aim of transformations (comparing or calculating), some manipulations can be more useful. In the case mentioned above, the usage of associative property of sum ($3+4=7$) and distributive property ($7+8\times7+7=10\times7$) allows to notice that $7+8\times7+3+4$ is equivalent to 10×7.

This definition is coherent with and specifies those given by Mason et al. (2009) and Linchevski and Livneh (1999). Some of the competences listed above can be activated by pupils to compare and to transform arithmetical expressions: thus, the presence of these abilities can witness the emergence of structure sense, however sometimes this same abilities can lead to mathematical inconsistency that pupils might be not aware of.

This paper aims to provide evidence of this type of phenomenon, that can be considered as a an indication of an incomplete development of structure sense due to a lack of control on the numerical interpretation of a specific structure.

Data collection and analysis

The results that we are going to present are part of broader research study (Maffia & Mariotti, 2016) aimed at investigating the teaching/learning of multiplication properties in the primary school. The empirical design included long-term teaching experiments involving, among others, a group of second graders. The results presented in this paper concern data coming from this specific group. Grade 2 was chosen to promote structural thinking in the case of multiplication since the very first introduction of this operation, that usually takes place at this school level in Italy. Among others, following Linchevski and Livneh's suggestion to "promote the search for decomposition and recomposition of expressions" (1999, p. 191), we designed and implemented activities aimed at introducing the pupils to the distributive property as a transformation of numerical expressions (Maffia & Mariotti, 2015). The rectangular model of multiplication was introduced: activities of cutting and pasting rectangles with the same height (or width) were proposed to explore the relationship between different arithmetical expressions, eventually generalized and symbolically expressed in the distributive property. Examples of such cutting and pasting are given in Figure 1.

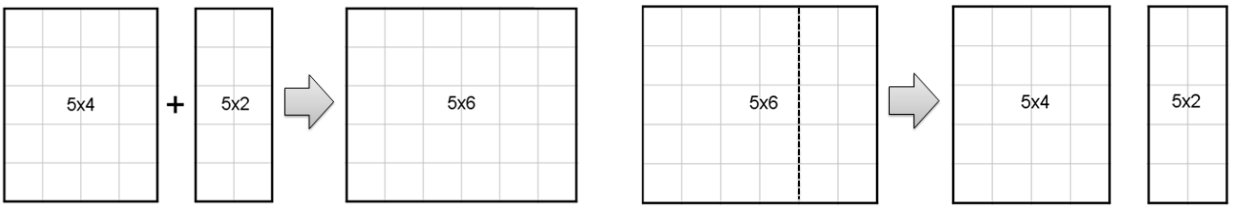

Figure 1: Composition and decomposition of rectangles

Starting from the activities with paper rectangles, the teacher realizes a mediation process to guide students till the usage of conventional arithmetical symbols to represent the relation between multiplication and sum according to the distributive property (*ibidem*).

In this paper, we show data from semi-structured interviews conducted one year after the end of the teaching experiment. Thus, at the moment of the interviews the children are third graders (aged 8-9).

The interviewer shows an image of two children who are writing the equalities $4\times7=7\times4$ and $6\times8+6\times3=6\times11$ on a blackboard (Figure 2) and he asks if what these children are writing can be considered correct. During the teaching experiment, children were asked to produce compositions and decompositions of multiplications, using paper rectangles and then writing them with arithmetical symbols. This is the first time that they have to validate or refute an already written equation.

After the equalities shown in Figure 2, three other numerical sentences are shown and the interviewee is asked to comment about their correctness. These sentences are $5\times6=5\times2+5\times4$; $5\times6=5\times3+5\times4$; $5\times4+5\times3=5\times2+5\times5$. The structure of the first one is similar to the one shown in the image, but the position of expressions is inverted in respect to the equal symbol. The second one is like the first one, except for one number (so it is wrong), and the last one has a different structure but it relates two expressions with the same structure – a sum of products – and specifically, the structure of one of the members of the other equalities. So, the different sentences are designed to allow the child to compare or contrast the structures in the different equalities and, eventually, to apply arithmetical properties. During the interview, paper and pen are provided.

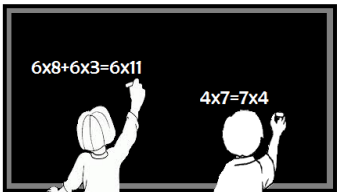

Figure 2: Image showed to students at the beginning of the interview

The interviews have been videotaped and then fully transcribed. Students' transcribed utterances were analysed seeking for evidences of structure sense, through identifying instances of the characterizing abilities. In the following section we discuss some examples, showing specific aspects emerging from this analysis. In the analysis, the six competences of the list are indicated through the corresponding number in the list that is indicated between square brackets.

Seeking symmetry in distributive property

We begin with some excerpts, starting from the end of an interview: Francis comments about the equality 5×4+5×3=5×2+5×5.

53	Interviewer:	Now I will show you a very long one. What do you think about this one [he shows the equality 5×4+5×3=5×2+5×5]?
54	Francis:	[he writes the equality on his paper and then he answers quickly] It's right!
55	Interviewer:	Did you already do it?
56	Francis:	Yes.
57	Interviewer:	Tell me how. I am not as fast as you are.
58	Francis:	Wait. I'll write it. 5×4, is 20. [he writes 20 under the first multiplication. Then he writes the results of the other multiplications; second line in Figure 3] If I would put the 3 and I put 2 [he circles the 3 and he writes a 2 above it] and here I put a 5 [he circles the 4 and he writes a 5 above it] it would be the same operation.

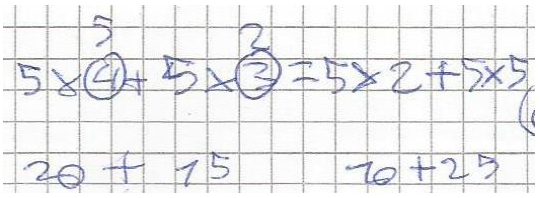

Figure 3: Francis' inscriptions for the last equality

In his explanation (line 58) Francis recognises the possibility of decreasing one of the factors of the second multiplication and increasing one of the factors of the first one, still maintaining the same result (he says "it would be the same operation"). We can recognize an occurrence of the first component of structure sense because Francis is jointly and consistently acting on each part of the arithmetical sentence to maintain its value: he is recognising that the transformation of one multiplication affects the other one, thus he is considering the arithmetical expression on the left side as a unique entity [1].

The expression 5×4+5×3 is transformed in 5×5+5×2 to show the equivalence with 5×2+5×5; so Francis recognises a useful transformation for his purpose [6]. However, in the obtained expression 5×5+5×2, the order of the two multiplications is inverted in respect to 5×2+5×5. Stating that the two

expressions are equivalent, Francis is considering the expression as a sum of two multiplications [4] and so – according to addition's commutative property – the order of the addends 5×2 and 5×5 can be inverted [5]. The child is also recognising that the expression is composed of two multiplications [3]; this interpretation is strengthened by the written operations in the second line of Figure 3.

So far, we have instances of five of the competences that characterize the structure sense; we can say that Francis is showing some evidence of structure sense. As a matter of fact, Francis' explanation not only shows his awareness of structure regularities, but it is completely consistent in terms of the mathematical meaning of the expressions.

However, this has not always been the case. At the very beginning, when the image (Figure 2) was firstly showed, he recognized the equality 6×8+6×3=6×11 as incorrect and stated that the equivalence would have been true if 6×11 was replaced with 12×12. Here is his explication:

11 Interviewer: Wait. Tell me how did you get twelve and twelve.

12 Francis: Six times eight plus six times three [he writes it] I would do six plus six [he draws circles around the 6s, as shown in Figure 4a] that makes twelve.

13 Interviewer: I understand. So you get the first twelve.

14 Francis: And eight plus three [he circles 8 and 3, Figure 4a] that makes twelve.

15 Interviewer: I don't agree. How much is eight plus three?

16 Francis: Eight plus three... eleven [he corrects the second 12 writing a 1 over the 2].

17 Interviewer: Eleven. Ok.

18 Francis: So it wouldn't be twelve times twelve but twelve times eleven.

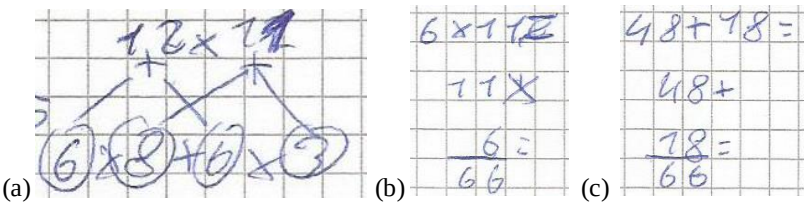

Figure 4: Francis' inscriptions for the first equality

Francis seems to recognize the expression 6×8+6×3 as relating two parts [3], two multiplications connected by an addition [4], and he elaborates this structure according to a *syntactic rule* clearly respecting some kind of "structure sense", but unfortunately it is inconsistent from the mathematical point of view. The transformations he operates (Figure 4a) are strictly at the syntactical level: he is transforming the expression as if the addition would operate in the same way on both the first and second factors of the two multiplications.

The interviewer asks Francis to check the correctness of his conjecture. Francis proposes to calculate the operations' results.

27 Interviewer: How can we get the result of this thing?

28 Francis: We calculate forty-eight plus six times three that is… eighteen. Forty-eight plus eighteen is… and six times eleven is… [he performs the written calculation in Figure 4b]. Forty-eight plus eighteen… is… [he performs the written calculation in Figure 4c] sixty-six. So it's right!

29 Interviewer: Is it? So, what was wrong here? [he points Francis inscription in Figure 4a] In your initial check. Because you said that it wasn't right.

30 Francis: I thought we had to do 12×11.

31 Interviewer: And is 6×11 enough?

32 Francis: […] Yes, because we have to calculate the results of the two multiplications, to calculate the result of the third one and see if the first two ones equal that… their result.

In line 28, Francis is able to divide the expression into its parts: he recognizes that it is composed of two multiplications [3], then he recognizes that he has to sum the two products, so he is recognizing the connection between the two parts [4]. This interpretation is made explicit again in line 32. Francis is showing two of the competences that characterize structure sense: number 3 and 4 in the list. This time, though using his structure sense, the pupil is interpreting the equivalence between the two expressions in a different way. Previously he considered the expressions 6×8+6×3 and 12×11 to be equivalent because one could be transformed into the other according to a syntactical manipulation. In the following, he recognizes two expressions to be equivalent when they give the same result (lines 28 and 32). We consider the first case an occurrence of a *syntactical interpretation* of the equivalence between numerical expressions, the second one as an occurrence of a *semantic interpretation*. Though not yet well harmonized, both types of interpretations seem to be available to Francis, at the same time, the semantic interpretation seems to maintain its primacy.

When the other two equalities are shown, Francis resorts again to the semantic interpretation. He calculates the results of the expressions on the two sides of the equal sign and then he checks if the results equal each other:

43 Interviewer: What if I show you this one? [he shows 5×6=5×3+5×4]

44 Francis: Thirty [he writes 30]. Fifteen, [he writes 15] twenty [he writes 20 next to 15 and then he puts a + sign between the last two numbers. Then he writes =35 obtaining the inscription shown in Figure 5a]. It doesn't work.

45 Interviewer: Can we modify it to make it correct? [Francis doesn't answer] If I would keep this as it is [he points the right side of the equality] what should I write on this side? [he points the left side of the equality]

46 Francis: Ehm… [he puts the pen on the sheet of paper]

47 Interviewer: Let's write it on the paper [Francis writes the equality] Ok. Let's say that I want this [he points the right side of the equality in Francis' inscription] as it is, but I would change the other to make it correct.

48 Francis: We should change the 6 [he circles it] into a 7 [he writes 7 above the 6, Figure 5b]

In this excerpt the interviewer tries to push Francis to go back to a syntactical interpretation. However, though Francis responds in a mathematically consistent way, it is impossible to determine if the proposed modification depends on a syntactical transformation (3+4=7) or on a comparison of the expressions' results. His behaviour in lines 27-32 and 53-58 suggests that both the interpretations are plausible.

(a)  (b) 

Figure 5: Francis' inscriptions for the third equality

Discussion and conclusion

As discussed in the introduction of this paper, the development of what we have called "structure sense" can be considered a main objective of the teaching and learning of algebra.

Starting from adapting the definition given by Hoch and Dreyfus (2004) to the case of arithmetic expressions, we set up a list of competences characterizing structure sense and we used it to evaluate students' behaviours as evidences of the presence of structure sense. The aim of this paper is not to discuss about the effectiveness of the classroom intervention; indeed, it presents a recurrent phenomenon that was possible to identify in the development of the structure sense: it is characterized by an unstable relationship between the syntactic and the semantic level in treating numerical expressions. The case of Francis can be considered a paradigmatic example.

The pupil shows all the competences we used to characterize structure sense but in order to check the correctness of an equality he adopts a syntactical manipulation of operations that is not mathematically consistent: an expression as $a\times b+a\times c$ is transformed into $(a+a)\times(b+c)$. We interpret this behaviour as coherent with a structural sense, but also as a case of corrective action aimed at overcoming what can be seen as a structural flaw, a *seeking for symmetry* in the distributive property. The perceived lack of symmetry could be twofold. On the one hand there is no symmetry in the role of the terms: the common factor in the multiplications plays a different role than the others. On the other hand the structure of the equality $a\times b+a\times c=a\times(b+c)$ is asymmetrical because there is a sum of multiplications on one side of the equal sign and just one multiplication on the other side. This interpretative hypothesis is reinforced by the fact that the student does not show difficulties in treating an equality like $5\times4+5\times3=5\times2+5\times5$, which has a symmetrical structure. This urgent demand of symmetry may be based also on the experience with other properties, such as the commutative and associative properties, and can be considered as a particular source of difficulty in dealing with distributive property.

The wrong transformation $(a+c)\times(b+d)\rightarrow a\times b+c\times d$ is well known in the context of school algebra and it is found also in the arithmetic context in equalities like $18\times26=10\times20+8\times6$ (Larsson, 2015; Lo et al., 2008). In this paper we have evidence of the application of the *opposite* transformation $a\times b+c\times d\rightarrow(a+c)\times(b+d)$ in the arithmetic context. As far as we know, this particular transformation has not been documented in literature before. It has to be stressed that this transformation is shown by four students out of nineteen pupils who were involved in our research. So, we have a too small sample to state anything about its spreading.

In any case, the emergence of this kind of erroneous transformation appears relevant from the didactic point of view: if we expect teachers to promote structural thinking they have to know the potential difficulties that students could meet. Literature shows that this is not always the case (Lo et al., 2008). One clear suggestion emerging from our study is that an approach privileging pure syntactical

transformations seems risky, whilst educating pupils on the danger of losing the semantic interpretation of an expression can help them to reach mathematical consistency.

Fostering structural thinking requires the development of semantic control assuring that any syntactic transformation has a consistent arithmetical interpretation. Further investigation is needed in order to fully describe how such a semantic control can be efficiently developed.

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Models' use and adaptation aiming at conceptual understanding of decimal numbers

Cristina Morais[1] and Lurdes Serrazina[2]

[1]Externato da Luz, UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal
cristina.morais@campus.ul.pt

[2]Escola Superior de Educação de Lisboa, UIDEF, Instituto de Educação, Universidade de Lisboa Portugal; lurdess@eselx.ipl.pt

In this paper, we report part of a study carried out within a design research methodology. An initial conjecture was made that included the importance of the hundred square model to facilitate the discussion about decimal number system features and connections among and within different rational numbers' representations. We present how this model was used and why it led to changes into different models, 10x100 grid, and decimat, during the teaching experiment. Finally, we reflect on how these changes inform the initial conjecture.

Keywords: Models, decimals, elementary education, design research.

Introduction

Currently, in the Portuguese official curriculum (Ministério da Educação, 2013), rational numbers are first approached in Grade 2 (7/8 years old students), in its fraction representation and measure meaning. This meaning has a central role and the use of line segments in the number line is recommended. Decimal numbers[1] are first introduced in Grade 3 and operations with both representations are highly valued. Given the students' age, we believe that these guidelines are too focused on procedures, and, instead, a learning path aiming at conceptual understanding should be privileged.

We will focus part of a broader study in which we follow a design research methodology. In this paper, we discuss how the decimal number system features can be addressed and discussed in the hundred square model, and why this representation was changed into other representations, during the teaching experiment. In order to better frame this paper, we first present an overview of the characteristics of design research methodology, because the use of this approach allowed the constant analysis and adaptation of the model explored during the teaching experiment. We then organize the paper according to the different stages of design research: in the preparation phase section, we present how the literature informed the design principles and initial conjecture; in the experimentation phase, episodes regarding the use and adaptations of the hundred square model will be presented, and in the third phase a deeper analysis of the episodes will be made as well as its impact on the initial conjecture.

Design research methodology

Design research is a methodology that has gaining ground in Mathematics Education research. It can allow the construction, or "engineering", as Cobb, Jackson, and Dunlap (2016) describe it, of

[1] Term used in this paper to identify positive rational numbers written accordingly to the decimal system notation, using the decimal comma or point.

instructional means to promote the learning of a particular topic, while constantly studying the development of that learning, considering all elements of the instructional means, not only the designed tasks but also the context in which they are carried on. Therefore, in design research theoretical and pragmatic components are highly dependent on each other.

There are five crosscutting features of design research that, together, distinguish it from other methodologies: (i) the purpose is to develop theories about both the learning process and the means designed to support it; (ii) it has a highly interventionist nature, since it can be a powerful methodology to design an approach to promote the development of a particular content or form of practice, in a real classroom; (iii) it has two interrelated components, a conjecture is made regarding students' learning (prospective component), that is constantly confronted to the actual learning (reflective component); which can lead to changes in the initial conjecture that is tested again, giving design research its (iv) iterative design; and finally there is an attempt to (v) develop humble theories that address the learning of a particular topic (Cobb et al., 2016).

One of the main characteristics of this methodology is its cyclic nature. Each cycle develops in three phases: (i) teaching experiment preparation and design; (ii) teaching experiment; and (iii) retrospective analysis that can lead to revisions and a new cycle (Cobb et al., 2016). The conjecture can be refined during the teaching experiment, resulting in micro cycles, or in between experiments, in macro cycles, or both types of refinements can happen (Prediger, Gravemeijer, & Confrey, 2015). The present study builds on micro cycles.

Study's rationale *(preparation and design phase)*

Decimal numbers in the research field

Studies related to decimal numbers reveal important evidence regarding difficulties that arise when dealing and operating with this number representation (e.g., Steinle & Stacey, 2003), whole numbers' knowledge influence (e.g., Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989) or about knowledge of the decimal number system, specific of this representation (e.g., Baturo, 2000). Studies present evidence that those difficulties and whole number interference can remain in adulthood (e.g., Vamvakoussi, Van Dooren, & Verschaffel, 2012), which reveals the demanding conceptual understanding needed regarding this representation of rational numbers.

According to Post, Cramer, Behr, Lesh, and Harel (1993), the development of rational number understanding is related to (i) flexibility with translations between rational number representations; (ii) flexibility with transformations within a representation; and (iii) progressive independence from concrete embodiments of rational numbers. In a broader perspective, representations can be a tool to or reflect students' development of mathematical ideas of a particular concept. Therefore, representations that can become models assume an important role, establishing a link between knowledge connected to reality and the mathematical knowledge to be developed (Van den Heuvel-Panhuizen, 2003). Initially, models named as models of (Gravemeijer, 1999), are closely connected with the task context and cannot be used in other situations. This type of model should evolve to a model for (Gravemeijer, 1999), that can be applied in various situations, focusing mathematical connections and not the situation described in the task. In order for this development to happen, the model should have certain characteristics that make it suitable to be used in diverse situations. As

Van den Heuvel-Panhuizen (2003) highlights, to support learning, the model should allow students not only to progress into the mathematical ideas but also to go back to the reality context if needed.

Research show evidence of decimal numbers' learning fostered by the appropriation of different models. We will now focus two of the models considered in the teaching experiment: the hundred square and the decimat. The hundred square is a model often used when teaching decimal numbers, until hundredths, that allows connections between the iconic representation of the grid and symbolic representations, not only decimals but also fractions and percentages. The grid also facilitates students to understand the meaning of each digit and, consequently, its decimal number system notation. It is also an important model to compare and order decimal numbers and, later on, can be used to add and subtract decimal numbers (Cramer, Monson, Wyberg, Leavitt, & Whitney, 2009). Decimat is a similar model, however, by being rectangular and divided into two rows and five columns, allows a clear visualization of each part again divided by 10, emphasizing the multiplicative structure of the decimal place value in the decimal number system (Roche, 2010).

Structuring the cycle

Based on literature review, the following six design principles were elaborated to guide the conjecture and instructional means: (1) use of tasks which context appeals to the use of rational numbers in its decimal representation; (2) promote movements among decimal numbers and other rational number's representations highlighting their relations; (3) promote the use of representations that support their transformations into models to think about rational numbers in its decimal representation; (4) encourage the use of prior knowledge; (5) promote the discussion of whole number interferences and common misconceptions; and (6) establish a learning environment where students are encouraged and feel confident to share and discuss their own mathematical ideas.

Supported by these principles, an initial conjecture was made: A teaching experiment comprised by different types of sequenced tasks, explorations and exercises, focusing decimal numbers in measure and part-whole meanings and the use of number line and hundred square model, considering students' whole number and informal knowledges and evoking the need for the use of decimal numbers, as well as its connections with other rational numbers representations, in a learning environment where students have an active role and small group work and whole-class discussions are privileged, will promote a meaningful understanding of decimal numbers.

A set of tasks was planned (some new and others adapted from existing materials) and students' understanding was anticipated. Tasks were open to adjustments or to be completely revised depending on the understanding students revealed along the way. The teaching experiment was intended to be carried out in Grade 3, from February to June 2014, however, the last tasks were conducted at Grade 4. The teaching experiment was, generally, carried out once per week, in one 90 minutes lesson, involving a total of 16 weeks over the two school years.

The participants were 25 students and their teacher. A diagnostic study was made with the same students, in Grade 2, that provided information about students' ideas of different rational numbers' representations, which supported the design of the initial tasks and also help to gather information concerning the teacher's role. Consequently, the classroom teacher asked for a detailed plan for each task. The plan was made by the researcher (first author) and discussed previously with the

teacher, and included suggestions to support teacher inquiry, possible students' answers and solutions and potential students' difficulties.

In the data presented in the next section, students will be referred to with fictitious names. The tasks were solved in small group work or in pairs, and whole-class discussions were privileged. Records of all the students' written work, along with participant observation by the researcher supported by audio/video recordings and field notes, constituted the main data sources. Meetings between the researcher and the teacher, prior and after each lesson, were also audio-recorded.

One of the expected products of the broader study is a set of indicators of decimal number understanding that can be helpful both for teachers and researchers. We intend that these indicators address two different levels: (i) what is specific of this rational number representation, and, (ii) the intertwinement between this representation and other rational number representations. In this paper, we outlined some indicators of students' understanding of decimal numbers to be supported by the appropriation of the models here presented. As an ongoing research, these indicators are preliminary and open to revision. Regarding the first level, we consider identifying the partitioning and grouping by powers of ten to create units of tenths, hundredths, and thousandths, and reveal an understanding of the decimal numeration properties (positional value, multiplication and addition properties, in addition to base ten property). In relation to the second level, we consider recognition of a decimal number in different representations; identifying the unit, and establish equivalences between numbers represented as decimals, percentages, and fractions.

Classroom episodes (*teaching experiment phase*)

We present three illustrative episodes of the use of three models throughout the teaching experiment, focusing on part-whole meaning. The examples presented concern the use of each model by students (representations as models of). The first two occurred in Grade 3 and the third in Grade 4. We focus our analysis on the indicators of decimal number understanding as mentioned above.

The hundred square was presented to students as a towel, divided into tenths and hundredths. After some exploration of this model, a task was presented to promote the discussion of common misconceptions, such as the comparison of decimal numbers based on its number digits. The hundred square model was showed to help students explain their answers. One of the questions was "Do you think 0,67 is bigger than 0,9?". In whole-class discussion, Jorge revealed how he used the hundred square model to compare both numbers:

> Jorge: Initially I thought that 0,67 was bigger than 0,9 because at first sight 67 seems bigger than 9. . . but then I realized that I could think in a different way. So, if we think that each column has ten-hundredths, we would have to paint six of these columns, without the seven (in 0,67) it would be only sixty. And the other one (0,9) would be 90, it was bigger, nine columns are 90 hundredths, so it was bigger than painting 67 hundredths.

Due to the appropriation of the hundred square, Jorge could visualize and compare the quantities represented by both numbers (Figure 1). The hundred square has shown to have great potential, as its use helped Jorge to overcome the initial, and expected, interference of whole number knowledge.

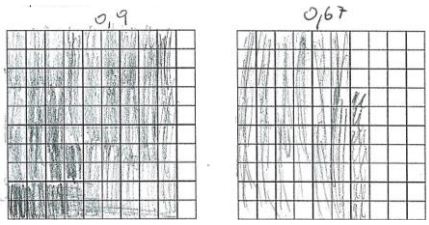


Figure 1: Jorge's work record at a task with the hundred square model

When preparing the teaching experiment, it was anticipated that students could visualize each small square in the model (one hundredth) divided into ten equal parts, each representing ten thousandths. However, it was important that students, in fact, saw the thousandths, instead of inferring that from this model. Given its shape, the hundred square doesn't allow further divisions into thousandths, in the same manner, thus another model was thought. Later, in the classroom, this model started to be called as "thousandths bar". With a rectangular shape, a bar represents the unit that is divided into ten large squares, representing tenths, and each one is then divided into ten columns, the hundredths, that are again divided into ten equal parts, the thousandths (Figure 2).

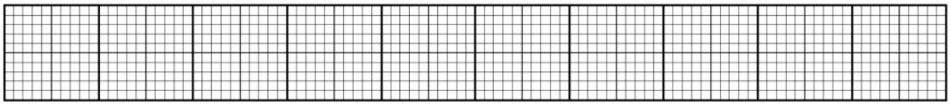
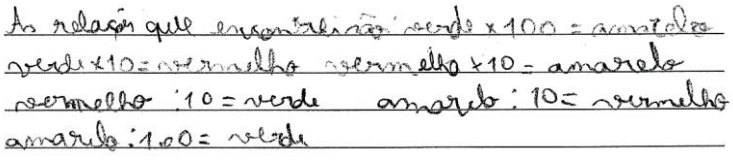


Figure 2: Thousandths bar model

At first, students were encouraged to find out how many "small squares" were in the whole bar. Many looked into one "big square" divided into quarters and calculated 4x25, which was 100, and then multiplied 100 by 10, reaching 1000. Initially, students thought that each "big square" was like the hundred square, representing a hundred hundredths, or one. This was probably due to the fact that each tenth in this model was similar to the hundred model. A unit change was implied: before, one big square represented one unit, now a similar but smaller square represents one tenth. Nonetheless, the model allowed students to relate tenths, hundredths, and thousandths.

An example of these connections made by students is shown in Figure 3. Artur's answer relates to a question where students were asked to paint in the thousandths bar 0,001 with green, 0,01 in red and 0,1 in yellow. After, they were asked about what connections they could find among these parts.



"The relations we found were:
green x 100 = yellow, green x 10 = red,
red x 10 = yellow, red : 10 = green,
yellow : 10 = red, yellow : 100 = green"

Figure 3: Artur's work record at a task with the thousandths bar model, with translation

Artur, like other colleagues, could clearly state the partitioning and grouping by powers of 10 that create the units of tenths, hundredths, and thousandths. It is important to refer that it was the first time that these relations were clearly stated by the students. We believe that visualizing each painted unit in the same model promoted the establishment of these connections. On the other side, and even though some students correctly identified the tenth and the thousandth in the bar, they had painted one hundredth as one-quarter of one-tenth. Students said that 0,1 was the biggest square in the bar, then 0,01 was the "middle" square and 0,001 was the smallest square (Figure 4).

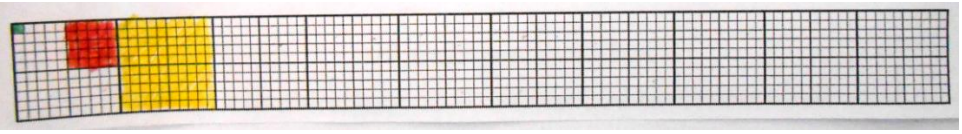


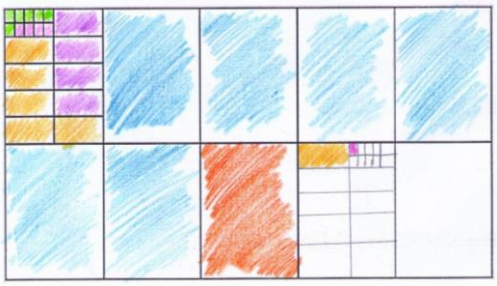
Figure 4: Mafalda's work record at a task with the thousandths bar model

We weren't expecting this response. It can be linked to two factors: due to the strategy used to count the total of "little squares" in the bar, or probably due to the bar layout that misleads students to think about the different decimal units in terms of squares. If a big square is 0,1 and a small square is 0,001, the middle square will be, incorrectly, 0,01.

Thus, together with the classroom teacher, we felt the need to adapt the model again. We needed a model that, like the hundred square, clearly allowed to see the connections between different units, and like the thousandths bar, allowed the extension to the thousandths and students' inference of further partitions by powers of 10, to develop the idea of density. After scratching a model with such features, followed by searching for a similar model in the research field, we came across the decimat model, as described by Roche (2010). Therefore, we included some tasks adapted from this author's work in the teaching experiment.

The model was presented with one tenth divided into hundredths, and one of which divided again into thousandths. When students first saw it, they called it "towel", relating it to the hundred square. They immediately recognized the model shown tenths, one of which divided into hundredths and thousandths. It was said that the model could be further divided if they wanted or needed.

One of those tasks was a game adapted from the one proposed by Roche (2010). In groups of about four students, two dices were given: one regular dice with dots and other with different symbolic representations, specifically 0,01; 0,001; $\frac{1}{1000}$, $\frac{1}{10}$, 1%, and 10%. The students had to roll both dices and multiply the numbers represented in them. Then, they had to color that part in the decimat and say which part of the decimat had already been painted, altogether. Figure 5 shows the record of the game played in Maria's group.



Número	Dices rolled		Part of board game painted
Move number	Number of dots	Representation	
1	4	1%	0,04
2	6	0,001	0,046
3	6	10%	0,646
4	1	10%	0,746
5	6	0,01	0,806
6	6	$\frac{1}{1000}$	0,812
7			

Figure 5: Maria's group record of game plays using the decimat model, with translation

Besides the flexibility in the movement within symbolic representations (decimal, percentage, and fraction) and the operations with decimal numbers as multiplication and addition, this example illustrates the potential of this model. Only one tenth is further divided, however, when needed, the students easily did the divisions on another tenth, revealing an at-ease use of the model.

Looking back and adjusting *(retrospective analysis phase)*

We addressed the use of part-whole models to promote students' understanding of rational number in its decimal representation. Both the hundred square and the thousandths bar models can foster connections between the unit partitioning and grouping by powers of 10, and the decimal number system. However, the hundred square only extends to hundredths and the features of the thousandths bar can hinder the idea of partitioning the unit by powers of 10.

We want to highlight that students had already worked with the hundred square and thousandths bar when the decimat was introduced, which influenced its successful use. We also need to refer that we weren't seeking for a single and perfect model. In fact, students should explore different models. In the present study, the students continued to use all models. Nevertheless, a model should promote the visualization of specific mathematical connections to support student's learning (Van den Heuvel-Panhuizen, 2003), thus, the model's features should allow its evolving alongside the development of students' understanding. The adjustment of the initial model, done to highlight the decimal number system properties, such as the partitioning and grouping of units and place value, led us to the revision of our initial conjecture, in which we will now emphasize the decimat as an important part-whole model.

However, the use of a model by itself is not enough for students to establish mathematical connections, so the connections intended by the use of models should be focused (Prediger, 2013). The results help us to understand that the decimat can be the first model approached, initially divided into tenths, then fully divided into hundredths, in the same manner that the hundred square was also first approached, and, finally, divided into thousandths. We believe that such an approach can promote the understanding of partitioning by powers of ten connected with decimal place value. Besides that, it will allow to order and compare different representations, promote the development of a benchmark number system, unit conceptualization and support decimal numbers' operations. All these connections are strong foundations for the development of decimal number understanding.

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Middle school students' difficulties in proportional reasoning

Oznur Soyak[1] and Mine Isiksal[2]

[1]Middle East Technical University, Faculty of Education, Ankara, Turkey; oznrsoyak@gmail.com

[2]Middle East Technical University, Faculty of Education, Ankara, Turkey; misiksal@metu.edu.tr

The purpose of this study was to investigate 6th, 7th and 8th grade students' common difficulties regarding rate and ratio problems. The data was collected from 149 sixth, seventh, and eighth grade students enrolled in public middle schools in Balikesir in response to three typical proportionality questions. Data analysis revealed that the confusion of unit rate identification and algorithmically based mistakes were identified as major difficulties in solving the missing-value proportion and comparison problems. To address the potential difficulties students have regarding rate and ratio problems, students should be exposed to different types of proportional problems.

Keywords: Proportional reasoning, unit rate identification, algorithmically based mistakes.

Introduction

Proportional reasoning is considered to be a keystone of students' mathematical development and is required to access more advanced high school mathematics like algebra, geometry, probability, and statistics (Lesh, Post, & Behr, 1988). Lesh and his colleagues identify the proportional reasoning involving multiple comparisons between quantities (Lesh et al., 1988).

Previous studies have shown that children can manipulate part-whole relationships within sets of data, but they tend not to employ a systematic approach or to use categories (Inhelder & Piaget, 2013). Although children's mathematical development contains preproportional reasoning knowledge, which has always been seen as the hallmark of the formal operations stage (Inhelder & Piaget, 2013), this knowledge is insufficient for an understanding of proportional reasoning and the solution of ratio and proportion problems (Lamon, 1993).

In order to promote a conceptual understanding of ratio as the first step of proportional reasoning, it is necessary to create comparisons, beginning with additive and moving on to multiplicative comparisons. Additive comparisons compare two quantities to decide how much greater or less one quantity is than another by finding their difference, which is not the same thing as ratio. On the other hand, multiplicative comparisons provide a ratio by comparing two quantities to decide how many times larger one quantity is than another. Therefore, providing some key activities carefully designed to link these two concepts will be useful to help students to construct the aspect of ratio (Kaput, 1985).

Difficulties in proportional reasoning

Teachers have focused on teaching how to arrive at correct answers by applying rules instead of explaining the reasons behind those rules (Hart, 1988). While most of the students will be able to do computation properly, they are not encouraged to build links between concepts. Furthermore, some of the methods used by students lead to problems due to incorrect solution strategies or incorrect use of correct strategies (Hart, 1988).

Regarding proportional difficulties, algorithmically based mistakes are usually the result of lack of attention during the learning process or weak conceptual understanding (Hart, 1988). According to

researchers, problems may also arise from the attempt to find a fast way to compute ratio problems. In other words, students may be rushing to solve the problem quickly without thinking about the relationship between the given quantities and tend to simply copy the procedures identically. In addition, even though they may be able to construct a correct algorithm for cross multiplication, some students may not correctly explain the reasoning behind the algorithm (Lobato, Ellis, & Zbiek, 2010). Routine problems may lead students to assume that they can mimic solution procedures, but when they come across different problem types or non-standard language, they may struggle. Literature review showed that a number of studies have dealt with the reasons for mathematical difficulties in the field of ratio (Ellis, 2013; Hart, 1988; Lamon, 1993; Misailidou & Williams, 2003; Sarwadi & Shahrill, 2014). However, few studies have examined common difficulties among 6th, 7th and 8th grades students connected with unit measures approach for the different types of proportionality problems.

Statement of the problem

This study was conducted to investigate 6th, 7th and 8th grade students' difficulties regarding rate and ratio problems. To put it another way, study aimed to answer the research question: "What are the common difficulties encountered by 6th, 7th and 8th grade students' while the missing-value proportion and comparison problems involving rate and ratio concepts?"

Method

Sample

The sample of this study, which included 66 males and 83 females, consisted of students from sixth to eighth grade at a public middle school in Balikesir, Turkey. The school addresses a wide variety of neighborhoods and income levels ranging from low income to upper middle class.

Measuring tool

Three proportional word problems in real world contexts were used to investigate students' proportional reasoning difficulties by grade levels (see Table 1). When we reviewed the literature on proportional reasoning, these three questions were cited as the most widely known problems. The first and the second questions were chosen from Lamon's (1993, 1999) studies. The last question was an adapted version of an orange-juice task identified by Noelting's (1980). In terms of problem types in the domain of ratio, the first and third questions were comparative, and the second question was missing value problem. While comparison problems provide four values and the aim is to specify the order relation between the ratios, in a missing-value problem three of four values are given and the last value is asked (Karplus et. al, 1983b; Lamon, 2012).These adapted questions were applied to sixth, seventh and eighth grade students, and the students were given one class hour to complete the written test. The questions are given in Table 1 below.

Ratio Achievement Test

Please solve the problems by using appropriate strategies

1. Ayse bought four bananas paid 3.6 liras from Market A. Berna bought three bananas paid 3.3 liras from Market B. Where would you buy your bananas to make profit?
2. Derya, Ahmet and Kaan bought three helium-filled balloons and paid 1.5 liras for all three. They decided to go back to the store and buy enough balloons for everyone in the class, How much did they pay for 24 balloons?
3. Zeynep and Sinan tested three juice mixes. Which juice will have the stronger lemon flavor?

Mix A	Mix B	Mix C
2 cups lemon concentrate	1 cup lemon concentrate	4 cups lemon concentrate
3 cups cold water	4 cups cold water	8 cups cold water

Table 1: Questions about proportional reasoning

Data analysis

The solution strategies were analyzed along the strategies as unit rate, scale factor, ratio tables, and cross multiplication (e.g., Bart, 1994; Hart, 1988; Lesh, Post, & Behr, 1988).Then, incorrect solutions were separated from correct solutions, and then qualitative analysis was conducted to capture the difficulties behind the incorrect answers. The students' solutions were categorized with regards to mistake (error) strategies that have been stated in the literature: misusing a correct strategy (e.g., Hart, 1988; Karplus et al, 1983), using additive strategy (e.g., Hart, 1988; Inhelder & Piaget, 2013) and faulty application of a correct results that deviates from the unit rate (e.g., Tourniaire & Pulos, 1985). These mistake strategies were used to characterize students' difficulties while solving the given proportional problems (see Table 1) involving rate and ratio concepts. Table 2 showed the types of difficulties.

Types of difficulties	Explanation	Example
Confusion of the unit rate identification	Using an arbitrary unit value (it is the guessing method without adjustment/ the unit value is the number of objects the problem starts with.	The student assumes that each banana is 3.6 liras fin Market A (Q1)
Algorithmically based mistakes	Having computational mistakes	$\frac{4\ cups}{8\ cups}$ =2 Mixture C (Q3)

Table 2: Classification of Student' difficulties on Ratio Achievement Test

Findings

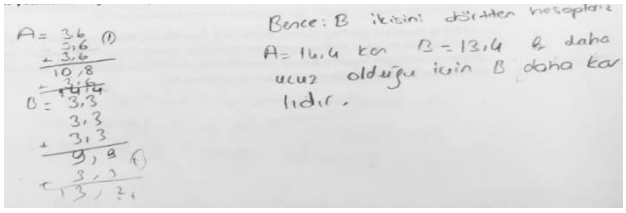
Students' incorrect solutions in solving rate problems were analyzed through category building (see Table 2) to reveal their difficulties and grouped under two headings: "confusion of the unit rate identification" and "algorithmically based mistakes". These two difficulties were identified as the major problems in three typical proportionality questions. Data analysis revealed that of 149 students, 45 (27%) for the Q1, 25 (15%) for the Q2 and 94 (56%) students for the Q3 gave either no answer or an incorrect solution, as recorded in Table 3.

	6 Grade (69)			7 Grade (39)			8 Grade (41)		
	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3
Incorrect Solutions	13(19%)	4(6%)	30(44%)	8(16%)	3(8%)	17(44%)	8(20%)	2(5%)	23(56%)
Confusion of the unit rate identification	10 (77%)	2(50%)	30(100%)	5(63%)	1(33%)	12(71%)	4(50%)	2(100%)	20(87%)
Algorithmically based mistakes	3 (23%)	2(50%)	0(0%)	3(37%)	2(67%)	5(19%)	4(50%)	0(0%)	3(13%)
No answer	7(7%)	8(12%)	9(13%)	5(13%)	2(5%)	8(16%)	4(10%)	6(15%)	7(17%)

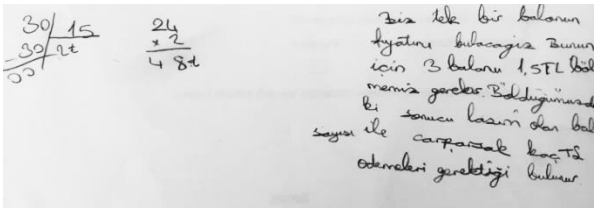
Table 3: Number and percentages of students' incorrect solutions

Confusion of the unit rate identification

Unitizing is identified as a cognitive process that occurs after identifying the unit. This process allows subjective preference by composing two quantities to create a new unit called composed unit (Lobato et al., 2010). Students employed a number of different measurement units. However, some types of questions require the use of a standard measurement unit; the use of any other unit than this one was not allowed (Lamon, 1999). The main difficulty in answering these problems was the identification of the unit/rate. To answer the Q1 (see Table1), it was necessary to measure the amount of stuff using the concept of unit. Students can also find a different number of measures with regards to their measuring unit. When solutions to the first problem were analyzed, confusion about the unit rate could be plainly seen, indicating that the student was not able to conceptualize the unit of measurement.



(a)



(b)

I think the answer is B. We can compute [corresponding two prices] for 4 [bananas]. They [corresponding prices] are Market A=14.4 and Market B= 13.4. Therefore Market B is more economical.

(c)

We need to find price for one balloon. Therefore we should divide 3 balloons by 1.5 liras. And then we should multiply the result by the number of balloons that they need to get how much money is required.

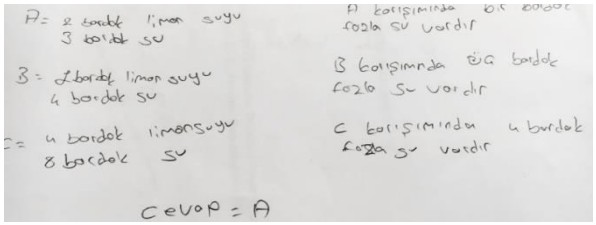
(d)

Figure 1: Student's original (a) and translated (c) solution on ignoring the number of objects, Student's original (b) and translated (d) solution on finding the amount of balloons for one lira as original versions

As can be clearly seen from Figure 1-a, although the unit was defined explicitly (single banana cost) the student thought that a banana cost 3.6 liras in Market A and 3.3 liras in Market B. In other words, the student ignored the number of objects. Then two prices 14.4 and 13.2 were compared and this solution did not contribute to the correct answer. 11 (16%) of 69 sixth grade students, 6 (15%) of 39 seventh grade students and 7 (17%) of 42 eight grade students stated that the best place to buy a banana was Market B because they spend 13.2 for four bananas and they get profit.

Another misunderstanding regarding the unit rate given in the question can be seen in Figure 1-b. This student could not decide the exact and correct numbers to find the unit rate in Q2. Therefore,

instead of finding the price for one balloon, as can be seen here, he calculated the amount of balloons for one lira by dividing 3 by 1.5. Then this value was multiplied by 24 to find the price of whole balloons. Indeed, it makes sense to divide 3 by 1.5 to find the amount of balloons per unit lira. This student seems to follow the directions provided in terms of finding single units. Even though in her written explanation on the right hand side of Figure 2 she stressed finding the value of one balloon, she noted that three would be divided by 1.5.



A=2 glasses of lemon concentrate 3 glass of water	Mixture A has one more glass of water,
B=1 glass of lemon concentrate 4 glasses of water	Mixture B has three more glasses of water
C=4 glasses of lemon concentrate 8 glasses of water	Mixture C has 4 more glasses of water
Answer: A	

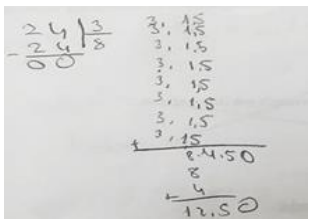
(a) (b)

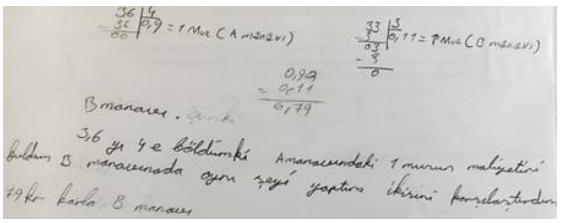
Figure 2: Student's original (a) and translated (b) solution on comparing the amount of lemon juice with the amount of water

The Q3 provides more evidence for students' difficulties regarding typical ratio comparison unit rate problems. Findings revealed that 56% of 149 students think about the unit rate with given quantities in a reasonable way. For instance, as can be seen in Figure 2 without thinking about fractional relativity, the student compares the amount of lemon juice used (2 glasses, 1 glass and 4 glasses) with the amount of water to find the stronger tasting lemonade. Even though this student came to the correct solution that Mix A has a stronger taste than the others, this solution contains misconceptions. He matched the glass of water with the lemon concentrate, and decided that there is one extra cup of water in Mix A. However, if this student was asked to sort the concentration of juice for each mix, the answer would likely be wrong, because, according to his solution, the order would be Mixture A and then Mixture B and then Mixture C. Thus, the less tasty mixture would have been Mix C.

Algorithmically based mistakes

The other difficulty which emerged from the data was algorithmically-based mistakes and emerged in the computation process. Ashlock (2001) identifies these errors as 'buggy' algorithms that involve more than one incorrect step in the procedure and do not attain the desired purpose. When the details of this difficulty were examined, it was revealed that 8% of the students made basic fact errors and conducted incorrect operations while dividing the decimals.





I divided [corresponding to price) 3.6 by [corresponding to amount] 4 to get the price of one banana for Market A. And I did the same computation for Market B. I made a comparison [between Market A and Market B]. Market B is more economical with 79 cent (kuruş).

(a) (b) (c)

Figure 3: Students' original (a) solution on an algorithmically based mistake, Student's original (b) and (c) translated solutions on a difficulty with division

Figure 3-a illustrates these algorithmically based difficulties for Q2. First, by dividing 24 by 3, this student got 8 groups. As shown above, this student attempted to add 1.5 eight times. She wrote 3 and left a space and then wrote 1.5. It can be seen in the calculation that the student first multiplied the whole part by 8 and then the decimal part by 8. However, she did not recognize the multiplicative structure even in a familiar computation and she started with an incorrect computation. Then it seems she apparently lost track and then made algorithmic error by adding .50 to 12. Another example for algorithmically based mistakes can be seen in Figure 3-b. This student divided 3.3 by 3 and found 0.11 as a cost of one banana. However, the correct answer was 1.1 and this mistake led the student to make the wrong comparison between the profits of two markets. According to this solution 0.9 is bigger than 0.11, so Market B is the best place to buy a banana.

Discussion and conclusion

The aim of this study was to investigate the common difficulties faced by 6th, 7th and 8th grade students while solving typical proportionality problems involving rate and ratio concepts. The first common difficulty resulted from confusion of unit rate identification. The second major finding that algorithmically based mistakes were another common difficulty in computation process. These findings show that current difficulties are consistent with the Turkish context beginning from 6th grades (Kaplan, Isleyen, & Ozturk, 2011).Unitizing is a different process from determining the unit, because different systems of units are based on different choices of base units. The most obvious finding to emerge from this study is that students mostly preferred the unit rate method as a solution strategy. The reason might be related to their tendency to retreat to more familiar strategies in their solutions.

The cognitive process emerges after making a decision about the unit and misunderstanding arises when students think about the unit, from their computations especially while explaining unitizing (Lamon, 1999). The results of this study showed that sixth, seven and eight graders had problems about conceptualization of the unit of measurement. More specifically, the lemonade juice problem showed that students from all grades used unit rate strategy as a part of faulty application of a correct result by comparing the numerical differences additively rather than the multiplicatively. As Noelting (1980) states, they focused on the basis of the number of glasses of orange juice instead of proportional relations between given quantities. Especially sixth graders who offered a rich repertoire of unit rate mistakes tended to apply unit rate for the lemonade problem, and their incorrect results seems to be deviated from unit rate method (see Table 3). Based on this point, lemonade problem juice experiments might be beneficial for students in terms of experiencing their own strategies in real contexts instead of just explaining them verbally. Findings suggest that they might be performing this operation without realizing the difference between additive and multiplicative reasoning (Lamon, 2012; Lobato et al., 2010). This situation can make a noteworthy contribution in terms of providing some indications of the complexity of these mixture problems not only for sixth but also for seventh and eighth graders. This suggests the need for more in-depth investigation into student thought-processes when making these specific mistakes (Ashlock, 2001; Son, 2013). These results are consistent with other studies regarding emergent difficulties within Turkish context and suggest that proportional reasoning should involve more than just applying rules, and that there is a need for more information about what students perceive the unit to be (Sarwadi & Shahrill, 2014). Teachers should be more aware of student conceptions about unit rate while teaching proportional reasoning. This

corroborates with Lamon (1993) and Singh (2000) who state that teachers must encourage the spend time to connect composed units with multiplicative comparisons by setting different types of problems for students as much as possible to enable them to build flexible and complex unit structures develop thinking strategies.

Findings of the present study showed that algorithmically based mistakes are commonly seen in the student computation process. Son (2013) identifies three categories of error in the responses received in her study: concept-based errors, procedure-based errors and diagnosis errors. As Ashlock (2001) states, one of erroneous steps lead to emerge these kind of mistakes and then as a result the intended purpose is not systemically accomplished. On the other hand, the lack of clinic interviews with students in this study suggests caution in identifying the exact causes of these fundamental mistakes. In another words, it is difficult to decide whether students' algorithmically based mistakes stem from limited conceptualization of the problems or whether there was a mistake in their algorithmic procedures. It is possible that some of the difficulties revealed in this study may result from concentration on algorithmic computation. This suggests the need for further investigation into the exact reasons for algorithmically-based mistakes through clinical interviews.

The findings of the present study and previous research suggest several implications. As Lamon (1999) states, textbooks don't provide the flexibility of using unit rates, and under these limited conditions students will not be able to compose units. Turkish mathematics textbooks do not provide sufficient examples of these types of problems which might promote the development of an understanding of unit rate. Besides, exposing students to a variety of proportionality problems can help them to develop multiplicative reasoning skills, and to promote flexibility in unitizing.

Even though the scope of this study includes only determining the potential difficulties students have regarding rate and ratio problems, we might make some implications and suggestions to overcome those difficulties. Future research might continue to investigate students' verbal reasoning process and provide more detailed insight into the reasons for incorrect solutions. Such exploration might yield more informative insights into the reasons for student identification of unit-rate confusion and algorithmically-based errors, and provide valuable implication for the development of students' multiplicative reasoning ability.

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Number line as a conceptual and didactical resource for teaching fractions using applets

Carlos Valenzuela García[1], Olimpia Figueras[1], David Arnau[2] and Juan Gutiérrez-Soto[2]

[1]Centro de Investigación y de Estudios Avanzados del IPN, Departamento de Matemática Educativa, Ciudad de México, México; carvaga86@hotmail.com; figuerao@cinvestav.mx

[2]Universitat de València, Departamento de Didáctica de la Matemática, Valencia, España; david.arnau@uv.es; juan.gutierrez-soto@uv.es

Results of secondary school students' performance who participated in a pilot study of a general research project whose purpose is to contribute to building better mental objects for fractions are described. A Local Theoretical Model for fractions is used as a theoretical and methodological framework to design and develop a seven stages teaching sequence based on the use of applets created with GeoGebra and the number line as a conceptual and didactical resource. In this paper details of the design, development, and results of the first two stages are given. Results show children's preferences to represent fractions on the unit segment, that is, they think of proper fractions. A majority of the participants paid more attention to the graphical aspects of the applet.

Keywords: Fractions, local theoretical models, applets for teaching, number line, improper fractions.

Introduction

The teaching and learning of fractions and rational numbers have been studied during several decades by researchers such as Freudenthal (1983), Behr, Lesh, Post and Silver (1983), Kieren (1988) and Figueras (1988). In the last decade, fractions have been considered as one of the most complex concepts studied in basic education (see for example López-Bustamante (2009) in Mexico, Contreras (2012) in Spain and Petit, Laird & Marsden (2010) in the United States of America).

The construction of better mental objects of fractions during elementary school is considered important because that concept is widely accepted as an integral part of mathematics curriculum. Moreover, Siegler, Duncan, Davis-Kean, et al (2012) characterized the knowledge of fractions and division as unique predictors of students' mathematics performance from elementary to high school.

On the other hand, the technological tools as a resource for teaching mathematics have been incorporated into the curriculum because there is evidence that those are cognitive resources (e.g., Kieran & Yerushalmy, 2004). In this sense, an attempt is made to design a teaching model that includes these tools to promote the building up of students' better mental objects of fractions.

Research objectives

The principle aim of the general study is to construct a Local Theoretical Model for fractions in order to enrich the actual teaching model (in Figueras' sense, 1988, pp. 21- 22) for Mexican elementary school. To achieve this aim a seven stage teaching sequence based on the use of applets was designed. The focus of this paper is the design and results of the first two stages of a pilot study. The purposes of those stages are: 1) to characterize the type of fractions students keep in mind to represent them on the number line displayed on the screen, 2) to make an exploration inquiry about students' ideas

related to density and order of fractions, and 3) to identify if the students relate numerical aspects of fractions with their graphical representation.

Theoretical framework and related literature

The idea of Local Theoretical Model (LTM) developed by Filloy (see Filloy, Rojano & Puig, 2008) is used as a theoretical and methodological framework. From the theoretical point of view, the LTM serves to focus on the object of study through four interrelated components: (1) formal competence, (2) teaching models, (3) models for cognitive processes, and (4) models of communication. The construction of these components allows having an interpretative framework to identify the different aspects of fractions, which according to Freudenthal (1983) appear as fracturer, comparer, measurer, fraction operator and numbers.

The building up of the teaching models component enabled the detection in Mexican and Spanish curricula, that the study of properties of order, density, the equivalence of fractions, and proper and improper fractions were considered in the last years of primary school. Bright, Behr, Post, and Wachsmuth (1988) and Saxe, Taylor, McIntosh, and Gearhart (2005) support the idea that working with the number line gives students an approach to the notion of above concepts. However, in some sixth-grade' textbooks used in Spain (e.g., González et al, 2015) continuous and discrete models are more often than not used for teaching fractions, and activities that include the number line are lacking. These are some reasons for using the number line in this research as a didactical resource.

A revisiting of specialized literature was carried out to construct the cognitive processes' models component. Some of the papers analysed report students' difficulties faced when they learn with the number line. Michel and Horne (2008) mentioned three principal misconceptions: (1) *Instrumental part-whole knowledge* -difficulties with unit-forming, that is, children consider any line segment as a unit-. (2) *Counting lines, not spaces* -to represent a fraction on the number line some students count the lines or points (considering the zero point) instead of counting intervals-. (3) *Decimalising the count* -to represent any fraction, some children always divide the unit segment in ten parts-.

Finally, the communication component is formed by the observation of communication processes between student-applet interactions. The building up of the four components grounds the design of the teaching sequence and its trial. The latter is detailed in the following sections.

Methodology and methods

From the methodological point of view the LTM serves to organize the research project in two main parts: (1) the building up of the four components of an initial LTM for fractions as a reference framework of the general research project and (2) an experimentation with students.

Three phases comprise the experimental part. The first one relates to the design of a pre-test, a post-test and a seven-stage teaching sequence. The second phase is the application of the teaching sequence and is structured as shown in Figure 1. Each stage is composed of two parts, one is a GeoGebra applet and the other a series of questions posed with the purpose that students show the ideas they bring into play about fractions when interacting with the applets. Applets and questions are set up on a Webpage that is associated with a database to record students' responses and interactions with the applets. The third phase corresponds to data analysis and characterization of students' performance.

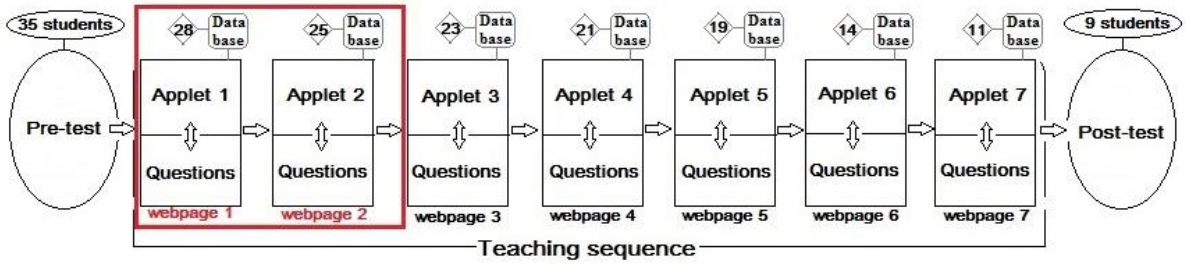

Figure 1: Structure of the teaching sequence design and its trial

The first two stages of the teaching sequence were done in one 45-minute session. Student individual interactions with applets were collected in a non-invasive manner. Answers given were stored in a database and collected with computers provided by researchers. At the end of the experimental phase, stored information was joined and organized to proceed with its analysis. The applets are used as a resource to teach fractions and to collect data.

Setting and participants

The pilot study was carried out with 45 students from 12 to 14 years old in a secondary school located in a troubled urban area of Valencia in Spain. According to their mathematics teacher, participants have a large history of difficulties in mathematics. The students have serious problems of truancy, for this reason, not all of them completed the trial of the teaching sequence. Due to this fact, only data from students that completed sequential steps were considered, that is, 28 students made the first stage and 25 completed the second stage and so on (see Figure 1). The students worked alone during the teaching sequence trial. In this study, the teacher applied the pre-test and post-test.

Applets' design and results

The applets were constructed in a learning environment for fostering the development of conceptual understanding of fractions, taking into account the didactical functions of technology in mathematics education adapted by Drijvers (2013, p. 3). As aforementioned, each stage of the teaching sequence has an applet with an exploration/interaction component (Figure 2 and 4) and a list of questions for students to reflect on what they observe during the interaction. To respond, students can turn to applets and observe the animation or representations of fractions.

To characterize the answers given by students, schemes that enable a codification have been constructed (Figure 3 and 5). Answers given by pupils were grouped in different types determined by the form in which questions are posed. Type i are answers to questions that are general statements with diverse interpretations. Type ii collects answers that can be classified solely as correct or incorrect. Type iii groups answers to questions where a justification is required and Type iv are answers to questions that requires information students must write on the applets' windows.

Applet design for the first stage. The applet's tasks for the first stage were developed considering two parallel lines of action. One directed towards the student's familiarity with the interactive environment. The other leads the student to represent different aspects of fractions on the number line and to introduce them to a proper use of the fractions' mathematical sign system (Figure 2). Three indications appear on the screen in Figure 2. The first one -'Move the sliders and watch what happens on the number line'-, has the purpose to focus students on the effects of the numerator and

denominator sliders that appear at the upper left corner of the screen and to relate those to the graphical and symbolic representations of fractions also shown on the screen.

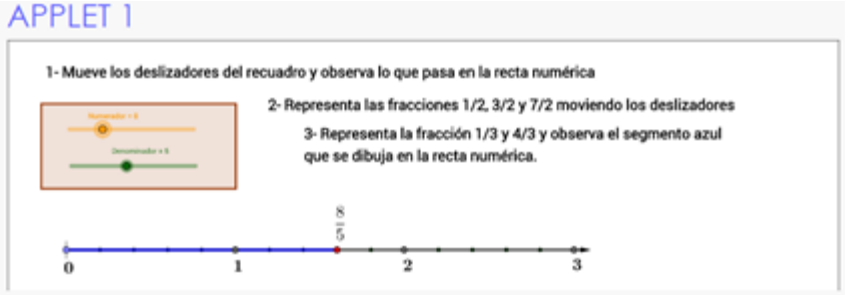

Figure 2: Screenshot of the applet for the first stage

With the second indication –"Represent fractions 1/2, 3/2 and 7/2 by moving the sliders"-, students are asked to represent the first two fractions to see them in the line segment on the screen. The main idea for asking students to represent 7/2 is to promote reflection regarding characteristics of fractions that can be visualized on the screen and of those that cannot.

The third indication -'Represent the fractions 1/3 and 4/3 and observe the blue segment that is drawn on the number line'- is provided in order to identify the point representing the fraction or the fraction as number, but also to focus students' attention on the magnitude representing the fraction, that is the length of the segment that represents the fraction. Thus, fraction as a measurer emerges, taking into account the part-whole relationship.

In addition to the above information, seven questions (Figure 3) are posed to make students write their ideas about the observations made during exploration/interaction period. Students can read questions and explore the applet as many times as necessary to answer them.

Data analysis and results of the first stage. For applet 1 there are only questions of types i and ii. The codification of students' answers is done using the scheme shown in Figure 3.

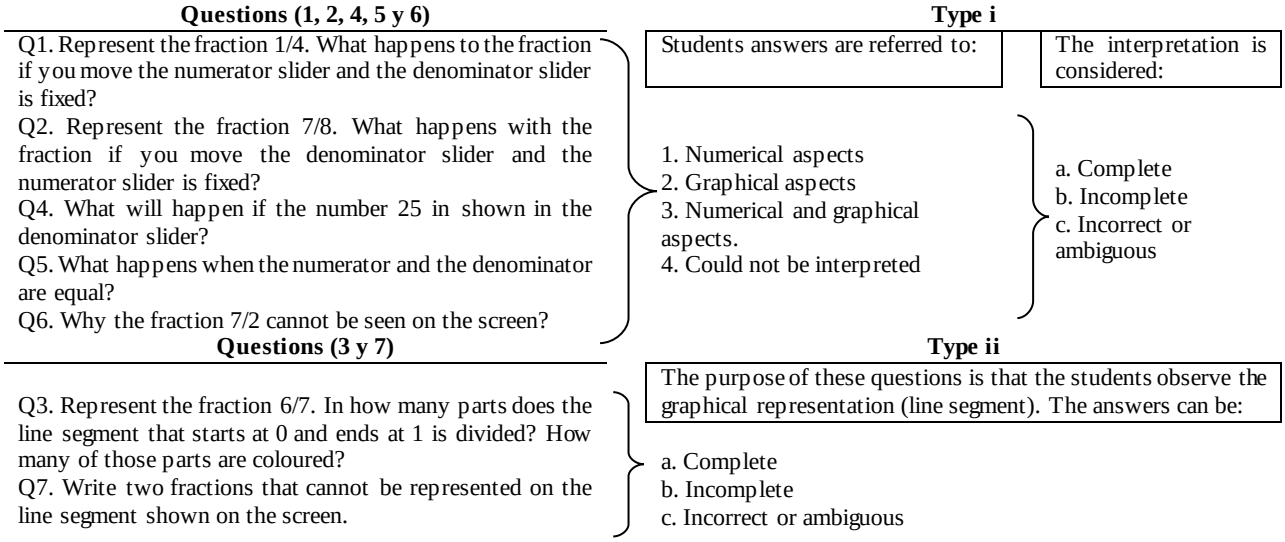

Figure 3: Scheme to characterize the answers to questions posed in the first stage

One of the most common mistakes to respond Q3 was that students focused on counting the lines or points considering the zero point instead of counting spaces. 22 students identified at least one fraction greater than three to respond Q7. In the answers to questions type i (for example, Q1, Q2,

Q4, Q5 and Q6), pupils orient their attention on what happens in the line segment that appears on the screen (Table 1). Of the 140 responses (28 students x 5 questions), 63 (45%) were classified in this group (code 2, in grey). Eight of these answers are considered complete (code 2a), since a consistent explanation is offered, 25 incomplete (code 2b) and 30 incorrect or ambiguous (code 2c). Two answers related to these results are included to exemplify the way the coding is done.

(1) The answer given by student S1 to question Q1 is: "the bar is moving to the right". This response is coded as 2b (see Table 1) because the focus is posed on what happens on the line segment. Specifically, it refers to the movement that occurs in the blue segment ("the bar" named by the student) representing the fraction when the slider moves to the right, that is when the value of the fraction is increased. The response is not considered complete because the student centres his attention on the movement to the right and presumably does not move the slider to the left. (2) The answer given by S3 to question Q1 is: "there are more points between numbers (denominators), in particular, four points between numbers". In this case, the answer also reveals a focus on the number line; the student observed the partition of the line segment, but the interpretation is incorrect or ambiguous because the student did not explore using the sliders (code 2c).

The focus on numerical aspects of the fraction (code 1, in green) was observed in 38 of the 140 responses to the questions of type i (27.15%); two answers were classified as complete (1a), 14 as incomplete (1b) and 22 as incorrect or ambiguous (1c). The answer given by student S17 to question Q6 is: "because the denominator is 2, and this is smaller than 7, so the numerator is bigger, so it is not possible". The student observed the values of the numerator and denominator, makes a comparison between the numerical values and justifies his answer; it was coded as 1c (see Table 1).

	S1	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**	**S8**	**S9**	**S10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
Q1	1b	2c	2c	3b	2b	3b	2c	1c	1c	1c	2c	2b	2b	2b	3b	2c	1c	2b	1b	2c	4c	3b	2c	1c	2b	2b	2c	4c
Q2	1b	1c	2c	2b	2b	3b	3b	1c	1c	1c	2c	2b	2c	1c	3b	1b	3a	2c	3b	2b	1b	3b	2c	2c	3a	2b	2c	2c
Q4	2b	2c	2c	2b	2b	2c	2c	4c	4c	4c	4c	2c	4c	4c	2b	2b	1b	2b	2a	1c	1c	4c	4c	2c	1c	4c	4c	4c
Q5	2a	3a	2a	2c	4c	2b	1b	1c	1c	1b	4c	1a	1b	4c	2a	2a	1b	2a	3a	1b	4c	1b	1a	2b	2b	2a	4c	2a
Q6	3b	1c	2c	1c	1b	1c	3b	4c	2c	1c	2c	1c	1c	2b	2b	2b	1c	3b	3a	4c	2c	4c	2c	2c	4c	2b	2c	1b
Q3	c	a	b	a	b	a	c	b	b	b	a	b	b	b	a	c	c	c	a	c	b	b	c	c	c	b	a	a
Q7	b	c	a	a	c	c	c	b	b	a	b	b	a	b	b	b	a	a	a	b	a	b	b	b	c	c	b	b

Table 1: Characterizations of students' answers in the first stage

Only in 17 of the 140 (12.14%) answers to the questions of type i (in purple) the focus on numerical and graphical aspects is made evident. Five of them were classified as complete (3a) and twelve as incomplete (3b). The answer given by student S19 to Q6 is: "because in the line, one can only represent numbers between 0 and 3, and 7/2 is greater than 3". The student observed the structure of the line segment; his focus is posed on the graphical representation but also refers to the fraction as a number. For this reason, the answer was coded as 3a.

Applet design for the second stage. A new form of symbolic representation is introduced in this applet (see Figure 4). In this case, students are asked to write five fractions in a pop-up window when they click the start button (INICIO in Spanish). This button is associated to a JavaScript subroutine that offers feedback and stores the student actions.

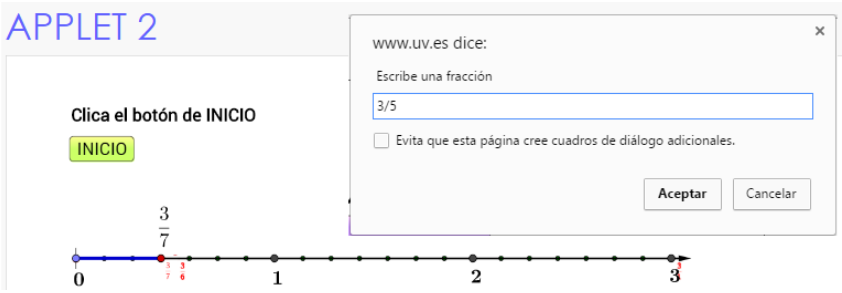

Figure 4: Screenshot of the applet for the second stage

Fractions are limited to those with denominator between 2 and 10 and numerator between 0 and 40. If a pupil writes a fraction that does not meet these conditions, a pop-up window appears with a message advising to take into account the characteristics of the numbers involved. When introducing a fraction greater than three, another alert window appears to indicate that the fraction cannot be seen on the number line. Fractions that are written by the user leave a trace in the form of red dots and the value of the fraction on the number line (see Figure 4). The visible trace on the screen helps the students answer questions posed in the Webpage in which the applet is embedded.

Data analysis and results of the second stage. In the applet that corresponds to the second stage, there are questions of type i, ii and iv. The codification is done using the scheme in Figure 5.

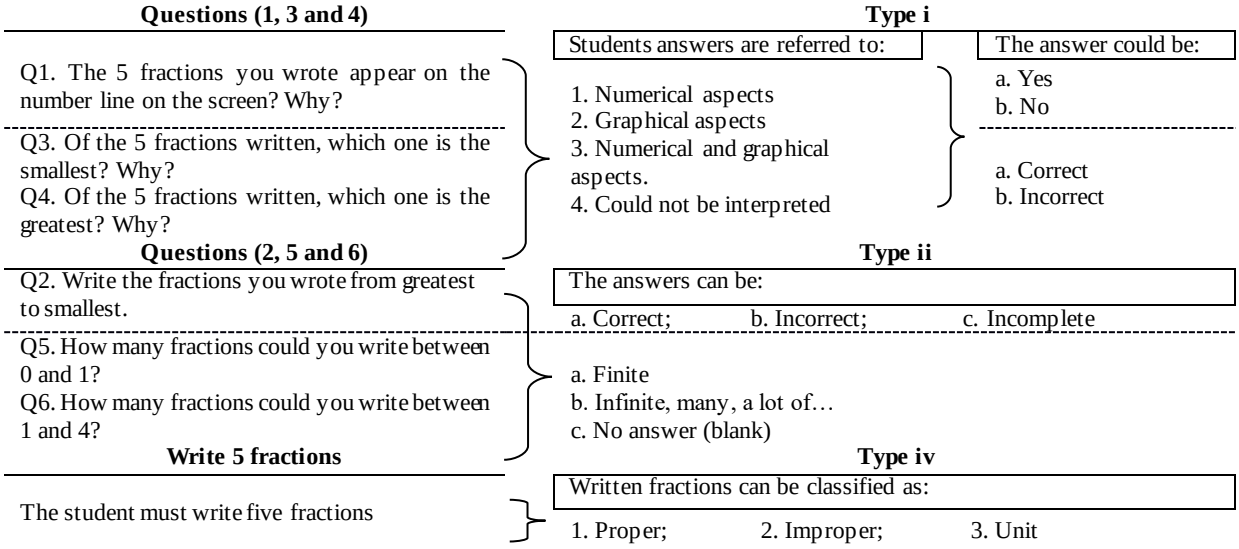

Figure 5: Scheme to characterize the questions of the second stage

Only 12 of 25 students were able to answer correctly Q2 (code 1). This result has an effect on the answers to questions Q3 and Q4, as shown in Table 2. The students who order fractions correctly chose correctly the greater or smallest fraction for questions Q3 and Q4 respectively. To justify the order of those fractions S1 considered the length of the blue segment that represents the fraction on the number line (2a). S2 considered the position of the point representing the fraction on the number line (2a), i.e. graphical aspects of fractions. Students S5 and S18 considered the characteristics of the numerator and denominator of the fraction, that is, numerical aspects of fractions (code 1a).

The justification for most students who do not respond correctly to questions Q3 and Q4 is based on comparing the numerators and denominators of the fractions. Two of these cases are the following: (1) Student S8 choose 8/3 as the smaller fraction, "because the denominator is the smallest." The

comparison made with fractions 4/10, 3/9 and 12/5. He also chose 4/10 as the greater fraction "because the denominator is the largest." (2) Student S7 chose 2/3 as the smaller fraction between 5/4, 3/7, 3/4, "Because that [fraction] has the smaller numbers."

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15	S16	S17	S18	S19	S20	S21	S22	S23	S24	S25	
Q1	2a	1a	2b	1b	4a	4b	2b	4b	4b	4b	4a	4a	4b	4b	4a	2a	4b	2b	1a	1b	4b	4a	4b	2a	4b	
Q2	1	1	0	0	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	0	1	0	0	0	1	
Q3	2a	2a	4b	1b	1a	4a	1b	1b	2b	4b	1a	1a	2a	4b	4b	2b	2b	1a	1a	1b	1a	4b	2b	2b	2a	
Q4	2a	1a	4b	1b	1a	4a	4b	1b	2b	4b	4a	2b	4a	4b	4b	4b	2b	1a	2a	1b	1a	4b	2b	2b	2a	
Q5	b	a	a	a	a	b	a	a	a	a	a	a	a	a	a	a	a	a	b	a	c	a	c	a	b	
Q6	b	a	a	a	a	b	a	c	a	c	a	a	b	a	a	a	a	a	b	a	c	a	c	a	b	
P	3	5	3	3	1	2	3	3	1	3	0	1	2	3	1	3	3	0	2	3	1	3	1	4	2	56
Im	0	0	0	2	1	3	1	2	2	1	4	2	2	1	4	1	1	2	1	0	3	0	3	1	1	38
U	0	0	0	0	2	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	5

Table 2: Characterizations of students' answers in the second stage

Of the 50 answers (25 students x 2 questions) related to the density of fractions (questions Q5 and Q6), 35 were classified with code a, because the answer refers to a finite number of fractions, 6 were blank (code c), and only in 9 answers the density property was mentioned in some sense (code b), because some of the responses were "infinite fractions" or "as much as one wants". Finally, Table 2 shows that students wrote more often proper (56) than improper fractions (38).

Conclusions

Results of the first two stages described before allow to highlight the fact that few students were able to relate the numerical and graphical representation of fractions. However, students who are able to relate these two representations can give complete and correct answers. To justify questions related to fractions' order, some students relied solely on the numerical aspect, for example, comparing the numerator and the denominator, which led to incorrect or ambiguous answers. Whereas answers in which graphical aspects are used, for example, the position of the point on the number line or the length of the blue segment, led to correct answers. Although most students focused on the latter aspect, they encountered difficulties in representing a fraction on the number line. The most common mistakes are instrumental part-whole knowledge and counting lines, not spaces, also reported by Michel and Horne (2008).

On the other hand, the idea of density that students had seems to be strongly related to the number of fractions they represented during their interactions with the applet, these results are a warning to continue investigating these ideas in later stages and reflect the influence of applets. The students wrote more proper fractions even though the number of proper fractions with small denominators as 2 or 3 are few compared with the number of improper fractions. These results can be related to fact that teaching models favouring the recognition of proper fractions are widely used.

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How middle-grade students explain ordering statements within real life situation? An example of temperature context

Aysenur Yilmaz[1] and Mine Isiksal-Bostan[2]

[1]Middle East Technical University, Faculty of Education, Ankara, Turkey; akubar@metu.edu.tr

[2]Middle East Technical University, Faculty of Education, Ankara, Turkey; misiksal@metu.edu.tr

The purpose of the study is to examine the extent to which middle-grade students agree on statements about the ordering of two negative integers given within a real-life context, and what kind of procedural and conceptual strategies they generate to order those numbers. Data is collected through a questionnaire including two statements about ordering integers from fifty-seven middle-grade level students in a public school. The results show that even though students agreed with both of the statements, they did not explain the concept of ordering in daily life with regard to conceptual meanings and that they have problems in their procedural knowledge repertoire.

Keywords: Integers, contextual problems, negativity.

Introduction

Integers is one of the main topics that students have to understand to be successful in later contents such as algebra, geometry, or data analysis. While integers are required to solve algebraic expressions, sometimes it is required to understand the number system (Levenson, 2012). In particular, negative integers are hard for students to understand because of difficulty in representing those numbers physically (Davidson, 1992), locating them on the number line, performing four operations (e.g.: Ojose, 2015), or ordering them (e.g. Schindler & Hußmann, 2013). Ball (1993) exemplifies the dilemma of the case of teaching negative numbers regarding the ordering of two integers. Furthermore, according to Ball (1993), the transition among direction and magnitude aspects of negative integers are the heart of understanding negative integers.

Students' solution strategies

Linchevski and Williams' (1999) study reports on a disco game supported by an abacus model that helps students make sense of net-change, zero situations, and develop strategies such as compensation and cancellation within the everyday life situations requiring operations of integers. In addition to this, students have solution strategies of four operations such as the tendency of using the first addend's sign, ignorance of the second addend's sign, or generalizing the statement by rote as 'two negatives make a positive' and so on (Ashlock, 2010). Considering the literature, it is seen that many studies focus on student solution strategies about operations on integers. However, in the available literature, there are limited studies that focus on students' strategies about ordering integers (Schindler & Hußmann, 2013; Ojose, 2015). Furthermore, what kind of pieces of information students have regarding ordering integers, is rare.

Conceptual and procedural knowledge

According to Hiebert and Lefevre (1986), pieces of information might be separately located or are in closely related within the knowledge network in students' minds. Once students start learning, they can make links among those pieces or students keep in mind the necessary steps to solve problems and they apply the steps to solve the problems without questioning. The pieces of knowledge can be

constructed with appropriate links to make meaningful understanding and that can be accomplished by creating relationships among the separately existing pieces of knowledge. Hiebert and Lefevre (1986) define conceptual knowledge as '... knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network' (Hiebert & Lefevre, 1986, pp. 3-4). The other kind of knowledge which is used by students in learning mathematics while solving mathematical problems is procedural knowledge. Hiebert and Lefevre (1986) describe procedural knowledge through including parts as 'one part is composed of the formal language, or symbol representation system of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks' (Hiebert & Lefevre, 1986, p.6). Both procedural and conceptual knowledge is necessary for students in order to finish the process with a correct answer and to understand the relationships among concepts. Students' backgrounds include procedural and conceptual knowledge and both of them are required for mathematical proficiency (Kilpatrick, Swafford, & Findell, 2002). They are constantly interacting with each other and in light of instructions given to students, the kinds of knowledge can be revealed. In this regard, this study will examine students' explaining strategies while ordering two negative integers through conceptual and procedural knowledge descriptions of Hiebert and Lefevre (1986). It gives information about how middle school students conceptualize ordering of integers, how they give meaning to integers in real life contexts, and what kind of procedural and conceptual strategies they have while ordering integers. With this study, possible explanations of the reasons behind middle-grade students' difficulties and their understandings in integers, specifically about ordering integers are revealed. Thus, the aim of this study is to explore answer for the following questions: (1) To what extent do middle-grade students agree on statements about the ordering of two negative integers? (2) What kind of strategies do middle-grade students generate to order two negative integers given within a real-life context? Related with the second research question, this study examines how those strategies can be classified as procedural or conceptual.

Context of the study

The context of the study might be understood better via curricular guidelines of ordering integers in Turkish middle school mathematics curriculum. The topic starts at the beginning of middle-grade levels. Objectives related to this topic are about interpreting integers, the meaning of absolute value, operations of integers, the meanings of the operations, comparing and ordering integers, solving integer-related problems, and relate exponential numbers to integers. Specifically, in learning to order integers, the objective says that: "Students should be able to compare and order integers" (MoNE, 2013, p.14). The curriculum advises teachers that the largest number is located to the right side in reference to the small number on the number line while ordering numbers. Therefore, it is seen that teachers are supported to use the link between number line location and integers' ordering. In other words, the curriculum supports 'direction' aspects of negative integers while the magnitude aspect of negative integers within real life situations (e.g.: temperature, asset and debt, elevator and so on.) are an application of objective presented in the middle school curriculum (MoNE, 2013). In this sense, in their instruction, students are given real-life contexts to understand how integers are represented within real life contexts in a meaningful way (MoNE, 2013). For ordering integers, the location of numbers on a number line is the basis of teaching the content (MoNE, 2013). In this respect, ordering

integers is not interpreted on the basis of real life examples. Rather than that, interpretation of the ordering integers mostly depends on the location of numbers on the number line.

Method

In this study, qualitative research method, specifically single case study approach, is used in order to reveal students' strategies and investigate students' understanding of contextual statements. The research method enables researchers analyzing data through creating a theme and codes (Creswell, 2005).

Sampling

Fifty-seven middle-grade students participated in the study. These students are selected under the purposeful sampling and they are studying at the middle-grade level in a public school in Ankara, Turkey. The school could be regarded as successful on the basis that it accepts students with having higher grades in nationwide held examinations. Before conducting the main study, a pilot study was conducted in two classrooms together with another fifty-four middle-grade students at the same grade level as the school the main study was conducted. The purpose of the pilot study is to make the statements understandable to the participants and to minimize the misunderstandings derived from the format of the questionnaire. The objectives which are aimed to be evaluated in the study require middle school students interpret integers within real life situations and to compare and order integers. Within this context, the content validity of the questionnaire is validated with mathematics teachers and one mathematics education instructor in the university.

Data collection tools

In this study to collect data the questionnaire given in Figure 1 is used. In order to understand middle grade students' general tendency of ideas about the different nature of ordering statements, students were asked whether they agree with the two ordering statements given in real life context. In addition, they were required to give details about the reason of agreement status in order to understand their reasoning in ordering two negative integers of strategies for explanation. As it is seen, there are two ordering statements given in the context of temperature. In this regard, questions were generated based on the context which middle grade students are familiar with and use the knowledge of ordering two negative integers by relating the direction and magnitude (quantity) aspects. The questionnaire has two statements pointing out two kinds of relations about ordering two negative integers. The first statement says that "As -10 degrees are less hot than -5, -10 is smaller than -5." The statement allows middle grade students to order two negative integers considering temperature as a quantity and direction. In other words, they can order two negative integers considering the mentioned degrees as similar to ordering two positive quantities. In line with this, -5 is more hot (more quantity of hot) than -10. In addition, temperature context enables students ordering integers considering direction integers. The second statement says that "As -10 degrees is situated on thermometer, lower than -5, -10 is smaller than -5. The statement allows students to interpret ordering of two negative integers based on their locations on the thermometer and their distance from zero. In this regard, whether they are using any conceptual or procedural strategies for using the nature of 'direction' and 'magnitude' meaning of two negative integers is examined.

"As -10 degrees are less hot than -5, -10 is smaller than -5. "	AGREE NOT AGREE NEUTRAL ○ ○ ○
	Why do you think so?
"As -10 degrees situated on thermometer lower than -5, -10 is smaller than -5.	AGREE NOT AGREE NEUTRAL ○ ○ ○
	Why do you think so?

Figure 1: Questionnaire items of the study

Data analysis

Similar and different categories of answers are grouped through content analysis. Explanation strategies are revealed by examining words or group of words students use in their answers (manifest content) and following this the underlying meaning of those wordings are revealed by investigating their explanation strategies deeply (latent content) (Fraenkel, Wallen, & Hyun, 2011, pp.483-484). Answers of the questions are analyzed for the agreement status, students' explanation strategies of the agreement status regarding conceptual and procedural strategies while explaining their responses. Data of the study are analyzed within Hiebert and Lefevre (1986)'s conceptual and procedural knowledge framework.

Findings

Agreement status of students about the ordering statements

Based on the analysis, it is seen in Table 1 that participants are agree with both of the statements.

Temperature Context	A	NA	N	NM	Total
1.statement	75	16	7	2	100
2.statement	66	14	16	4	100

*A: Agree, NA: Not Agree, N: neutral, NM: No marking

Table 1: The percentage of agreements to the ordering statements

The analysis of the results show that middle grade students agree with the idea that ordering two negative integers can be explained by using 'hot' concept together with the word 'less'. In this regard, they agree on the idea that two negative integers can be ordered by using the words 'more' and 'less' which express the quantity of something and came to the conclusion that 'hotter' is bigger than 'less hot'. Thus, for the first statement "As -10 degrees is less hot than -5 degree, -10 is smaller than -5", most of the students (75%) selected the choice *agree*. Parallel with this, for the second statement most of the students (67%) also agree on the idea that two negative integers can be ordered based on the location of the integers on the thermometer. In other words, the idea accepted by most of the participants is that the number which is below the other is much smaller.

The explanation strategies of middle grade students

Table 2 shows that the most preferred explanation strategies for the given statements are related to the network of hot and cold (37%), and rule-based explanations in reference to zero or positive numbers (18%). The strategy of *network of hot and cold* represents a relationship between the concepts of temperature, coldness and integer in the minds of students. In line with this, students make transitions among those concepts. For example, students form a link between negative integers and coldness saying if negative numbers increase the weather gets cold. The strategy of rule-based explanation in reference to zero or positive numbers is about rules with which students are familiar and which are created using zero and positive numbers like negative numbers are [ordered as] opposite to positive numbers. This table also depicts that students do not tend to use *rule-based explanations* in both of the statements. Put differently, *rule-based explanations* are not a dominant strategy for explaining those two ordering statements. They are used in the second statement, which mentions the location of numbers on the thermometer, compared to the first statement which is about interpreting the coldness, hotness, and their relationship. Another unexpected result is that although not many, some students did not consider this sort of order and they agree with the idea that "the number -10 is more than -5". In other words, they reject the order and think that -10 is bigger than -5 or -10 is hotter than -5. Similar to this, some students criticize the statements saying 'they are illogical' or 'they [both of the statements] are the same' and so on. Besides, some students used *the copy of the statement* strategy which is related to writing the same statements given in the questionnaire. In this regard, a substantial portion of students (44% for the first statement; 62% for the second one) do not employ any kind of explanation strategies for ordering integers in a given context. Students who copy the statement, use no strategies (e.g.: I don't know), use unclear statements (e.g.: I don't know why I am saying that I agree with the statements), and left the answer blank did not give reasons in writing as if they could not interpret the situation.

Conceptual and procedural strategies

Examination of students' written responses showed that students used conceptual and procedural strategies while explaining their agreement. Whether the strategy a student is used is related to procedures or concepts is determined considering how the concepts are interpreted in the statements, how transitions are made among the statements, and the content of the definition of Hiebert and Lefevre (1986). Students' responses of conceptual strategy were analyzed based on conceptual knowledge definition of Hiebert and Lefevre (1986). In this regard, *network of hot and cold* category was created when students interpret hot and cold concepts and make transitions between them saying that 'more hot' means 'less cold' etc. In addition to this, as Hiebert and Lefevre's definition for conceptual knowledge supports the relationship between pieces of information, *the network of hot and cold* was appropriate for this category. On the other hand, procedural strategy category was created based on Hiebert and Lefevre's (1986) procedural knowledge definition which supports the repertoire of basic factual knowledge and symbolic representation without interpretation of those facts and representations. In line with this, *rule-based explanation reference to zero or positive numbers* was categorized as procedural knowledge. *Rule-based explanation reference to zero or positive numbers* was related to the facts which are presented as context independent relations including facts of ordering two negative integers regarding their magnitude and direction.

Explanation Strategies		Example of students ' statements	Statements (%)	
			1.	2.
Conceptual strategy	Connected knowledge of zero	Closer to zero is connected to being hotter	2	6
	Network of hot and cold	The big number (-10) is colder than the smaller number (-5)	21	4
		As numbers increases hotness increases	5	2
		If negative numbers increase the weather gets cold	4	-
		-5 has more quantity of hot than -10	7	2
		Quantity of quicksilver is smaller at -10 degree	-	2
		Total	*39*	*16*
Procedural strategy	Rule-based explanation reference to zero or positive numbers	Bigger number is closer to zero (and the reverse of the statement)	9	14
		Numbers get smaller if you move to the left side of zero	4	4
		Negative numbers are ordered as opposite to positive numbers	5	-
		Numbers above the number line are bigger than the numbers below the number line	-	2
		Total	*18*	*20*
Other statements	Criticism of the content of the statement	No relationship between the given statements, they are contradictory to each other	5	9
	Rejecting the order	-10 is not less than -5	8	9
	Copying the statement	Student write the same statements given to them	7	14
	Unclear statements	For the reason that this is appropriate	7	1
		Drawings help ordering two negative integers	2	9
	Blank	-	5	4
	No strategy	I don't know	12	18
		Total	*46**	*64*
		Total answers*	100*	100

*One student suggested more than two ways for explanation

Table 2: Explanation strategies and conceptual and procedural strategies of students

Table 2 indicates that for the first statement, students used more conceptual strategies than the second one, and students used more procedural strategies for explaining the second statement than the first one. This might be derived from the nature of the statements which allow students to focus on the conceptual nature of the word 'less hot' and of allocating numbers on a number line. However, it seems that a considerable number of students used the procedural and conceptual strategies (20% and 16% for the first and second statement, respectively) regardless of the nature of the problem.

Most of the students used the *network of hot and cold* strategy with the conceptual strategy of ordering two negative integers using coldness (e.g.: -10 is colder than -5). In other words, most of the students made a transition from the word less hot' to the word 'colder'. As opposed to the expected interpretation of students, they explained ordering two negative integers considering the quantity of hotness concept (e.g.: -5 have more hotness than -10). Students used *connected knowledge of zero* strategy, for instance, saying that 'being closer to zero is connected to being hotter'. Similarly, in the first statement, the procedural strategy of 'bigger number is closer to zero' is used (e.g.: -5 is closer

to zero and so it is bigger and vice versa). Parallel with this, for the second statement, the same procedural strategy was mostly used by the participants.

Most of the participants supported the idea that two negative integers can be ordered by using *rule-based explanation in reference to zero or positive numbers*. In this regard, it might be said that students have pieces of knowledge about the nature of magnitude and direction of integers. Participants agreed that 1) the number which is located below the other on the thermometer is the smaller (direction) or (2) the number which is closer to zero is bigger than the other negative integer, which is less close to zero (magnitude).

Discussion and conclusion

In this study, the two statements about ordering allow students to see how they interpret those statements and what kind of strategies they use. Moreover, this study helps reveal the difficulties students encounter in their learning process. As seen in Table 1, students agreed with the given statements; however, Table 2 showed that middle grade students have variety of conceptual and procedural strategies that might not support their agreement. The results of the study show that their dominant strategies are spread over the sample of the students and are rarely used. Table 1 shows that most of the participants supported the idea that negative integers can be ordered by the nature of the amount of hotness that each integer is assigned. In other words, each negative integer is assigned to the concept of being hotter or being less hot. A possible explanation might be that students conceptualize ordering with the help of quantity or cardinal conception of numbers (Davidson, 1992). It means that students agree that ordering negative integers can be thought as similar to ordering positive integers when giving meaning to hotness concept. In this regard, for the first statement, students changed the word *hotness* to the word coldness and explained the statement based on coldness. Most of the students interpreted the situation by transferring hotness to the cold. While comparing two negative integers, they used the 'the hotter is less cold' or 'less hot is colder' relationships. Thus, these kinds of explanations indicate students' lack of interpretation of ordering as an amount of hotness because they might have a potential for interpreting the smaller number (-10) as a bigger number while ordering the concept of coldness a quantity. However, the relationship which indicates what being less cold or being colder means needs to be established within ordering context. Otherwise, it causes misconceptions or errors about misinterpreting what the integer statements or symbols mean (Ashlock, 2010). Moreover, this finding supports the idea that it is not easy to infer order relations from context-related statements, but teachers can integrate those strategies to classroom activities to establish a relationship between real life situations and negative numbers (Schindler & Hußmann, 2013). It is worth emphasizing that the procedural strategies illustrated in Table 2 might create faulty decision while comparing two negative integers. Students' understanding of negative numbers might be achieved by using the procedural strategies carefully being aware of the overgeneralization. For instance, the procedural strategy of 'bigger number is closer to zero' might be problematic when the number is a whole number.

Taken together, students have a variety of conceptual or procedural strategies that can be used for interpreting ordering in real life situations. Those strategies are important to make instruction better and to facilitate student understanding. In future studies, the meaning of ordering negative integers within real life contexts and in mathematics can be examined to explain some students' lack of interpretations of the given statements.

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TWG03: Algebraic thinking

Introduction to the papers of TWG03: Algebraic thinking

Dave Hewitt[1], Reinhard Oldenburg[2], Valentina Postelnicu[3], and Heidi Strømskag[4]

[1]University of Loughborough, UK,
[2]University of Augsburg, Germany,
[3]Texas A&M University-Corpus Christi, USA,
[4]Norwegian University of Science and Technology, Trondheim

Keywords: Algebra, early algebra, functions, research frameworks, teaching and learning algebra.

In CERME-10, the Thematic Working Group 3 "Algebraic thinking" continued the work carried out in previous CERME conferences. There were a total of 16 papers and 5 posters with a total of 29 group participants representing countries from Europe and other continents: Canada, Finland, Germany, Greece, Ireland, Norway, Portugal, Spain, Sweden, Tunisia, Turkey, UK, and USA.

Recurring issues

While the importance of algebra education is universally acknowledged, the problem of teaching it successfully to most students is not yet solved. Thus, there is a need to go back to basics over and over again and a lot of issues occur repeatedly in the history of CERME working groups on algebra. A broader overview is given in Hodgen, Oldenburg and Strømskag (2017). The discussion during CERME 10 brought up the following fundamental issues:

- What is algebra? There is still no uniform definition of what is the particularity of this field and what are the relations to other mathematical fields like combinatorics or geometry (that use letters as well). Moreover, many notions are not fully defined.
- How can it be empirically determined what works? We still have no universal measures of algebraic competence. Hence, many ad hoc tests are used.
- What should be taught? Too little is known about how knowledge builds up in the long term. For instance, it may be that certain concepts and metaphors that work well in some grades will give raise to obstacles later on.

Regarding the first point in this list, the group discussed the question of whether it would be sensible to rename the group's title just to "Algebra", because the word "thinking" gives the cognitive a higher weight than it might deserve. But this was resolved by the shared understanding that "algebraic thinking" is interpreted to include language, affect and possibly further factors.

The second point was taken up in a series of discussions about the quality of research and communication. Conceptual validity is seen often to be a problem. To rely just on Cronbach's alpha to ensure internal consistency seems not adequate. Perhaps the community should ensure that whole tests, measurement instruments and data are made accessible for other participants? Still, it will be difficult to ensure a common understanding of notions, given the plurality of theories and terminology used.

Despite these questions, there are substantial areas where a consensus has been reached: It is accepted that early algebra "works", in the sense that it is possible to develop algebraic thinking using, or just beginning to use, formal symbolic notation. Furthermore, most researchers see structure as a guiding principle in algebra and especially the structure of equations and the role of the equal sign is identified as central. The context/environment of each research event is relevant and especially the tasks and its implementation by the teacher are crucial together with the role of the researcher. Regarding ideas

for the curriculum, we agreed that equation solving should not start with too simple equations. Filloy and Rojano's (1989) distinction between arithmetical and algebraic equations is important to exemplify the domain in which algebraic methods can show there power to students.

Some comments on issues dealt with in the papers

Functions have been identified my many colleagues as central issue in algebraic thinking and hence we have seen several papers (Isler et al., Pinto & Cañadas, Postelnicu, and Weber) that deepen the understanding of functions.

Isler et al. report results from a quantitative study in the US of Grade 6 students' written work on a functional thinking assessment item. The results show that students who experienced an early algebra intervention during Grades 3-5 were more likely to successfully represent a function rule in words and variables than students who did not. Also, both comparison and intervention groups of students were found to be more successful representing a function rule in variables than in words. The results underscore the impact of early algebra on students' later success in algebra, and challenge the view that the concept of variable should not be introduced until secondary school.

Pinto and Cañadas report from a study of 24 Spanish Grade 3 students' functional thinking during engagement with a contextualised linear problem (placing tiles). Two types of functional relationships were identified—*correspondence* and *covariation*—and the ability to generalise was observed in some of the students. The study was part of a broader teaching experiment, and the data were collected through a task-based questionnaire.

Postelnicu conducted a study of 58 US high school students' difficulties with writing equations of parallel and perpendicular lines (in the context of Algebra 1). Chevallard's theory of didactic transposition was employed to account for the relativity of the mathematical knowledge with respect to the institutions where the knowledge was created. The analysis shows that the mathe-matical knowledge (through the didactic transposition) lost its essential feature—the *proof*—with serious consequences for the curriculum. What remained to be learned was how to execute tasks.

Weber presents a theoretical paper, where vom Hofe's construct of 'Grundvorstellungen' and Sfard's distinction between operational and structural conceptions are used to analyse structural and operational models of logarithmic functions. Weber claims that logarithmic functions should not be introduced structurally, as inverse exponential functions. Instead, several operational models of the logarithmic concept are proposed, and their explanatory power for graphing is expounded.

Zindel presents a model for conceptualizing the core of the function concept, which is made up of those facets that are equally important for all types of functions and common to all representations. The so-called *facet model* enables the identification of potential obstacles and a detailed description of students' learning processes when connecting representations (e.g., verbal and symbolic representations when solving word problems). In total, 19 design experiments with overall 96 learners (mainly Grades 9-10) were conducted and qualitatively analyzed.

A focus on the thinking in algebraic thinking has been laid by four papers: Palatnik and Koichu; Twohill; Soneira, González-Calero and Arnau; and, Proulx.

Palatnik and Koichu took a detailed view on how students make sense of formula they found on various ways. The authors found that the process of sense making is consists of formulating and

justifying claims, making generalizations, finding mechanisms and established coherence among the explored objects.

Twohill investigated number sequences from geometric patterns and the path of students to general terms. It turned out that between figural and numerical aspects of the patterns there is a whole continuum of ways that students think about these sequences. It is not easily said what aspects students should look at to be successful in finding a proper generalization.

Soneira et al. investigated in details the well-known error that students might produce expressions in which different occurrings of the same variable have different (but often related) reference. They explain this by idiosyncratic semiotic systems used by the students. The process of translation between algebra and natural language is highly complex.

Proulx investigated how teachers and students solve algebraic problems mentally. Forcing them not to use paper and pencil or other techniques allows to get close to their thinking. This revealed a wide variety of approaches and students and teachers differed in these. In the end, a sense for the diversity should be developed especially by the teachers.

Röj-Lindburg et al. considered the transition from informal to formal methods of equations solving in Grade 6 (12 years old) in Finland. The approaches taken by three teachers were analysed. One teacher used the image of a balance scale; another used uncomplicated 'real-world' situations; and the third had an emphasis on formal methods, in particular the need to 'do the same thing on both sides'. The third teacher's lesson was analysed and concluded that the discussion focused strongly on memorizing the procedure and did not develop an algebraic understanding of equality. In fact, it was concluded that in none of the teachers' lessons was there a need for students to adopt an algebraic way of thinking about equality.

Steinweg brought out the fact that the mathematics teaching units in Germany primary education lack explicit algebra learning environments. She offered ways in which key algebraic ideas can be used as guiding principles to rethink 'arithmetic' topics in six German primary school classes so that they can be used as learning environments for algebraic thinking. She focused on work from a pupil who was working on a task to decompose the area of a given rectangle and who appeared to show an awareness of the inherent distributive structures. Pre- and post-tests showed an increase in the percentage of children giving answers deemed to be algebraic in nature.

Papadopoulos and Patsiala studied the use of a particular learning environment called "Father Woodland" with seventy Grade 3 students (8-9 year olds) from two different primary schools in Greece. The approaches taken by the students were categorized into four types and it was noted that over the course of eight tasks, there was increased use of approaches which were classified as either 'combining words and symbolic language' or 'using symbolic language to express relationships". An argument was made that the environment helped develop the students' algebraic thinking.

As mentioned above, several researchers were concerned with the issues of "what should be taught" and what constitutes proficiency from the learners' points of view. Pinkernell, Düsi and Vogel proposed a way to construct validity for the concept of proficiency in elementary algebra, and presented the methodology of constructing a "model" of proficiency, together with the resulting

product – the "revised model." Wladis and colleagues described an instructor-generated "concept framework" for elementary algebra in the tertiary context.

Chimoni and Pitta-Pantazi addressed the issue of determining empirically "what works" for teaching algebra. They conducted a study with 96 early algebra students and compared two intervention courses. The first intervention course included real life scenarios and semi-structured tasks, while the second intervention course involved mathematical investigations and structured tasks. The results showed that the first course had better learning outcomes.

Two papers reported on structural aspects of algebra, at the elementary and university level, respectively. Strømskag and Valenta addressed the issue of justifying the commutative property of multiplication of natural numbers for Grade 6 students. At the heart of the study were the limitations of the visual representation used by the observed student teacher to help her students justify the property of commutativity of multiplication. Mutambara and Bansilal investigated the understanding of the concept of vector subspace. Participating students were 84 in-service teachers enrolled in a mathematics course at a Zimbabwean university. The action, process, object schema (APOS) theory, based on Piaget's genetic epistemology, was proposed for the analysis of two tasks. The results highlighted the teachers' difficulties with the concepts of sets, matrices, and vector subspace.

Outlook

The synopsis of papers given above shows the wide variety of theories, topics and methods used in this group. Such a pluralistic situation is highly welcomed as it allows to test the validity of research results from multiple perspectives. Thus, the consensus described above, can be viewed as solidly grounded and form the base for further research that can and should address questions that are not yet understood well enough. One such area is the domain of high school algebra. Weber's paper has shown the potential of better understanding such concepts. Another point to be developed further is the perspective of teachers and teacher education.

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The effect of two intervention courses on students' early algebraic thinking

Maria Chimoni and Demetra Pitta-Pantazi

University of Cyprus, Department of Education; chimoni.maria@ucy,ac,cy, dpitta@ucy.ac.cy

The aim of this study is to investigate the nature and content of instruction that may facilitate the development of students' early algebraic thinking. 96 fifth-graders attended two different intervention courses. Both courses approached three basic content strands of algebra: generalized arithmetic, functional thinking, and modeling languages. The courses differed in respect to the characteristics of the tasks that were used. The first intervention included real life scenarios, and semi-structured tasks, with questions which were more exploratory in nature. The second intervention course involved mathematical investigations, and more structured tasks which were guided through supportive questions and scaffolding steps. The findings, yielded from the analysis of pre-test and post-test data, showed that the first course had better learning outcomes compared to the second, while controlling for preliminary differences regarding students' early algebraic thinking.

Keywords: Early algebraic thinking, teaching intervention, tasks.

Introduction

In response to calls for spreading the teaching and learning of algebra throughout K-12 grades (e.g. NCTM, 2000), a wealth of studies focused on the design and implementation of instructional interventions that facilitate the development of early algebraic thinking (e.g. Blanton & Kaput, 2005; Irwin & Britt, 2005; Warren & Cooper, 2008). These studies offered strong evidences that students are able to develop algebraic thinking as early as the primary grades. Moreover, these studies highlighted the key role of teachers in providing their students with rich opportunities to investigate and understand algebraic ideas from elementary school.

As Kieran, Pang, Schifter and Ng (2016) highlighted, a large number of past studies identified pattern generalization and functional thinking as important routes that foster the development of early algebraic thinking; however, little research has involved other aspects of algebra, such as properties of numbers and operations. This bring us to the question of the effectiveness of intervention courses that might involve a range of algebra content strands, such as functional thinking and generalized arithmetic. There is, therefore, still a need to extend our understanding of supportive instruction that aims to improve students' early algebraic thinking and further clarify the content of a corpus of lessons that capture the core content strands of algebra.

The current study addresses this issue. Furthermore, considering the suggestions of recent literature regarding the impact of different types of tasks on students' learning (e.g. Swan, 2011), this study raises the question of whether the nature and content of the tasks used in instructional interventions, regarding their structured or semi-structured nature, and the association of their context to real life scenarios or not, might affect students' early algebraic thinking.

Theoretical framework

The notion of early algebraic thinking

Several research studies addressed the multidimensional nature of early algebraic thinking. Kaput (2008) claimed that there are three fundamental content strands of algebra: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of modeling languages. Generalized arithmetic, involves generalizing rules about relationships between numbers, manipulating operations and exploring their properties, transforming and solving equations, and understanding the equal sign in number relations. Functional thinking, refers to the identification and description of functional relationships between independent and dependent variables. Modeling, refers to the generalization of regularities from mathematized situations or phenomena inside or outside mathematics.

The notion of early algebraic thinking has also been associated through literature with several mathematical processes. For example, Kieran (2004) suggested that early algebraic thinking is linked to problem solving, modeling, working with generalizable patterns, justifying and proving, making predictions and conjectures, analyzing relationships, and identifying structure.

Early algebraic thinking, therefore, is expected to emerge through intervention courses that capture a variety of areas and contexts related to algebra and assist students to use a range of mathematical processes.

Sources of meaning in algebraic problems and the importance of the nature of tasks

Radford (2004) specified that there are three main sources of meaning within algebraic problems that trigger the development of early algebraic thinking: (a) the algebraic structure itself (e.g. the letter-symbolic representations, graphical representations), (b) the problem context (e.g. word problems, modeling activities) and (c) the exterior of the problem context (e.g. social and cultural features, such as language, body movements, and experience). Hence, the specific characteristics of these sources might facilitate or not the construction of meaning when students participate in algebra lessons.

Additionally, existing literature on the importance of tasks that students are engaged with, has shown that the nature and features of mathematical tasks influence learning, since they direct students' attention to specific content and specific ways of processing information (Jones & Pepin, 2016). For example, Sullivan, Clarke and Clarke (2012) suggested that problem-like tasks have a positive effect on students' mathematical thinking rather than step-by-step procedures. In this perspective, the extend to which a task involves problems that are more or less structured, is associated with an open question or a series of scaffolding questions, and represents situations related to students' experiences within real life contexts or not, may influence students' development of early algebraic thinking.

Aim of the study

The aim of this study is the investigation of the effect of two different intervention courses in improving students' early algebraic thinking. Both courses involved the three content strands of algebra suggested by Kaput (2008), had the same duration, and were based on the inquiry-based learning approach. Nevertheless, they were implemented through different types of tasks. The first course, which we named "Semi-structured problem situations", used semi-structured tasks connected to real life scenarios and required students to identify the mathematics involved in order to answer to their main question. The second course, which we named "Structured mathematical investigations",

used more mathematical tasks that were supported by scaffolding questions. Hence, the two teaching interventions were compared in relation to the types of the tasks through which algebraic thinking was expected to emerge.

Methodology

Participants

The participants were 96 fifth-graders from 4 classes in 2 urban schools. The classes were selected by convenience. Two of the classes (one class from each school) formed the group that participated in the first course and the other two classes formed the group that participated in the second course.

Test on early algebraic thinking

The same test was administered to the students before and after the conduction of the courses in order to measure their early algebraic thinking. The test consisted of 22 tasks that were accordingly categorized into three groups which reflected Kaput's (2008) three content strands of algebra. Table 1 presents examples of the tasks in each category. The first group (generalized arithmetic) involved tasks, such as determining whether the sum of two numbers will be odd or even, using the properties of operations, describing movements in the hundredths' table, and solving equations and inequalities. The second group of tasks involved finding the nth term in geometrical and numerical patterns, interpreting graphs, and describing co-variational and correspondence relationships among quantities. The third group of tasks (modeling) required the generalization of regularities by observing the relationships involved in realistic situations. The internal consistency of scores measured by Cronbach's alpha was satisfactory for the test (a=0.87).

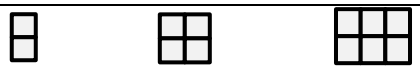
Algebraic Thinking as Generalized Arithmetic	Is the sum ***245676 + 535731*** *an* odd or even number? Explain your answer.
Algebraic thinking as functional thinking	 *Figure 1* *Figure 2* *Figure 3* Bill is arranging squares. How many squares there will be in the 16th figure? Show your work.
Modeling as a domain for expressing and formalizing generalizations	Joanna will take computers lesson twice a week. Which is the best offer? Justify your answer. **OFFER A**: €8 for each lesson **OFFER B**: €50 for the first 5 lessons of the month and then €4 for every additional lesson

Table 1: Examples of tasks included in the early algebraic thinking test

Teaching experiments

Both intervention courses addressed the same concepts and objectives, and were developed through ten lessons of 80-minutes duration each. The first researcher taught all the lessons. Table 2 presents the objectives of the lessons in each strand.

Lessons	Content strand	Objectives
3,4	Generalized arithmetic	Apply properties and relationships of whole numbers, apply properties of operations on whole numbers, treat numbers by attending structure rather than computations
1,2,6,7	Functional thinking	Encode information graphically for analyzing a functional relationship, identify correspondence or co-variation relationships, identify numerical and geometrical patterns
5,8,9,10	Modeling languages	Generalize regularities from mathematized situations inside or outside mathematics

Table 2: Structure of Instructional Interventions and Objectives for each Lesson

The "Semi-structured problem situations" course used semi-structured problems arising from real life situations. Students were confronted with a general question and were given time to explore the problem situation, analyze and combine information and apply their own strategies to solve the task. These tasks employed some features of modeling-like tasks. Specifically, modeling-like tasks were considered as appropriate for enhancing the development of algebraic thinking because they involve the description and interpretation of complex systems of information through the application of processes such as, constructing, explaining, justifying, predicting, generalizing, conjecturing, and representing (English, 2011).

The "Structured mathematical investigations" course reflected mathematical contexts that aimed to direct students to identify structure and relationships in mathematical concepts. Specifically, these tasks were more mathematical in nature, involved scaffolding steps and pathways which guided students to the extraction of an explicit conclusion. This kind of activities were considered as relevant and important for enhancing algebraic thinking since they apply fundamental processes, such as formulation and expression of relationships and generalizations, and progressive symbolization.

In order to ensure the content validity of the tasks we used for both interventions tasks of previous studies (e.g. Blanton & Kaput, 2005) or online resources (e.g. https://illuminations.nctm.org/) which seem to be well accepted by researchers and mathematics educators. Moreover, the authors consulted two other mathematics education experts about their judgment regarding the content validity of the tasks until consensus was reached. Figure 1 presents examples of tasks from each intervention course. The task on the left was adapted from a lesson presented in the website https://illuminations.nctm.org. Using a context of arranging chairs around tables, students were exposed to two different linear patterns. As specified in the website, this activity leads to an intuitive understanding of how to extend and describe a pattern using words or symbols. The task on the right was adapted from a lesson presented in the website www.explorelearning.com. Students studied different patterns of squares in a grid. Each new pattern was more complex compared to the previous pattern (The pattern presented in Figure 1 was the third pattern). As stated in the website, this activity aims to the extension of figural patterns and the extraction of a general rule. In this sense, both tasks targeted on the description and generalization of figural and numeric patterns. However, the first task introduced from the beginning a complex pattern; the second started from a simple pattern and moved to more complex patterns.

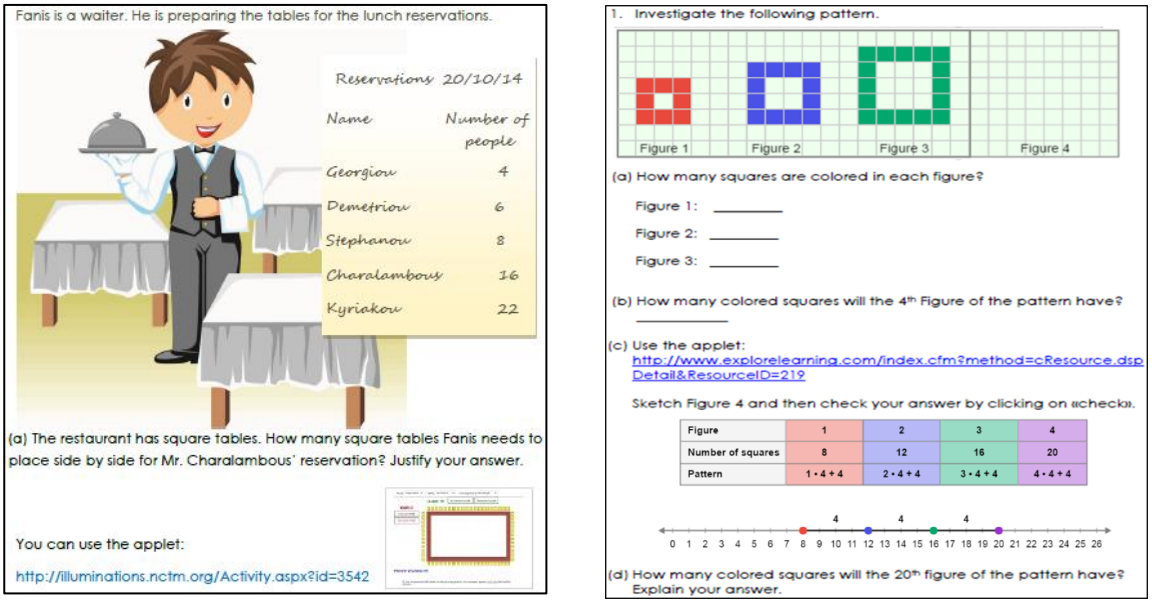


Figure 1: Semi-structured problem situation (left) and Structured mathematical investigation (right)

Analysis

The SPSS statistical package was used to analyze the results. Since the tasks in the pre-test and post-test were the same, gain scores were used (the difference between post-test and pre-test scores) as the dependent variable. The Kolmogorov-Smirnov and Shapiro-Wilk tests showed that the gain scores were normally distributed ($p>.01$). The P-P and Q-Q plots did not show crucial variations. In order to compare the early algebraic thinking abilities of the two groups prior to the intervention, a multivariate analysis of variance (MANOVA) was conducted. MANCOVA was used to examine the impact of the intervention courses on participants' early algebraic thinking. The type of intervention was the independent variable, students' performance in early algebraic thinking pre-test was considered as the covariate, and the performance differences between the pre- and post- tests as the dependent variables. Moreover, paired-sample t-test was performed in order to measure the differences in the performance of students of the same group in the pre- and post-tests.

Results

The results of the MANOVA analysis suggested that the two groups did not have any statistically significant differences in their early algebraic thinking abilities prior to the intervention ($F=.576$, $p>.05$). Table 3 presents the results of the MANCOVA analysis, regarding the comparison of the impact of the two teaching experiments on the groups' performance in the early algebraic thinking post-test, controlling for their pre-test scores.

The analysis indicated significant overall intervention effects, controlling for pre-test scores in the early algebraic thinking test (Pillai's $F=9.586$, $p<.05$). The students in the "Semi-structured problem situations" group had a significantly higher overall performance in early algebraic thinking to students in the "Structured mathematical investigations" group. The effect size indices for the overall algebraic thinking ability (partial $n^2=.088$) suggested that the effect of the "Semi-structured problem situations" course over the "Structured mathematical investigations" course was moderate. The performance of the "Semi-structured problem situations" group in the generalized arithmetic tasks did not have any significant difference in relation to the performance of the "Structured mathematical investigations"

group (Pillai's F=.081, p>.05). The "Semi-structured problem situations" group had significantly higher performance in the functional thinking tasks (Pillai's F=26.845, p<.01) and the modeling tasks (Pillai's F=9.804, p<.05) in comparison to the "Structured mathematical investigations" group. The effect size indices for the functional thinking tasks (partial n^2=.286) and the modeling tasks (partial n^2=.128) suggested that the effect of the "Semi-structured problem situations" course over the "Structured mathematical investigations" course was moderate.

	Structured		Semi-structured					
Ability	Mean[1]	*SE*	Mean[1]	*SE*	*df*	*F*	*p*	n_p^2
Overall Performance	.452	.206	.570	.179	1	6.452	.013*	.088
Generalized Arithmetic	.663	.213	.647	.246	1	.081	.777	.001
Functional Thinking	.369	.225	.547	.270	1	26.845	.000**	.286
Modeling	.291	.291	.509	.319	1	9.804	.003*	.128

[1] Estimated Marginal Means, *p<.05, **p<.01

Table 3: Results of the Multiple Covariance Analysis between the Two Intervention Groups Post-test Performance in Early Algebraic Thinking

Table 4 presents the results of the paired-samples t-test regarding the differences in the pre- and post-test scores within the same group.

Ability		Pre-test		Post-test			
		M	SD	M	SD	T(df)	p
Overall Performance	Structured	.337	.195	.452	.206	-5.519(33)	.000**
	Semi-structured	.368	.151	.570	.179	-10.147(34)	.000**
Generalized Arithmetic	Structured	.467	.326	.663	.213	-4.112(33)	.000**
	Semi-structured	.473	.235	.647	.246	-4.818(34	.000**
Functional Thinking	Structured	.302	.263	.369	.225	-2.774(33)	.09
	Semi-structured	.404	.228	.547	.270	-5.663(34)	.000**
Modeling	Structured	.223	.241	.291	.291	-1.231(33)	.227
	Semi-structured	.183	.202	.509	.319	-9.926(34)	.000**

**p<.01

Table 4: T-test Comparisons between Pre-test and Post-test Performance of the two groups

The results showed statistically significant differences between the pre- and post-tests performance means of the "Structured mathematical investigations" group. Students in this group had a significant increase in their overall early algebraic thinking ability and in the generalized arithmetic tasks. The results also showed that no statistically significant differences existed between pre- and post-tests

performance means in the functional thinking and modeling tasks. Regarding the "Semi-structured problem situations" group, the results showed statistically significant differences in the mean difference between the pre- and post-tests means of performance. These students had a significant increase in their overall ability and in all types of tasks.

Discussion and conclusion

This study compared the effect of two intervention courses on students' early algebraic thinking. The results showed that instruction with "Semi-structured problem situations" had better learning outcomes compared to instruction with "Structured mathematical investigations", while controlling for preliminary differences regarding students' early algebraic thinking. Specifically, students who received instruction through the "Semi-structured problem situations" outperformed students who received instruction through the "Structured mathematical investigations" in the early algebraic thinking post-test. Nevertheless, more detailed results regarding the effect of the two types of courses have shown that both of them had positive impact in the generalized arithmetic strand. What seems to have influenced the overall outcome of the comparison between the two courses is the fact that students involved in the "Semi-structured problem situations" course had significantly higher performance in the functional thinking and modeling strands.

A possible explanation for this result seems to be the fact that the two intervention courses involved different types of tasks in respect to the way algebraic thinking was expected to emerge. While both interventions had high cognitive demands and were developed through activities that entailed cooperative learning, use of manipulatives, and technological tools, it appears that the nature and type of the tasks used had a significant role regarding the learning outcomes. As suggested by Stein and Lane (1996), the tasks determine not only the concepts and knowledge that students acquire but also the way students will come to process, use and make sense of those concepts and knowledge.

On the one hand, the tasks that were included in the "Semi-structured problem situations" course shared common features with modeling approaches to mathematical problem solving. As English (2011) described, modeling-like tasks offer enriched learning experiences that require students to extract meaning from open situations by mathematizing the situations in ways that are meaningful to them. This kind of processes are linked to early algebraic thinking. As Kieran (2004) supported, early algebraic thinking is related to several processes, including problem solving, modeling, justifying, proving, and predicting. Hence, modeling-like tasks seem to involve the majority of the processes that are related to early algebraic thinking. On the other, "Structured mathematical investigations" tasks appeared to be effective in helping students to notice the structure in arithmetical contexts and engage students to learning experiences that are mostly focused on the generalized arithmetic strand.

As Radford (2004) argued, the algebraic structure of a problem (e.g. the letter-symbolic representations), the problem context (e.g. word problems, modeling activities) and the exterior of the problem context (e.g. social and cultural features, such as language, body movements, and experience) constitute basic sources that students utilize in order to extract meaning. The results of the current study indicated that the "Semi-structured problem situations" tasks encompassed all of these sources in an effective way and enabled students to construct their own meaning and develop understanding of various algebra aspects. Thus we may say that the positive effect of an intervention course is in a great extend related to the design and implementation features of the tasks involved.

Future research might further investigate whether the effect of "semi-structured" or "structured" tasks is different with younger or older students. The effect of an intervention course that makes use of both "semi-structure" and "structured" tasks might also be addressed. Moreover, the qualitative characteristics of students' behavior while participating in this kind of intervention courses needs to be investigated in detail, in order to better understand the nature of thinking they develop.

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Grade 6 students' abilities to represent functional relationships

Isil Isler[1], Susanne Strachota[2], Ana Stephens[3], Nicole Fonger[4], Maria Blanton[5], Angela Gardiner[6], and Eric Knuth[7]

[1]Middle East Technical University, Ankara, Turkey; iisler@metu.edu.tr

[2]University of Wisconsin-Madison, Madison, WI, U. S.; sstrachota@wisc.edu

[3]University of Wisconsin-Madison, Madison, WI, U.S.; acstephens@wisc.edu

[4]University of Wisconsin-Madison, Madison, WI, U.S.; nfonger@wisc.edu

[5]TERC, Cambridge, MA, U.S.; maria_blanton@terc.edu

[6]TERC, Cambridge, MA, U.S.; angela_gardiner@terc.edu

[7]University of Wisconsin-Madison, Madison, WI, U.S.; knuth@education.wisc.edu

This paper reports results from Grade 6 students' written work on a functional thinking assessment item. The results show that students who experienced an early algebra intervention during Grades 3-5 were more likely to successfully represent a function rule in words and variables than students who did not. Also, both comparison and intervention groups of students were found to be more successful representing a function rule in variables than in words. The results underscore the impact of early algebra on students' later success in algebra, specifically with functional thinking, and challenge the notion that variable as a varying quantity should not be introduced until secondary school.

Keywords: Early algebra, algebraic thinking, secondary school.

Background of the study

Algebra in the U.S. has long served as a gatekeeper to future academic and employment opportunities (Ingels, Curtin, Kaufman, Alt, & Chen, 2002). Thus, recent reform efforts have sought to integrate aspects of algebra into the elementary curriculum (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010). By algebra in the elementary grades (hereinafter, *early algebra*), we do not mean an add-on to the existing curriculum or a pre-algebra course that is typically taught at the secondary level. We follow instead the definition put forward by Blanton el al. (2007):

> [Early algebra is] a way of thinking that brings meaning, depth and coherence to children's mathematical understanding by delving more deeply into concepts already being taught so that there is opportunity to generalize relationships and properties in mathematics. (p. 7)

Through a series of interrelated projects, Project LEAP [Learning through an Early Algebra Progression], has developed and investigated the efficacy of a learning progression for early algebra in Grades 3-5 (Grade 3 is the fourth year of elementary/primary school in the U.S. Students in this grade are typically 8-9 years old; Grade 5 students are typically 10-11 years old).

In the first project, we focused on developing the learning progression and its components: a curricular framework and progression, instructional sequence, written assessments, and levels of sophistication describing students' strategy use (see Fonger, Stephens, Blanton, & Knuth, 2015 for

more information about the development of the learning progression). This project also examined the effectiveness of the early algebra intervention (see Blanton et al., 2015 for the results of this study).

In the second project, from which the data for this paper came, we used a quasi-experimental design to follow two groups of students—one designated comparison and one designated intervention—across Grades 3-5. The intervention students were taught about 18 one-hour weekly early algebra lessons each year. These lessons replaced their mathematics instruction thus the total time spent on mathematics instruction remained unchanged. The lessons were taught by a member of the project team, a former Grade 3 teacher. Each lesson started with a "jumpstart" at the beginning of the class that included a review of previously-discussed topics. The lessons continued with group work and whole-class discussions centered on research-based tasks. The algebraic concepts were often revisited in the "jumpstarts" so that the intervention students were provided opportunities to revisit several concepts throughout the intervention. The comparison students did not receive any intervention and were exposed only to their traditional mathematics curriculum throughout the three years. Written assessments were administered to measure students' early algebra understandings and skills at four time points: at the beginning of Grade 3 and at the end of Grades 3, 4 and 5 (see Blanton, Isler, Stephens, Gardiner, et al., 2016 for the preliminary results of this study). A year after this longitudinal study ended, both the intervention and the comparison students were administered a written assessment in Grade 6 (when students were about 11-12 years old). This paper reports results from a functional thinking item that was common across Grade 5 and 6 written assessments.

Theoretical framework

Kaput (2008) conceptualized algebra as: (1) the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and quantitative reasoning, (2) the study of functions, relations, and joint variation, and (3) the application of a cluster of modeling languages both inside and outside of mathematics. Building on Kaput's content strands, we identified three "big ideas" (Shin, Stevens, Short, & Krajcik, 2009) of early algebra: generalized arithmetic; functional thinking; and equivalence, expressions, equations, and inequalities. In addition to the content strands for algebra, Kaput (2008) also identified algebraic thinking practices. We organized the content strands around four algebraic thinking practices: (1) generalizing, (2) representing, (3) justifying, and (4) reasoning with mathematical structure and relationships.

The focus of this paper is on students' abilities to engage in the algebraic thinking practices of generalizing and representing in the context of the "big idea" functional thinking. Blanton, Levi, Crites and Dougherty (2011) described functional thinking as "generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior" (p. 47). In this paper, we explore students' abilities to represent functional relationships in words and variables.

Methods

Participants

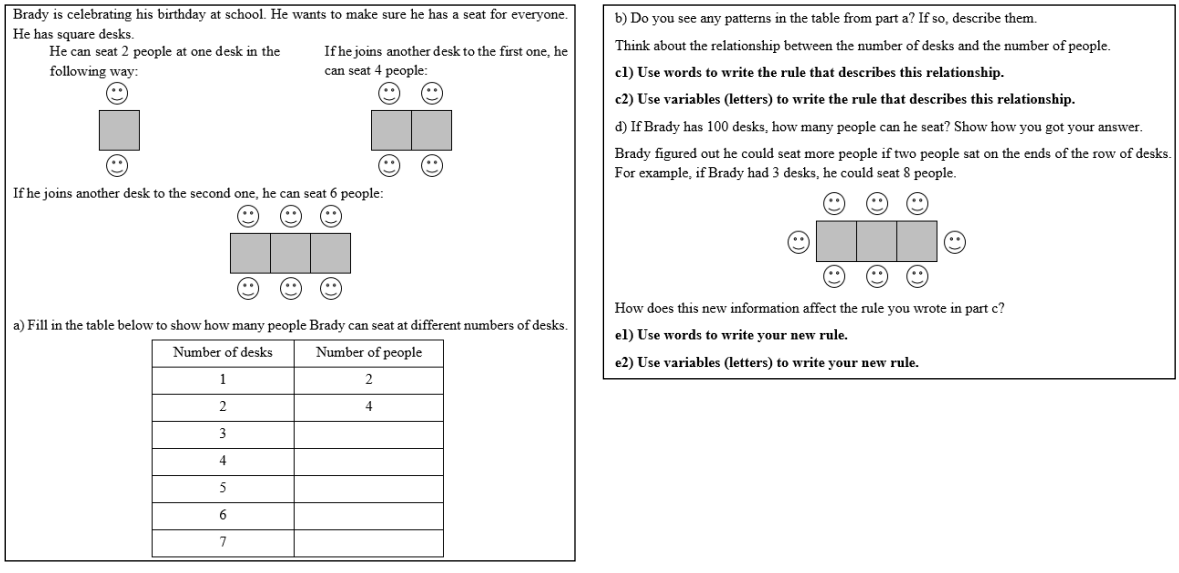
The participants were 80 Grade 6 students, 46 of whom were part of the intervention that took place during Grades 3-5 and 34 of whom were part of the comparison group in those grade levels. All of the students were from the same middle school. The teachers of these students reported using the *Connected Mathematics Project* (CMP3) curriculum for their mathematics instruction. The demographics for the district are 8% non-white, 5% English Language Learners, and 20% low socioeconomic status students.

Data collection and analysis

An early algebra assessment was administered to students at the end of Grade 6. The assessment items were developed and validated in a prior project. The assessment consisted of 11 items, most including multiple parts that addressed the aforementioned big ideas, and took students approximately one hour to complete.

We focus on one item, the Brady task (parts c1, c2, e1, and e2) (see Figure 1), which was designed to assess student's functional thinking, one of the big ideas, fundamental to our learning progression. The student responses were coded for both correctness and strategy use (correct or incorrect). The strategies that are the focus in this paper are listed in Figure 2. For more information about the coding scheme for strategy use for this item and the levels of sophistication observed in students' written work, see Blanton et al. (2015), Stephens et al. (in press), and Strachota et al. (2016).

A second coder conducted reliability coding for all items and any disagreements were discussed and resolved until 80% inter-rater agreement score was reached for all items.


Brady is celebrating his birthday at school. He wants to make sure he has a seat for everyone. He has square desks.

He can seat 2 people at one desk in the following way:

If he joins another desk to the first one, he can seat 4 people:

If he joins another desk to the second one, he can seat 6 people:

a) Fill in the table below to show how many people Brady can seat at different numbers of desks.

Number of desks	Number of people
1	2
2	4
3	
4	
5	
6	
7	

b) Do you see any patterns in the table from part a? If so, describe them.

Think about the relationship between the number of desks and the number of people.

c1) Use words to write the rule that describes this relationship.

c2) Use variables (letters) to write the rule that describes this relationship.

d) If Brady has 100 desks, how many people can he seat? Show how you got your answer.

Brady figured out he could seat more people if two people sat on the ends of the row of desks. For example, if Brady had 3 desks, he could seat 8 people.

How does this new information affect the rule you wrote in part c?

e1) Use words to write your new rule.

e2) Use variables (letters) to write your new rule.

Figure 1: The Brady task

Strategy Code	Description	Example
Parts c1 and e1		
Functional-Condensed in Words	Student identifies a function rule in words that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.	Part c1: *The number of people is 2 times the number of desks.* Part e1: *The number of people is 2 times the number of desks plus 2.*
Parts c2 and e2		
Functional-Condensed in Variables	Student identifies a function rule using variables in an equation that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.	Part c2: $2 \times d = p$ Part e2: $2 \times d + 2 = p$

Figure 2: Functional-Condensed in Words and Functional-Condensed in Variables strategies for the Brady task

Next, we focus on the results regarding the Grade 6 students' abilities to represent the function rules in words and variables (parts c1, c2, e1, and e2 of the Brady task) by comparing the performance of students who were exposed to the early algebra intervention during Grades 3-5 (n = 46) to the performance of students who were not part of any early algebra intervention (n = 34). We also compare Grade 6 results to the results we obtained at the end of the three-year intervention in Grade 5 (n = 90 for the intervention group and n = 61 for the comparison group).

Results

Results for parts c1 and c2 showed that by the end of Grade 6, the intervention students used the functional-condensed in words strategy in part c1 and the functional-condensed in variables strategy in part c2 more frequently than the comparison students (48% vs. 26% for part c1 and 65% vs. 41% for part c2) (see Figure 3). Moreover, both the intervention and the comparison students were found to be more successful representing the function rule in variables than in words (65% vs. 48% for the intervention group and 41% vs. 26% for the comparison group, respectively stating the rule in variables vs. words).

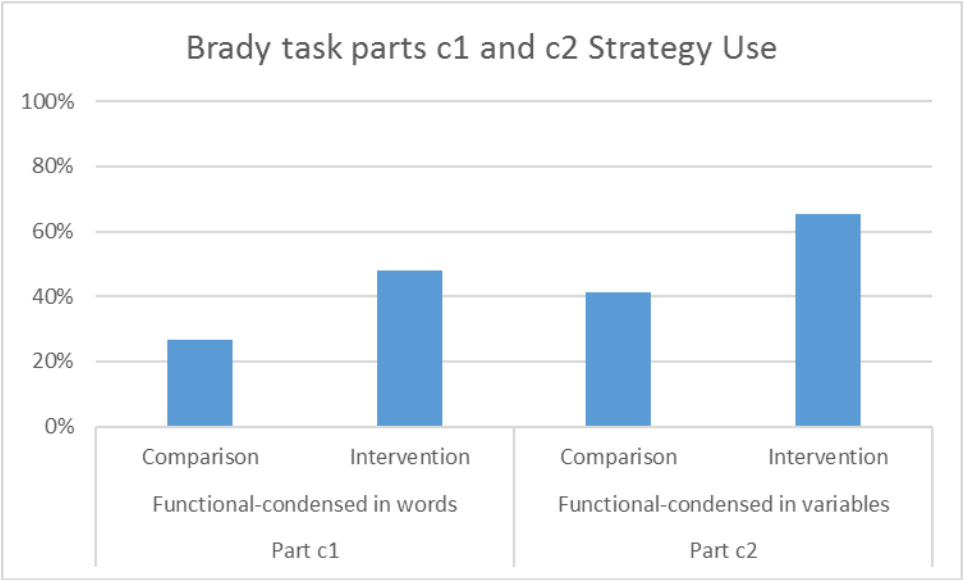


Figure 3. Percentage of Grade 6 students using the Functional-Condensed in Words strategy in part c1 and Functional-Condensed in Variables strategy in part c2

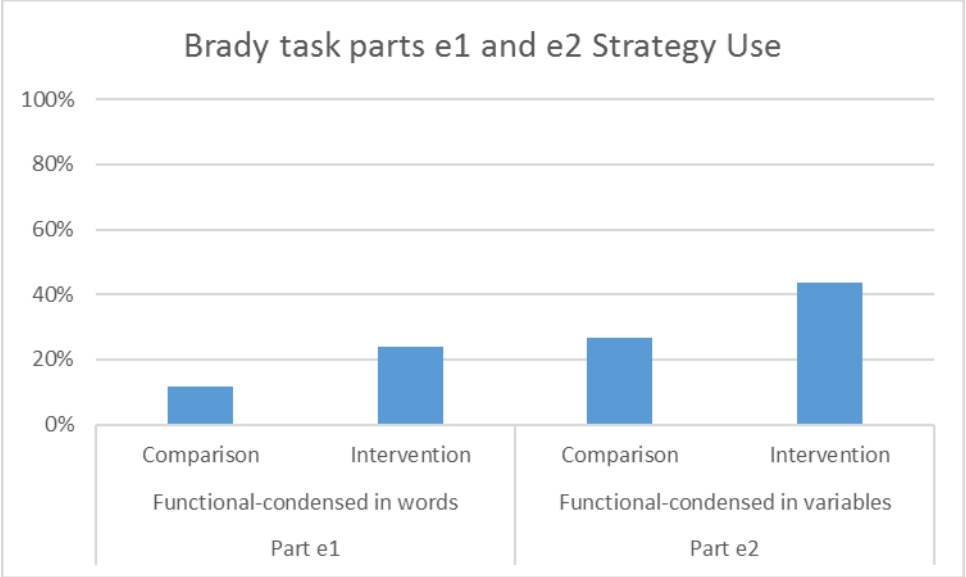


Figure 4. Percentage of Grade 6 students using the Functional-Condensed in Words strategy in part e1 and Functional-Condensed in Variables strategy in part e2

Results for parts e1 and e2, which asked students to extend the rule, showed patterns similar to the results for parts c1 and c2, which asked students to write the rule. The intervention students used the functional-condensed in words strategy more frequently than the comparison students in part e1 (24% vs. 12%) and functional-condensed in variables strategy more frequently than the comparison students in part e2 (43% vs. 26%) (see Figure 4). Similarly, both groups of students were found to be more successful representing the rule in variables than in words (43% vs. 24% for the intervention group and 26% vs. 12% for the comparison group, respectively stating the rule in variables vs. words).

When we compared the Grade 6 results to the results we obtained at the end of the three-year intervention in Grade 5 (n = 90 for the intervention group and n = 61 for the comparison group) on the same item parts, we found out that the percentages of responses in which the intervention students represented the functional relationship in words and in variables stayed about the same from Grade 5 to Grade 6. The percentages of responses in which the comparison students represented the functional relationship in words and in variables increased from Grade 5 to Grade 6 (see Table 1 for students' percentages in Grade 5 and 6). However, the intervention students still outperformed the comparison students in Grade 6.

	Representing the function rule in words (part c1)		**Representing the function rule in variables (part c2)**		**Extending the function rule in words (part e1)**		**Extending the function rule in variables (part e2)**	
	Grade 5	Grade 6	Grade 5	Grade 6	Grade 5	Grade 6	Grade 5	Grade 6
Comparison	16%	26%	21%	41%	3%	12%	8%	26%
Intervention	50%	48%	67%	65%	27%	24%	40%	43%

Table 1. Percentage of students using the Functional-Condensed in Words strategy in parts c1 and e1 and Functional-Condensed in Variables strategy in parts c2 and e2 in Grades 5 and 6

Fisher's exact tests revealed that the students' performances significantly differed by group (intervention and comparison) in all parts in Grade 5 and in part c2 in Grade 6 ($p < .05$). Although the intervention students outperformed the comparison students in all other parts in Grade 6, there was no significant association between performance and group. We discuss the results next.

Discussion and conclusion

Results across items showed that a year after the conclusion of the early algebra intervention, the intervention students remained more successful in generalizing functional relationships and representing them in words and variables the comparison students. These results emphasize the impact of our Grades 3-5 early algebra intervention on students' success in algebra in the secondary school, and the importance of early algebra in helping students develop algebraic thinking practices, specifically, representing and generalizing (Kaput, 2008), as early as elementary school.

The results also showed that the comparison students' performance in representing functional relationships increased from Grade 5 to Grade 6 while the intervention students' performances stayed about the same. Based on our Grade 6 curriculum analysis (Connected Mathematics Project 3 [CMP3]), we suspect that the increase in comparison students' performance might be due to the focus on patterns and functions in the curriculum, which is covered in a unit called "Variables and Patterns" in CMP3 in Grade 6. In our analysis of CMP3, we found that the "Variables and Patterns" unit covers some of the same content that was addressed in the early algebra intervention during Grades 3-5 (e.g., analyzing relationships among variables, filling in tables, making graphs, investigating expressions, equations, and inequalities). Thus, while the CMP3 curriculum may have contributed to the intervention students' ability to retain knowledge and skills learned in the Grades 3-5 intervention, it might have helped the comparison students to "catch up" a bit with the intervention students in Grade 6.

Another important finding is that both groups of students were more successful in representing the function rule in variables than in words. This finding challenges the notion that variable as a varying quantity should not be introduced until Grade 6 (see the Common Core Grade 6 Expressions and Equations Standard #9, NGA Center & CCSSO, 2010), and suggests that an earlier introduction may support students in developing functional thinking. During the LEAP intervention, we observed that elementary students were successful using variables to represent generalizations in multiple contexts including representing unknown, varying quantities in algebraic expressions, representing equations with a fixed unknown, and using variables to represent and generalize a functional relationship (see Blanton, Isler, Stephens, Knuth, et al., 2016 for further details). Blanton et al. (2011) stated:

> learning to express functional relationships in symbolic form not only strengthens understanding and facility in the use of a symbolic language—a skill that is so essential to algebra—but as the study of functions deepens, flexibility with symbolic rules also supports analysis of changes in the behavior of complex functions through more sophisticated techniques. (p. 63)

We therefore underline the importance of introducing functional thinking in the elementary school and using variables as varying quantities in functions to help students represent and generalize functional relationships. This study is also a first step towards measuring the impact of the early algebra instructional intervention on students' success in algebra in the secondary school.

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An APOS perspective of the understanding of the concept of vector subspace

Lillias H.N Mutambara and Sarah Bansilal
University of KwaZulu-Natal, School of Education, South Africa; tendaimutambara@gmail.com

The content of Linear algebra is often considered to be difficult because of the formal logic required as well as the lack of connections to previous courses such as calculus. The purpose of this study was to explore the conceptions of vector subspace concepts, of 73 in-service mathematics teachers as revealed in their written responses to two tasks. The action, process, object, schema (APOS) theory was used to structure the analysis of the responses. The findings revealed that the teachers struggled with the vector sub-space concepts mainly because of poorly developed conceptions of prerequisite concepts of sets and binary operations.

Keywords: APOS, vector subspace, binary operations, vector space.

Introduction

Linear algebra is considered to be one of the most widely applicable subjects for students in the field of mathematics in that it can be applied to many different content areas, such as engineering and statistics and can be studied for mathematical abstraction. However, we noted that when the students take their first linear algebra course, they seem to encounter cognitive barriers. Dorier, Robert, Robinet and Rogalski (2000) noted that the teaching of vector spaces have completely disappeared in the secondary schools and the teaching has become less formal as there are no studies on algebraic structures. Some criticisms given by students about linear algebra concern the use of formalism and the lack of connections with what they already know since this is not done at secondary level. Dorier et al., (2000) elaborated that the formalism is experienced when students need to learn new definitions, symbols, words and theorems. Stewart and Thomas (2010) noted that many students in the first years cope well with the procedural aspects of solving systems of linear equations but struggle to understand the crucial concepts underpinning the material involving the study of vector space concepts such as subspace, linear independence and spanning. The teachers complain that the students have no skills in elementary cartesian geometry, and display an inconsistent use of the basic tools of logic or set theory (Dorier et al.,2000).

An APOS study set up to explore pre-service teachers' mental constructions of matrix algebra concepts, found that most of the participants were operating at the action and process level, with a few operating at the object level (Ndlovu & Brijlall, 2015). The authors argued that the lack of background knowledge of basic algebra schema hampered the teachers efforts to develop adequate schemas at the object level. Many preservice teachers could not manipulate numbers correctly when multiplying matrices and some of them failed to use notations correctly. They confused the notations A^T and A^{-1}. The goal of mathematics teaching is that students understand mathematical concepts presented to them or information that they discover for themselves. This is also supported by Hiebert and Carpenter (1992) who asserted that one of the most widely accepted ideas in mathematics education is that students should understand mathematics. The research described in this paper is concerned with students' difficulties with conceptual understanding of vector subspace.

Britton and Henderson (2009) in studying student's conceptual understanding of a subspace argued that the abstract "obstacle of formalism" and the theoretical nature of linear of linear algebra are the root cause of the difficulties experienced. They believed that lecturers teach students for procedural rather than conceptual understanding and students have poor backgrounds of the concepts on proofs, logic and set theory. One of the questions asked the students to show that the set scalar multiples of a vector formed a subspace of R^3. Most of the students could show that the set, V, is non empty but failed to prove the aspect on the closure property. Students chose particular vectors instead of arbitrary vectors, and some student worked out the sum of two vectors and assumed that the sum belonged to the set V. Some students had misconceptions about the definition of a subspace, while others mixed up concepts and showed rote learning of the concepts on vector subspace. The researchers also noted that students had problems with logic and set theory, moving from abstract to algebraic mode algebraic mode and failing to write a convincing proof.

Theoretical framework: APOS theory

We use the action–process–object–schema (APOS) theory as a framework to make sense of the data. According to Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, Trigueros and Weller (2014) APOS theory is based on the extension of Piaget's principle that an individual learns mathematics by applying certain mental mechanisms to build specific mental structures. The main mental mechanisms for building the mental structures include interiorisation, coordination and encapsulation. The mental structures refer to the action, process, object and schema. As actions are repeated and reflected on, the student moves from relying on external cues to having internal control over them. This is characterized by an ability to imagine carrying out the steps without necessarily having to perform each one explicitly. Interiorisation is the mechanism that makes this mental shift possible. Encapsulation occurs when an individual becomes aware of a process as a totality upon which transformations can act. At this stage the student can analyse properties of the object and compare objects arising from the same process. (Arnon et al., 2014)

Many actions, objects and processes are interconnected in the individual's mind and these will be organised to form a coherent framework called a schema. An object can be assimilated by an existing schema, thus extending the span of the schema. According to Piaget, schema development also passes through stages of development. The Intra level is the preliminary level and is characterised by analysing particular events or objects in an isolated manner in terms of their properties, where explanations are local and not global and relationships between objects may not be perceived. At the Inter level, comparison and reflection upon properties of objects lead to the establishment of relationships. The individual can coordinate two different interpretations of the concept to mean the same thing. During the Trans stage, the student reflects upon and coordinates the relations and is aware of the complete structure. Using these definitions, we now present a genetic decomposition of the vector space concept.

Genetic decomposition of the Vector Space concept

We draw upon the work of Parraguez and Oktac (2010) and Arnon et al., (2014) to present a summarised description of the genetic decomposition of the vector space concept. The construction of the vector space concept is developed as the coordination of the prerequisite concepts of set and

binary operations. Hence, we refer to the set and binary operations Schema as abstraction layers of the vector space concept.

Set Schema. At an Action level, an individual conceives of a set when given a specific listing if a particular condition of set membership. The Action of gathering and putting objects together in a collection according to some condition is interiorised into a Process. This is encapsulated into an Object when an individual can apply actions or processes to the Process such as compare two sets, consider a set to be an element of another and analyse properties of the set. (Arnon et al., 2014)

*Binary operation Schema***.** A binary operation is a function of two variables defined on a single set or on a Cartesian product of two sets. At an Action level, given a binary operation, an individual can take two specific elements of the sets and apply the formula. The individual interiorises the action into a Process that takes two objects (elements) and acts on these to produce a new Object (element) that is the result of the binary operation. At the Object level, an individual can distinguish between two binary operations, check whether a binary operation satisfies an axiom and compare Objects arising from two different binary operations. (Arnon et al., 2014)

Parraguez and Oktac (2010) describe how these two schema are drawn together to form the concept of vector space:

> The Objects that are sets with two kinds of operations (addition and multiplication by a scalar) can be coordinated through the related processes and the vector space axioms that involve both operations, to give rise to a new Object that can be called a vector space. At the Intra level the object of vector space stays isolated from other actions, processes, objects and schemas. For example the student can verify different sets as being vector spaces or not, but does not see the vector space structure inherent in all of them. At the Inter level the object of vector space starts having relationships with other concepts such as subspace, linear transformations, basis, etc. When the student reflects upon these relations, through synthesis they can be recognized as part of a whole structure that makes up a vector space schema. This implies that the Trans level is reached and the student can recognize and work with non-standard examples of vector spaces and can invoke her/his schema when needed. (Parraguez & Oktac, 2010, p. 2116)

This description emphasises the complexity of the construction of the vector space concept which is built upon layers of abstraction. Firstly, the binary operation and set concepts are developed through to higher levels of abstraction via the Action-Process-Object path. The vector space concept is then constructed on these layers, forming an even higher layer of abstraction and as the vector space schema develops, at each stage the previous layer is re-organised as increasing coordination and coherence across the objects and relationships develops. The vector subspace concept is built upon this schema – students will not be able to see the connections between a vector space and vector subspace if they have not developed the vector space schema up to at least an Inter level.

Methodology

This study was conducted with 73 underqualified mathematics teachers who were enrolled in a part-time in-service course at a Zimbabwean university that was designed to upgrade them. The design of the program was such that the teachers would complete the equivalent of an undergraduate three-year degree program. However, the lectures were offered in two intensive block sessions for each

semester and held from 8 Am to 6 Pm every day. The participant teachers have already taken a first course in matrix algebra, a course in mathematical discourse and structures together with two courses in calculus. The second course in linear algebra (during which they participated in the study) includes the concepts of vector spaces, linear independence, linear transformation and diagonalization, eigenvalues and eigenvectors. The research question that underpins this study is: *What does an APOS perspective suggest about the conceptions of vector subspace concepts held by 73 in-service mathematics teachers as revealed in their written responses to two tasks?.*

The data was collected from the teachers' written responses to an activity sheet consisting of nine items which were intended to probe their understanding of vector spaces and vector sub-spaces. In this short paper, we focus on two tasks which were set within the vector space of 2×2 matrices. These appear below.

Item	**Comments**
1. Let **V** be the vector space over of all 2×2 matrices over the real field $\mathbb{R}$. Show that **W** is not a subspace of **V**, where **W** is the set of 2×2 matrices which have a zero determinant.	For this, teachers were expected to find a counter-example to show that the set W is not closed under vector addition.
2. Show that the set of all $M_{2\times2}$ matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a vector space.	For this, teachers could argue that since $M_{2\times2}$ is already a vector space, then it was only required to show that the given subset formed a vector subspace of $M_{2\times2}$ or, they could show that the eight axioms for a vector space were satisfied.

Table 1: Research Tasks

Results and discussion

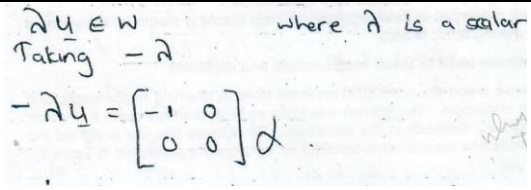
The teachers' responses were numbered from one to 73, where the order held no significance, for example, R11 is the response of Teacher number 11 on the list. We found that the teachers displayed different levels of engagement with the vector space concept. On the one hand some teachers held extremely limited conceptions of binary operations and of set that were not even at Action levels, while on the other hand some teachers had developed strong enough conceptions of vector space schema at an Inter level. The details of some of these levels of engagement are described now.

Binary Operation Layer

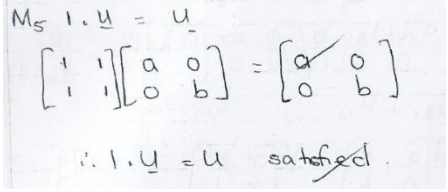
The teachers' responses to the two tasks, suggested that the teachers were reasoning at various levels about binary operations, ranging from those who did not show evidence of even Action- level conceptions, while some displayed Object-level conceptions. Responses which illustrate different levels of reasoning are discussed below.

Not yet at Action levels

It was clear that some teachers had not developed an Action conception of binary operations, as illustrated by the responses of R13 to Task 1 and that of R1 in Task 2. The response by R13 below shows that the teacher has not been able to carry out the Action of scalar multiplication because he has been sidetracked by thinking about the sign of the scalar.



Response R13 to Task 1



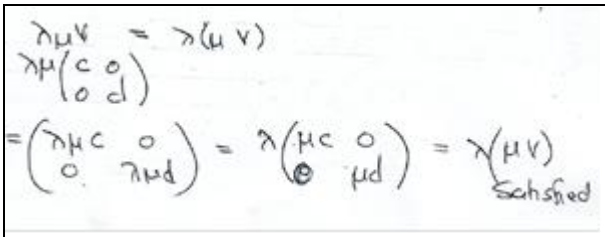
Response R1 to Task 2

Figure 1: Responses suggesting action level conceptions of the binary operation have not developed

The response R1showed that the teacher was not clear about what "1" in the axiom referred to in the scalar multiplication 1.**u** and took it as the "identity" matrix, consisting of 1's in all the entries. The teacher then proceeded to carry out a pairwise multiplication of the corresponding elements in the two matrices. This is shown in the response of R1 in Figure 3.

Process- level engagement with the binary operations

Some teachers' responses suggested that they were able to engage in Process- level reasoning with the binary operations. However, for Task 2, if they had not developed an Object-level conception they were unable to apply the axioms correctly to the binary operations, as illustrated in Figure 2 below.



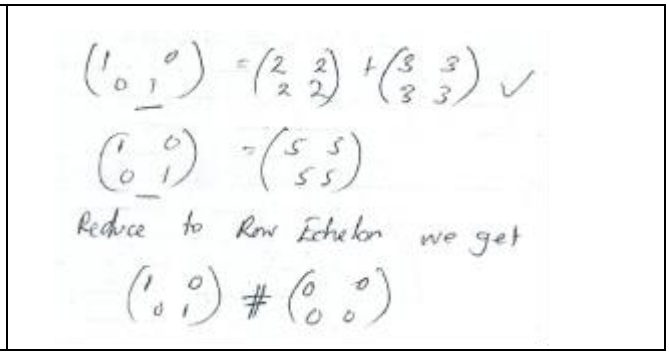


Figure 2: Response R11 working with scalar multiplication and R12 showing confusion about closure

In Figure 2, R11 is trying to show the associativity property of the operation of scalar multiplication $(\lambda\mu)\mathbf{v}) = \lambda(\mu\mathbf{v})$**.** The teacher is able to multiply the scalar into the vector (2×2 matrix) without any problems, suggesting that she has developed a Process conception of the binary operation of scalar multiplication. However, her expression on the left hand side of the first line does not have any brackets. This indicates that the teacher has problems with distinguishing between Objects arising from the different binary operations ($(\lambda\mu)\mathbf{v}$ as opposed to $\lambda(\mu\mathbf{v})$**.** This suggests that the teacher has

not yet developed the necessary Object- level conception of the binary operation of scalar multiplication.

Similarly some teachers showed evidence of Process-level reasoning about the binary operation of addition but this was not sufficient to enable them to respond correctly to Task 1, which required them to show that the closure condition was not satisfied as in the case of R12 which also appears in Figure 2.

The response R12, shows firstly that the teacher has considered a particular type of 2×2 matrices, that has identical entries. These matrices belong to the given set, because their determinants are zero. The teacher has done the vector addition of the two matrix elements, but was unable to show that the set was not closed under the binary operation. The sum of the matrices does satisfy the condition of having a zero determinant. Although he is able to reason at a Process-level about the binary operation of vector addition of 2×2 matrices, the teacher was confused about what he needed to do, with the result of the addition (sum). He seemed to be trying to show that the sum should be equal to the identity matrix. This shows that he is unable to work with the sum of the vectors as an Object.

Set Layer

Similar to the binary operations layer, teachers' responses to the two tasks, showed reasoning at various levels in the Set layer. Examples of responses which illustrate this are discussed below.

Not yet at Action levels

Many teachers were not clear about what the elements of the subset of Task 1 was. One teacher (R17) used a general 2×2 matrix, **U**, with variable entries and the zero matrix and tried to show that the closure condition was not satisfied as shown in Figure 3. The teacher has assumed that the matrix **U** belonged to the set W. This suggests that the teacher has not developed an Action conception of the subset **W**, of all 2×2 matrices which have a zero determinant.

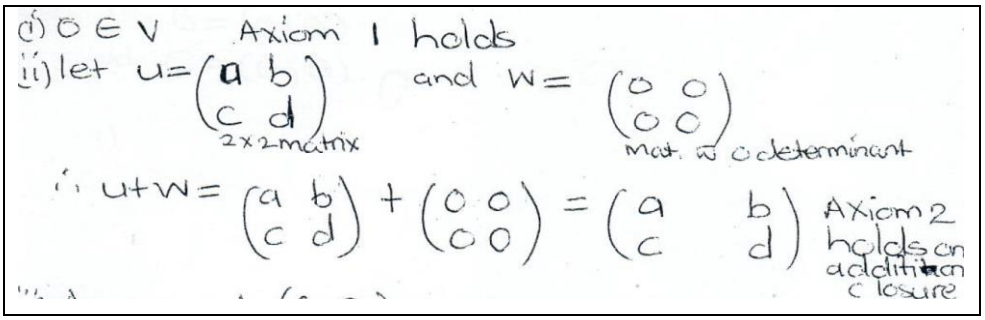


Figure 3: Response R17, unable to identify non-zero elements of the given Set

Process- level reasoning

Some teachers showed that they were able to recognize whether elements belonged to the set of 2×2 matrices with zero determinants by considering the condition which characterized the set One teacher (R6) considered a matrix with elements 1, 2, 3 and 4 and showed that the determinant of the matrix is not equal to zero as shown in Figure 4. That is, he identified a 2×2 matrix which did not belong to the given set. This shows that he can work out that the element does not belong to **W**

suggesting that he has developed a Process conception of the Set of all 2×2 matrices which have a zero determinant, however he could not prove that the set **W** was not a subspace.

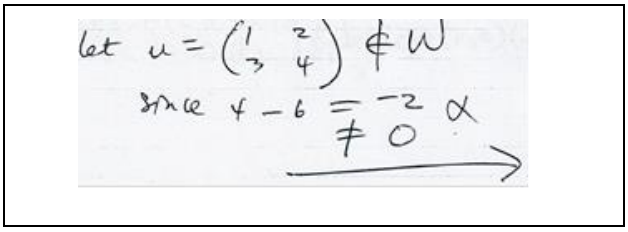


Figure 4: Response R6 considering an example of a 2×2 matrix which did not belong to the given set.

Indications of Inter level conceptions of vector space schema

There were six teachers whose responses suggest that they have may developed an Inter level conception of the vector space schema. R3, for example, considered two 2×2 matrices **x**, and **y** where **x** had 0's in the first column and 1 in the first row of the second column, while **y** had had 1's in the first column and 0's in the second column. The sum (**x** + **y)** was therefore a 2×2 matrix with three entries being 1. So the determinant was 1 and hence did not satisfy the zero- determinant condition. The proof involved generating a counter-example of two elements and then showing that the sum did not belong to **W**, hence implying that **W** was not closed under the binary operation of vector addition. This argument suggests that the student has developed Object-level conception of both binary operation as well as Object-level conception of the set of 2×2 matrices with zero determinants set and the relations between these are perceived. Being able to show that the set **W** did not satisfy the conditions for being a subspace of the vector space **V** , suggests that these teachers have developed Inter level conceptions of the vector space concepts. These teachers also presented appropriate responses to Task 2.

Conclusion

The use of APOS theory suggests that most of the teachers have not developed the necessary mental constructions that would enable them to reason about the properties of vector subspaces, which requires an Inter level conception of vector space schema. The study showed that for many teachers, their conceptions of the prerequisite schemas of set and binary operations hampered them from developing the vector space schema at an Inter level. Most teachers showed that they were able to generate the vector addition of two matrices, in their responses to Task 1. This suggests that they had developed a Process conception of the binary operation of vector addition using 2×2 matrices. However, to show that the set was not closed under the binary operation, the teachers needed to have encapsulated the process of vector addition into an Object

whose properties they could analyse further. Some teachers had not developed even Action level conceptions of the binary operations. Many teachers struggled with the basics such as trying to identify the elements of the subset and were unable to reason about the kind of elements that produced a zero determinant. These teachers had basic problems with working with the set itself of matrices. These struggles indicate that their conceptions of the set of 2×2 matrices with zero determinants had not progressed past an Action level so they did not have access to the sophisticated Object–level reasoning about properties of 2×2 matrix with zero determinants. It is however important to note that the classification is based on written responses and if interviews

were used, the inferences about their levels of reasoning may differ somewhat depending on how the teachers' responded to questions about their reasoning.

Unlike simpler concepts which require coordination between few Objects and Schema, developing a robust understanding of vector subspace is dependent on a sufficiently strong conception of the various layers underlying the concept. Each of these layers is built upon previous ones and becomes increasingly abstract, requiring the coordination and connection between the various objects and relations. Hence very few teachers were able to cope with both tasks. For the teachers to develop the insight that was necessary, they needed to have access to Object-level conceptions of the set (of 2×2 matrices with zero determinants as well as Object-level conceptions of both binary operations. It is therefore no surprise that only six teachers seemed to be able to cope with the abstraction required to present proofs about why a subset did not form a vector subspace of the vector space.

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Four-component decomposition of sense making in algebra

Alik Palatnik[1] and Boris Koichu[2]

[1] Shaanan Academic Religious Teachers' College, Haifa, Israel; umapalatnik@gmail.com

[2] Weizmann Institute of Science; boris.koichu@weizmann.ac.il

This article presents a case in which a pair of middle-school students attempts to make sense of a previously obtained by them position formula for a particular numerical sequence. The exploration of the sequence occurred in the context of two-month-long student research project. The data were collected from the students' drafts, audiotaped meetings of the students with the teacher and a follow-up interview. The data analysis was aimed at identification and characterization of events and algebraic activities in which the students were engaged while making sense of the formula. We found that the students' conviction, by the end of the project, that the formula "makes sense" emerged when they justified the formula, checked its generality, discovered a geometry mechanism behind it, and found that it came to cohere with additional formulas. The findings are summarized as a suggestion for a four-component decomposition of algebraic sense making.

Keywords: Algebraic sense-making, problem-solving, project-based learning, integer sequences.

Introduction

Sense making has long been a focal concern of the mathematics education research community (e.g., Kieran, 2007; NCTM, 2009). NCTM (2009) recognised sense making as a means to know mathematics as well as an important outcome of mathematics instruction. To review, NCTM (2009) refers to *sense making* in mathematics "as developing understanding of a situation, context, or concept by connecting it with existing knowledge" (p. 4). Nevertheless, NCTM (2009), as well as many additional mathematics education publications, is rather inexplicit as to what sense making comprises of and how it occurs. Moreover, it has been broadly acknowledged (e.g., Schoenfeld, 2013) that empirically-based knowledge about the processes involved in sense making, as well as knowledge about the processes involved in learning through mathematical problem solving, is insufficient.

The case presented in this article occurred with two 9th graders, Ron and Arik (pseudonyms) who participated in the Open-Ended Mathematical Problems project, which was conducted by the authors (an abridged version of Palatnik and Koichu, 2017). The initial part of Ron and Arik's project lasted for three weeks and resulted in an insight solution to the problem of finding a position formula for a particular sequence. This part is analysed elsewhere (Palatnik & Koichu, 2015). The insight gained was celebrated as an important highlight of the project. The students told us, however, that they found the formula "by chance" and that it did not make sense for them. As a result, making sense of the obtained formula became an explicitly chosen goal and the main theme of the second part of the students' project. This part had lasted for four weeks and ended when the students succeeded, in quite an idiosyncratic way, to make sense of the formula.

The goal of our study was to discern the activities and processes involved in the sense making effort. Specifically, we pursued the following research questions:

In which events and algebraic activities were the students engaged while attempting to make sense of a formula?

What were some of the processes involved in the students' explicitly expressed conviction, by the end of the exploration, that the formula "makes sense"?

Theoretical background

In empirical studies, the notion of sense making frequently denotes ways by which learners of mathematics act upon a particular entity in the context of particular mathematical activity. The expression "to make sense of…" is attributed in different studies to such entities as proofs, instructional devices, concepts, solution methods and problem situations (e.g., Smith, 2006; Rojano, Filloy, & Puig, 2014).

The idea of algebra as an activity was elaborated by Kieran (1996, 2007). Kieran identifies three types of activities in school algebra: *generational*, *transformational*, and *global/meta-level* activities, and argues that each type has special affordances to meaning construction. The *generational activity* involves the forming of the objects of algebra (e.g., algebraic expressions or formulas) including objects expressing generality arising from geometric patterns or numerical sequences. The *transformational activity* includes various types of algebraic manipulations. Transformational activity can involve meaning construction for properties and axioms on which the manipulations rely. A related point is highlighted by Hoch and Dreyfus (2006), who proposed the notion of *structure sense*, which is related to algebraic manipulations and aspects of *symbol sense* (Arcavi, 2005) in relation to *friendliness* with symbols as tools, an ability to switch between attachment and detachment of meaning, and an examination of the meaning of symbols. Finally, Kieran (2007) argues that meaning construction is associated with *global/meta-level mathematical activities* (e.g., problem solving, working with generalizable patterns) in a sense that "these activities provide the context, sense of purpose, and motivation for engaging in the previously described generational and transformation activities" (p. 714). It is essential for the forthcoming analysis that when the learners are engaged in a global/meta-level activity, they can carry it out in a variety of ways, and the decision to use the algebraic apparatus arises as learners' choice.

Treatment of sense making as an inseparable part of mathematical thinking makes the MGA model of creating mathematical abstractions (Mason, 1989) particularly important for our study. The main operational categories of the model are Manipulating, Getting-a-sense-of, Articulating (hence MGA). The MGA model elaborates on the processes of creating abstraction as a helix, in which each cycle includes its own, local, sense-making act. Briefly, the model presumes that **m**anipulating familiar mathematical objects (M) leads to the formation of a sense of **g**enerality or regularity based on properties of these objects (G), and then to the **a**rticulation of that general property or regularity (A), which in turn forms new objects for further manipulations. Mason (1989) suggested that the driving force behind the process of creating abstractions is the gap between expected and actual results of manipulations.

To summarize, in our study we adapt NCTM's (2009) perspective on sense making, and elaborate on it in an algebraic context. Our theoretical framework is built upon the idea of algebra as an activity (Kieran, 1996, 2007) and on analytical apparatus of Mason's (1989) model of mathematical thinking known as Manipulating – Getting-a-sense-of –Articulating (MGA).

Method

Learning environment, participants and the mathematical context

The Open-ended Mathematical Problems project, in the context of which the case of Ron and Arik took place, is being conducted, since 2010, in 9th grade classes for mathematically promising students. The learning goal of the project is to create for students a long-term opportunity for developing algebraic reasoning in the context of numerical sequences. It is of note that 9th graders in Israel, as a rule, do not possess any systematic knowledge of sequences; this topic is taught in the 10th grade.

The project is designed in accordance with the principles of the Project-Based Learning (PBL) instructional approach (e.g., Blumenfeld et al.,1991). Specifically, the organizational framework of the project is as follows. At the beginning of a yearly cycle of the project, a class is exposed to 8-10 challenging problems. The students choose one problem and work on it in teams of two or three. They work on the problem at home and during their enrichment classes. Weekly 20-minute meetings of each team with the instructor (the first author) take place during the enrichment classes. When the initial problem is solved, students are encouraged to pose and solve follow-up problems. At the end of the project, all teams present their results to their peers. Then 4-6 teams, chosen by their classmates, present their work at a workshop at the Technion – Israel Institute of Technology, attended by academic audience (for more details see Palatnik, 2016).

Ron and Arik chose to pursue the Pizza Problem (Figure 1) which is a variation of the problem of partitioning the plane by n lines (e.g., Pólya, 1954).

> Every straight cut divides a pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts?
>
> A. Solve for $n = 1, 2, 3, 4, 5, 6$.
>
> B. Find a recursive formula for the nth term of the sequence.
>
> C. Find a position formula for the nth term of the sequence.

Figure 1: The Pizza Problem

Using Kieran's (2007) terminology, we expected the PBL environment and the Pizza Problem in particular to afford students to be engaged with generational and transformational activities in the context of a global/meta-level activity. In this way, the students were provided with opportunities for developing algebraic sense-making and we – with an opportunity to study their sense-making effort.

Data sources and analysis

We audiotaped and transcribed protocols of the weekly meetings with Ron and Arik (eight 20-minute meetings), collected written reports and authentic drafts that the students prepared for and updated during the meetings (more than 40 pages) and interviewed the students by the end of the project. These data were used to create a description of the students' exploration and for dividing it into events.

In accordance with the presented above methodological principles for exploring sense making, we discerned the activities the students chose to be engaged in: proving, generalizing, pattern-seeking,

and question-generating. We also applied the MGA model to trace mathematical objects manipulated by the students in a sequence of activities potentially contributing to sense-making.

Findings: Ron and Arik make sense of the obtained formula

We present here four main events that occurred during students' sense-making pursuit.

Event 1: Choosing new goals

The following conversation took place just after the students presented their solution of the Pizza Problem to the instructor:

> Instructor: Now you have a lot of work to do, and this is great. First of all, you see that the formula works. Now we have to think why it works, and try proving that it works.
>
> Ron (to Arik): Write it down. "Why it works, and prove that it works" (laughs), it is interesting!

Ron accepted instructor's suggestion. In his words: "When we have a formula, but don't know its meaning, it is not interesting. If we knew how the formula is constructed, we would know it 100%. We got it by chance. So we do not know what it means." In addition, both students proposed to explore a more general problem, that of plane partitioning (see Figure 2).

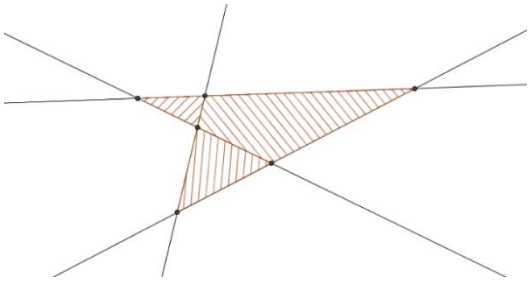

Figure 2: Division of the plane: there are three closed (hatched) and eight open pieces

The students also suggested additional objects to explore: the points of intersection of the cutting lines with and within a circle representing a pizza and the number of segments on the cutting lines.

Event 2: Simultaneous exploration of several sequences and first manipulation with a formula

Having chosen the above goals, the students started making sketches and counting: segments within the circle, closed and open parts of the plane and points of intersection of the cutting lines, for different numbers of lines (see Figure 3a-c). As a by-product, the students noticed that the sum of the first n odd numbers also equals n^2. They also began exploring the connections between different sequences (see Figure 3d-3e). In particular, Ron noticed that the differences between the corresponding terms of the sequences form a sequence 0, 1, 2, 3... (see columns X,Y at Figure 3d).

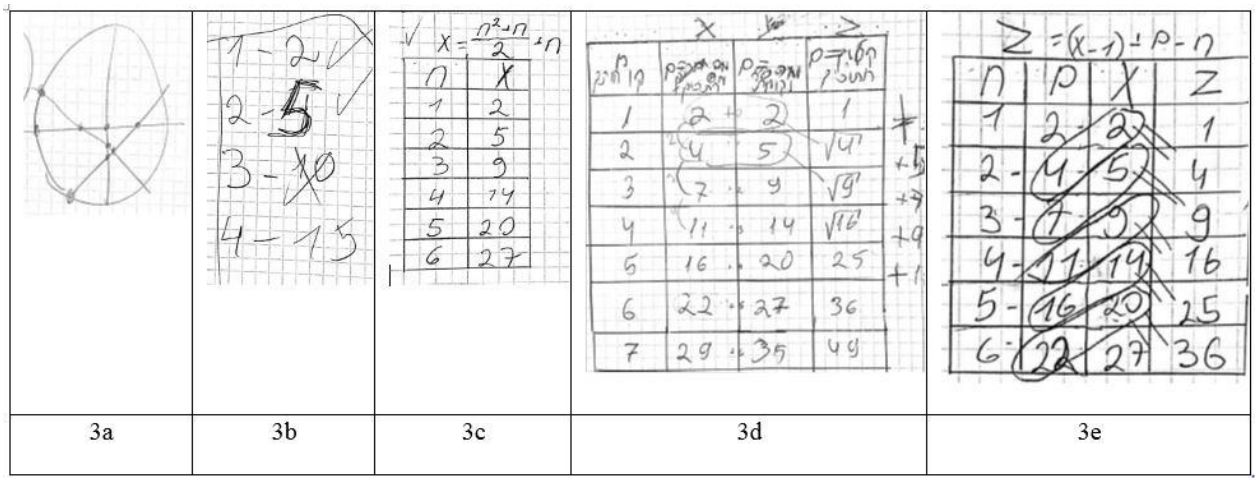

Figure 3: The strategy employed in Events 1 and 2

To obtain an explicit formula for the sequence 2, 5, 9, 14 … (the numbers of intersections of the cutting lines with and within the circle), Ron *adjusted* the formula $P_n=\frac{n^2+n}{2}+1$ into the formula $X=\frac{n^2+n}{2}+n$ (Figure 3c) in the following way: "I thought it would be like the previous formula, but it did not fit. So I got rid of 1 and added *n* [to the right side of the formula], and it was right."

Event 3: Producing an explanation of why the target formulas worked

The wish to understand why the formula returns the maximum number of pieces was a repeated theme in weekly meetings with the instructor. The students eventually answered this query in the following way. After exploring of new drawings Ron and Arik realized that the maximal number of pieces is obtained when a new cutting line crosses *all* the previous lines in new points. As a result, the students concluded that a new cutting line added *n* new intersection points to the existing configuration of lines. For the students, it was an explanation of why the formula $X=\frac{n^2+n}{2}+n$ returned the *maximum* number of the intersection points. They further asserted that this idea also explained, for them, why the target formula $P_n=\frac{n^2+n}{2}+1$ returns the *maximum* number of pieces.

Event 4: "Proving" the target formula

As mentioned, the need to prove the correctness of the formula for the Pizza Problem was an additional driving force for the students. First, Ron suggested: "We thought of a way to prove it [the position formula]…[in order to do so, we wanted] to connect all the formulas we had, every table we've made… may be it will give us the formula, then we will know that it is a true formula indeed. Then we'd have a proof". Ron and Arik built upon the following inference: for any number of cuts, the sum of the number of open and closed pieces (see Figure 2) equals the overall number of pieces into which a plane is divided. They explored the sequences for open and closed pieces. The number of open pieces for *n* cuts, 2*n*, was easy for them to find and explain: adding a new cutting line adds exactly two open pieces to the drawing. For the closed pieces the students empirically (i.e., by counting on the drawings) obtained a sequence 0, 0, 1, 3, 6 for 1, 2, 3, 4 and 5 cuts, respectively. They perceived it as "quite close" to the target sequence (2, 4, 7, 11, 16…) and began manipulating

the target formula ($P_n=\frac{n^2+n}{2}+1$) in a way similar to *adjustment* in Event 2. Eventually Ron and Arik obtained the correct expression $\frac{(n-2)^2+n}{2}-1$. The last piece of the puzzle came when Ron and Arik and their classmate with whom they consulted devised and realized the following plan. Since the formula $P_n=\frac{n^2+n}{2}+1$ represents the total numbers of pieces and since Ron and Arik have obtained the formulas for the numbers of closed and open pieces, the three formulas should match. After several unsuccessful attempts, Ron and Arik implemented this idea and algebraically connected the three formulas. In their final presentation, they showed a slide with the following transformations:

$\frac{(n-2)^2+n}{2}-1+2n=\frac{n^2-4n+4+n-2+4n}{2}=\frac{n^2+n+2}{2}$; $\frac{n\cdot(n+1)}{2}+1=\frac{n^2+n+2}{2}$. This and validation of all three formulas by means of Excel tables were presented as "the proof" of the target formula, and the formula itself was treated as "making sense" by the students.

Discussion

The four-weeks-long exploration of two 9th grade students working on a particular project has been presented. The answer to the first research question (about events and algebraic activities in which the students were engaged while attempting to make sense of the previously obtained position formula) straightforwardly follows from the above exposition. Briefly, the students were engaged in generational and transformational activities in the context of the global/meta-level activities of explaining to themselves why the formula worked and of proving the formula. It is of note that Ron and Arik's persistence to make sense of their formula is unusual. We suggest two circumstances contributing to the emergence of the students' self-imposed sense-making goal. First, the students' activities were organized and driven by their interest to a particular mathematical phenomenon and not merely to generation of some patterns (cf. Hewitt, 1992, for train spotters metaphor). Second circumstance is the organizational setting of the project, which was in accordance with project-based learning instructional approach (Blumenfeld et al., 1991). In such an environment the students had a chance to get used to the long-term, open-ended explorations, to the high level of expectations and to having room and time to spend with a problem.

Our second research question concerned the processes involved in student sense making. To answer the query "why the formula works" the students examined *the geometric mechanism* behind the formula. In the course of generational activity the students experimented with concrete drawings (i.e., drawings with 4-6 cutting lines), which apparently served as a visual tool to reveal a generic process that occurs when a line is added to a system of *n* existing lines. Accordingly, the multi-stage process of abstracting, at each stage of which an MGA cycle occurred, seems to be the central process underlying the why-part of the students' sense-making effort.

The query "how to prove the formula" turned to be the thorniest part of the project. The students addressed this query when they succeeded to show how the target formula came to *cohere* with two geometrically related formulas. These formulas were obtained by means of exploration of the connections between the sequences chosen by students. The connections were found in the process that featured counting on the drawings, pattern-sniffing in the tables and manipulating the previously obtained formulas by adjusting them. Eventually, the target formula was inserted in a cloud of related formulas, which did not exist when the students began the sense-making pursuit.

Thus, the process of generating a cloud of formulas and checking it for coherence seems to be an important process in the proving part of the students' sense-making effort (cf. Rohano, Filloy & Puig, 2014, for sense making by connection of a new mathematical text to a system of texts). It is of note that the coherence was achieved not only among various objects, but also by means of a coherent exploration strategy.

As argued, Ron and Arik constructed meaning of the target formula in a sense-making process consisting of sequence of generational and transformational algebraic activities in the overarching context of global, meta-level activity, long-term problem solving. In this sense-making process, the students: (1) formulated and justified claims; (2) made generalizations, (3) found the mechanisms behind the algebraic objects (i.e., answered why-questions); and (4) established coherence among the explored objects. We now take the liberty of formulating this summary as a proposal for a four-component decomposition of sense making (see Figure 4).

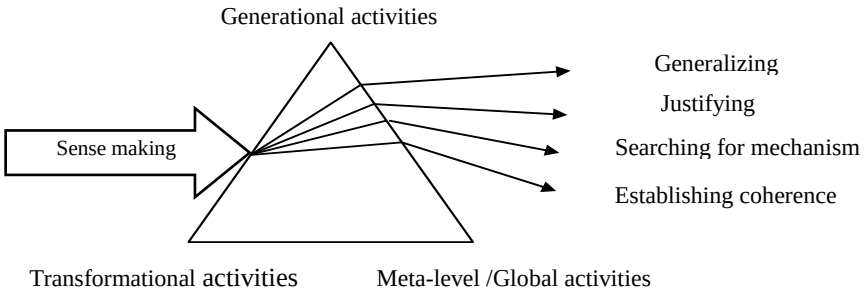

Figure 4: Four aspects of an algebraic sense making through algebraic activities

The aspects of *generalizing*, *justifying* and *search for mechanism* in sense making are in line with the main attributes of symbol sense (Arcavi, 2005) as well as findings about the role of generalizing and justifying in meaning construction (e.g., Lannin, 2005; Radford, 2010). However, *establishing coherence* has not yet been considered as part of sense making.

The four-component decomposition elaborates NCTM's (2009) definition of sense making in the following way. First, it presents sense making as a conjunction of processes. Second, it highlights the potential of algebraic activities to provide students with means to make sense of algebraic objects.

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‘Father Woodland’: A learning environment to facilitate the development of algebraic thinking

Ioannis Papadopoulos[1] and Nafsika Patsiala[2]

[1]Aristotle University of Thessaloniki, Greece; ypapadop@eled.auth.gr

[2]Aristotle University of Thessaloniki, Greece; nafspats@eled.auth.gr

In this paper, the contribution of the use of the “Father Woodland” learning environment in Grade-3 students’ algebraic thinking is examined. Four types of thinking were identified indicating a progressive movement towards the use of symbolic language that seems to have a rather developmental character. In their solutions the students induced rules for solving equations that will later be introduced formally to them.

Keywords: Early algebraic thinking, puzzle-like environments.

Introduction

Given the central position of algebra in the secondary curriculum which led to a separation between arithmetic and algebra with the first being the main focus of elementary mathematics curriculum, mathematics educators try to cope with the challenge of managing the transition from arithmetic to symbolic algebra. Numerous researchers admit that this separation deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years (e.g., Kieran, 2007). One way to address this issue might be to study the impact of certain learning environments in the students’ development of algebraic thinking. Papadopoulos, Kindini and Tsakalaki (2016) working with a mobile puzzle environment found that sixth graders exhibit a progressive movement towards algebraic thinking. In this context, we try to explore the potential contribution of another specific learning environment called ‘Father Woodland’ in young students’ algebraic thinking that would allow us to identify certain steps in this movement from arithmetic to algebra. This is based on two of the algebra goals specified by NCTM (2000) standards, i.e., (i) represent and analyze mathematical situations and structures using algebraic symbols, and (ii) use mathematical models to represent and understand quantitative relationships. Therefore, we try to examine whether this environment facilitates the achievement of these goals through the identification of the types of thinking that the students followed in order to cope with the given tasks.

Early algebraic thinking and ‘Father Woodland’ environment.

Cai and Knuth (2011) do not limit algebraic thinking in earlier grades to simply mastering arithmetic and computational fluency but it goes deeper in identifying the underlying structure of mathematics which includes the development of particular ways of thinking, analysis of relationships between quantities, noticing structure, generalization, problem solving, justifying, proving and predicting. Cai et al. (2005), in a cross-cultural comparative study talk about multiple representations (pictures, diagrams, tables, graphs, and equations) that are used to represent functional relationships between two quantities and more specifically about ‘pictorial equations’ used to represent quantitative relationships providing thus a means for developing students’ algebraic ideas. This raises the necessity to make the distinction between the external and internal representations in the sense of considering at a minimum configurations of symbols or objects external to the individual learner or

problem solver (i.e., concrete materials, pictures/diagrams, spoken words, written symbols) that can be described mathematically and configurations internal to the individual (i.e., mental models and cognitive representations of the mathematical ideas underlying the external representations) respectively (Goldin, 2002). Such internal representations are inferred from the way the students express their aspects of the process of mathematical thinking in their written responses. It seems that certain learning environments can be in favor of introducing young learners to these aspects of algebraic thinking (Papadopoulos et al., 2016). In the current study a specific learning environment has been chosen. It is called 'Father Woodland' and is about a Czech fairy-tale figure owning a farm who organizes tug-of-war games among the animals living in the farm (Hejný, Jirotková, & Kratochvílová, 2006). The weakest animal is the mouse. Two mice are as strong as a cat, a cat and a mouse equal a goose and a goose and a mouse equal a dog (Fig. 1). The strength of each animal is represented by a picture and an icon (symbol) and the students are asked mainly to decide between two groups the stronger one or to add some animals to the weaker group in order to create two equivalent groups, or to reveal the identity of hidden animals so as to obtain equity.



Figure 1: Equivalences in the Father Woodland environment

It is a rich environment. Hejny and his colleagues use it in a series of textbooks they produced. The relevant tasks within these textbooks are connected with the development -among others- of an early number sense, pre-concept of divisibility, the lowest common multiple and greatest common divisor as well as the solving of equations. Hejný, et al. (2006) used this environment with Grade 1-3 students focusing on how the environment facilitated the identification and acceptance of the association between animals and quantities. Marchini and Back (2010), used also a modified version of the environment to fit in the Italian schools and worked with Grade-1 students focusing on how the variety of "ways for representing the same mathematical concept together with *treatment* inside a register and *conversion* between registers facilitate pupils' understanding and the construction of concepts" (p. 55). In this study we focus on the use of this environment as a way to smooth the transition from arithmetic to algebra (in the sense of using pictorial equations as a means or developing algebraic ideas, see Cai et al., 2005) by considering the various types of students' thinking that would show a progressive movement towards algebraic thinking.

Design of the study

Seventy 3rd graders (8-9 years old) participated in the study. They were the total population of three classes from two primary schools and they represent a sample of an ordinary Greek primary school. They had no previous experience working with this kind of environments and they had not been taught any of the basic concepts of algebra such as equations or variables. When the students were introduced to the 'Father Woodland' environment, each tug-of-war game was presented using both the pictorial and symbolic representations of the animals. The whole study (part of a broader one) lasted five weeks. The students were initially introduced to 'Father Woodland' and then on a regular basis they were given tasks to solve individually. The whole project took part in parallel to the normal

teaching and was not integrated in the content of their math lessons. This paper focuses on the first 10 tasks due to the limited number of pages. There are 3 collections of tasks. In the first, there were two groups of animals in each task and the students were asked to add a mouse to the weaker group in order to make both groups equivalent (Fig. 2). This demands comparison and relational thinking connected to the notion of equality as an equivalence. In the second, the tug-of-war game took place but one (or some) of the animals wore a mask. The students were invited to find the animal(s) behind the mask (Fig. 3, left and middle). The aim was to exploit relational thinking in the form of using alternative ways for representing the unknown quantity. Finally, in the third, the students were asked to create two equally strong teams using any combination of the farm animals (Fig. 3, right). The aim was to see whether the students exploit the experience gained before and how intuitive mathematical ideas are embedded in their creations. For each task, the students were asked to explain their answer in a separate textbox. During the study, no feedback was given to the students about their answers. The students' worksheets constituted the data for this study. These data were examined in order to identify evidence of early algebraic thinking and possible formal mathematical concepts, which are informally used in the students' answers. In the context of qualitative content analysis, inductive category development was used to organize the categories.

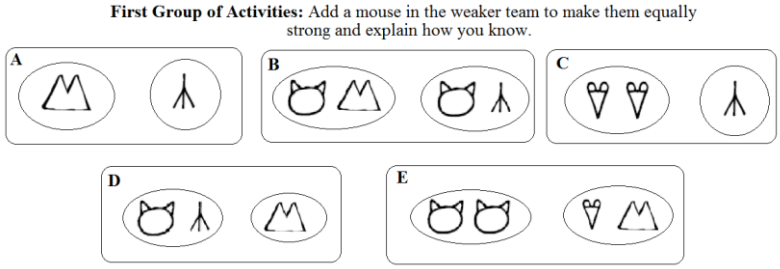


Figure 2: The first group of activities

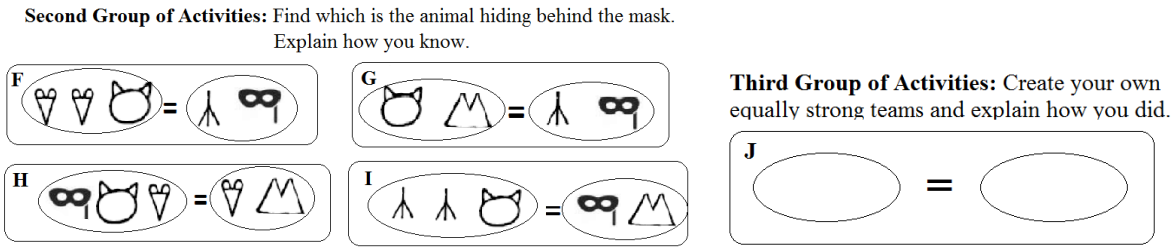


Figure 3: The second (left, middle) and the third (right) group of activities

Results and discussion

After the data examination, the answers were categorized in four types. The criteria for this categorization were the ways students chose to express the underlying structure in each task (i.e., using pictures, words or symbols), the relationships among the given quantities (i.e., using the given or new (invented) relationships), and the mathematical information contained within the pictorial representation (i.e., identifying a basic unit, substituting animals with their equivalents, adding/subtracting the same quantity in both sides, etc.). The four types are: (i) using pictorial language, (ii) using words to express relationships, (iii) combining words and symbolic representations and (iv) using 'symbolic' language to express relationships. Obviously not all the

students applied all the types. This is why it was decided to choose a proper sample of students to show the diversity of the approaches taken.

Type 1 – Using pictorial language

This type refers to the students who preferred drawing pictures in detail rather than using a symbolic representation (Task-E, Fig. 4). The answer is correct. The missing mouse must join the group on the left to get two equal teams but the reasoning is weak since it is limited to merely transfer the 'abstract' information into a more 'realistic' version and it lacks an explicit explanation of the 'underlying' thought. It seems that the student fails to shift the focus to the existing relationships between the values of the participating animals.



Figure 4: Use of drawings

Type 2 – Using words to express relationships

This type was used by the majority of the students and it proved more convenient for most of them to express their solution of the problem. Actually, in this type, the students made a step forward by trying to use words for expressing relationships among the quantities as it can be seen in the next two examples. This choice in many of the cases was described by the students in detail revealing thus their line of thought. In Task-J, one student created two equivalent groups by placing 5 dogs on the left and 20 mice on the right (Fig. 5). His explanation was: "*I thought that 5 dogs are as strong as 5 geese and 5 mice. But, 5 geese = 5 mice and 5 cats and 5 cats are as strong as 10 mice. So 20 mice*".

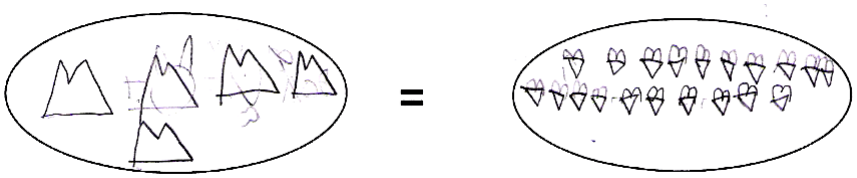


Figure 5: Use of mouse as the basic unit

The student exploited all the default information given by the pictures in Figure 1, e.g., 1 d(og)=1g(oose)+1m(ouse), 1g(oose)=1c(at)+1m(ouse), and 1c(at)=2m(ice). Then, the whole process can be presented on a more formal way as 1d=1g+1m ⟹ 5d=5(g+m)[*multiply both parts by the same number*] ⟹ 5d=5g+5m [*distributive property*]⟹ 5d=5(c+m)+5m [*substitute with equivalent*] ⟹ 5d=5c+5m+5m ⟹ 5d=5x2m[*substitute with equivalent*] +10m ⟹ 5d=10m+10m ⟹ 5d=20m. It must be said that this is not explicitly outlined by the child. But this enables us to identify in the student's explanation the seeds of the mathematical reasoning described above. In the same way we will try to see the possible formal way of expressing the students' answers in the remaining part of the paper. Actually, we make inferences about students' internal representations on the basis of their production of external representations (Goldin & Shteingold, 2001).

The second example concerns Task-H which asked the students to identify the animal hidden behind the mask. The student's answer was (Fig. 6, left): "*There is a cat hiding behind the mask. Because, a mouse and a dog are as strong as a cat and two mice. I figured it out because a mouse and a dog are as strong as 5 mice so behind the mask is a cat since 2 cats and 1 mouse are as strong as 5 mice too*".

The student's starting point was the right part of the equation and she chose the mouse as the building block to replace all the involved animals. It is necessary to mention here that the students did not restrict themselves on the given relationships that were given during the first session (Fig. 1). They were able, during the next sessions, to identify new relationships based on the given ones. This student made use of one of these invented relationships by claiming implicitly that a dog has the same strength with 4 mice. This claim results to the total amount of 5 mice in the right part. Given that a cat has the same strength with 2 mice, it means that there are 3 mice in the left part of the equation plus the hidden animal. Two mice are needed to obtain equality; therefore, a cat must be placed behind the mask.

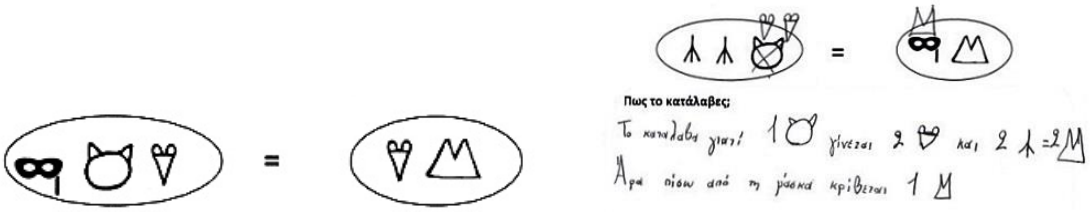
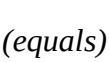


Figure 6: Use of words (left) and combination of symbols and words (right) to explain relationships

If we translate the reasoning of the student into its formal version, considering x the unknown, we obtain the following series of equations. $x + c + m = m + d \xrightarrow{d=4m} x + c + m = 5m\ [substitute\ with\ equivalent] \xrightarrow{c=2m} x + 3m\ = 5m[substitute\ with\ equivalent] \Longrightarrow x = 2m\ [subtract\ the\ same\ amount\ from\ both\ sides] \xrightarrow{c=2m} x = c\ [substitute\ with\ equivalent]$

The strategy that led to successful solution here was: (i) choose the basic unit (e.g., the mouse), (ii) translate the picture to equation, (iii) substitute the dog by its equivalent number of mice and execute the operation, (iv) substitute the cat by its equivalent number of mice and execute the operation, (iv) find the unknown (x).

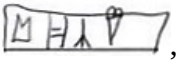
Type 3 – Combining words and symbolic language

This type starts -as a first step towards symbolic language- to combine words and symbols to show relationships between the participating animals. This is one answer for Task-I: "*I figured it out because*  *becomes (equals)*  *plus*  *. So there is*  *hiding behind the mask*".

The student in the first half of her answer used the word 'becomes' to denote the equality between cat and mice (Fig. 6, right). But, in the second half she used the sign of '=' to denote again the relationship between mice and geese (left part) and dogs (right part). Firstly, she substituted the cat (c) with 2 mice (m). Now, the left part consists of two identical sub-groups (a goose and a mouse per subgroup) which if substituted by their equivalence in terms of dogs reveal the identity of the unknown. So, starting from the left part: $2g + c = x + d \xrightarrow{c=2m} 2g + 2m = x + d\ [substitute\ with\ equivalent] \Rightarrow 2(g + m) = x + d\ [distributive\ property] \xrightarrow{g+m=d} 2d =$

$x + d[substitute\ with\ equivalent] \Rightarrow x = d\ [subtract\ the\ same\ quantity]$. The knowledge that 2 mice plus 2 geese is the same as 2 dogs -which is based on the given relationship that a mouse plus a goose is the same as a dog- reveals an implicit understanding of the above mentioned distributive property.

Type 4 – Using symbolic language to express relationships

The last type used by the students abandons the use of words and the reasoning is mainly symbolic. The first example is an answer from Task-F dealing with the animal behind the mask. The student started with the left group, using a symbolic expression to show its substitution by a dog (). Then, based on this expression she wrote another one to show the solution (, see Fig.7 left). The first claim of the student seems arbitrary but if seen carefully it makes use of known relationships in order to obtain new ones. The left group represents the sentence m+m+c. Given that m+c=g(oose) the sentence becomes m+g which, according to the given relationships, equals with a dog. Then, it is obvious that what is needed in the right part of the equation is a mouse.

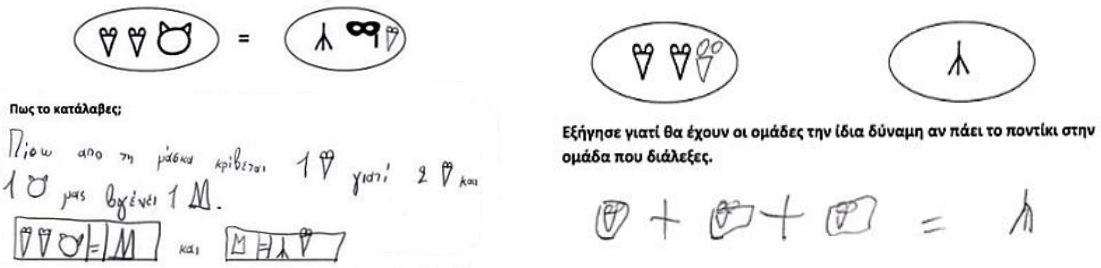


Figure 7: Usage of symbolic representations (left and right)

The second example shows a solution for Task-C. This solution is considered more elaborated since the student made use of the sign of the required operation to show the equivalence. This is indicative of understanding both the operation that takes place and the correct use of the sign of this operation. The missing animal is the mouse that must join the left group to get equality (Fig. 7, right). The sum of the strength of the three mice equals the strength of the goose. Again, this is a relation different than the given ones and it is important that so young students exhibit this ability, to use and combine given situations in order to get new ones. So, it is interesting to follow the thought of the student. Two mice equal with a cat. Then, a cat and a mouse have the same strength with a goose. Consequently (transitivity) three mice are equivalent with a goose. Expressing relationships using this symbolic language to solve a problem constitutes an important step towards the development of algebraic thinking. Besides, all the answers show an explicit focus of the students to the underlying structure of each equivalence in order to reach a solution.

It is intersting to examine now the findings of this study in the light of a previous one. Papadopoulos et al. (2016) in their study based on the use of mobile puzzles with 6th graders, distinguished mainly four types of students' thinking (translating the picture to equality expressions, using words to show the relationship, using symbolic language to show the relationship, and combination of more than one of the previous types). This means that there is a match between the types of thinking in these two studies and this stregnthens the possible positive contribution of puzzle-like learning

environments to the development of young students' algebraic thinking. The feeling from the first study (no numerical data available) was that the order of these types is rather developmental in the sense that types 3 and 4 are more advanced (and thus less frequent), and are met at the end of the project (an indication that they are connected with the accumulated experience). One could attribute this finding to the teachers' appreciation of the symbolic answers instead of textual ones. However, this is not the case, since the teachers were not involved in the project and in the meanwhile the students did not receive any feedback about their answers.

	Type-1	Type-2	Type-3	Type-4
Tasks A-E	5 (1.6%)	253 (83%)	14 (4.6%)	33 (10.8%)
Tasks F-I	8 (3.74%)	157 (73.36%)	24 (11.22%)	25 (11.68%)
Task J	2 (2.94%)	56 (82.35%)	8 (11.77%)	2 (2.94%)

Table 1: Frequency of Types 1-4

In this study an effort was made to get arithmetical evidence that would shed light on this issue. Table-1 confirms that Types 3 and 4 are indeed the less frequent ones. The low frequency of Type-1 was expected since the pictorial language was already included in the statement of the tasks.

Table-2 presents the distribution of the last two types across the range of the tasks. As it can be seen to a great extent the number of instance for each Type is increased as we move towards the last tasks indicating that these types are connected with the accumulated experience.

	Task-A	Task-B	Task-C	Task-D	Task-E	Task-F	Task-G	Task-H
Type-3	3	2	4	3	3	7	7	7
Type-4	5	6	3	8	10	8	8	8

Table 2: Distribution of Types 3-4 across tasks

Conclusions

The findings of our research indicate that the 'Father Woodland' environment might contribute to the development of students' algebraic thinking. The four types of thinking mirror the rules induced by the students in order to solve the posed problems. Starting from certain external representations of equality sentences the students made an attempt to express their internal representations through the shift from pictorial to symbolic language. Obviously, this is not all that matters with the development of algebraic thinking with young children. However, it cannot be considered trivial. The students had to add or remove the same animal (quantity) from both sides, to substitute certain animals with their equivalence, isolate the unknown animal (variable) trying to maintain the same strength between both groups of animals applying at the same time the distributive law or transitivity. Despite that lack of explicit knowledge about operations and relations hinders a good approach to algebra (Gerhard, 2013), it seemed that there were instances of an implicit knowledge of certain rules for solving equations which will be later introduced formally to the students. However, it still remains to answer questions like: In what way the transition from the animal symbols back to the arithmetical or algebraic equations will be possible? Additionally, the findings support the developmental character of these types of thinking when the students use puzzle-like learning environments aiming to support

algebraic thinking. However, this does not mean that some students did not occasionally move backwards to previous types of thinking. This is in itself a significant finding we aim to explore further since the relatively small number of participants does not allow to generalize our findings.

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Aspects of proficiency in elementary algebra

Guido Pinkernell[1], Christian Düsi[2] and Markus Vogel[1]

[1]University of Education Heidelberg, Germany;
pinkernell@ph-heidelberg.de, vogel@ph-heidelberg.de

[2] Cooperative State University Baden-Württemberg, Germany;
christian.duesi@mosbach.dhbw.de

This paper provides insight in the process of developing a comprehensive and concise summary of the most prominent aspects of proficiency in elementary algebra at the end of secondary grade. It will serve as a theoretical frame of reference for devising or validating instruments for diagnosing the mastery of elementary algebra at the transit from school to university. A first draft had been based on literature, which then was presented to experts for further evaluation. The model now comprises ten aspects of proficiency which are allocated into a table of two dimensions, one referring to elements of algebraic language, the other referring to the cognitive actions performed on them.

Keywords: Elementary algebra, basic skills, secondary school mathematics, STEM Education.

Introduction

For being successful in subjects from science, technology, engineering or mathematics at high school or university (STEM), a good mastery of formal algebra is indispensable. But what is a good mastery of algebra, and how does it show? To provide a theoretical base for devising an instrument for diagnosis, we have been working on a comprehensive yet concise overview of the important aspects of proficiency in elementary algebra as it has been covered by relevant research. For this, the model is meant to be summative, i.e. it contains aspects of proficiency that are expected by the end of secondary school. And, when referring to elementary algebra we mean symbolic algebra as it appears at the end of secondary school. A first draft of the overview was based on relevant literature from national and international publications, which was then presented to experts for validation. In the following we will first give a short report on the main outcomes of the literature review of the first stage, after that we will present the main outcomes of the expert survey. This paper's main part then contains a detailed description of the present state of our model.

Model development

First draft based on literature

The aim of this first stage of model development was to give a structured compilation of the important aspects of proficiency in elementary formal algebra as it is present in research publications of nearly 40 years, starting with Küchemann's paper on children's understanding of variables (1978). A loose series of unconnected aspects would not be of much help where a concise summary is needed. So we decided to build categories from the various aspects found in literature, based on two a-priori dimensions that served as a first theoretical frame for categorization: It starts with listing all relevant mathematical objects of the domain in question, which here are variables and expressions as elements of the algebraic language. The second dimension focusses on the mental and real activities associated with these objects that all represent some stage of "making

sense of algebra" (Arcavi, 1994). Arcavi's approach to describing what is meant by understanding algebra seemed a suitable frame, or better: attitude for collecting and arranging research findings from the past years. "Symbol sense" is, in his own words, "a complex and multifaceted 'feel' for symbols" (Arcavi, 1994, p.31) which invites to searching for aspects of quickly grasping a situation where symbolic algebra is involved. Hence the second of the two dimensions of our model comprises a range of various activities that in some way "make sense" of variables and expressions, ranging from stating correct manipulation rules and identifying expression to which these rules can be applied to modeling realistic situations by means of algebra.

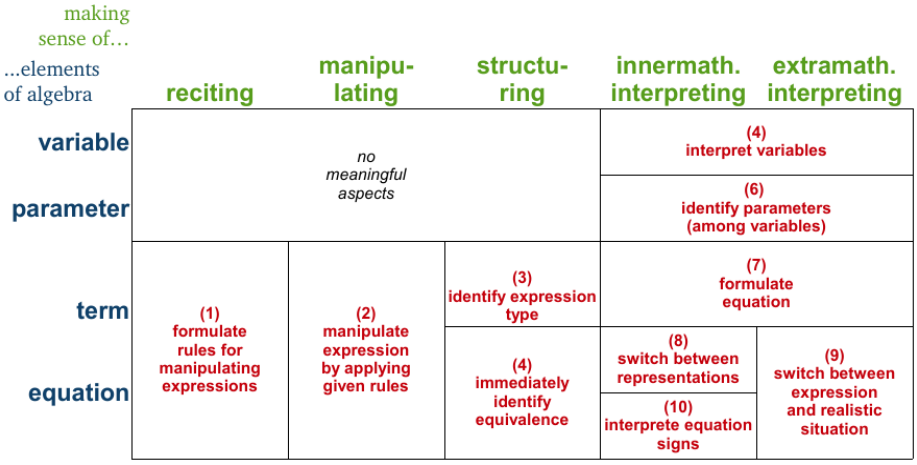

Table 1: Literature based draft of important aspects of proficiency in elementary algebra

This first draft resulted in formulating ten aspects of proficiency, arranged in a tableau along the two dimensions (table 1) introduced above. Here, these ten aspects were defined as single elementary "blocks" of which more complex activities could be constructed by combination. For example, combining three abilities which we then called "(1) formulate rules for manipulating expressions", "(2) identify expression type" and "(3) manipulate expression by applying rules" would sum up to what is generally meant by the ability to manipulate algebraic expressions correctly. While this approach seemed sensible for a detailed diagnosis, it proved to be unhandy in discussions even between experts as the following sections shows.

Expert survey

For validating the results of the literature review an expert survey was conducted. By choosing the experts we focused on professors and seminar teachers of mathematics education in German speaking areas who published primarily about teaching and learning elementary algebra within recent years. Twenty-four colleagues were contacted via e-mail, one of them declined but recommended a colleague as a substitute who, after checking our requirements, was contacted too. All experts were asked to complete an online-questionnaire which was implemented in the free online-tool SoSci Survey (www.soscisurvey.de). The experts were expected to comment on two fields: firstly, on the contents and the structure of the model, while each of the ten aspects was illustrated by one exemplary task, and, secondly, on a battery of additional 48 items that was devised to operationalize the ten aspects. Because giving feedback on all 48 items would have taken too much time we created 6 different versions of the online-questionnaire and reduced the extent of each version by an overlapping multi-matrix-design (cf. Zendler, Vogel & Spannagel, 2013). All in

all, an expert had to view 22 or 23 tasks. Since the response rate was 50%, each task was processed and judged by an average of about 6 randomly assigned experts.

In the first section of the survey the experts were presented an interactive version of Table 1 where explanations of the entries were shown together with exemplary tasks in a pop-up window. Here, they were asked to familiarize themselves with the table entries and explanations while making notes on which aspects they thought needed to be reworked or were unessential for basic mastery of algebra, or which aspects were missing in their view. There also was place for free comments on the model as a whole. In the second section of the survey the experts were asked to assign each of the 22 or 23 task assigned to them to three of the ten aspects of the model and comment on their decision. Here, a smaller version of the interactive table was present with each assignment question. By combining assignment questions and open question types the question format can be considered as being half-open. This type of questions is suitable for experts who are considered to have a differentiated self-awareness, expressiveness, motivation and fidelity (cf. Gerl & Pehl, 1983). The open question format is to be seen as the qualitative part of the online-questionnaire which complies with a written survey in sense of an expert interview (Bogner, Littig, & Menz, 2002). Qualitative methods are especially suitable for purposes of theory-based exploration (cf. Bortz & Döring, 2006). In the following we focus on the results of this qualitative part of the survey.

Data analysis and results of the expert survey

The aim of the expert survey was to validate the first, literature based draft of the model. For data-reduction by clustering analogous comments we followed an open-ended approach to qualitative evaluation as provided by Grounded Theory (Corbin & Strauss, 1990) along these four steps: (1st) All comments from the open question parts of the survey were rephrased by the authors into single conclusive statements about the model or a task. (2nd) All these statements were pasted into the free digital mindmap programm xmind (www.xmind.com) where, for each expert, they were arranged in the order as they appeared in the survey. (3rd) Categorisation took place first with the aim of clustering comments of analogous content – now renamed as “contributions to model revision” and then, (4th), with the aim of categorizing these contributions according to how each of them would contribute to modifying the model or the test battery.

Since many of the statements – though sometimes commenting on single items of the battery – were of a general kind, the process of categorizing eventually detached from the original structure of the questionnaire, so that the following four categories emerged: (a) contributions to clarifying the theoretical frame of the model, (b) contributions to restructuring the model, (c) contributions to reformulating definitions or exemplary tasks of single aspects of the model, (d) contributions to reformulating or deleting a task from the test battery. Among the contributions assigned to category (a), some experts asked how the various aspects of “making sense of algebra” relate to existing models of mathematical understanding. Other experts expressed their uncertainty of how our model relates functions. And other experts were missing activities of preformal algebra. Among the contributions assigned to category (b), one expert pointed out that the activities of transforming or interpreting algebraic expression would imply the activity to identify the structure of an expression so that “structuring” needs to be given a more prominent role in our model’s layout. Some experts mentioned “substituting” as one of the central activities in doing algebra. Additionally, from their task assignments it became apparent, that many experts were misinterpreting the elementary activity

"to transform with given rules" from our first draft as meaning "being able to manipulate expressions correctly" regardless whether a rule is given or not. This contribution also was assigned to category (c), as it would not only lead to introducing a new aspect but to reformulating existing elementary activities too. There were also further contributions to reformulating definitions or exemplary tasks, and to reformulating or deleting tasks from the test battery.

The revised model

For model revision, the authors discussed each contribution as to whether to incorporate it and how. Among contributions that were accepted was that the activity of structuring needed a more prominent position within the model. Here we followed recent findings of Rüede (2015) as to which structuring can be understood as an activity of making sense of an expression by identifying relations between parts. Relations are identified by substitution, i.e. parts of an expression are seen as entities that can be related to each other. Thus the activity of "substituting" was accepted as part of the central activity of structuring. Together with Musgrave's et al. (2015) definition of substitution and Kieran's (1989) distinction between systemic and surface structure it helps to refine activities of recognising the applicability of transformation rules or the operational ordering of an expression. Now, these two activities cover the activity of substituting which, from a cognitive perspective, means to construe parts of an expression as meaningful entities, esp. to replace variables and terms by other variables or terms in writing or in thought.

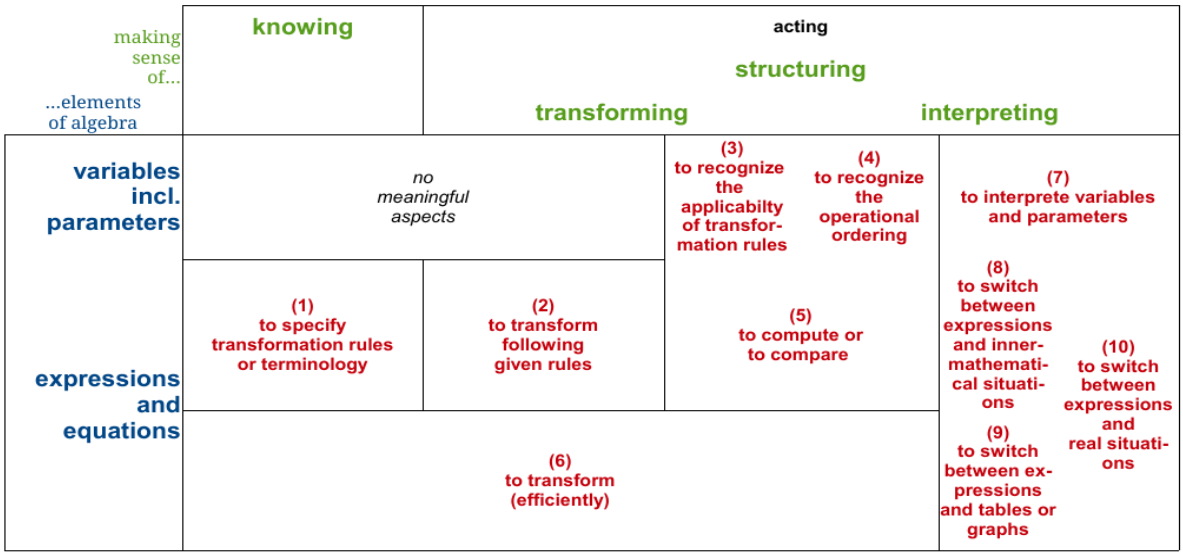

Table 2: A concise summary of the important aspects of proficiency in elementary algebra

Further amendments were applied to the model so that Table 2 now shows its present state. In the following each table entry is explained in detail.

Elements of algebra

- Variables including parameters: Variables are signs that represent numbers or quantities. Parameters are variables that vary over sets of values of other variables (Veränderliche vs. Einzelzahl: Malle, 1993, variable vs. metavariable: Drijvers, 2001, values taken by a variable: Bardini et al., 2005). This discrimination arises from the context of the task.

- Expressions and equations: Algebraic expressions are compositions of variables and arithmetic operation signs. When a variable is viewed as representing a range of number values or quantities (variable object: Schoenfeld & Arcavi, 1988; Bereichsaspekt: Malle, 1993) the value of the expression is interpreted as a function of this variable (Malle, 1993; Heid, 1996). Equations are expressions where two terms are compared with regard to their values, symbolized by an equation sign. An equation differs from a computation or transformation of a term in that it is used in a relational sense (notion of equivalence: Kieran, 1981; operational vs. relational view: Baroody & Ginsburg, 1983; Zuweisungs- vs. Vergleichszeichen: Malle, 1993).

Making sense of elements of algebra

- Knowing and acting: a first level of differentiation that differs between declarative knowledge about rules and various forms of "making sense" of algebraic objects. The latter is further differentiated into transformational and generational types of activities following Kieran (2004). Kieran's third class of "meta-level" type of activities has been omitted as it describes a higher level of mastery that is not considered being a part of basic mastery.
- It seemed appropriate to formulate within the range of Kieran's (2004) transformational and generational types of activities three central activities: Transforming (to transform an algebraic expression into an equivalent expression of different structure (transformational equivalence: Musgrave et al., 2015; treatment: Duval, 2006)), structuring (to transform or interpret an algebraic expression while maintaining its structure (substitutional equivalence: Musgrave et al., 2015, Rüede 2015)) and interpreting (to describe a non-algebraic situation by formal algebra and vice versa (conversion: Duval, 2006)). Among these, structuring takes on a fundamental role. It describes an activity of recognizing the structure of a present expression or formulating an expression that is structurally equivalent to relations between quantities in a given situation.

Ten aspects of proficiency in elementary algebra

(1) "To specify transformation rules or terminology" – Important technical terms for expressions and rules for manipulating expressions or equations are identified or specified, e.g. names for classes of terms or equations, or rules for simplifying expressions, binomial rules, rules for solving quadratic equations, etc.

(2) "To transform by following given rules" – Expressions and equations are transformed into equivalent expressions or equations by applying given rules (manipulation skills: Hoch & Dreyfus, 2006).

(3) "To recognize applicability of transformation rules" – An expression is identified as a representative of a class of structurally equivalent expressions and transformation rules that are associated with this class. This is done by, mentally or explicitly, substituting variables or terms by terms or variables (systemic structure: Kieran, 1989; structure sense: Hoch & Dreyfus, 2006)

(4) "To recognize the operational ordering" – The logical ordering of the operations within an expression is recognized. This is done by, mentally or explicitly, substituting terms by variables (surface structure, Kieran, 1989; Rechenschema: Vollrath & Weigand, 2006; Rechenhandlung: Malle, 1993)

(5) “To compute or to compare” – An expression with an equation sign is interpreted in an operational or a relational sense, as it is appropriate in the context (Malle, 1993; operational vs. relational view: Baroody & Ginsburg, 1983; Knuth et al., 2006).

(6) “To transform (efficiently)” – Expressions and equations are being transformed into equivalent expressions or equations (2,4), by activating existing knowledge about transformation rules (1) which are identified as applicable to the present problem (3). Also, two expressions or equations are identified as being equivalent „on the spot“ without applying rules explicitly (algebraic expectation, Pierce & Stacey, 2001). A transformation is „efficient“ if, among various rules of transformation that are applicable, one is chosen that allows relatively few steps and few computations (strategic flexibility: Rittle-Johnson & Star, 2009; structural relations of second order: Rüede 2015, cp. Malle 1993).

(7) “To interpret variables and parameters” – Variable signs are interpreted or used as representations of numbers (Einsetzungs-, Gegenstandsaspekt: Malle, 1993; Küchemann, 1978). Within given contexts, appropriate variables are identified or used as parameters.

(8) “To switch between expressions and innermathematical situations” – A non-algebraic but innermathematical situation (e.g. dot patterns or geometric configurations) is described by a term or an equation, and vice versa (Bauplan: Vollrath & Weigand, 2006).

(9) “To switch between expressions and tables or graphs” – An expression or equation is translated to a value table or a graph, and vice versa (McGregor & Stacey, 1995), e.g. when viewed as a function (Duval, 2006; Nitsch, 2015), or for solving an equation (Arcavi, 1994).

(10)“To switch between expressions and real situations” – An expression or equation is translated to a realistic situation, and vice versa (McGregor & Stacey, 1995; Heid, 1996), e.g. when viewed as a function (Nitsch, 2015). This activity involves a higher gradient of abstraction than activities (7,8,9) that results from the need to replace the concrete mental model of the given real situation by an abstract mental model before formulating an expression (Malle, 1993).

Summary and outlook

In its present state, the model intends to be a concise summary of aspects of proficiency in elementary algebra, based on relevant literature and a survey of maths educators from the German speaking community. It represents a normative view on what ideal schooling can provide at the end of secondary grade, thus serving as a theoretical base for devising instruments for a summative and differentiated diagnosis of proficiency in elementary algebra at the transition from school to university. The model does not cover all aspects of school needs to consider, but it is restricted to

- symbolic algebra, not generic: at the end of secondary school maths, an individual's proficiency in algebra must have reached a stage of being competent with symbolic representations of indeterminate number values and quantities and relations between them,
- a summative view, not formative: the model is meant to comprise all aspects of proficiency at the end of secondary school maths, not while they are being taught,

- algebra, but not functions: a model about algebraic proficiency cannot cover all aspects of the concept of function, but does cover some which are only present in the form of a functional interpretation of an algebraic expression.

Additionally, based on this frame of reference, a test battery is presently being prepared for large scale application. While the model helps to devise tasks that cover most important aspects of proficiency in elementary algebra, the data raised from applying the battery will be used to generate an empirical cognitive model that adds to the theoretical normative view of the present one.

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Functional thinking and generalisation in third year of primary school

Eder Pinto and María C. Cañadas

University of Granada, Spain

epinto@correo.ugr.es / mconsu@ugr.es

This paper focuses on functional thinking as an approximation to algebraic thinking in third-year primary-school students. It describes a study with a class group of 24 Spanish pupils displaying functional thinking to solve a contextualised problem, identifying the type of functional relationships distinguished by these students and the ability to generalise observed in some of them. It contains an analysis of the information collected in one questionnaire, which is part of a teaching experiment. The students distinguished two types of functional relationships, correspondence and covariation, predominantly the former. Three students generalised as well.

Keywords: Algebraic thinking, generalisation, functional relationship, functional thinking.

Introduction

The idea of introducing algebraic notions in the elementary and even in the pre-school curriculum began to gain acceptance in the early nineteen nineties, when the emphasis was on what students were able to learn (Kaput, 2008). That led to the early algebra proposal, which seeks to further and enhance algebraic thinking among the younger pupils through approximations by working on classroom algebra-associated elements intended to help secondary school students perform the tasks expected of them. Generalisation lies at the heart of algebraic thinking: arithmetic operations can be viewed as functions and algebraic symbolism supports such thinking (Blanton, Levi, Crites, & Dougherty, 2011).

Functional thinking is a type of algebraic thinking, which is focussed on functions, regarded as the relationship between two co-varying quantities. The growing research interest in this type of thinking is attributable to its many advantages as an introduction to algebra (Blanton & Kaput, 2011).

Studies on functional thinking address different aspects. Some of the foremost include: (a) functional relationships drawn by students (Cañadas & Morales, 2016), (b) patterns and generalisation (Brizuela & Lara-Roth, 2002), and (c) representational strategies and systems (Carraher & Schliemann, 2007).

This paper addresses a topic not covered by previous studies concerning the different types of functional relationships identified by third year primary school students (hereafter, P3). The objectives pursued are: (a) to identify P3 pupils exhibiting functional thinking, (b) to describe the generalisation observed, and (c) to describe the functional relationships identified by students who generalise.

Functional thinking and functional relationships

Consensus has yet to be reached around the definition of algebraic thinking (Cañadas, Dooley, Hodgen, & Oldenburg, 2012). Algebraic thinking is regarded as an educational objective that affords, for instance, opportunities: (a) to generalise; and (b) to enable students to use symbols to

represent ideas, which helps them solve problems, communicate and justify their ideas (Kaput, 2008).

Functional thinking is regarded as a cognitive activity "that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationship (individual incidences) to generalisations of that relationship across instances" (Smith, 2008, p. 143). Such thinking involves the construction, description and reasoning with and about functions and includes generalizing about inter-related variables (Blanton, 2008).

Based on the functional relationships established from a mathematical perspective, Smith (2008) proposed three types of approximation for working with functions: (a) recurrence, which entail finding the variation or pattern of variation in a series of values for a variable in a way such that a specific value can be obtained based on the preceding value or values; (b) correspondence, and (c) covariation. We particularly focus on the correspondence and covariation relationships because the recurrence does not involve values of more than one variable. Correspondence, stresses the relationship between the pairs (a, $f(a)$) for the variable; and covariation focuses on how a change in the values of one variable entails a change in the values of another. We show an example of these two functional relationships in Figure 1.

Correspondence. "The focus would be on the relation between x and y, which might be described as twice x plus six, or algebraically as: $2x + 6$".

Variation between two quantities	
x	y
1	8
2	10
3	12
…	…

Covariation. "The focus is on corresponding changes in the individual variable". Example, in the table 1, when x increases by 1, y increases by 2.

Figure 1. Example of functional relationship (Smith, 2008, p. 146-147)

These relationships not only concern the generalization as the representations of the general relationship, they can also refer to the pattern observed in particular cases of the two variables. In recent studies, Blanton, Brizuela, Murphy, Sawrey, and Newman-Owens (2015), and Stephens, Fonger, Knuth, Strachota, and Isler (2016) described the types of functional relationships and the levels of sophistication with which subjects think about such relationships. One indicator for establishing such levels is the kind of functional relationship. Some findings showed that students in the early years evolve from the ability to establish recurrent relationships, the most basic area worked with, primarily in pre-school and early primary education, to the understanding of correspondence and covariation. Cañadas & Morales (2016) observed correspondence and covariation relationships in P1 pupils. These pupils' replies showed no evidence of the recurrence relationship. Moreover, as pupils generalise, correspondence and covariation relationship were observed more frequently, particularly the former.

Generalisation

Generalisation is one of the core processes in algebra (Kaput, 2008). All pupils can generalise and abstract from specifics, for this activity is "entirely natural, pleasurable, and part of human sense-making" (Mason, Graham, & Johnston-Wilder, 2005, p. 2). Generalisation is said to have been attained when a statement is made that applies to all the instances in a given class.

Although algebraic symbolism is the characteristic representation for algebra, there are other ways of representing the generalization, specially when concerns elementary students. Carraher, Martinez, & Schliemann (2008) focused on third year primary school students' generalization and how they express it. These students generalise functional relationships (correspondence and covariation). The authors highlight that students should learn to generalise solving mathematical problems that allow them to look for and observe patterns, relationships, and structures. In this way, students have the possibility to get new informations and reflect about the generalisations produces by themselves and their partners.

Method

This study forms part of a broader teaching experiment focusing on Spanish P3 students' functional thinking. The contextualised problem posed in each session involved a linear function. The fourth and last session is discussed hereunder.

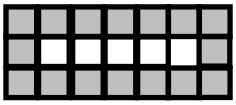
Subjects and data collection

The subjects were 24 P3 pupils (8-9 years old), intentionally selected on the grounds of school and teacher availability. These students had not worked with problems involving functional relationships prior to the study, except in the first three sessions of the same project in which they were introduced to problems involving two linear functions: $f(x)=x+5$ and $f(x)=x+3$. All the sessions were guided by a teacher-researcher.

In the first part of the session, we introduced the tiles problem[1] to the students, asking them questions concerning particular cases, in order to assure that they understand the situation and the questions. In this problem, the function involved is $f(x)=2x+6$. This paper focuses on the results from a written questionnaire that had to be answered individually in connection with the problem posed. The way in which the problem was posed and questions used are presented in Figure 2. In questions Q1, Q2, Q3, Q4.A and Q4.B pupils were asked about specific non-consecutive cases, whilst the fifth (Q5) asked the pupils to generalise the relationship between the dependent and independent variables (white and grey tiles, respectively). The students were furnished with manipulative material: white and grey paper tiles.

[1] The well-known tiles problem used here has been applied by a number of researchers in the context of classroom algebra (e.g., Küchemann, 1981).

A school wants to renovate the ground of all its corridors because it is already very damaged. The management team decides to pave the corridor with white tiles and grey tiles. All tiles are squares and have the same size. The tiles are being placed in each corridor so that you can see in the picture below.



The school ask a company for renovating the different corridors of the school. We ask you to help the workers to answer some questions that they need to answer for their work.

Q1. How many grey tiles are needed for the floor of a corridor in which 5 white tiles are placed? How do you know that?

Q2. Some corridors are longer than others. Therefore, the workers need different number of tiles for each corridor. How many grey tiles are needed for a floor corridor in which 8 white tiles are placed? How do you know that?

Q3. How many grey tiles are needed for a floor corridor in which 10 white tiles are placed? How do you know that?

Q4A. How many grey tiles are needed for a corridor floor in which 100 white tiles are placed? How do you know that?

Q4B. Now, do it in a different way and explain it below.

Q5. The workers always place the white tiles and then the grey tiles first. How do you know how many grey tiles you need if you have already placed the white tiles?

Figure 2. Tiles problem

Analytical categories and data analysis

Following our research objectives, we used information from the theoretical framework and previous studies to design the categories used in data analysis. Moreover, we were aware of possible modifications needed as long as we performed a preliminary data analysis in order to adapt them to our specific data. Two categories were established: (a) functional relationships, and (b) generalisation. Finally both categories were related because the generalisation involved at least one of the functional relationships.

The category of functional relationship covered the type of functional relationships identified by the pupils: (a) correspondence, and (b) covariation. Functional thinking was deemed to be present in pupils' replies when at least one functional relationship was drawn in at least two of the questions posed. This criterion pursued to avoid those students who used a computation strategy but not neccesarily a relationship between variables.

The second category dealt with the presence or absence of generalisation and how it was reached and expressed in any of the five questions posed. More specifically, it focused on the students' replies to Q5 (regarding generalisation), because is the only question in which students generalised.

Results and discussion

The 24 pupils' written responses to the questionnaires were analysed. All the students answered the first three questions, 23 the fourth one, and 16 the fifth one (generalisation). Those findings were

interpreted to mean that more students answered the first four questions because they involved specific, non-consecutive instances and small numbers. Similarly, the high rate of blank answers to Q5 was conjectured to be due to the complexity involved in generalizing the functional relationship.

The findings set out below are organised in keeping with the objectives pursued. The students are referred to with the letter S followed by a number, from 1 to 24.

Functional relationships

Eleven students exhibited functional thinking, identifying a functional relationship in at least two of the questions asked. The other 13 students, in contrast, did not.

Among the students exhibiting functional thinking, seven students distinguished only correspondence relationships in their replies, and four identified both functional relationships (correspondence and covariation).

A representative example of students who used only the correspondence relationship is S22. We show this student's responses to the first four questions in Figure 3.

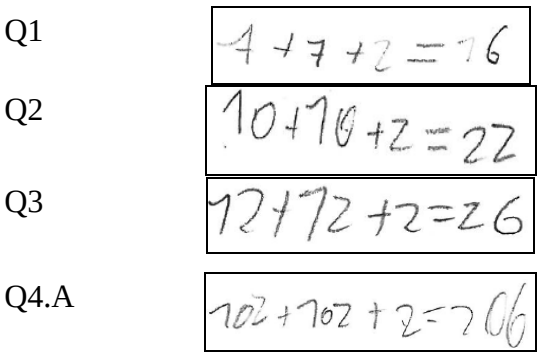


Figure 3. S22's responses

For instance, in Q4.A, he took the number of white tiles (100) and added 2 (100+2). Then he found the number of grey tiles needed for the bottom and top rows by adding 102 twice (102 + 102). Lastly, he added 2, the ones on the right and left of the white tiles (102 + 102 + 2). He used that same functional relationship for 5, 8 and 10 white tiles in Q1, Q2, and Q3, respectively. In all three cases, this student related pairs of values (a, f(a)) to the a values in each specific case and established a relationship with the number of grey tiles: 16, 22 and 26, respectively.

Four pupils identified two functional relationships in their answers to the questions on the questionnaire: correspondence and covariation. None of the students recognised more than one functional relationship in their answers to a given question.

S3 is a student who identified correspondence and covariation relationships. In Q2, she answered that 22 grey tiles are necessary for 8 white tiles, using a counting strategy. In Q3, S3 answered, "if 8 [white tiles] need 20 [grey tiles], there are 20 + 2 = 22". Although this answer is wrong, she used the previous response to work on (adding two to the previous response). We observe that the student focused on the variation between the number of white tiles (between 8 and 10, there is an increase of 2 white tiles) in order to calculate the number of grey tiles, considering that such increase is also 2. Therefore, she focused on how variation in values of the number of white tiles influence in a variation in values of grey tiles, which is the notion of covariation relationship.

Generalisation

We find generalization evidence in Q5. In previous questions, students referred to the relationship between variables through particular cases involved.

Three of the 11 pupils who exhibited functional thinking showed the generalisation in their replies to Q5. One of them, S9, generalised appropriately to the problem posed. In contrast, the other two students —S11 and S22— generalised incorrectly. In what follows, we present examples of the students who generalized, describing when they got it and what kind of relationship generalised.

S9 used a numerical representation to calculate the number of grey tiles in the first four questions. In Q5, he stated "you double the number white tiles and then you add 6". He used different representation to the verbal one in other questions. This fact evidences the importance of the verbal representation in the development of functional thinking in the same way as Kieran, Pang, Schifter, and Fong Ng (2016) noted. Student S9 generalised the correspondence relationship that he also identified in questions concerning particular cases.

S11 found a correspondence relationship in the questions concerning particular cases. In his reply to Q5 he noted: "if there are 50 tiles, then I add 50+50 and then the ones on the sides, 3+3, 106 in all". The student used a particular case to answer the question but he evidenced that he recognised the fixed number of grey tiles (3+3). This fact shows generalization at an initial stage: although his answer is not complete, he is approaching to the generalization of the relationship because he identified the function constant (6). S11 used different relationships to determine the number of grey tiles in Q2, Q3, and Q4, focussing on a correspondence relationship between the variables involved. He "generalise" the correspondence relationship in Q5.

S22's reply illustrates another way to generalise in Q5: "add 6". This generalisation was incomplete, for she recognised the number of grey tiles that remains constant (left and right sides), but not the number on the top and bottom rows, even though in the preceding questions she distinguished the pattern for determining the number of grey tiles given a certain number of white tiles (see Figure 1). Moreover, S22 used the correspondence relationships to answer the first four questions (see Figure 3). On the contrary, this student used a co-variation relationship in Q5 because he identified the neccessity of adding 6 to calculate the number of grey tiles given any number of white tiles.

Conclusion

The students exhibiting functional thinking (those recognizing at least one functional relationship in at least two questions) could be detected on the grounds of the relationships they identified.

The correspondence was the functional relationship predominantly observed in the students' answers, followed by covariation. This holds particular significance, specifically by: the pupils' age, the specific demands of the tiles problem and the functional relationships distinguished. The prevalence of the correspondence relationship in the first four questions, which involved familiar specific cases, seemed to be connected with the pupils' broader experience with areas such as numerical patterns. Additionally, we conjecture that this functional relationship could be induced by the problem context because each particular case involved in one question is not connected with other particular cases.

Covariation was observed in Q5, which sought to induce the pupils to express the general relationship between the variables involved; the preceding questions could be answered with no need to generalise.

According to Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015), functional thinking involves (among others) drawing general patterns from relationships between quantities that co-vary and representing and justifying such relationships in different ways with a number of representational systems. The results of their study are supplemented by the present findings, further to which P3 pupils naturally (for they had not worked on this area in the classroom) identified more than mere recurrence, establishing relationships (correspondence and covariance) involving the values of both variables. Whilst influenced by the type of problem, these P3 students were found to be able to distinguish correspondence and covariation relationships, even though they were not always able recognise a general pattern.

In a future line of research the way generalisation is expressed will be studied in greater depth, along with pupils' arguments and explanations. Student interviews are regarded as a suitable tool for obtaining a fuller description of how inter-variable relationships are expressed.

Acknowledgements

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Didactic transposition in school algebra: The case of writing equations of parallel and perpendicular lines

Valentina Postelnicu

Texas A &M University-Corpus Christi, USA; Valentina.Postelnicu@tamucc.edu

A study was conducted with a high school teacher and her 58 Algebra 1 students, with the purpose of gaining insight into students' difficulties with writing equations of parallel and perpendicular lines. Chevallard's theory of didactic transposition was employed in order to account for the relativity of the mathematical knowledge with respect to the institutions where the knowledge was created. During the process of didactic transposition, the mathematical knowledge lost its essential feature, the proof, with dramatic consequences for the school algebra curriculum. What remained to be taught and learned was how to execute tasks. As predicted by mathematicians, this utilitarian view of the curriculum affected the actual process of teaching and learning by focusing on executing basic tasks, and resulted in teachers and students having difficulty executing those tasks.

Keywords: Didactic transposition, school algebra, writing equations of lines.

Introduction

Students' difficulties with aspects of linear functions, like the rate of change, slope, *y*-intercept, or writing equations of parallel lines have been studied in the United States by many researchers, among them the group from Berkeley led by Schoenfeld (Schoenfeld, Smith, & Arcavi, 1993). In spite of a wealth of research, the issue of students' difficulties with linear functions has remained relevant (Postelnicu, 2013). The study reported here has the purpose to account for the students' difficulties with writing equations of parallel and perpendicular lines, and to advance a plausible explanation on the persistence of students' difficulties. We propose another way to look at this issue, by paying attention to the difference between the mathematics as a body of knowledge (scholarly mathematics created by mathematicians) and mathematics as a subject matter to be taught and learned (school mathematics created by textbook authors or taught by teachers) (Bosch & Gascón, 2006).

About the theory of didactic transposition

For this study we employ Chevallard's theory of didactic transposition because it takes into account both the mathematics as a scholarly body of knowledge and as a subject to be taught and learned. By *didactic transposition* of knowledge, we mean "the transition from knowledge regarded as a tool to be put to use to knowledge as something to be taught and learnt" (Chevallard, 1988). As pointed out by Bosch and Gascón (2006), when studying any didactic problem, like the teaching and learning of writing equations of parallel and perpendicular lines, we must account for all the steps of the process of didactic transposition:

i) from scholarly knowledge created by mathematicians (e.g., analytic geometry) to school mathematics written by textbook authors (e.g., high school textbooks); ii) from textbook knowledge (e.g., Algebra I textbook written by Larson et al., 2007) to school mathematics taught by teachers in classrooms (e.g., the mathematics taught by the teacher in the study reported here); and iii) from school mathematics taught by the teacher in this study to mathematics learned by her students.

Each society and institution has a certain way of creating mathematical knowledge and using it, thus bringing to life a *praxeology*, "an organized way of doing and thinking contrived within a given society" (Chevallard, 2006). A praxeology is composed of *praxis* and *logos*, each with two components (Bosch & Gascón, 2014):

i) *praxis* - **tasks**/problems that can be solved employing a certain **technique** ("ways of doing"/executing the tasks/solving the problems), and

ii) *logos*- **technology** (the discourse of the technique, justifies the technique), and **theory** (general discourse or abstract set of constructs and arguments, justifies the technology).

A praxeology can be "*point praxeology*" with only one type of task, "*local praxeology*" with a set of tasks sharing a technological discourse, and "*regional praxelogy*" with all the point and local praxeologies sharing a theory (Bosch & Gascón, 2014). For example, a point praxeology may contain only the task of sketching a "quick" graph of a line with an equation written in slope-intercept form. A local praxelogy may contain all the tasks requiring writing equations of lines given certain conditions, including tasks like Tasks 1 and 2 given to students by the teacher from our study:

Task 1*:* Sketch the graph of the line that is parallel to $y = \frac{1}{2}x - 3$ and goes through $(2,-1)$.

Task 2: Sketch the graph of the line that is perpendicular to $y = \frac{3}{4}x - 2$ and goes through the point $(2,-6)$. What is the equation of the new line you created?

Tasks 1 and 2 have the same target knowledge, an algorithm for solving the class of problems that require the writing of the equation of a line passing through a given point and parallel or perpendicular to a given line. Such tasks are similar to those proposed in the textbook used by the participants in this study. The technique used to execute Task 1 may employ algorithms like the one used by the teacher in this study:

1. Draw a Cartesian systems of coordinates and plot the given point $(2,-1)$.
2. Identify the slope of the line parallel with the given line, $m = \frac{1}{2}$.
3. Use a "quick graph" to obtain a second point on the line (from the point $(2,-1)$ move 1 unit up and 2 units to the right, and obtain and plot the point $(4,0)$).
4. Draw the line passing through the given point $(2,-1)$ and the newly found point $(4,0)$.

Part of the technological discourse for this local praxeology may include the justification for the fact that the second point $(4,0)$ obtained in the way described above is indeed situated on the line with the slope $m = \frac{1}{2}$ and passing through $(2,-1)$, or the justification that the line parallel to the line $y = \frac{1}{2}x - 3$ has the same slope, $m = \frac{1}{2}$. A theoretical justification of the fact that parallel lines have the same slope and, conversely (i.e., a theorem and its proof in analytic geometry, based on similarity of triangles) may belong to a regional praxeology.

Method

Participants in this study were an Algebra 1 teacher from a public high school in the United States and her 58 students who agreed to participate in this study. The teacher had a Bachelor degree in Mathematics and five years of teaching experience. Classroom observations (Erickson, 1985) were conducted by the researcher/author of this paper for all six Algebra 1 classes of 50 minutes each, taught by the participating teacher, during the same day of school. The researcher took notes referring to the way the teacher and her students interacted with the mathematical content of the tasks. The teacher taught the same lesson, about writing equations of parallel and perpendicular lines, to each of her classes. Prior to the day of observation, the teacher introduced to her students the equation of a line in slope-intercept form and point-slope form, and the notions of parallel and perpendicular lines. The teacher started each lesson with Task 1 (described above), discussed it with her students, and then administered Task 2 (described above). Two raters scored students' answers to Task 2 ("1" for correct answer, and "0" for incorrect or incomplete answer). Using techniques from grounded theory (Strauss & Corbin, 1998), students' algorithms were split into two categories, algorithms with a graphical approach (teacher's approach, described in the section referring to the theory of didactic transposition, and in the section referring to teacher knowledge), or algorithms with an algebraic approach (textbook approach, described in the section referring to textbook knowledge). The inter-rater agreement (Cohen, 1960) was very high, k = .92 (p < .001), 95 % CI [.83, .98].

Analysis and results

Figure 1, below, is adapted from Bosch & Gascón (2006) and illustrates the steps of the process of didactic transposition specific to the study presented in this paper.

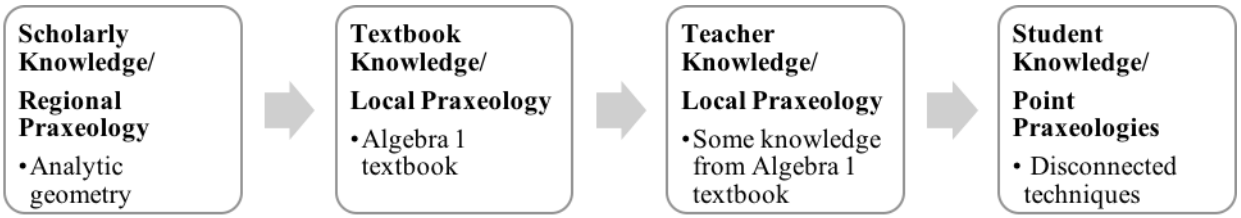

Figure 1. The process of didactic transposition

We will describe each of the types of knowledge/praxeologies in Figure 1, starting from the scholarly knowledge, in the direction of the arrows. Worth mentioning, in determining the regional, local or point praxeologies, one starts from the task and what the task entails, in our case writing linear equations of parallel or perpendicular lines to a given line, and passing through a given point.

Scholarly knowledge/Regional praxeology

As can be seen in Figure 1, we chose analytic geometry as our regional praxeology. Given the space constraints, we refer here only to perpendicular lines. When writing equations of perpendicular lines, we use the following **theorem**:

> Two nonvertical, nonhorizontal lines l_1, l_2 with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$ (Kay, 2001, p. 303).

Worth mentioning, the proof chosen for the above theorem is specific to the participants' institutionalized knowledge. Our regional praxeology/theory contains all the axioms, definitions, theorems and their proofs needed to prove the above theorem. An example of a path through the theory is given by Kay (2001) in his textbook, College Geometry, where he starts constructing the geometry with the foundations of absolute geometry (points, lines, segments, angles, triangles, quadrialterals, circles), and continues with the Euclidean geometry (trigonometry, coordinates, vectors). This path of knowledge includes the definition of a right angle, the definition of perpendicular lines, the Pythagorean theorem and its converse together with its proof based on similarity of triangles, and the distance formula between two points, given their coordinates. Within this regional praxeology, "a right angle is any angle having measure 90" and "two (distinct) lines l_1, l_2 are perpendicular if and only if l_1, l_2 contain the sides of a right angle" (Kay, 2001, p. 97). A proof of the theorem stated above is simple (see Figure 2).

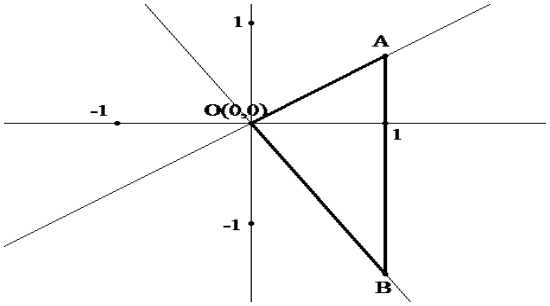

Figure 2. Perpendicular lines and their slopes

As can be seen in Figure 2, the system of coordinates has been specially chosen, without loss of generality, so that its origin, *O*, coincides with the point of intersection of the two lines, l_1 (with slope m_1, containing the segment *OA*, with *A* chosen such that its *x*-coordinate, $x_A = 1$) and l_2 (with slope m_2, containing the segment *OB* with *B* chosen such that its *x*-coordinate, $x_B = 1$). The line l_1 passes through $O(0,0)$ and has the slope m_1, hence all its points (x, y) satisfy the equation $y = m_1 x$. Similarly, all the points (x, y) situated on l_2 satisfy the equation $y = m_2 x$. As such, the points *A* and *B* have the coordinates: $A(1, m_1)$ and $B(1, m_2)$, respectively. To prove the direct implication of the theorem, we assume that the lines are perpendicular therefore they contain the sides of a right angle, hence the triangle AOB is right, and according to the Pythagorean theorem we have $OA^2 + OB^2 = AB^2$. Using the distance formula to calculate *OA*, *OB*, and *AB* function of their coordinates, we obtain $m_1^2 + 1 + m_2^2 + 1 = (m_2 - m_1)^2$ and after we simplify, we have $m_1 m_2 = -1$. Conversely, if $m_1 m_2 = -1$, then $m_1^2 + 1 + m_2^2 + 1 = (m_2 - m_1)^2$, therefore $OA^2 + OB^2 = AB^2$. Using the converse of the Pythagorean theorem, it follows that the triangle AOB is right, hence the lines l_1 and l_2 containing its legs, *OA* and *OB*, respectively, are perpendicular.

The observation that any vertical line with the equation $x = a$ is perpendicular to any horizontal line with the equation $y = b$ takes care of the exception stated in the theorem ("nonvertical, nonhorizontal lines").

Textbook knowledge/ Local praxeology

In the United States, the Algebra 1 course, usually taught in the first year of high school, contains topics like linear equations with one and two variables, linear functions, linear inequalities, systems of linear equations, quadratic equations and functions, and introductions to polynomials, exponential equations and functions. For historical reasons (Kilpatrick & Izsák, 2008), the American students take the Algebra 1 course before Geometry, thus they learn about the slope of a line, or slopes of parallel and perpendicular lines before learning about similarity or how to prove the Pythagorean theorem. As such, there is no expectation for justifications or proofs for the "key facts" stated in Algebra 1 textbooks. The textbook knowledge for this study comes from the Algebra 1 textbook used by the participants' school district (Larson et al., 2007). The lesson referring to the writing of parallel and perpendicular lines offers some techniques and the technological discourse to justify the techniques (e.g., definition of perpendicular lines and a "key concept" without proof - the theorem referring to the slopes of perpendicular lines):

> PERPENDICULAR LINES. Two lines in the same plane are perpendicular if they intersect to form a right angle. Horizontal and vertical lines are perpendicular to each other.
>
> KEY CONCEPT. If two nonvertical lines in the same plane have slopes that are negative reciprocals, then the lines are perpendicular. If two nonvertical lines in the same plane are perpendicular, then the slopes are negative reciprocals (Larson et al., 2007, p. 320).

A similar definition and a similar theorem about parallel lines precede the definition and theorem about perpendicular lines. The technique/algorithm described in the textbook (Larson et al., 2007, pp. 319-321) for executing tasks similar to Task 1 and 2 has an algebraic approach (Knuth, 2000):

1. Identify the slope *m* of the new line based on the "key concepts" referring to the slopes of parallel or perpendicular lines (parallel lines have the same slope, perpendicular lines have slopes negative reciprocals).
2. Use the slope-intercept form of an equation $y = mx + b$, the newly found slope *m*, and the given point (x_1, y_1) to find *b* (the *y*-intercept).
3. Write the equation of the newly found line $y = mx + b$.

Teacher knowledge/Local praxeology

We continue describing Figure 1 with teacher knowledge, as observed. After discussions with her students, the teacher executed Task 1 using the algorithm with the graphical approach described above, in the section about the theory of didactic transposition. She drew a system of coordinates, plotted the point $(2,-1)$, identified the slope of the parallel line $m = \frac{1}{2}$ and used it to obtain the second point $(4,0)$ (from $(2,-1)$ moved 1 unit up and 2 units to the right), and drew the line connecting the points $(2,-1)$ and $(4,0)$. To check the execution of Task 1, the teacher proposed to her students to graph the first line $y = \frac{1}{2}x - 3$ as well, on the same system of coordinates like the newly obtained line, and make a judgment regarding their parallelism, based on visual inspection. When one of the students proposed a technique with an algebraic approach for Task 1, the teacher allowed the student to carry it out, but then she asked the class for another way to execute Task 1, and led the students to use her graphical approach technique. Part of the observed technological

discourse for the teacher's local praxeology also included the definitions of parallel lines ("lines that do not meet") and perpendicular lines ("two lines in a plane that intersect at a 90° angle"), and the "key concepts" that parallel lines have the same slope and perpendicular lines have the slopes opposite reciprocals. In short, the teacher's local praxeology, as observed, consisted of some basic tasks, a technique with a graphical approach, and some of the technological discourse necessary to justify the technique. In the teacher's view, as expressed during the class discussions, Tasks 1 and 2 were part of the same class of problems, requiring the same technique, and Task 2 could be approached in the same way as she approached Task 1, i.e., graphically.

Student knowledge/Point praxeologies

Executing Task 2 with the algorithm from the textbook implies: i) identify the slope of the perpendicular line, $m = -\frac{4}{3}$; ii) substitute $(2,-6)$ and $m = -\frac{4}{3}$ in $y = mx + b$, and determine b from $-6 = -\frac{4}{3}(2) + b$; iii) write the equation of the newly found line $y = -\frac{4}{3}x - \frac{10}{3}$. Only two students (3.4 %) of those 58 participating in the study executed Task 2 successfully, and both employed the technique with an algebraic approach described in their textbook. About one out of five students (20.6 %) tried to solve the problem employing the technique with the graphical approach used by the teacher – they determined the slope of the perpendicular line, $m = -\frac{4}{3}$, started from the given point $(2,-6)$ and used the "quick graph" technique to obtain the second point on the line, then drew the line connecting those two points and determined incorrectly, by visual inspection, the value of the *y*-intercept of the newly created line, and finally wrote the equation of the line (see Figure 3).

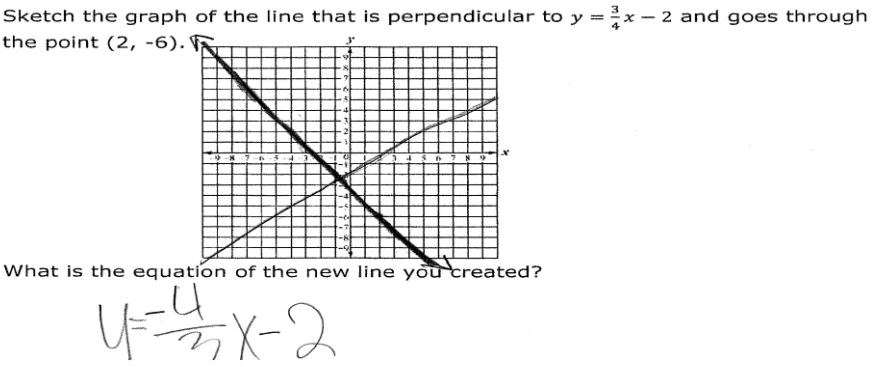

Figure 3. Example of student work – Task 2

It can be seen in Figure 3 that the student incorrectly determined the *y*-intercept of the newly created line as -2, while the correct value is $-\frac{10}{3}$, a value hard to determine with the graphical approach. Moreover, the student tried to check/evaluate her work, and graphed the original line together with the newly created line and it seems that she was satisfied with her visual inspection of the perpendicularity of those two lines she graphed. The rest of the students (76 %) tried to use a graphical approach (strongly suggested by the teacher's approach and the presence of the grid), but could not identify the slope of the perpendicular line, or determine the *y*-intercept. When writing the equation of the perpendicular line, students tried to substitute the point $(2,-6)$ or its coordinates in

the slope-intercept form equation of a line $y = mx + b$, irrespective of their graphical representations or the meanings of the coordinates of the point, slope of the line, and *y*-intercept of the line. They obtained equations, like "$y = -\frac{4}{3}x + (2,-6)$", "$y = -\frac{6}{2}x - 2,$" or "$y = 2x - 6,$" showing a great disconnect with the technique required to execute Task 2. As observed from their written assignments, the students' praxeologies were point praxeologies with only one task and technique (successful or not), without any justification of the technique.

Discussion

The scholarly knowledge constructed from axioms, definitions, and rigorously proven theorems has been replaced by other definitions and axiom-like "key concepts" considered true without proof. In our case, the theorem regarding the slopes of perpendicular lines and its proof has been replaced by a "key concept" without proof, and several "solved examples" presenting the technique for executing basic tasks. What remained to be taught and learned was how to execute basic tasks. The change was dramatic from scholarly mathematics to textbook knowledge, since mathematics was stripped of its essential feature – the proof. The mathematicians warned that this dramatic change in our curriculum would lead to generations of teachers and students with increasing difficulty executing tasks requiring more than one step (Wu, 1997). The textbook knowledge shrank to teacher knowledge, but remained the same in nature, i.e., a set of tasks, techniques, and justifications for techniques. As observed, the teacher from this study used only basic tasks that were similar with the examples solved in the textbook, employed only the graphical approach, and supported her technique with some justifications. The teacher may have favored those tasks and techniques for various reasons like, the end-of-course exams contained mainly basic tasks for which the techniques were appropriate, her students' weak competency with symbolic manipulations, and time constraints. The technique with a graphical approach imposed by the teacher in this study was inadequate for Task 2. Almost all the students (96.6 %) relied on the teacher's knowledge – a subset of the textbook knowledge. The students failed to connect the graphical and symbolic representations of points and lines, and the techniques necessary to carry out Task 2. As predicted by mathematicians (Wu, 1997), the students had difficulty executing tasks with more than one step.

The observed *logos* from the local praxeologies (textbook and teacher knowledge) contained only the technological discourse. There was no observed *logos* in the case of point praxeologies (student knowledge). Without theory based on proof, the knowledge advancement can only be obtained by learning to execute new tasks with new techniques that are not necessarily connected to old ones. As seen in our study, this type of knowledge advancement did not lead to teacher's or students' success.

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Students' and teachers' mental solving of algebraic equations: From differences to challenges

Jérôme Proulx, Marie-Line Lavallée-Lamarche, & Karl-Philippe Tremblay

Université du Québec à Montréal, Canada; proulx.jerome@uqam.ca

This paper presents students' and teachers' strategies for mentally solving algebraic equations. The enactivist notions of problem-posing offer conceptual grounds to engage in analysis of students' and teachers' strategies, and in their comparisons, leading to the exploration of differences in the nature and origin between the solving processes of students and those of teachers. Final remarks reflect on the potential of being sensitized to the nature of these differences in solving processes.

Keywords: Algebraic equations, mental mathematics, problem-solving.

Introduction

This paper is in continuity with the one presented at CERME-8 in WG3 about the mental solving of algebraic equations (Proulx, 2013a). Work has been conducted with secondary-level students and with teachers on mentally solving algebraic equations of the form Ax+B=C, Ax+B=Cx+D, Ax/B=C/D without paper and pencil or any other material aid. The main research objective is to gain better understandings of the nature of the strategies developed when solving these algebraic equations mentally. From our analysis, students' and teachers' ways of solving highlight significant differences to which it is worth paying attention in terms of their meaning and nature as well as the issues that they raise about the teaching and learning of algebraic equation-solving. This paper reports on analyses of the strategies developed by students and teachers for similar if not identical algebraic equations, and grounds it in a discussion of the nature of solving processes.

Solving processes in mental mathematics: Emergence and problem-posing

As mentioned in Proulx (2013a), recent work in mental mathematics points to a continued need for investigating and conceptualizing how students develop mental mathematics strategies. Researchers have begun criticizing the notion that students "choose" from a toolbox of predetermined strategies to solve problems in mental mathematics. Threlfall (2002), for example, insists rather on the organic emergence and contingency of strategies in relation to the tasks *and* the solver (e.g., what s/he understands, prefers, knows, experienced, is confident with; see Butlen & Peizard, 1992). This view is aligned with Lave's (1988) situated cognition perspective that conceives of mental strategies as flexible emergent responses, adapted and linked to specific contexts and situations.

In mathematics education, the enactivist theory of cognition has been concerned with issues of emergence, adaptation and contingency of learners' mathematical activity (e.g., Maturana & Varela, 1992; Varela, Thompson, & Rosch, 1991). In particular, Varela's (1996) distinction between problem-posing and problem-solving offers insights for conceptualizing the generation of strategies. For Varela, problem-solving implies an understanding that problems are already in the world, lying "out there" waiting to be solved independent of us. He explains, in contrast, that we specify – we pose moment by moment – the problems that we encounter through the meanings we make of the world. We do not "choose" or "take" problems as if they were lying out there objectively and independent of our actions: we bring them forth. In short, for Varela, we *pose* our problems.

This perspective underlines Simmt's (2000) argument that it is not tasks that are given to solvers, but rather prompts that are taken up by solvers, who by posing them in a specific mathematical context create tasks with them. Prompts become tasks when solvers engage with them, when they pose them as tasks. Hence solvers make the prompt a multiplication task, a ratio task, a function task, an algebra task, and so forth, and solve it in relation to this posing. And, this posed task is not static or fixed once and for all, because the posing triggers a solving process that in turn transforms the posed task in an ongoing and dynamic process. It is with/in this process that the task emerges, organically, constantly becoming, being re-solved and re-posed (see Proulx, 2013b). By way of an example of this interaction between the posing and the solving, here is a strategy taken from a study on mental calculations (Proulx et al., 2014). To solve 741–75, one solver explained:

(a) *741 – 75 is like 700 – 75 + 41*.
(b) *700 – 75 is like having 7 dollars and subtracting 3 quarters. I am left with $6.25.*
(c) *6.25 is six-twenty-five, so I add 41 to 625. I do 5+1 is 6, 4+2 is 6, and I have 600, so 666*.

When 741–75 was given, the first step was to find a way of solving, of entering, of posing it as a task. This prompt was then posed as a decomposition task, leading the solver to decompose 741 in 700 and 41, in order to subtract 75 from 700. This decomposition produced in return a new prompt for the solver, that is, to solve 700–75, which was posed as a monetary task (7 dollars minus 75 cents). This other solving step led to another prompt, 625+41, which the solver posed anew as a decomposition task of each digit in each number in relation to their position (hundreds, tens, units) and its successive addition. Hence each solving step led to the posing of a task to solve, necessitating a way of entering into it, continuing to solve it, etc., producing an entire solution path.

A posteriori, in light of this entire solution path, one can assert the presence of a strategy, but this is an assertion after the fact because all this unfolds one step at a time when advancing in the solving of the posed task. Interacting with the prompt, engaging in the task, is to take a step and another, and these steps emerge in the solution path. All this happens in continuity, step by step, with each step leading to another posing of the task, to another solving process contingent on and emerging from previous steps, leading in return to another posed task, leading to another step, etc.

> [E]ach solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. […] The 'strategy' (in the holistic sense of the entire solution path) is not decided, it emerges. (Threlfall, 2002, p. 42)

The entire solution path, or strategy in Threlfall's sense[1], is not predetermined, but generated for solving, emerging from the interaction with the prompt. Thus the solver *transforms* the prompt as a mathematical task generating a strategy for the posed task for solving it. It is this dynamic entry on strategies, on solution paths, that characterizes the conception that grounds the analysis of solving processes in this study, that is, of students' and teachers' strategies for solving algebraic equations.

[1] It is in Threlfall's sense that the expression *strategy* is used in this paper, that is, not as a fixed and reified entity, but as the *entire solution path*, in its totality and dynamic nature as it unfolds through the diverse *solving steps*.

Solving algebraic equations without paper and pencil

The context in which the mental mathematics sessions are conducted is simple. A group of solvers, students or teachers, sit at their table without any paper or pencil and attempt to solve the prompts given. The organization takes the following structure: (1) an equation is offered in writing on the board; (2) solvers have 15-20 seconds to solve the equation mentally (without paper-and-pencil or material aid); (3) at the signal, strategies are shared orally and explained to the group.

The data collected come from the strategies explained orally by solvers, taken in note form by at least two research assistants (RAs). These notes are refined with on-the-spot discussions between the research team members (PI and RAs) following data-collection sessions, to produce a report on the various solution paths developed by participants. In this report, each strategy is given a name that describes it for matters of classification (descriptive level of analysis). With these descriptions, various analyses are conducted, depending on the purpose aimed for (meaning making engaged with, nature of strategies produced, difference with paper and pencil, etc.). In this case, as explained in the introduction, the analysis of strategies is conducted with the precise intention of establishing comparisons between students' and teachers' solving processes. Thus the analysis in this paper is not conducted for/on the strategies themselves, but mostly to establish this comparison ground between the solution paths being laid down by students and by teachers.

Students' posed tasks and solution paths

Work was conducted in two Grade-8 classrooms with about 30 students for 75-minute periods. In each classroom, the same five equations were given. The two used in the analysis are "Solve for x the equation $2x+3=5$" and "Solve for x the equation $\frac{2}{5}x=\frac{1}{2}$"; the solution paths emerging for these are illustrative of the solving processes developed by students. As such, the analysis is not focused on the occurences of strategies, but on the nature of the solving processes and the functionality and meaning of the strategies developed. For this, strategies in both classrooms are grouped to offer and set up comparative grounds with teachers' solving processes.

For "Solve for x the equation $2x+3=5$", students produced the following:

Inversing operations. One student explained that he did "minus 3" on the right side of the equality to obtain $2x=2$, which directly gave $x=1$ (without needing to divide by 2).

Balancing. One student explained having done the same actions on both sides of the equality, thus subtracting 3 on both sides to obtain $2x=2$ and then dividing by 2 on both sides to get $x=1$.

Direct reading. Some students explained that with $2x+3=5$ he knew right away that x must equal 1, because $2+3=5$. Another explained having first taken the x out of the equation, leaving $2+3=5$. So, when 2 was multiplied by x, it has to remain a value of 2 to fit in the equation, so x needed to be 1. Another student explained that this meant that $x=0$, because in $2+3=5$ the x is uneccessary.

For "Solve for x the equation $\frac{2}{5}x=\frac{1}{2}$", students produced the following:

Transforming in equivalent fractions and decimals. One student explained having transformed ½ in 10/20 to make the ½ divided by 2/5 simpler, then repeating the same thing for 2/5 to obtain $\frac{20}{50}x=\frac{10}{20}$. He explained that this is equivalent to $0.4x=0.5$ and thus the response is $x=0.5/0.4$.

Inversing and transforming in decimals. One student explained inversing the equation to $\frac{5}{2}x = 2$. He then transformed in decimals to obtain 2.5*x*=2, and dividing by 2 got 1.25*x*=1 so that *x*=1.25.

Cross multiplying. After making the equation $\frac{2x}{5} = \frac{1}{2}$, the student explained having cross multiplied, where 5 times ½ gave 2*x*= 2.5 and thus *x*=1.25.

Halving. One student explained that half of 5 is 2.5, and because one looks for ½, so *x* is 1.25.

Finding a scalar. One student explained looking for the value of *x* that made 2/5 equal ½. Placing fractions over 10, he explained that *x* is 1.25 because 4 times 1.25 is 5 and 5/10=½.

Finding common denominator and adding. The student explained having placed fractions over 10, obtaining $\frac{4}{10}x = \frac{5}{10}$. Subtracting 4/10 and 5/10 gave –1/10, so then *x* is worth 1/10.

All these solution paths, and their underlying solving steps, are not necessarily "adequate" or "standard", but illustrate an emergent solving process geared toward finding a value for *x* that satisfies the equality. Through these examples of solution paths emerges a diversity of solving steps, of entries for solving. As explained in Proulx (2013c), the "mental" dimension provokes the search for an entry point, a way of posing the prompt, of getting in, of making a step. The solving context is thus created by the solver, producing an adapted way of entering into the problem. This diversity, which is translated in a variery of solution paths for solving the "same" equation, transforms that "same" equation, which is differently contextualized or posed differently from one solver to the next as each develops his/her own ways of posing and solving. The diversity of solution paths illustrates well how the various ways of posing the task led to varied ways of solving by solvers, hence diverse strategies or solution paths. In other words, the "same" equation gives rise to the emergence of a variery of posing, which leads to the development of a variety of strategies.

Teachers' posed tasks and solution paths

The work with the group of 20 secondary-level teachers was conducted during a day-long session, where the solving of algebraic equations was carried out in the first half of the day. Similar equations given to students, even identical ones, were given to teachers to solve (about 10).

In general, most if not all teachers' strategies can be described as efficient and errorless for solving the algebraic equations, e.g., through balancing strategies and isolating *x*. However, at specific moments during the sessions, some strategies of a more arithmetical nature were proposed. For example, some teachers explained having done what they described as "recovering", where, e.g., for "Solve for *x* the equation $x^2 - 4 = -3$", one teacher explained:

> [Hiding x^2 with his hand] I look for the number which, when subtracted 4, gives –3. I know it is 1. So then, what number squared gives 1? It is +/–1.

This solution path is similar to "inversing" methods discussed in Filloy and Rojano (1989), or Nathan and Koedinger's (2000) unwinding, where operations are undone to obtain a value for *x*. As Filloy and Rojano explain, in order to solve an algebraic equation this way "[i]t is not necessary to operate on or with the unknown" (p. 20), because it comes back to an arithmetical context of operating on numbers. However, these arithmetic strategies were occasional for teachers. For example, for "Solve for *x* the equation $\frac{2}{5}x = \frac{1}{2}$", teachers produced the following:

Equating middle and extreme products. One teacher explained having acted like with ratios, multiplying middle and extreme terms together, obtaining $4x=5$, hence $x=5/4$.

Multiplying by the inverse. One teacher explained having divided by 2/5 on each side of the equality, leading to multiply by 5/2 to get the same answer, giving $x=5/4$.

Isolating x *in two steps*. One teacher explained having multiplied by 5 on each side of the equality, obtaining $2x=5/2$, and then dividing all by 2 to obtain $x=5/4$.

Following these solution paths, one teacher offered another entry:

Simplifying the equation. The teacher explained having aimed to get rid of ½ by multiplying the entire equation by 2, giving "4 fifths of *x* equals 1". He then multiplied by 5/4 to find *x*.

Here the teacher simplifies the equation, eliminating the ½, in order to find the value of *x* through multiplying by the inverse coefficient. Numerous teachers were intrigued by this solution path and questioned the teacher about it. He explained that his intention was to get rid of ½ to obtain "1 on one side" of the equation and because "multiplying by 2 is easy here". Asked about the numbers present in the equation, he also explained that it was not clear for him if other numbers like 3/2 or 1/6 would have led him toward similar solving steps and that it was the presence of ½ that triggered his activity. This is thus an example of a local strategy, affected by the concrete "data" in the equation: the solving steps are produced on the spot for this equation and not as a general strategy applicable to all cases (as well as not being a strategy for isolating *x*, but about simplifying ½). The entry in the solving is done locally with the ½, the task is posed as one implying a ½, and not by the equation taken in its totality independent of its concrete values as could be the case in a cross-multiplying product. Although local, this strategy underlines an entry directly grounded in the data of the equation. The teacher simplifies this equation by doubling ½, because it was "simple" to do so, and then solves it. However, this kind of solution path diverges from most of the strategies that teachers have produced.

Whereas in students one sees more local solving steps of this sort, directly sensitive to the data in the equation to solve, the strategies developed by teachers appear more decontextualized and general, less centered on the direct data of the equation. In short, faced with the same prompts, teachers posed problems different from those posed by students. This difference between teachers and students is well reflected in a comment made, after sharing the "doubling the ½" strategy, by one of the teachers about what he perceived as the optimal strategy to solve this equation:

> In Grade-8 the winning strategy is really the ratio one [multiplying extreme and middle values]. We work at it so much with them and I encourage them to use it in front of these sorts of equations. […] I am not against the other strategies, but with my students [waiving his hand in discouragement], I am not sure that it would come out much, especially if we ask them to solve without paper calculations. In mental mathematics it is not obvious, whereas with ratios I think that 2 times $2x$ gives $4x$, and 1 times 5 gives 5, and $4x$ over 5 they know afterwards that they have to divide by 4, these are rules of transformation of the equations.

This comment on the winning strategy and students' (un)ease, supported by the other teachers, contrasts with the students' solution paths displayed above. Shortly afterwards, this prompt was given to teachers: "Find the value of $2t$ in $3(2t)+6=18$", for which they produced three strategies:

Balancing. One teacher subtracted 6 on each side of the equation, obtaining $3(2t)=12$, and then dividing by 3 on each side to obtain $2t=4$.

Undoing of operations. One teacher explained having done the opposite operation, by subtracting 6 from 18 and then dividing it by 3.

Recovering. One teacher explained hiding $2t$ to find the number which when added 6 is worth 18. This number is 12. Then he asks which number multiplied by 3 gives 12. This number is 4.

Here again, this recovering strategy provoked questions from teachers, not stemming from misunderstanding but mostly from curiosity about having used this kind of strategy to solve the equation. One teacher explained that even if he himself solves algebraic equations in a variety of ways, he does not teach this variety to his students because he considers it important to proceed step by step in a structured and linear fashion for each side of the equality: something with which other teachers strongly agreed. However, after this comment, another teacher raised the following:

> I have a question. I would show it like that to my students [step by step, operating on each side of the equation in the same manner]. However, as a secondary student I was never shown this "balancing" way. It is one of my colleagues who told me "Listen, I teach it like that". Then, when I taught Grade-7, I started doing this "you do the inverse operation, bing, bang". And I wonder if it has not become an automatism. Is it because we always do it like that, that students themselves begin doing it also rapidly? Is it OK if they do so, or do they need to continue with their personal strategies? […] Should we encourage varieties of strategies in students?

This comment is reminiscent of Freudenthal's (1983) one about automatism in teaching:

> I have observed, not only with other people but also with myself […] that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why [students don't understand them] is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question. (p. 469)

What Freudenthal underlines as much as the teacher is not about misunderstandings of non-usual solution paths, but about well-ingrained habits of solving that (1) prevent one from stepping outside them, and (2) question the relevance of alternate solution paths for solving equations. The teacher's question is about this, that is, the relevance or legitimacy of alternate solution paths: Should they be taught? Should they be accepted? These questions sensitize one to the variations in ways of solving, but also to the challenges that this raises for teaching-learning situations.

Discussion and reflections on algebraic equation solving processes

In what ways do these differences in solution paths raise challenges? The discontinuity, the distances between the various strategies developed by students and teachers are without any doubt important sources of challenges for the teaching-learning of algebraic equation solving. One way of addressing these questions is to probe the solving processes as much in students as in teachers. How can this variety be explained, as well as the differences in solution paths of teachers and students?

If we take into account Threlfall's (2002) views, we can consider that the nature of what emerges for the student is quite different from what emerges for a teacher. A student's experience is quite different from that of a teacher. For the teacher, one has the impression that some earlier

experiences intensely orient future ways of solving. The teacher's comments above, as well as Freudenthal's, insist on the challenge of stepping out of the frame pretraced by earlier repetitive experiences of teaching the solving of algebraic equations along specific solution paths. Faced with having to teach students and make them learn, teachers make choices that in turn orient the nature of their own mathematical experiences of this specific mathematics thematic. By insisting on some solution paths seen as fruitful for students to solve equations, these solution paths in turn orient teachers' solving processes; a phenomenon that the above expression *automatism* describes well.

It is this experience that plays a major role for teachers. When facing the "same" equations, teachers and students pose quite different tasks. Although all solving steps leading to what are seen as different strategies arise from the posing of different problems by different persons when interacting with the prompt, the solving steps do not share the same origins. Teachers are expert solvers in the sense of what could be called an "overspecialization". They can perceive these equations through the same algebraic lens, steering almost all equations to the same kind of task, posing them as the same tasks. Students are not non-experts, but have, however, not yet lived the same repetitive experiences that cause this overspecialisation. Think of the riverbed metaphor: teachers' riverbed is well dug, quite specialized, and the river runs through it comfortably. That of students looks much more like a stream that deviates at the least change in scenery, but this does not prevent it from unfolding (sometimes in the same place, sometimes elsewhere, unpredicted but adequate, or not).

Final remarks: On differences and challenges

Despite its efficiency in solving, teachers' overspecialisation limits the variety of problems that they pose, being less sensitive to variations (in numbers, unknowns, operations, etc.). This makes it difficult for them to act differently, as one of the teacher expressed, but also to appreciate the variety of students' solution paths; a variety that teachers themselves no longer experience much in their day-to-day mathematics teaching (a situation that also leads to question the validity of these solution paths). These different poses provoke a distance between solution paths in students and in teachers. The challenge for the teaching and learning of algebraic equation-solving can then be seen in teachers' overspecialization – which implicitly imposes a particular pose of the task and thus the strategy – which brings about a distance with students' solution paths. Also, this overspecialisation seems to be generated by the belief that a specifically guided and structured experience of solving will be helpful to students: teachers believe, as expressed above, that students need these structured and specific experiences. This belief may be an important source of reflection, because the challenge of this overspecialisation points to the necessity of not discarding the less common or local solution paths, but of developing a sensitivity toward them and toward the role played by this overspecialisation or these automatisms.

This resonates with Anghileri's (2001) literature review on mental calculations, pointing to students locally tailored strategies, linked to their understanding of problems. However, she explains that these strategies do not last because they are often substituted by standard algorithms taught in classrooms, where students attempt to conform to what happens in everyday mathematics lessons. Anghileri adds that this situation is complex, because without negating the power of standard algorithms for solving, they conflict, provoke a "distance", in the development of students' aptitudes in problem-solving. For her, this amounts to a matter of intentions:

> The emphasis in teaching arithmetic has changed from preparation of *disciplined human calculators* to developing children's abilities as *flexible problem solvers*. This change in emphasis requires new approaches in teaching that will develop children confidence in their own methods rather than replicating taught procedures, and that will enable them to understand the methods used by others (Anghileri, 2001, p. 79).

The transition toward flexible problem-solvers represents an invitation to think about algebraic teaching and learning, an invitation directly aligned with the above argument on the importance of developing sensitivities about distances between teachers and students' algebraic solving processes.

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Introduction to equation solving for a new generation of algebra learners

Ann-Sofi Röj-Lindberg[1], Ann-Maija Partanen[2] and Kirsti Hemmi[1,3]

[1]Åbo Akademi University, Faculty of Education and Welfare Studies, Finland; aroj@abo.fi,
[2]University of Lapland, Finland; anna-maija.partanen@ulapland.fi

[3]Uppsala University; khemmi@abo.fi

Understanding of equality and solving equations are some of the big ideas in algebra. They have been in focus in early algebra research for some decades and in many countries it is now usual to work with equalities and solving equations using informal methods from early school years. However, it is not clear how the transition to formal methods of equation solving could be conducted in order to maintain students' interest and enhance their algebraic understanding. We shed light on the issue by reporting on what happens when three teachers introduce equation solving with formal methods in Grade 6 (age of 12) in Finland. We especially consider how the introduction could support students' development of an algebraic understanding of equality and their engagement in more formal mathematics.

Keywords: Equations, primary school mathematics, equality, algebra.

Introduction

Learning to solve linear equations in a more formal manner may be a critical moment in a student's school mathematical experiences. The student is perhaps able to figure out the missing value in an equation with one unknown written as an open number sentence, for instance solving $x - 10 = 15$ by arguing 'for x minus 10 to become 15, then x has to be 25 because when 10 is taken away from 25 the answer is 15'. But the student does not understand how and why the standard algorithm works for solving the very same equation, that is, looking at the equation $x - 10 = 15$ as equivalent to the equation $x - 10 + 10 = 15 + 10$ and equivalent to the equation $x = 25$. This particular situation is shown to be associated with how the student understands equality (Kieran, 1981; Knuth, et. al., 2006, 2011; Vieira, Gimenez & Palhares, 2013). In arithmetic it is often enough to understand the equal sign as an operator, as a do something -signal (Kieran, 1981). In algebra, however, the student should understand equality between two expressions as an equivalence relation that does not change. Such algebraic understanding may not be supported if the student is taught to solve equations by memorized procedures as 'move terms from one side to the other side of the equal sign and change the corresponding signs' or 'do the same thing on both sides'. Successful equation solving is connected to a relational meaning of the equal sign, and understanding the notion of equation as a statement about an equivalence structure (Knuth, et. al., 2006, 2011; Stacey & MacGregor, 2000).

Scholars have only recently started to theorize about possible learning progressions with respect to algebra in different school years (Cai & Knuth, 2011). Although students may be taught to consider the equal sign as a relation by working with equivalent expressions and by solving simple equations with informal methods already in early school years, it is not sure that they can operationalize the

meaning of their experiences when moving to more formal methods of equation solving in later school years. In Finland, Grade 6 teachers are expected to focus on a progression from equation solving by informal methods to equation solving by more formal methods. The general aim of this paper is to shed light on this progression by analyzing how students in three Grade 6 classrooms in Finland are introduced to solving linear equations with one unknown.

Learning to solve linear equations

A distinction can be made between an arithmetical and an algebraic notion of equality and a corresponding difference in arithmetical and algebraic understanding. Following Filloy and Rojano (1986) and Vlassis (2002), if the unknown in a linear equation appears only on one side of the equal sign, e.g. $x + 5 = 8$, $13x = 39$, $6(x + 3) = 48$, the student has less need to operate on or with the unknown, or to deal with the equivalence structure of the expressions on both sides of the equal sign. For equations of this arithmetical type the student probably manages to find the value of the unknown by applying known number facts or inverse operations. When the unknown appears on both sides of the equal sign arithmetical understanding is however no longer enough. Neither is arithmetical understanding enough in the abstract type of arithmetical equations where certain algebraic manipulations are needed, for example, because of the presence of negative integers (e.g. $2 - x = 7$) or several occurrences of the unknown (e.g. $6x + 5 - 7x = 27$) (see Vlassis, 2002, p. 351). When solving such more abstract equations the student who has an algebraic understanding of equality first of all acknowledges that the expressions on both sides of the equal sign are representing equal values, next that the solving process involves mathematical actions, which preserve this balance and produce equivalent equations. Vlassis (2002) noted that concrete representations of equalities, like the two pan balance model, might act as good tools for developing students understanding of equality, but Vlassis also pointed at their limitations. For example, the balance model cannot represent the negatives in an equation. More generally, a true algebraic understanding of equation solving implies that the student considers the equation as representing a problem situation, and starts to understand the equation as an equivalence structure maintained by the operations one has to apply on both sides to solve for the unknown.

If a real-life problem is used as a tool to introduce students to solving equations and the unknown is solved for through the syntax of algebra, the student needs to refrain from an arithmetical understanding of the solution to the problem. This situation may be at odds with the student's knowledge and intuitions about arithmetic because the meaning of the equal sign changes from announcing a result to stating an equivalence relation (e.g. Carraher, Schliemann, Brizuela & Earnest 2006). Furthermore, an algebraic interpretation of the solution to the real-life problem implies that the student should be able to refrain from immediately attributing a concrete meaning to the letter appearing in the corresponding equation. Instead the student should understand the letter as an unknown number, the value of which is not significant at the moment the equivalence relation is set up and manipulated (Vlassis, 2002).

Although the equal sign (=) is focused together with minor than (<) and major than (>) signs already from Grade 1 in order to enhance students' understanding of equality, primary school students in Finland most often use the equal sign in their mathematical practices in order to show the result of an arithmetic problem. It is also usual to concretize the simple equations appearing in primary

school with a balance model and encourage students to solve equations with testing, using inverse operations and other informal methods.

Methodology

The material was gathered in the spring 2012 from three Grade 6 classrooms in the Swedish-speaking community of Finland as part of the international VIDEOMAT-study (see Kilhamn & Röj-Lindberg, 2013). Four consecutive lessons on equation solving, and a fifth lesson on problem solving were video-taped and imported into Transana, an open source transcription and analysis software for audio and video data (www.transana.org). The teachers answered a few clarifying questions immediately after each lesson and participated in formal interviews after the last (fifth) videotaped lesson. The teachers Anna, Bror and Cecilia have a similar educational background as certified generalist teachers and Masters of Pedagogy. At the time of the study their teaching experience varied from Bror's five years to Cecilia's seven and Anna's 15 years of experience. They used the same textbook and teacher guide. In this paper we report on a close attention to what happened during the first videotaped lesson when the three teachers introduced solving linear equations in one unknown with formal methods. We especially considered how the introduction might have supported students' development of an algebraic understanding of equality. First we briefly present the characteristics of all three teachers' introductions to solving equations and continue by presenting the case of Anna.

Three entries to equation solving

In line with the Grade 6 textbook all three teachers introduced equation solving with one-step equations were the unknown appeared once on the left side and with an integer on the right side.

Cecilia started from a strong emphasis on the equal sign as stating an equivalence relation. Her aim was to help students unlearn their earlier use of the equal sign to represent a string of calculations. She focused on the equal sign as representing equivalence in several ways: by discussing its meaning explicitly with her students, by referring to a solution a student had made in the test, and by representing both inequality and equality with a balance scale. These situations were familiar to the students, who also participated in the corresponding discussions in seemingly relevant ways. However she did not utilize the balance scale analogy to support the emergence of algebraic understanding of equation solving. The first lesson her students solved only equations of the addition type, e.g. $x + 8 = 15$, by subtracting, in this case $x = 15 – 8$, without any further discussion related to a structural meaning of the equal sign.

Bror started from four uncomplicated real-world situations that the students solved mentally by stating the answer. The first one was "There are seven fruits in a basket altogether. And four of the fruits are apples, the rest are pears, followed by the question: How many pears are there in the basket?" Each situation was then represented with an equation, e.g. $7 + x = 12$, solved and checked with arithmetical means like in Cecilia's classroom. Next Bror did a rapid switch in his teaching to an algebraic interpretation of why the subtraction $12 – 7$ in the solution can be thought of as 'the number 7 is moved to the right side of the equal sign and the corresponding sign is changed'. However, the students' activity showed no explicit signs of an emerging algebraic understanding of equality. In their verbal answers students continued to refer to the equal sign as a do something –

signal, and in their notebooks they applied inverse operations to find the value of the unknown for all equations.

The third teacher, Anna, started in a similar way as Cecilia and focused on the equal sign as stating an equivalence relation, however in a more formal manner. Anna started from defining an equation as equal expressions. She then did a quick transition to the algebraic approach of 'doing the same thing on both sides' to maintain equality and find the value of the unknown. Her message to the students was clear: you must isolate the unknown number step by step by operating on both sides of the equation.

Here we present episodes from the first lesson in Anna's classroom. The episodes show how the pattern of communication funneled the students (Wood, 1998) into memorizing 'do the same thing on both sides' rather than focusing on why this strategy works.

We have to do the same thing on both sides

Anna opens her lesson by writing an open number sentence, 4 + _ = 9, on the white board. The students give the value of the open number, and they name the object an equation. Anna reads the following aloud from the white board and then fills the placeholder in 4 + _ = 9 with the letter x.

> Anna: An equation is an equality relation between two mathematical expressions, which are called left side and right side. It includes one or more unknown numbers. If there is one unknown number, you normally use the letter *x.*

Next, Anna continues to read aloud: "An equation is an equality relation between two sides. The two sides are separated by an equal sign". She illustrates the statement with an arithmetical equality, 4 + 2 = 7 - 1, 6 = 6, and with the equation 4 + x = 9 where she emphasizes that both sides of the equation must be equal. The students are asked to solve for two open number sentences and she stresses that the placeholder for an unknown can be replaced by the letters x, y or z. All the equations she has shown to her students so far include only one number on the right side, except for the arithmetical equality she used to indicate a new understanding of equality: the equal sign as a signal of an equivalence structure.

Before the start of the following episode Anna refers to solving equations as a stepwise mathematical strategy. She writes x + 12 = 18 on the white board. By stating, "we know that x should be six" she then indicates that the students' attention should not be on finding the value of the unknown. She then starts funneling her students to discover "how to do it": how to preserve the equality while simultaneously finding the value of the unknown.

> Anna: If I have an equal sign in the middle, then I aim at having x alone on the left side (…) But now I have plus twelve there, what do you think, the way of thinking, how can I get this plus twelve away from there? I want to have x alone on the left side of the equal sign. How can I get it away? Janne.

Janne's answer "eighteen minus twelve" shows that he attends to finding the value of the unknown. Anna's attention is however on the mathematical actions to preserve the equality and she does not develop his answer any further. She repeats the question "How can I make plus twelve to zero?" several times, but does not get the answer she wants. She then gives the students a hint by drawing a

minus sign after 12 on the left side of the equation. When she starts getting answers she accepts, the funneling accelerates and ends by her statement "I have shown how it actually goes step by step".

Anna: How shall I get plus twelve to zero? I helped you a little bit on the way. Nelli.

Nelli: Minus twelve.

Anna: Minus twelve. But twelve minus twelve is zero, isn't it. But now the matter here, when I do something on the left side so what do you think I should do on the right side? Tor.

Tor: Take away from there, that twelve.

Anna: Exactly. I have to do the same thing here, now I have got eighteen, what should I also do then, here, on the right side? Well, now, Mimmi.

Mimmi: Minus eighteen.

Anna: No, not minus eighteen, the same thing as on the left side. Mimmi.

Mimmi: Minus twelve.

Anna: Minus twelve. Well let's check, x, twelve minus twelve is zero, so then, now I've got x on the left side, eighteen minus twelve is (…) Quickly Mimmi.

Mimmi: Six.

Anna: Six. Now I have, stepwise, through mathematical steps, done this equation. You could quickly see that it must be six. You could do it just like that. But now I have shown how it actually goes step by step. I want to have x alone on the left side, so that I get what x equals to. And then, I just have to look what I have on that side, what I need to do. In this case, I had plus twelve, then I have to take minus twelve so that it becomes zero. But when I do something on the left side, I also have to do the same thing on the right side. Do you understand? Did you follow?

Immediately after the previous episode, Anna and her students started solving the equation $y - 6 = 11$. The "mathematical steps" are repeated and it becomes clear that some students, as Janne, now know that "to do the same on both sides" is the name of the game the teacher wants to hear.

Anna: Quickly, I know you know the answer. But we shall think about the mathematical steps. What do I want, I'll put the equal sign here, what do I want to have alone on this side of the equation? What am I aiming at? (Suggestions from students: x and y) Y, okay, I've got y there. But it is not ready yet. I've got the minus six, what shall I do then, what do I want to do then? Now I've got minus six. Karin

Karin: You want to make it zero.

As Anna then continues by asking how she can get a zero out of $y - 6$ she gets the answer "plus six" from both Karin and Janne. But she is not yet satisfied.

Anna: Plus six. Okay. And then on the right side I have eleven. Are we ready with it or shall I still do something? /…/ Why Janne, plus six there too?

Janne: We have to do the same thing on both sides.

Anna: The same thing on the left and right sides. What I do on the left side, the same thing on the right side, or on the right and left sides. It's plus six, now, because I had minus six. Okay. Then I have got y. Those two cancel each other out. Then I've got y there. And what will be on the right side? Vanja.

Vanja: Seventeen

Anna: Seventeen. And I know that you could have been able, you could find it already in a few seconds, but now we did the mathematical steps, again. Are you following? [SS: Yeah, yeah.] Beginning to understand this, although these are easy numbers /…/ Now you have solved equations, easy equations. Later, there will be a little bit harder ones, but now we'll begin with these.

When Anna starts teaching the steps of solving the equation x + 12 = 18 in the first episode, the students do not contribute with the answers she seems to expect. After receiving a hint from Anna in the form a minus sign, x + 12 –, Nelli gives the expected answer, "minus twelve". In the second episode we can notice how the students and Anna use the same wordings as when solving the equation x + 12 = 18 in the first episode. Earlier Anna stated her expectation very clearly when she said "I want to have x alone on the left side of the equal sign" and she repeats the question "How shall I get plus twelve to zero?" many times. Now her expectation is reformulated as questions, "What do I want to have alone on this side of the equation?" followed by "What do I want to do then? Now I've got minus six", and the student Karin finally gives Anna the expected answer "You want to make it zero".

In her summary in the first episode Anna reminded the students "when I do something on the left side, I also have to do the same thing on the right side". In the second episode the student Janne repeats her words in his answer "We have to do the same thing on both sides" and Anna confirms that he remembers correctly by saying "The same thing on the left and right sides. What I do on the left side, the same thing on the right side, or on the right and left sides". Anna's discussion with her students focused strongly on memorizing the procedure she called the "mathematical steps" and, hence, did not serve well in supporting their development of an algebraic understanding of equality. Moreover, solving the equation by algebraic means, and with an algebraic interpretation of equality, was of no use to the students who already knew the value of the unknown x. Nevertheless they tried to fulfill Anna's expectations and answer her questions.

Discussion

The three teachers in this study took some initiative in leading their students forward, from an arithmetical to an algebraic understanding of equality, but none of the teachers confronted the students with situations where mathematically more powerful approaches were needed than those students were already familiar with. Cecilia emphasized the need of understanding the structural meaning of the equal sign. Bror and Anna focused the strategy of doing the same thing to both sides of the equation. It seems, however, that neither the teachers nor the authors of the Grade 6 textbook were aware of the underlying conceptual differences between solving equations within an arithmetical understanding of equality and, on the other hand, within an algebraic understanding.

The students had encountered missing value problems in the textbooks every now and then from Grade 1 onwards. They were familiar with the logic of that type of tasks. However, in the videotaped lessons the students did not have any real need to adopt algebraic ways of thinking about equality. For instance, one can wonder whether the students in Bror's classroom were motivated at all to make sense of the uncomplicated real-life situations with a new complicated way of thinking as the solutions to the problems were obtained more economically by arithmetical means. At best, solving equations by adding or subtracting the same term from both sides of an equation was used by students as a memorized procedure and applied for one particular type of equation, only. The students did not need an algebraic understanding of the equal sign to solve neither the real-life problems nor the equations, and the book did not explicitly expect students to expand their mathematical knowing into operating with or on the unknown (cf. Filloy & Rojano, 1986).

Balacheff (2001) recommends that students should experience a clear rupture between arithmetic and algebra. The rupture might be a strong emphasis of the newness of a situation, for instance to give more complex equations to students to be solved by including numbers beyond students' arithmetic capacity and hereby support a need for an algebraic approach. Another way forward could be an algebraic use of numbers (Blanton & Kaput, 2003), for instance to investigate series of arithmetic tasks with a pattern, and support the students to express and justify the pattern. In many cases an algebraic sense of equality can be developed from just small changes and extensions in the types of tasks and questions textbooks or teachers present to students, and by encouraging discussions. The students' arithmetical understanding of the equal sign can be confronted in versatile problem situations where a structural meaning of the equal sign is in focus whilst the situations are represented and made sense of within an algebraic syntax. In Anna's classroom, for instance, the students general thinking about numbers, and hereby their emerging algebraic sense of equality, could have been supported in several ways (cf. Blanton & Kaput, 2003; Carraher et al., 2006). A focus on transforming the arithmetical equality $4 + 2 = 7 – 1$ into equivalent expressions like $4 + 1 = 7 – 2$ or $14 + 2 = 10 + 7 – 1$ at the start of the lesson, as well as on the simple questions "How do you know that the procedure 'doing the same on both sides' is true and does the procedure *always* work?" could probably have promoted an algebraic understanding of equality by means of students' own justifications of the transformations and the procedure. Moreover, the funneling effect we saw in Anna's classroom could have been avoided (cf. Wood, 1998).

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An insight into the sources of multiple referents for the unknown in the algebraic solving of word problems

Carlos Soneira[1], José Antonio González-Calero[2] and David Arnau[3]

[1]Educational Sciences Faculty, University of A Coruña, A Coruña, Spain; carlos.soneira@udc.gal

[2]Faculty of Education, University of Castilla-La Mancha, Albacete, Spain; jose.gonzalezcalero@uclm.es

[3]Faculty of Teacher Training, University of Valencia, Valencia, Spain; david.arnau@uv.es

The term "multiple referents" (MR) is used to describe the error of using the same letter to refer to two different quantities when translating word problems to equations. MR can be related to the presence in the statement of what are termed indexical expressions so that the presence of indexical expressions in statements may be associated with a greater number of MRs. In this paper we analyse students' performances when solving word problems algebraically with the aim of determining the causes of this kind of error. Results from this research indicate that there is not an unique factor that accounts for the commission of MRs. At least the error may be associated either with the tendency for students to use personal idiosyncratic sign systems when translating from the natural language to the algebraic language or with the construction of a wrong problem model.

Keywords: Algebraic language, word problems, secondary school, indexical expressions, multiple referents.

Introduction

The relevance and utility of solving word problems is widely accepted by the educative community as an issue interesting by itself and as medium to develop general skills applicable to everyday life and other academic subjects. Actually, its relevance is reflected in many research agendas on educational mathematics (e.g., Kieran (2006) or Mason, Graham & Johnston-Wilder (2005)).

In the mathematical activity in general and in problem solving in particular, it becomes essential the internal and external representation of the mathematical ideas, in such a way that different registers are simultaneously combined. As word problems are enounced in natural language, the role of this register is a determinant factor in the task difficulty and, indeed, a correlation between reading comprehension and mathematical ability exists (e.g., Capraro, Capraro, & Rupley, 2012). In this regard, the influence of syntactic (e.g., Abedi & Lord, 2001) and semantic (e.g., Riley, Greeno, & Heller, 1983) variables has been studied. In the algebraic solving, another register that necessarily arises is the symbolic one, which works differently than the natural language. For instance, in the algebraic language there is a univocal correspondence between expressions and symbolised quantities, whereas it does not happen in the natural language, where the same name can be given to different objects (Filloy, Rojano, & Puig, 2008). This fact may cause errors when solving word problems algebraically. Without prejudice of the usefulness of other methods, the algebraic resolution constitutes a useful tool that can expand the potential of the subjects as problem solvers.

Theoretical framework

The Cartesian Method describes the process of obtaining and solving the equation (or equations) that represents the situation depicted in a problem statement (Filloy et al., 2008; Polya, 1981). This process implies a cyclical process of sense production. In order to explain it, we use the notions of Textual Space and Text (Filloy et al., 2008). The Textual Space encompasses the semantic content communicated by the linguistic structure of the text in the explicit context to which it refers. The Text is the result of a personal re-elaboration of the Textual Space carried out by the subject, by means of which he/she gives sense to these semantic content. The resulting Text becomes the Textual Space of a new reading and so on. In a global way, the Cartesian Method requires a transformation from the original Textual Space expressed in natural language to a Text expressed with one equation (or a system of equations). The first step is an analytical reading of the statement (Filloy et al., 2008), which involves a reading/transformation process where the solver extracts the set of quantities and arithmetical relationships among them from the original Textual Space. Hence, its result may be interpreted as the identification of the problem model in the terminology of Nathan, Kintsch and Young (1992). By way of example, let us consider the problem P2 of Table 1. If we use the letter x, to represent the son's age at present, and y for the father's age at present, then the problem model can be expressed, for example, by means of the system $y = 2x$; $y - 17 = 3(x - 17)$. The resolution of the system provides the solution $x = 34$, $y = 68$.

However, a solver can eventually perform this process in successive cycles of transformation from Text to Textual Space, and use several registers in order to externally represent it. Since many non-linguistic signs are commonly used, we adopt a semiotic perspective and consider the notion of Mathematical Sign System (MSS) in the sense of Radford (2000). In this way, we account for the construction and use of idiosyncratic MSS (MSS_{Idi}), specific for each individual, with its own syntactic features and ways of assigning meaning to symbols. Such MSS_{Idi} are used as intermediate registers when the student attempts to translate a Textual Space stated entirely in natural language (MSS_{Nat}), and to produce a Text expressed in algebraic language (MSS_{Alg}).

Regarding the change of MSS, we follow the approach of Duval (2006) and use the term conversion to name the change of semiotic representation register where the signifier is located, but without modifying the represented meaning. Such conversion can be congruent or non-congruent, depending on whether the syntactic structure and/or semantic segmentation are the same or not in the source and in the image representations. In this paper, we focus our attention to one kind of linguistic expressions called indexical expressions. These are expression whose meaning may shift depending on the context. They are common in MSS_{Nat} (e.g., "here", "today", or "somebody's age" at different temporal moments). However, in MSS_{Alg} the designation of quantities is functional and indexical expressions do not exist. Hence, if the statement of a word problem contains an indexical expression, then the conversion to equations is necessarily non-congruent. In this paper we study the phenomenon consisting on the violation of such functional designation. We will refer to it as use of multiple or shifting referents for the unknown (MR), following the terminology of Stacey and MacGregor (1997, 2000). Although MR is valid in MSS_{Nat} or certain MSS_{Idi}, it represents an error in the context of MSS_{Alg}. In the present work we use a family of word problems known as age problems, and where indexical expression s arise naturally. Specifically, they are used to express that the age of the characters evolve over time (Table 1). As Stacey and MacGregor (2000)

reported, the MR may appear due to distinct causes: "i") the letter refers to different quantities in one equation or within a system of equations; ii) the letter refers to different quantities at different stages (ages); and, iii) the letter is a general label for any unknown quantity or a combination of quantities" (p. 10). It should be noted, however, that we follow the Radford (2000) semiotic approach, where signs are seen as tools of the mind to perform actions in a particular context, so they must be studied in terms of the practice they mediate. In this sense, the tasks we propose in this work are essentially different from that of Stacey and MacGregor (1997, 2000), because we require the students to pose an equation (or equations) but not to solve it. The latter is relevant because, as Stacey and MacGregor (1997, 2000), or Radford (2000) have pointed out, it is usual that, when a solution is required, the meaning students give to letters has an arithmetic resemblance. The task we propose focuses on the conversion from the natural language to the algebraic language. As a consequence, in concordance with Radford's (2000) approach, the idiosyncratic semiotic registers that might potentially emerge would be different from those of Stacey and MacGregor (1997, 2000).

Another key aspect that must be taken into consideration is the moment, specific mathematical activity or step of the Cartesian Method where the MR has its origin. In particular, if a solver did not build a problem model correctly and two different quantities are erroneously considered as equals, then an MR would be committed although the subsequent steps were done correctly. On the other hand, even though a correct problem model was identified during the analytical reading, an MR can happen due to conversion errors when posing equations. A lack of command of MSS_{Alg} or coordination between distinct MSS (especially between MSS_{Nat} or MSS_{Idi} and MSS_{Alg}.) may cause MRs. The ideas exposed above seem to point out the existence of various sources of MRs and that MRs may be due to different causes for each person, this being relevant in order to design a didactic intervention. Note also that the error by MR during the application of the Cartesian Method has a conceptual nature, because it relies on the unawareness about the way meaning is given in the MSS_{Alg} and the MSS_{Idi}. This may reduce the potentiality of the MSS_{Idi} as a tool, and obstruct the progress in the use of the algebraic method.

Research aims and research methodology

The aim of this work is to search for and to document possible sources of MR when solving word problems using the Cartesian Method. Subsequently, we pay special attention to determine in each case whether the MR arises due to the inability to build a correct problem model or due to a lack of command to perform conversions between different MSSs. Regarding to the last possibility, we also study the emergence of MMS_{Idi}s during the conversion process from MSS_{Nat} to MSS_{Alg}.

Participants

The sample consists of 54 students (15-16 years) in their fourth grade of secondary school (in a modality oriented to a subsequent BSs) in three Spanish Public High Schools. According to the Spanish curriculum, they were familiar with the Cartesian Method, having used it over the two previous years.

Materials and procedures

This work consists of two phases: a written phase and a case study, although in this paper we will exclusively focus on a qualitative study of students' written productions from the written phase.

Prior to that, we carried a pilot study with 36 participants, aimed to calibrate the difficulty of the problems and the time required to complete them. In addition, based on the pilot study, we stipulated a series of criteria to code the students' productions.

In the written phase we used a questionnaire consisting of six age problems but in this manuscript we only employ students' resolution of three problems from the whole collection (Table 1). All the problems were versions from others in textbooks corresponding to two previous grades (13-14 years old). The statements were entirely expressed in MSS_{Nat} and they all contained indexical expressions. The written test was conducted in the students' usual classroom. Each statement was shown on a screen at the front of the classroom for three minutes. The participants were given explicit instructions prior to beginning the test. They must try to pose the equation (or equation) that leads to the solution of each problem but solving it was, however, not mandatory. We focus on the raising conflict due to the structural differences in MSSs. In order to stress these differences and also to delimit the study, we turn to the indexical expressions.

Students' productions were analyzed on the basis of the above developed theoretical framework, taking into especial consideration the ideas contained in Duval (2006), Filloy et al. (2008), Radford (2000) and Stacey and MacGregor (2000).

P1	One sister is 3 years older than the other sister, and their father is 43. In 7 years time the father's age will be twice the sum of the sisters' ages. How old is each sister?
P2	A father is twice the age of his son. Seventeen years ago, the father was three times the age of his son. How old is each one?
P3	Eight years ago Ana's age was four times Maria's age. In 12 years time Ana's age will be only twice Maria's age. Find the age of each one.

Table 1: Problems from the written test

Results

In the example shown in Figure 1 the student performs some cycles from Textual Space to Text and represents a Text in a MSS_{Idi}, which is less abstract than the MSS_{Alg}. It consists of a table with two columns separated by a straight line, and also a system with two equations. In each column of the table the relations between the ages of the characters at a particular temporal moment are correctly represented. The letter *x* stands for Maria's age in both temporal moments, and its value evolves automatically when switching between columns. This means that indexical expressions are valid in this MSS_{Idi}, so it shares some features with the MSS_{Nat}. A spatial reorganization takes place, but, at a deeper structural level, there is not a discontinuity between the rules of both systems regarding the acceptance of indexical expressions. Here we interpret that the subject's analytical reading produces a correct problem model, and that it is correctly represented within the frame of her/his MSS_{Idi}. The subject properly coordinates both MSSs, performing a correct conversion from MSS_{Nat} to MSS_{Idi}. However, when posing the system of equations the student produces an MR because the letter *x* stands for the age of both problem characters. In this case we interpret that the MR takes place in subsequent cycles due to a lack of command of the MSSAlg and/or a deficient coordination between her/his MSSIdi and the MSSAlg. Indeed, the error may be a consequence of the student's

lack of awareness of the structural differences between her/his MSS_{Idi} and the MSS_{Alg} regarding the validity of indexical expressions.

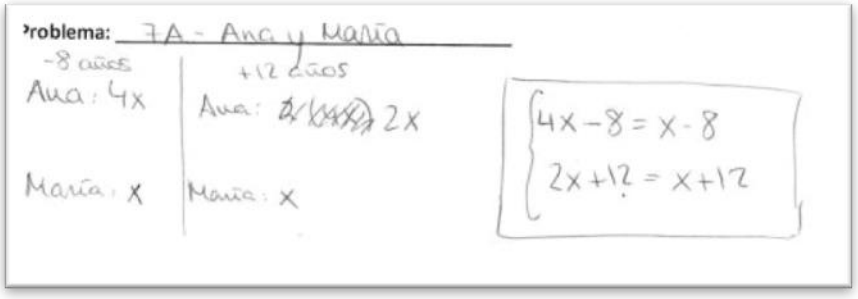

Figure 1: Student's resolution of the problem P3 ("años" means "years")

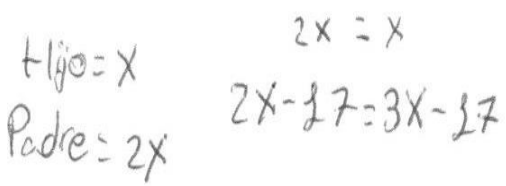

Figure 2: Student's resolution of the problem P2
("Hijo" and "Padre" mean "Son" and "Father", respectively)

Figure 2 shows another student's performance in which we interpret that a correct problem model has been built and again the MR seems to be caused by a lack of ability to use the MSS_{Alg} in subsequent steps of the Cartesian Method. The first equation seems to indicate an overlapping between MSS_{Alg} and the student's MSS_{Idi}. Indeed, the letter *x* does not stand for the actual age of one character, but as a general label that refers to any unknown quantity. Actually, one of the possible causes of MR suggested by Stacey and MacGregor (2000). In addition to that, in light of the first equation, we interpret that the meaning and interpretation of the equals sign in this MSS_{Idi} differs from those in the MSS_{Alg}. The subject uses symbols from the MSS_{Alg}, but the way of giving meaning to them is completely wrong in the frame of the MSS_{Alg}. Finally, the second equation seems to reveal that the student has difficulties related to the syntactic rules of MSS_{Alg} because he/she does not perceive the necessity of using brackets.

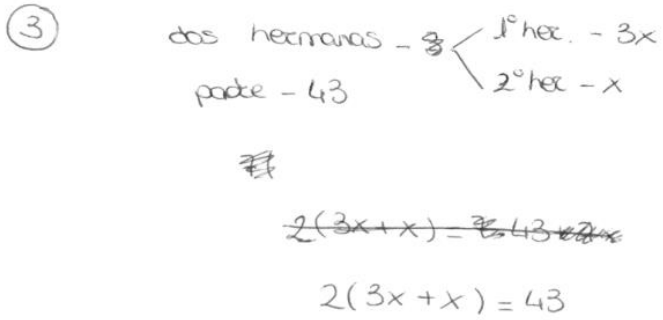

Figure 3: Student's resolution of the problem P1
("hermanas" and "padre" mean "sisters" and "father", respectively)

Unlike in previous examples, in the case shown in Figure 3 the student does not to identify a correct problem model since the passage of time is not considered for any character. In addition to that, the additive relation between the siblings' ages is wrongly symbolised because a multiplicative relation is used. Regarding the multiplicative relation that should link the future ages of the three characters, the student seems to use the actual ages instead of the futures ages. From a conservative view it may be argued that the letter x could refer to one quantity at different times (e.g., the current and the future ages of one sister). However, the use of the known quantity father's current age in the equation reveals the existence of an incorrect problem model. Here, the MR already arises during the analytical reading, because it seems that the student do not perceive the necessity to involve in its model two different quantities for each character (one referring to the current time and another to the future).

Conclusions

Personal idiosyncratic representations are usually thought to be helpful to students in order to understand and solve problems. It is commonly accepted that these representations can work as a bridge to more formal systems, as is the algebraic language. However, as Weinberg, Dresen and Slater (2016) suggest, the differences between such idiosyncratic systems and institutional semiotic systems can be also a source of conflict. In particular, these authors claim that the way in which such systems differ may greatly influence the students' mathematical activity. In this paper we report and analyse an example (Figure 1) where the student firstly represents a problem model in an idiosyncratic system that contains algebraic symbols but structurally different to the algebraic language, and only following this, a system of equations is posed.

This and other examples of students' outputs reported in this work provide evidences of the tendency to manage algebraic symbols according to a set of personal rules that are not coherent with those of the algebraic language. Indeed, the analysis of such outputs allow us to affirm that some of the characteristics of these idiosyncratic semiotic systems are more close to the natural language rather than the algebraic register. This fact is not trivial and can lead to the commission of errors when mathematical relationships are translated between different mathematical sign systems. But such translation has to be unavoidably performed during the algebraic solving of word problems. Thus, a semiotic perspective on the students' performances makes possible to locate possible sources of the error consisting on using multiple referents for the unknown (MR). Specifically, the examples presented in this paper strongly support the fact that the commission of MR may be due to quite some different factors. On the one hand, an erroneous understanding of the problem may lead to the construction of a wrong problem model, what in turn would lead to commit a MR. In this case, however, the error is not necessarily related to the students' ability to use the algebraic language. Instead, difficulties seem to emerge when conceptualising the arithmetical relations expressed in the statement in natural language.

On the other hand, we provide examples that show how the appearance of MR can be a consequence of students' misunderstandings and overlapping between the structural semantic and syntactic rules of natural language, idiosyncratic systems, and algebraic language. In some cases, the student build a correct problem model but represent it using an idiosyncratic sign system that shares important features with natural language (e.g., indexical expressions are valid in both systems). Since these idiosyncratic sign systems employ algebraic symbols but do not obey the

same rules than this language, the MR may appear due to the incongruity between both mathematical sign systems. In other examples, similar causes prompt errors when the student directly translates from natural language to algebraic language. From a didactic point of view the different nature of MRs is transcendent because teachers need to know the sources of the MR for a particular student in order to design an appropriate remedial instruction.

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Key ideas as guiding principles to support algebraic thinking in German primary schools

Anna Susanne Steinweg

University of Bamberg, Germany; anna.steinweg@uni-bamberg.de

German mathematics teaching-units in primary school lack explicit algebra learning environments; however, the topics which are taught address algebraic thinking if seen from a new perspective. Teachers and children are mostly unaware of the algebraic potentials of tasks –especially in the scope of the content area patterns and structures. The project presented here submits a suggestion of algebraic key ideas as guiding principles to rethink 'arithmetical' topics and to design learning environments on algebraic thinking. Additionally, effects of implementing and evaluating such tasks are illustrated by one example.

Keywords: Algebraic thinking, awareness, key ideas, patterns and structures, properties.

Introduction

The approach presented here is worked out for the particular situation in German primary schools and accordingly addresses one of the borderlines in the scope of algebraic thinking pointed out by Hodgen, Oldenburg, Postelnicu, & Strømskag (2015), i.e. "differences between teaching cultures in different countries (and within countries) are enormous and restrict generality of results very much" (p. 386). In German primary school algebraic topics have no tradition and still no explicit place in curricula, textbooks, and teaching-units on the one hand (KMK, 2004). On the other hand, recent research on early algebra or algebraic thinking is emerging (e.g. Akinwunmi, 2012; Gerhard, 2013). Furthermore, the daily classroom interaction and common teachers' beliefs reveal bright opportunities for early algebra (e.g. Krauthausen & Scherer, 2007). Yet, implementing promising approaches and algebraic tasks in daily school life still is a great issue.

The main aim of the project is making algebraic learning chances possible for children. Opportunities to get to know algebraic ideas and ways of algebraic thinking depend on tasks presented in the classroom. These tasks are offered by teachers. Hence, the focus has to be on tasks and on teachers' awareness of the potential of these tasks. In so doing early algebra can be supported via a detour that influences classroom interaction and therefore children's awareness and abilities.

Theoretical framework

Algebraic thinking and core areas

Algebraic thinking is assigned to special thinking habits. Current research identifies mainly four algebraic thinking practices, which lay in generalising, representing (incl. symbol use), justifying, and, reasoning with generalisations or relations (e.g. Kaput, 2008; Kieran, Pang, Schifter, & Ng, 2016; Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015). The focus of attention needs to shift from numerical solutions to mathematical structures behind the given patterns or equations. This shift allows seeing the generality as reification (Sfard, 1991) and therefore creating new objects (Mason, 1989). "By attending to relations and fundamental properties of arithmetic operations (what we call relational thinking) rather than focusing exclusively on procedures for calculating answers" (Carpenter, Levi, Franke, & Zeringue, 2005, p. 53) procedural thinking is not erased but expanded.

Algebraic thinking as conceptual (Tall & Gray, 2001), relational or structural thinking can be applied to various topics or as Sfard puts it, "any mathematical activity may be seen as an intricate interplay between the operational and the structural versions of the same mathematical ideas" (Sfard, 1991, p. 27). Although in most of the research studies certain topics are outlined to be particularly relevant for algebraic thinking, different content-orientated registers identifying the strands or core areas of algebra can be found in many of them (Table 1).

		pattern and formulas	patterns	patterns (& structures)
generalised arithmetic	generalised arithmetic	restrictions	arithmetical structures	property structures
	equations			equivalence structures
functional thinking	functional thinking	functions	relations	functional structures
	proportional reasoning			
application of modelling languages	variable	symbolic language		
Kaput 2008	Blanton et al. 2015	Drijvers, Goddijn & Kindt 2011	Kieran et al. 2016	Steinweg 2016

Table 1: Algebraic strands and core areas

These lists overlap in various topics as shown in the re-assigned order in Table 1. The core areas therefore are more or less universally acknowledged and differences can be identified in details only. For instance, some authors differentiate between generalised arithmetic and equations whilst others exclude patterns or list symbolic language (variables) within its own section. My suggestion (Table 1 rightmost column) and basis of this paper takes into account the current situation in Germany, because it "is important to indicate that any curriculum has a complex relationship to what actually occurs in classrooms" (Cai et al., 2005, p. 14). The German primary curriculum includes no algebra area, even though a single door for the implementation of algebraic thinking opens up. This possible link is the content area 'patterns and structures', which is given in the national standards (KMK, 2004). Moreover, patterns and structure can be regarded as generic field for the different algebraic topic strands (Drijvers et al., 2011).

Owing to these two reasons my own suggestion stresses the terms patterns and structures. In the following paragraph these terms are theoretically analysed in more depth.

Patterns and structures

Often mathematics itself is described as the science of patterns (Devlin, 1997). In this view, all mathematical theories arise from patterns spotted. Even axioms characterise patterns to build on. Not surprisingly, teaching and learning about patterns and structures is not a special topic but is fundamental for all mathematics lessons:

> Pattern is less a topic of mathematics than a defining quality of mathematics itself. Mathematics 'makes sense' because its patterns allow us to generalize our understanding from one situation to another. Children who expect mathematics to 'makes sense' *look* for patterns. (Brownell, Chen, & Ginet, 2014, p. 84)

Becoming aware of patterns allows us to see sense in mathematics and to appreciate its beauty. This awareness is at least twofold. On the one hand, seeking patterns can be classified as meta-cognitive, on the other hand, there is a cognitive component of awareness which is characterised by "knowledge of structure" (Mulligan & Mitchelmore, 2009, p. 38).

Patterns can be described as "any predictable regularity, usually involving numerical, spatial or logical relationships" (Mulligan & Mitchelmore, 2009, p. 34). Constructing a pattern of numbers or shapes by making up a rule or a certain operative variation (Wittmann, 1985) of a given number or task is an individual creative process. If, for instance, the pattern of a number sequences is creatively made up, the regularity then is fixed and can be used, continued, and described (Steinweg, 2001).

In the approach presented here *structure* is understood as mathematical structure and not as a category system to describe the individual pattern awareness of children (Rivera, 2013). Mason, Stephens & Watson (2009) recommend "to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced" (p. 10). They point out the difference between the spotting of (singular) relations and the use of the given example as paradigmatic for certain properties of a general structure (Mason et al., 2009). Thus, detecting structures, in contrast to patterns, requires mathematical knowledge about objects and operations. The relation between mathematical objects is essentially determined by mathematical structures (Wittmann & Müller, 2007). Awareness of structures often suffers from the fact that structures are mentioned only briefly and only formulated in rules, like a+b=b+a, in mathematics lessons. Unfortunately, these condensed statements are not an appropriate tool to become aware of the logical structures and properties of mathematical objects and relations which are fundamental for mathematics. In summary structure is the crucial term in the twosome patterns and structures. The approach described in this paper therefore puts emphasise on structure, and mentions structure purposefully in each key idea in order to call attention to it.

Algebraic key ideas – a suggestion

The key ideas outlined in the approach presented here focus on (1) patterns (& structures), (2) property structures, (3) equivalence structures, and (4) functional structures (Steinweg, 2016). The first idea differentiates between patterns and structures. Patterns are not a priori structures but may eventually generate products following mathematical properties and relations. Hence, the expression 'structures' is given in brackets to indicate this substantial difference. The second key idea lies in the properties of numbers and operations: Numbers can be divided into odd and even, divisibility can be explored, etc. Daily used –supposedly arithmetical– operations follow structures because of their properties (commutativity, associativity, distributivity). One example of this key idea is presented below. The third key idea holds learning opportunities in evaluating, preserving or construing equivalence in given correct or incorrect equations by sorting terms, etc. The main issue here is to overcome the urge to calculate the given terms and to solely compare the results but to focus on the relation of given numbers, sums, differences, products, or quotients (Kieran, 1981; Steinweg, 2006). This key idea goes hand in hand with the currently commonly used and fostered individual strategies in arithmetic, which can be found in Germany (also cf. Mason et al., 2009). The last key idea sums up learning environments on functional structures, (i.e. mainly proportional), relations, and co-variation aspects. One example is a task called 'number & partner number'

(Akinwunmi, 2012). The structural relationship can be described by a rule (functional term) which assigns a partner number to each given number.

As mentioned above, the key ideas presented here are ordered by mathematical core areas and put emphasis on structures as one of many feasible approaches. Sufficient knowledge of mathematical structures is crucial for both teachers and children. Only well trained teachers are able to understand the mathematical structures and to make them accessible for children. One possible strategy to get access to mathematical structures lies in implementing especially designed tasks which enable children to explore, use, describe, and even prove mathematical structures (Steinweg, 2001).

Methodology

In the research project learning environments suitable for the four key ideas outlined above are designed (Wittmann, 1995) and evaluated in order to uncover the algebraic potential of common tasks and to give tangible examples in the algebraic core areas within the field patterns and structures. Each learning environment includes various tasks in a booklet to be handed out to the children and information for teachers in a teacher's guide (Steinweg, 2013). The teachers participated in an introductory meeting in which the tasks and possible teaching arrangements –given in the guidelines– were discussed. They committed themselves to implement all of the tasks in daily classroom work with the intensity and depth of the use of the learning environments being in their hands. This means that there was no specific focus on the child-teacher-interaction while working on the tasks –with the exception of some mathematics lessons randomly visited by the author. The research therefore focusses on the question: *Does the implementation of the designed tasks show any effects on children's algebraic competencies?* Six German primary school classes with 144 children from 2nd to 4th grade (on average 7- to 9-year-olds) participated in the project.

Research results on the example of distributivity

This paper exemplarily illustrates the research idea on distributivity as one element of the key idea 'property structures'. The main challenge is to see the structure of equations and terms in a meta-perspective way. For instance in the term $2 \cdot 8 + 5 \cdot 8$ children have to spot the specific 'internal semantic' (Kieran, 2006, p. 32). Only if the equal factor is identified as an important component in the products can the 'variable' factors be summed up. The two products have to be identified as objects in a sum and then the two different factors can be added to create a new product $(7 \cdot 8)$. The additive combination of products and the decomposing of products into a sum of two products with one equal factor in each case seems a tough challenge for the children. The shift of attention to elements of the equation as objects and to identify the mathematical structure is essential for algebraic thinking. Most likely, the children participating in the project had already experienced derive-and-combine-strategies solving multiplication tasks in class. The actual approach to the multiplication tables in German mathematics in primary school is peculiar. There is no longer 'doing tables' but working on core tasks (e.g. doubles, times 5, times 10) and derive-and-combine-strategies to solve other multiplications. Only core tasks should be known by heart as facts (sometimes known as 'helping facts' in Anglo-Saxon literature). Unfortunately, an arithmetical perspective –calculate terms to determine the specific result– is normally supported by teachers in primary mathematics. The out of the common change of perception of the structure of equations is

therefore challenging. Only by a shift of attention can the structure of the maths behind the equation, i.e. distributivity, be recognised.

The tasks implemented in the project try to support the identification of structure. For instance columns of variations of one equation are given to allow focussing on both the constant and the changing elements, like 3·2+6·2=…·2, 3·3+6·3=…·3, 3·4+6·4=…·4, etc. Alongside tasks in symbolic representations, rectangle areas as representation for multiplications (length by width) can be used as well. If rectangles are accepted as multiplication representation, manipulating these rectangles by cutting and re-interpreting the two part-rectangles as multiplications can be the next step to explore and understand distributivity. The children were given one example and then asked to find three more possibilities to decompose the product 7·5 (Figure 1). Such rectangles can be provided by the teachers as representations on worksheets or 'actively' made up by the children by cutting out sections of grid paper. As an instance of possible developments in algebraic thinking by simply working on the tasks Philipp's solution of one exemplary task of one worksheet given in the booklet is interpreted in Figure 1.

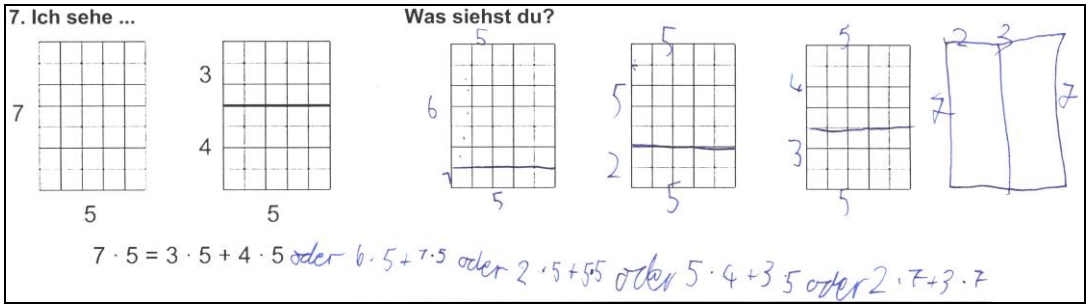

Figure 1: Philipp explores distributive structures by interpreting rectangles as multiplications

Philipp's solution is stunning in some ways. The task asks him to find three further decompositions of the given product. He marks his ideas in the given three rectangle areas and writes down matching symbolic representations of the product-sums. The little dots in the first rectangle grid point at the fact that Philipp might have counted the number of squares. The other solutions do not show these presumed counting dots. Philipp apparently becomes aware of the structure and the main idea. The relation between the terms is understood individually. Philipp connects the different solution by using the verbally form of 'oder', which means 'or'. This habit indicates that he is aware of the possibility of different compositions of the product but not yet sure about the equality-relation between these terms. The worksheet invites the children to find three products by giving three blank rectangles. Philipp extended the task spontaneously by drawing a fourth rectangle. This add-on again is remarkable. Philipp sketches a rectangle without drawing the grid. The countable squares obviously are no longer necessary for him. The fourth solution is the only one which decomposes the factor 5 instead of the factor 7. Philipp applies the main structure of splitting up the factor flexible for either factor now. The example of Philipp shows the multi-facetted possibilities to gain access to mathematical structures by working on challenging tasks.

As the main research question aims at evaluating effects of the implementation of the learning environments, results of a pre- and post-test are of interest. In this paper the results of the test item $10 \cdot 5 - 4 \cdot 5 =$ ___ · ___ (corresponding to distributivity) are documented exemplarily (Table 2).

Category	pre-test (n = 135)	post-test (n = 133)
Algebraic	1.5 %	32 %
Procedural	32.5 %	35 %
no answer given	66 %	33 %

Table 2: Results solving $10 \cdot 5 - 4 \cdot 5 = ___ \cdot ___$

The task is quite hard to handle for the participating children in the pre-test even so the curriculum expects teachers to work on derive-and-combine-strategies in multiplication. Two thirds have no idea what to fill in the blanks. Only in very few cases are children able to combine the two multiplications into 6 x 5 and thereby make use of the structure (algebraic perspective). After participating in the project one third is now able to give this answer. Another third places a multiplication like 3 x 10, which is fitting because of an equivalent result (procedural, arithmetical perspective). The figures suggest a developmental step of the procedural thinkers to the algebraic thinkers apparently. This assumption actually cannot be confirmed by the data. The developments are very much individual. For instance, some children who gave no answer prior to the project are now able to see the structural relation or calculate to find matching terms and others still have no answer at all. Despite the fact that these results are still far from being satisfactory, the increase in numbers of children using an algebraic perspective is considerable.

Discussion

The project gives an initial indication that it is possible to foster algebraic thinking by providing sound learning environments without explicit variable use in the scope of the content field of patterns and structures. The challenges offered support effects on understanding and on performance in algebraic tasks. Yet, the impact of learning environments alone is not enough to support all children. As mentioned above, the project provides no binding specifications to teachers of how, for example, to focus on distributivity, but offers different opportunities to explore this mathematical structure via the designed tasks. As a "good balance between skill and insight, between acting and thinking, is […] crucial" (Drijvers et al., 2011, p. 22), further effort should focus on exploring the differences between procedural and structural work on tasks. Teachers' instructions and interaction in classroom discussions as well as the specific role of representations have to be focused on in further studies.

The hope is that the developed key ideas function as bridges between arithmetical topics and algebraic ones and also as guiding principles for classroom interaction. If common arithmetical strategies –like derive-and-combine– are seen from a different angle, they actually are algebraic ones. From a meta-perspective view the procedures performed are determined by mathematical structure and the properties of operations. The shift of attention towards structures has to be made explicit to both teachers and children. Only if teachers appreciate algebraic structures can they offer effective support and take up children's algebraic ideas. In the particular situation in Germany awareness of the multi-facetted potential of the usually underestimated core area patterns and structures is crucial. Last but not least, the sensibility implies a win-win-situation: "Awareness of structure of expressions helps students understand these better, thus leading to a better understanding of rules and procedures" (Banerjee & Subramaniam, 2012, p. 364).

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Constraints to a justification of commutativity of multiplication

Heidi Strømskag[1] and Anita Valenta[2]

[1]Norwegian University of Science and Technology; heidi.stromskag@math.ntnu.no

[2]Norwegian Centre for Mathematics Education; anita.valenta@matematikksenteret.no

In this paper we investigate a pre-service teacher's whole-class discussion at Grade 6, where she attempts to justify the general claim that multiplication is commutative in ℕ. *Our analysis points at two conditions that constrain the discussion in class: The first is that the diagram used to represent a multiplicative situation—and the way the diagram is used—is inadequate because it does not illustrate the meaning of multiplication. The second is that the diagram enables the students to respond "adequately", even if they may not have understood why multiplication is commutative. The latter is a constraint because it deprives the pre-service teacher of the opportunity to get feedback that might have let her understand that she would need to revise her intervention.*

Keywords: Multiplicative situation, commutativity, representation, justification, teacher education.

Introduction

Addition and multiplication in ℕ (or, more generally, in ℝ) are commutative and associative. Moreover, multiplication is distributive over addition. These basic properties of addition and multiplication play a crucial role in abstract algebra, but also in arithmetic and algebra in elementary school. The work on fluency of the multiplication table gets considerably easier if the students make use of the basic properties. In fact, all arithmetic strategies can be shown to have origin in the basic properties of commutativity, associativity and distributivity. Promoting these properties in elementary school is important for integrating arithmetic and algebra (Carpenter, Franke, & Levi, 2003). Involved in this is the activity of generalising and formalising relationships and constraints, which is one of three strands of algebra identified by Kaput (2008). Furthermore, the basic properties of addition and multiplication are also important in "transformational activities" (Kieran, 2004), which involve syntactically guided manipulation of symbols (e.g., simplifying expressions, exponentiation with polynomials, and solving equations).

At the background on the above, the question raises as to how it can be discussed with students that addition and multiplication are both commutative and associative, and that multiplication is distributive over addition? If we want these properties to emerge from students' mathematical reasoning in the classroom—and not from the teacher's just presenting them as rules—how could it be done, and what challenges may arise? In this paper we discuss this question with data from teacher education, where a pre-service teacher has designed and implemented a lesson, aimed at whole-class discussion with 12 year-old students on basic properties of multiplication. The research question set out to answer is: *What conditions constrain a pre-service teacher's whole-class discussion with Grade 6 students about commutativity of multiplication in* ℕ***?***

Theoretical framework

Multiplicative structures—modelling situations involving equal-sized groups

From a mathematical point of view, multiplication and division by natural and rational numbers may appear easy. However, from a psychological point of view, it is more complex. In a teaching situation,

these operations are dealt with not only as abstract binary operations, but also in terms of how they model different situations. Vergnaud (1988) claims that "[m]athematical concepts are rooted in situations and problems" (p. 142). Because a single concept does not refer to only one type of situation, and a single situation cannot be studied with only one concept, Vergnaud proposes that researchers study "conceptual fields". This is defined as a set of situations, the mastery of which depends on mastery of a conceptual structure. For instance, the conceptual field of *multiplicative structures* consists of all situations that can be analysed as simple and multiple proportion problems, where the necessary operation is multiplication or division (Vergnaud, 1988).

According to Greer (1992), the most important types of situations where multiplication of integers is involved are:

- equivalent groups (e.g., 6 tables, each with 4 children)
- multiplicative comparison (e.g., 3 times as many girls as boys)
- rectangular arrays/areas (e.g., 4 rows of 7 students, or area of a rectangle)
- Cartesian product (e.g., the number of possible trousers-sweater pairs)

Fishbein, Deri, Nello and Marino (1985) investigated how 623 pupils enrolled in 13 Italian schools (Grades 5, 7 and 9) responded on 26 multiplication and division word problems. Their findings confirmed the impact of *repeated addition* as an intuitive model on multiplication, in which a number of groups of the same size are put together—that is, equivalent-groups situations.

In Norwegian schools, multiplication is usually introduced through situations with equivalent groups, where $4 \cdot 7$ means $7 + 7 + 7 + 7$, while $7 \cdot 4$ means $4 + 4 + 4 + 4 + 4 + 4 + 4$. Here, it is not obvious that multiplication is commutative. The first of the factors—the number of equivalent groups—is taken as the operator (termed *multiplicator*); the other factor—the size of each group—is taken as the operand (termed *multiplicand*). In this model, the multiplicand can be any positive quantity, but the multiplicator must be an integer (Fishbein et al., 1985).

Justification in the elementary classroom

Algebraic thinking is a term used to describe particular ways of thinking applied when we are looking beyond quantities and operations on quantities. It includes analysing relationships between quantities, noticing structure, studying invariance and change, generalising, problem solving, modelling, conjecturing, justifying and proving (Cai et al., 2005). In this section we concentrate on justification. There are several ways of approaching justification and proof in school mathematics. Balacheff (1988) has identified four types of reasoning in 14-15 year-old students' practice of proving a conjecture that applies on infinitely many examples:

- *Naïve empiricism* is where students think that some examples (even one or two) are sufficient to justify a conjecture.
- The *crucial experiment* is where students think that the validity of a conjecture is accomplished by testing it on an instance that has some complexity—the reasoning being "if it works here, it will always work". The crucial experiment is different from naïve empiricism in that the generality at stake is explicitly articulated.
- The *generic example* involves making explicit the reasons for the truth of an assertion by means of operations on an object that is a representative of the class of elements considered. A generic example is an example *of* something—the validity of a hypothesis is argued for by the characteristic properties of this example.
- The *thought experiment* requires that the one who produces the proof distances him from the actions of solving the problem—he must give up the actual object for the class of objects on

which relations and operations are to be represented in formalised symbolic expressions. Proof by induction is an example of a thought experiment.

Proofs by naïve empiricism, the crucial experiment, and the generic example are based on actual actions and references to examples—these proofs are referred to as *pragmatic* proofs (Balacheff, 1988). The thought experiment is based on abstract formulations of properties and of relationships among properties—this proof is referred to as a *conceptual* proof. Balacheff emphasises that a proof by naïve empiricism or by a crucial experiment does not establish the truth of an assertion, and the reason why he refers to them as "proofs" is because they are recognised as such by the students who produce them. He asserts, further, that the generic example and the thought experiment are mathematically valid proofs. They involve a fundamental shift in the students' reasoning because the nature of the truth of a claim is established by giving reasons. When using a visual representation to justify a claim of generality, the representation needs to have some properties. Schifter (2009) has identified three criteria for a representation in elementary grades to be adequate: (1) the meaning of the operation(s) involved is represented in diagrams, manipulatives, possibly complemented by story contexts; (2) the representation is accessible for a class of instances; and, (3) the conclusion of the claim follows from the structure of the representation (p. 76).

Justification in the elementary classroom of the commutative property of multiplication can be done through a generic example based on a rectangular-area situation. In this situation, multiplication in $\mathbb{Q}^+$ is commutative because the area of a rectangle is the same regardless of the order in which its side lengths are multiplied. However, given the impact of the equivalent-groups situation, we consider it important also to be able to build on this intuitive interpretation in justifying that multiplication is commutative. Then of course, the asymmetry of this situation is a challenge. In the following we explain how a justification of commutativity of multiplication in $\mathbb{N}$ can be constructed, taking the situation of equivalent groups as a starting point. We discuss a justification first by a generic example, then by a thought experiment.

Let $4 \cdot 3$ be interpreted as the total number of discs when we have 4 equivalent groups of 3 discs, as shown in the upper row of Figure 1. The discs can be regrouped into 3 equivalent groups of 4 discs, which corresponds to $3 \cdot 4$ (illustrated in Figure 1[1]).

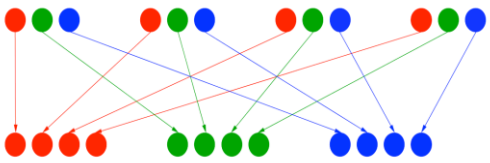

Figure 1. A generic example illustrating the symmetrisation of an asymmetric situation

The total number of discs is not changed, and consequently we have that $4 \cdot 3 = 3 \cdot 4$. This process of regrouping can be imagined with an arbitrary number of groups *a*, and an arbitrary number of discs *b*—that is, for any $a \cdot b$ where *a* and *b* are natural numbers. The number of discs in the equivalent groups in the original grouping transforms into the number of groups in the new grouping. The

[1] The discs are coloured to make the process clearer. The arrows signify the movement of the discs.

example with the transformation of $4 \cdot 3$ into $3 \cdot 4$ is thus used as a generic example in the justification.

Justification by a thought experiment can be rather similar to the generic example presented above, though the reasoning is done in general terms: Given natural numbers *a* and *b*, then $a \cdot b$ can be interpreted as the total number of discs when we have *a* groups with *b* discs in each group. Regrouping of discs by taking one by one disc from each group to make a new group gives *b* groups with *a* discs in each group—that is, a situation in which the total number of discs can be represented by $b \cdot a$. Since the number of discs is not changed in the process of regrouping, we can conclude that $a \cdot b = b \cdot a$ for all natural numbers *a* and *b*.

Methodological approach

The pre-service teacher (henceforth PST) participating in the research reported here was in her first year of a 4-year undergraduate teacher education programme for Grades 1-7 in Norway. The investigation has been done within a compulsory mathematics course in the programme, which involves an integration of mathematics and didactics. The data were collected at the end of the second semester. The main content of the mathematics course previous to data collection was multiplicative thinking, and the emphasis was on different strategies for, reasoning with, and properties of, multiplication and division. The second author, together with a colleague, taught the mathematics course and carried out the data collection.

During the mathematics course, the PSTs in the class worked on several assignments that involved practice of teaching in school (Grades 4-7), all concerned with strategies for and properties of multiplication. In the fourth assignment, from which the data analysed in this paper emerged, the PSTs were asked to plan and carry out a discussion with students concerning a given strategy or property of multiplication or division. The PSTs video-recorded and transcribed their discussions with students. The transcript analysed here is from one of 25 classroom discussions that were carried out and analysed.

Our research question concerns basic properties of multiplication, and we are interested in PSTs´ handling of general justifications—that is, justifications for an infinite number of cases. In most episodes where the PSTs discussed properties of multiplication with the students, there was no attempt to generalise and justify. There were discussions of particular examples, usually succeeded by a conclusion along the lines of "this will apply for all numbers". In this paper, we present an analysis of one of the discussions, the case of Janet (a PST). This case is chosen because Janet actually tried to discuss with students *why* a property of multiplication applies in general (in $\mathbb{N}$) and it demonstrates challenges thereof, which are also traced in some of the other transcripts (not analysed here). We will explain what conditions that prevent the case of Janet from being successful in the sense of including a valid argument for the claim of generality.

Results

Establishing a situation to interpret a multiplication problem

Janet and the students have discussed the products $12 \cdot 10$ and $10 \cdot 12$, and the students have come to the recognition that the products are the same. David says it is because "the numbers have simply changed places" (turn 10). Then Janet provides another example:[2]

11. Janet: We will get the same answer, it's just the calculation that is reversed… If we take another arithmetic problem, will that be similar, too? $13 \cdot 17$ and $17 \cdot 13$ [writes on the blackboard]. [Pause 7 sec.]. Will this be the same, or are they different? [Pause 11 sec.]. What do you think? Do you think it will be the same answer, or are they different?
12. Brian: I think it will be the same, because you have just exchanged the numbers.
13. Janet: You think it will be the same? Mary, do you think it will be like Brian said?
14. Mary: Yes.
15. Janet: I don't know whether you have done this before, made a story or a drawing. Is there anyone who would try to make a story for $13 \cdot 17$, if we just concentrate on this [product]? Can someone make a story or drawing that might explain $13 \cdot 17$? Is there anyone who dares to do that? [Pause 5 sec.].
16. Janet: What does it mean? Could it mean that we have 13 of something that we shall have 17 times? If we imagine having a baking tray with muffins for instance. If we imagine having a baking tray [draws on the blackboard]. Can someone try to figure out how the drawing will be, if we have a baking tray with $13 \cdot 17$ muffins? [Pause 3 sec.].
17. Janet: Where should we place 13 for instance? Should we just draw them all over the place, or should we place them across or down? Anyone who dares to try? Trying is allowed. Remember, no answer is silly. [Pause 5 sec.].
18. Janet: Nobody dares to try? Well, OK. If we imagine that we have 13 muffins across here [draws on the blackboard], and we have 17 down. We fill out the whole tray, but I don't bother to draw them all. You understand that we have 13 across and 17 down. If we were to calculate this instead of counting all the muffins, how could we do that? You may want to take $13 \cdot 17$. Then we can think that we have 13 across and 17 down [points at the blackboard]. However, if we had $17 \cdot 13$ [points at the blackboard], can someone figure out what the drawing would look like? Carl?
19. Carl: It will be 17 across and 13 down.
20. Janet: Uh-huh. Anne, can you repeat what Carl said?
21. Anne: It will be 17 across and 13 down.
22. Janet: Yes, we would have had 17 here and 13 down [points at the blackboard and explains]. Have we changed how many muffins we have on the tray? [Tim shakes his head].

In turn 15, Janet invites the students to give an interpretation of the product $13 \cdot 17$. Nobody responds, after which Janet (turn 16) introduces multiplication in terms of equivalent groups: $13 \cdot 17$ is explained as the number of objects we will get when "we have 13 of something that we shall have 17 times". This is a non-commutative situation, where 17 is the multiplicator and 13 is the multiplicand.

Then there is a shift to a rectangular-array situation, when Janet introduces a context of muffins on a baking tray to interpret the product $13 \cdot 17$ (turns 16-17). In turn 18 she explains how the product can be placed on the tray: 13 muffins across and 17 down. She says that the whole tray should be filled out, but draws only the first row and first column. The resulting diagram is reproduced in Figure 2, and we will refer to it as a "degenerated" array. With a proper (13x17)-array, it would have been possible to interpret the multiplication problem as an equivalent-groups situation in correspondence

[2] The transcript has been translated into English by the authors. Names are pseudonyms.

with Janet's initial explanation of multiplication: 13 muffins in a row could be interpreted as a group, and 17 rows could be interpreted as equivalent groups of 13 muffins. But the degenerated array and Janet's use of spontaneous concept ("across" and "down") instead of the scientific concepts "row" and "column", makes it unclear how the presented situation should be interpreted as the product $13 \cdot 17$.

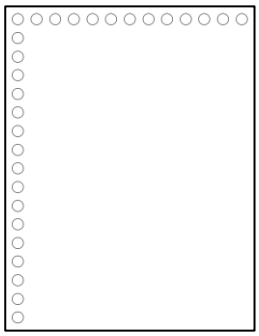

Figure 2. The diagram used by Janet to illustrate the product $13 \cdot 17$

The operation aimed at is just declared by Janet (turn 18): "You may want to take $13 \cdot 17$". When she asks how it would be if they had $17 \cdot 13$, Carl gives the desired answer (turn 19, repeated by Anne in turn 21). Nevertheless, this does not imply that Carl has understood what $17 \cdot 13$ means—it indicates only that he is able to substitute the numbers used by Janet. The diagram in Figure 2 does in fact represent two numbers (one across and another down), but the diagram does not represent the operation of *multiplying* these numbers.

Justifying that multiplication is commutative in $\mathbb{N}$

After having indicated how the products $13 \cdot 17$ and $17 \cdot 13$ should be interpreted as (degenerated) array situations in terms of muffins on a tray (as presented above), Janet sets out to justify that multiplication is commutative for all numbers in $\mathbb{N}$:

23. Janet: When we have $17 \cdot 13$, the tray would look like this, and if we have $13 \cdot 17$, we can just imagine that we rotate the tray. Then the arithmetic problem will be different. We may also think that we have a sheet of paper. If we imagine having $13 \cdot 17$ like this, and $17 \cdot 13$ like this [demonstrates on the sheet]. Then we can see that these arithmetic problems will be the same.

Janet uses the products $13 \cdot 17$ and $17 \cdot 13$—represented as drawings of muffins on a tray—to exemplify that multiplication is commutative. In turn 23, she refers to these products as being different arithmetic problems. The imagined rotation of the tray (possibly 90 degrees) is used to show that the arithmetic problems have the same result, the reasoning being that the rotation of the tray does not change the total number of muffins—hence $13 \cdot 17 = 17 \cdot 13$. We interpret the sheet mentioned in turn 23 as a representation enabling Janet to actually show the rotation and its effect on the arrangement of the muffins (a feature not afforded by the representation on the blackboard).

Having established that $13 \cdot 17 = 17 \cdot 13$, Janet then asks whether this property applies for all numbers:

29. Janet: How do you think it will be? Does it apply only for these numbers, or does it apply for all numbers? When we multiply two things… [Pause 5 sec.]. Mary?

[Mary says that she thinks that it applies for all numbers, and exemplifies by $1 \cdot 2$ and $2 \cdot 1$]

33. Janet: Uh-huh. Do you think it applies for all numbers, all whole numbers? [Pause 4 sec.]. Or are there numbers for which it doesn't apply? [Pause 5 sec.].

Several students respond that they think it applies for all numbers, and Janet asks why they think so.

37. Mary: I think it has to apply for all numbers. Because it's about the same [pair of] numbers.
38. Brian: It can be a little demanding when you have very large numbers, like 1 million times 2 millions. It will be challenging to draw.
39. Janet: Uh-huh. Well, it will indeed be much to draw if we were to draw a million. But if we imagine that we take away all the muffins. If we imagine that we have only one sheet of paper [erases the muffins on the blackboard drawing]. We can imagine that we have 1 million times 2 millions, then we can place it like this [points at an array-model on the blackboard]. So, does anyone dare to formulate a rule for multiplying two numbers. When we use what we have just seen, which applies on those [points at the blackboard drawing]. [Pause 10 sec.]
40. Mary: It will be the same if we swap the numbers.
41. Brian: It is possible also to check out with this tray in case one is insecure.
42. Janet: Uh-huh. A rule can be that, when we do multiplication problems, the order does not matter. Whether we take $13 \cdot 17$ or $17 \cdot 13$ does not matter. We can see this [property] if we make such a drawing. If we rotate the drawing, the [total] number has not been changed, we just rotate the drawing.

The multiplication problem $13 \cdot 17$ is used as a generic example in the dialogue to justify the commutative property of multiplication in $\mathbb{N}$. The property that the factors in a multiplication problem commute is based on the idea of rotating a tray (or sheet) with muffins arranged in a rectangular array—this is Janet's intention, even if the diagram used is not a proper array. The generic properties of the example are, however, vaguely expressed: Janet suggests that the total number of muffins on the tray is not changed by a rotation (turns 23 and 42), but she does not express in clear text what the commutative property means in the actual situation (i.e., exchanging row and columns). When Brian (turn 38) provides an example that involves the product 1 million times 2 millions, it can be considered a crucial experiment (supplemented by Janet in turn 39): the validity of the conjecture of commutativity is accomplished by testing it on an instance that is quite complex (and impossible to draw). In turn 42, Janet utilizes the generic example of $13 \cdot 17$ when she articulates the conclusion of the claim—an important, last step in a justification process.

Discussion

The decision not to draw all the muffins (possibly because it would take too long) prevents the diagram in Figure 2 from representing the *meaning of the operation* at stake (even if Janet says that the whole tray should be filled out). Hence, Schifter's (2009) first criterion for a representation to be adequate is not met. It can be noticed that the other representation used, the sheet, does neither illustrate the meaning of multiplication, but it affords the rotation to be demonstrated physically. For the meaning of multiplication to be represented in a diagram, the total number of objects—the result of the operation—needs to be displayed. This entails that the numbers involved must be of manageable size, thus enabling them to be represented in diagrams or manipulatives. That Janet failed to draw the complete array indicates that the numbers she used in the generic example (13 and 17) were too big, as she possibly conceived of it.

It is possible to represent any pair of natural numbers in the diagram used by Janet, and hence, it seems as if Schifter's (2009) second criterion is met. Yet, this is irrelevant since the meaning of the operation is not represented in the diagram. Further, since the diagram does not represent

multiplication at the outset, it is useless to check if Schifter's third criterion is met (i.e., whether the conclusion of the claim follows from the structure of the diagram).

The discussion in class (based on Figure 2) enables the students to evidence possession of some knowledge. This knowledge is, however, different from the knowledge aimed at by Janet: The students were able to say that the result—in the general case—would be the same even if the numbers in the multiplication problem were reversed. Yet, there is no indication that the result they refer to is the *product* of the two numbers. It is likely that the students imagine a diagram with objects in a formation similar to the one in Figure 2, and that they see that rotation does not change the total number of objects in the diagram. This is basically an aspect of the principle of number conservation, and it is doubtful whether the students have understood why multiplication is commutative for any pair of natural numbers, which was the aim of the lesson.

In conclusion, there are two conditions that constrain Janet's discussion with the students about commutativity of multiplication in ℕ: The first is that the diagram, as used by Janet, is inadequate because it does not illustrate the meaning of multiplication. The second is the matter of fact that the diagram (and the way it is used) enables the students to respond "adequately" (i.e., as expected by Janet), even if they may not have understood why the commutative property applies for multiplication. The latter is a constraint because it deprives Janet of the opportunity to get feedback that might have let her understand that she would need to change her approach.

The case of Janet can be used in teacher education to discuss with pre-service teachers criteria for, and impact of, generic examples (or representation-based proofs) used to justify general claims about properties of arithmetic operations. It is relevant to extend the research reported here by analysing written material from students' justification of properties of arithmetic operations.

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Shape patterning tasks: An exploration of how children build upon their observations when asked to construct general terms

Aisling Twohill

Dublin City University, Ireland; aisling.twohill@dcu.ie

Generalisation is a key element of algebraic thinking, and children's developing thinking is supported by engagement with algebraic activities when they attend primary school (Kaput, 2008). Shape patterns provide a context within which children may identify structure, and construct general terms. Within my research I presented children with shape patterns and observed their interactions as they discussed the structure of the patterns and sought to construct general terms. In this paper, I discuss the elements of the patterns to which they attended, and how their focus supported some children in constructing general terms, while other children experienced more challenge in their engagement with the patterns. Specifically I focus on children who observed both numerical and figural aspects, and I examine the varying levels of success they experienced.

Keywords: Algebraic thinking, generalisation, shape patterns, reasoning, task-based interview, group interaction.

Introduction

Kaput (2008) defined generalising, representing generalisations, and syntactically guided reasoning on generalisations as core aspects of algebraic thinking. Providing a context for functional thinking, shape patterning tasks may facilitate children in engaging with these core aspects. Strømskag (2015) defines a shape pattern as a sequence of terms, composed of 'constituent parts', where some or all elements of such parts may be increasing, or decreasing, in quantity in systematic ways. While a limited number of terms of a shape pattern may be presented for consideration, the pattern is perceivable as extending until infinity. In order to construct a general term for a shape pattern, children must "grasp a regularity" in the structure of the terms presented, and generalize this regularity to terms beyond their perceptual field (Radford, 2010, p. 6). In seeking to construct general terms for shape patterns, children's opportunities for success improve when they attend to both the spatial and numerical aspects of the structure of the pattern (Radford, 2014). In seeking to extend a pattern, children identify the position of pattern elements by attending to the spatial structure, and in order to quantify such elements they must attend to the relevant quantities within the pattern terms presented.

In the Irish context, the Primary School Mathematics Curriculum (hereafter PSMC) includes an 'algebra' strand, and there are content objectives prescribed for teachers to address in their teaching, from when children commence primary school at four or five years of age (Government of Ireland, 1999). The curriculum does not, however, include the construction of general terms in any context, and beyond the first two reception years the patterns prescribed are strictly numerical in nature. During task-based group interviews, I asked children to construct general terms for each of three shape patterns. In this paper I will explore the tendency of children participating in my research to attend to the spatial and numerical aspects of pattern structures, as they worked towards the construction of general terms. Of the children who attended to both numerical and spatial elements, some children experienced success in applying their observations in order to construct general

terms, whereas some children seemed to experience a challenge in translating their observations into meaningful understanding of the structure of the pattern.

The construction of general terms

Building upon the core aspects of algebraic thinking outlined by Kaput (2008) and mentioned above, Blanton, Brizuela & Stephens (2016) identified four key practices of algebraic thinking as: a) generalising, b) representing, c) justifying, and d) reasoning with generalisations, emphasising that these practices must focus upon structures and relationships. Blanton et al. (2016) also presented the following three content domains within which children may apply the key practices of algebraic thinking: a) generalised arithmetic, b) equations, and c) functional thinking.

In particular, Blanton et al. (2015) highlighted the relevance of functional thinking to the algebraic thinking of young children, stressing the connection between functional thinking and the four key practices of algebraic thinking highlighted above. Blanton et al. (2015) emphasised the role of functional thinking in young children's algebraic thinking by stating that functional thinking includes generalisations of co-varying quantities and their relationship; representations of these relationships, and reasoning with the relationships in order to predict functional behaviour. As functional thinking, and specifically shape patterning, is absent from the PSMC, it is therefore highly improbable that most children attending Irish primary schools will have engaged with activities designed to develop functional thinking.

Rivera and Becker (2011) identified two approaches utilised by children in seeking to construct general terms for geometric patterns. Some children in Rivera and Becker's longitudinal study focused solely, or primarily, on numerical aspects of the terms provided, and sought to use the numerical patterns observed in order to identify a commonality, extend the pattern and construct a general term. Such children the authors described as adopting a 'numerical' mode of generalising. In comparison, Rivera and Becker considered a child to have adopted a 'figural' approach if he/she used figural aspects of the pattern such as the shape of terms, or the position of elements, both within the term and relative to each other. A figural approach may include attention to numerical elements, but not in a manner that supersedes the child's perception of the spatial aspects of the pattern structure.

Along with observations of the structure of terms, children may observe various relationships within patterns which support their understanding. A 'recursive' strategy involves an examination of the mathematical relationship between consecutive terms in a sequence, and if using an 'explicit' strategy, a child identifies a rule for the relationship between a term and its position in the pattern (Lannin, 2005). Such observations work in tandem with children's numerical or figural approaches, and in this paper I will refer to children's explicit, or recursive thinking when relevant.

Radford asserted that children's constructions of generalities may be factual, contextual or symbolic. Factual generalisations involve instantiating a general structure to specific terms, whereby children do not express a generalisation as applicable to all terms, but apply an "operational scheme" which allows them to calculate a value for particular terms (p. 82). Many of the children involved in this research project applied factual generalisations when they described the near and far terms for the patterns, as in the example of Emily for Pattern 3 when she said "You'd need a hundred and twelve horizontal and then fifty-seven vertical". Emily applied her understanding of the

2n horizontal poles, and n vertical poles in order to calculate the number of poles for this far term. Equally she could apply this thinking to any term of this pattern, and could therefore be said to have factually generalised, even though there is no abstraction evident in her expression, and she is describing a specific term.

Contextual generalisations, by comparison, involved the consideration of non-specific terms (ibid.). While contextual generalisations are not completely abstract, or general to all terms, they indicate a distancing from the specific, whereby children may make reference to "the next term" or to a generic term. As an example, Grace could be said to have constructed a contextual generalisation for Pattern 1 when she said "whichever number it is at the top it will just be one more than it, and at the bottom". Symbolic generalisations involve the abstract expression of disembodied mathematical objects, wherein children express the algebraic concepts with no reference to the method of their calculation, or to any specific term. Throughout this paper I make use of 'n' to refer to a general term number, but all such expressions were generated by me, as representations of children's verbal utterances, constructions and gestures. The children were not prompted to construct symbolic generalisations.

Methodology

In seeking to explore children's constructions, I facilitated their engagement with the patterns in groups of four in a 'task-based interview' setting. Goldin (2000) emphasised that task-based interviews, involving individuals or groups, have become an essential tool within mathematics education research, as the goal of mathematics education has moved from the transmission of disconnected facts, to the development of children's rich understanding and "internal constructions of mathematical meaning" (p. 524). In order to explore children's complex understandings, and to observe their approaches to the solution of problems, it is necessary to adopt a research approach designed specifically for this purpose. In this way, it may be possible for educators and policy makers to assess whether the application of progressive approaches within classrooms are working to develop children's mathematical understanding and robust problem-solving skills.

Sixteen children (with a mean age of 9.6 years) participated in the task-based interviews, when they were presented with three patterns (see Figure 1). When presented with the patterns, the children were asked to describe what they observed, to extend the pattern to subsequent or previous terms, and to construct near and far terms, as factual generalisations. The children were provided with concrete materials, and asked to construct pattern terms during their discussions. Having considered far terms, I asked the children to describe "any term" in the pattern to facilitate their articulation of a contextual generalisation. At the commencement of each interview, and at regular intervals throughout, the children were encouraged to work collaboratively, by constructing terms together, by sharing their ideas, whether they agreed or disagreed, and by asking questions of each other. The children did not receive any instruction before participating in the interviews.

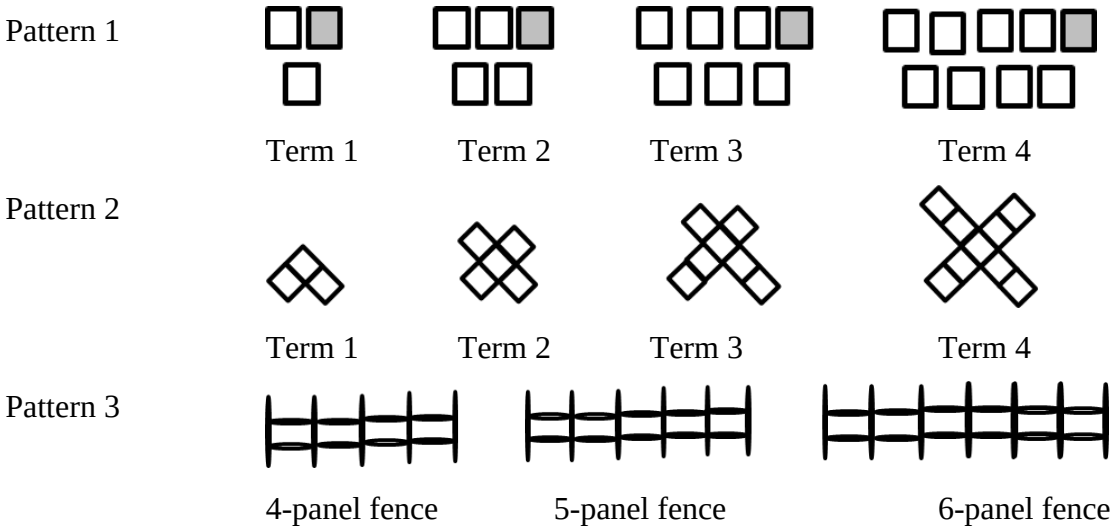


Figure 1: The three patterns presented to children.

Seeking to explore the reasons for children's strategy choice, as well as the strategy chosen, the research approach I considered most appropriate was phenomenological, as a phenomenological approach to research seeks to explore a phenomenon where it occurs, and acknowledges the many factors which influence how the phenomenon plays out within the given setting (Creswell, 2013). In exploring the children's thinking, I not only analysed the children's actions, but I also sought to analyse the contributing factors that impacted on the strategies children employed.

Goldin (2000) advises that "by analysing *verbal and nonverbal* behaviour or interactions, the researcher hopes to make inferences about the mathematical thinking, learning or problem-solving of the subjects" (p. 518, my emphasis). In seeking to explore children's mathematical constructions, I was conscious throughout that my inferences from children's comments were approximations of their true meaning. As Van Manen (1990) attests "a *good* phenomenological description is an *adequate* elucidation of some aspect of the lifeworld" (p. 27, my emphasis) and while I sought to unpick as best I could how and why children thought about the mathematical tasks, I posit that it is not possible to feel a sense of completion, or closure, in relation to the children's thinking, but rather that interpretation is ongoing (Postelnicu & Postelnicu, 2013).

In exploring the children's verbal utterances, I coded comments as referring to figural or strictly numerical aspects of a pattern. Rivera and Becker (2011) state that adopting a figural approach is to "figurally apprehend and capture invariance in an algebraically useful manner (p. 356). When coding children's statements, I identified a statement as indicating a figural approach if it included reference to the position of an object within a term, by using words such as 'top', 'bottom' or the deictic 'there' along with an associated gesture (Radford, 2006). An object in this context referred to a square, a diamond, a tile, a line, or any item which formed a constituent part of a term. Comments were deemed to indicate a numerical approach if no reference was made to the position of objects within a term. The term 'growing' was used regularly by children, and required some thought with regards to whether it indicated a figural approach. Typically, when mentioning growth children were referring to a sense of the terms' shape growing in size, that is "selectively attend[ing] to aspects of sameness and difference among figural stages", but I could not assume that this was always the case

(Rivera & Becker, 2011, p. 356). Rather, it was necessary to attend to some term, or deictic within a child's comment, and to seek to determine the referent, which would indicate whether the child was referring to the shape as growing, or the quantity of constituent elements.

Findings and Discussion

In seeking to explore the strategies used by children, and the reasons underpinning their strategy choice, I firstly considered an overview of each child's approach across all three patterns. As the length of this paper does not allow for a complete explication of the approach of each child, Figure 2 presents an overview of my interpretation of the children's thinking, as it pertained to the balance between numerical and figural observations. In collating and analysing this data, I referred to the comments children made in the context of the exchanges they participated in. I also referred to field-notes made during the interviews, photographs of the children's constructions, and the children's drawings and jottings which I had retained as artefacts of their thinking during the interviews. I had video recordings for three of the four interview groups, and audio recording for the fourth, as not all participants of this group had assented to video recording.

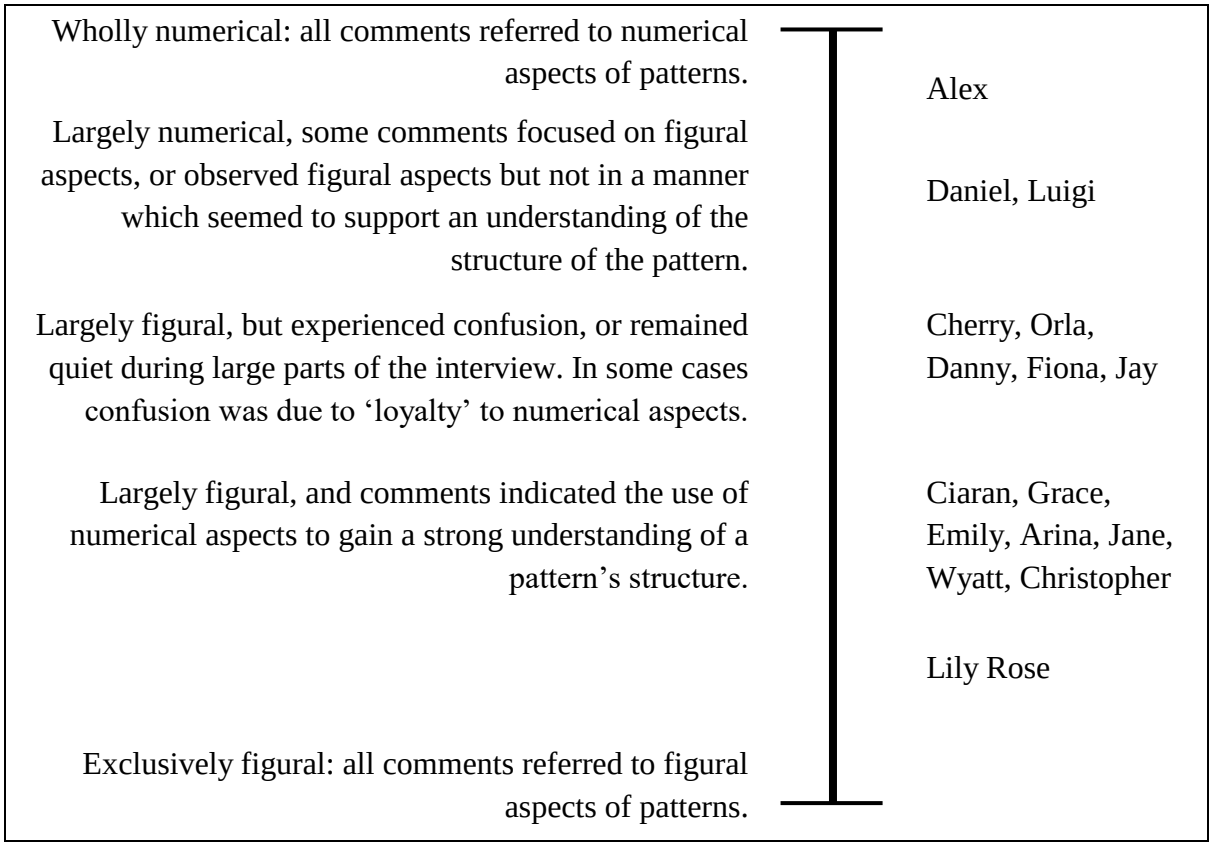


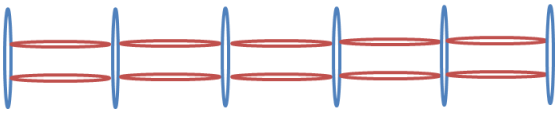
Figure 2: An overview of my interpretation of the children's thinking, as it pertained to their focus on numerical and figural aspects of patterns

As can be seen from Figure 2, rather than demonstrating a dichotomy between children who approached all patterns figurally and children who approached all patterns numerically, this small group of children (n=16) span a continuum from children who made comments focusing largely on numerical aspects, through children who commented on both numerical and figural aspects in meaningful ways, to children who referred largely to figural aspects. In considering the aspects children were attending to, I sought to distinguish between children's observations which supported

their thinking, and observations which they made and didn't build upon or apply in order to construct a general term. Later in this paper I will further explore how two of the children applied their observations of figural aspects.

While attending to figural aspects may have supported some children in constructing general terms, figural observations did not lead inevitably to generalisation. In order to compare children's tendency to observe figural aspects with their success in constructing factual or contextual generalisations, I generated a scoring rubric for children's progress towards the construction of a general term. A score of 0 indicated no progress, a correct extension of the pattern scored 1, some description of general terms scored 2, factual generalisation scored 3 and a contextual generalisation scored 4. I generated a total score for each child and calculated the mean score for each cluster of children identified in Figure 2 above. I found that the cluster of children identified in Figure 2 as making "largely figural" observations while applying numerical observations succeeded well, achieving a mean score of 8.6, where a score of 9 would equate to, for example, factual generalisation of all three patterns. In contrast the group of children who also made many figural generalisations, but expressed some confusion fared considerably less well, achieving a mean score of 3.2, where a score of 3 would equate to extending each pattern correctly, but not making any progress in describing a general term.

To explore what other factors may have impacted on children's progress, I chose to contrast the thinking of two children, Cherry and Arina, who worked together, and who both articulated figural observations, but made strikingly dissimilar progress. Arina achieved a score of 9 overall, while Cherry achieved a total score of 2. In this section I will discuss the girls' observations of Pattern 3, a 'fences' pattern presented in Figure 1 above. Arina had demonstrated strong figural thinking on two previous patterns, and succeeded in describing factual generalisations for both. When the children began their deliberations about the pattern discussed here, Arina remained very quiet, making few comments, but succeeded in constructing the 56th term using an explicit approach. She didn't verbally articulate her thinking enough to confirm whether this factual generalisation was based upon a numerical or figural mode of generalising, but she could be seen on the video footage counting up in twos to quantify the horizontal posts, and adding on a number equivalent to one more than the term number. Figure 3 presents an illustration of how Arina may have been quantifying the number of posts for terms in this pattern.



5th Term in the Pattern: n=5. 2n horizontal Posts and n+1 Vertical Posts

Total number of posts: 2n + (n+1) = 3n+1

Figure 3: A representation of Arina's perceived structure of the Fences pattern.

By comparison, Cherry's verbal articulations of her thinking indicated that she observed within each fence one panel containing 4 posts, and every other panel containing three posts. When finding the number of posts needed for the nine-panel fence, she used a recursive approach, counting on six posts from the seven-panel fence she had constructed with match-sticks. I interpreted Cherry's perception of this pattern as including figural aspects, as she referred regularly to 'posts' and

grouped the posts into groups of three or four, as appropriate. However, Cherry's approach to this pattern was dominated by a counting strategy, an analysis of which is beyond the scope of this paper. This counting strategy did not support Cherry in constructing far generalisations, and in seeking to construct a far generalisation, Cherry drew 56 panels, and began to count the number of posts needed.

Conclusion

Rivera and Becker (2011) suggest that when children only attend to numerical aspects of a pattern, they are grasping the commonality within the structure of the pattern at a superficial level. In analysing the comments made by the children in this research study, it may be seen that one cluster of children attended to figural elements but did not succeed in generalising. I would suggest therefore that some children who attend to figural aspects of the pattern, may persist with a limited and superficial understanding of the structure of the pattern, and that a figural perspective may not lead inevitably to successful construction of a general, or generic term. Other perceptions of the pattern structure seemed to be required in tandem with observations of both figural and numerical aspects. In the examples given here, Arina's explicit approach supported her thinking, and in marrying an explicit approach with observations of both numerical and figural aspects, Arina grasped the structure of the pattern and successfully constructed a factual generalisation. In contrast Cherry demonstrated a consistent tendency to use counting as a strategy in seeking to quantify the number of elements of components of far terms of this pattern. While she could describe the figural structure of the pattern, her thinking seems to have been hindered by difficulties she encountered in seeking to conceptualise an expression which would allow her to calculate the number of posts without counting them. Even though this was the third pattern, and other children in her group had described explicit approaches during the interview, Cherry might not have made sense of the explicit thinking articulated by others. Equally, limitations in her multiplicative thinking may have contributed to this, and restrained her from exploring the explicit relationship between each term and its position in the pattern. Further research is merited into the interplay between the many aspects of patterns to which children may attend in seeking to grasp the structure.

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Multiple models for teaching logarithms - with a focus on graphing functions

Christof Weber

School of Education, University of Applied Sciences Northwestern Switzerland, Liestal, Switzerland; christof.weber@fhnw.ch

This argumentative paper focuses on graphing logarithmic functions and presents some mathematical knowledge for teaching, employing vom Hofe's construct of basic models (or 'Grundvorstellungen') and Sfard's distinction between operational and structural conceptions. On the assumption that difficulties students have with graphing might be a consequence of the standard interpretation of logarithms as inverse exponents, it claims that logarithmic functions should not be introduced structurally, as inverse exponential functions. Instead, several operational models of the logarithmic concept are proposed, and their explanatory power for graphing logarithmic functions is expounded. These models are intended to serve students as a meaningful basis for argumentation.

Keywords: Logarithmic functions, basic models, operational—structural, knowledge for teaching.

Introduction

Logarithms as functions differ from logarithms as numbers (e.g. determining $\log_2 1024$) or operators (e.g. using logarithms to manipulate expressions and solve equations), especially when it comes to teaching and learning (Smith & Confrey, 1994). To make logarithms as numbers and operators meaningful and thus more accessible to learners, some subject matter knowledge for teaching has been developed by Weber (2016). This was achieved by combining the theoretical construct of basic models (or 'Grundvorstellungen', vom Hofe & Blum, 2016) with the construct of operationality and structurality (Sfard, 1991; 2008). The present paper focuses on the functional side of logarithms and its teaching, namely on the sketching of logarithmic graphs. After recalling various conceptualisations of logarithms and analysing some students' difficulties in graphing logarithmic functions, four basic models are discussed with regard to their explanatory power for graphing logarithmic functions. Because of the mathematical analogy between logarithms and division (cf. Weber, 2016), some arguments used here are analogous to those familiar from the teaching of division (Ball, Thames, & Phelps, 2008; Greer, 1992; vom Hofe & Blum, 2016).

Background

Various conceptualisations of the logarithmic concept

As we are interested in different ways of viewing logarithms, it is worth taking a quick look at the history of mathematics. Several properties of logarithms have been discovered since their invention in the early 17th century (for details, see Weber, 2016):

(P1) Napier and Bürgi conceptualised logarithms as numbers that count divisions. For example, $\log_2 8$ equals 3 because 8 has to be divided by 2 three times (to yield 1). In analogy to the standard division algorithm, this interpretation can be extended to a *logarithm algorithm* for the manual calculation of logarithms proceeding by repeated division down to 1 instead of re-

peated subtraction down to 0: one divides instead of subtracting, exponentiates by 10 instead of dividing by 10, trying to yield 1 instead of 0 (ibid., pp. 79–80).

(P2) The conception found in current collections of formulas only became possible after Descartes had invented the symbol for powers, a^n. This notation enabled logarithms to be seen as particular exponents, a property that Euler used in his 1765 definition $\log_a b = x :\Leftrightarrow a^x = b$. Moreover, the left-hand equation can be read not only arithmetically (focusing on the number x), but also functionally, taking "$\log$" as a function of the argument b.

(P3) Another conceptualisation was provided by Cauchy in the 19th century. He proved that logarithmic functions are, apart from a factor, the only continuous solutions F of the functional equation $F(x \cdot y) = F(x) + F(y)$ ($x > 0$, $y > 0$). Specifying the logarithmic function in implicit form, this conceptualisation is a purely functional one.

Each of the properties (P1) to (P3) highlights a particular aspect of logarithms. Because, from a mathematical perspective, they are all equivalent, each property could serve as a definition of the concept. From an epistemological perspective, however, their qualities are distinct, as discussed below.

Logarithmic functions, their graphing and some of the difficulties students face

Logarithmic functions are essential in calculus and for modelling processes, which is why they are taught at secondary level and in undergraduate courses. As with other types of functions, there are several challenges for students, such as when it comes to interpreting graphs verbally, or deciding whether or not a given graph represents a certain function (e.g. Leinhardt, Zaslavsky, & Stein, 1990; Markovits, Eylon, & Bruckheimer, 1986; 1989). For instance, one well-known student misconceptions says that any function should be linear (e.g. Sfard, 2008, p. 21). An issue specific to logarithmic functions is the confusion of the logarithmic graph with the "combined graph" (Kastberg, 2002, p. 129), i.e. with the exponential and the logarithmic graph merged into a single image. A similar misconception reported is students viewing the graphs of $y = 2^x$ and $y = \log_2 x$ as being "exactly the same" (Williams, 2011, p. 54). Misconceptions like these might be caused by the standard introduction of logarithmic functions as inverse exponential functions.

As space precludes other aspects, this paper deals with the graphing of logarithmic functions only, i.e. with manually sketching the graph of a function in a Cartesian plane, based on its logarithmic equation $y = f(x)$. To analyse some of the difficulties that can arise here, the following steps (S1) to (S4) involved in graphing logarithmic functions are identified:

(S1) *Determining and calculating an appropriate number of pairs* (table of values): This step involves selecting a finite series of x-values that lie within the domain of the function, which in the case of logarithmic functions is a proper subset of the real numbers. In particular, one has to determine a first x-value, a last x-value, and the pattern the series follows (e.g., if there is a pattern, whether the difference or the ratio of two consecutive x-values is constant).

(S2) *Drawing the axes and scales, and plotting the corresponding points* ("local construction," Leinhardt et al., 1990, p. 13): The plotted points form the supporting points of the graph to be drawn.

(S3) *Connecting the supporting points with a line segment* ("prediction," ibid., p. 13): This step involves making a conjecture based on the visual characteristics of the plotted points. It will therefore be referred to here as "graphical interpolation". It includes decisions regarding the curvature and the degree of smoothness of the line (e.g. differentiable at the points).

(S4) *Extending the graph to the right and left of the line segment*: This is a second prediction to make, based on a part of the graph, referred to here as "graphical extrapolation": Does the graph straighten out to a straight line? If not, in what way does its curvature change? Is it bounded, does it have vertical asymptotes, intercepts with the axes, etc.?

In each of these steps, the lack of a meaningful basis for argumentation can result in difficulties: Firstly, determining the domain in (S1) is a known issue (e.g. Markovits et al., 1986, 1989). As an illustration, Figure 1 shows two students' graphs of the function $y = \log_2(x) - 3$. Student A's graph extrapolated to the left intersects the x-axis and thus exceeds the domain. Moreover, his x-values form an arithmetic progression, which is not optimal in terms of the corresponding y-values.[1]

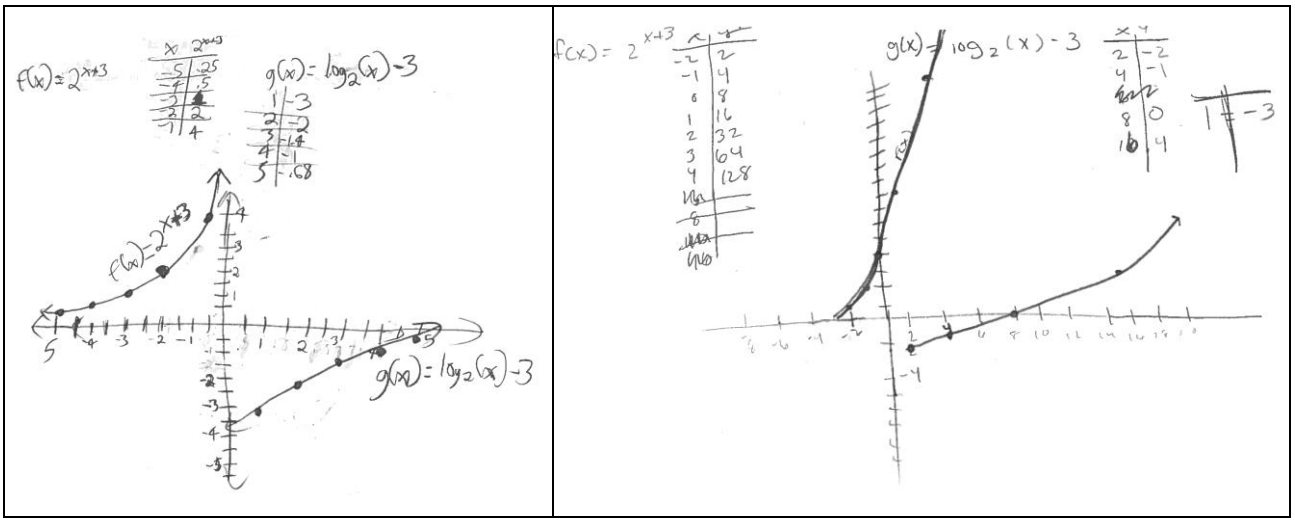

Figure 1: Students' documents showing their tables of values for $f(x) = 2^{x+3}$ and $g(x) = \log_2(x) - 3$, together with the corresponding graphs (left side: student A, male; right side: student B, female)

Secondly, graphs are sometimes thought of as isolated points (Leinhardt et al., 1990), or the supporting points may be interpolated with a straight line (Markovits et al., 1989). Figure 1 shows that in their third step (S3), both students chose the graphical interpolations to be a more or less straight line, at least piecewise. And thirdly, in (S4), the graph is sometimes not extended beyond the range of the supporting points at all, or only by a little; or there may be an extrapolation to one side which suggests a progressive growth of the logarithmic function (cf. student B in Figure 1).

[1] The two first- and second-year undergraduate students (18 and 19 years old) had attended a precalculus course held at a public university on the east coast of the USA in spring 2016. Logarithmic functions had been introduced in the traditional way, i.e. as inverse exponential functions. The documents shown in Figure 1 are from their final examination. The exact wording of the task was: "Consider the two functions $f(x) = 2^{x+3}$ and $g(x) = \log_2(x) - 3$: a) For each function, create a table of values, choosing your x-values carefully. b) Graph both functions on the same set of axes."

Students' difficulties like these give rise to the following question: *What mathematical knowledge for teaching logarithmic functions could endow learners with a meaningful basis for argumentation in order to potentially reduce their difficulties?*

Understanding functions and logarithms

There are many ways to conceptualise what it means to understand the concept of a function in general (e.g. Lauritzen, 2012; Markovits et al., 1986; Sfard, 2008), or the concept of logarithmic functions in particular (Berezovski & Zazkis, 2006; Kastberg, 2002). Interestingly enough, graphs are rarely included in conceptualisations of how functions are understood; when they are, they are used to gauge whether it is possible to derive the equation of a certain function (Markovits et al., 1986).

Graphing as a vital aspect of understanding functions

Focusing on the opposite — graphing equations — here, Lauritzen's (2012) conceptualisation of procedural and conceptual knowledge of functions is useful, because he attaches importance to the construction of graphs. To measure the ability to perform "graphic procedures" (ibid., pp. 52–53), he asks students to sketch the graph of a function, thus subsuming graphical interpolation (S3, see above) and extrapolation (S4). On the other hand, he considers calculating values (S1) a type of "algebraic procedure" (ibid., pp. 54–55). In Lauritzen's theoretical framework, graphic and algebraic procedures together operationalise the *procedural knowledge of functions*. In other words, graphing can be seen as a vital part of understanding functions.

Operational and structural conceptions of functions, and the discourse on functions

Sfard's analysis of how mathematical notions are formed shows that conceiving mathematical notions (1991) and talking about them (2008) can happen in two fundamentally different ways: as processes (*operationally*), or as objects (*structurally*). For instance, learners tend to read equations of functions and tables of values operationally, whether as prescriptions of how to calculate values of the function, or as a covariation between two quantities (Sfard, 1991, p. 15). By contrast, they tend to perceive graphs of functions structurally, as "[…] infinitely many components of the function […] combined into a smooth line, […] as an integrated whole […]" (ibid., p. 6). According to Sfard (1991), firstly, operational approaches are more accessible to learners when forming new concepts. Secondly, concept formation, for instance of functions, means subsuming the discourses on equations and graphs in a new discourse. For example, the concept "logarithmic function" is reified as soon as the discourse on logarithms as numbers and operators is merged with the discourse on logarithmic graphs, as soon as they "become mere representations" (Sfard, 2008, p. 122).

As such, the history of the logarithmic concept reminds us of the reification of rational numbers (Sfard, 1991, 2008): On one hand, because the conceptualisation (P1) of logarithms as numbers that count divisions can be transferred to a set of computational rules that make it possible to calculate logarithmic values step by step, it expresses *an operational view* (similar to the division $1 \div 2$). This is why Sfard's findings on functions could apply to logarithmic functions as well, as her analysis of understanding functions relates to polynomials only, and their operational character. On the other hand, a historically more recent conceptualisation such as inverse exponents (P2) expresses *a structural view* (similar to the fraction $1/2$).

Basic models for logarithms as numbers and operators

To help students access and understand a certain mathematical concept, it is sometimes embedded in a context that is realistic or, if this is not possible, in a context that is at least familiar to the students (cf. the Dutch "realistic mathematics education", van den Heuvel-Panhuizen, 2003). In the German-speaking countries, *basic models* is a theoretical construct to capture what is meant by making concepts accessible, or understanding them (referred to as "Grundvorstellungen", vom Hofe & Blum, 2016). Put simply, a basic model for a concept must have two characteristics: Firstly, it is an interpretation of that concept in a context in which students are likely to have more experience, and secondly, it has a certain explanatory power, that is, it is flexible enough to be applicable to different mathematical situations. For instance, when division is seen within the everyday context of fair-sharing, an equation such as $30 \div 1/2 = 60$ is difficult to follow or perform. However, within the context of splitting-up or measuring, it can be explained as "$1/2$ fits into 30 sixty times". Both basic models of division, fair-sharing and splitting-up, are thus indispensable for understanding division (referred to as "partitive" and "quotative division", Greer, 1992). For Ball and colleagues, they constitute the *specialized content knowledge* for teaching division (Ball et al., 2008, p. 400).

For the teaching of logarithmic functions, no basic models are known thus far. For logarithms as numbers and operators, however, I have previously identified four models (for details, see Weber, 2016):

(BM1) *Logarithms as multiplicative measuring*: The logarithm of a number b (to base a) indicates how often the base a fits into the number b as a factor. This interpretation derives from the algorithm mentioned above (P1), or from the relation $b / a^{\log_a b} = 1$. Example: $\log_2 1024$ can be simplified to 10 because 2 as a factor fits into 1024 ten times. As it generates *result*, multiplicative measuring emphasises the *operational* side of logarithms most strongly.

(BM2) *Logarithms as counting the number of digits*: The (common) logarithm of a number b finds the number of digits of b needed to represent b in positional notation, minus one. This interpretation derives from the fact that the number of digits of any natural number n (in decimal notation) is equal to $\lfloor \log_{10} n \rfloor + 1$. Example: The number 2^{2000} has 603 digits when written out in decimal notation because $\log_{10} 2^{2000} \approx 602.06$. In describing the *effect* it has on numbers and thus dealing with a specific application, this interpretation could be used to support the *operational* explanation of logarithms in the case of numbers.

(BM3) *Logarithms as decreasing the hierarchy level*: The logarithm of an expression reduces third-level operations (powers, roots) to second-level operations (multiplications, divisions), and it reduces second-level operations to first-level operations (additions, subtractions). This interpretation derives from property (P3). Example: The expression $\log\sqrt{cd}$ can be expanded to $1/2\left(\log c + \log d\right)$ because the taking of square roots, as a third-level operation, becomes dividing by two, and multiplication of the variables becomes addition of their logarithms. In describing the *effect* it has on expressions and thus dealing with another specific application, this interpretation could support the *operational* explanation of logarithms for expressions.

(BM4) *Logarithms as inverse exponents*: The logarithm of a number (or expression) to base a is the exponent by which the base a must be raised to yield the number (or expression). This

derives from property (P2), and is useful for solving exponential equations. Example: 40 as a power of 2 is approximately $2^{5.32}$ because $\log_2 40 \approx 5.32$. Because this interpretation *relates* logarithms to another object (exponents), it reflects the *structural* view of experts.

In the next section, this collection of operational and structural basic models will be shown to have the potential to serve as a basis for argumentation for the graphing of logarithmic functions as well.

Basic models for logarithmic functions and their explanatory power

From a mathematical standpoint, every property of logarithmic graphs can be derived from exponential graphs, using (BM4) as a basis for argumentation. From an epistemological viewpoint, however, conceiving logarithmic functions as inverse exponential functions reflects the structural view of experts who have reified their experiences, and not the view of learners. Perhaps, as Sfard (1991) and others suggest, operational conceptions should instead *precede* structural ones as consecutive steps to be passed through when teaching a new concept such as logarithmic functions? As basic models (BM1) to (BM3) do not replace logarithmic functions with another class of functions, they could be more appropriate for learners than model (BM4). Instead, they inform students about what logarithms "do" and what logarithms "are good for", interpreting them within contexts in which learners are likely to have some experience (counting, long division, having an effect on …, etc.).

In what respect, then, could the three basic models described offer students a meaningful basis for argumentation when they are introduced to graphing logarithmic functions? And in what way could they potentially reduce the difficulties described above? Here are some arguments:

1. *Operational conceptualisation*: The logarithm algorithm, which stems from property (P1), "logarithms are numbers that count divisions", and is captured by basic model (BM1), can allow students to conceive a logarithmic equation such as $f(x) = \log_2(x) - 3$ operationally, much like a polynomial one: "First, calculate how often the base 2 fits into x by repeated division, then subtract 3."

2. *Domain of the function*: In order to graph a logarithmic function, it is essential to determine its domain (cf. (S1) and Figure 1). With reference to the model "logarithms as multiplicative measuring" (BM1), values such as $\log(0)$ and $\log_2(-8)$ can be recognized as incalculable because there are no reasonable answers to the corresponding questions "How many times does 10 as a factor fit into 0?" and "How many times does 2 as a factor fit into –8?"; neither 0 nor –8 can be converted to 1 through repeated division. This is why logarithms of 0 and of negative values cannot be defined.

3. *Pattern of the finite series of x-values*: Another choice to be made for graphing a logarithmic function easily is the pattern that the series of x-values follows (S1). In view of the basic model "logarithms as decreasing the hierarchy level" (BM3), the x-values should follow a geometric series, with the ratio of two consecutive x-values equalling the base. The logarithm would then transform the geometric series into an arithmetic one, resulting in equidistant y-values.

4. *Growth of the graph:* As we have seen above, both graphical interpolation and graphical extrapolation can cause many problems (cf. (S3), (S4)). Referring to the basic model "logarithms as counting the number of digits" (BM2), the growth of logarithms can be recognized as non-proportional: In general, doubling a number does not double its number of digits. Furthermore, it

is strictly increasing and unbounded above because this is how the number of digits behaves. Thus neither the interpolated nor the extrapolated graph can be a straight line, but must increase monotonically, growing degressively.

Discussion

This work builds on my earlier paper about the basic models for logarithms as numbers and operators (Weber, 2016). There, the supposition was discussed that the standard textbook explanation $\log_a b = x :\Leftrightarrow a^x = b$ could be too compact or "dense" for many learners to serve as a meaningful basis of argumentation, which may be why dealing with logarithms often turns into mere manipulation of formal symbols, causing students' difficulties (ibid., pp. 85–86). If this applies to logarithmic functions as well, an alternative, broader way of introducing and teaching logarithms is required. For this reason, this paper discusses some content knowledge for the teaching of logarithmic functions. The guiding theoretical construct is that of basic models (vom Hofe & Blum, 2016), combined with the construct of operationality and structurality (Sfard, 1991, 2008). The four basic models, developed previously for logarithms as numbers and operators (Weber, 2016), are shown here to have some explanatory power for logarithmic functions and their graphing, that is, that they could potentially help to make logarithms meaningful and reduce some common difficulties that students encounter.

This paper lays some theoretical foundations for future research. To what extent an approach with multiple basic models can facilitate more meaningful teaching and understanding in the actual classroom will have to be investigated carefully. There has been a first encouraging episode from my own teaching, where a student who in general struggles with mathematics realized why logarithmic functions cannot be proportional: Making use of basic model (BM2), not the standard interpretation (BM4), she argued precisely as in point 4 in the previous section. A teaching experiment is therefore planned in the near future to study the affordances and limitations of the basic models, exploring the discourse of students who are taught not just one but four interpretations of the logarithmic concept. In analogy to the teaching and learning of division with multiple models, a crucial point will be the students' shift from the multiple basic models proposed here to the object of logarithmic functions. Or, to cite Freudenthal (1975, as quoted in van den Heuvel-Panhuizen, 2003, p. 15, italics in original): "Models *of* something are after-images of a piece of given reality; models *for* something are pre-images for a piece of to-be-created reality".

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Hierarchical and non-hierarchical clustering methods to study students algebraic thinking in solving an open-ended questionnaire

Benedetto Di Paola,[1] Rosario Battaglia[2] and Claudio Fazio[3]

[1]Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Palermo, Italy; benedetto.dipaola@unipa.it

[2]Dipartimento di Fisica e Chimica Università degli Studi di Palermo, Palermo, Italy; onofriorosario.battaglia@unipa.it

[3]Dipartimento di Fisica e Chimica Università degli Studi di Palermo, Palermo, Italy; claudio.fazio@unipa.it

Keywords: Algebraic thinking, clustering, k-means method, dendrograms.

Introduction

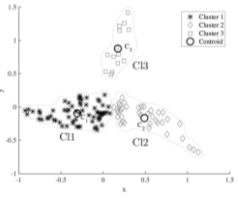
The problem of taking a data set and separating it into subgroups, where the members of each subgroup are more similar to each other than they are to members outside the subgroup, has been extensively studied in science and mathematics education research. Student responses to written questions and multiple-choice tests have been characterised and studied using several qualitative and/or quantitative analysis methods. However, there are inherent difficulties in the categorisation of student responses in the case of open-ended questionnaires. Very often, researcher bias means that the categories picked out tend to find the groups of students that the researcher is seeking out. In our contribution, we discuss an example of application of hierarchical and non-hierarchical analysis method, to interpret the answers given by 118 Tenth Grade students in Palermo (Italy), to six open-ended questions about algebraic thinking. We show that the parallel use of the two quantitative analyses allows us to interpret in deep way the reasoning of students solving different mathematical problems using Algebra. These clustering methods also allow us to highlight different students groups, that can be recognised and characterised by common traits in their answers, without any prior knowledge on the part of the researcher.

Methodology

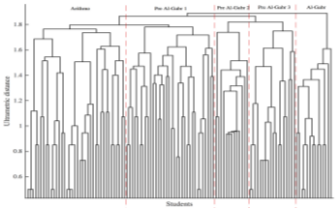
In recent years, some papers have tried to develop detailed models of the reasoning competences of the student populations tested, or to subdivide a sample of students into intellectually similar subgroups, by using quantitative or qualitative analysis methods. (Everitt, Landau, Leese & Stahl, 2011; Prediger, Bikner-Ahsbahs & Arzarello, 2008) It is worth noting that research papers using quantitative analysis methods to study student responses to open-ended questionnaire can be found in Science education; not many research work can be trace in Mathematics education (Di Paola, Bataglia & Fazio, 2016), especially on the application of clustering analysis. In this paper we focus on the application of hierarchical and non-hierarchical clustering methods referred to dendrograms representation and k-means algorithm (Everitt, Landau, Leese & Stahl, 2011), trying to make sense to answers given by 118 Tenth Grade Italian students to six open-ended questions on algebraic thinking. The questionnaire was administered to the students at the beginning of the school year, before any discussion about Algebra had taken place. They answered in 45 minutes.

In particular we discuss the results on the study of typical students' behaviour in tackling the algebraic resolution of word problems and, at the same time, at understanding how the student semantically and syntactically control questions containing symbolic algebraic expressions (Radford & Puig, 2007). Our decision to refer to word problems, according to the PISA test, can allow us to study student literacy in using algebra (Bohlmann, Straehler-Pohl & Gellert, 2014) and in the transition from arithmetic to the modelling of problems expressed in a not-symbolic language, called "natural language (NL)" (Prediger, Bikner-Ahsbahs & Arzarello, 2008). K-means and dendrograms approach allowed us to partition and characterize our student sample, without making any a-priori assumptions and giving interesting output about student's behaviour.

Clustering results



The k-means method (showed in Figure 1) allowed us to simply group and characterize the common students traits related to their solution strategies of the open-ended questions about algebraic thinking (procedure choices, mistakes, failings etc.). This gives us the opportunity to safely partition students into three groups: these are characterized by centroids C_i (called *Arithmo, Pre Al-gabr and l-gabr*) that represent the answering strategies given with maximum frequency by the students who are part of the cluster. The Hierarchical clustering method (showed in Figure 2), obtained using the Weighted Average linkage, identified five groups of students (called *Arithmo, Pre Al-gabr 1, Pre Al-gabr 2, Pre Al-gabr 3 and Al-gabr*) allowing us to better highlight their difficulties in the answering strategies related to the transition between the NL (typical of word problem) and the symbolic one. The results we found are largely coherent with the ones already reported in the literature obtained by means of qualitative methods. For this reason, we can consider the use of both hierarchical and non-hierarchical clustering a valid tool to complement the use of qualitative analysis to study a large number of students with respect to the way they give answer to the questionnaire.



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Conceptualizing the core of the function concept – A facet model

Carina Zindel

TU Dortmund University, Germany; carina.zindel@math.tu-dortmund.de

This paper presents a model for conceptualizing the core of the function concept, which is made up of those facets that are equally important for all types of functions and common to all representations. The so-called facet model enables the identification of potential obstacles and a detailed description of students' learning processes when connecting representations (e.g. verbal and symbolic representations when solving word problems). In total, 19 design experiments with overall 96 learners were conducted and qualitatively analyzed.

Keywords: Function understanding, concept formation, functions, word problems, lower secondary school.

Theoretical background

Conceptualizing function understanding

Functions are regarded as an important learning content in school. Consequently, many theoretical and empirical studies have investigated the teaching and learning of functions since the beginning of the 20th century, working with very different conceptualizations of "function understanding" (Niss, 2014). For examples, cf. Oehrtman, Carlson, & Thompson (2008); Leinhardt, Zaslavsky, & Stein (1990); Vinner & Dreyfus (1989); common in German didactics is the concept of basic mental models (GVs), cf. vom Hofe (2016). Niss (2014) emphasizes the complexity of the function concept and the necessity of "intentional and focused work on designing rich and multifaceted learning environments" (Niss, 2014, p. 240). Most studies share the overarching claim that students should really *understand* functions, even if this aim is interpreted in different ways, e.g. being able to identify a function (Vinner & Dreyfus, 1989), taking a process view of functions (Oehrtman et al., 2008), and more. Thereby, some of the conceptualizations are more specific than others with regard to their incorporated aspects of single representations or types of function. One example is the role of slope, which is only important for linear functions. This paper now presents a conceptualization of the core of the function concept as a dependence relation that is independent of (1) specific aspects of representations, (2) types of functions, and (3) different perspectives which all still can and should be taken. The facets making up this core were reconstructed empirically and are summarized in Figure 1.

The different function types (linear, square, exponential …) are introduced one after another within the German syllabus for example. Hence, when students deal with square functions it is not surprising that they tend to incongruously adopt specific attributes of linearity to square functions (e.g., Leinhardt et al., 1990). One reason might be that square functions are presented as "new" learning content without making explicit commonalities (especially the facets of the core of the function concept) and differences (e.g., the relevance and usability of constants for interpreting functional equations) to linear functions. This problem also occurs when connecting representations. The aim of this study is to develop teaching-learning material that makes these common core facets explicit.

Connecting representations – learning medium and learning content

An important activity when dealing with a functional relationship is connecting its representations. The notion "connecting representations" resonates with the activity Duval (2006) describes as the conversion of semiotic registers. Studies consider this activity in two different roles: On the one hand, connecting representations is regarded as useful medium for concept formation processes (cf. Duval, 2006). On the other hand, studies emphasize that connecting representations proves to be demanding for learners (Niss, 2014; Leinhardt et al., 1990). Therefore, connecting representations is not necessarily a resource one can build on when designing teaching-learning material, because it already requires some kind of conceptual understanding itself. Hence, before using it as a resource for higher concept formation processes the students' understanding of the core facets needs to be supported. Many studies focus on the symbolic, numerical and graphic representations (e.g. Moschkovich, 1998; Duval 2006). In this paper, the verbal representation will be used to explicate the common core facets. Dealing with functions means dealing with their representations (e.g. Duval, 2006; Leinhardt et al, 1990; Swan, 1985). However, the representations can be considered under different lenses or perspectives. Some perspectives are more obvious in one representation than in the other. Niss explains this fact as follows:

> One important issue that arises in this context is the fact that functions can be given several different representations (…), each of which captures certain, but usually not all, aspects of the concept. This may obscure the underlying commonality – the core – of the concept across its different representations, especially as translating from one representation to another may imply loss of information. (Niss, 2014, p. 240)

Considering the activity of connecting representations as a learning content raises the question of which detailed aspects form the core of the function concept that students have to understand. The core of the function concept shall include those aspects that are common in all representations and equal for all types of functions. This requires an adequate conceptualization to describe this core explicitly and in a differentiated way.

Facet model

Conceptualizing the core of the function concept

This study approaches the question which facets form the core of the function concept by using the construct of „comprehension elements" (Drollinger-Vetter, 2011), which is based on cognitive psychological theory. Comprehension elements (further called: facets, indicated by ||…|| in the text and designated in the boxes of the model) of a concept are defined as central mental schemes, which are mirrored differently in different representations. This theory draws on Aebli's (1981) conception of understanding as a network of facets that are com-

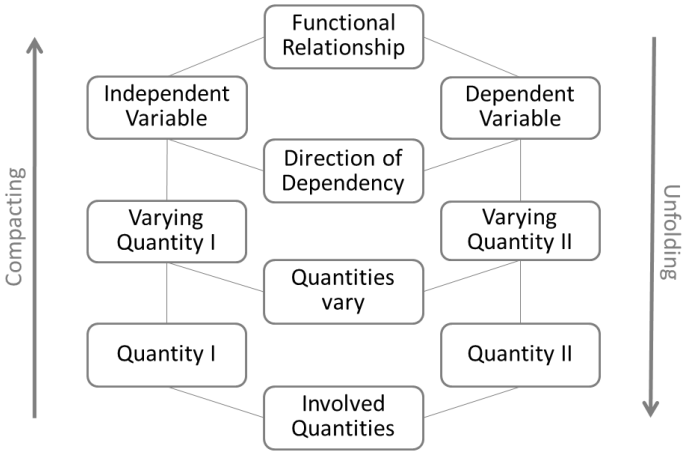

Figure 1: Facet model

pacted into denser concepts: concept formation processes require the acquisition of single facets of the concept and then the relation between the facets. The most compact facets are in the top region of the model while the more unfolded ones are located in the bottom region. Processes of understanding are initiated through processes of unfolding and compacting (Drollinger-Vetter, 2011). Depending on the situation, the edges of the model can be interpreted either as a process of compacting or as one of unfolding.

This construct is now adopted for the function concept. The facets have been reconstructed in the first design experiment cycle. When considering the facets common to linear and square functions for example, one can identify that first it is important to know that there are two ||involved quantities||. General facets like this are shown in the middle column of the model, the concrete manifestations in the situation are shown in the outer paths. Having identified the concrete ||quantity I|| and ||quantity II||, students have to realize that these ||quantities vary|| and that the ||direction of dependency|| matters. These are the facets necessary to finally identify the two quantities as ||independent variable|| and ||dependent variable|| in the concrete situation. Considering the ||independent variable|| and ||dependent variable|| by describing the whole ||functional dependency|| is the most compact way to talk about the core of the function concept. But when dealing with word problems it is equally important to be able to unfold compacted facets. Other "facets" as the ||meaning of the slope|| for example, are only helpful when dealing with linear functions. When dealing with square functions, the constants can only be interpreted in the graphic representation. Accordingly, using the facet model allows the following conceptualization of understanding the core of the function concept:

> "Conceptual understanding of [the core of] functional relationships can be defined as the ability to adopt different perspectives in different [representations] and to coordinate them by flexibly and adequately addressing the facets from [here: Figure 1]. The adequate addressing comprises flexible compacting and unfolding of conceptual facets, thus moving upwards and downwards in the facet model." (Prediger & Zindel, in press, p.9)

This model has proven successful to identify and describe potential obstacles (for examples cf. Prediger & Zindel, submitted). Of course, learners might address other additional facets than the normatively expected ones. The model is sensitive for these individual facets which can also be noticed and combined with other facets.

Research questions

Connecting representations is not necessarily a resource that can be used to support conceptual understanding, because it already requires some kind of conceptual understanding itself, namely flexibly unfolding and compacting the associated facets (Figure 1). This is a starting point to focus on the question of how to support conceptual understanding. In the overarching study teaching-learning material has been developed and empirically tested. In this paper the focus is on the following research question:

How can the facet model be used to describe and visualize learning processes (especially processes of connecting representations)?

Design

The methodology of this project is Topic-specific didactical Design Research (Prediger & Zwetzschler, 2013), which relies on an iterative interaction between designing teaching-learning material, conducting design experiments and analyzing the processes. In the overarching project, three design experiment cycles in laboratory setting and a fourth design experiment cycle in classroom setting were conducted. In total 39 learners participated in 16 design experiments in laboratory setting and further 57 learners participated in 3 design experiments in classroom setting (usually grade 9-10). The overall 42 sessions were videotaped (1890 minutes), partly transcribed and qualitatively analyzed.

Facet model as methodical framework to describe learning processes

This facet model, which has just been introduced, can be used now as a starting point for supporting conceptual understanding by explicitly addressing its facets. The teaching-learning material intends to give the opportunity to get to know, address and combine facets from the facet model. The following part starts dealing with the design element of varying phrases and proceeds with a presentation of the empirical insights regarding its effects.

Varying phrases – a design element

Due to limitations in length of this paper Figure 2 shows only an excerpt of activities from the learning arrangement, realizing the design principle of connecting representations and including the systematic variation of phrases.

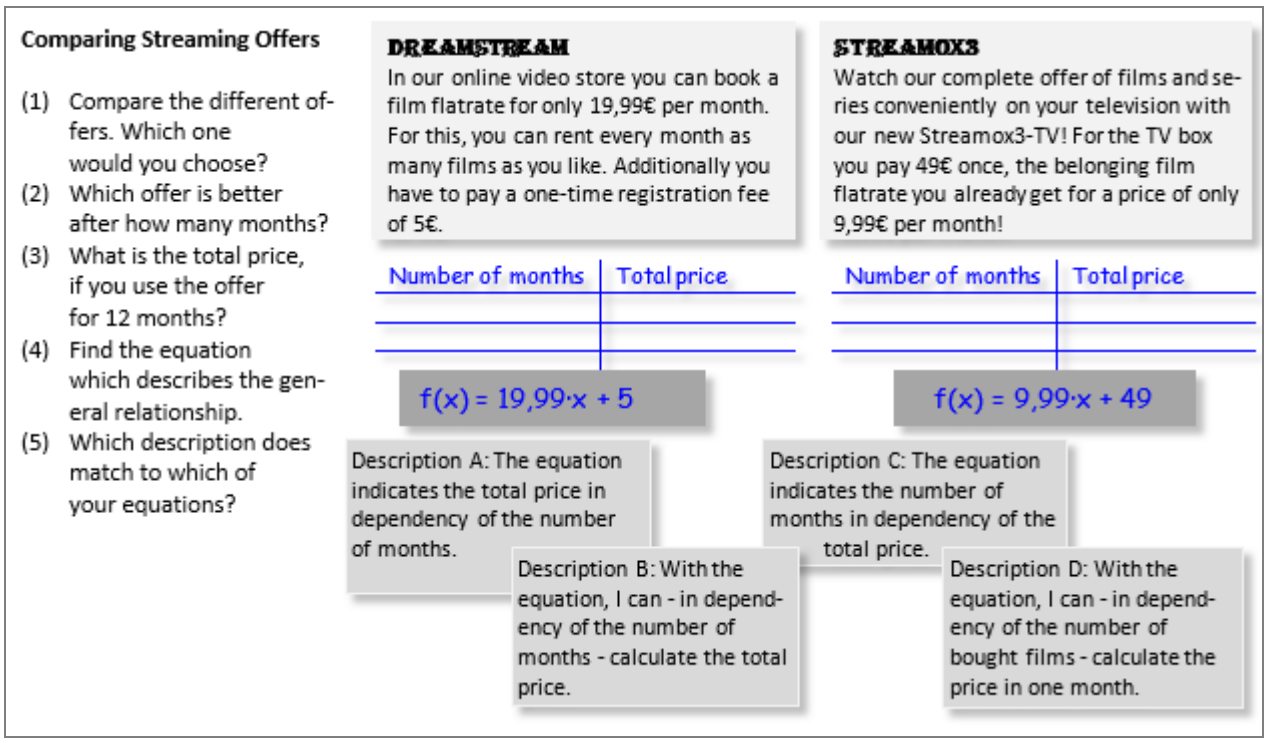

Figure 2: Excerpt from the learning arrangement (Descriptions A-D literally translated from German)

From a normative perspective, different facets should be addressed by dealing with varied phrases. To achieve this, all the phrases vary in at least one of the facets. No sequence of phrases to be considered is given to the students.

Empirical insights: Describing learning processes by using the facet model

In order to connect representations it is necessary to address the same facets in both representations adequately. Both representations (each visualized by one model) refer to the same functional relationship (here: verbal representations on the left, symbolic representation of the DreamStream offer on the right). Each model visualizes the facets that are addressed in the respective representation. Adequately addressed facets or connections are framed by green lines, inadequately addressed facets or connections by red dashed lines. Depending on the situation the lines can be interpreted as process of either unfolding or compacting.

A brief insight into the case study of Tatjana (15) and Alexandra (14) illustrates how dealing with varied phrases makes students aware of the facets from the model. Tatjana starts with description D.

100 Tatjana *[3s]* Well, the first one definitely fits *[points to "With the equation, I can - in dependency of the number of bought films - calculate the price in one month"]*.

101 Alexandra Yes, I think so, too. *[laughs]*

102 Tatjana *[laughs] [3s]* Because actually it doesn't matter how many films one takes. One still pays the same per month anyway.

First Tatjana (Figure 3) misjudges the matching of description D and explains it in 102 with the argument that it does not matter how many films one buys. She does not consider the two quantities in the phrase as ||varying quantities|| that are connected by a dependence relation. Instead she focuses on the two ||involved quantities|| and creates a connection between them by herself. This connection corresponds neither to the phrase nor to the functional equation. She identifies ||quantity I|| and ||quantity II|| in the equation, but she does not realize that these quantities and the ones in the phrase are not the same. Following this thought, they pay attention to the next phrase.

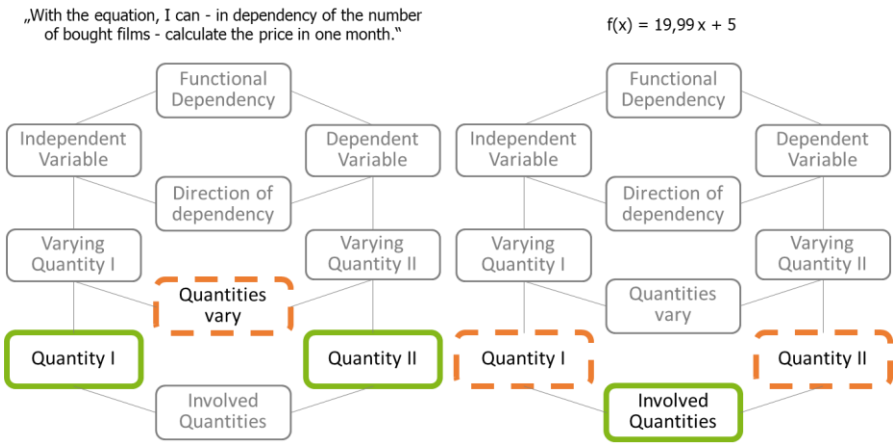

Figure 3 Tatjana (100 and 102)

104 Alexandra *[12s]* I think the second one is right, too *[points to "With the equation, I can - in dependency of the number of months - calculate the total price"]*. Because with the number of months, this would be x indeed – hum – calculate the total price, how much (…)

105 Teacher Mhm.

106 Tatjana *[11s]* This is the same like this *[points to "The equation indicates the total price in dependency of the number of months"]*, right?

107 Alexandra Yes.

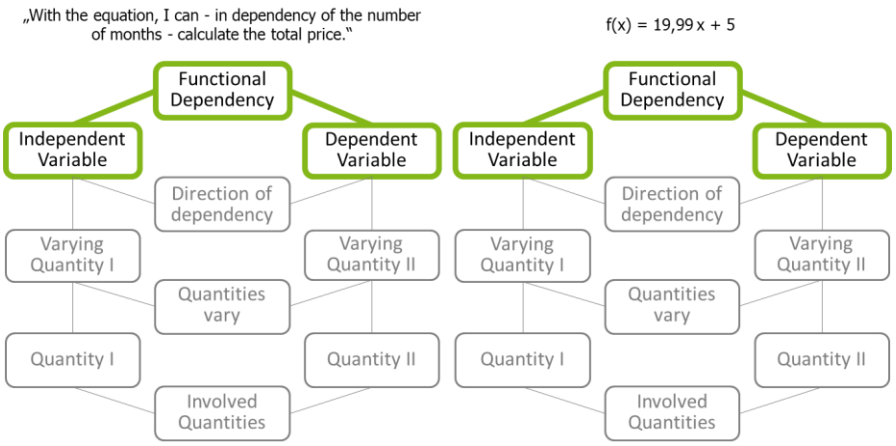

Figure 4 Alexandra (104)

Alexandra (Figure 4) correctly explains in 104 the matching of description B by identifying the same ||independent variable|| in the phrase and in the functional equation. Moreover, she addresses the ||functional dependency|| by identifying that one can calculate the total price (||dependent variable||) with the identified ||independent variable|| (number of months). Afterwards, Tatjana determines that the descriptions B and A mean the same and only vary linguistically (106).

Finally in this scene the tutor asks Alexandra's opinion to description D.

141 Alexandra Yes. I think so, too, that this is right, because one – x is indeed – are indeed the films and thereby one can just – no these are the months! So I don't think that this is right. I think this is wrong.

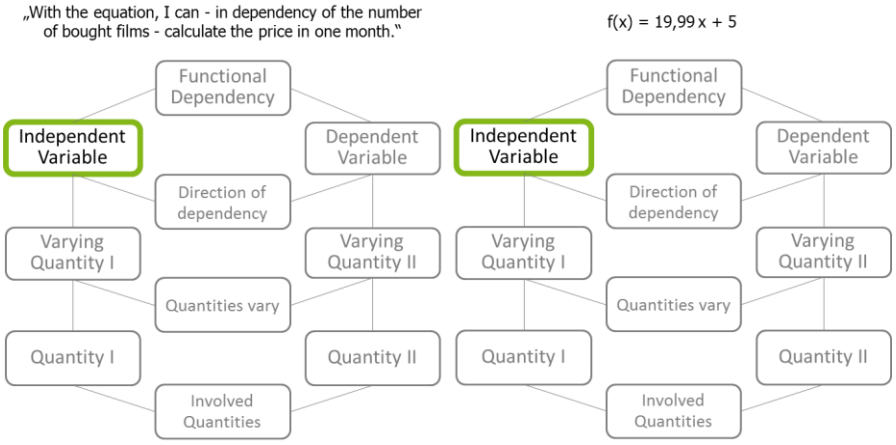

Figure 5 Alexandra (141) and Tatjana (146 and 148)

In 141 Alexandra (Figure 5) reasserts her approval to Tatjana's judgment in 100 that the phrase fits to the DreamStream offer. She starts to explain this decision by reasoning about the meaning of the ||independent variable|| in the functional equation and falters. Beginning with interpreting the phrase she identifies that the ||independent variable|| would be the number of bought films. But then she correctly states that this fact does not apply to the functional equation because the ||independent variable|| is the number of months. She concludes that the phrase does not fit after all, which suffices for a non-match.

145 Teacher *[laughing]* If you like to say something about this, here you are.

146 Tatjana Yes, now I think, this is wrong.

147 Teacher Why?

148 Tatjana Well, because it is put on the number of bought films there.

Then Tatjana revises her first judgement, too. In 148 she explains this fact by referring to the different ||independent variables||.

Summing up, contrasting the varied phrases initiated the addressing of different facets. The learning process is visible in the increased precision and explicitness in students' utterances. In the begin-

ning, Tatjana's utterance was not precise enough to match the phrase to the equation adequately because she could not identify the differences in the functional relationships described. This fact becomes visible in the model through the non-adequately addressed facets (Figure 3). One reason may be that she could not interpret the verbal representation. In contrast, Alexandra is able to address the facets of the core adequately and precisely. This fact becomes visible in the model due to the same adequately addressed facets in both representations (Figure 4). When Alexandra deals with the first phrase, she first approves Tatjana's assessment. However, in her explanation she struggles and realizes that the ||involved quantities|| are not the same in the verbal and symbolic representation. She adequately concludes that these representations do not belong together. At the end of the scene, Tatjana revises her first judgement (Figure 5). Thereby, she focuses more on the meaning of the given phrase than on the situation itself.

Of course, this excerpt is only an illustrating example of such a learning process. In other cases the developments look very different. One reason for this fact is that the sequence and number of considered phrases varied due to the fact it was not preset in the material, but adopted for each process by the teacher. Overall, the empirical analysis of students' learning processes indicates the analytic power of the facet model and that dealing with varied phrases can support the process of addressing facets as well as unfolding and compacting them.

Conclusion

This paper presented a conceptualization of function understanding focusing on the core of the function concept, which is based on cognitive psychological theories. It provides not only the identification of potential obstacles but also a normative framework for supporting function understanding. These facets of the core of the function concept are not specific to single representations or function types. Nevertheless, it is of course important to learn specific knowledge about representations and different types of functions. However, the core facets should be emphasized whenever students get to know new aspects of functions and should be addressed consistently and repeatedly in order to make students aware of the commonalities of every functional relationship.

Dealing with varied phrases stimulates addressing the core facets as well as unfolding and compacting them. This has been illustrated in the empirical insights. The facet model enables both, visualizing and describing processes of connecting representations by contrasting the facets that are addressed in each representation. An adequate connection of representations requires adequately addressing the same (core) facets in both representations. However, using the model not only enables describing these normatively prescribed core facets but it is also sensitive for individually activated facets.

Within these brief empirical insights, the model enabled the investigation of the connection of verbal and symbolic representations of functional relationships when dealing with word problems. Presumably this is compatible to other connections of representations. Furthermore, the conceptualization presented here focuses on the core facets of the function concept. To what extent this conceptualization can be combined with others in order to form a broader understanding of the function concept ought to be subject of further analysis.

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A classical dot pattern-based approach to algebraic expressions in a multicultural classroom

Marc Sauerwein[1] and Malte Mink[2]

[1]University of Bonn, Germany; sauerwein@math.uni-bonn.de

[2]Otto-Kühne-Schule, Bonn, Germany; maltemink@googlemail.com

Keywords: Algebraic expressions, secondary school mathematics, refugees.

Introduction

In February 2016, the Otto-Kühne Schule in Bonn, Germany, established an International Preparation Class (IVK, Internationale Vorbereitungsklasse) for 20 foreign pupils from different conflict and war zones all over the world. In this class, they learn the German language as foreign language adequate to their skill level (12 hours per week) and other subjects such as mathematics (5 hours per week). Their educational background is very heterogenous and therefore their mathematic class is usually split into at least three different groups covering topics from multiplication tables to quadratic functions. This project focuses on learning material developed for one group in their first lessons about algebraic expressions and their manipulation. The above-mentioned restrictions and conditions led to the necessity to develop a special approach[1] with few lingual prerequisites and the potential to support the development of mathematical language.

Theoretical background

According to an analysis of German textbooks by Prediger & Krägeloh (2015, p. 91), variables are usually introduced by lingual means. In particular, for the generalising aspect (Arcavi et al., 2016), it is referred to the everyday language. Taking important literature on structure sense resp. structuring (Hoch & Dreyfus, 2004, 2006; Rüede, 2012) into account, the learning material was constructed with a twofold goal: on the one hand the material should be easy accessible (in a linguistic way) in which the students can broaden their notion of variable while on the other hand the material may foster the activity of structuring on a beginner level. For this, dot patterns (or: figurative numbers) were chosen. By a figurative number, we mean a sequence of pictures consisting of dots (Figure 1) and the related number sequence.



Figure 1: The "filling glass" with the sequence 6, 10, 14, …

[1] For more details on the current status of the project, we refer to: http://www.math.uni-bonn.de/people/sauerwei/

Actual setting

The class started with a discussion of the dot pattern in Figure 2. The leading questions were: How many dots are in each picture? Can you continue the pattern? How many dots are in the 4th picture? How many dots are in the 100th picture? How many dots are in arbitrary picture?

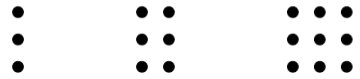

Figure 2: Adding three dots with the sequence 3, 6, 9, …

This very basic introductory example could catch the attention of every pupil. Even pupils with a usually low motivation for mathematics participated actively. All the questions were answered promptly and correctly and the only reasonable formula was found (3x). At this point, we did not introduce the formula (x+x+x) for adding the rows since we did not want to lead the pupils in any direction. From there on, the class worked individually or in smaller groups with the same leading questions on other dot patterns. It was stressed by the teacher that there can be many correct expressions for each pattern, but that each expression requires its own justification. Moreover, it was agreed on that two expressions are only equal if they yield the same result for every number plugged in. Hence, the only chance to verify equality was via the dot patterns and their structure. Thus, the dot patterns became a tool for argumentation.

This project is ongoing and more cycles of implementation in different regular and international classes are in preparation.

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An instructor-generated concept framework for elementary algebra in the tertiary context

Claire Wladis[1,2], Kathleen Offenholley[1,3], Jae Ki Lee[1,4], Dale Dawes[1,5], Susan Licwinko[1,6]

[1]Borough of Manhattan Community College at the City University of New York, Mathematics Department, New York, USA

[2]Graduate Center at the City University of New York, Urban Education Department, New York, USA

cwladis@bmcc.cuny.edu

[3]koffenholley@bmcc.cuny.edu

[4]jkl2105@tc.columbia.edu

[5]ddawes@bmcc.cuny.edu

[6]slicwinko@bmcc.cuny.edu

Keywords: Elementary algebra, conceptual understanding, algebra concepts, tertiary education.

Elementary algebra and other developmental courses have consistently been shown to be barriers to student degree progress and completion in the United States. Significant research has been done in the primary and secondary context, but little research has been conducted with students enrolled in elementary algebra courses in the tertiary context, despite the fact that there is significant evidence to suggest that mathematics learning is likely somewhat different in this context (Mesa, Wladis, & Watkins, 2014).

Teacher beliefs and expertise

There is significant research suggesting that teacher beliefs are often strongly related to the teaching practices that teachers implement in the classroom, and therefore are also related to student beliefs and learning experiences (see e.g. Fang, 1996; Maggioni & Parkinson, 2008). In addition, teacher expertise also has the potential to benefit the research community by contributing important information about what teachers have learned while teaching.

Theoretical framework

This study uses Vygotsky's (1986) theory of concept formation: learners begin to use algebraic symbols, graphs and other representations before they have "full" understanding of them, and through this experimentation and communication with "more knowledgeable" others, they internalize more formal and correct meanings for the objects that the representations symbolize.

Methodology and results

Five elementary algebra instructors collaborated on this action research project, some of whom are also educational researchers. This included faculty with doctorates in both mathematics and mathematics education, with varied backgrounds and different teaching styles. This study used the *Action Research Spiral* Framework (Kemmis & Wilkinson, 1998) to guide the process of collaborative exploration into student thinking about elementary algebra concepts. This framework outlines a cyclical practice in which practitioners go through the following steps repeatedly: 1) plan; 2) act and observe; 3) reflect; 4) revised plan, etc… In a cyclical process of experimentation, instructors developed assignments and assessment questions intended to assess student

understandings on the framework (see Table 1) that they had initially developed through discussion based on prior teaching experience. An example of one type of assessment question is below:

Assume that $a \neq 0$. Dale simplifies the expression a^3a^{-2} and gets the correct expression a. Which of the following must be true? There may be **more than one** correct answer—select **ALL** that are true.

a. $a^3a^{-2} = a$
b. If Dale lets $a = 10$ in both the expressions a^3a^{-2} and a, he will get two different answers.
c. Dale can substitute a for a^3a^{-2} anywhere it appears in an algebraic expression.
d. If Dale lets $a = 20$ in both expressions, he will get the same value for each expression.
e. Dale needs to know the value of a before he can say whether a^3a^{-2} and a are equal.

This question was designed to test the extent to which students understand 4.a. in the framework. Based on student responses, instructors probed students about their understanding of specific components of item 4.a. in order to better understand what those are and how they relate to one another. Based on this process, the framework was revised: The first draft contained only item 4.a.ii.1; after repeated cycles the other items under 4.a. were added and structured hierarchically.

1. **Algebraic Symbolism**
2. **Algebraic Structure**
3. **Properties/Generalizing Arithmetic Operations**
4. **Equality/Equivalence:** Understands equality/equivalence. The student understands: a. what it means for two expressions to be equal i. that two expressions are equal iff they are equal for all possible variable values ii. that if two expressions are equal, one may be substituted for the other in any context 1. that rewriting expressions is a process where an expression is replaced by an equivalent one b. what it means for two equations to be equivalent
5. **Equations as Relationships between Variables**
6. **Thinking Graphically**
Note: Because of space constraints, not all details of the framework could be reported here

Table 1. Elementary Algebra Concept Framework, with details for one sample sub-concept

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TWG04: Geometry

Introduction to the papers of TWG04: Geometry education

Keith Jones[1], Michela Maschietto[2], Joris Mithalal-Le Doze[3]

[1]University of Southampton, School of Education

[2]University of Modena e Reggio Emilia, Dipartimento di Educazione e Scienze Umane, Italy

[3]Université de Lyon, S2HEP, ESPE de Lyon

Keywords: Geometry education, instruments, proof and argumentation, spatial skills, visualization.

Introduction

Around 25 researchers of various geographical origins (from across Europe and also from North America, the Middle East, East Asia) participated in Working Group 4 on Geometry Education. Some 20 contributions (15 papers and 5 posters) informed five discussion sessions and two further sessions dedicated to debates and the preparation of a final report that was presented at the end of the conference. Each discussion session was structured around a selection of contributions, each of which was briefly introduced and followed a reaction from a pre-arranged reactor to inform the collective discussion.

The name of this group was previously *Geometrical Thinking*, and this was modified for this CERME to emphasize the focus on the teaching and learning of geometry. During the previous CERME, four competencies were used to describe geometrical thinking: reasoning, figural, operational and figural. The group took these dimensions as a background that was very helpful to understand each other and to compare our approaches to the issue of what is at stake in the teaching and learning of geometry.

This choice was all the more crucial given that many approaches and issues were discussed during the group sessions. Three main issues were addressed during the working group:

- The role of material activity in the construction of mathematical concepts, including using instruments, manipulation, investigation, modelling…
- Visualization and spatial skills;
- Language, proof and argumentation.

In comparison to the previous CERME, this time psychological points of view, among others, were represented. This raised new questions, often with very different theoretical and methodological backgrounds. As rich as the discussions were, mutual understanding was a great issue. Consequently, we did not focus, during the discussions, especially on one single topic at a given school level. In each of the three issues aforementioned, we tried to identify the interest of various theoretical or methodological approaches, of different cultural or institutional contexts, and the ruptures or continuity during the education process.

It is important to note that almost all the papers addressed ‘classical’ issues in this WG: this means teaching geometry to young children, the impact of specific contexts, geometrical activities in pre-service teacher training, moving from practical to theoretical geometry, using Digital Geometry Environments, and so on. Nevertheless, the main part of the discussions were about confronting,

sometimes in passionate ways, the theoretical and methodological approaches (for instance, didactical engineering was a 'classical' element for this WG) of the phenomena being studied. We try here to give an overview of these debates.

Topics of rich debate in the group

Role of manipulation and thoughtful experiment

This very broad topic has been a great field of study and experiment during the previous CERMEs. The discussed papers concerned students at all level, from kindergarten to university and included the use of instruments for investigation, manipulation and modeling. As these subtopics were strongly linked in the papers, we decided not to split the topic.

The use of two kinds of instruments was evident. One kind of instrument was in the form of ‘material manipulatives’ used as ways to enhance the didactical potentialities of the manipulation by pupils: such manipulatives include the protractor, paper-pin, mathematical machine, 2D and 3D shapes, miniature, compasses, and so on. The other kind of instrument the participants studied comprised various technological tools, including DGEs, videos, IWB, tutorial system, touch-screen tablet. Some papers described the use of only one kind of instruments, while other ones proposed educational environments in which the two kinds of tools were used by students and teachers within complementary and synergistic approaches.

Those papers had different approaches and theoretical backgrounds. For instance, there was discussions about papers that aimed at fostering the use of tools to mediate mathematical meanings (e.g. geometric reflection, Pythagorean theorem), with explicit reference to the theory of semiotic mediation. In papers that used DGEs or manipulation (of shapes, 3D models, geometric miniatures) to pass from the global spatial perception (iconic visualization) to an analytic visualization and to identify proprieties (non-iconic visualization), the main references were to the instrumental approach, the works of Duval and Van Hiele's levels.

Some of the papers examined how the use of tools give opportunities for new experiments that can be useful in teaching. These tools included images used as a way to stimulate dialogic talk amongst student, or technological tools used to change the way of teaching. In this last case, the double approach (didactic and ergonomic) was used.

Two papers focused on teacher education (pre-service and in-service) and reported on the use of DGE to improve generalization and geometrical construction (with their justifications within Euclidean geometry). Here it seems that DGEs are no longer ‘new’ and specific in the classes but nevertheless remain somewhat complicated within teachers’ education.

The group noted, as detailed later, that there is a true need for improving the ‘networking’ between the didactical approach and the psychological approach concerning the use of tools.

Visualization and spatial skills

Some 8 contributions mentioned visualization or spatial skills as a keyword. This topic has been raised over the three previous CERMEs and continues to be an important and autonomous subject in our discussions. We chose to use the word *skills* rather than abilities, capacity or capability, as it can be that these latter terms induce pejorative interpretations, seeing it as something innate that cannot

be changed or trained. The research questions were multiple and intertwined: What are the children spatial skills? How can we evaluate or train it? What is the role of spatial skills in the teaching and learning of mathematics? Visualization: what are we talking about? How to train visualization in geometry? What for? What are the links with language issues? We first had to clarify the relations between visualization and spatial skills: are these referring to the same thing?

In terms of spatial skills, these are related to a psychological point of view. They are linked to the perception, representation, (mental) manipulation of objects, orientation (following a path...), spatial knowledge, location in space. Spatial skills have many facets, and from a psychological point of view visualization is one of these (but it is not very precisely defined in the literature). Spatial skills are very important in mathematics education and has various meanings: sometimes it is not specific to geometry (STEM education), and sometimes it is linked to spatial problems and spatial knowledge (Berthelot & Salin). We pointed some mutual understanding issues between the two fields: for instance, micro/macro space (Berthelot & Salin) are similar to small/large scale (Montello).

What we called visualization is more specific to geometry, and involves combination of perception, interpretation and reasoning. It links perception to reasoning, and helps back and forth between practical and theoretical matters in geometry, so that it depends on spatial skills, mathematical constraints and language. Then, the precise meaning of visualization depends on the topics: visualization is not the same when drawing plane projections of 3D models or when trying to prove a result. From a didactical perspective it has a double nature – psychological and mathematical – and, in this case, spatial skills are a part of visualization. We will keep this acceptation in this text.

Spatial skills are very important for early geometry, as most of the tasks are related to the perception of the space: role in the learning of geometric characteristics of the shapes (Douaire & Emprin), need for the coordination of small/large scale, micro/macro space, local/holistic perception (Vendeira, Papadaki, Klaren)… A psychological point of view is crucial to make more clear general cognitive difficulties of the tasks, and a didactical one links it to the teaching of mathematics. Visualization is more a mathematics education issue, so it is related both to spatial skills and to mathematical knowledge. In a general way, the question is "*How to get enough information using a drawing to solve a given problem?*". It is declined, with very different aims, in every context: as an obstacle (prototypical shapes or too obvious results), using DGS, differences with Autistic Syndrom Disorder students, identification of geometrical properties or characteristics on a drawing… It is a great issue for early geometry, but it is often neglected when students get older, and we suggest this should be studied.

Language, proof and argumentation

The former topics are linked to proof and argumentation by langage. Argumentation and formal proof are linguistic activities about abstract objects, but they involve working on material objects (and then manipulation and visualization). Many works pointed this out. For instance Fujita's dialogic process involves both visualization and social interaction, Klaren's work on ASD students suggests that not seeing a square as a rectangle could be lined to the dutch word for rectangle, and we worked on Duval's dimensional deconstruction which is a discursive process and visualization at the same time.

Some five contributions addressed proof and argumentation, not necessarily about proof itself but about ways of motivating proofs or argumentation. One topic for discussion was the influence of prototypical images on the reasoning process. Another topic was how teachers can have different concept images of a geometric figure (such as a rhombus) and different conceptions of a valid geometric construction of the shape. A third topic for discussion was the design of tasks that can provoke surprise, uncertainty or cognitive conflict, and tasks that can provoke the reconsideration of conjectures or proofs. This last consideration was strongly linked to the visualization issues, as for instance using non-euclidean geometry was seen as a way to give less visual information and to provoke the use of mathematical proof.

A particular focus for discussion was the digital environment *QED Tutrix* which is being designed to provide hints to the student user, while taking into account a judgment of the student's cognitive state based on the way they are using the system.

Perspectives and conclusion

As might be discerned from the introduction, and as can be found in the papers that follow, there are a number of topics that continue to be of great interest to this topic working group. These include the role of instruments, manipulation, representations, proof and argumentation, and initial geometrical knowledge, in geometry education. We also note that the variety of the teaching and learning contexts increased: young children, secondary school, pre and in-service teachers training, but also university, specific education (ASD students), clinical studies… The synthesis of this numerous points of view required intense and rich debates. At this CERME, in the continuity to the former ones, a number of topics became more important. These include visualization and spatial skills which had already been discussed in the last CERMEs, and language in doing geometry, whose role has increased during this session.

In conclusion, the working group continues to feature great diversity: in cultural backgrounds (curricula, school culture, teaching culture, research culture …), research questions, theoretical backgrounds and methodology. This continues to present some challenges in people understanding each other, sometimes linked to language and sometimes to what can be implicit meanings due to different research backgrounds.

A very visible benefit of the great diversity is that it invariably leads to very fruitful discussions and to attempts (and success) to clarify participants' points of view. In taking forward the work of the group, there is an increasingly important need for combining the frameworks, both theoretical and methodological.

Learning to use a protractor: Touch screen tablet or not?

Cécile Allard[1] and Edith Petitfour[2]

[1]University Paris Est Créteil, Créteil, LDAR (EA4434) Créteil, France; cecile.allard@u-pec.fr

[2]University of Rouen, Rouen, LDAR (EA4434) France; edith.petitfour@univ-rouen.fr

In the present study, we compare the practice of one teacher in two 5th-grade classes for the same teaching concept (about angles) according to different working arrangements. In one of her classes, the teacher combines lecturing and interactive class: she talks and she exposes knowledge in front of her students. In the second class, she decides to try a new working arrangement: the flipped classroom. We compare the knowledge at stake by studying actions, gestures and language when video lectures are used on the one hand and when such devices are not used on the other hand. Thus, we compare the knowledge, which was shared and discussed in the classroom.

Keywords: Practice, protractor, language, angle, touch screen tablet.

Introduction

Our goal is to study the impact of the working arrangements' modifications (use of the touch screen tablet or not) about knowledge exposure (Allard, 2016). More specifically, we study the geometric knowledge exposure when the students learn to use an instrument (the protractor). We think that the use of artifacts have a contribution at the cognitive level (Mariotti & Bartolini Bussi, 2008).Thus learning to use a protractor contributes to the construction of the angle's concept.

This paper focuses on the teaching and learning of the angle's concept. It involves one teacher, Marie, and her grade 5 students in their use of the protractor. In France, at primary school level, students learn how to compare angles and how to reproduce a given angle using templates or tracing paper. In 5th grade they are then trained to use a protractor to measure angles and to construct an angle with a given measurement.

For several years, ministry of education recommends the use of new technologies and more recently emphasizes the need to individualize teaching: the flipped classroom may provide the means to meet these demands. This model of classroom instruction can be defined as an educational technique that consists of two parts: interactive group learning activities inside the classroom, and direct computer-based individual instruction outside the classroom (Bishop & Verleger, 2013).

Theoretical framework

This qualitative study is built on prior work on the Double Approach by Robert & Rogalski (2005), current work on the concept of angle and on the study of actions with instruments Petitfour (2015).

About practice

To analyse and interpret teacher's practices, we use the theory of the Double Approach defining five components: personal (teacher's choices, beliefs), social (teachers' working place, their colleagues and the social environment of their students, in both disadvantaged and advantaged areas), institutional (curriculum, relationship with supervisors), cognitive (choices about mathematical contents, tasks, organization and forecasts on how to manage the session) and mediative (improvisations, speeches, motivation of the students' participation, devolution of instruction and

knowledge exposures). The study of the students' activities allows to give information on cognitive and mediative components (what students did, what they said about their activities and what they learnt). Consequently, we have to clarify the elements of knowledge at stake, misconceptions and difficulties in the learning of the concept of angle.

About angle

Angle appears to be a very complex and multifaceted concept, which can be defined in three different aspects: turn (an amount of turning between two lines meeting at a point), ray (a union of two rays with a common end point) and region (the intersection of two half-planes) (Mitchelmore & White, 1998).

Students encounter numerous difficulties in the learning of angles and misconceptions have been pointed out in several experimental studies (Berthelot & Salin, 1995; Mitchelmore & White, 1998; Devichi & Munier, 2013). Many students for instance consider that an angle's size depends on the length of its arms; or that one arm must be horizontal and the direction always counterclockwise; or that the angle is a sector in a circle (i.e. a "slice of pizza"). Tanguay and Venant (2016) hypothesize that this misconception is a possible effect of the systematic use of the protractor, when measures in degrees are at stake. Moreover, the confusion between the mathematical concept and the shape that represents it generate mistakes when students consider the spatial characteristics of the design representing the angle (Balacheff, 1988).

Students face some difficulties in using a protractor, which have been stressed in the literature. For example, Close (1982) observed mistakes due to a lack of mental representation of the protractor's angle to superimpose to the angle to be measured or due to the complexity of the dual scale. Tanguay (2012) summarizes some well-known difficulties encountered by students: they align the edge of the protractor body itself along one of the angle's arm instead of the baseline; or they don't place the protractor origin over the vertex of the angle to be measured. They can also read the measure on the wrong scale or on the right scale but in the wrong direction, for example by reading on counterclockwise graduation 39° left to 40° instead of 41°.

About actions with instruments

We consider four categories of knowledge according to a theoretical framework to study actions with instruments (Petitfour, 2015)[1]:

- *geometric knowledge* about geometric objects, relations and properties
- *graphic knowledge* about representation of geometric objects (symbolic signs, drawings)
- *technical knowledge* about the instrument functionality, its use to obtain the graph and the theoretical relationship between the graphical trace and parts of the instrument
- *practical knowledge* about a given artifact, connected to its concrete handling and to a concrete organization of the action with it

Methodology

We study the practice of one teacher, Marie, who wanted to experiment with a new style of instruction: the flipped classroom. Thus, her students should learn some mathematical content online,

[1] This framework is based on an instrumental approach (Rabardel, 1995).

by watching video lectures at home before the class session, whereas in the classroom, they should solve more exercises. Consequently, the teacher would be freer in the class session for discussing with the students: that should lead her to a more personalized guidance and to greater interactions with students than traditional teaching.

In our study, three components are common to the two situations that we analyze here: institutional, personal and social. Marie's components can be defined as follows: she is appreciated by her supervisors, colleagues and students. She thinks that in order to learn (and to teach) well it is necessary to handle and to solve a lot of exercises. She finds that she has never enough time. She is concerned about the learning of her students. She teaches in a rural school without social difficulties. In her two 5th-grade classes, she volunteers to include gifted students. These three components are stable and the same in the two situations. Thus, we can compare the practice of the same teacher in two 5th-grade classes in the same area. Consequently, we can focus only on the mediative and cognitive components. In order to foster the comparison, we asked Marie to teach the same content and to propose the same tasks in her two different classes, one with video lectures and the other one without.

Data

In order to describe practices and inform mediative and cognitive components involved in the Double Approach, we focused on:

- one video for the 'flipped classroom' and two for the ordinary class that we have transcribed
- two short video lectures which we have transcribed
- the notebook lessons, exercises book and topics of assessment
- the teacher's interviews conducted before and after the sessions

We have split each session in episodes. We have identified different types of episodes in relation with the teacher's specific goals and exposure of knowledge (Allard, 2016). For example, in the lecturing session, the teacher remembers specific words (how to call angles according to their openings measured in degrees) at the beginning of the class session. The teacher also recalls to the students the specific symbol to note an angle ($\widehat{ABC}$). These moments are reminder episodes (type 1): the teacher recalls of previous knowledge. Regulation episodes (type 4) are moments when the teacher intervenes to explain and to anticipate difficulties. The main goal of the regulation episodes is to provide students' progress. So we have identified six types of episodes.

For lecturing sessions

The two lecturing sessions last 45 minutes. The aim of the first session is to measure an angle with the protractor. The aim of the second session is to draw an angle of a given measurement. They follow the same organization in six types of episodes (table 1).

1	Reminder episodes: - categorizing angles according to their opening measured in degrees (1st and 2nd sessions) - noting and naming the angles (1st and 2nd sessions) - defining the center of the protractor, based on a description of the artifact (1st and 2nd sessions) - measuring angle with a protractor (2nd session) - estimating and controlling the measure (2nd session)
2	Methodological episode: estimating and controlling the measure
3	Presentation of the new knowledge: presenting and discussing the methodological sheet about measuring (first session) and drawing angles (second session) with a protractor.
4	Regulation episode: - anticipating difficulties (in relation with the dual scale or extend ray) - reviewing any personal concerns or difficulties that are raised during the session
5	Exercises episode: providing activities and handing out methodological and exercises sheets
6	The correction of the exercises: exposing knowledge and the difficulties encountered

Table 1: the six types of episodes for ordinary session class

The teacher talks to the entire class during certain types of episodes: 1, 2, 3, 4 and 6. Students are then facing the blackboard. It is only during the episode of type 5, that students work individually to solve exercises and the teacher interacts with them about their errors or difficulties.

For the classroom with tablet

Some students did not have access to the learning platform at home (because of one technical problem), as a result they did not watch the video session before the session in the classroom. Therefore, the teacher improvised and gave time during classroom session to watch and to listen to the video-class on the touch screen tablet. So, it's an opportunity for us to observe students discovering by themselves a mathematic lesson. This session lasts 45 minutes. We have then identified three types of episodes (table 2).

3	Presentation of the new knowledge by the video sessions about measuring and drawing angles with a protractor
4	Regulation episode: reviewing any personal concerns or difficulties that are raised during the session
5	Exercises episode: providing activities and handing out methodological and exercises sheet

Table 2: the three types of episodes for video session

Students worked in groups (by pairs) and watched the video lessons (episode of type 3). After that, they completed some exercises (the same ones that in the ordinary session class). The teacher walked around and talked with students to regulate their work one by one. For example, once more she explained how to handle the protractor.

Data analysis

Comparison of the episodes

Ordinary class requires two sessions instead of only one for the class with the touch screen tablet in order to learn the same content. For the teacher, it seems to be an efficient method. But our study shows that less knowledge is going around in the class in the case of touch screen classroom (episodes of type 1, 2, 6 are absent), there is no link between previous knowledge and the new one, no collective reminder by the teacher and a lack of formulation of the knowledge by the students.

In the ordinary classroom, students listen to each other and listen to the teacher, they raise their hand in order to come to the blackboard, they do individual exercises and sometimes ask questions or answer question the teacher's ones. They are facing the blackboard, which promotes discussions between themselves and the teacher. In the touch screen tablet session, students watch the first video session several times and they solve the exercises individually. While they work on the exercises, many students listen to the touch screen tablet: they look at the gestures in the screen and they try to do the same, they stop, observe and copy. After that, they watch the second video and do exercises. Meanwhile, the teacher walks around the pair groups, corrects mistakes, rectifies the bad handling of the protractor.

In the ordinary classroom, the teacher leads her students to use an appropriate language during the presentation of the new knowledge episode: for example, "center" for the center of the protractor instead of "hole". We can see that the students appropriated this formulation in the ordinary classroom whereas they did not in the other classroom even if they had listened to the session video several times. We can see in this comparison to what extent the working methods seems to change what is said and what is shared.

Now we compare what changes at the level of the potential learning about angle and the use of the protractor.

Knowledge at stake

Actions with a protractor in order to measure an angle or to draw an angle with a given measurement involve different pieces of knowledge. We study the knowledge at stake in the types of episodes where this knowledge is exposed to all the students, that is to say where the whole class has the opportunity to hear or to see the same thing. It concerns episodes of types 1, 2, 3, 4, 6 in the sessions without a tablet and type 3 in the session with a tablet.

Some geometric knowledge appears explicitly in the session without a tablet whereas it does not in the session with a tablet. On the one hand, the classification of angles based on the degree measurement is recalled: angles are categorized according to their openings measured in degrees: an angle can be right, acute, obtuse, straight, zero or reflex. On the other hand, "angle" is defined as formed by two rays (sides) that have the same endpoint (vertex), and "ray" is defined as a line extending indefinitely in one direction from a point. In the same way, graphic knowledge appears explicitly only in the session without a tablet: first, the symbol Λ enables to distinguish an angle and a triangle; second, an angle can be named with three letters, the second letter names the vertex, the first names a point on one side and the third a point on the other side. The fact that the line representing the ray can be extended in one side is mentioned in both sessions but only explained referring to some geometric knowledge in the session without a tablet.

As far as the representations of an angle are concerned, drawings are more numerous in the session without a tablet. Indeed, in this latter session, five angles are drawn on the whiteboard, one is freehand traced and the others are traced with one of the edges of a set square. Besides, one is obtuse and four are acute with similar opening around 45° but which one with the vertex at right (it allows to use both protractor's scales) (Figure 1). At last, the five drawings have a horizontal side and are named with three points.

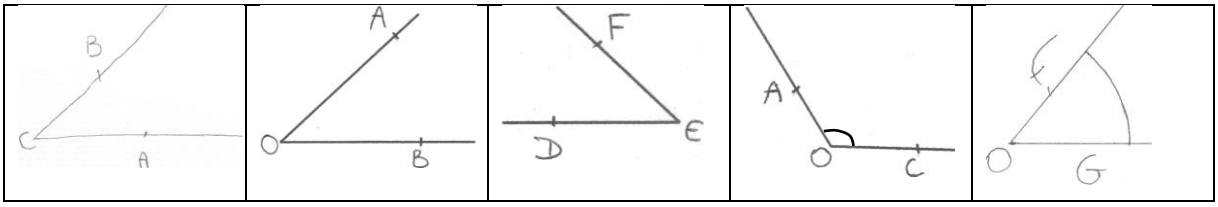

Figure 1: Drawings of angles in the session without tablet

The drawing of the angle $\widehat{FOG}$, 52° had been presented by two students on the blackboard and each time, they have drawn an arc of a circle along the semi-circular edge of the protractor despite the fact that the teacher mentioned the uselessness of this curved line. This representation has led some other students to speak explicitly about "cake slice" which revels the misconception of the angle as a sector in a circle.

Only two acute angles are drawn on a paper in the session with tablet. They are oriented in a non-prototypical way and are named with the letter of the vertex (upper case) and the letters of the direction of each ray (lower case) (Figure 2). It is not this latter notation that is used in the application exercise but the notation with three points.

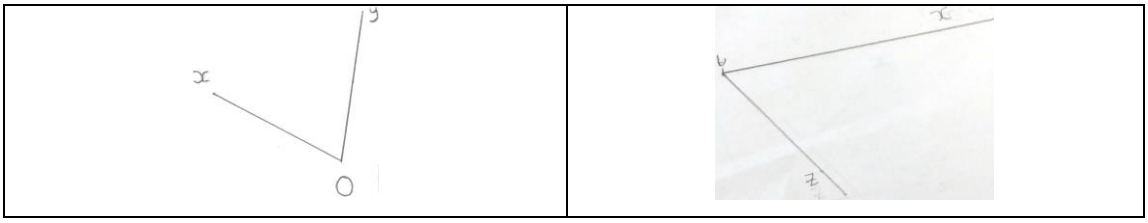

Figure 2: Drawings of angles in the session with tablet

In both sessions, drawn angles are named both spoken and written. In the session without a tablet, the teacher sometimes used gestures too (Figure 3) expressing either the "ray" aspect or the "region" aspect of an angle.

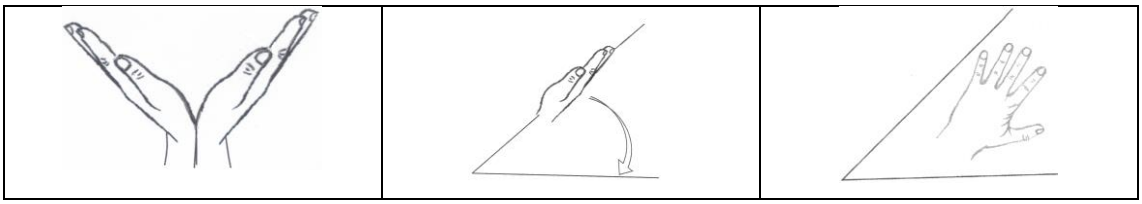

Figure 3: Gestures about angles (Left, "ray" aspect and middle and right, "region" aspect)

Moreover, in the case of the obtuse angle $\widehat{AOC}$ (Figure 1), she used a symbolic sign – a small arc of a circle – in order to indicate what angle is to be considered. In the session with a tablet, there are only deictic gestures pointing the letters "x", "O" and "y" when they are uttered in order to name the measured angle.

Some technical knowledge is recalled only in the session without a tablet. This knowledge concerns the protractor's functionality: a protractor is a measuring instrument, the measurement is expressed in degrees; and a description of the protractor parts to link with the graphical representation of an angle. Indeed, the localisation of the protractor's centre is first given in a general way, by the use of

language accompanied by gestures, then the graduation 0° and the dual scale (inner and outer) from 0° to 180° are named and pointed on the protractor by the teacher.

The steps to measure an angle and the steps to draw an angle using a protractor are formulated in the same way in the two sessions. What is said is exactly what is written on the summary sheet given to the students. For example, here are stated the three steps to measure an angle: first, place the protractor so that the centre is over the vertex of the angle to be measured; second, place the graduation zero degree over one side of the angle; third, follow the graduations of zero degree, ten degrees, twenty degrees, … until you reach the other side of the angle. It is also stated that perceptive evaluation of the opening of the angle – greater or smaller than the one of the right angle – enables to control possible measurement errors.

In the session with a tablet, the students can hear and see the operating filmed sequence as many times as they wish, pause and go back whereas in the session without a tablet, the operating sequence is presented several times by different students on the blackboard (for example, three times to measure an angle before doing the application exercise). The teacher helps the students to formulate the method they implemented.

Practical knowledge appears only in the session without a tablet. For example, some protractors have a hole to show the centre near the bottom of the protractor whereas the blackboard's protractor has his centre on the bottom; the semi-circular edge of the protractor can be damaged so that it is better to avoid tracing along this edge; if the protractor goes beyond the lines represented sides of an angle to be measured, then the lines must be extended. Regarding organizational aspects, the teacher gave students the advice to store the protractor in a pocket in their workbook to avoid breaking it.

Results and conclusion

Our analysis of one teacher's practice about the use of the protractor allowed us to point out the following results. Some of the students 'difficulties and mistakes are the same in both sessions, with and without tablet, when the students trace or measurement by themselves during the exercises episodes: wrong localization of the measure on the protractor, measurement without extending the line representing ray when it is necessary, clumsy handling of the protractor. Errors to note and to name points and rays appeared only in the session with tablet (students didn't manage to adapt what was presented in the video lecture). Moreover, there are inaccurate wordings that show confusion between length and angle. The correction of the arising errors is private in the both sessions but also public and shared in the session without a tablet. Finally, whatever the case, there is no difference between the assessment results of the two classes, according to the teacher and the collected data cannot inform us about the arrangement that would better foster learning.

This study confirms that the modifications tied to the mediative component, in particular in terms of working arrangements, have a very important impact on the knowledge exposure (Allard, 2015). Indeed, in the session with tablet, the only exposed knowledge is the knowledge of the tablet, without possible links with difficulties met by the students. In the lecturing session, there is more knowledge exposure thanks to the interactions between the students and their teacher.

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Constructing draggable figures using GeoGebra: The contribution of the DGE for geometric structuring

Lina Brunheira and João Pedro da Ponte

ESE - Escola Superior de Educação, Instituto Politécnico de Lisboa and UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; lbrunheira@eselx.ipl.pt

Instituto de Educação, Universidade de Lisboa, Portugal; jpponte@ie.ulisboa.pt

This study is a research on teaching practice, developed in the context of an elective course on Dynamic Geometry for prospective kindergarten and elementary school teachers taught by the first author of this paper. We aim to analyse the role of GeoGeobra in the development of geometric reasoning, particularly the way individuals geometrically structure figures. The participants are a class of six future teachers. Data was gathered from the participants' portfolios and classroom observation while working on an exploratory task, which focuses on constructing draggable figures. The results show that this type of activity promotes spatial and geometric structuring, beginning with the perception of elements and relationships that enable the dynamic construction, and moving on to the description of the construction using formal concepts associated to the tools of the DGE.

Keywords: GeoGeobra, geometry, structuring, visualization, teacher education.

Introduction

Research has shown the interest in engaging students of different ages in activities using dynamic geometry environments (DGE), for the improvement of concept learning and the development of reasoning (Sinclair & Yerushalmy, 2016). In particular, Hanna and Sidoli (2007) indicate that the DGE are a promising way "in enhancing the students' ability to notice details, conjecture, reflect on and interpret relationships and to offer tentative explanations and proofs" (p. 77). Also, in education programs, relevant research confirms the effectiveness of the use of technological tools, including DGE, to improve the knowledge of teachers and future teachers (Jones & Tzekaki, 2016). In this study, we aim to analyse the role of GeoGeobra in the development of geometric reasoning, in a context of a geometry course for future kindergarten and elementary school teachers, based on exploratory tasks. In particular, we which to understand how the construction of draggable figures in the DGE contributes to geometric structuring (Battista, 2008)?

Theoretical framework

Prospective elementary teacher education in geometry

The knowledge necessary for teaching includes mastery of mathematical reasoning, ways to solve problems and communicate mathematics effectively, understanding of concepts, procedures and the process of doing mathematics (Albuquerque et al., 2005; NCTM, 1991). Concerning geometry, kindergarten and elementary school teachers should understand how it is used to describe the world; analyse two and three dimensional figures; use synthetic geometry, coordinates and transformations; improve skills in producing arguments, justifications and in visualization.

Some researchers have claimed that there are few studies about teachers and future teachers' knowledge in geometry (Chapman, 2013; Clements & Sarama, 2011; Steele, 2013). However, the

existing literature provides reasons to believe this is a problematic area. As Clements and Sarama (2011) state, in many countries teachers from every level are not always provided with adequate preparation in geometry and lack of knowledge and confidence in this area. Concerning kindergarten and elementary prospective teachers, many only recognise and categorize shapes by their overall similarity to prototypes, instead of charactering them by their properties (Clements & Sarama, 2011; Fujita & Jones, 2006) a problem we also identify in Portugal (Menezes, Serrazina & Fonseca, 2014). Overall, as Jones & Tzekaki (2016) recently stressed, studies on teachers' geometric knowledge and teacher education programs indicate that we still need to give attention on how prospective teachers build their understanding of geometrical objects. Also, we should take into account the effectiveness of approaches such as the use DGE.

Developing geometric reasoning and the use of DGE

King e Shattschneider (2003) present eight reasons for a teacher to use a DGE: (i) to take advantage of the accuracy of geometric constructions and measurements, leading to confident results; (ii) to promote visualization; (iii) to encourage exploration, investigation and discovery leading to the formulation of questions, conjectures, and their test; (iv) to encourage demonstration because the experimental evidence offers the necessary conviction for such enterprise, and may provide clues; (v) to support the understanding of geometric transformations; (vi) to support the understanding of loci; (vii) to provide simulation opportunities for a wide variety of situations; and (viii) to allow the creation of microworlds, using new tools and allowing exploitation of non-Euclidean geometry.

A major emphasis on using DGE concerns the constructing of figures. Laborde (2001) compares this type of activity when performed using a DGE versus using paper and pencil. In her view, when we draw a figure using paper and pencil, the activity is often controlled by perception rather than being driven by the properties of the figure. Instead, in a DGE is not possible to construct a square in a similar way ("led by eye") and it requires more knowledge about the figure. But if the students are able to apply the properties correctly, we can ask ourselves, as does Battista (2007), what have they learned from the activity? For this researcher, "perhaps no *new* knowledge was acquired, but instead, the students' knowledge and reasoning were deepened and enriched . . . Or perhaps connections between properties were newly constructed or extended" (p. 878).

In order to analyse this reasoning we draw on the framework developed by Battista (2008). This researcher established a categorization of reasoning using three levels, corresponding to increasing degrees of sophistication: spatial structuring, geometric structuring and logical/axiomatic structuring. Spatial structuring is a special type of abstraction corresponding to the mental act of constructing an organization or form for an object or set of objects by identifying its components, combining them into spatial composites, and identifying the way they combine and relate. Spatial structuring enables a person to imagine manipulating an object, reflect, analyse and understand it. Geometric structuring describes spatial structuring using formal concepts such as congruence, parallelism, angle, transformations or coordinate systems. Geometric structuring is based on spatial structuring, that is, to be able to structure geometrically an object, it is necessary that one has interiorized the corresponding spatial structure. Logical/axiomatic structuring formally organizes geometric concepts in a system so that their relationship can be established through logical deduction. To operate at this level, it is necessary that verbal or symbolic statements can replace mental models. The research of Battista (2008) in a DGE (the *Shape Makers* microworld) with fifth

graders showed that the manipulation of shapes and the reflection on that manipulation may enable the pupils to move from thinking holistically to thinking about the geometric properties of the figure, that is, to progress from spatial structuring to geometric structuring. However, he also points that there is a need for guidance, reflection and experimentation in order to construct formal geometric conceptualizations of the DGE constraints.

Methodology

The first author of this paper designed and taught an elective course on Dynamic Geometry in 2015/26, as a new offer in the teacher education program for prospective elementary school teachers in her institution, in Portugal. The course was divided into two phases: (i) 10 lessons dedicated to solving geometry tasks organized into four topics – problem solving, constructing, investigating and creating; and (ii) 5 lessons dedicated to didactics of geometry, projecting the work of the DGE with children from kindergarten to 6th grade. In the classroom, there was one computer for each participant, but they were encouraged to discuss with their colleagues. Regarding the assessment, each participant built a portfolio containing a task from each topic, detailed solution and a reflection on the activity, and also constructed a GeoGeobraBook with the files used to solve the tasks.

This study is a research on teaching practice based on the observation of the activity of the participants and their solutions of a task and aims contributing to their professional and organizational development, "as well as to generate important knowledge about educational processes, useful for other teachers, for academic educators and the community in general" (Ponte, 2002, p. 13). Data was gathered mainly from the portfolios and GeoGeobraBooks of the participants, complemented by the field notes taken by the first author, while observing the participants and supporting their work. There were only six participants: five females who were in the 2nd year of the program (also attending a compulsory course of Geometry) and a male in the 3rd year, the only one who had some experience with DGE. Since the Dynamic Geometry course is elective, the choice the participants may be considered an indicator that they like geometry and do not feel strong difficulties in this area, which was confirmed in this group. The task (Figure 1) was proposed in the 5th lesson, within the topic *Constructing*. It was intended that the participants would reproduce draggable figures, or families of figures, in GeoGeobra from the properties visually identified, thus corresponding to one of the major emphases reported by Battista (2007).

1. Construct both stars. Describe briefly the process.
2. For each of the stars, find another building process and describe it.
3. Construct other stars of this family with a larger number of points. Generalize one of the construction processes you used.
4. Establish relationship between the number of star points and other elements.

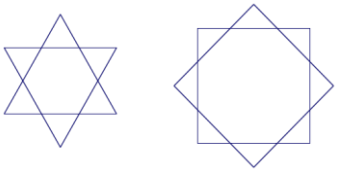


Figure 1 – Task *Constructing stars* (adapted from Johnston-Wilder and Mason, 2005)

The data was analysed using a framework (Table 1) built by the first author of the paper (Brunheira, 2016), based on the concepts of spatial and geometric structuring (Battista, 2008). The table does not include the logical/axiomatic level, since it means that one operates at a symbolic level, which is not the purpose in this task. We use the framework to analyse the solutions, looking for the evidence

of the descriptors in order to characterize the level of structuring of the participants. However, we stress that despite the attribution of a level to a solution, this does not mean that we can characterize the level of structuring for an individual solely based on a solution of a task, so this must be seen as an indicator. Also, we cannot consider that solving a single task is enough to improve significantly, but this analysis may enable us to recognise it's potential.

<table>
<tr><td rowspan="2">Levels</td><td colspan="2">Geometric structuring</td></tr>
<tr><td>Spatial structuring</td><td>Knowledge of concepts</td></tr>
<tr><td>N0</td><td>Does not establish geometrical relationships between figures and their elements, or does not provide most of the times.</td><td>Does not know most of the basic concepts and the language is very limited in terms of geometric vocabulary.</td></tr>
<tr><td>N1</td><td>Perceives geometric relationships involving visible elements of figures, but it may depend on the position of the figures, their elements or the context.</td><td>Knows the concepts of side and angle, congruence, perpendicularity and parallelism in the plan; in space, knows the concept vertex, edge, and face.</td></tr>
<tr><td>N2</td><td>Perceives geometric relationships involving visible elements of figures in any positions or context.
Perceives geometric relationships involving invisible elements of figures, but it may depend on the position of the figures, their elements or the context.</td><td rowspan="2">Knows the concepts as axe of symmetry, diagonal, bisector, midpoint and the geometric transformations in the plan; in space, knows the concept of congruence, parallelism and perpendicularity.</td></tr>
<tr><td>N3</td><td>Perceives geometric relationships involving visible or invisible elements of figures in any positions or context.
Produces generalizations of geometric relations for a family of figures.</td></tr>
</table>

Table 1 – Descriptors of the levels of spatial and geometric structuring

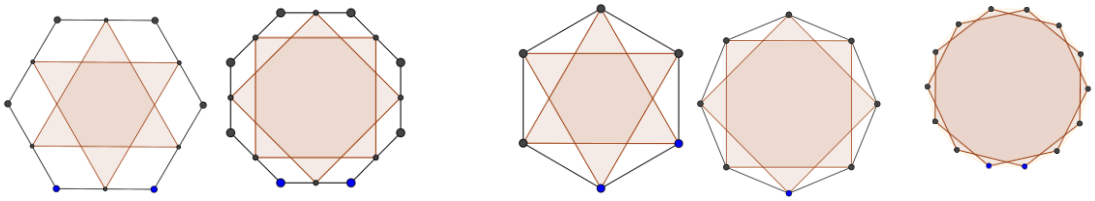
Results

Next we present an analysis of task solutions from prospective teachers Maria, Carla and Louise taken from their portfolios, which we consider to be representative of all the solutions presented.

Maria's solution of the task

In figure 2 we present an excerpt containing two processes presented by Maria. Process A was used to build the two initial stars and process B was used for the same purpose, as well as to generalize. Both constructions begin with the image of the star as a whole figure and a regular hexagon where the star is inscribed in two different ways.

Process A Process B



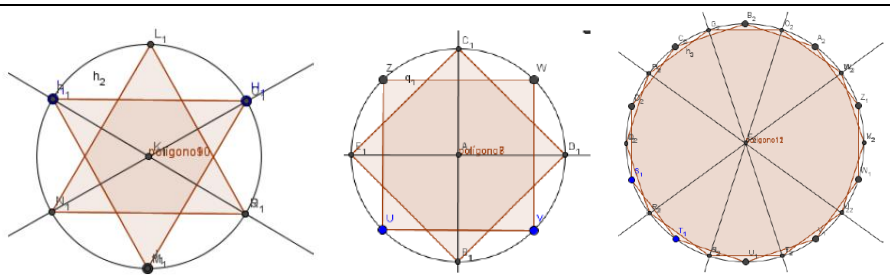
Construct a polygon with a certain number of sides and then two polygons from the union of non-consecutive vertices. The number of the star points corresponds to the number of the vertices of the polygon used for its construction. It is not possible to do this based on regular polygons with an odd number of sides, since there are not two sets of non-consecutive points to be connected.

Figure 2 – Excerpt of Maria's solution of the task

Maria looks at the star as a whole figure inscribed in a regular hexagon in two different ways. She draws on invisible elements that were created to assist the construction. Regarding the generalization, Maria presents a process which can be applied to any star and establishes a relationship between the initial polygon and the number of points of the star. Finally, she identifies that this polygon cannot have an odd number of sides and justifies her finding. Thus, Maria's solution shows a very good geometric structuring for this family of figures, corresponding to Level 3 of the framework.

Carla's solution of the task

Carla uses a procedure similar to Maria' process B and another process, shown in Figure 3.



1. Construct an initial figure in accordance with the number of points of the star (this polygon should be a regular polygon in which the number of vertices is half the number of points of the star). 2. Trace the perpendicular bisectors for each side of the polygon to find its center. 3. Draw a circle centered at the intersection point of the bisectors and a radius to reach a vertice of the figure. 4. The intersection points between the bisectors and the circumference will be the vertices of the second figure that makes up the star.

The number of points of the star is twice the number of sides of the inicial figure.

Figure 3 – Excerpt of Carla's solution of the task

She looks at the star decomposing it into two congruent regular polygons, one of which constitutes the starting point for construction. The determination of the second polygon involves visualizing the star inscribed in a circumference, and the vertices of the second polygon on the perpendicular bisectors (a concept that she did not know). Thus, she identifies that the consecutive vertices of the

star are equidistant from each other and also equidistant from the centre of the star. Regarding the relations established, Carla identifies that the number of vertices of the initial polygon is twice the number of points of the star, but does not justify this. Therefore, Carla identifies various relationships between their elements, using visible and invisible elements and adequate concepts, such as the circle and the perpendicular bisector, thus showing a very good geometric structuring of the family of figures, which also corresponds to level 3.

Louise's solution of the task

Louise builds the stars initially as Carla (draws the first polygon, traces the bisectors and finds the point of intersection). However, while Carla seems to look at the star in a static point of view, Louise visualizes the "movement" of the first polygon to obtain the second. The participant had an intuitive idea that rotating the initial triangle in a certain way, it would be possible to obtain the second triangle and form the star, although she did not know the formal concept of rotation and that we should define the rotation by a centre and an angle. She asked for help to find out if the GeoGeobra could run this "movement" and the teacher explained how the "Rotation" tool worked. Next, Louise presented the following relationships: "For regular polygons with even number of sides, amplitude = 180°/(number of sides); For regular polygons with odd number of sides, amplitude = 180º". Thus, we consider that her solution also reflects level 3.

Discussion

All participants were successful in the task. They presented different and valid constructions mobilizing a variety of elements of the figures (visible and non-visible), relations between them, transformations and properties, some of them were unknown to them. So, the main conclusion we want to emphasize is that the construction of figures using GeoGeobra significantly enhances the geometric structuring by promoting the identification of properties and relations between elements, as Battista (2008) reported in his study. This improvement stems from different features and strengths that we recognize in the DGE, some of them indicated by King e Shattschneider (2003). We start with two features – easiness of use and accuracy of the constructions – which we associate the two strengths – promoting intuition and exploration. In fact, sometimes participants started the construction from an insight of the properties and elements of the figure (or auxiliary figures) that could be useful, but they were not sure. The possibility to easily test the conjectures through a quick and accurate construction was a key aspect, as Maria explains:

> With GeoGeobra it was possible to explore different forms of construction of the stars using polygons, lines, midpoints, parallel lines, among others, easily, simply and accurately. If we didn't have this software this would be a long and relatively difficult process, especially the construction of regular polygons used as a basis for the construction of stars. (Portfolio)

Another potential of GeoGeobra that emerged was the promotion of justification, which we did not ask for in the task. In fact, the ability to test the construction validity, as in a trial and error process, does not mean that participants do not reflect on their actions, as we note in Louise's comment:

> I had to stop and think why the rotation angle depends on the number of sides, as well as to find a mathematical answer to for the correct value. (Portfolio)

In this case, we see a need to reflect on the value of the angle, which led to the justification of the chosen value and the understanding the generalization. So, although the DGE played an important role in the user's belief that a relationship is valid, did not lead to underestimate justification, instead promoted the search for it (Hanna & Sidoli, 2007).

Another feature of GeoGeobra is that it leads the user to work with the formal concepts associated with its tools. In this way, we may think that we can only take advantage of the DGE when operating at the level of geometric structuring. In fact, as Battista (2007) suggests, we cannot make geometric constructions without reaching some level of "conceptual and representational explicitness". However, this investigation shows that GeoGeobra can facilitate the transition from spatial to geometric structuring. An example that supports this conclusion is the use of new concepts, like perpendicular bisector or rotation, that participants had just a vague memory from middle and high school, but were correctly applied as the DGE promoted their appropriation.

Finally, in connection to the nature of the task which favours different solutions, GeoGeobra supports this diversity through a set of tools available, which also stimulates creativity. As Peter says:

> The choice of this task reflects on the freedom it gives us to construct the figures using different processes . . . [which] depend on our ability to imagine overlapping figures, guidelines for the construction and other key points of the figure . . . improves the ability to find relationships between figures and their elements and encourages creativity. (Portfolio)

Conclusion

This research was based on a construction task for which we recognize the potential mentioned by Laborde (2001). Besides, we corroborate the claims of King and Shattschneider (2003) regarding the reasons that support the use of the DGE, particularly the use of rigorous constructions and the promotion of visualization, exploration, investigation, discovery and demonstration, to which we would like to add creativity and intuition. However, the data also shows that constructing draggable figures in GeoGeobra contributes to spatial and geometric structuring. The main contribution of this study concerns the importance of this work in prospective teacher education. From a mathematical point of view, the data shows the relevance of the exploratory work involving geometric constructions using a DGE, promoting the evolution in the way they structure the geometrical figures by identifying relationships and properties. Apart from this perspective, the comments of participants also show the relevance of reflecting on mathematical activity itself. This reflection – here enhanced by the portfolio – enables prospective teachers to become aware of their own learning in relation to the task, which can be an important contribution to their didactical knowledge.

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Geometric constructions in a dynamic environment (GeoGebra): The case of in-service teachers

Dorit Cohen[1], Anatoli Kouropatov [2], Regina Ovodenko[3], Maureen Hoch[4] and Sara Hershkovitz[5]
Center for Educational Technology, Tel Aviv, Israel; [1]doritc@cet.ac.il; [2]anatolik@cet.ac.il; [3]reginao@cet.ac.il; [4]maureenh@cet.ac.il; [5]sarah@cet.ac.il

A unit on dynamic geometric construction was included in a professional development course for in-service mathematics teachers. As a final task in that unit 28 teachers were required to construct a rhombus based on their own choice of given objects and tools, using the dynamic geometry software GeoGebra. Their responses were analysed according to: the choice of given objects; the choice of tools; the explanation and validity test; and the number of different rhombuses they claimed to have obtained. The teachers were found to have different concept images of a rhombus and different conceptions of what constitutes a valid geometric construction. While many claimed to have obtained an infinite number of different rhombuses, differences were observed in the "type of infinite". Recommendations are given for improving the task design to strengthen teachers' mathematical and pedagogical knowledge.

Keywords: Geometric construction, dynamic geometry, GeoGebra, concept image, in-service teachers.

Introduction

From the use of a dynamic environment for geometric construction arise new frames of reference for the idea of construction. In this paper we present data accumulated during an in-service teachers' professional development course on dynamic geometric construction. We report on the data from two viewpoints:

1. The practical viewpoint – we will describe teachers' construction methods in the final task in which they were required to construct a rhombus in different ways (each construction according to different given objects). We will discuss issues arising from the construction, both those concerning the nature and validity of the construction and those concerning the teachers' images of a rhombus.
2. The pedagogical viewpoint – we will suggest possible ideas for improving the task design in order to hone and strengthen teachers' mathematical and pedagogical knowledge.

Theoretical background

Construction in geometry has a specific meaning: the drawing of geometric figures using only compass and straightedge without measuring angles or lengths (Hartshorne, 2000a). Geometric constructions have been a popular part of mathematics throughout history. Euclid documented them in his book entitled "Elements", which is still regarded as the authoritative geometry reference. In that work, he uses these constructions widely and extensively, and so they have become a part of the

geometry field of study. Geometric constructions also provide insight into geometric concepts and give us tools to draw figures when direct measurement is not appropriate (Hartshorne, 2000b).
For about a decade, geometric constructions were removed from the Israeli curriculum. In 2014 geometric constructions were again included in the geometry curriculum for middle school in Israel. Curriculum decision makers claim (INMMC, 2013) that geometric construction integrates geometric content knowledge with a deductive way of thinking that is essential for geometry proofs.

The professional literature indicates that teaching mathematics using technological tools helps in the process of constructing an abstract knowledge of mathematics, and geometry in particular (Lagrange et al., 2003). GeoGebra is a powerful dynamic environment that allows testing of countless number of examples; provides effective and convenient tools for confirming or contradicting conjectures; and provides a wide selection of different tools for geometric constructions – from digital analogs of compass and straightedge, to shortcut tools, such as automatically drawing a line parallel to a given line (Fahlberg-Stojanovska & Stojanovski, 2010). GeoGebra is also a tool of assessment (Bu et al., 2012); thinking processes can be observed by examining the construction protocol which can replay step-by-step onscreen.

Vinner and Hershkowitz (1983) focused on the cognitive development of mathematical concepts, and proposed a model of two components: the *concept definition* – the verbal description of the mathematical concept, which characterizes the concept mathematically; and the *concept image* – the cognitive structure that includes all the examples and the processes related to the concept in the learner's mind. Geometric concepts have a special status: Fischbein (1993) coined the term "figural concepts" and argued (Fischbein & Nachlieli, 1998) that geometrical figures are characterized by both conceptual and sensorial properties. A geometrical figure is a mental abstract which is governed by a definition. At the same time, it is an image. In geometrical reasoning the two categories of properties should merge absolutely.

In the context of this research, geometric constructions, which are carried out in a GeoGebra dynamic environment, can be seen as figural concepts. We used our personal, not formal definition (Vinner, 1991) for geometric construction. To construct a geometric figure means to draw the figure on a computer screen using GeoGebra digital tools in such a way that essential properties of the figure remain invariant under dragging.

The purpose of the study

The aim of the present study is to:

1. Characterise the in-service teachers' geometric constructions on the final task of the unit.
2. Improve the design of the construction tasks in order to increase their contribution to the development of in-service teachers' mathematical and pedagogical knowledge.

Methodology

The research population

This research was carried out in the framework of a 30-hour in-service professional development course whose aim was to acquaint middle-school and high-school teachers with a digital program for ninth-grade mathematics students. A part of this course was the geometric construction unit. Since our assumption (which subsequently proved unfounded) was that the teachers were already acquainted with traditional geometric constructions using straightedge and compass, the unit dealt mainly with the use of the dynamic geometry software GeoGebra. When necessary, traditional methods were referred to. The focus of the research was on the responses of 28 teachers to the final task of the unit.

About the final task

Throughout the course the in-service teachers tried out different geometric construction activities using GeoGebra and thus learned to create a valid construction using dynamic tools. Instructions for the final task included the following reminder.

> Constructing with dynamic tools is not the same as drawing on a page since the objects (points, segments, etc.) are moveable: you can drag them and observe how other objects change accordingly. Each figure must be constructed so that it retains its characteristics even after other objects are dragged. For example, on constructing a rhombus according to its diagonals, check that after changing the lengths of the diagonals the figure remains a rhombus. This type of construction is called a valid construction.

In the final task the teachers were required to:

1. Construct a rhombus by three different methods, each construction according to different given objects (according to your choice), for example, according to its diagonals.
2. Describe each construction process and prove that it results in a rhombus.
3. State and explain how many different (non-congruent) rhombuses can be obtained by each method.

Note that no instruction was given as to whether the given object is fixed or dynamic. A fixed side will yield an infinite number of rhombuses (same side length, different angles) while fixed diagonals will yield only one rhombus. Clearly if the objects are dynamic an infinite number of rhombuses are possible, but this is a feature of the software and not of the underlying mathematics.

Data collecting and analyzing

Each teacher sent a solution which comprised a picture of the final construction of the rhombus, the GeoGebra file, and explanations and justifications for the construction process. We used interpretative methods for analyzing these data.

Findings: Characteristics of geometric constructions

A qualitative analysis of the teachers' constructions yielded four main categories.

1. The given objects on which the construction of the rhombus was based
2. The construction process itself according to tools used

3. The teacher's explanation and validity test
4. The number of different rhombuses the teacher claimed to have obtained

We present some examples of teachers' constructions, according to the above categories. The teachers' choices of given objects on which to base their construction were varied; for example, one side, two diagonals, one diagonal, an angle, a combination of the already mentioned objects, and other objects (such as area). We will present just two of these choices: one side and two diagonals. Here are examples from three different teachers who constructed a rhombus according to its side.

Tzila's construction

1. "The construction is according to 4 equal sides, each 5 cm long."
2. Used segments of fixed length, parallel line through a point, intersection point.
3. "I created a rhombus from two adjacent sides each 5 cm long, using parallel lines; that is a parallelogram with all 4 sides equal 5 cm."
4. Infinite number. "Using a circling movement, with point A fixed, in a sort of circling round each time getting another rhombus whose diagonals are changing."

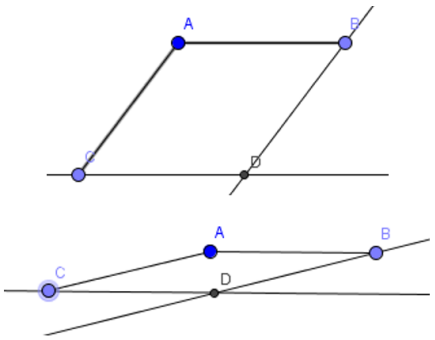

Figure 1: Tzila's construction before and after dragging

Instead of just writing that the side of the rhombus was given, Tzila added a definition of rhombus, which, like every definition, provided a sufficient condition. Possibly she misunderstood the task, and thought that she had to state the conditions for creating a rhombus, or perhaps her concept image of rhombus is a parallelogram with four equal sides, or perhaps she misunderstood the components of deductive geometry.

Anat's construction

1. A quadrilateral of equal sides – each side of length 2 cm.
2. Used segment of fixed length, circle with fixed radius.
3. "According to the theorem: a rhombus is a quadrilateral with all sides equal."
4. No answer.

Anat's explanation shows that she, like many of the teachers, was confused about the components of deductive geometry (stating she was using a theorem when in fact she was using a definition).

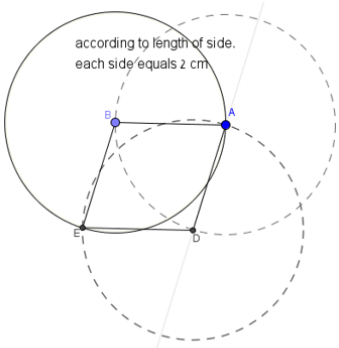

Figure 2: Anat's construction

Yaron's first construction

1. Rhombus with side of length **a**.
2. Used circle with given radius, intersection point.

3. "The length of the side of the rhombus is **a** since all the sides and the radius are length **a**. The construction is valid according to the dragging test. The construction relies on the principal of the length of the radius of a circle".
4. "There can be an infinite number of rhombuses since although the sides remain equal lengths the diagonals and angles can be changed."

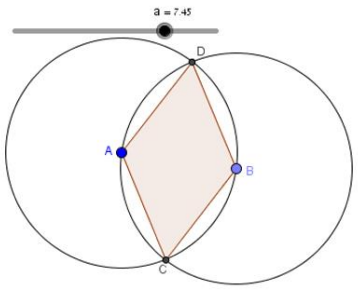

Figure 3: Yaron's first construction

Despite Yaron's claim that the diagonals and angles can be changed, he in fact built a rhombus in which the shorter diagonal was equal to the side and so the acute angle of the rhombus was 60º. He claimed that there are an infinite number of possible rhombuses given the length of a side, which is correct, but does not correspond to what he constructed. This error was repeated by many of the teachers. Interestingly, in another construction – described below – Yaron erroneously claimed an infinite number of possible rhombuses because of the dynamic nature of the given attributes.

Here are examples of two teachers who constructed a rhombus according to its diagonals.

Yaron's second construction

1. Rhombus according to two diagonals.
2. Used segment, perpendicular bisector, circle, intersection point.
3. "I based this on the fact that in a rhombus the diagonals are perpendicular and bisect each other, and on the principal of the length of the radius of a circle. The construction is valid according to the dragging test."
4. "There can be an infinite number of rhombuses since the lengths of the diagonals are not fixed. can lengthen or shorten them (or even only one them). The rhombus can change – both its sides its angles."

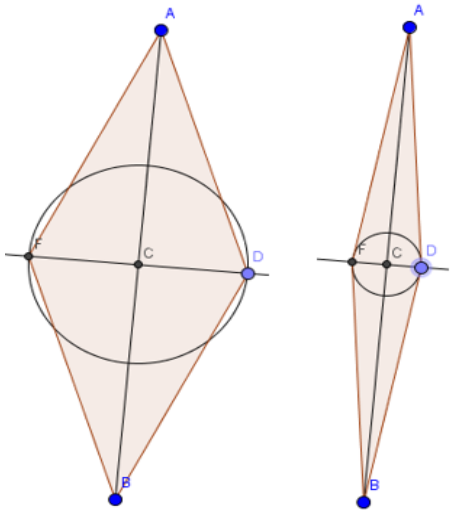

Figure 4: Yaron's second construction before and after dragging

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Yaron constructed diagonals whose length changed dynamically. He did not relate to the fact that each pair of diagonals determines only one possible rhombus.

Nora's construction

1. According to two diagonals.
2. Used segments of fixed length. By trial and error changing the angles between the segments.
3. Did not check validity.
4. No answer.

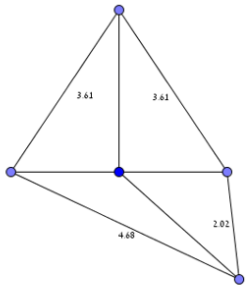

Figure 5: Nora's construction after dragging a vertex

If Nora had checked her construction she would have realised that it did not pass the validity test, as seen in figure 5.

Discussion and conclusion

In the discussion we will relate to the two viewpoints mentioned earlier: the practical and the pedagogical.

The practical viewpoint

In the context of the practical viewpoint we will characterise the findings in each of the construction categories. The first category is: **Given objects for constructing a rhombus**. We have presented two of the teachers' choices of given objects: one side or two diagonals. All the teachers in the course chose the size of their given objects in one of two ways – either fixed size (a number) or dynamic (a parameter). This choice seems to be related to the teacher's conception of geometric construction – what is permitted in such a construction. It should be noted that these teachers had not previously learned constructions in a rigorous manner, and in particular, not dynamic constructions. Therefore we relate to their answers as based on intuitions about what is a construction and on geometric knowledge relevant to the specific topic. We suggest that teachers who chose fixed numerical givens have a concrete conception of construction that is related to particular sizes during the construction process, and did not refer to the characters of the figure which is constructed with dynamic tools. Teachers who chose dynamic givens have a broader conception of construction, not connected to size. This choice of dynamic givens corresponds more closely to the instructions – see above description of the final task. In addition, there were some teachers who were confused about defining the given objects, for example, Tzila, as described above.

The second category is: **Choice of construction tools**. We identified three types of construction process: measuring; using the extensive range of tools supplied by GeoGebra; and using a limited range of tools (cf. Fahlberg-Stojanovska & Stojanovski, 2010).

Construction by measuring involves fixing the size of the required object (side, diagonal, or angle). For example, in Nora's construction she moved two rays to form a right angle. Another teacher created four equal segments using co-ordinate axes.

An example of construction with the extensive range of tools is using the parallel line tool to ensure a parallelogram, as in Tzila's construction. Another example is using the perpendicular bisector tool, as in Yaron's construction.

Construction with a limited range of tools imitates construction with straightedge and compass. For example, Yaron built a circle of variable radius. Anat used both measurement and the limited range of tools (using a circle of radius equal to the given segment of length 2 cm).

The third category is: **Explanation and validity check**. None of the teachers gave formal proofs of their constructions. However their explanations provided us with an idea of their concept images (Vinner, 1991) of a rhombus. They explained their constructions according to their concept image and not according to the definition as given in the national school curriculum: a quadrilateral with all sides equal. A problematic concept image seems to create confusion between the different components of a deductive geometric argument, for example, between a theorem and a definition and between a theorem and a property. This confusion can be seen in the explanations of Tzila and Anat.

We discerned two problematic areas in the teachers' explanations: an incoherent concept image related to mathematical knowledge; and an intuitive conception of construction in a dynamic environment, which seem to be related to the construction process in everyday life, for example

Nora's construction which produced a rhombus. This construction did not fulfill the demand that the construction will remain a rhombus also after dragging.
This conception resulted in most of the teachers producing an invalid construction. There is not enough information in the data to pinpoint the main causes of these problems – are they connected to missing technological knowledge, or to missing mathematical knowledge, or to a faulty conception of construction, or to a combination of all three? More research is required to clarify this.

The fourth category is: **Number of solutions**. We identified three types of responses, each claiming an infinite number of solutions: relating to the dynamic nature of the given object, as seen in Yaron's second construction; relating to the mathematical properties of the given object, as seen in Tzila's construction; relating to the position of the given object in the plane. This latter type was seen in the response of a teacher who wrote: "In the plane it is possible to position the rhombus in any place you want".

The pedagogical viewpoint

In the light of these research findings we recommend investigating some aspects of task design in a dynamic environment. On one hand the task design can assist teachers to execute the task, and on the other hand it can persuade them to use this task in their classroom (Bu et al., 2012). We suggest that it would be worthwhile to design the task on three levels: mathematical-pedagogical, technological-pedagogical, and reflective.

The mathematical-pedagogical level requires the teacher to define the concept and to construct the figure accordingly. In line with Fischbein (1993) we relate to a geometrical concept as a figural concept and thus its construction has practical meaning. Subsequently the teacher is required to construct the figure based on different sufficient conditions. Such a construction task would enable identification of the teacher's mathematical knowledge and her concept image, and could help to bring that concept image closer to the concept definition.

The technological-pedagogical level requires the teacher to construct the figure at first using the limited range of GeoGebra tools (imitating straightedge and compass) and subsequently using the extensive range (enabling short cuts). We would like to investigate the connection between these two types of construction and the connection between the teacher's image of the geometric concept and her conception of geometric construction.

The reflective level requires the teacher to check the validity of her construction, while considering the meaning of validity. Giving a detailed account of the results of the validity test should be an inseparable part of the task. Such an execution of the task should strengthen the connection between the mathematical and technological aspects. For example, if the construction "collapses" – after dragging the dynamic objects the properties of the required figure are not preserved – it is important to understand whether the "collapse" is due to a mathematical failure or a technological failure, or a combination of both. Such an analysis should contribute to the development of mathematical knowledge, pedagogical knowledge, and technological knowledge.

We believe that the above characterisation may provide a starting point for further research and may contribute to the technology, pedagogy, and content knowledge framework (Koehler & Mishra, 2009) – to help teachers integrate technology into their teaching.

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Transitioning from definitions to proof: Exploring non-Euclidean geometry in a college geometry course

Rebecca Dibbs and Janessa Beach

Texas A&M University-Commerce, United States

rebecca.dibbs@tamuc.edu; janessa.beach@tamuc.edu

Geometry is the subject where U.S. students are weakest on international assessments, but college geometry is an area of proof that is understudied. Since geometry is secondary students' only exposure to proof, it is vital our secondary teachers can prove effectively in this content area. However, one obstacle to developing deeper understanding of geometric concepts in college geometry courses is that students tend to try recalling prior geometry instruction instead of engaging in any new material within a Euclidean geometric context. A document analysis of student portfolios revealed that although pre-service teachers in this document study began the semester with limited abilities to work with formal definitions, by the end of the semester all were able to propose and justify conjectures on novel surfaces.

Keywords: Geometry, inquiry-based learning, pre-service teachers.

Theoretical background

Geometry arises from a set of undefined terms and axioms through which all other theorems and definitions are constructed. Hence, a thorough understanding of geometry involves a deep understanding of proof; yet, teachers possess a narrow understanding of proof. Studies indicate that pre- and in-service teachers believe proof only helps explain ideas used in mathematical concepts, and they do not recognize the ability of proof to systemize results (Mingus and Grassl, 1999; Knuth, 2002b). Teachers lack the geometry content knowledge required for geometry proofs, and they are convinced by empirical evidence as well (Jones, 1997; Knuth 2002a). Consequently, teachers with inadequate proof and geometry understanding cannot be expected to impart adequate proof and geometry knowledge to students. Furthermore, college geometry is the only undergraduate proofs course that has not been studied in any systematic manner (Speer and Kung, 2016).

Pre-service teachers in undergraduate proof courses do not adequately understand what arguments qualify as proof (Weber, 2001). They lack comprehension of the mathematical language and concepts necessary to proof (Selden, 2012), and they possess an incomplete understanding of definitions and theorems (Weber, 2001; Selden and Selden, 2008). In typical direct-instruction, lecture proof courses, students are expected to develop proficient proof skills with little to no guidance. Without guidance, students will fail and likely cultivate ineffective strategies (Weber, 2001). These ineffective strategies are typically proof schemes dependent upon external and empirical convictions, such as the authoritarian, ritual, and perceptual proof schemes (Harel and Sowder, 2007). In order to successfully write a proof, students need to employ effective strategies or proof schemes with arguments based on axioms and logical deductions, which in turn requires understanding of definitions and the idea of conditionally true statements.

The typical instructor-centered learning environment, is the dominant paradigm, but may not induce the logic and proof techniques needed to construct a proof in all students (Fukawa-Connelly,

Johnson, & Keller, 2016). Alternatively, a proof course should consist primarily of student-student and student-teacher interactions (Selden and Selden, 2008). Given the prevalence of lecture-based proof courses in the United States (Fukawa-Connelly, Johnson, & Keller, 2016) and preservice teachers' continuing struggles with geometry, one potential approach to help improve pre-service teachers' is an inquiry-based learning pedagogy where students are active learners, and the instructor is responsible for facilitating students' exploration of the content, particularly definitions (Padraig and McLoughlin, 2009).

However, one of the additional challenges of college geometry is that the material is familiar to students. Rather than investigating the ideas presented in the current assignment, many students rely on recollections from previous geometry courses, especially if the problems seem familiar. Hence, an inquiry-based college geometry course in non-Euclidean geometry seems more likely to help pre-service teachers develop their proving skills and deepen their understanding of the geometric concepts they will eventually teach. This study was guided by the question: how, if at all, an inquiry based non-Euclidean geometry class helped deepen students' understanding of definitions and Euclid's postulates? We argue that the deep exploration of a limited number of non-Euclidean geometry problems helped students to move from primitive geometric knowledge to formalizing definitions and successfully posing and solving conjectures on novel surfaces.

Methodology

An adaptation of Pirie and Kieren's (1994) model for student understanding was used to code students' written assignments for their understanding of definition. Although this eight-level model was originally created to model students' understanding of fractions, it adapts well to geometry, as the purpose was to describe the transition from concrete to abstract reasoning to justification to problem posting. One level, image having, was not used when coding, since students always had access to physical models of whatever non-Euclidean surface they were working with that week, so we could not determine students' facilities for understanding similar exercises without physical models. We also did not include looping back within our standards of evidence because it was not something we observed in the data.

This study took place at a midsized, rural, Hispanic-serving research university in the South, and the students who participated were those enrolled in a ten student college geometry course. The data collected was part of a larger study; this study presents a case study of five pre-service teachers Lindsey, Bradyn, Alexis, Mackenzie, and Chase. We also analyzed the work on one non-preservice teacher, Florencio, because Florencio's papers were different from the other participants. While we wanted to maintain a purposeful sampling of pre-service teachers, Florencio was enough of a disconfirming case that we felt his inclusion was necessary (Patton, 2002). Florencio was an engineering major with one prior proofs class. Lindsey has no formal proof experience, Bradyn had completed discrete mathematics in one attempt with a B, Alexis had completed discrete mathematics and abstract algebra with A's, and Chase, who has a learning disability, had completed discrete mathematics with a C after two attempts and failed another upper level proof class. Mackenzie was a non-traditional student in her final semester; she had completed all other upper level proof classes with a mixture of A, B, and C grades.

Students in the course were provided with course notes (Miller, 2010) that presented open-ended problems related to a specific learning goal. There were fifteen assignments; five of which were focused on formal axiomatic proof, eight on definitions and axioms in various non-Euclidean situations, and two assignments (the midterm and final project) which combined both strands in the same assignment. Four of the assignments were formal (one revision allowed), and the other ten were informal (unlimited revisions). On the midterm (F3), students were given new but similar problems to their assignment and asked to work through them individually, and on the final project (F4), students were asked to discover as many things as they could about the geometry of the surface a cone. For each new assignment, students were assigned a specific problem from the provided course notes and a group. If a group appeared to be making little progress or moving in an unproductive direction, the teacher would use guided questioning to redirect students' thoughts. If multiple groups stopped progressing, the teacher would initiate a whole class discussion.

To determine students' understanding of definitions and postulates, researchers examined the first submission students turned in for each assignment. Researchers also used observations to gain further understanding of students' proof comprehension. As students discussed their ideas within their group, a researcher sat behind them listening and taking notes on their interactions for about ten minutes. The submissions were analyzed by assignment and all the drafts from an individual participant were analyzed at the same time. After this initial reading of blinded assignments, researchers would journal their impressions of the coding and the overall trajectory exhibited in the multiple submissions. These journals were used to operationalize concepts in the literature review, and then they were compared to the standards of evidence table (Table 1).

Level of Understanding	**Identifiers**
Primitive Knowledge (1)	Students are applying prior knowledge of Euclidean geometry, stating given definitions, or providing empirical proofs
Image Making (2)	Students make distinctions and reclassify prior knowledge or use prior knowledge in a new manner
Property Noticing (3)	Students can apply a definition on a previous surface to a novel surface or situation by recognizing commonalities in the learned and novel situation
Formalizing (4)	Students can abstract a method, formula, or common property from previous property noticing
Observing (5)	Students can propose conjectures and provide justification or counterexample
Structuring (6)	The argument is logical and made up of systematic application of axioms and theorems/If any portion of the argument could be clarified, the clarification is not necessary for the argument's validity.
Inventising (7)	Students can pose new questions and solve them, creating new (to the student) knowledge

Table 1: Standards of evidence (Modified from Pirie and Kieren, 1994)

Findings

With the exception of Mackenzie, all pre-service teachers struggled to complete the initial assignments with correct arguments; primitive knowledge from a previous high school geometry course was applied to the problem instead of an argued solution. However, by the fourth inquiry-based task, all participants were able to formalize definitions, and all students were at least able to successfully use definitions and postulates in novel situations to construct proofs. All students followed a similar trajectory throughout the semester and improved, on average, four levels of understanding (Figure 1).

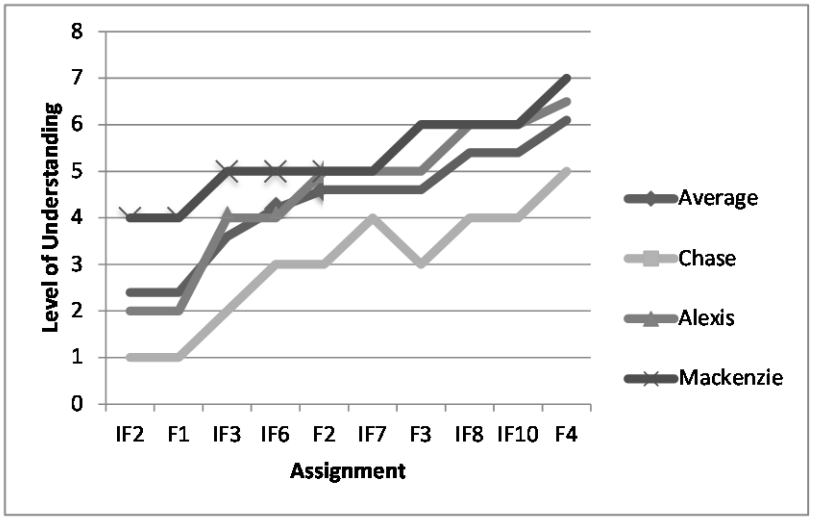


Figure 1: Student levels of understanding throughout the semester

The first definition centered-assignment of the semester was an inverse categorization problem. After finding all possible symmetries on the square, students were asked to use Geogebra to start with a subset of these symmetries, construct all possible quadrilaterals with that set of symmetries, and justify why they had found all cases. Mackenzie was able to categorize on first assignment, but the other preservice teachers either listed the symmetries of each quadrilateral or could not justify if/why they had found all cases (Figure 2). Both of these difficulties indicate students making partial reclassifications of their prior knowledge.

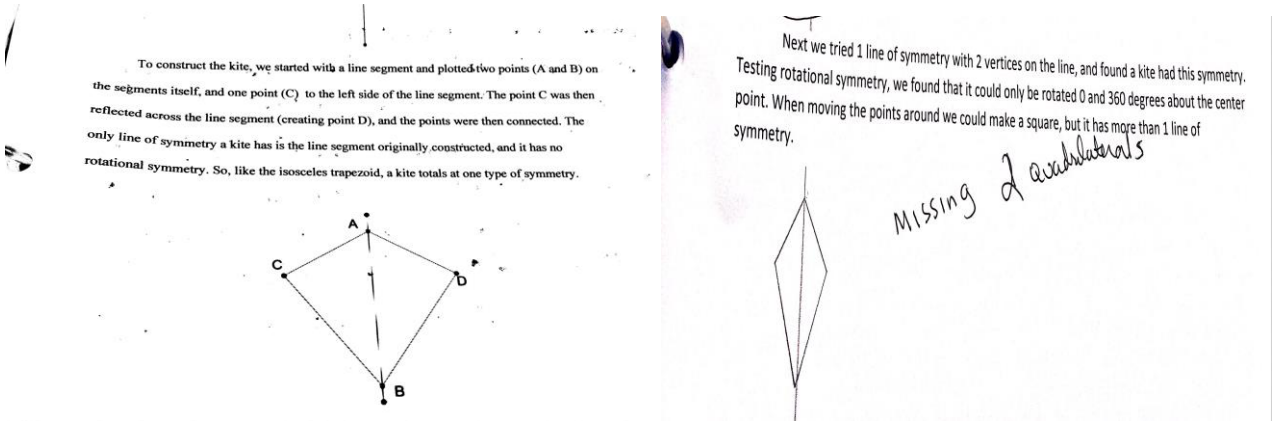


Figure 2: Typical solutions on IF2 Chase (left, primitive knowledge); Florencio (right, image making)

During the middle third of the semester, the two assignments that helped pre-service teachers move towards formalizing their understanding of definitions and counterexamples were IF3 and IF6. In both assignments, students were asked to justify which, if any of Euclid's postulates held on a sphere (IF3) and the hyperbolic plane (IF6). Students were also asked to prove the existence of asymptotic geodesics on the hyperbolic plane. On IF3, the first exposure to the postulates, Alexis was not able to work in the spherical context and reasoned through the justification of the postulate in terms of the more familiar planar geometry (Figure 3). However, by IF6 Alexis was able to provide a counterexample for false postulates (Figure 4).

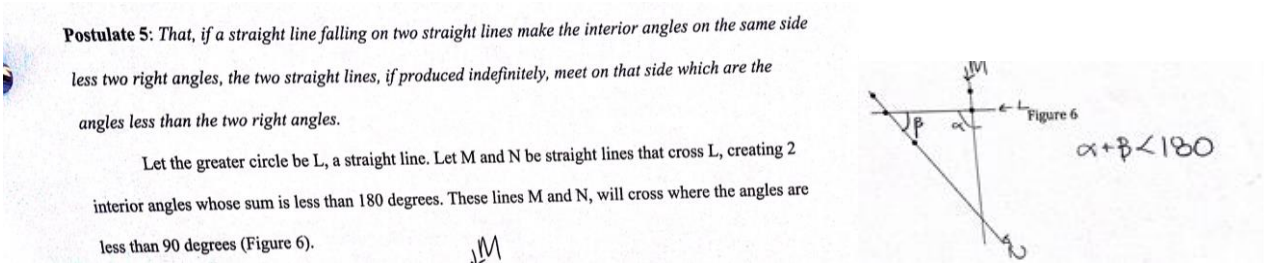


Figure 3: Alexis' IF3 (Image Making) Postulate 5 solution

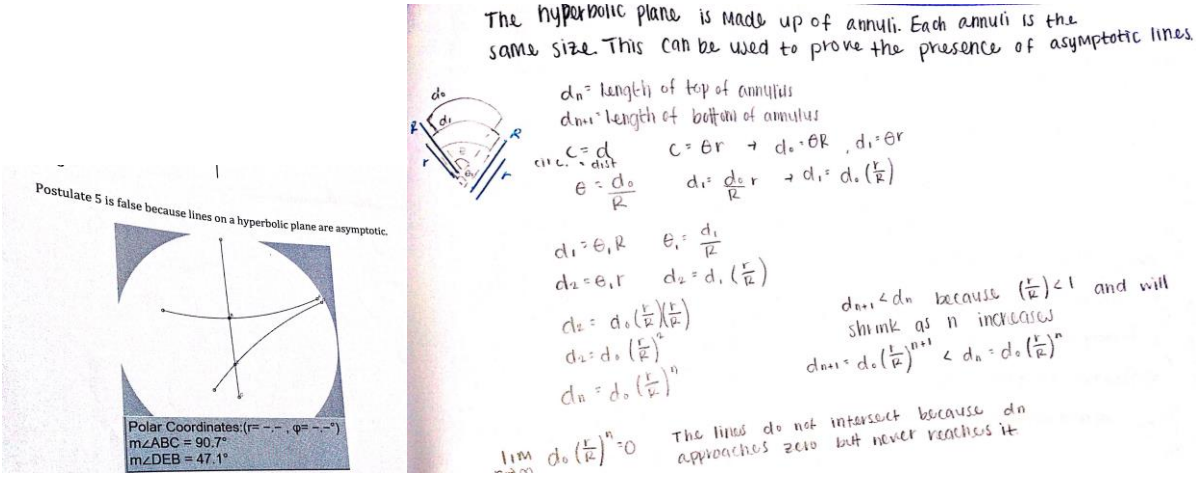
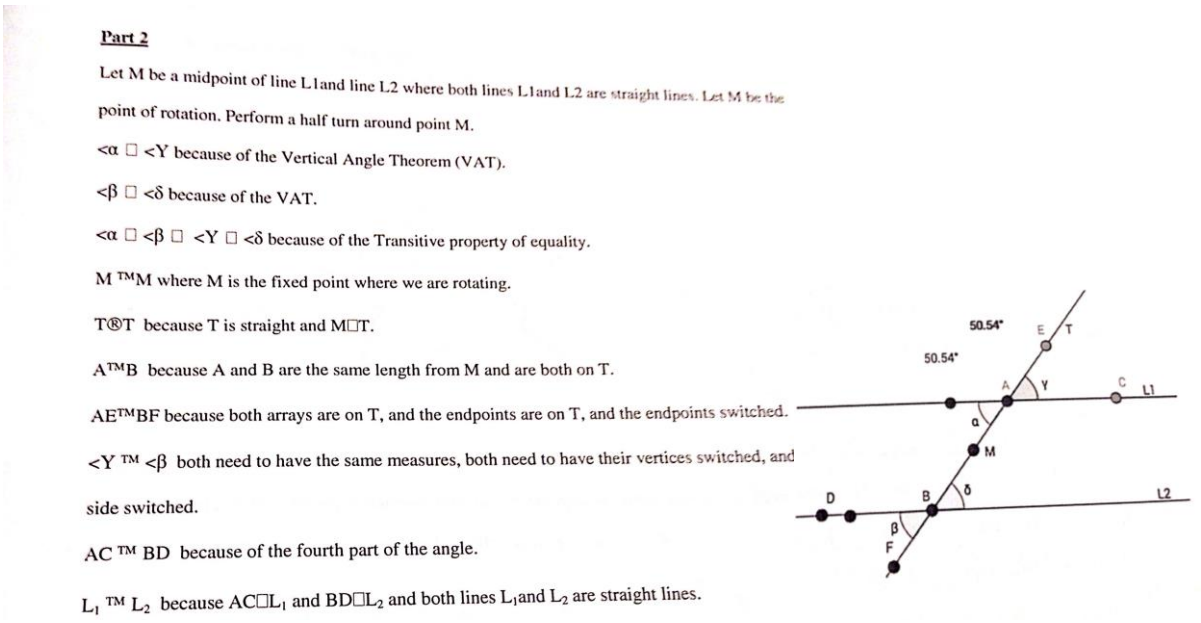


Figure 4: IF 6 (Formalizing) Postulate 5 solution

In the final third of the semester, the goal of all assignments was to integrate pre-service teachers' improved proof schemes with more formal uses of definitions. IF5 and F3 were major proofs assignments that took students most of the middle third of the semester. With their improved proof schemas, and understanding of the surfaces, were more easily able to construct proofs for parallel lines that were independent of the surface upon which the lines were drawn (IF8 and IF10). Most students were able to construct a generally correct proof, with some minor disordering of steps and missing justifications. This shows participants possessed a more structural understanding of symmetry than the understanding demonstrated in IF2. Bradyn, like Florencio, was image making on the first assignment related to symmetry, but by the time symmetry was used to construct proofs related to parallel transported lines, Bradyn was much more successful (Figure 5). Although Bradyn's language is not quite standard and he had trouble typesetting his proof, his overall structure is systematic and he has a transformational understanding of symmetry not present in his initial write-ups.



Part 2

Let M be a midpoint of line L1and line L2 where both lines L1and L2 are straight lines. Let M be the point of rotation. Perform a half turn around point M.

<α □ <Y because of the Vertical Angle Theorem (VAT).

<β □ <δ because of the VAT.

<α □ <β □ <Y □ <δ because of the Transitive property of equality.

M ™M where M is the fixed point where we are rotating.

T®T because T is straight and M□T.

A™B because A and B are the same length from M and are both on T.

AE™BF because both arrays are on T, and the endpoints are on T, and the endpoints switched.

<Y ™ <β both need to have the same measures, both need to have their vertices switched, and side switched.

AC ™ BD because of the fourth part of the angle.

L_1 ™ L_2 because AC□L_1 and BD□L_2 and both lines $L_1$and L_2 are straight lines.

Figure 5: Bradyn's second proof in IF8 (observing)

The final formal assignment asked students to discover (and prove) as many things as they could about the geometry of an infinite cone. Groups were expected to prove 2-4 conjectures. Given the open nature of F4, one group chose to only investigate properties of a cone where group members had successfully revised an assignment on another surface. This limited their levels of understanding to observation. The other two groups each had at least one investigation of a conjecture about a novel concept with at least one new or newly-modified definition, which is summarized in Table 2.

Participant	Project Summary	Code
Alexis (+3 others)	Using a novel group-invented definition of straightness to investigate self-intersecting lines on cones (conjecture: no formula possible), angle sums of triangles with self-intersecting sides	Structuralizing, Inventising
Mackenzie, Lindsey (+1 other)	Holonomy, internal angle sums of a triangle with no self-intersecting sides, triangle congruence theorems, non-intersecting lines that are not parallel transports	Inventising
Bradyn, Chase (+1 other)	Postulates (some cone angles), collected data for self-intersecting lines	Observing

Table 2: Summary of final project

Discussion

Regardless of prior proof course grades or experience all pre-service teachers struggled to complete the initial assignments with correct arguments; primitive knowledge from a previous high school geometry course was applied to the problem instead of an argued solution. However, by the fourth inquiry-based task, all participants except for Chase were able to formalize definitions, and all students were at least able to successfully use definitions and postulates in novel situations to construct proofs by the end of the semester.

By centering the college geometry course around understanding core geometric concepts on several different surfaces, participants were forced to engage in understanding each new situation rather than simply applying their prior Euclidean geometry knowledge to a more familiar problem. As a result, students developed more advanced understanding definitions and counterexamples. All participants got to at least formalizing definitions and seven of the ten students in the class ended the semester at either the structuring or Inventising level.

The structure of the course maximized students' opportunities to reify their understanding of definitions and postulates. The use of multiple non-Euclidean contexts was key to helping students develop better understanding of formal definitions. By switching surfaces, operationalizing definitions and determining if they were applicable stayed a problem and not an exercise. Further, the repetition of the postulates and determining geodesics in particular were important because these concepts were presented in enough slightly different concepts to allow students to develop deeper levels of understanding of the definition than a single context would allow. Students also reported that the chance to revise their written work was a valuable way to help them reflect on the current surface and how it compared to their prior work.

There are two potential limitations for this study, which could be remedied by further inquiry. First, we did not collect students' interview data; our analysis is a document study coupled with observations of students' in-class discussions. Teaching experiments or task-based interviews of students' understanding of definition in similar inquiry based classes, and comparative data to

students in axiomatic Euclidean courses would also be of use. We also had non-native English speaking students in this class, and further research on their experience with proof and geometry is still needed. Finally, although most students followed the same trajectory, Chase was about two weeks behind everyone else. However, Chase has dyslexia; and there is a dearth of literature about undergraduates with learning disabilities; more work is needed to understand how to support such students' learning.

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Teaching geometry to students (from five to eight years old)
"All that is curved and smooth is not a circle"

Jacques Douaire[1] and Fabien Emprin[2]

[1] Ermel Team-IFé, France, jacques.douaire@wanadoo.fr;

[2] Ermel Team-IFé, University of Reims Champagne-Ardenne, CEREP, France, fabien.emprin@univ-reims.fr

The ERMEL team (IFé, ENS-Lyon) in France, builds complete engineering to teach math in elementary school. For several years, we have been experimenting with teaching situations on space and geometric learning for 5-8 years old students. In this paper, we focus on the results about the relationship between actions on objects, graphs and first geometric properties about curved line. Our methodology involves an analysis of the student's way of solving problems and thus their abilities, but it also allows us to make a hypothesis on what it is that teachers need in order to carry these learning situations out.

Keywords: Geometry, teaching, learning, primary education, spatial ability.

Issue of this paper

Our research takes place in the French context of geometry teaching in primary school. Our goal is to build a proven, complete and reliable teaching engineering and thus to improve geometry teaching.

In CERME 9, we reported our results about the knowledge of straight line for the same kind of students, not as an element of geometrical figure but as a usual or new component of the pupil's practice. Several point of view in the WG4 showed the similar approach and questions were shared:

- The relation between everyday and geometrical concept, perception, language and manipulation
- How to start with low level, and long planning time
- What kind of tools for the research and tools for teaching geometry are there? (How to help teacher to know what student are able to do)

In this paper, we aim to make our contribution to the three first topics of the WG4 group: what is doing, learning and teaching geometry at school?

Why working on geometry teaching?

A starting point of our research is a finding of deficiencies in the geometry teaching practices in the early grades of elementary school. A short analysis of easily available French resources for teaching geometry to young pupils shows that two mains goals are pursued: learning geometric words and drawing abilities. Spatial activities also exist but without much problem solving, and also unrelated to the pupil's initial knowledge or mathematical activity. This research is also based on the idea that students' abilities are insufficiently taken into account in geometry teaching in primary school. Thus we have to identify the knowledge at stake in this learning and take the students' already acquired knowledge into account.

About our team

ERMEL is the research team on mathematics education in primary school (in French "Équipe de Recherche en Mathématiques à l'École Élémentaire") belonging to the French institute of education (IFé). ERMEL team is made up of primary school teachers, teachers' trainers and researchers working in different regions of France. Results of these researches lead to comprehensive books publications including complete teaching engineering. Studies on teaching and learning conducted since 1999 are about geometry teaching more precisely on the analysis of spatial and geometric skills that students from primary school to GS (5-6 years old) CE1 (7-8 years old) can build. Key issues of these researches are knowledge creation and resources production for teachers and teachers' trainers as well as the study of the appropriation of these resources. In CERME 9 we have clarified our theoretical framework and the steps of our methodology: a didactic engineering, based on an experiment conducted in many classrooms for several years (Douaire & Emprin, 2015). In this paper, we try to explain the transition between space and graphic knowledge which is often underestimate.

Purpose of the study

Our previous studies (Douaire & Emprin, 2015) showed that the knowledge developed by pupils in meso-space by solving problems does not necessarily build a geometric knowledge usable on paper. This lead us to ask many questions: in particular, what are the opportunities for pupils to understand the underlying patterns of drawings on a sheet of paper?

We have to clarify the relationship between two types of knowledge, spatial and geometric, and especially the discovery of the meanings of lines on a sheet of paper, and how these plots can provide information about objects in space. If several works address the distinction between drawing or figure at the beginning of the "college" (Parzsyz, 1988) or the apprehension of the components of a figure at the end of primary school (Duval, 2005) (Perrin & al, 2013), our research concerns the emergence of graphic representations a few years earlier.

In this working group we will identify the contributions of space experiments in the construction of geometrical knowledge. Our goal is that pupils overcome the overall perception of a figure and develop the analysis of its components.

We analyze learning involved by a problem solving of closed curves figures construction, the characteristics of different implementations, as well as the needs of teachers. We present insights of learning situations for space experiments and we question the relevance of a resource based on the needs of teachers for its implementation identified in the context of ongoing Ermel research.

Presentation of the experimentation

Experimentation concern procedures (graphics, practices, discourse ...) that can be developed by 7-year-olds pupils to distinguish circles from ellipses and other rounded shapes. We are not trying to develop early knowledge of the circle, but to promote the passage from a global perception of drawings and shapes to a geometric analysis of geometric shape underlying.

Some questions concern the comparison procedures: what are the abilities of students of CP (6 years old) or CE1 (7 years old) to distinguish a circle from another closed curve (an "almost circle")? What use of superposability as a validation procedure for that?

We present an experiment in progress: students have to produce closed rounded shapes, and must prove if they are different from others.

Proposed situation

The problem is to build closed shapes using four circular or elliptic arcs (quarter of a big circle, little circle or an ellipse figure 1). Identifying that shapes built are different lead students to develop and formulate analytical geometric criteria.

Two major phases of this situation are analyzed successively, the first concerns the problem solving phase to produce shapes: we briefly analyze the productions. The second concern comparison of production. We also present some exchanges during the validation of the solutions.

Finally, a brief summary will address the explicit needs of teachers.

Presentation of the activity

Each pair of students has a deck of 16 cards: 4 quarts of a small circle, 4 quarters of a large circle and 8 quarters of an ellipse (shapes figure 1 are cut following dotted lines like in figure 2). The major axis of the ellipse is the diameter of the large circle. The minor axis of the ellipse is the diameter of the small circle. Thus, shapes can be linked.

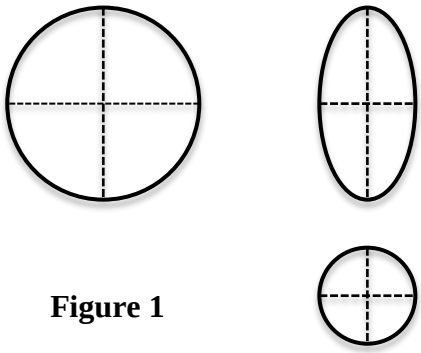

Figure 1

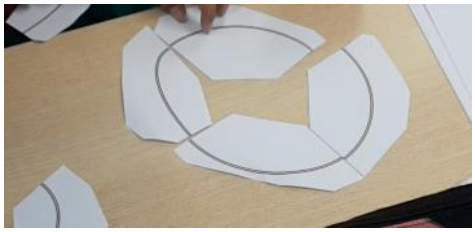

Figure 2

Several successive phases in this situation:

1- To ensure the pupils' appropriation of the constraints of the problems they are experimented one by one. Each pupil must first assemble pieces to form a closed shape. Then they verify that these solutions are really closed. These discussions lead to clear assembly instructions: "Are only accepted shapes that are joined edge to edge (assembly like in figure 3 are rejected)". In this first phase, the students do not need to draw, but to assemble pieces of heavy paper. Then pupils have to make closed shapes using exactly four arcs. The findings, confirmed by teachers and observers in classes are that all students are aware of the goal, namely produce closed curves ("tracks") consisting of four arcs at the end of devolution phase. Students have understood the problem's rules.

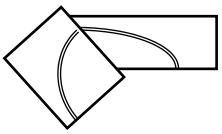

Figure 3

2- Then each student looks for new solutions. In order to save their shapes and be able to make new assembly pupils are asked to draw on tracing paper (or lite paper) the outline of each new shape found. When students believe they cannot find new shapes the search stops, and solutions are pooled: are they different? If a student thinks he has found another solution, it is displayed and compared with previous. Students explain why they think it is different or it's the same as another already displayed. The goals of the pooling are to identify products that meet or not the constraints and identify the identical solutions.

Possible shapes

The solutions are:

- reconstruction of three basic shapes (large circle, small circle, ellipse) solutions 1-3
- combining two half ellipses contiguous or half a small circle (ovoid: solution 4) a large circle (such as "roly poly" or "roller" or solution ... 5). Radius or half axes being concurrent.
- combining alternative quarters of small circle and large circle (solution 6)
- combining two quarters of alternating ellipses with a quarter of a small circle and a quarter wide circle (solution 7)
- combining two quarter ellipses and a small circle (8, 9) or large circle (14,15)
- juxtaposing four quarter ellipses (10, 11)
- juxtaposing two quarters ellipse with on one side a quarter of large circle, and on the other side a quarter of small (12,13).

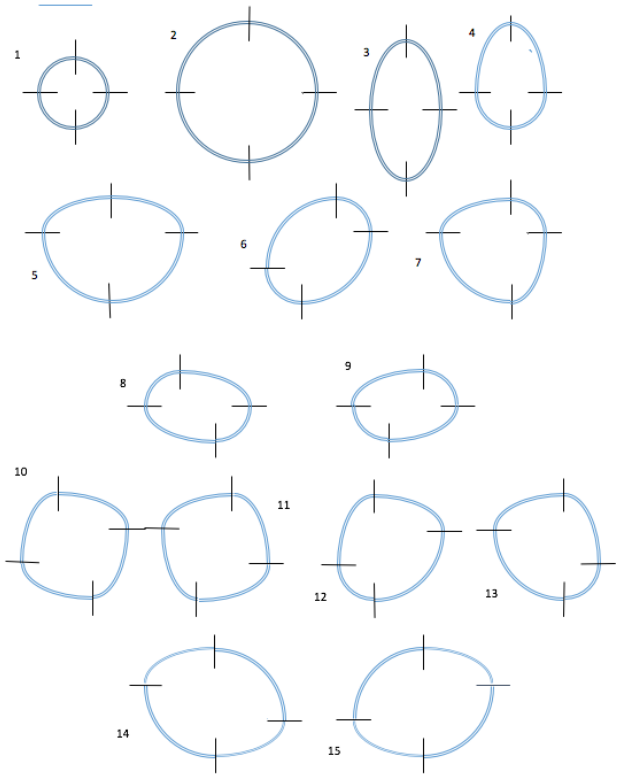

Figure 4: 15 different shapes were found

Of course, the goal of this situation is not to find all solutions but finding shapes form 1 to 5 with also one or two shapes they cannot name globally is enough for pupils to learn.

Description of pupils' strategy to produce shapes

- choose pieces of random way;
- if unsuccessful start from scratch;
- put two pieces, then try the other two;
- replace one or two pieces in an assembly already achieved;
- place the fourth piece by estimating its size.

Description of the comparison procedure

Solutions are shown on the blackboard (some of them may be identical but differently oriented).

We describe comparison strategy used by pupils:

1. Use of the overall look of the drawing (perceptual validation):
    a. recognition of known shape (circle, round, egg ...);
    b. rely on variables: overall size, width;
    c. rely on differences of regularity in curves.
2. Identification of the elements that compose the shape (analytical aspect).
3. Recognition of identical shapes by rotation or reversal.
4. Use of construction processes, with the possibly of remaking to the class.
5. Use of symmetry properties of the shape, mention of the folding ...
6. Use of a practical validation by overlapping.

Strategies based on perception are meanly used.

Those results confirm that spatial abilities are often neglected and that it is a great challenge of our current research.

This pooling highlight some questions linked with the drawing:

- the impact of the thickness of the lines on strategy using perception,
- the acceptable tolerance to judge the compliance with constraint, for example the fact that the curve is closed.

The fact is that those questions emerge are important because it lead pupils to progressively give up perception in favor of the analysis of the shape. This dialogue during the validation of solutions illustrated this aspect.

Exchanges during the validation

The solutions are exposed to the blackboard and students check if the set is suitable.

Student (S2) I believe we cannot do it ...

Teacher (T) The others, are you sure?

S2 appears to confirm

T This drawing there you think that doing it is impossible?

S2 show the drawing of the shape and try to express something.

T So, what could you do to know?

S2 Take the pieces

M So, the team who have made this track, you come back to the dashboard.

Resuming the question of drawings validity

S It looks like …

T With four parts

S2 But then there are bumps

S …but in all there are bumps ... because we do not succeed in drawing….

T Yes, there is the problem of drawing, we will not be going back to that. Is this circuit possible to do? ”

(Many S.) Yes

We can see in this transcript that pupils manage to move away from the drawing to analyze the shape. plutôt un”.” que “;”

After a pupil has the drawing of his shape on one hand and follows the lines of the set of pieces to check is the shapes are the same (we will show a video of this moment).

Analysis of learning at stake

Firstly, the effects of the situation on pupils learning we can observe are:

1- A change in perception of the role of vocabulary:
   a. The familiar vocabulary for describing the known forms, is not effective for others. Many different shapes can look like an egg ...
   b. Since drawing on paper is not always successful students have to describe the shapes by analyzing the way they have been built: the arcs used and their sequence of use.
2- The progressive understanding of the role of sheet layouts to work on the lines (here curves, but straight in other situations)
3- A better knowledge of the circle, based on the development of procedures compared with other forms (Artigue & Robinet, 1982)
4- The practice of displacement to produce new solutions and to recognize identical figures arranged differently
5- Transition from a practical validation based on the superposition to a validation based on the analysis of the properties
6- The perseverance in research: the students have to rely on perseverance and go to the end of the task, and of course explain, justify, criticize, debate.

We think that, at the end of this situation, students are able to explain, with their words, several of their learnings.

Secondly, we have analyzed what are the difficulties, and what is possible to propose to pupils. We think, after several experiences, that the main obstacles are not epistemological; they do not come from the inability of students to analyze forms or working on their uncluttered representation. But it is rather didactic obstacles created by neglecting their knowledge and solving capacity. There is a necessary transition from a perceptive approach of the shapes to the analysis of their geometrical characteristics (size, composition, curvature…). In our progression of learning we also offer other situations which contribute to this passage from "global" (for instance, students perceive the regularity of form) to "analytic".

Thirdly we try to find condition of a real use in classrooms. For those of teachers who are not satisfied with their way of teaching geometry we think it is important to propose activities that are real problems (where pupils have to produce new strategies) and to clarify learnings at stake. But we think that it would be unrealistic for a teacher whose main objective would be to set a vocabulary to embrace the process. Potential learning is related to the possibility for the teacher to understand the issues: allow students to implement the different comparison procedures.

This workshop proposes to enlighten those issues.

In this presentation we have not detailed the many changes in the description of the situation, related to successive and necessary experiments for students to produce the specified procedures. On this aspect of the construction of a teaching situation, we simply discuss the question of the context: is it necessary for such young students to evoke a familiar context? The way quarter-circles or ellipses are drawn have also been choose different during the experiment: either simple lines (parts of the cited figures), double lines to evoke real-world objects: railway circuits. But does the latter choice provide a better understanding of the constraints (in particular the continuity of lines)? We currently believe that, on the contrary, this approach makes the problem harder to understand.

Conclusions and prospects

Let us return briefly to the issues addressed:

On learning targeted: what learnings can be developed based on the perception (regularity of a shape) to contribute to the analysis of geometrical properties? We mentioned a shift from a parts assembly problem to a drawing problem (a graphical problem); to that extent, the sensitive space has changed. First it is the space of action on the shapes to assemble, which has been among the first personal procedures of the students in the resolution phase. During the pooling, pupils focus on curves continuity and sensitive space become the place where graphical plots are questioned: they are new geometrical objects.

On taking initial knowledge of students into account: how knowledge, language, gesture, participate in the apprehension of the common elements to the diverse types of spaces?

How are first procedures and knowledge of the "graphical-space" combined with previous experience on objects? In particular, these prior knowledge is not primarily "declarative" but rely on gestures (eg the difference between rotate and reverse or the use of drawing with instruments without necessarily aiming to represent a geometric objects ...)

They are also expressed in the language forms, as part of a language used by the student to control his actions or to communicate about a production (formulation of a procedure, checking of constraints validation of a solution ...). The importance of this learning is often underestimated in education in favor of the use stereotyped and offline vocabulary.

Our research aims to analyze and develop, not only in this specific example, general student abilities via experience and actions on objects with graphical plots. Thus they evolve from a spatial perception to geometric characteristics of the shapes. Our concerns are also those of Swoboda (2015) about problem solving mainly with older students: “Therefore, the problem of bringing students to the ability of making mental transformations I treat as an educational task. In the literature, there is no explicit opinion on what educational level there is possible to create such skills. »

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Dialogic processes in collective geometric thinking: A case of defining and classifying quadrilaterals

Taro Fujita[1], Jonathan Doney[2] and Rupert Wegerif[3]

[1]Graduate School of Education, University of Exeter, UK; T.Fujita@exeter.ac.uk

[2]Graduate School of Education, University of Exeter, UK; J.Doney@exeter.ac.uk

[3]Graduate School of Education, University of Exeter, UK; R.B.Wegerif@exeter.ac.uk

In this paper, we report on how a group of UK 12-13-year-old students work with hierarchical defining and classifying quadrilaterals, an area which students find very difficult to understand. We implemented a geometry test to 9 groups of students. Quantitative data suggest that the students found very difficult to undertaking hierarchical defining and classifying quadrilaterals even in collaborative learning situations. Our qualitative video-recorded data from the four groups suggest that we find that even in collaborative learning settings prototypical images of geometrical figures strongly influence students' ways of hierarchically defining and classifying quadrilaterals. In addition, groups often had opportunities to examine their ideas, but they did not explore these opportunities because each member did not see what others were saying 'as if through the eyes of another'

Keywords: Dialogic, collective geometric thinking, defining and classifying quadrilaterals.

Introduction

Geometry has been recognised as of one of the most important topics in school mathematics as it provides students with learning opportunities for developing their spatial thinking, reasoning and sense making of this world. Sinclair et al (2016) reviewed over 200 research papers in geometry published since 2008, and identified six themes, including the understanding of the teaching and learning of definitions. They state that one of their research questions is about students' understanding of hierarchically defining and classifying shapes (p. 706). Our paper is concerned with this issue, in the context of hierarchically defining and classifying shapes in collaborative learning settings which can be productive ways to develop mathematical thinking and understanding (e.g. Martin and Towers, 2014). We chose this topic because the research has reported that students find the understanding of hierarchically defining and classifying shapes difficult, but how students tackle this in collaborative settings has not been sufficiently investigated.

The purpose of this paper is to examine the following research question 'What obstacles will be identified when students are working together with geometrical problems?' In order to answer this question, we first propose our theoretical framework for emergence of collective geometrical thinking in the context of hierarchical defining and classifying quadrilaterals. We then investigate students' collaborative learning process in defining and classifying quadrilaterals. In this paper, we focus on collaborative group work where teachers or instructors' interventions are minimal. Therefore, while we acknowledge that teachers' roles are highly important to support learners' mathematical thinking and understanding (e.g. Martin and Towers, 2014), we do not consider this issue in this paper.

Theoretical framework

Concept images and definitions and prototypical examples of geometrical figures

In order to study students' thinking with geometrical shapes, the terms 'concept image' and 'concept definition', introduced by Vinner and Hershkowitz (1980) are useful. A concept definition is defined as 'a form of words used to specify that concept' and concept image as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process' (ibid, p. 152). When considering a parallelogram, at least two types of concept images are considered, i.e. one is 'conceptual images' such as 'parallelograms have two sets of parallel lines', 'opposite sides of a parallelograms are equal', and the other is 'visual images' (e.g. ). In relation to learners' concept image and definition, another useful idea is the prototype phenomenon (Hershkowitz, 1990). This theoretical idea claims that students' difficulty in seeing geometrical shapes flexibly is caused by the prototype example, which students often encounter in their initial stages of learning of geometrical figures. For example, as a concept definition, parallelograms are introduced as 'a quadrilateral with two pairs of parallel sides', but a typical 'slanted' visual image is often used. This 'visual images' will stay strongly in students' minds, and as a result their 'conceptual images' become "the subset of examples that had the "longest" list of attributes – all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics." (ibids., p. 82). Thus, for many students, when the hierarchical relationships between quadrilaterals are required, they cannot accept that rectangles can be a member of parallelogram group as, on the one hand, rectangles have 90 degree angles, and on the other hand, parallelograms should be 'slanted' ('visual geometrical images') and not have such angles, and therefore rectangle are not member of parallelograms, stating 'rectangles have 90 degree angles and so they are not a member of parallelograms' as their 'conceptual image' (e.g. Fujita, 2012).

Collective geometrical thinking process

Collaborative learning has been recognised as a key topic in mathematics education research, and the difficulties in geometric thinking described above might be overcome if students' undertake problems collaboratively. Martin and Towers (2014) apply Pirie and Kieren's model (1994), which describes the growth of mathematical understanding with eight potential layers; Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventing. The learners' developmental paths from Primitive knowing to Investing would not be straightforward. For example, when an individual/a group of learners had difficulty in noticing properties during problem solving, they might examine their already made images, and as a result they re-make new images for exploring new paths for problem solving. This is what Pirie and Kieren call Folding back. Martin and Tower also suggest that this process is crucial in collective thinking process. For example, suppose a group of students are discussing whether a 'rectangle' can also be seen as a 'parallelogram'. In order to solve this (under a curriculum hierarchical relationships of geometrical shapes are assumed), they have to collectively make and have their conceptual and visual images of rectangle and parallelograms including their definitions, examine their properties collectively, and then formulate their reasoning etc. In this process, they might fail to collectively have useful conceptual and visual images of parallelograms and in this case they have to fold back to their collective image making stage in order to continue to examine this problem.

Dialogic process in collective geometrical thinking

The framework for collective thinking process by Martin and Towers (2014) discussed above can offer useful ways of analysing collective thinking process which collectively made or had conceptual and visual images, held back, noticed properties and so on, but this approach can be strengthened by considering some of the dialogic processes involved in thinking. For example, Mercer and Sams (2006) studied how certain types of talk, which mediate conceptual knowledge, affect students' ways of collective thinking and problem solving. They particularly consider that the roles of exploratory talk, described as being critical friends each other and using explicit reasoning during problem solving, showing how it is crucial for developing understanding, comparing to the other types of talk such as disputational (being competitive or disagreeing with each other in egoistic ways) or cumulative talk (agreeing each other without constructive criticisms). Extending this talk type approach further, Kazak, Wegerif and Fujita (2015) report that an 'Aha!' moment occurred after learners had engaged in productive 'dialogues'. 'Dialogues' to which we refer include more than exchanging recognisable utterances, but it is in a Bakhtinian sense, which Barwell (2016) recently described as "a theoretical idea that defines the nature of many aspects of the relationality of language." (p. 6). Our view is that such 'dialogues' elucidate differences and gaps, and encourage learners to see their learning from a different perspective, which is based on Bakhtinian dialogic theory (1963; 1984).

From this point of view, in addition to effective collaborative practice such as building effective collective conceptual and visual images of geometrical figures for problem solving, seeing a problem 'as if through the eyes of another' is important for emergence and development of collective group thinking and understanding. This includes, for example, recognising multiple 'voices' in mathematical concepts, seeing ideas from an 'outside' perspective, establishing dialogic space, learners' attitudes to each other, laughter, and so on. This is what Wegerif (2011) refers to as dialogic process of conceptual growth. Barwell (2016) also states, in the context of the development of the concept 'polygon': "the process of making sense of the word and the concept 'polygon' arises through the differences between the two groups of shapes on the blackboard, between the different ways of classifying shapes that preceded this moment, and so on." (p. 9). Let us take again the example whether a 'rectangle' can also be seen as a 'parallelogram'. Here, in their utterances students will use 'rectangle' or 'parallelogram', but they will contain 'multiple perspectives and agencies, i.e. rectangles for their own definitions and conceptual and visual images, for peers' definitions and images, for the formal definitions which appear in the textbook or for definitions used by teachers, and so on. In their talk they might agree or disagree with their thinking and if the group of students do not see a rectangle from an 'outside' perspective, they might not be able to reach mutual agreements or reasonable answers, or extend their discussions and apply other contexts such as 'is a square a type of parallelogram?', etc.

Methodology

The participants of our study were 27 Y7 students (12-13 year old) in a lower secondary school in South West England. Their abilities are recognised by their class teacher as the second highest group in the year group, meaning that their achievements are higher than the average students in the UK school context. They have also studied formal definitions of basic 2D shapes including parallelograms. The participates undertook the following tests in 2015-16, summarised in fig. 1.

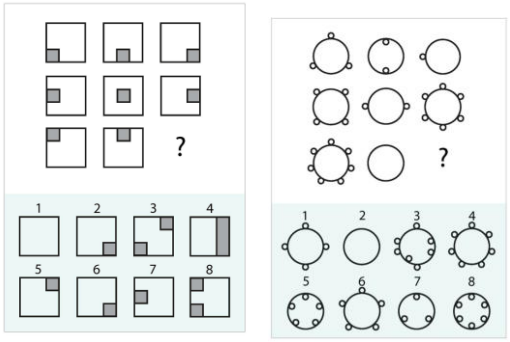
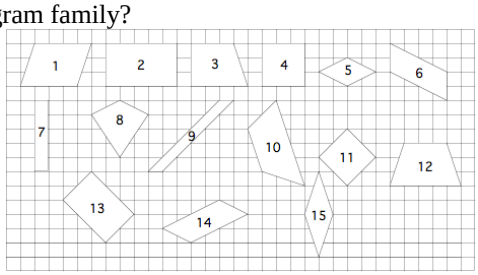
Group thinking measure test	Geometry group test
General group thinking test: We first identify groups' thinking by using the group thinking test which consists of two tests A and B of 15 graphical puzzles each, carefully matched to be of equal difficulty. We used this test as it was reported that the test provides useful insights into how groups work well in general. Students do one test working in groups of three and the other test working individually, assuming a measure of individual thinking correlated to a measure of group thinking with a measure of the difference between the individual scores and the group score (Wegerif et al, in press). In our study, the half of students in the chosen class individually undertook Test A, and then we formed groups of three in accordance with their test scores. Then each group undertook Test B. The other half did Test B as their individual and Test A as group test. All groups' work was video-recorded.  For each question, students have to choose which graphical image should fit into '?' based on patterns and properties of the other 8. For the left the answer is 5, and the right one it is 4 (by seeing 'outside' as addition and 'inside' as subtraction).	Geometrical thinking tests: The same groups of students undertook geometry test which are derived by Fujita (2012) about hierarchical relationships between parallelograms. 4 of 9 groups were video-recorded for further analysis, informed by their group thinking test performances. Students undertake the following questions related to inclusion relations between quadrilaterals. For each question, 3 points will be given if students' choice is based on the hierarchical classification, but if it is prototypical then 1 point will be given. For example, for Q1, 3 points for '1, 2, 4, 5, 6, 7, 9, 11, 13, 14, 15' but 1 point for '1, 6, 9, 14' or for Q3, 1 point for choosing (b) and (c) are correct. Q1. Which of the quadrilaterals 1-15 above are members of the Parallelogram family?  Q2. What is a parallelogram? Please write its definition. Q3. Read the following sentences carefully, and circle the statements which you think are correct. (a) There is a type of parallelogram which has right angles. (b) The lengths of the opposite sides of parallelograms are equal. (c) The diagonally opposite angles of parallelograms are equal. (d) There is a type of parallelogram which has 4 sides of equal length. (e) Some parallelograms have more than two lines of symmetry. Q4. Is it possible to draw a parallelogram whose four vertices are on the circumference of a circle?

Figure 1. Tests for group thinking and geometrical thinking

In the data analysis, we first examined general relationships between general group thinking test and geometry test, focusing on whether geometrical thinking can be predicted from group thinking test performances. We then analysed the video data by considering what types of talk (disputational, cumulative, or explorative) can be recognised in their group work, what kind of 'voice' can be recognised in their collective Image Having/Making, Property Noticing and Folding Back processes (see the next section for the examples).

Findings and analysis

Overall performance in their group thinking test and collective geometric thinking

The test results for our sample of 27 students are as follows:

Test	Mean	S.D.
Group thinking test (Individual, N=27)	9.3 (out of 15)	2.07
Group thinking test (Group, N=9)	10.4 (out of 15)	1.24
Geometry test (N=9)	3.67 (out of 12)	1.66

There is no statistical significant difference between individual and group scores in the Group thinking tests (Mann-Whitney U test, p-value is 0.1063, $p > .05$), indicating in this class in general collaborative learning did not positively affect test scores. Also, low scores from geometry tests indicate that the students' collective geometric thinking are also governed by prototypical examples of parallelograms, despite being given opportunities to share their ideas and to work collaboratively to solve the geometry test. Furthermore, statistical analysis of the data, using linear regression modelling, showed that the ability to predict geometry test scores from individual thinking scores, and group maths test scores was very weak (R^2 of 0.046). Likewise, the relationship between individual group thinking scores and geometry test scores was very weak (Spearman Correlation 0.224). This might suggest that collective geometric thinking as 'measured' by the geometry test is different from the thinking ‘measured’ by the group thinking test at least in our sample (we will explain relationships between general group thinking and mathematical thinking in more detail in our presentation.)

Examples of students’ collective thinking process

Although quantitative analysis did not suggest strong relationships between general group thinking and collective geometric thinking, the video data suggest some interesting features relating to why students could not do well in the geometry test in their collaborative work. In total 340 interactions from students were examined in terms of stages of collective thinking process and dialogic theory. In this section we select examples from Group 1 and 5, whose obstacles were particularly related to not only their conceptual and visual images of quadrilaterals but also their dialogic relationships in their collaborative learning.

In the individual test, the three students BS, AC, and JC in Group 1 scored 14, 10, 11 but their group score was 12. This means their group work did not benefit very positively (in the context of group thinking measure test). In their geometry test, their interactions were rather disputational and they could not see their peers’ ideas from the others’ point of view in addition to influences from prototypical examples. For example, in their collective Image Making/Having stage, they discussed what a parallelogram was conceptually and visually, one of them questioned if rectangle or square can be a parallelograms based on the statement voiced by BS (line G1 49), but immediately after AC said “And a square and a rectangle. It’s trash”. This indicate in the line 54, the word ‘square’ or ‘rectangle’ by AC were very personal, and not accepting the ‘voice’ by BS or JC:

G1 47. JC What is a parallelogram? Write the definition.

G1 48. AC A squashed up rectangle.

G1 49. JC No both sides are parallel.
G1 50. AC A squashed up rectangle.
G1 51. BS So that would mean thirteen as well and two.
G1 52. AC: And a square.
G1 53. BS: And one and…
G1 54. AC: And a square and a rectangle. It's trash.

They then continued their discussion, and it is evident that their understanding is influenced by the prototypical image of parallelogram (line G1 59, G1 60 or G1 70). In addition, it seemed that they could not see each other's positions. In the line 61, AC aggressively said 'That's what we got told in…', referring to authoritative voices. In the line 64, BS again held back to a definition "all the sides are parallel" and suggested rectangle can be a parallelogram (line G1 67). There was a dialogic gap between BS and AC/JC. However, JC and AC again referred to a (wrong) definition based on the prototypical image (line G1 68 and G1 69), and BS's voice was dismissed, and BS disappointingly said 'Oh no', and their collaborative explorations stopped.

G1 58. AC I think it's…
G1 59. BS A squashed up rectangle.
G1 60. JC No if it's a squashed up I need to know squeeze it.
G1 61. AC That's what we got told in…
G1 62. BS All sides are the same.
G1 63. AC No they're not.
G1 64. BS No, no, all the sides are parallel.
G1 65. JC Yeah.
G1 66. AC Yes so is a square.
G1 67. BS So that will do one, two (pointing a rectangle image).
G1 68. JC No because that's a quadrilateral not a parallelogram. A parallelograms are like…
G1 69. AC A squashed up rectangle.
G1 70. JC No parallelograms are like that they're like that they're messed up.
G1 71. BS Oh no.

Let us see another group, Group 5. In the individual test, the three students JM, BH, and TF in Group 5 scored 9, 10, 7, but their group score was 11. This means their group work did not benefit either positively or negatively. In their collective Image Making stage of the geometry test, JM first voiced his own image and definition (line G5 12) which was influenced by the prototypical image and then TF agreed. Then BH added 'Two pairs of parallel sides" (line G5 14). This made JM question "a rectangle has two pairs of parallel sides as well?" (line G8 17), but after a moment he said "But it (parallelogram) doesn't have right angles" (line G5 18), indicating he could not see BH's point of view. TF then agreed with JM. BH did not argue back from here (a kind of cumulative talk), and they now had parallelogram as 'a rectangle without 90 degree angles' as their collective image of parallelogram.

G5 7. JM …Ok what is a parallelogram?
G5 8. BH Oh.
G5 9. JM It's rectangle but…

G5 10. BH It's like…
G5 11. TF Erm like…
G5 12. JM It's like, it's a rectangle but it doesn't, we're not, it doesn't have all ninety degrees. It doesn't have all right angles.
G5 13. TF Yeah, yeah yes so it's a rectangle but it doesn't have…
G5 14. BH Two pairs of parallel sides.
G5 15. JM It's a rectangle.
G5 16. BH Two pairs of parallel sides.
G5 17. JM Yeah but that erm that a rectangle has two pairs of parallel sides as well.
G5 18. JM … (a moment) But it doesn't have right angles so it's rectangle without …
G5 19. TF Ninety degree angles.

After this, this shared definition used throughout the problem solving process in their Collective Property Noticing stage, resulting they only chose (b) and (c) of Q3 as true or in Q4 they formulated it would be impossible to draw a parallelogram whose four vertices are on the circumference of a circle because "the obtuse angles would not touch the circumference of the circle" (G5 line 56). The other groups (Group 2 and 8) also showed similar processes, i.e. definitions based on prototypical images were collectively made and had uncritically at first and then these were used to examine properties and formulate their answers.

Discussion

In this paper we examined what obstacles will be identified when students are working together with geometrical problems. By answering our research question, our findings suggest that even collaborative learning settings prototypical images (Hershkowitz, 1990; Fujita, 2012) strongly influence when students were making/having conceptual and visual images of geometrical figures collaboratively, i.e. collective Image Making and Having stages (Pirie and Kieran, 1994; Martin and Towers, 2014). Also, when learners collectively had definitions based on prototypical images and missed opportunities to dialogically examine these (Bahktin, 1964; Wegerif, 2011; Barwell, 2016). Even if they shared ideas during their problem solving processes well, these students could not reach the correct answers by examining different ideas voiced in their collaborative work (Barwell, 2016). It is interesting to see that groups often had opportunities to examine their collective definitions (e.g. line G1 67 or G5 18-19), but they did not explore these opportunities because each member did not see what others were saying 'as if through the eyes of another' (e.g. line G1 67-71 or G5 16-19). Thus, in conclusion, in addition to prototype phenomenon, in collaborative learning settings it is necessary for students to dialogically examine their starting points of problem solving (in this case the definition of parallelogram).

In our research context, we did not find strong relationships between general group thinking and geometric thinking. As the sample size is relatively small, we would like to pursue this topic in our future research, together with developing effective pedagogical models for better collective geometric thinking.

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Double perspective taking processes of primary children – adoption and application of a psychological instrument

Cathleen Heil

Leuphana University, Lüneburg, Germany; cathleen.heil@leuphana.de

Perspective taking can be conceptualized in the framework of mental transformations in terms of subsequent egocentric transformations. Kozhevnikov and Hegarty's (2001) PTSOT is a test instrument for adults that investigates double perspective taking processes. Both perspective taking processes can be defined by certain egocentric transformations. An adoption of this test for primary children reveals that they are able to understand the test and verbalize easily their thinking processes. 8 items were solved by 254 fourth graders. Results show a variety of typical difficulties that can be interpreted in detail using the egocentric transformations framework. The theoretical framework and the straightforward way of item construction allow us to systematically generate items for various applications in psychology and mathematics education.

Keywords: Egocentric transformations, geometric thinking, perspective taking, spatial abilities.

Theoretical background

Spatial abilities have widely been debated within mathematics education, with a multitude of meanings and definitions (Mulligan, 2015). Discussions within WG4 in CERME highlight implicitly the implication of those abilities on all four geometrical competencies to support geometrical thinking that were proposed by Manschietto et al. (2013).

Psychometric studies on individual differences have shown that the construct of spatial ability is multidimensional and consists of several spatial ability factors (e.g,, McGee, 1979; Linn & Petersen, 1985). Although the exact factor structure of spatial abilities remains a subject of intensive debate, some of these factors were studied in detail, both for adults and children. One of these factors, spatial orientation, also denoted as perspective taking in the literature, was proposed to measure the ability to imagine the appearance of a set of objects from different orientations (perspectives) of the observer (McGee, 1979).

In the experimental cognitive literature, spatial abilities have been conceptualized as the ability to engage different mental transformations that require the subject to update an encoded visual stimuli with respect to three different frames of reference: the intrinsic reference frame of objects, which encodes relations among objects, the egocentric frame of reference, which encodes object locations with respect to one's body, and the environmental frame of reference (Huttenlocher & Presson, 1973; Zacks et al., 2000). Perspective taking ability has been defined as the ability to perform a set of egocentric transformations on objects, in which the relationship between the environmental coordinate frame and those of the objects remain fixed, while each of their relationships with the observer's egocentric reference frame are updated. Although the conceptualization of perspective taking within the theoretical framework of mental transformations is logically equivalent to the qualitative description of the factor spatial orientation, it adds value for the deconstruction of complex perspective taking into separable, analyzable transformation processes.

Perspective taking abilities have been studied extensively in the developmental psychology literature (see Newcombe, 1989, for a full review). It has been highlighted that children demonstrate first perspective taking abilities even in infancy, show first achievements on more advanced tasks at around 4 or 5 years and improve performance considerably between the age of 6 and 8 (Frick et al., 2014). Complex perspective taking tasks that involve typically conflicting frames of references such as Piaget and Inhelder's (1956) *Three Mountains Task* are not fully mastered until the end of primary school. Although the involvement of conflicting frames of reference in perspective taking tasks, e.g. the involvement of a to-be-imagined frame of reference that conflicts with the child's direct relation to the visual stimulus, has received criticism (see Huttenlocher & Presson, 1973, for a detailed discussion), it is valuable from a spatial cognition point of view. Tasks with conflicting frames of reference are of high interest, because they demand for the ability to represent, maintain and coordinate multiple frames of references within one coherent spatial framework. This ability is meaningful and predictive for a whole range of everyday life spatial abilities, such as misaligned map reading (Lobben, 2004), environmental learning and wayfinding (Allen, 1999; Hegarty et al., 2006). A deeper understanding of perspective taking processes involving conflicting frames of references allows us therefore to discuss spatial abilities of primary children in a much broader context. This is consistent with the mathematics curriculum but also curricula of applied sciences and geography.

A qualitative re-analysis of typical markers of perspective taking ability such as the Guilford and Zimmerman (1948) Spatial Orientation Test for adults turned out not to be construct valid as being solved mostly by mental rotation strategies (Barratt, 1953). Kozhevnikov and Hegarty (2001) proposed a novel, psychometric paper and pencil perspective taking/spatial orientation test (PTSOT) for adults in order to overcome the drawbacks of the Guilford-Zimmermann task. The test instrument consists of 12 items that display an arrow of seven 2D-objects. On each item, the participant is asked to imagine being at the position of one object (anchor point), facing another object (defining the imagined perspective within the array) and is asked to indicate the direction to a third object (target). Item formulation stimulates therefore complex perspective taking processes with conflicting frames of references. The answer is noted on an "arrow circle" (see example item in Figure 1, left side, item adopted for better readability). Participants are neither allowed to rotate physically the object array nor the "arrow circle".

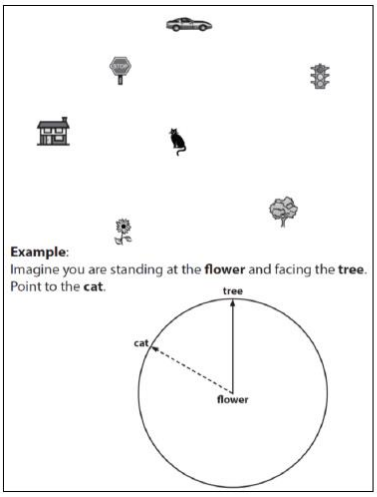
The PTSOT has been shown to be construct valid by the authors themselves, involving mostly self-reported perspective taking processes in adults (Kozehevnikov & Hegarty, 2001). Due to its accepted validity the test has been used to underline perspective taking abilities to be predictive for environmental layout learning in real world and virtual contexts (Hegarty & Waller, 2004).

The present study aimed to address the development of an instrument that stimulates complex perspective taking processes with conflicting frames of reference. One goal of the study was an adoption of the original PTSOT for primary children. A second goal was to describe the instrument with respect to test characteristics and typical error patterns that are caused by problems or failure in a set of mental transformations that are necessary to solve the items. Finally, an overall goal was to conclude potential applications of the instrument within the field of psychology and mathematics education.

Adoption and item construction

Design of an adopted instrument

The adoption process was conducted throughout a qualitative study with 25 fourth graders in a bachelor thesis project. We adopted the PTSOT with respect to 12 design parameters that are listed in Figure 1. We will elaborate in detail on the literature background in a following publication.



PTSOT	Design parameter	Adopted instrument
Investigation of perspective taking abilities	Goal of the instrument	Investigation & stimulation of perspective taking
Adults	Target group	Children
In Groups	Test procedure	Individually
Speed Test (5min)	Format	Power Test
2D	Dimensionality	3D
concrete, practical	Object specification	Concrete, practical
Directed & undirected	Object directionality	Exclusively directed
Seven	Number of objects	Six
unsorted	Object arrangement	unsorted
Written	Item Formulation	Verbally
"Arrow Circle"	Documentation of the solution	Circular Disk with 12 numbered sections
Angle difference	Scoring	Dichotomous (0-1)

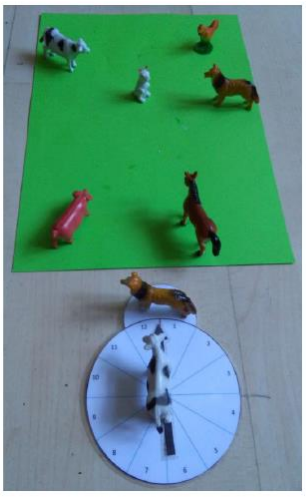


Figure 1: Comparison between the PTSOT and the adopted instrument with respect to 12 design parameters. One item of the PTSOT is shown on the left, the whole adopted instrument is shown from a quasi-bird perspective on the right side, showing the object field and the solution answering disk

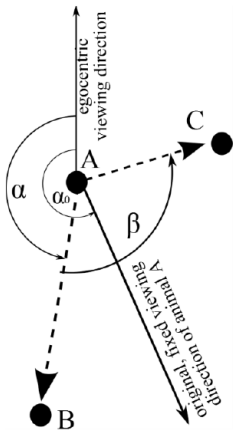
The adopted instrument consists of a 3D, small scale array of six farm animals that are placed on a green sheet of paper (the "meadow"). The child is sitting in front of the fixed array of objects, taking an oblique view on the whole scene. Just in front of the child there is a fixed, circular disk with 12 numbered sections and a mobile arrow on it. Animals can be stuck on the disk using glue dots. Verbal item formulation is standardized as following:

Tutor: "Imagine that you are animal A (sticks animal A in the middle on the arrow of the disk) on the meadow and are facing animal B (places animal B on on the semicircle attached to the disk). In which direction do you have to turn in order to see animal C?"

The child turns the arrow of the answering disk at the section which corresponds to the right direction and the tutor notes the answer. During the solution process, the child is allowed to gesture but not to turn the array, the answering disk or itself.

Item construction & framework for item analysis

We constructed eight initial items with the help of two parameters that describe two perspective taking processes within one item. In Figure 2 you can see that each item is defined by a set of four subsequent mental transformations of the egocentric frame of reference, which – pairwise – define one perspective taking process within the item.



Process	Item construction parameters	Defining mental transformations
1	**Shift of egocentric viewing direction - angle *α*** *"Imagine that you are animal A and you are facing animal B…"*	$\mathcal{T}_1$ transformation of the egocentric frame of reference from an opaque view into the field view of animal A (adoption of viewing angle α0) $\mathcal{T}_2$ rotational transformation of the egocentric frame of reference by α-α0 towards animal B (adoption of viewing angle α)
2	**Localization angle *β*** *„…in which direction do you have to turn in order to see animal C?"*	$\mathcal{T}_3$ rotational transformation of the egocentric frame of reference by β towards animal C (adoption of viewing angle α+ β) $\mathcal{T}_4$ transformation of the egocentric frame of reference from the resulting field view to an egocentric opaque view

Figure 2: Item analysis for an arbitrary item showing the two mental viewing directions (dashed lines), the original viewing direction of animal A (black line), the egocentric viewing direction of the observer as well as the item construction parameters α and β

Exploratory Studies

We studied the range of strategies that children use to solve different items in a qualitative interview study with 16 fourth graders in the context of a second bachelor thesis. Interviews served as an aid for the interpretation of results of the exploratory study as well as verification of the goal of the instrument.

In a main study, we performed the eight items of our adopted test instrument with 254 fourth graders (mean age was 9.17 years; 116 boys and 138 girls) out of 11 classes in Lüneburg; forming a heterogeneous sample in terms of scholar achievement and social background. The test was administered in a separate room in a 1:1 situation with the experimenter. We documented children's solutions, but we did not film the children.

Results and discussion

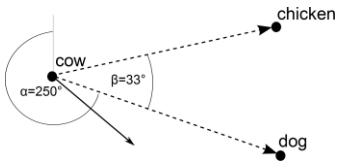
Test theoretical considerations

In a first approach, Item Response Theory (IRT) analysis, we scaled the data using a Rasch-model[1] in *Conquest*. The characteristics prove that data fit well with dichotomous data from the exploratory study with a MNSQ within 0.95 and 1.07 for all items. EAP reliability is poor, 0.456, yet might be influenced by the small number of items. Discrimination values show poor discrimination (0.24 and 0.33) for two items and acceptable (0.42- 0.56) discrimination for the other items. Item difficulties are between -0.52 and 2.8 (0 being medium difficulty), yet showing a tendency towards a selection of very difficult items. We conclude that from a test theoretical point of view, our selection of eight items is still far away from been applied as a psychometric measure of perspective taking. However, a first analysis pointed out a set of items with good characteristic values.

Quantitative results

[1] Rasch-models are one specific class of measurement models in IRT in which latent trait estimates depend on both persons' responses and the properties of the item (difficulty, discrimination).

In a second analysis, we interpreted the number of answers per section in the answering disk within the mental transformation framework that was presented in Figure 2.



(Item 6) *"Imagine that you are the cow and you are facing the dog. In which direction do you have to turn in order to see the chicken?"*

Item 6 is characterized by $\alpha=250°$, thus demanding the child to rotate its egocentric frame of reference by more than 90° to the right while indicating the direction of the chicken ($\beta=33°$), thus asking for a clear left/right decision at this point of view. IRT analysis showed that the item is difficult (1.06) and has an acceptable discrimination value (0.55).

We performed the analysis of item solutions within our mental transformation framework. Errors were classified into *"problems"* (task is basically understood but there are a few inaccuracies within one transformation), *"failure"* (one transformation is not performed at all, but the item is solved within the general item structures) and *"neglect"* (the goal of the item is changed due to a misunderstanding/heavy problem with one of the two perspective taking processes) in order to stress the amount of difficulty that a child showed during the solution of an item.

A detailed analysis in Figure 3 demonstrates the depth in which item solutions might be interpreted with the mental transformation framework. Figure 3 shows typical error patterns, such as

- neglect of the first perspective taking process, thus solving the item from a fixed egocentric viewing direction
- failure at the last transformation $\mathcal{T}_4$, thus having left-right problems
- failure at $\mathcal{T}_2$, the item is thus solved by taking the initial, fixed heading of the first animal (cow) and the child fails to shift the viewing direction from α_0 to α
- problems with estimation of the angle β in $\mathcal{T}_3$
- neglect of the first perspective taking process and projection of the egocentric viewing direction on the first animal (cow), thus pointing towards the relative position of the dog

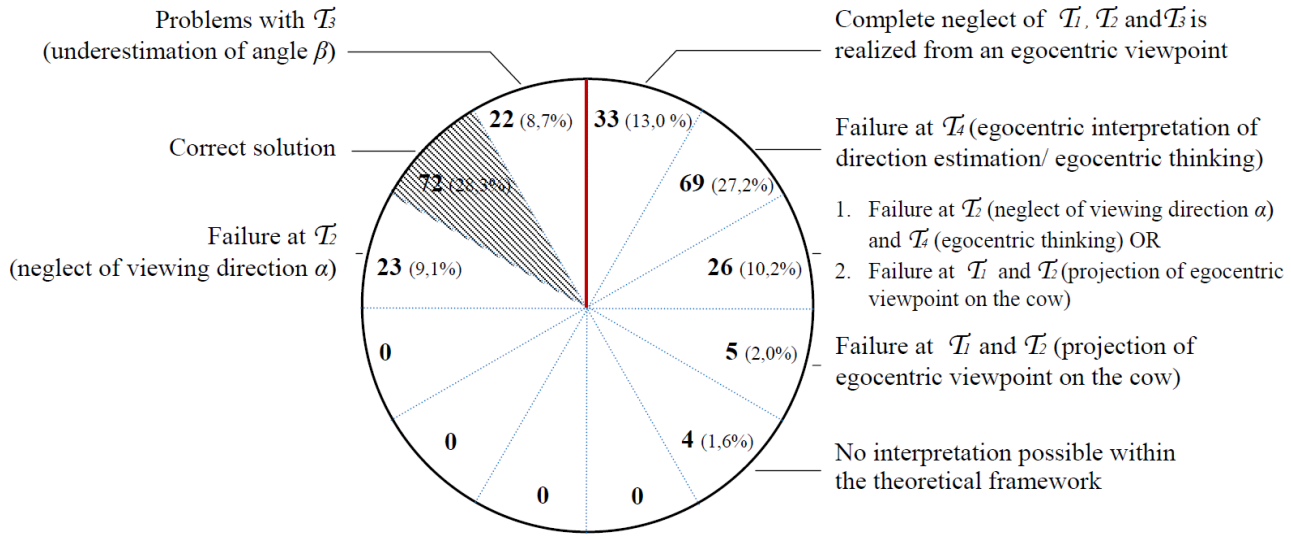
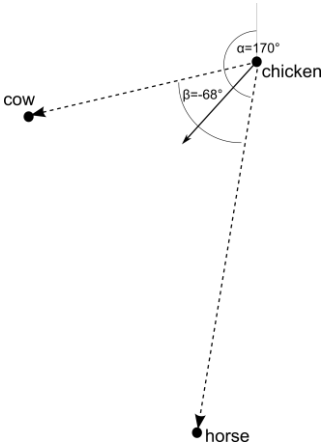


Figure 3: Analysis of solutions for item 6.

(Item 8) *"Imagine that you are the chicken and you are facing the horse. In which direction do you have to turn in order to see the cow?"*



Item 8 is characterized by $\alpha=170°$, thus demanding the child to transform its egocentric frame of reference by almost 180° while indicating the direction of the cow ($\beta=68°$), thus asking for clear left/right decision at this point of view. IRT analysis showed that the item is very difficult (2.48) and has an acceptable discrimination value (0.53). The high difficulty of this item results from the need to take almost an opposite perspective while indicating to the front right.

For the analysis we expected therefore a large percentage of children to fail at the last transformation $\mathcal{T}_4$ (thus producing right-left-errors) as well as a high number of children to neglect the first perspective taking process, thus solving the item from an egocentric viewpoint. Figure 4 demonstrates that the last item shows a whole range of typical difficulties.

However, although the solution rate is for this item was low, many children managed to perform most of the transition processes correctly. Almost 77% of the children succeeded on performing at least $\mathcal{T}_1$, 47% succeeded in doing at least $\mathcal{T}_1$ and $\mathcal{T}_2$ correctly, 30% managed $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ and almost 10% managed to do all the transformations correctly. Figure 4 shows that problems with the first perspective taking process may occur due to egocentric behavior, the failure to perform $\mathcal{T}_2$ or due to the projection of the egocentric viewing direction on the chicken. We explain errors in the first perspective taking process by difficulties that are inherent to children at this age (see Huttenlocher & Presson, 1973) but also by problems in understanding the item formulation and the item structure itself. The latter might be improved by doing multiple examples with the children (we explained the item structure with only one example).

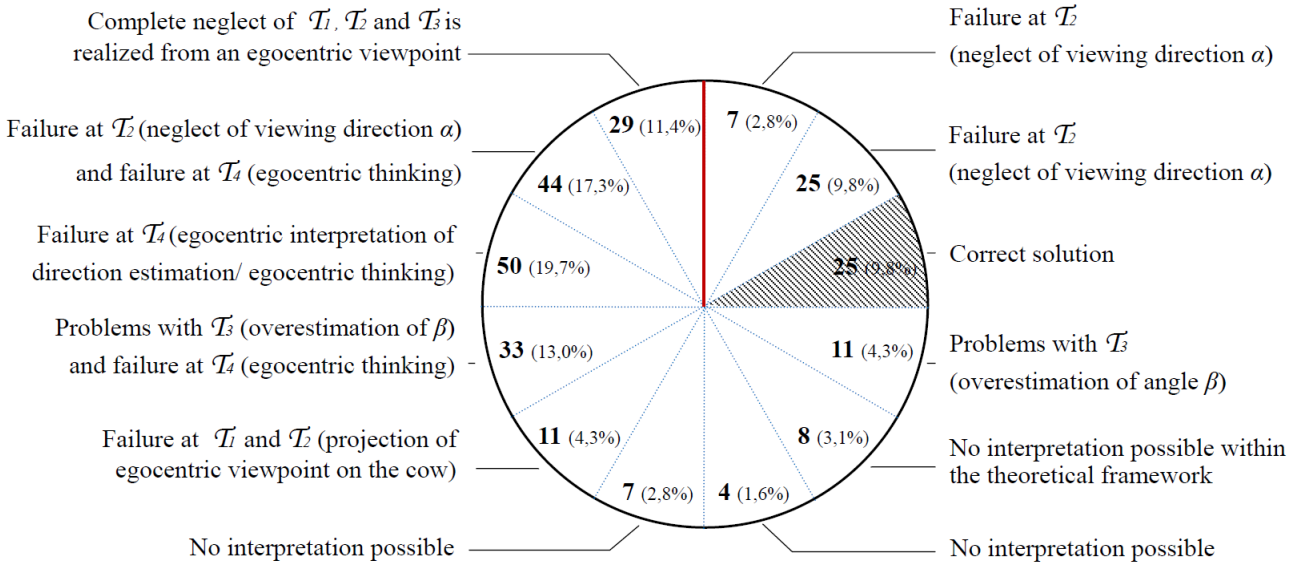


Figure 4: Detailed analysis of item 8, revealing problems in the $\mathcal{T}_4$ transformation process. Answer patterns that are not interpretable within our framework might be explained by counting in the field of animals or arbitrarily guessing the answer.

Applications of the instrument in different contexts

We analyzed performance on our adopted version of the perspective taking test at two different levels in order to underpin the argumentation on possible applications of the instrument.

Diagnostics

We demonstrated that item construction is straight forward using the construction parameters α and β. Our instrument allows therefore purposeful item construction in order to investigate individual differences and developmental issues in complex double perspective taking abilities. Children verbalized easily their thinking processes in our qualitative study, using gesture for showing viewing directions and the direction of the third animal. A combination of item solution and explaining aloud the solution process might help to diagnose the transformation processes that are still problematic for each child.

During construction and evaluation process, we identified two problems with our instrument that should be considered during item formulation: First, we measured angles from head to head of each animal. As the animals are quite large, the correct estimation of the angle β might depend on whether the child focuses on the head or the tail of each animal. Second, as the rabbit was placed at the center of the animal array, children had problems with taking the viewing direction of the rabbit. Instead, they projected their egocentric viewing direction onto the rabbit and answered items as giving relative positions of animal C to the rabbit.

Learning environment

Our adopted instrument consists of easily purchasable, inexpensive material. Again, our study showed that children are able to verbalize their spatial thinking processes with ease, using a whole range of gestures. Our instrument might be used for teaching of complex perspective taking processes (children solve pre-formulated items by the teacher), in communicative settings (children formulate items on their own), in discussions that address the transformation processes explicitly ("Can you explain me why this item is so complicated?"), or in creative settings (formulation of own items within constraints, e.g. difficulty, or re-configuration of all animals on the meadow).

Psychological test instrument

The original PTSOT has gained much attention concerning psychometric measurement of perspective taking ability because it is construct valid and reveals the predictive nature of perspective taking abilities for environmental learning. Another wishful application of our adopted version of the PTSOT is therefore in psychometrical measurement of children's perspective taking abilities. IRT analysis revealed a poor reliability and pointed out some inappropriate items with low discrimination values. In a further study, an exploratory analysis on a larger set of items of intentionally different difficulties is planned. IRT analysis might then reveal good items for a psychometric test of spatial ability in children. An IRT analysis of the test instrument might then be linked to our analysis technique based on egocentric transformations in order to develop a typology of complex perspective taking in children.

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Traveling in spatiality, in spatial sense

Catherine Houdement

Université Rouen Normandie, Laboratoire de Didactique André Revuz (EA4434), Rouen, France

catherine.houdement@univ-rouen.fr

What attention is given to spatial sense in Geometry? The outcome (2015) of a special ZDM issue, Geometry in the Primary School *and the CERME conference is a good opportunity to think about and compare different approaches or frameworks regarding this topic.*

Keywords: Spatial sense, spatial knowledge, geometrical thinking.

Introduction

A travel abroad enriches our way of thinking and furthers our understanding of the values and behaviours of his/her own culture. For us just the same an intellectual trip outside the country (or the language) helps deepen our knowledge and analysis of our way of thinking. The study initially aims to compare the perspectives of educational researchers regarding the notion of *spatial sense.* Along the way, it documents the conceptual frameworks which support research.

In ZDM issue 47, *Geometry in the Primary School,* the theme of *spatial sense* is common to all papers (Mulligan, 2015). Could it be that the study involves primary school or that the imagery in thought is a topical subject in recent neuroscience research? In the 36 papers of the ICMI Study *Perspectives on Teaching Geometry for the 21st Century* (1998) the word *spatial* was mentioned 353 times but never in a title. Our ongoing study is based on (not yet completely extensive) reading of ZDM 47 and CERME texts on Geometry since the first time (2003) that a specific group about Geometry exists. Such a group has persisted for all the CERME under the name *Geometrical Thinking* (except 2011 *Geometrical teaching and learning*).

In short, this paper provides an overview on spatial sense the difference theoretical frameworks regarding it, based on literature published in ZDM issue 47 and CERME papers of the last decade (2003-2015).

First definitions and motivations for studies about spatial sense

Many expressions (*spatial reasoning, spatial sense…..*) are related to *spatial* when considering English papers from cross- and interdisciplinary fields of mathematics education, psychology, child development and neuroscience: In a first approach we consider these expressions as equivalent and agree the large definition quoted by Mulligan (2015), from Spatial Reasoning Group (2015)

"*Spatial reasoning* (or *spatial ability*, *spatial intelligence*, or *spatiality*) refers to the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects. Examples of spatial reasoning include: locating, orienting, decomposing / recomposing, balancing, diagramming, symmetry, navigating, comparing, scaling, and visualizing." (Mulligan, 2015, p. 513).

It should be noted that this definition considers *spatial* in a broader sense than 3D- situations or 3D-geometrical thinking. **In this paper we are interested in *spatial skills*, which are not associated with objects 3D**.

Many researchers stress the utility of *spatial reasoning* for mathematical learning and problem solving (Owens & Outhred, 2006; Sinclair & Bruce, 2015), and to Science, Technology, Engineering and Mathematics Education (Mulligan, 2015). Tahta (1980) and others note the ability to mobilize wisely spatial skills in mathematical and scientific thought, which Battista (1999, quoted by Mulligan, 2015, p. 514) named *spatial structuring* and defined as: "the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new objects" (Battista, 1999).

Spatial structuring is also an important component of the early learning of numbers, of measurement units, and well of geometrical thought (Van den Heuvel-Panhuizen et al., 2015; Mulligan, 2015). Van den Heuvel-Panhuizen et al. (2015, p. 347) outline the strong relationship between spatial and mathematical abilities. Mathematical performance and spatial abilities are positively correlated, not only in mathematical domains that are ostensibly spatial.

French scholars as Berthelot & Salin – with a long story of research on geometrical education – agree on the place of *spatial abilities* in geometry learning, also for geometrical proof. But they have always emphasised (since the 1990s) their interest for everyday life: "the treatment of spatial abilities is the main source not only of many of the further learning difficulties met by secondary school pupils, but also of some of the main deficiencies in spatial representation needs in everyday life" (Berthelot & Salin, 1998, p. 71). This point of view has encouraged them to define and elaborate, unconstrained by geometry (or dissociated by classical geometry), what they called *spatial knowledge*.

In conclusion it appears that spatial abilities are critical for learning mathematics and beyond. As these abilities are naturally associated with geometry, geometry seems the ideal niche for their teaching. It could even be read between the lines that it would be, nowadays, the main reason for teaching geometry in compulsory school education. Indeed, the present trend in some countries is to marginalize geometry in curricula in favor of probability and statistics (Maschietto et al., 2013).
Obviously *spatial reasoning* nourishes *geometric reasoning* (Mithalal-Le Doze, 2015), but *geometric reasoning* needs other abilities, like defining and classifying (in the sense of Brunheira & Ponte, 2015), axiomatic reasoning, and doesn't take in account non-mathematical forms of deduction... Furthermore spatial problems, like finding one's way in an unknown town, cannot be assimilated to geometrical tasks. Studying *spatial reasoning* for itself seems to be interesting.

Spatial sense in CERME papers (2003 to 2015)

Let us first examine how the successive CERME (2003 to 2015) working groups named *Geometrical Thinking* deal with spatial sense.

The Group *Geometrical Thinking* worked within the continuity in the CERME3, 4 and 5 (2003, 2005, 2007). In CERME5 the topic *Spatial abilities and Geometrical tasks* was considered; it is

noted: if it seems possible to agree about Geometrical tasks, it is necessary to precise what *spatial abilities* mean. We try to do it in the next section.

In CERME6 (2009) *spatial abilities* is not a specific topic of the Group even though it could be present in the sub-theme *Teaching, thinking and learning 3D Geometry*.

In CERME7 (2011) *spatial abilities* is connected to *diagrammatic reasoning.* Deliyanni et al. (2011) explore spatial abilities in relation with 2D-geometrical figure understanding and consider the influence on reasoning of the different diagram's apprehensions (Duval, 1995): perceptual, sequential and operative. Braconne-Michoux (2011) proposes to intertwine the Geometrical Paradigms (Houdement & Kuzniak, 2003; Houdement, 2005) and the Van Hiele levels which integrates visualization, a spatial ability.

In CERME8 (2013) the Group introduction proposed four competencies (see Figure 1) to support geometrical thinking: reasoning, figural, operational and visual (Maschietto et al., 2013) and assume that the links between these competencies are more important for geometrical work. *Spatial abiliies, spatial sense* are not explicitly mentioned but it seems (see above in Mulligan, 2015) to "have a place" in each competency. Two papers assume a *spatial* entry (other than 3D activities): in Sevil & Aslan-Tutak (2013) and particularly de Freitas & Mc Carthy (2013) emerges a new face of *spatial abilities*, the gestural / haptic ones.

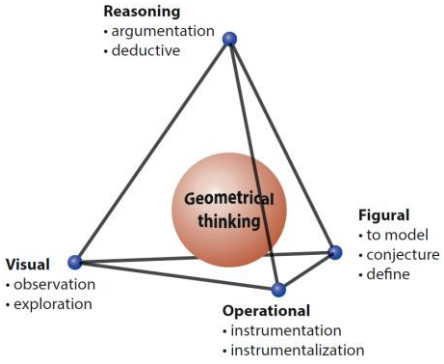

Figure 1: The geometrical competencies (Maschietto et al., 2013)

CERME9 (2015) in the continuity of CERME8 is supported by the same model (Figure 1). The topic quoted in relation with spatial abilities is *visualization.* The authors (Ceretkova et al., 2015) stress the influence of geometric knowledge on visualization, beyond perceptive and psychological aspects.

How do educational researches deal with the issue of *spatial abilities*? More precisely what kind of theoretical frames do they use or construct for their research?

Theoretical frameworks for spatial abilities

Specifying the frame the authors use in order to analyse spatial abilities is quite rare: only once in ZDM, only a few times in CERME papers before CERME9. Let us give some examples of such frames.

Example 1

In Panaoura et al. (2007) *spatial abilities* are commonly addressed by three major dimensions s**patial visualization, spatial orientation and spatial relations.** These researchers use an analysis model for *spatial abilities* (Demetriou & Kyriakides, 2006) with three components, namely **image manipulation, mental rotation and coordination of perspectives** to investigate whether or not and to what extent primary and secondary school students' spatial abilities are related to their performance on geometry tasks involving 2D figures, 3D figures, or nets of solids.

Example 2

In Berthelot *&* Salin's research (quoted in Douaire & Emprin, 2015) *spatial knowledge* is knowledge which enables to control one's relations to the surrounding space, the sensible world. This control may consist in recognizing, describing, manufacturing or transforming objects; moving, finding, communicating objects' position; recognizing, describing, constructing or transforming a route (Berthelot & Salin, 1999, p. 38). Children begin to integrate *spatial knowledge* before going to school while experimenting, and sharing with adults about their actions. Spatial knowledge cannot be reduced to geometrical knowledge but can be necessary to solve a geometrical problem.

It should be noted that this definition relies on problems (what Berthelot & Salin name spatial problems). In Brousseau's theory (the theoretical frame of Berthelot & Salin's research) knowledge is what enables to solve problems, and problems solving is a condition for learning. For instance how to define knowledge to be taught to use efficiently a map when lost in an unknown town? First identifying situations in which using plans and maps are necessary; second analysing the spatial interactions to solve them and thus indentifying the necessary knowledge.

Daily life interactions take place in space of different sizes which exert different constraints on the actions. *Microspace* is very close to the subject, like a sheet of paper, a computer screen, a touch screen; in this space objects can be moved, touched, turned; it corresponds to the usual grip relations. *Mesospace* is the surrounding space, inside a room, a building; the subject can move inside it, mesospace is the space of usual domestic spatial interactions. *Macrospace* is the broader space, unknown city, rural or maritime spaces; the subject has only local views, he had to conceptualize (Berthelot & Salin, 1998, p. 72; Douaire & Emprin, 2015, p. 532). Thus s*patial knowledge* is structured into three main conceptions, microspatial conception, mesospatial conception, macrospatial conception. For instance, following this frame, a straight line can be conceived as a print trace produced with a ruler, the edge of a door, or a set of trees properly aligned in a orchard.

This framework has different functions: in Berthelot & Salin, as in Douaire & Emprin (2015) the framework allows them to construct situations as means to teach students spatial knowledge (as to alignment and straightness, through the good use of a map to navigate). In other described cases, the frame allows to evaluate and compare performances of students, or to map spatial abilities.

Example 3

Following Newcombe et al. (2013), Van den Heuvel-Panhuizen et al. (2015, p. 346) distinguish between two kinds of spatial skills: **between**-objects representation and transformation skills (for

example in Perspective-Taking tasks –PT–-, like the Three Mountains of Piaget & Inhelder, 1956) and **within-**objects representation and transformation skills (for example a mental rotation).

The aim of their research is to assess children's PT-skills focusing on the difference between two components of what they named IPT (Imaginary Perspective-Taking): **visibility** and **appearance.** These two competencies are highlighted by the items they proposed to the children.

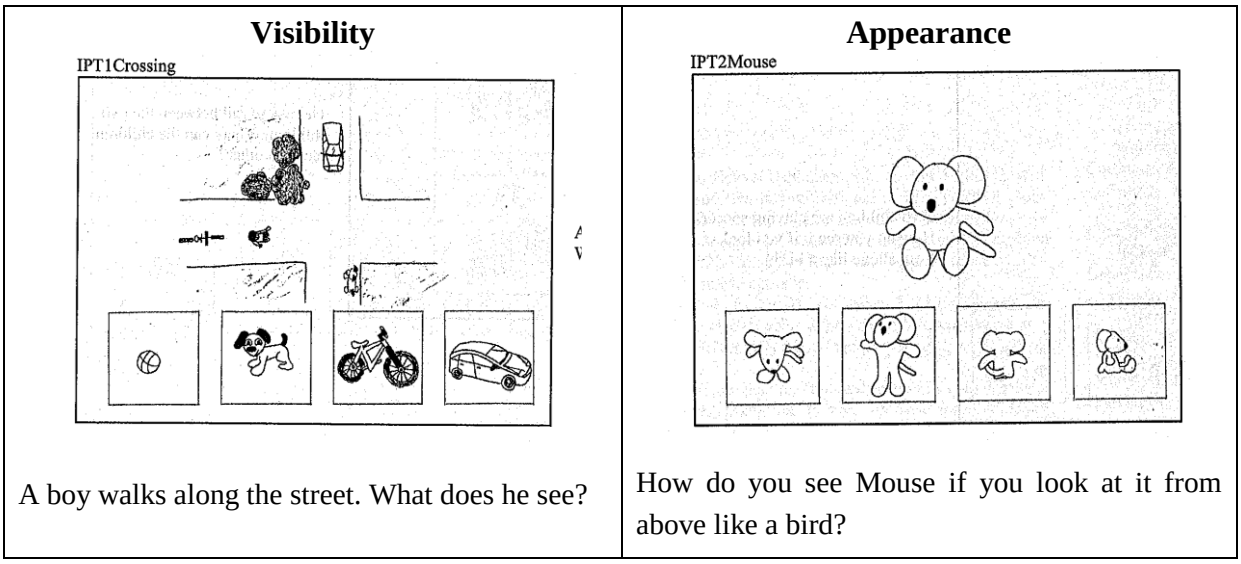

Figure 2: Examples of drawings and questions in the test (Van den Heuvel-Panhuizen et al., 2015)

It could be noted that the 3D situation (meso- or macro-spatial) is communicated to the 2D representation (micro-spatial).

The analysis of their tests with more than 300 children of Netherlands and Cyprus (age 4-5) shows that kindergartners of the two countries can answer correctly: on average respectively 70% and 55% of the visibility items, and 40% and 30% of the appearance items; that the development of the IPT competence visibility precedes the development of the IPT competence; that specific item characteristics of the evoked context could also influence the difficulty level of the item.

Example 4

In the 1990s Duval (1995, 2006) has brought an important contribution to necessary visualization of the drawing (implicitly in a microspace) for geometric reasoning. Some authors rely particularly on Duval's research, e.g., Mithalal-LeDoze (2009, 2015), Papadaki (2015) and Swoboda (2015).

With iconic visualization "the drawing is a true physical object, and its shape is a graphic icon that cannot be modified. All its properties are related to this shape (…)" (Mithalal-LeDoze, 2009, p. 797). With non iconic visualization "the figure is analysed as a theoretical object represented by the drawing, using three main processes: *Instrumental deconstruction*: in order to find how to build the representation with given instruments), *Heuristic breaking down of the shapes*: the shape is split up into subparts, as if it was a puzzle), *Dimensional deconstruction*: the figure is broken down into figural units — lower dimension units that figures are composed of —, and the links between these units are the geometrical properties (…)" (Mithalal-LeDoze, 2009, p. 797; Papadaki, 2015). Duval quoted two other processes *Change of Scale and Change of Orientation.*

Studying visually impaired students moving 2D objects to imagine 3D objects, Papadaki (2015) introduces a new source for mental images, the kinesthetic one. For these students visualization integrated many repetitions of the same gesture, cross-checking it with one's everyday life tactile experience and geometrical knowledge as objects' definitions. Visualization is clearly more than vision, what Duval wrote for a long time describing different ways (iconic and non iconic) of visualizing a figure. But Papadaki (2015) introduces a new dimension in visualization, a dynamic one; conceiving a figure 3D as the result of a reproducible movement of a figure 2D.

Swoboda (2015) stresses the rotation as a natural transformation for young students; mental rotation is a fundamental component of the frame of Demetriou & Kyriakides (2006), and of non iconic apprehension of Duval too. Maybe the rotary motion and the rotation as transformation could be studied sooner (than the line symmetry) in compulsory school to enrich visualization skills.

Conclusion

Spatial sense and the different frames regarding it is now better mapped. We will not go back on what was discussed above, but just stress some difference between the frames regarding it.

1) The framework of Berthelot & Salin in relation with the size of space enables to realize that almost all the mentioned situations or items of the ZDM and CERME papers (except Douaire & Emprin, 2015) are located in the microspace; a priori they only request microspatial knowledge[1]. The existence and importance of mesospatial and macrospatial knowledge seem underestimated. For example in the paper of Van den Heuvel-Panhuizen et al. (2015) spatial apprehension of a picture (or a drawing, microspace) is considered as an apprehension of the evoked real world (meso- or macro-space).

2) Some papers try to isolate "basic" skills or "basic" items which could be predictive of spatial sense and serve for students assessment, analysing spatial situations (for example Van den Heuvel-Panhuizen et al., 2015) or geometrical problems (Duval 1999, 2006). On the other hand Berthelot & Salin, in coherence with their support framework, don't attempt to describe finely "the" spatial skills but ask students to solve real spatial problems in which spatial skills are at work.

3) Papadaki (2015) and Swoboda (2015) allow us to realize that gestures can be a powerful help to mentally construct geometrical images and facilitate visualization. Among the gesture the rotary motion (rotation) could play an important role.

4) Visualization is enriched with new entries as gestures and motion; it has gained increasing importance on the spatial skills. But what is meant by this term? Another inquiry to lead…

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[1] It is not always the case: situations with use of DGS often incite students to use mesospatial conception.

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Autism and mathematics education

Michiel Klaren, Birgit Pepin and Marieke Thurlings

Eindhoven University of Technology, The Netherlands; m.j.klaren@tue.nl

Inclusive education urges educational research to deepen its understanding of students with special needs. The aim of this theoretical paper is to provide an overview of the didactical aspects in the field of autism and mathematics education. First we review literature on autism spectrum disorders (ASD) from a broad psychological perspective, and second, we focus on three cognitive theories which are used to explain the behavioral symptoms of ASD. Next, we discuss mathematics specific didactical issues that relate to these cognitive theories. Finally, we elaborate on an example of research on mathematics education for ASD students based on the Van Hiele model of thinking in geometry. In the conclusions we bring together the findings and give suggestions for future research.

Keywords: Autism Spectrum Disorder, mathematics, geometry, Van Hiele, secondary education, didactics, inclusion.

Introduction

Autism is a neurodevelopmental disorder characterized by impairments in social interaction and communication, and by restricted and repetitive patterns of behavior (American Psychiatric Association, 2013). Different combinations of the impairments occur, and together form a continuum, which is called the autism spectrum of disorders (ASD; Wing, 1988). Both the severity of the symptoms and the intellectual capacities of people with ASD vary widely. There is a growing understanding that people with autism can also provide a substantial contribution to society (e.g. Mottron, 2011). In most countries, and also in the Netherlands, inclusion policies require schools to develop support for children with autism. In this article we will focus on ASD students with higher intellectual abilities[1] and who require limited support, and hence can participate in inclusive secondary education.

Mathematics is a school-subject that builds on logic, is well structured, uses symbolic language with well-defined meanings and deals, at least in its purer forms, with unambiguous questions. These characteristics make mathematics a subject that is relatively easy to access for people with ASD. Research shows that autism is found more often amongst mathematics students than those of other disciplines (e.g. Baron-Cohen, Wheelwright, Burtenshaw, & Hobson, 2007). However, other studies showed that the majority of ASD students and higher intelligence have an average mathematical ability compared to the normal population, while only some have mathematical giftedness (e.g. Chiang & Lin, 2007).

In secondary schools that include ASD students, teaching mathematics is not an easy task. Based on research and our own experiences of teaching in inclusive schools, we claim that ASD students learning mathematics require specific subject-didactical support. A review of research on autism

[1] In literature on autism this also indicated as High Functioning Autism and Asperger Syndrome or HFA/AS (Whitby & Mancil, 2009).

showed that most studies are about medical or behavioral aspects, and less than 20% targets education (Graff, Berkeley, Evmenova, & Park, 2014). Research on autism in inclusive education is mainly about pedagogic topics concerning the ASD students (Ravet, 2011). However, there is little research on didactical aspects of autism in specific subject fields.

The aim of this theoretical paper is to provide an overview of the didactical aspects in the field of autism and mathematics education. Our research question is the following:

Which didactical issues are related to ASD students learning mathematics?

Literature review

Autism spectrum disorders

After the first case studies by Kanner in 1943, and Asperger in 1944, it took until the end of the 1970's, when Wing and Gould (1979) provided their classic description of the triad of impairments of autism: first, the absence or impairment of social interaction, especially with peers; secondly, the absence or impairment of the development of verbal and nonverbal language; and third, repetitive, stereotyped activities of any kind. Confusion about different subtypes lead Wing (1988) to the conclusion that there must be an autistic continuum, which she coined the Autism Spectrum of Disorders (ASD) with diagnoses such as classic autism, PDD-NOS (Pervasive Developmental Disorder – Not Otherwise Specified) and Asperger's syndrome. With the latest version of the diagnostic manual DSM-5 (American Psychiatric Association, 2013) the diagnosis of ASD is based on persistent deficits in social communication and social interaction, and on restricted, repetitive patterns of behavior, interests or activities. The subtypes of PDD-NOS and Asperger are no longer official diagnoses, and DSM-5 describes three levels of severity for ASD (i.e. requiring very substantial support; requiring substantial support; and requiring support).

The psychological research into autism has been dominated by three cognitive theories (Rajendran & Mitchell, 2007): the Theory of Mind deficit; Executive Dysfunction; and the Weak Central Coherence accounts.

The Theory of Mind deficit

Children with ASD experience their social environment as unpredictable and incomprehensible (Baron-Cohen, Leslie, & Frith, 1985). They seem to treat people and things the same way. Observations showed, strikingly, that children with Down syndrome and a low intelligence developed a normal social competence, whilst children with ASD and higher intelligence did not (Baron-Cohen et al., 1985). To explain this, Baron-Cohen, et al. did research on the *Theory of Mind*: neurotypical children (non-ASD) are able to impute mental states to themselves and others (in other words, they have a "theory of mind"), whilst children with ASD fail to do so. This "mind-blindness" was shown by the false belief test: a story, played out for the child with dolls, where one doll has a belief about the location of an object that is incongruous with its real location. The test subject is then asked where the doll will look for the object. To answer correctly the test subject should infer the mental state of the doll ("I think she thinks"). A large proportion (80%) of children with ASD incorrectly assumed the doll would look on the real location. To explain the 20% of children with ASD who answered correctly, second-order false belief tests ("I think she thinks he thinks") were developed, and based on those results, it was assumed that a Theory of Mind is not always lacking completely but may be

not fully developed in children with ASD. Another problem was that Theory of Mind can be used to explain impairments in play, social interaction and verbal and nonverbal communication, but not for explaining the other characteristics of ASD, such as the restricted interests, obsessive desire to keep things unchanged (rigidity and inflexibility), and so on (Frith & Happé, 1994).

Executive dysfunction

Early in the 1990's, Ozonoff, Pennington and Rogers (1991) suggested that deficits in the executive functions could explain symptoms of autism such as narrow interests, rigidity and inflexibility.

> Executive function is defined as the ability to maintain an appropriate problem-solving set for attainment of a future goal; it includes behaviors such as planning, impulse control, inhibition of pre-potent but irrelevant responses, set maintenance, organized search, and flexibility of thought and action (Ozonoff et al., 1991, p. 1083).

The research by Ozonoff et al. (1991) showed that deficits with executive function where found in both children with classic high-functioning autism and those with Asperger's syndrome (who succeeded on the second-order false belief test). This suggested that deficits in executive function form a primary cognitive deficit in ASD.

Weak Central Coherence

In neurotypical children (non-ASD) the development of information processing is oriented towards extracting the overall meaning from the sensory input. This inclination is called 'central coherence'. Frith (1989) described how this development is different in children with ASD, and she proposed the weak central coherence theory to explain the symptoms of autism. Psychological tests later showed that children with ASD have superior performance on local information processing, but were less inclined to global information processing (Happé, 1999). In people with ASD this is also observed as a preoccupation with details and parts and a failure to understand the meaning of the whole.

Implications of ASD for mathematics education

Although the focus of this paper is on mathematics education, we first address some approaches for ASD students that apply to education in general.

General education

A general pedagogic approach for students with ASD is *structured teaching* (Mesibov & Shea, 2010). Structure can be provided in the physical environment (e.g. arrangement of the room and the use of visual clues), the sequence of events during the day (e.g. an understandable schedule), the individual tasks (e.g. provide specific information of the goals and the completion criteria) and the grouping of tasks into a work system. Many of these approaches in autism have not been well researched, and research is now addressing the determination of evidence-based practices (Reichow, Volkmar, & Cicchetti, 2008).

Mathematics education

Based on a review of 18 studies of mathematical abilities of ASD students with AS/HFA, Chiang and Lin (2007) found that the majority of the ASD students have average mathematical capabilities and only some ASD students have a mathematical giftedness. Based on these results, Chiang and Lin

concluded that an age-appropriate mathematical curriculum can be used, but individual adjustments may be needed to support both relative strengths and weaknesses.

In a review of the literature on academic achievement profiles of ASD students, Whitby and Mancil (2009) report that more than half (52%) of individuals diagnosed with ASD have IQs above 70 and for these children, academic goals come within reach. There is a need for appropriate interventions to allow these children to perform up to their potential and obtain meaningful employment. For mathematical abilities, Whitby and Mancil found that computational skills were intact, but applied mathematics capabilities were impaired. Issues with the application of mathematics are possibly due to executive functioning deficits with their organizational and attentional skills that have a negative effect on multi-step problem solving. Deficits in comprehension (both listening and reading) relate to contextual understanding (e.g. word problems) and conceptual understanding (e.g. abstract concepts).

With word problems, ASD students have difficulty choosing the right approach because they have, due to their weak central coherence, difficulty seeing the similarities and the common structure of different examples and exercises. An ASD adjusted didactical approach for solving word problems should address improving reading comprehension, mathematics vocabulary, computation, and everyday mathematical knowledge (Bae, Chiang, & Hickson, 2015, p. 2206).

In solving mathematical problems ("a question that exercises the mind"; Schoenfeld, 1985) ASD students are impaired by executive dysfunction (Ozonoff et al., 1991). In mathematical problem solving ASD students have similar issues as with word problems, but are also expected to face issues with cognitive flexibility, the use of heuristics and the use of meta-cognitive strategies. Positive results have been reported on the effects of cognitive strategy instructions for students with learning disabilities (Montague, Krawec, Enders, & Dietz, 2014), and these results may also be obtainable for ASD students. In classroom practice we see that some ASD students develop problem solving procedures of their own, which may work on the initial (simple) problems but cannot be generalized to later extensions. Upon receiving feedback, the rigidity of these ASD students sometimes inhibits them from accepting the time-proven approaches. Feedback is also known to play an important role in self-regulated learning (Butler & Winne, 1995). Improving feedback seeking strategies of ASD students, can support their self-regulated learning, and help to overcome barriers in problem solving.

Conceptual understanding can be a challenge for ASD students: they have, due to their weak central coherence, difficulty integrating information and generalizing previously learned concepts (Klinger & Dawson, 2001). Minshew and Goldstein (2002) found that individuals with high-functioning autism had impaired concept formation. Furthermore, these individuals had difficulty with cognitive flexibility and showed incomplete understanding of learned concepts. Temple Grandin, diagnosed with ASD, describes (2006) how she memorizes as much facts and experiences as possible, and uses an internal search engine to retrieve visual images of prototypes to understand a concept. In general, ASD-students benefit from visualization of abstract concepts, and with their strong root memory, they remember the visualizations as prototypes of the concept.

An example: A research study on learning geometry by ASD students

To illustrate research on subject-specific didactic problems for ASD students, we describe an example from geometry and the cognitive theory of weak central coherence. This example is based on a master thesis of the first author (Klaren, 2012). The theoretical/ analytical framework is concerned with Van

Hiele's theory, which defines five levels of the learning process that learners are said go through when learning geometry. The five levels are described by Hoffer (1981) as:

- Level 1, Recognition: the student learns some vocabulary and recognizes the shape as a whole;
- Level 2, Analysis: the student analyzes properties of the figures;
- Level 3, Ordering: the student logically orders figures and understands interrelationships between figures and the importance of accurate definitions;
- Level 4, Deduction: the student understands the significance of deduction and the role of postulates, theorems, and proof;
- Level 5, Rigor: the student understands the importance of precision in dealing with foundations and interrelationships between structures

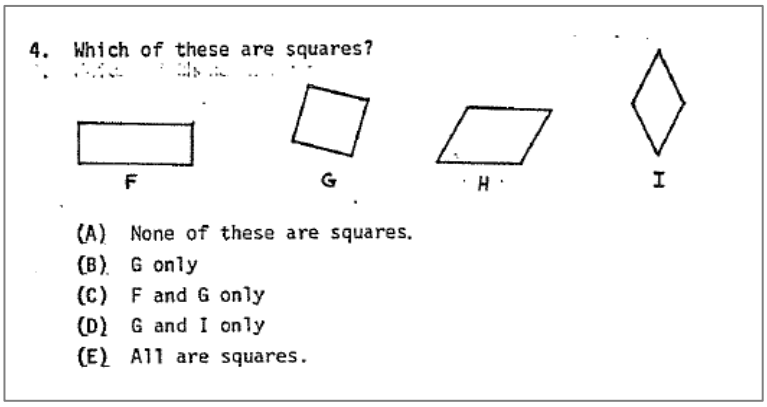

Figure 1: Example of question on level 1 (recognition)

Van Hiele posited that learners master the levels in a stepwise manner and always in the same sequence. In other words: the difficulty of questions specific to the successive levels, will rise. Usiskin (1982) confirmed the ability of the Van Hiele theory to describe and predict the performance of students in secondary education on geometry. The fifth level was not well operationalized by Van Hiele, and left out of further analyses.

Students with weak central coherence (operationalized as students with ASD), are hypothesized to have an inversion of the difficulty of level 1 and 2. In other words: they are expected to find questions at level 1 ("shape as a *whole*"; see Figure 1) more difficult than questions at level 2 (analyses of properties).

To test the hypothesis, 81 children with ASD and higher intellectual abilities, age 12 to 17 years, were tested with the geometry test of Usiskin (with the texts translated in Dutch and the original diagrams). Rasch analyses was used to estimate the average difficulty (on a logit scale) of questions at each level.

The results were compared with results found for non-ASD students by Wilson (1990) in a reanalysis of the data of Usiskin. The combined results (Figure 2) show that the estimated average item-difficulty of questions on level 1 and 2 were indeed shifted for ASD students (level 1 more difficult and level 2 easier) compared to non-ASD students. However, level 1 was not found to be more difficult than level 2 for ASD students. Based on these findings, the (strong) hypothesis of inversion must be rejected, but the test gives some support for the anticipated differences between ASD and non-ASD students

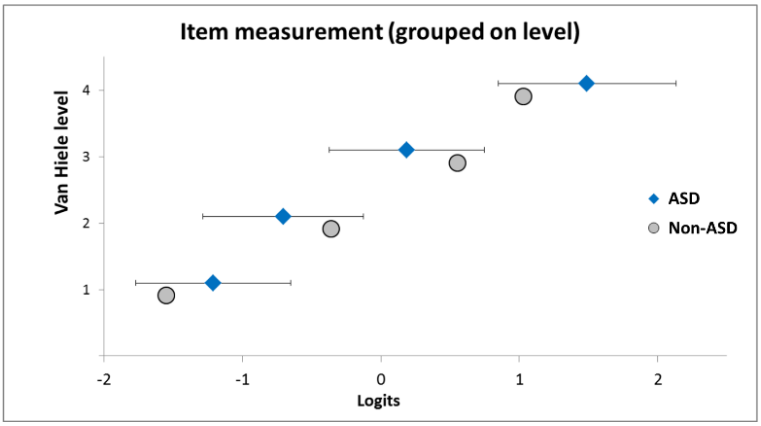

Figure 2: Average item-difficulty per level

regarding the difficulty of the first two levels of Van Hiele. In the didactical practice of geometry teaching, this should raise awareness that visual recognition by ASD students is to be linked to explicit analyses of properties in order to support concept formation.

A peculiar observation was that some students who performed well on questions on higher levels, made unexpected errors in questions on level 1: on questions where the squares were to be pointed out (e.g. Figure 1), they included the rectangles. A possible explanation of this type of errors is the ambiguity of the mathematical language: in Dutch the translation of square is "vierkant" (literal translation: "four sides"). In interviews after the test, students explained their answers by stating that the rectangles had four sides. Possibly these students where not relying on the shape as a whole, but applied rule-based logic in combination with literal understanding of the mathematical concept.

Conclusions

The three cognitive theories described in this paper represent theoretical frameworks that can guide research in autism and mathematics education.

As described by the (lack of) Theory of Mind account, ASD students experience their social environment as unpredictable and incomprehensible. Research on ASD and general educational will address the development of evidence-based practices that support ASD students in their social interactions. Research on mathematics education can develop interventions that aim for using (understanding) and seeking feedback with respect to learning mathematical concepts and skills.

Related to feedback is the use of self-regulated learning. Students with ASD, with their typical weak executive functioning, can be supported by interventions that improve their metacognition and use of strategies and heuristics, especially in the field of mathematical problem solving.

Recent research on perception by people with autism (e.g. Pellicano & Burr, 2012; Hohwy, 2013) is deepening the neurocognitive understanding of the weak central coherence account. Research on mathematics education may benefit from these results, and improve the understanding of concept learning by ASD students.

To summarize, students with ASD have deficits in their social interaction, their contextual and conceptual understanding, and the self-regulation of their executive functioning. In order to allow successful inclusion of ASD students in education, teachers have to apply effective instruction methods to overcome the "mind-blindness" of these students. Research can help define design guidelines for instructional methods in mathematics, which are attuned to the specific needs of ASD students and allow them to see the beauty of mathematics.

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Learning sine and cosine in French secondary schools

Ratha Loeng

Paris Diderot-Paris 7 University, France; loengratha@hotmail.com

The purpose of this paper is to present a questionnaire devoted to trigonometry and its results. More specifically, the study focuses on curriculum and cognitive aspects of learning and teaching sine and cosine over the final five years in French secondary schools, from grade 8 to grade 12. We ask: what are the difficulties of students in learning sine and cosine concepts and making the transition from the geometric setting to the functional setting. We identify four major types of error committed by students. This raises the question of how to make students effectively assimilate the concepts of sine and cosine.

Keywords: Trigonometry, sine and cosine, secondary school, available knowledge.

Introduction

In secondary schools in France, trigonometry notions are introduced progressively from grade 8 to grade 11 (science option), and end with sine and cosine functions in grade 12 (science option). We ask the key questions: How to articulate the passage from trigonometry in right triangles to the trigonometric circle and to sine and cosine functions? And, how to make the students assimilate these notions? To answer these questions, we investigate the status of the educational system: curriculum and learned knowledge.

In connection with our analysis of institutional texts and manuals for the study of the mathematical organization (Anthropological Theory of the Didactic, Chevallard, 1999; Bosch & Gascón, 2014), we elaborated a questionnaire for grade 12 students. The purpose of this questionnaire is to test the knowledge acquired by students in the final five years of study on trigonometry and specifically the sine and cosine functions in grade 12 (science option), to identify implicit notions in learning and teaching of trigonometry, to clarify the set of issues in our research work on what the students have learned, and to generate further research questions.

The questionnaire consists of six exercises, mainly concerning the cosine and sine of an acute angle in a right triangle, the cosine and sine of an oriented angle in the trigonometric circle, and some properties of the cosine or sine function like periodicity. In this paper, we analyze ***the first three exercises of the questionnaire*** and describe its results.

Methodology

For the curriculum study, we use the Anthropological Theory of the Didactic (Chevallard, 1999; Bosch & Gascón, 2014) as a theoretical framework in order to identify the mathematical organization (MO) of the French institutions that is developed in French mathematics textbooks. It is rather an exploratory analysis of French textbooks which gives us an overview of the teaching of trigonometry and of trigonometric functions in secondary schools.

We do not present in details the curriculum study but the identified MO allowed us to construct a questionnaire. It is grounded in the most used types of tasks and techniques in the French teaching institutions, from grade 8 to grade 12 related to trigonometry in right triangles (Exercise I, grades 8-

9), trigonometric circle (Exercise II, grades 10-11) and trigonometric functions (Exercise III, grade 12).

To analyze the tasks in the questionnaire, we choose tools from the Theory of Double didactic and ergonomic Approach (Robert, 2008; Robert & Hache, 2013) which provides us with fine cognitive tools to analyze students' knowledge. The questionnaire is a test given to grade 12 students outside the context of any particular chapter in order to avoid any didactical contract influence. We study the knowledge adaptation in these tasks (see the following section) and especially, in this paper, the available level of knowledge application. This will allow us to have an idea of the cognitive complexity of the questionnaire and what students have learned (particularly in terms of available knowledge).

> When tasks require adaptations of knowledge that are at least partly indicated, we speak of the level of knowledge application that can be mobilized. Students' work is not effectively analogous, depending on whether they must look for the knowledge to use (questions of why or what), or apply and adapt the indicated knowledge (question of how). If it is up to the student to recognize the knowledge to use, we speak of the available level of knowledge application. […] We also distinguish combinations, links, or changes among elements such as frameworks, and work further on different types of intellectual activities that are specific to mathematics. (Robert & Hache 2013, p.37)

The questionnaire was given in two versions in order to avoid any influence of neighbor students. Variations between them were cognitively irrelevant to our study: different lengths in Exercise I, different coordinates in Exercise II, and cosine vs. sine functions in Exercise III.

Presentation of the questionnaire, a priori analysis

In this section we specify the aims of each question (of each exercise), the available knowledge (AK), the correct methods[1], possible erroneous methods and possible errors made by students. Most of the tasks are not "simple and isolated tasks" in the following sense:

> We first distinguish simple and isolated tasks, or immediate applications of piece of knowledge without adaptation or combination. A single piece of knowledge is used, potentially with simple replacement of general inputs by the given information in the context of the exercise. (Robert & Hache 2013, p.36)

Exercise I: Trigonometry in a triangle

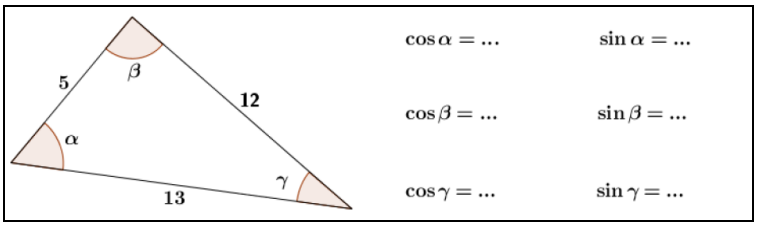

Figure 1: Exercise I – Trigonometry in a triangle

Exercise I (see Figure 1) asks for the values of sine and cosine of the angles of the given triangle.

[1] We focus on the knowledge effectively used by students, for example, we do not discuss methods using scalar product of vectors (seen in grade 11).

Note that in this figure, there is no coding denoting that β is a right angle. We want to know the available knowledge of students, and particularly if the generalized Pythagorean theorem effectively is mobilized by students.

The **first Method, M-I.1**, relies on the definitions of the cosine and sine of an acute angle in a right triangle (seen in grades 8 and 9) which require first the reciprocal of Pythagorean theorem (AK, seen in grade 8) or the visual recognition of the right angle of the given triangle without proof.

Note that trigonometry in a right triangle only allows to calculate the cosine and sine of an *acute angle* whose measure is strictly between 0° and 90° but does not give $\cos\beta$ nor $\sin\beta$, which require another knowledge (AK).

There are three steps in M-I.1, where steps a and b form a non-simple task, and step c reinforces the complexity:

a. Recognition of a right triangle with the reciprocal of Pythagorean theorem (AK) or by visual inspection.

b. Definitions of the cosine and sine of an acute angle of the right triangle to find independently $\cos\alpha$, $\sin\alpha$, $\cos\gamma$ and $\sin\gamma$.

c. Property of the cosine and sine of the right angle to find $\cos\beta$ and $\sin\beta$.

Note that there are two other possible pieces of knowledge that could be used: finding, for example, $\cos\alpha$ with $\sin\alpha$ via the formula $\cos^2+\sin^2=1$ (AK, seen in grade 9) or using a right triangle property (AK, seen in grade 9) that ensures the equalities $\cos\alpha=\sin\gamma$ and $\sin\alpha=\cos\gamma$.

Possible errors of the student are confusion between cosine and sine or an error in formulas.

The **second method, M-I.2**, relies on the generalized Pythagorean theorem (or Al-Kashi's formula, *it allows to calculate the cosine of an angle whose measure is strictly between* 0° *and* 180° *or between 0 and* π *radians*, seen in grade 11) to find the cosine of an angle of any triangle. In this case, what is the reaction of the student when finding $\cos\beta=0$? Does he/she conclude that β is a right angle? Noticing that the given triangle is in fact a right triangle, will he/she change the strategy to M-I.1 method to determine the remaining values?

There are two related tasks, consisting in finding straightaway cosine, then sine. Apply the generalized Pythagorean theorem (AK) to independently find the three cosines and then, with the results obtained, use the fundamental relation $\cos^2+\sin^2=1$ (AK, seen in grade 11) to find the three sines. It is a non-simple task to find the cosine of an angle of a triangle. And, it is a non-simple and non-isolated task to find the sine of angle: there is the introduction of steps and also a combination of settings (numerical and algebraic) - transformation into an equation of the type $x^2=a$, determining the sign of the sine of the angle and deducing its value.

Possible errors of the student are in the application of the formulas, in algebraic transformation, and in numerical calculations.

Exercise II: Trigonometry in the trigonometric circle

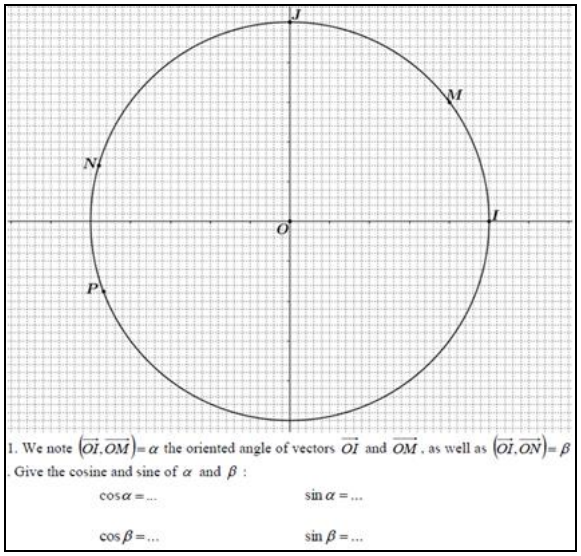

Figure 2: Exercise II – Trigonometric circle

From Exercise II, we only discuss here the question 1 (see Figure 2) which consists in asking for the cosine and sine of α (an acute angle) and of β (an obtuse angle). In the context, we gave the coordinates of the points M(4/5; 3/5) and N(-24/25; 7/25). We want to know the available knowledge of students such as definitions of the cosine and sine of an oriented angle (seen in grade 11). It is asked similarly to the Exercise I for the cosine and sine values but in another setting, and moreover, with different notions of angles.

Note that in textbooks corresponding to the 2010 program in grade 11 (science option), one begins by defining measures of an oriented angle, then the cosine and sine of an oriented angle in this way: *The cosine and the sine of an oriented angle are the cosine and the sine of any of its measures*.

The **first method, M-II.1**, relies on the expression of the Cartesian coordinates of a point of the trigonometric circle with the cosine and sine. *As* $M(a;b)$ *is a point of the trigonometric circle in a direct orthonormal frame* $(O;I,J)$ *of the plane and as* α *designates the oriented angle* $(\overrightarrow{OI},\overrightarrow{OM})$, *so we have* $\cos\alpha=a$ *and* $\sin\alpha=b$ (AK). It is a non-simple task. As α does not designate a measure of the oriented angle $(\overrightarrow{OI},\overrightarrow{OM})$, do students use the M-II.1 method, namely that the coordinates of the point M are $(\cos\alpha;\sin\alpha)$? If not, with the graph, will they think of using other methods, for example, like M-II.2 in the following?

The **second method, M-II.2**, relies on the relations between algebraic writing of a complex number and its trigonometric writing (AK) and on the characterization of equality of two complex numbers via equality of real and imaginary parts (AK, seen in grade 12). It is a non-simple task - introduction of steps: mark, in the given graph, the angles α, β - consider the points M and N as respective image points of complex numbers of module 1, then write the two complex numbers in two forms: algebraic writing and trigonometric writing - deduce the exact values of the cosine and sine of α

and of β. Compared to M-II.1 method, M-II.2 method requires the change of settings, of registers and of point of view in the reasoning (Duval, 2006).

Possible errors of the student are the confusion of sine and cosine in the expression of Cartesian coordinates of a point of the trigonometric circle with cosine and sine, error in the application of formulas and in numerical calculations.

Exercise III: Graph of trigonometric functions

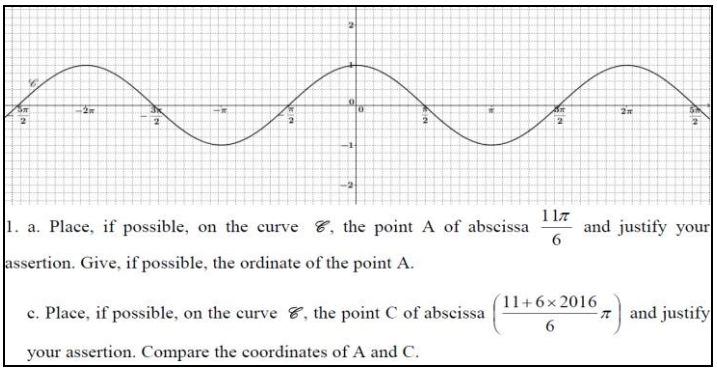

Figure 3: Exercise III – Graph of cosine function

From Exercise III, we only present here the question 1.c. We want to know the available knowledge of students like the existence and the nonexistence of a point on the curve of a trigonometric function (cosine or sine) and the possibility of placing a point on the curve in the given graph, and particularly the property of periodicity.

The **third method, M-III.**1: Placing the point C of abscissa $\frac{11+6\times2016}{6}\pi$ on the curve $\mathscr{C}$ *within* the graphic is an impossible task: recognize the existence of a point on a curve justifying it using the domain of a function and the given graph. It is a non-simple and non-isolated task - combination of graphical and functional settings.

Comparing the coordinates of A and of C relies on the property of a point on a curve and that of the cosine of a real number (AK) (or rather that of periodicity of the cosine function). It is a non-simple task. Besides, one could use the geometric property of periodicity of a function (AK, using graph of the function with translation).

Note that for comparing the coordinates of A and of C, there is another possible method based on the property of a point on a curve and the calculator in radian mode (an incomplete method with an erroneous conclusion starting from the numerical results from the calculator).

Possible errors of the student are a numerical error in the calculation; the periodicity of cosine function might not be an available knowledge.

A posteriori analysis of the three exercises shown and their results

The questionnaire was given to 40 students in March 2016 in two grade 12 classes of two different high schools.

Exercise I: 37 students used M-I.1 method (22 by reciprocal Pythagorean theorem (see Figure 4) and 15 by visual inspection), only 1 student used M-I.2 method, and 2 students did not answer this exercise.

We identify three major types of error committed by students, denoted TE1 (5 students), TE2 (9 students), TE3 (18 students). TE1 and TE2 are two types of more general errors, and TE3 consists in attempting to give different values of the ***cosine and sine of the right angle of the right triangle*** in the geometric setting.

TE1: "*Confusion of the values of the cosine and sine of an angle with the value of the angle*".

For instance, $\cos\alpha = 5/13 = 67.4$ (with or without the sign "°" designating the measure of the angle α obtained in degree; here, the second " = " sign would designate "it gives"). We did not meet the error committed like $\alpha = 5/13$ signifying that the angle α was the value of the cosine of α.

TE2: "*Confusion of the value of the cosine (or sine) of an angle with the value of the cosine (or sine) of a real number*". Students mobilize correctly, for example, the definition of the cosine of an acute angle of the right triangle but they do not stop there, they continue putting the sign " = " and conclude with an approximate value of the cosine of the real number which designates the value of the cosine of the acute angle obtained using the calculator either in degree mode or in radian mode. In this case, their result would be an approximate value of the "cosine of the ***cosine of the angle***". For instance, $\cos\alpha = 5/13 = 0.99$ with the calculator in degree mode, and in radian mode, $\cos\alpha = 5/13 = 0.93$ (see Figure 4).

TE3: "*Inventing values of the cosine and sine of the right angle of the right triangle*". It seems that students do not remember the values of the cosine and sine of the right angle which are the respective particular values 0 and 1. Thus, they invent these values in the right triangle committing TE3 (see Figure 4).

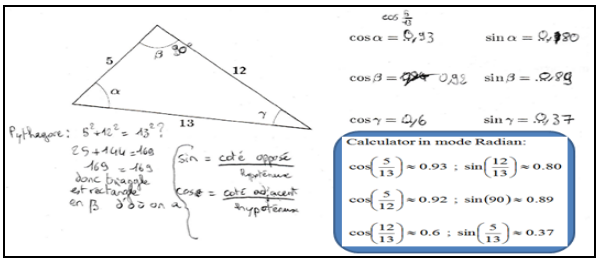

Figure 4: Exercise I – TE2 and TE3

We can conclude that the cosine and sine of an acute angle (α or γ) of the right triangle are available knowledge only for half of the students while those of the right angle (β) are not available knowledge because only about a quarter of students correctly gave the values 0 and 1.

Exercise II: 11 students out of 40 did not do this exercise. 17 students out of 29 correctly gave the *values of the cosine and sine of the angle* α: 11 students recognized the cosine and sine of the angle α as the *x* and *y* coordinates of the point M (M-II.1); 2 students exploited the properties of complex numbers (M-II.2); 4 students used previous knowledge seen in the 8th and 9th grades (M-I.1). 13 of these 17 students also correctly gave *those of cosine and sine of the angle* β (11 used M-II.1 & 2 used M-II.2) while 4 other students who used M-I.1 method to give the cosine and sine of the angle α (***acute angle***) had any difficulties to calculate the cosine and sine of the angle β (***obtuse angle***). 3 students out of 12 who incorrectly gave the values of the cosine and sine of the angles α and β committed TE2, and 5 students did not calculate those of β. Thus, cosine and sine of an oriented angle are not available knowledge because only about a quarter of students effectively mobilized this knowledge.

Exercise III: Among 36 students who did this exercise, 11 committed TE4 (see below) and 14 did not answer question 1.c. And, 6 students out of 22 who answered question 1.c committed TE4 (see Figure 5) and they placed inside of the graphic the point C on the curve either at the point A or elsewhere.

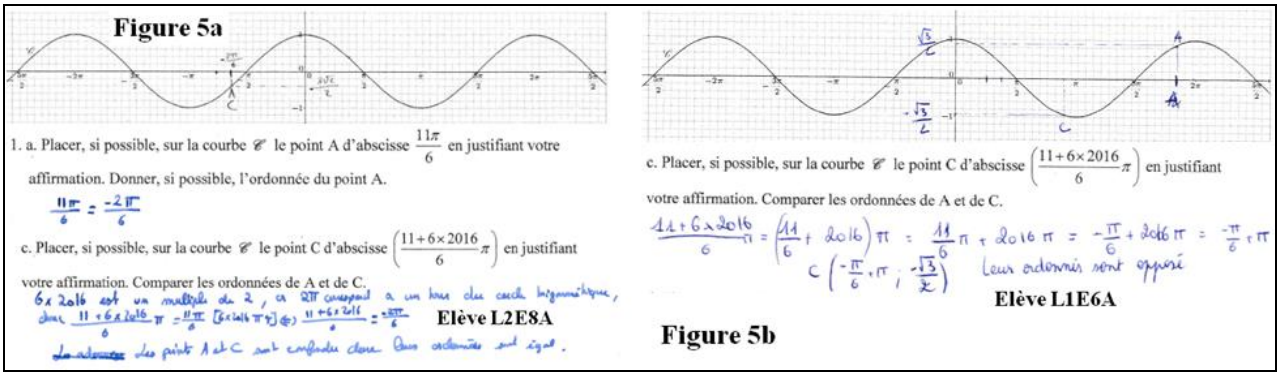

Figure 5: Exercise III-1.c – TE4

TE4: "*Confusion between the position of the two points on the trigonometric circle, images of two real numbers of the difference* $2k\pi$ *(k is an integer) and that of the points on the cosine/sine curve with abscissas these two real numbers*". The TE4 is an error amounting to say, for instance, that the points A and C might have the same abscissa while the two points have in fact the same ordinate by the periodicity of the cosine function (see Figure 5a).

Conclusion

Through our analysis of this questionnaire, we clearly see the difficulties of students in using their knowledge on cosine and sine of an angle (seen in grades 8, 9, 11) and on those of a real number (seen in grades 10, 12) in the geometric and functional settings. Considering French institutional texts and manuals, and the work produced by students in answering the questionnaire, we identify the implicit notions related to the learning and teaching of trigonometry as follows.

In the case of **trigonometry in right triangles** (grades 8, 9), the notion of cosine and sine of an acute angle is an available knowledge for students while that of cosine and sine of the right angle in right triangles is not an available knowledge. To give the values of the cosine and sine of the right

angle, about half of the students tried to use the ratio of two lengths of the right triangle but it is not adapted to the case of the right angle, and other knowledge is required (seen in grades 10, 11).

In the case of **trigonometry in the trigonometric circle** (grade 11, science option), a remarkable number of students have difficulties seeing the link between the coordinates of a point on the trigonometric circle and the cosine and sine of the oriented angle that is defined in this circle. This knowledge is an available knowledge for only about a quarter of the students.

In the case of **sine and cosine functions** (grade 12, science option), some students have difficulties distinguishing between two real numbers of difference $2k\pi$ (k is an integer) denoting two measurements in radian of the same oriented angle.

Overall, there is available knowledge for students to solve mathematical tasks on the trigonometry and on trigonometric functions, yet there is a blur or confusion in using their learned knowledge: between the value of cosine (or sine) and the angle or a measure of the angle, between an angle and its measurements, between a measure of an angle and a real number.

We can undoubtedly find epistemological, didactic, and curricular reasons; and this constitutes our research questions.

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The Pythagorean theorem in mathematics laboratory

Michela Maschietto[1], Stefano Barbieri[2], Francesca Scorcioni[2]

[1]University of Modena e Reggio Emilia, Italy; michela.maschietto@unimore.it

[2]IC "Marconi", Castelfranco Emilia, Italy; stefano.barbieri@tin.it, scorcioni.francesca@gmail.com

This paper refers to the Pythagorean theorem and the use of physical artifacts (called mathematical machines), which are related to one of the proofs of the theorem. It aims to discuss the didactical use of these kinds of artifact, paying attention to students' work with them and the role of the teacher. It presents a laboratory approach to this theorem developed within the Theory of Semiotic Mediation in mathematics education for 13-year-old students in Italy. The analysis shows that manipulation of the machine only does not imply the emergence of the mathematical meanings embedded in the machine. It also pays attention to the different graphical representations of the artifact and their role in the learning process.

Keywords: Artifacts, geometry, laboratory, lower secondary school education, Pythagoras.

Introduction

The Pythagorean theorem is a traditional content in the mathematics curriculum of the secondary school, not only in Italian school (Moutsios-Rentzos, Spyrou & Peteinara, 2014). This theorem is often proposed in the geometrical domain at the beginning, and it is soon converted into formulas and related to algebraic calculations. There exist several proofs of this theorem[1], some of them are proposed as visual proofs. On this topic, Bardelle (2010) analyses how university students in approaching a visual proof of that theorem try to look for the algebraic relation among sides starting from their knowledge of the theorem rather than getting the relationships between the components of the given figure. On the other hand, different exhibits are constructed basing on this kind of proofs, and they are also associated with and spread as gadgets (Eaves, 1954). In our work, we ask if and how it is possible to approach the Pythagorean theorem starting from artifacts which embed one of its proofs (Rufus, 1975), taking into account the role of manipulation, with 7-grade students (*13-year old students*). At the same time, we are interested in reinforcing the geometrical meaning of equivalent figures, which makes this theorem a particular case.

In this paper, we introduce the theoretical background for the didactical use of physical artifacts (called mathematical machine[2], Maschietto & Bartolini Bussi, 2011), then we present the teaching experiment.

Theoretical framework

In this section, we outline the theoretical framework of our work, based on the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) and the cognitive processes in geometry fostered by the

[1] http://www.cut-the-knot.org/pythagoras/index.shtml Accessed 20th March 2017.

[2] Mathematics Laboratory at the University of Modena e Reggio Emilia: www.mmlab.unimore.it. Accessed 20th March 2017.

task of reproducing artifacts. The teaching experiment is designed according to the methodology of mathematics laboratory (Maschietto & Trouche, 2010) with different kinds of artifacts.

Mathematics laboratory

The teaching experiment is proposed and analyzed within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008, TSM), grounded in the Vygotskian notion of semiotic mediation and role of artifact in cognitive development. Following to the TSM, the teacher chooses the artifacts evoking particular mathematical meanings and uses them to mediate those meanings, proposing tasks to be accomplished by those artifacts. The tasks are organized in terms of didactical cycles with group work, individual work and collective discussions (mathematical discussions) orchestrated by the teacher. The cycle usually starts with the exploration of the chosen artifact, above all in small group work, structured following fundamental questions as: "How is the machine made?", "What does the machine make?" and "Why does it make it?". In general, the first two questions try to take in account students' processes of instrumental genesis (Rabardel & Bourmaud, 2003). In the mathematics laboratory, students' processes of formulation of conjectures and argumentation are strongly motivated and supported by the third question. The mathematical meanings emerge from the use of the artifacts, the interactions among peers and between peers and the teacher, who has the role of an expert guide. In all the activities, students are involved in a semiotic activity (producing gestures, words, drawings, called artifact signs) that the teacher makes evolving into mathematical signs (i.e., linked to mathematical contents) by the means of pivot signs. In this sense, the teacher uses the artifact as an instrument of mediation for mathematical meanings.

The teaching experiment on the Pythagorean theorem is carried out with the use of two mathematical machines (M1 and M2 in Figure 1)[3]. They were analyzed in terms of their semiotic potential (Bartolini Bussi & Mariotti, 2008), corresponding to a semiotic relationship between an artifact and: on the one hand the personal meanings emerging from its use to accomplish a task; on the other hand, the mathematical meanings evoked by its use.

The analysis of the semiotic potential considers three components: mathematical content, historical references and utilization schemes (Rabardel & Bourmaud, 2003). This kind of analysis is essential for the choice of the artifact and the identification of mathematical meanings evoked by it.

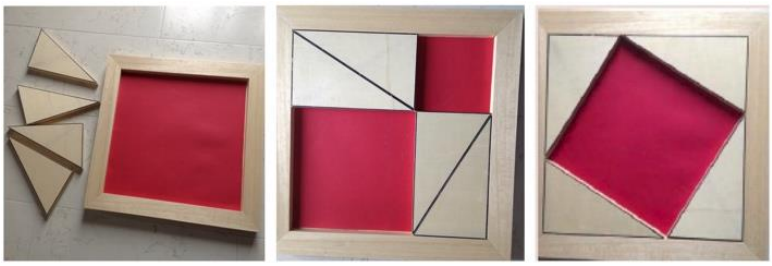
M1:  M2: 

Figure 1: The mathematical machines proposed to the classes (M1 on the left, M2 on the right)

[3] http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=162&Itemid=243&lang=it. Accessed 20th March 2017.

Semiotic potential of the artifacts

The mathematical machine M1 (Figure 1, on the left) is a wooden artifact, composed of a square frame and four triangular prisms, with right triangles as the base that are congruent each other. The fundamental relationship between the prisms and the square inside the frame (red square in Figure 1) is that the sum of the legs of the right triangles (base of the prism) is equal to the side of the square frame. This artifact shows a proof of the theorem (Rufus, 1975). For making evident the interior squares as figures, we have added a red paper into the frame.

The scheme of use of this mathematical machine is quite simple: shift the prisms into the square frame, without raising them from the base and without superposing them (this condition is evident because of the height of the prisms and the frame). The mathematical meanings involved in this artifact are: geometrical figures as right triangle and square, the area of those figures, and equivalence of area by addition/subtraction of congruent parts. The property of the triangles to be right-angled is obtained by the support of the square frame, and that represents the hypothesis of the theorem embedded in the machine itself. The movement of the prisms is bound by the frame, which ensures the invariance of the sum of the areas of the triangles and the squares or, in other words, the invariance of the area of the squares, whatever it is. Two tasks can be proposed: the first one is to place the prisms for obtaining square hole(s), the second one is to pass to a configuration (M1 in Figure 1, in the center) to the other one (M1 in Figure 1, on the right).

In our experiment, we asked the students to reproduce 1:1 the first mathematical machine on paper (four triangles and square corresponding to the interior of the frame), after its manipulation and description. This choice was due to the fact that we had only one wooden model in the classroom and we wanted to propose the task about the configurations with a model for each small group. In such a way, the students constructed a new artifact. We want to pay attention to the two elements that characterize the semiotic potential of the reproduction of the machine: the negligible thickness for all the components of the machine and the lack of the frame. The first element can force the students to transfer implicit constraints of the manipulation of the wooden machine into a control of the reciprocal position of the right triangles to avoid their superposition (Figure 2, on the right). The second element fosters to make evident the range of the movement of the right triangles on the big square base (Figure 2, on the left). In this way, making explicit the mathematical components of the utilization schemes is supposed to reinforce the link to mathematics evoked by the machine.

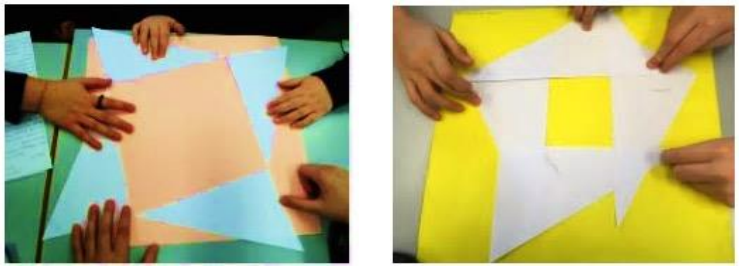


Figure 2: configurations by manipulating the paper machine

Drawings and geometrical figures

In the first activities with the artifact, the students are asked to answer the question “how the machine is made”, with the request of representing it. As we have written above, in this case, the students had

to physically reproduce 1:1 the machine (while, in general, they should draw the machine in their homework or worksheet, which often is not squared paper). In the TSM framework, drawing the artifact corresponds to individual production of artifact signs, strictly dependent on student's knowledge and his interpretation of the artifact. However, with respect to the TSM, we aim to pay more attention to our request of drawing. Following Duval (2005), this is a task of geometrical construction involving student's visualization and how geometrical properties are identified (see also Vendeira & Coutat, 2017). Our tasks involve the two kinds of visualizations that Duval distinguishes as iconic and non-iconic:

> A visualisation is iconic when, for instance, it represents positions or shape of real-world. It is non-iconic while it is organised to internal constraints and gives access to all cases possible. (Duval, 2008, p.49)

Concerning the role of visualization as an argument in proof, Duval (2005) analyzes the proof of the Pythagorean theorem corresponding to our first mathematical machine (as given by Rufus, 1975). He claims that the visualization is not complete if it only considers the two configurations (see Figure 1), because the relationship between the big square and the hypotenuse of the right triangles on one hand, and the two other squares and the legs of the same right triangles on the other hand are supposed known for the reader. This is grounded on the relationship between a conjecture and a figure. But if an arrow from left to right, for instance, connects the two representations, the transformation from one representation to another is realized. Nevertheless, the comparison of the areas of the squares is not directly possible, but it has to consider a computation (i.e., the difference between the big square and the four triangles) for paying attention to invariant elements in that transformation. In our machines, the transformation of representations corresponds to the movement of the four triangles, nevertheless with the loss of their simultaneous view.

Research questions

In this paper, we are interested in the didactical use of the mathematical machines for the Pythagorean theorem. Our research questions are:

1. Is it possible, and how, to approach the Pythagorean theorem with the mathematical machines described above?

2. Does the sequence of movements with the machines give a sufficient representation of the theorem for its understanding?

3. Which kinds of visualization are related to the tasks of drawing M1?

Methodology

According to our theoretical framework, the didactical methodology is the mathematics laboratory with artifacts. The tasks for students are organized in didactic cycles (Bartolini Bussi & Mariotti, 2008), consisting of small group work (GW), individual activities (IW), and collective mathematical discussions (CW). In the classrooms, other technologies are available, such as the Interactive Whiteboard with its software for making animations of the machines, and the simulations of the second machine made with Dynamic Geometry Software from the web. In the specific case of two classes involved in the experiments, the platform Edmodo was used. Therefore, the teaching

experiment proposes a learning environment in which material and digital technologies are present. In general, it is structured in three phases, as follows:

Phase A: 1) GW: Exploration of the first mathematical machine M1 (Figure 1); 2) CW: sharing of the description of the M1; 3) GW: construction of the M1 by paper; 4) GW: study of the possible configurations of the four triangles of M1 (Figure 2); 5) IW: representation of M1 on workbook; 6) CW: identification of relationships (invariants) between the components of M1.

Phase B: 7) History of the Pythagorean theorem and Pythagorean triples; 8) GW: Generalization of the theorem by different puzzles.

Phase C: 9) CW: Exploration of the second mathematical machine M2 and its reproduction with paper; 10) GW: Preparation of posters on the two mathematical machines.

The teaching experiments have started in 2013, and have involved six Italian classes of 13-years old students and two teachers, co-authors of this paper.

The analysis is carried out on students' worksheets, videos, photos and IWB files.

Findings

In this section, we refer to phases A focusing on the task of drawing the machine M1.

Steps 1-3. Work with the material model in small group and its reproduction

During the first three steps, the students worked in small group with the task of describing the machine M1 and collecting the elements (for instance, the types of triangles, the length of the sides) useful for its reproduction with colored paper. Before the reproduction, a collective discussion allowed students sharing their explorations and agreeing on a written description of the machine, with the measure of its sides. In particular, the right triangles were described as equivalent and some students recalled the Tangram game. Then, the students obtained the reproduction scale 1:1 by measuring and using tools for drawing (above all, rules and set square).

After this, the students had to fill a worksheet with the properties of the two figures, square and right triangle, constituting the machine. The manipulation of this new paper machine was guided by the task of looking for "square holes". But this task requires being conscious of the two schemes of use: the triangles must remain in the big square and do not overlap each other (Figure 2). During students' work, the configuration with the two square holes (Figure 1, M1 in the center) often appears first with respect to the configuration with the square alone (Figure 1, M1 on the right). This could be because the sides of the square are not parallel to the side of the square frame.

Individual Work for representing the two configurations in paper and pencil (Step 5)

Although the students had correctly described the congruence of the four right triangles (and constructed those in the previous step) into the square, several representations were not correctly drawn. We summarize some elements of students' drawings:

1) Square base is not equal in the two configurations (Figure 3, on the right);

2) All the four right triangles are not all congruent: a) in one confirmation itself (Figure 3, left, drawing on the left); b) between the two configurations (Figure 4);

3) The "square with the hypotenuse as side" is not a square (Figures 3 and 4).

The review of all the representations shows an important invariant of the machine was not taken into account by the students: the side of the square base is equal to the sum of the two legs.

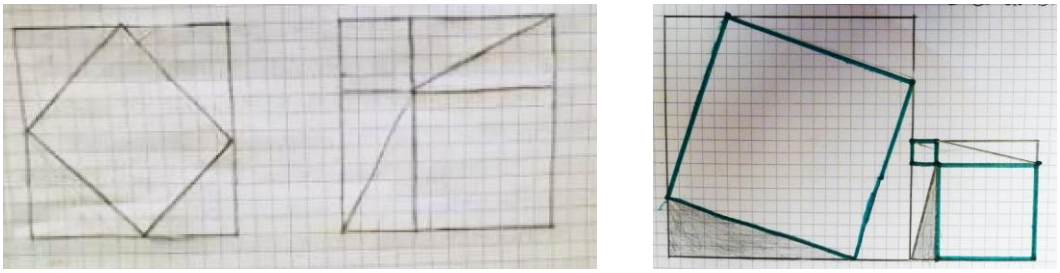


Figure 3: Students' representations of the two configurations of M1 on their workbooks

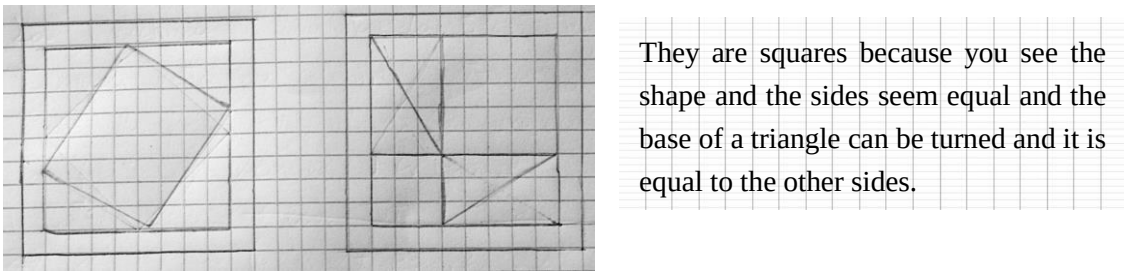


Figure 4: Student' representations of the two configurations of M1 on his workbook

Collective discussion with IWB

The collective discussion had two phases: the teacher paid attention to the wrong representations of the configurations; he took into account the passage from acting on the machine (both wooden and paper) to identify the relationship between the two configurations. First, the teacher used a checklist with the geometrical properties of the components of the machine that had been shared in the previous discussion for comparing the different representations. After, he asked to make new representations on the workbooks.

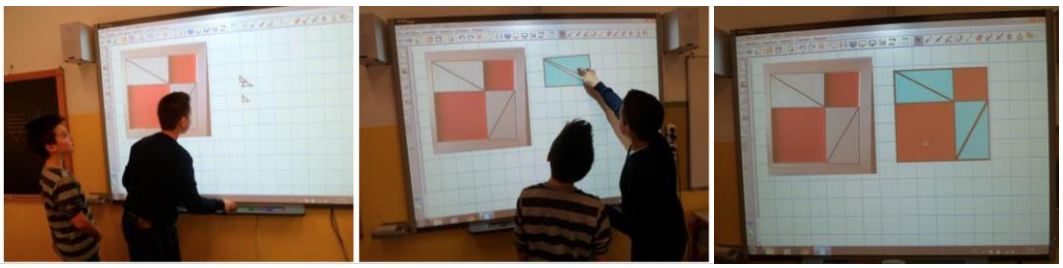
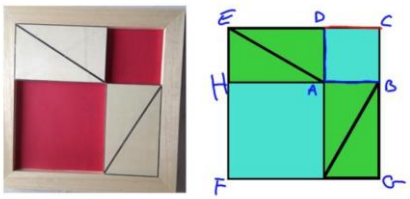


Figure 5: Collective work on IWB

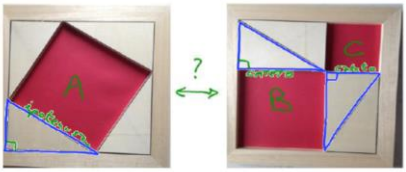
Then the machine is represented on the IWB from a photo (Fig. 5, on the left). The use of the IWB enables a new collective manipulation of the machine, in which the students passed from one configuration to another one by dragging the right triangles as they made with the material machine.

An important part of the discussion focused on the argumentation that the holes were squares (Figure 6, on the left). The collective use of digital machine allows students linking the manipulation of the triangles to the manipulation of Tangram pieces (Figure 6, on the right) and, so, emphasizing the conservation of the areas of the holes. The Pythagorean theorem becomes a particular case in the equivalence of areas.



If we consider the following characteristics:
- the blue base is a square
- the 4 green triangles are scalene, right triangles, congruent each others

Can we proof that the two blue "holes" are two squares?
(without mesuring them…)



We leave the same 4 triangles from the square base (as in the Tangram).
The square constructed on the hypotenuse is equivalent to the squares constructed on the legs.
Pythagorean theorem.

Figure 6: Question on proof and conclusion of the collective work on IWB (screenshots)

Discussion and concluding remarks

This paper aims to study the approach to the Pythagorean theorem using some physical artifacts that are material representations of that theorem. Students' answers to different task seem to confirm the assumption that the manipulation carried out by the students on the first mathematical machine is not enough for the emergence of mathematical meanings embedded in the machine. About our first research question, the analysis shows that those tasks allow fostering the production of signs, according to the theoretical framework of the TSM, and representations that can be used by the teacher for the mediation of mathematical meanings.

The scheme of use of shifting triangles for obtaining different configurations can support the emergence of personal signs and show the Pythagorean theorem in the context of equivalence of areas. For instance, in the first task of describing M1, some students recall the Tangram. If this meaning is not available for the students, the teacher has to focus on areas through a written, and/or symbolic calculation. With respect to our second research question on the feasibility of approaching the theorem with artifacts, we can argue that the Tangram, or meaning related to it, can be considered a prerequisite. In this case, Tangram means equivalent areas and manipulation of pieces for obtaining equivalent figures.

The comparison between the resolution of the tasks of making M1 by paper and representing M1 on workbook pays attention that the two tasks foster two different visualizations, as we have asked in our third research question. The first task solicits an iconic visualization of the two configurations, in which the shapes are drawn, but not their relationships inside the same configurations and between the two configurations. The second task seems to support a non-iconic visualization, because the students have to choose the measures of the sides of the figures (that are the parameters of M1) and make links between them. This choice has the potential of giving access to generalization to all the right triangles. However, it is not enough to draw twice a square and four triangles but the students have to represent their relationship, that is, an iconic visualization does not support the resolution as the wrong representations on workbooks show. Moreover, the students do not use the previous description of the components of M1.

Within the TSM framework, when the teacher proposes the discussion about those representations, the students' drawings are pivot signs for him. They are signs related to the artifact, but they are used for identifying and representing geometrical properties and invariants of M1. The potential of giving access to generalization is exploited by the teacher.

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The design of a teaching sequence on axial symmetry, involving a duo of artefacts and exploiting the synergy resulting from alternate use of these artefacts

Antonella Montone[1], Eleonora Faggiano[1] and Maria Alessandra Mariotti[2]

[1]Università di Bari Aldo Moro, Dipartimento di Matematica, Italy; antonella.montone@uniba.it; eleonora.faggiano@uniba.it

[2]Università di Siena, Dipartimento di Scienze Matematiche e Informatiche, Italy; mariotti21@unisi.it

This paper presents a teaching sequence conducted with 4th grade students, aimed at the construction/conceptualization of axial symmetry and its properties in which a crucial role is played by a duo of artefacts. This consists of a concrete artefact and a virtual artefact which address the same mathematical content. According to the Theory of Semiotic Mediation, both the artefacts have been chosen for their semiotic potential, in terms of meanings that can be evoked by carring out suitable tasks involving their use. The design of the teaching sequence is developed with the purpose of exploiting the synergy between the artefacts, in such a way that each activity boosts the learning potential of all the others.

Keywords: Axial symmetry, duo of artefacts, synergy of artifacts, semiotic mediation.

Introduction

The study of geometric transformations originates from the observation of phenomena and regularities present in real life, and takes on a particularly important role in the field of mathematics, both as a mathematical concept in itself and as a tool that can be used to describe geometric figures. For this reasons, it can offer an interesting lens through which investigate and interpret geometric objects, thus contributing to the development of students' reasoning and argumentation skills (Xistouri & Pitta-Pantazi, 2011). However, an effective use of transformational geometry in mathematics education requires a correct mathematization process of real life observations, ending with the mathematical formalization of concepts and properties (Ng & Sinclair, 2015). This process of construction of meanings could be fostered by the use of artefacts. But, the design of the teaching sequences needs to be developed according to a theory that can take into account the key transitions from the personal meanings, emerging from the activities, to the mathematical meanings, that are the aims of the teaching. This paper presents a teaching sequence aimed at the construction/conceptualization of axial symmetry and its properties, in which a crucial role is played by a *duo of artefacts* (Maschietto & Soury-Lavergne, 2013). This is composed of a concrete artefact and a virtual artefact which address the same mathematical content. The design of the teaching sequence is framed by the Theory of Semiotic Mediation and is developed with the purpose of exploiting the synergy between the artefacts. Both the artefacts have been chosen for their semiotic potential, in terms of meanings that can be evoked when carring out suitable tasks involving their use. The components of the concrete artefact are a sheet of paper and a pin, while the components of the virtual artefact originate from the components of a specific dynamic geometry environment (New Cabri - Cabrilog), in which microworlds focused on particular concepts can be created. The research hypothesis concerns the synergic action expected to develop when alternating the use of the concrete

artefact and the virtual artefact, so that each activity can boost the learning potential of all the others. The aim of the paper is to highlight key moments of the design of a teaching sequence and to underline, in particular, how the meaning emerges not only through the unfolding of the semiotic potential of the two different artefacts, but also strongly through the synergy activated by the alternate experiences gained using the duo.

Theoretical framework

The geometric concept addressed in this research is axial symmetry, in the sense of the isometric transformation of the plane itself, with a line of fixed points (the axis); from the definition it can be deduced that axial symmetry transforms straight lines into other straight lines, segments into other congruent, comparable segments, and it is an involutory function (Coxeter, 1969). Attention will therefore be paid to the symmetrical properties by means of which it is possible to construct the symmetrical point from a given point in regard to a straight line, in other words the perpendicularity of the axis with respect to the line joining the corresponding points, and the equidistance of the two points from the axis. Although geometric transformation is traditionally reserved for high school students, we believe that it becomes crucial already for the primary school students to move from a generic perception of regularity to that of correspondence between figures, and subsequently to the transformation (point by point) of the plane in itself (Sinclair & Bruce, 2015). The design we present is based on the theoretical framework of semiotic mediation. The Theory of Semiotic Mediation (TSM), developed by Bartolini Bussi and Mariotti (2008) in a Vygotskijan perspective, deals with the complex system of semiotic relations among fundamental elements involved in the use of artefacts to construct mathematical meanings: the artifact, the task, the mathematical knowledge that is the object of the activity, and the teaching/learning processes that take place in the class. The aim of the teaching is to guide the evolution of personal meanings toward mathematical meanings, recognized as such by the math culture that the teacher needs to mediate. In a long, complex interweave process the teacher can foster the shared construction of mathematical signs. Some recent researches have drawn on TSM focusing the interplay between static and dynamic reasoning in the teaching and learning of geometry (i.e. Bartolini-Bussi & Baccaglini-Frank, 2015). The main aspect that we focused upon in the design process of the teaching sequence was the semiotic potential. The semiotic potential of an artefact consists of the double relationship that occurs between an artefact and, on the one hand, the personal meanings emerging from its use to accomplish a task (instrumented activity), and on the other hand, the mathematical meanings evoked by its use and recognizable as mathematics by an expert (Bartolini Bussi & Mariotti, 2008). This potential is the basis underlying both the design of the activities and the analyses of both the actions and production of signs and the evolution of meanings.

The duo of artefacts involved

As stated above, a duo of artefacts is employed: concrete and virtual. The concrete artefact consists of a sheet of paper, with a straight line drawn on it marking where to fold it, and a pin to be used to pierce the paper at the right points in order to construct their symmetrical points. This artefact allows an axial symmetry to be created in a direct fashion because the sheet naturally models the plane and the fold allows the production of two symmetrical points using the pin. The virtual artefact has been designed by the Authors to exploit the added value conferred by technology to the use of the chosen concrete artefact. It is embedded in an Interactive Book (IB) created within the authoring environment

of New Cabri, which allows learning activities to be designed and created, including the objects and tools of a dynamic geometry environment. The IB appears as a sequence of pages including the designed tasks, together with some specific tools that correspond to specific elements of the concrete artefacts. In particular, among the tools available in the authoring environment, and in agreement with the general principles of dynamic geometry, the tools chosen are: those that allow the construction of some geometric objects (point, straight line, segment, middle point, perpendicular line, intersection point), the "Symmetry" and "Compass" artefacts and the "Trace" tool. A fundamental role is also played by the drag function, boosted by the "Trace" tool, that allows to observe the invariance of the properties characterizing the figures.

Research methodology

The study reported in this paper is inserted in a larger project that is aimed at validating the hypothesis regarding the possible synergic effect of the use of the two artefacts. The methodology employed is that of the teaching experiment (Steffe and Thompson, 2000). In this context the design of the teaching sequence plays a key role, because it is this sequence, designed in conformity with the chosen theoretical framework and the teaching hypotheses formulated, that constructs the teaching/learning environment where the observations will be made and, in general, the data collected on which to analyze the results of the experiment. In accordance with the TSM, the design of the teaching sequence follows the general scheme of successive "didactic cycles". The expression didactic cycle refers to the organization of teaching in activities. These consist of using the artefact, individually producing signs and then in the end collectively producing and absorbing signs through Mathematical Discussion activities (Bartolini Bussi, 1998). As regards the design of the activities using the artefact, in accordance with the study hypothesis that the two types of artefacts may be complementary, it was decided to alternate activities involving the use of one or the other artefact, formulating tasks that could exploit the complementarity of their semiotic potentials. The devised sequence was accompanied by an a priori analysis illustrating the semiotic potential expected to emerge during the activities.

Developing the sequence

In this paper we present the design of the sequence, addressed to 4th grade students, describing the six didactic cycles that make it up, and the tasks and semiotic potential of the artefacts involved. These are related to the conceptualization of axial symmetry as punctual transformation, and the properties that allow us to construct a symmetrical copy of an object with respect to an axis.

The first didactic cycle and the semiotic potential of the concrete artefact

The first didactic cycle involves three tasks (T1, T2 and T3). Given a figure (convex quadrilateral) drawn (in black) on a sheet, at the moment when handing over the sheet a red line is drawn on it. In T1 the pupils are asked to draw in red a symmetrical figure to the black one, with respect to the red line, by folding the sheet along the line and using the pin to mark the necessary symmetrical points by piercing the paper. After completing this task, on the same paper a blue line is drawn and in T2 they are asked to draw a blue symmetrical figure to the black one, employing the blue line. In T3 the pupils are asked to write an explanation of why and how they drew the red and blue figures and what looks the same and what looks different about them. In these first three tasks, folding the paper along the line evokes the meaning of axial symmetry, while the holes/points created with the pin evoke the

idea of symmetry as puctual correspondence. In addition, joining the points obtained with the pin is the process that yields as product the symmetrical figure, provided that the correspondence between the segments is preserved. This evokes the idea of symmetry as a one-to-one correspondence that transforms segments into other congruent segments. Finally, comparing what changes and what stays the same when drawing two symmetrical figures with respect to two distinct axes evokes the dependence of the symmetrical figure on the axis. The use of the pin can allow the meaning of the punctual correspondence to emerge without necessarily needing to explain the functional dependence between the points. In addition, folding the paper, so as to make one figure coincide with the other, can allow the intuitive meaning of line/axial symmetry to emerge as the element that characterizes the transformation. Finally, T3 makes the pupils reflect on the invariant aspects and the key role of the axis, when creating a symmetrical figure by folding the paper.

The second didactic cycle and the semiotic potential of the virtual artefact

The second cycle involves two tasks (T4 and T5) to be carried out using the virtual artefact: the button/tool "Symmetry" with the dragging function. In T4, the pupil is asked to build the symmetric point of a point A with respect to a given line, using the "Symmetry" tool and call it C. The second step is to activate the "Trace" tool on point A and point C, move A and see what moves and what doesn't, and explain why. In the next two steps, in the same way the pupils are asked to move the line and the symmetrical point, after having activated the "Trace" tool on A, and to watch what happens during the dragging. In T5 the pupils are asked to write down in a summary table the answers to the questions asked by the interactive book in T4. In T4 and T5 clicking on "Symmetry" evokes the meaning of symmetry as punctual correspondence and once more underlines the key role of the axis as the element that characterizes the application, because in order to obtain the symmetry it is necessary to click not only on the point but also on the axis. Moreover, dragging the point of origin and observing the resulting movement of the symmetrical point evokes the idea of the dependence of the symmetrical point on the point of origin; dragging the axis and observing the resulting movement only of the symmetrical point evokes the idea of dependence of the symmetrical point on the axis; dragging the symmetrical point and observing the resulting rigid movement of the entire configuration evokes the idea of the dual dependence of the symmetrical point both on the point of origin and on the axis: the effect of the various drags is made even more evident by the "Trace" tool and by the observation of the relations among the trajectories. The difference in the movements between the symmetrical point and the point of origin can be compared to the distinction between dependent and independent variable.

The synergy resulting from alternate use of the artefacts

The hypothesis formulated is that the observation the pupils need to do in T4 will cause the concrete experiences they have already had with the concrete artefact to reemerge, in other words, that the images on the screen can be better interpreted in the light of the previous acts of folding and piercing. In this way we expect that the meanings that have already emerged thanks to the use of the concrete artefact may be extended and completed by the specific meanings that should emerge using the virtual artefact. In short, the expected phenomenon is that a reciprocal boosting process will occur, in the form of a synergic process of mediation through the different types of artefacts. For example, after having constructed the symmetrical point using the button, the relation between the two points can be interpreted through the actions of folding, so the two points can be seen as two holes. But the

meaning of the relation, that is symmetrical, can be enhanced by the distinction between the original point and the corresponding point, thus contributing to the development of the mathematical meaning of a functional – asymmetrical – relation between a point (independent) and its symmetrical point (dependent).

The third didactic cycle and the semiotic potential of the concrete artefact

According to our hypothesis, the third didactic cycle involves three tasks (T6, T7 and T8) again using the concrete artefact. In T6 the pupils are guided as they see how correct folding yields the perpendicularity between the segment joining two symmetrical points and the axis, and the equidistance of the symmetrical points from the axis. In T7 they are asked to construct a symmetrical point without using the pin but just by correct folding. In T8, finally, they are asked to explain what two segments joining two distinct pairs of symmetrical points have in common and what is different about them. In the tasks of the third cycle, folding the paper along the line passing through the two corresponding points and then, without opening, along the axis and finally observing the superimposition of four right angles, evokes the properties of perpendicularity between the axis and the segment joining two corresponding points; observing that the two points are superimposed when folding along the joining line and then, without opening, along the axis, in other words that the segment joining the two corresponding points is cut in half by the axis, evokes the property of equidistance of each of the two points from the axis. The complex folding processes required in the accomplishment of these tasks can be compared to the symmetry of the relationship between perpendicular lines and evokes the idea that the perpendicularity and equidistance properties allow a symmetrical copy of points to be constructed with respect to a line without needing to use the pin but just by folding correctly. Comparing the segments to be created in T8 could allow to see the perpendicularity and the equidistance as being characterizing properties. Finally, from the mathematical point of view, the step that leads to the elimination of the pin is fundamental in order to bring about the evolution of the meaning of symmetry from the simple operative level of folding, to the mathematical meaning of geometric transformation identified by a line and the geometric properties that characterize it.

The synergy resulting from alternate use of the artefacts

We expect that, the interpretation of the actions and the configurations with the concrete artefact might be related to the experiences within the virtual environment. In particular, we may expect that two different points, of which to construct the symmetric points, can be interpreted as different positions adopted by a point that has been dragged, thereby contributing to the generalization of the two properties (perpendicularity and equidistance) and to the evolution of the status of these properties from being seen as contingent to being seen as characterizing.

The forth didactic cycle and the semiotic potential of the virtual artefact

The fourth cycle involves two tasks (T9 and T10) to be carried out using the virtual artefact composed by the buttons/tools “Perpendicular line”, “Compass” and the dragging function. In T9 pupils are asked to construct the symmetrical point of a point A with respect to the given line, without using the tool “Symmetry”, and call it C. Then it asks them to check whether the construction they have made is correct, using the tool “Symmetry” and moving point A. In T10 it asks them to explain how they found C and why what they did works. Clicking the button “Perpendicular line” and then on point A

and on the axis, evokes the idea of the perpendicularity between the segment for A on which the symmetrical point lies and the axis; clicking on the button "Compass" and then on the intersection point between the axis and the line through A perpendicular to the axis and on A, evokes the idea that the symmetrical point is obtained from the intersection between the circumference thus created and the perpendicular line, and so is at the same distance as A from the axis; constructing the line through A perpendicular to the axis and the circumference with the center at the intersection point between the axis and the perpendicular line and radius at the distance of A from the axis, evokes the idea that by using the properties characterizing the symmetry, already previously constructed, it is possible to identify the symmetrical point.

The synergy resulting from alternate use of the artefacts

In the same way as occurred for task T7 we expect that so as to construct C, without using the tool "Symmetry", the pupils will need to rely on the properties of perpendicularity and equidistance, already emerged from folding activities. However, this will bring them to recognize and to reuse these properties to construct the symmetrical point using specific buttons. These are quite complex notions and we do not expect the resolution process to be immediate but rather to be the result of trial and error. We also expect that the recognition of perpendicularity and so the possibility of using the button "Perpendicular line" may act synergically on the construction of the signs built up during the whole process, in terms of both images and words. We then expect a quite different complexity to present when transforming the properties of equidistance using the tool "Compass" (whose use should not be correctely linked to the mathamtical meanings embedded into it): the conceptualization of the configuration could consist of the relation between the segment joining the two points and the axis that divides it in half, rather than have been conceptualized in terms of distances and equalities among distances.

The fifth and sixth didactic cycles: inverting the order of the artefacts

In the fifth and sixth cycles the order of use of the two artefacts is inverted and they start with the virtual artefact. Both the cycles consist of the same two tasks (T11 and T12; T13 and T14), the difference is in the artefact. In T11 and T13 there are a pair of points A and C that must be interpreted as symmetrical points with respect to a symmetry where the axis is hidden. They are asked to identify and trace the axis. Finally, they are asked to check, using the button/tool "Symmetry" or with the pin, whether the symmetrical point of A with respect to the line is really C. In T12 and T14 they are asked to write down how the axis was identified and to explain why what they did works. In the tasks of these two last cycles, drawing the segment AC and then using the button "Midpoint", such as folding along the line through A and C, and then without opening the paper, folding so as to superimpose points A and C, evokes the idea that the middle point is a point that is equidistant between A and C and so must belong to the axis; observing that by folding so as to superimpose A and C you obtain the superimposition of four equal angles, evokes the idea that the line/fold for the middle point that allows the superimposition of A and C is perpendicular to segment AC; clicking on the button "Perpendicular line" and then on the middle point between A and C and then on segment AC, such as folding first along the line through A and C and then without opening, superimposing A and C, and seeing that four right angles are formed, evokes the idea that the axis is perpendicular to the segment joining A and C, as well as that it is perpendicular to the axis, as they had already seen. It should be noted that these tasks have been devised so that the same properties of symmetry used to

construct the symmetrical point with respect to a line (without using the artefacts “Symmetry” and pin) can be used to identify the axis that generates a pair of symmetrical points. But to draw up the construction the pupils need to invert the relation of perpendicularity between the axis and line through A and C. In addition, the property “the middle point of segment AC lies on the axis” must be redefined as “the axis passes through the middle point”. Also in this case it is a form of inversion of the belonging relationship, expressed in two different ways that have the same geometric meaning but that focus attention (by inverting the subject of the sentences) on one or the other element of the relation.

The synergy resulting from alternate use of the artefacts

The use of the same task (T11 and T13) with the two different artefacts, is not accidental but has been designed with the aim to bring out the common elements between the different schemes of use of the artefacts. We expect that this strengthens the idea that the two construction are both based on the use of the characteristic properties and are feasible only using them. In particular, what emerged in the previous activities, related to the double folding and to the properties of axial symmetry needs to be thinked over within the collective discussion aiming to bring out the development of the operational meaning of perpendicularity toward the geometric meaning of mutual relationship between lines. This is expected to recognize the geometrical meaning of the word perpendicular and of the configuration composed by two intersected lines so that four right angles are formed. In conclusion, two signs could be shared, a verbal and an iconc, defining the perpendicularity as “a property concerning two lines that by interesction form four equal angles”. It could be also noted that, this can be connected to the common routine to construct the “sample” of a right angle by means of a double folding. In T12 and T14, we expect that the pupils will describe the construction by listing, in the fifth cycle the used button and in the sixth the folding actions carried out and their effects. The relationships between a button and its embedded property such as the ones between the folding and its effects should emerge in the pupils’ descriptions.

Final remarks

The teaching sequence described above has already been experimented in a first pilot study. The analysis of results, based on videotapes and dialog transcriptions, has shown that the use of the duo of artefacts seems to develop a synergy whereby each activity enhances the potential of the others (Faggiano et al., 2016). Our research hypothesis concerning the synergy developed through using the artefacts has been validated. For instance, in the second cycle it was seen that the dynamic representation of the points and the observation of the coordinated movements of the points of origin and its symmetrical point, characteristic of the virtual artefact, recalled the meaning of correspondence between points that had previously emerged when piercing the paper with the pin using the concrete artefact. In this way, the dynamism of the virtual artefact enahnced the understanding of point-to-point correspondence, paving the way to making further considerations about the correspondence between segments and between lines. The study is still in progress but the results obtained encourage us to go ahead and develop a long term teaching experiment to confirm them.

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Hide and think: Limiting the obvious and challenging the mind in geometry

Chrysi Papadaki

University of Bremen, Germany; chrysi@uni-bremen.de

In this paper I will present the implementation and some preliminary results of a teaching experiment conducted in a 9th grade geometry class within the framework of my doctoral studies. For the teaching experiment I designed geometry tasks in a Dynamic Geometry Software (DGS), with specific characteristics: they involve transitions from 2D to 3D geometric objects (and vice versa), they are "Black Box" tasks and they are designed to cause surprise, uncertainty or cognitive conflict to the students. The main foci of the paper are the description of the tasks, examples of strategies that a pair of students followed while dealing with them and the influence that the characteristics of the tasks had on students' strategies, visualization and argumentation.

Keywords: DGS, argumentation, visualization.

Introduction

In the lesson of geometry students are often asked to solve a problem, which is usually accompanied by a drawing. In many cases the drawing acts as an obstacle for students' argumentation, as "since they are able to *see* results on the drawings, since they can work easily on it, mathematical proof seems to be useless" (Mithalal, 2009, p. 796).

The idea behind the study presented in this paper could be expressed by the following question: What would happen if we asked students to solve geometry tasks designed in a DGS, in which the drawing of a geometrical solid that they would have to identify was actually invisible to them? This is something that one couldn't do with physical objects, hence the designing of the tasks in a DGS. My hypothesis is, that when challenging the students with such a task, they will reach a point at which they will realize that when naive visualization is limited or fails, they have to turn to more efficient strategies in order to prove a conjecture or justify an answer. In this hypothesis the strategies are based on the use of geometrical properties of the figure. The aim of the study is to examine how students' argumentation can be promoted by using tasks with specific characteristics and how visualization and argumentation are interweaved in 3D-DGS tasks.

I will show how these two parts of the geometrical activity are linked theoretically and then present the teaching experiment that was conducted and some preliminary results.

Theoretical background

The interplay between visualization and argumentation in the frame of tasks designed in DGS

The notion of visualization has constituted a topic of research interest both for psychologists and mathematics education researchers. Nevertheless, until today researchers have not yet agreed on one unique definition regarding this notion. In this problem-solving context it is the cognitive part of visualization that will be taken into account, as it is related to argumentation.

Duval (2005) classifies visualization into two categories, *iconic* and *non-iconic visualization.* In *iconic visualization* "the drawing is a true physical object, and its shape is a graphic icon that cannot

be modified. All its properties are related to this shape, and so it seems to be very difficult to work on the constitutive parts of it" (Mithalal, 2009, p. 797). In *non-iconic visualization*

> the figure is analysed as a theoretical object represented by the drawing, using three main processes:
> **Instrumental deconstruction:** in order to find how to build the representation with given instruments.
> **Heuristic breaking down of the shapes:** the shape is split up into subparts, as if it was a puzzle.
> **Dimensional deconstruction:** the figure is broken down into *figural units* — lower dimension units that figures are composed of —, and the links between these units are the geometrical properties. It is an axiomatic reconstruction of the figures, based on hypothetico-deductive reasoning (ibid, p. 797)

It follows that argumentation isolated or visualization isolated, is not enough for students to solve a problem or prove a statement. There needs to be a continuous interplay between the two, for students' geometric reasoning to progress and evolve. And it is through the use of tasks designed in a DGS that the interplay between visualization and argumentation is made stronger.

Several studies that have been conducted show that the use of DGS can promote students' visualization skills (see for instance Christou et al., 2006). During the last three decades DGS, like GeoGebra and Cabri, are being used more and more in the teaching of mathematics in secondary education. In this paper, the focus regarding DGS will be on their use in geometry teaching.

As Laborde characteristically writes: "DGS contain within them the seeds for a geometry of relations as opposed to the paper and pencil geometry of unrelated facts" (2000, p. 158). Healy and Hoyles (2002) and Jones (2000) argue that DGS could play an important role in supporting students formulating deductive explanations and therefore also in the development of students' deductive reasoning, as they

> appear to have the potential to provide students with direct experience of geometrical theory and thereby break down what can be an unfortunate separation between geometrical construction and deduction (Jones, 2000, p. 81)

Before I proceed to the presentation of the study and the tasks, I would like to give some more insight regarding the theoretical perspectives on which the task-design was based.

Theoretical perspectives behind the task design in this study

The design (the characteristics) of the tasks, aims at challenging the students to produce conjectures and examine their validity using strategies that go beyond iconic visualization and naïve empirical justifications, engaging them naturally in mathematical activity that involves non-conic visualizations and argumentation.

Characteristic No 1 – D-transitional tasks

According to Markopoulos (2003), understanding the properties of a solid is equivalent to understanding the characteristic parts of a 3D shape, the comparative relations between the same or different structural parts and how the elements of the solid are interrelated. That is an idea which is very close to what Duval (2011) calls "figural units" of a figure. That means that the properties of

the component parts (figural units) of a 3D geometrical object are also properties of the 3D geometrical object itself.

This correlation of properties between shapes and geometric objects of different dimensions is vividly present in tasks that involve both 2-dimensional (2D) and 3-dimensional (3D) geometric objects. This thought generated the idea of what I call *D-transitional tasks*. These are tasks involving transitions from 2D to 3D (and vice versa) geometric objects. Such tasks provide students with the opportunity to think about an object through the manipulation of another, and thus combine their properties and identify the relationships, connections and dependencies between their properties. Furthermore, contrary to other studies which also use 3D DGS geometry tasks (see for instance Hattermann, 2009), the students are not asked to see 2D units in 3D shapes, but to identify 3D shapes by studying 2D sub-figures of them.

Laborde argues that:

> If properties of figures are not conceived as dependent, a deductive reasoning has no meaning. The question of the validity of a property conditional on the validity of other properties would not arise in a world of unrelated properties (2000, p. 157)

As Pittalis and Christou explain:

> Students should understand that each paradigm of a 3D shape has a number of invariant and variant geometrical properties based on the properties of the isolated component parts and its own properties as a unified structure. The invariant properties constitute the criteria that the 3D shape should meet in order to represent an example of a class of 3D shapes (2010, p. 194)

Characteristic No 2 – Cognitive conflict, uncertainty and surprise

As cited in Hadas et al.:

> Goldenberg, Cuoco, and Mark (1998) stated that: "A proof, especially for beginners, might need to be motivated by the uncertainties that remain without the proof, or by a need for an explanation of why a phenomenon occurs. Proof of the too obvious would likely feel ritualistic and empty" (p. 6)
>
> They concluded that DGS may provide opportunities for the creation of uncertainties, leading students to seek for explanations (2000, p.128).

But uncertainty is not the only possible motivation for students to seek for a proof of the validity of their conjecture. According to Healy and Hoyles (2002), and Laborde (2000) students feel the need for explanation when what they observe on the computer screen gives them a feedback that is surprising or is in conflict with what they expected.

Characteristic No 3 – Black Box: A way to create uncertainty and surprise

Black Box activities were designed by Laborde (1998) in the context of geometry teaching. Such activities have also been used by Knipping and Reid (2005) during a research in geometry teaching using Cabri Geometry. In such a task, a construction is already offered to the students but the properties and rules on which this construction is based are hidden. The Black Box activities give to students the opportunity of interesting and productive explorations. "When students' predictions

turn out to be wrong, this is a good opportunity for asking 'Why is it so?' and calling for an explanation or even proof' (Laborde, 2002, p. 311).

The teaching experiment

The teaching experiment was designed by the researcher and implemented by the participating teacher as part of his geometry lesson. The participants were 24 students of a 9th grade class. *Five D-transitional* tasks were designed in the *3D Graphics* environment of GeoGebra 5. Before the implementation of the teaching experiment, a 90' session was dedicated to introduce the software to the students. During the teaching experiment the students worked in pairs on the computers.

For the purposes of the data collection there were used 3 cameras. Each camera was focused on one pair of students. For the three pairs of students that were video recorded, a screen-recording program was used to record their work in GeoGebra. The analysis of the data is based on the transcribed video of the students' discussions, the screen recordings and their notes on the worksheet they were provided with for each task.

Description of the tasks

In each task the GeoGebra window was divided into two sub-windows. On the right sub-window (*3D Graphics*) there was a 3D coordinate system in which a solid was designed, and a blue plane defined by axes x and y. On the left sub-window (*Graphics*) there were three sliders (h for height, t for tilt and s for spin), which the students could manipulate in order to move the solid, and a 2D depiction of the cross-section that was created when the solid intersected with the blue plane. The decision for the use of this representation was made, based on the prior knowledge of the students, who had already worked on intersections of solids in previous geometry lessons.

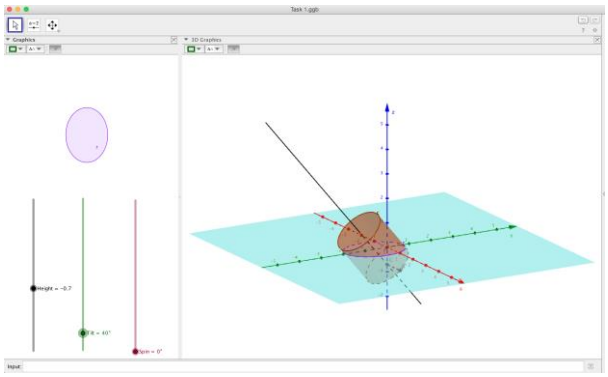


Figure 1: Snapshot of Task 1

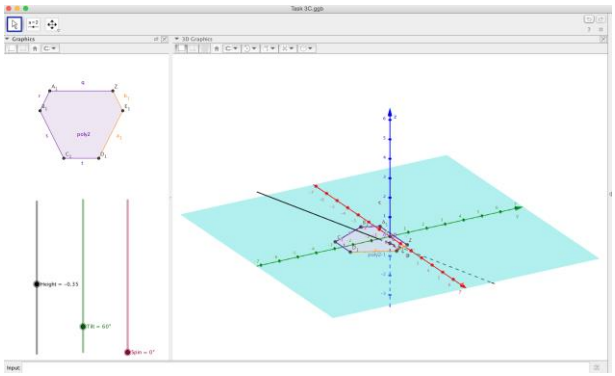


Figure 2: Snapshot of Task 3C

In Task 1 (see Figure 1), the solid (a cylinder) was visible. The students were first asked to experiment with the three sliders and describe what the function of each slider was and how it affected the position of the solid in space. Subsequently, they were asked to examine the shapes of the cross-sections that were created in some different (h, t, s) positions of the sliders.

In Tasks 2, 3A, 3B and 3C (see Figure 2) the solid was hidden. The question set to the students in each of these tasks was "Which solid do you think this could be, judging from its cross-sections?".

The teaching experiment was conducted in two phases. During each phase the students worked as shown in Table 1 below.

Phases	Steps of the Phases	Duration
1st Phase – Students work in pairs (1st day)	Task 1 – Visible Cylinder	22’
	➢ Classroom discussion	18’
	Task 2 – Invisible Sphere	15’
	➢ Classroom discussion	5’
2nd Phase – Work in parallel. Students work in pairs (2nd day)	Task 3A - Invisible Cone Task 3B - Invisible Pyramid Task 3C - Invisible Cube	50’
	➢ Classroom discussion	30’

Table 1: Phases of the teaching experiment

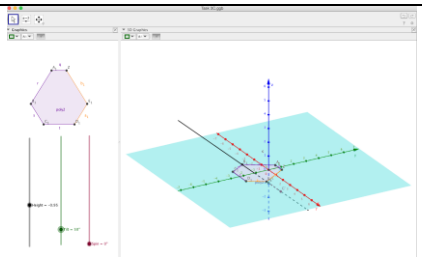
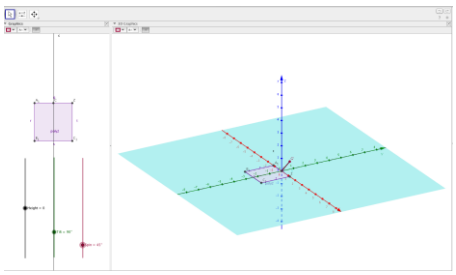
Before each phase, the teacher explained the lesson procedure to the students. He explained to them that the focus of these tasks should be the justification of their answers. That was both linked to the didactical contract and the tasks. The students were asked to be as precise and explicit as possible in their explanations, to use mathematical arguments in order to support their answers to the rest of the classroom, during the classroom discussions that would follow.

During the time the students worked on the tasks, the teacher and the researcher were only observers of the situation. During the classroom discussions the teacher acted as a facilitator of the discussion.

Preliminary results

As the analysis of the collected data is still in progress, I will here present an example from the data as part of the preliminary results of this study. I will present the analysis of some excerpts from Gabriel and Elbert’s discussion while working together on Task 3C (see Table 2). The students started by exploring freely the situation in the task, without using the worksheet that has been given to them as a helper.

Dialogue	Pictures and Analysis
G: Holy ****! What is this sh**? It should be symmetric! E: It could also be a cube. Change the height again. Make it zero (the tilt) and change the height. (They have set h=0 and d=0 and they change the tilt) E: Ay ay ay! G: We have here a diagonal cross-section, that is no... E: No cube (he laughs) G: No cube… Right?	The first sign of surprise is seen in the reaction of Gabriel, when he starts exploring the cross-sections performed, by the blue plane, on the solid when moving it with the sliders The students’ first conjecture, based on their visual perception, is that the hidden solid is a cube. Nevertheless, as soon as they find some unexpected cross-sections like pentagons and hexagons, their naïve visualization proves insufficient and they start questioning their initial conjecture.

	
[…] E: Quite…not a cube. But it should be a cube. G: Could be one (cube). E: The other face is definitely square. We have seen that. G: Mhm.. (affirmative) E: The top face is also square. So, it could be a cube, it could – G: Wait, wait, wait. Spin 90, 90. So, 45. So, now is this thing perfectly oriented. E: Or it could be a – what do you want to see? G: More than one, more than one. I have more than one, therefore lets try the tilt. Now it is straight, right? I have simply oriented the thing (the solid) on the coordinate system E: That would work, if it is a cube, but it could also be a, a thingy. How is it called again? Prism? [...] G: That would be a cuboid.	They start following a more organized exploration, by identifying some figural units of the hidden solid. They now start looking at the properties of the cross-sections (the 2D geometrical object), that is of the figural units, of the solid and relate and transfer them to the hidden solid (the 3D geometrical object). At this point, they discuss that according to the characteristics and properties of the cross-sections they have seen until now, they can only argue that the solid is a cuboid and not a cube yet. Although they have moved from iconic visualization to identifying figural units of the solid and some of their properties, they haven't yet reached the level of dimensional deconstruction.
[…] E: We know that the height and the width are equal. We know that it is square. That the faces are square. That means, it must – G: We check that, right? We are here at the smallest – E: No, this is so. Everything else makes no sense. We have a height of 2,35, a width of 2,35 and a length of 2,35. G: Well good. E: That is a square, eeh a cube! […] G: Yes, as we proved, the face is square. We have proved that. But	The students end up to the conclusion that their initial conjecture was true, basing their justification on deductive arguments linking together and relating the properties and the characteristics of both the visible cross-sections and the invisible solid. This process resulted to the reconstruction of the hidden solid by first achieving the dimensional deconstruction of its cross-sections. 

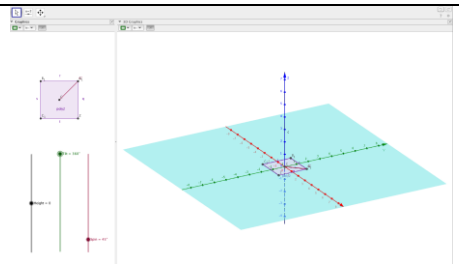
the length, the width and the height are equal, we have proved, because we changed the tilt and the object was always a square at 90, 180, 270 and 360 and 0. Just turned round the axis.	

Table 2: Excerpts from the collaborative work of Gabriel and Elbert on task 3C

Conclusion

Gabriel and Elbert wanted to come up with a justification of their conjecture that would be satisfying for them. The nature of the task and the students' initially "unorganized" exploration soon caused them surprise and cognitive conflict. The students started doubting their conjecture and they turned to a deeper exploration moving from iconic visualization to identifying figural units of the solid and their properties. They based their reasoning on the properties of both the cross-sections and the solid and also the dependencies between them. They achieved the reconstruction of a hidden solid, judging by its 2D cross-section and subunits. At the end they correctly identified the hidden 3D shape as a cube by studying its 2D sub-figures. They stopped and accepted their conjecture as true only when they had produced what for them constituted a valid argumentation.

There is still a long way to go until I can present some more general results from my study. Nevertheless, examples like the one presented here show that it is possible to support students' visualization, argumentation and deductive reasoning in geometry, by using tasks whose design is based on the idea of the interplay between visualization, argumentation and which have the characteristics presented in this paper.

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What influences grade 6 to 9 pupils' success in solving conceptual tasks on area and volume

Veronika Tůmová

Charles University, Faculty of Education, Prague, Czech Republic; veronika.tumova@pedf.cuni.cz

In this article, I investigate relationship of space structuring and multiplicative thinking with success in solving volume and area problems. The observations are made based on the analysis of written tests completed by 748 Czech pupils from grades 6 to 9. Phenomena observed in solutions of three volume problems are used as indicators of the level of space structuring and multiplicative thinking, which is then related to the pupils´ results in the conceptual part of the test. Relatively strong connections were identified for both factors.

Keywords: Volume measurement, multiplicative thinking, structuring of space, hypothetical learning trajectory.

Many Czech teachers mention measure in geometry as one of the critical parts of primary mathematics and Czech pupils scored in TIMSS well below their average in most of the tasks concerning area or volume. Moreover, our previous research (Vondrová, 2015; Tůmová & Janda, 2014) shows that pupils' problem often lies in conceptual understanding. However, the measurement in geometry is problem worldwide – the difficulty seems to be intrinsic to the topic itself (Duval, 2006; Finesilver, 2015; Kamii & Kysch, 2006).

The aim of my research has been to investigate how the conceptions of area and volume are built, what the major pitfalls and problems are, what skills and strategies are helpful for solving problems and what are the frequent unsuccessful strategies and pupils´ misconceptions.

Theoretical framework and review of existing literature

As Sáiz (2003) pointed out, it is helpful to distinguish between the concept of volume, which means a set of encyclopaedia knowledge of volume with its uses in various contexts (theoretical construct, or semantic field of volume) and conception of volume, which is its counterpart in the internal, subjective "universe of human knowing" (Pehkonen & Furinghetti, 2001, as cited in Sáiz, 2003, p.97).

The concept of volume is rather complex compared to area also because there are several ways it can be understood: e.g., as a capacity, a free space inside a closed surface or a space occupied by the solid (Potari & Spiliotopoulou, 1996). I will focus on the second perspective which I also understand like a space that can be filled with cubic units (limiting myself to rectangular prisms).

Further, I presume that pupils´ conceptions are built gradually and hierarchically – as described in the hierarchic interactionism theoretical framework presented by Sarama and Clements (2009, pp. 20nn). In order to describe a likely trajectory that pupils may take while getting familiar with the concepts of area and volume, I will use a tool called hypothetical learning trajectory (HLT). A learning trajectory describes a sequence of levels of thinking. It consists of a goal, a developmental progression, or learning path (i.e., ordered sequence of mental ideas and actions), and therefore it has also teaching implications (i.e., what instruction helps pupils to move along that path).

The starting point of my further work has become the HLT for volume as proposed by Battista (2007, p. 903) as it distinguishes two parallel streams: numerical reasoning and non-numerical reasoning

(Table 1). By non-numerical reasoning, I mean reasoning in the geometrical context about conservation, comparison, transformations, geometry motions, etc. The emphasis on non-numerical reasoning when building a conception of area or volume is in line with research (Kospentaris, Spyrou, & Lappas, 2011; Huang, 2011).

Seeing the HLT as two parallel streams, we can see another important feature of geometrical measurement: the importance of building connections between numerical and non-numerical aspects. For example, Huang (2011) showed that both the curriculum which stresses the numerical calculations for area measurement and the geometry motions curriculum aimed at developing the non-numerical reasoning have to be combined to improve pupils' performance.

<table>
<tr><th>Non-numerical reasoning</th><th>Numerical reasoning</th></tr>
<tr><td>1. Holistic visual comparison of shapes</td><td>1. Use of numbers not connected to unit iteration</td></tr>
<tr><td>2. Visual comparison of shapes by decomposing/recomposing</td><td rowspan="2">2. Unit iteration and enumeration (coordinated structuring of space into arrays), includes:
Units properly located only along the edges/sides
Units properly located without overlaps or gaps
Units organized into composites (layers) – repeated addition; multiplication
Operating with other units than cubes</td></tr>
<tr><td>3. Comparison of shapes by property preserving transformations/ decompositions
Connections</td></tr>
<tr><td></td><td>3. Operating on numerical measurement
Structuring becomes implicit – multiplication of measures</td></tr>
<tr><td colspan="2">4. Integrated numerical and non-numerical reasoning</td></tr>
</table>

Table 1: HLT used for the concept of volume

I will use this HLT as a roadmap, showing a hypothetical way how the conceptions of volume and area is built. In my work, I will try to find relations between selected elements described in the HLT (e.g. space structuration into arrays) and success in solving conceptual area and volume problems.

Battista (2007) distinguishes pupils' problems with space structuring of the situation and problems in connecting the space structuring with appropriate numerical procedures. The latter is apparently very difficult. Pupils often use a completely different formula (e.g., for the surface or area instead of volume) or substitute wrong measures into the formula (Vondrova & Rendl, 2015; Tan Sisman & Aksu, 2016; studies cited in Battista, 2007, p. 893). These problems can be found not only with pupils, but also with prospective teaches (Dorko & Speer, 2013).

Huang (2014) found a significant relationship between multiplicative thinking and performance at solving a certain type of area measurement problems. In our previous research (Tumova & Janda, 2014), we also identified a relation between multiplicative thinking and success in solving application tasks for volume and area: pupils who were able to calculate the number of tiles needed using division were much more successful than pupils who had to draw the actual tiling and use multiplication.

The structuring of space into arrays is one of the main building blocks in the HLT. Dorko & Speer (2013) hypothesize that the ability to structure space into arrays of cubes is related to the computational success in all their tasks. I wanted to explore these two issues on a larger scale.

Research questions

The article focuses on the part of my work dealing with volume and grade 6 to 9 pupils. The research questions are: (1) Is the ability to structure space into arrays related to success in conceptual tasks? (2) Can we see any relation between multiplicative thinking and success in conceptual tasks?

Methodology

Tasks

I constructed tasks roughly corresponding to the HLT of area and volume. They can be divided into three categories. The first category consists of *non-numerical tasks* that aim to find whether pupils can de-compose and re-compose 2D and 3D objects and manipulate these objects (or their parts) mentally. Results of these tasks are not used for the data analysis in this article.

The second type are *structuration tasks* – i.e., tasks that rely heavily on array structuration of space (manipulation). In 3D, tasks H12, H13 and H14 were used.

H12.3: The blue cube building has 20 cubes in the first layer. How many cubes do you have to add in order to get the smallest possible completely filled prism (in other words, if the building was in a tightfitting rectangular box, how many cubes you have to add in order to fill that box completely)?

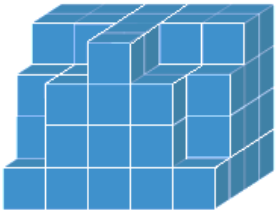

Figure 1: Blue cube building

H12.1. and H12.2. are similar, with different cube buildings with first layers of 9 and 12 cubes, respectively.

The aim of task H12 is to identify, how well the pupil can understand the array structure when the structure is depicted and unit cube is used. It can be solved by enumeration of the missing cubes, no calculation is needed.

Task H13*: You have exactly 59 cubes (with the edge of 1 unit) to build a cube building*[1] *on a plot of land which is 4 units long and 3 units wide. You have to use all of the cubes but the building has to be as LOW as possible. How many layers will there be? How many cubes will there be in the top layer? [No drawing provided.]*

In H13, two types of solving strategies are possible. The first is calculation-based: divide 59 by the product of 3 and 4 (i.e., the number of cubes in one layer). The resulting whole number means the completely filled layers and the rest is the number of cubes in the last incomplete layer: $59 : (3 \cdot 4) = 4$ (rem. 11). The other approach is partially manipulative: draw cubes in the bottom layer and see how high we can continue building until all 59 cubes are used (using repeated addition or multiplication). Based on the method the pupil will use and drawings he/she will make (e.g., if he draws just a rectangle, or structure consisting of rows and columns), we can hypothesize what kind

[1] This term was understood intuitively by pupils – it is a building made of unit cubes and a subset of a 3D array structure that can be physically built (all the columns must be built putting one cube on another).

of space structuring the pupil used and relate it to the levels in HLT (Table 1, rows 2 and 3 in the right column).

Task H14*: What is the maximum number of parcels measuring 2x1x1 dm that would fit into a cubic box with an edge of 6 dm? Justify your answer. [No drawing provided.]*

H14 also has two types of strategies. One is calculation-based: calculate the volume of the box and the volume of the small parcel and divide the two to get the correct number of fitting parcels. The manipulative strategy: draw or imagine how may parcels are in one layer and multiply it by the number of layers. H14 is more complex because the unit to be used is not a cube, also the volume of the box has to be calculated from the length of the side.

The third category of tasks are *conceptual tasks* (4 for area and 2 for volume) – these are non-routine or novel tasks that require more advanced understanding of the underlying principles and concepts than a simple use of formula (see also the definition of Tan Sisman and Aksu, 2016, p. 1298). After piloting the tasks, only those were selected as conceptual tasks to the main test in which the percentage of pupils that used structuration as their solving strategy was less than 10 %. Task H16 is a typical example.

Task H16. *A cuboid-shaped vase has a base of 9x12 cm. If I pour one litre of water inside, how high it will reach? (Hint: 1 l=1000 cm^2).*

Data collection

After a small scale pilot testing of the tasks and revision based on discussions with experts, the test was distributed to more than 1 300 pupils (grades 4 to 9) from 8 different ordinary primary schools in Prague (it is to be noted that this was a convenience sample). The pupils solved the tasks within the first three weeks of September 2015, after summer holidays to eliminate the influence of the mathematics topic currently taught and test pupils' long-term knowledge. The test was given by mathematics teachers who got instructions from me. The test took 55 minutes and it was completed by 735 pupils in grades 6 to 9.

Data analysis

Pupils' solutions were coded by the author and two more coders. Based on the above *a priori analysis* of tasks, we had some preliminary codes. The coders first coded for the preliminary codes and while doing so, assigned points for each task and noted other phenomena such as what mistakes appeared, what other calculations appeared in the written solution, etc. Examples of codes are in Findings section. The codes were further grouped and analysed both quantitatively and qualitatively. The bulk of the quantitative analysis consisted of looking for relations between phenomena and the total result in all the conceptual tasks (CONC total).

Note: Whenever I look at the performance of pupils, I always report the result of all conceptual tasks (CONC total) – i.e., for both area and volume problems (tasks H3-H6 and H15-H16). I wanted to assess the conceptual understanding of geometrical measurement therefore I included both concepts into the result. The results are reported in percentages calculated as the number of points actually achieved by the pupil divided by the maximum number of points.

Findings

Phenomena identified for structuration problems H12, H13 and H14 will be related to CONC total regardless of age of pupils. The findings will be grouped according to research questions.

Structuration of space and its relation to the success in conceptual tasks

H12 to H14 are hypothesized to form a series of tasks with a growing demand on space structuring abilities. H12 tests pupils' ability to structure space properly in a situation when the structure of cubes is given (shown in a picture) and the unit is a standard unit cube. H13 tests whether pupils are able to calculate the height of a cuboid based on the knowledge of its volume and base in a discreet situation. H14 diagnoses how pupils can structure space using non-cubic units.

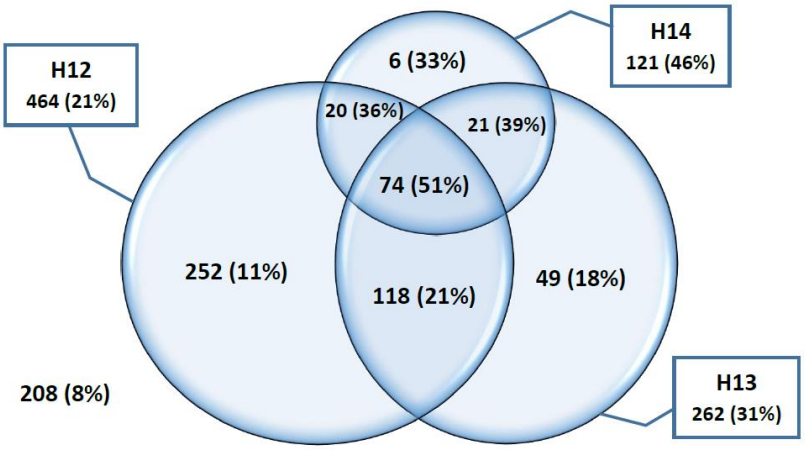

Figure 2: Venn diagram showing the overall test score and a number of pupils successful in H12-14

The Venn diagram in Figure 2 shows the number of pupils who scored at least 50 % in H12, H13 and H14. The number in brackets is the average CONC result (i.e., of all conceptual tasks) for relevant group of pupils. For example, 252 pupils successfully solved H12 but not H13 or H14, the average CONC result for these pupils is 11%. There were 74 pupils successful in all three tasks (in the centre) and their average result was 51%.

As expected, H14 proved to be the most difficult and the pupils who solved it, also scored the highest in the conceptual part of the test. As we can see, there seems to be a relationship between pupils' ability to structure space into appropriate structure (3D array) and their success in the conceptual tasks in the test. If we calculate correlation between the result of structuration tasks (H2, H12-14) and conceptual tasks (CONC total), we get a relatively high Pearson correlation of 0.66.

Further, in the solutions of H13, we identified almost all the strategies for space structuring mentioned in the HLT (pupils most often drew the first layer only, so the structuring is, in fact, a tiling). The strategies were coded as follows (see Table 2): "Unable to structure" (incorrect number of cubes in a layer, structure only along the edges – perimeter or structures 5x4, etc.), "Individual cubes" (each cube drawn separately), "Rows and columns" (partitioning drawn for whole rows and columns), "3D structure" (3D drawing), "Implicit structure" (no drawing or only the rectangle drawn and number of cubes correct), "Unable to identify" (where the structuring strategy was unclear from the test).

The difference in CONC total result is statistically significant between strategy "Implicit structure" and all other strategies[2]. This means that the pupils who can determine the number of cubes in one

[2] Data are shown in left part of Table 2. The independent sample T-test in SPSS was used for each pair of codes.

layer without having to draw the structure (according to HLT, this means that they have reached the level where the structure became implicit for them), perform significantly better than other pupils. Again, there seems to be a relation between the way the space structuring is depicted and success in the test – the higher level of structuring (according to HLT), the better average result in the conceptual part of the test.

To see connections with space structuration, we look at manipulative strategies in H14 in more detail (about half of the pupils solving this task used manipulative strategy) – see Table 2 right.

Structuring of area in H13	**Avg CONC total**	**No. pupils**
Unable to identify	24%	71
Unable to structure	11%	43
Individual cubes	19%	68
Rows and columns	21%	80
3D structure	27%	31
Implicit structure	38%	93
Total H13	**25%**	**386**

Strategy in H14	**Avg CONC total**	**No. pupils**
Strategy cannot be determined	16%	31
First layer incorrect	19%	41
No. of layers incorrect	24%	25
Numerical error	34%	5
Structure correct	47%	43
Manip. strategis total	**31%**	**114**

Table 2: Space structuring and strategies vs. success rates in tasks H13 and H14

First, we look at the manipulative strategies: 43 pupils were able to structure the space inside the box properly and solve the task (Structure correct) – they scored the highest in the conceptual tasks (the average result of 47%). These pupils performed significantly better in the conceptual tasks than pupils who used other strategies. Quite a few pupils (41) were not able to structure even the first layer (their average CONC result is only 19%). If we presume that the strategies represent the level of skill in space structuration, we can say that pupils who cannot fill even the first layer would form the lowest level, those who did one level correctly but did not get the number of layers right would be on the next level, followed by those who drew the correct structure. We can see that pupils who structured space better score higher in CONC total.

Multiplicative thinking and its relation to the success in conceptual tasks

To analyse the use of operation, we look at H13. The codes we used are (Table 3): “No operation mentioned”, “Unable to interpret results of operation” (includes 9 pupils who could not perform the operation of division correctly), “Repeated addition” (calculate the cubes one by one or by layers – i.e., 12+12+…), “Repeated multiplication” (such as $3 \cdot 12$, $4 \cdot 12$ until something bigger than 59 is reached), “Division and correct interpretation” (division $59 : 12 = 4$ (rem. 11) and correctly interpret results). All strategies were present in all grades – around 5% of pupils in all grades were not able to interpret results of the operation they used and 18% of all pupils in grade 9 still used repeated addition strategy to solve this problem.

Strategy_OPERATION	Average result of CONC and Structuration tasks	Average of CONC result	No. of pupils
No operation mentioned (NO)	25%	13%	126
Unable to interpret (UI)	23%	16%	33
Repeated addition (RA)	32%	22%	106
Repeated multiplication (RM)	38%	30%	53
Division and interpretation (DI)	47%	39%	68
Total	**33%**	**18%**	**386**

Table 3: Use of mathematical operation and success rates in H13

The independent sample T-test in SPSS showed that the combined result of Structuration tasks and CONC total result for strategy DI is significantly different from all other strategies, except from RM. The same is true for CONC result. This means that the pupils who can use division and correctly interpret results in this task, performed significantly better in all conceptual tasks and in all conceptual and structuration tasks than pupils who used repeated addition or pupils who could not interpret results of their calculations. The difference between RM and DI groups was not significant.

Quite a few pupils (33 in H13) were not able to interpret results of the operation they used. This is apparent in other tasks as well. In H14, there were 41 pupils who used what may seem to the observer like a random mathematical operation on given numbers and their interpretations of results were incorrect.

Conclusion and discussion

To sum up, the structuring of space seems to be related to the success in our conceptual tasks. We looked at space structuring from two perspectives. First, when pupils are able to apply the correct structuring (i.e., which of H12 to H14 are solved correctly). Second, how pupils depict the structure of a layer which is in a rectangular form with given lengths of sides. In both cases, the pupils who structure space better achieved better results in area and volume conceptual problems. This seems to confirm the hypothesis of Dorko and Speer (2013) above.

The relation between the use of mathematical operation (multiplicative thinking) and success in our conceptual problems seems to manifest itself even on a large sample which confirms our finding from the previous research (Tumova & Janda, 2014). From the observed number of pupils who were not able to interpret results of the operations they used we can see that connecting the appropriate mathematics operation with the geometrical situation is another major problem for Czech pupils as mentioned in (Battista, 2007). Not only how good is their multiplicative thinking, but what operation in which situation they decide to use and with what numbers might be the most important. Investigating this connection in more depth (how it is built, what promotes building it) remains an open question for future research. Also, connections between multiplicative thinking and space structuring seem to be worthwhile to investigate further. The connection seems to be bidirectional: structuring of space guides enumeration but also enumeration can help space structuring (Finesilver, 2015, p. 257).

One of the limitations of my research is the fact that the results cannot be easily generalised as the sample was not representative. Another limitation is in the set-up of research – I did inferences on pupils' thinking based only on what was written in the test. Some of my interpretations of the written solutions might need additional support in interviews with pupils.

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Shapes recognition in early school: How to develop the dimensional deconstruction?

Céline Vendeira[1] and Sylvia Coutat[2]

[1] University of Geneva, Faculty of Education, Geneva, Switzerland; celine.marechal@unige.ch

[2] University of Geneva, Faculty of Education, Geneva, Switzerland; Sylvia.coutat@unige.ch

In primary school there is, in geometry, an important rupture between primary (age 4 to 12) and secondary school. Some activities are already proposed for students from 8 to 12 years old to help to prepare for this rupture. Our research has the same aims, but for students from 4 to 6 years old by developing some pre-geometrical activities around shape recognition using dimensional deconstruction.

Keywords: Shape recognition, visualization, dimensional deconstruction, way of thinking.

Context

In all French speaking Switzerland we have a common program for all compulsory education. For the geometry curriculum, in the elementary division (from 4 to 8 years old), students use a physical space where «the shape is linked to the visual perception of the object»[1]. Then, from 8 years old, they use a conceptual space where objects are associated with figures as «unchangeable and ideal». These figures are independent of their graphical representation. We consider that there is an important gap between these two divisions, and we need to help students to overcome it.

Theoretical framework

Some researches like Berthelot and Salin (1993-1994), Houdement and Kuzniak (2000), Parzysz (2003), Braconne-Michoux (2008) show a rupture, in geometry teaching, between primary and secondary school where the focus is put on reasoning and deduction. Without reaching the theoretical level of the geometric objects and therefore of their properties, an intermediate work on the elements which compose the forms is possible and constitutes the heart of this research with students of cycle 1 and beginning of cycle 2. As Duval (1994) says, one of the aims of geometry in primary school is to emerge the operative apprehension of a figure parallel to the one, first and naturel, more perceptive. Therefore, to help, it "presupposes the dimensional deconstruction of the visual representations"[2] (Duval & Godin, 2005, p.11). The first visualization is global; the perception is centered on the closed contours of the shape. This visualization is called two dimensional visualization (2D element), which is referred to by Duval (2005) as the iconic way of seeing. The dimensional deconstruction considers the elements of the shape like the sides and the lines, which are one dimensional elements (1D element) and the vertices and the points, the zero dimensional elements (0D elements). This is what Duval (2005) distinguishes as the non-iconic way of seeing. The decomposition of shapes into figural unities is an essential stage prior to building the non-iconic visualization.

1 Our translation from the French speaking Switzerland curriculum « Plan d'études romand » (https://www.plandetudes.ch/per) : « la forme est liée à la perception d'ordre visuel d'un objet ».

2 Our translation.

Having as objective to evolve the students' visualization of geometrical shapes, we rely on the work of the Lille group (Duval (1994), Duval, Godin & Perrin-Glorian (2004), Duval & Godin (2005), Keskessa, Perrin-Glorian, Delplace (2007), Godin & Perrin (2009), Perrin-Glorian, Mathé & Leclercq (2013), Perrin-Glorian, Godin (2014), Perrin-Glorian (2015), Bulf & Celi (2016)) which consider the transition between the recognition of a form by global perception and the deduction from its axioms as the "identification of properties that are verified or produced with instruments"[3] (Houdement and Godin, 2014, p.28). Most of those researchers propose tasks of reproduction of figures in particular the reproduction problems called "restoration of figures (define by Perrin-Glorian and Godin (2009)). In our case, we work on tasks of forms recognition. To prepare students to the rupture pointed out between primary and secondary school, we introduce two new levels between the global perception and the non-iconic way of seeing. We call them the “hybrid thinking” and “thinking by characteristics”. Below, we associate each thinking according to the ways of seeing of Duval. We give an example based on the following shape: 

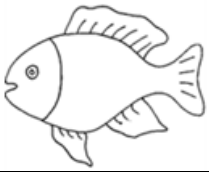
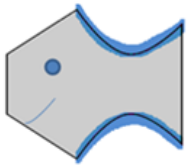
	Iconic visualization	Towards dimensional deconstruction		Non-iconic visualization
Way of thinking	Perceptive thinking	Hybrid thinking	Thinking by characteristics	
Associated vision				
Associated language[4]	« It looks like a fish»	« It looks like a fish with a flat nose and a curved body »	« It is a shape with holes (the convex character). It as rounds and straights (straight and curved edges) »	

Table 1: Levels of the different visions of the shapes

Those levels focus 1) on a global vision of the shape (2D) 2) on a hybrid vision of the shape (using the global vision and some characteristics of the shape (0-1-2D) 3) on the vison of only some localised areas of the global form (as the types of lines (straight or not, etc.)) considering therefore some of the characteristics of the shape (0-1D). This third level corresponds to pre-geometrical work. The last level “non-iconic visualization” relates to the definition of the properties of geometric figures. So, to think by the characteristics is more than a perceptive vision but does not yet correspond to the geometric properties. As for the hybrid thinking, it mobilizes in the same time the global vision of the shape, through its surface, and a more expert vision from its elements.

This table shows a wider process of dimensional deconstruction than usual. Thus, our goal, in this research, is to develop some pre-geometrical activities around shape recognition using these two new levels to provide students with a more harmonious transition in order to overcome the gap between primary and secondary school.

3 Our translation.

[4] In the next steep of our research we will also consider the “associated action”.

The developed material

In order to help 4 to 6-year-old students to pass from the perception of the shapes as they are worked in primary school to what is expected in secondary school, we propose to work on shapes recognition tasks with a collection of 36 shapes. Very often, the most common use of characteristics concerns the number of sides of the shape. However, students at this age are precisely building the concept of number which is therefore "fragile". Other characteristics of shapes are nevertheless affordable and interesting from Cycle 1. For example, the presence of straight or curved edges, symmetries, parallel opposite sides or the convex or concave character of the shape. Of course, students are not expected to use the correct mathematical terms. What is important is that they identify these characteristics, whatever the vocabulary used. The collection of 36 shapes takes into account the different characteristics cited. Figure 1 presents the collection of the 36 shapes. All the different tasks are built around this collection (tasks of classifications, associations and housing).

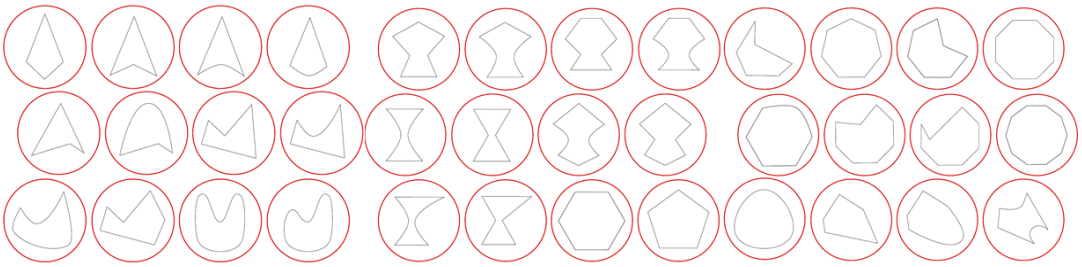


Figure 1: The collection of the 36 shapes

The chosen shapes are not nameable, at least not using "classical" shape names such as triangles, squares, rectangles, circles. To identify them, students have to focus on other aspects than their name. Either students recognize, in the shape, a resemblance to a well-known object (for example the fish mentioned above), or they are obliged to refer to its characteristics. This last point promotes dimensional deconstruction with the components of shapes.

Thus, the main objective aimed through the developed activities is that the student constructs a thinking of the objects based on their characteristics. This does not mean that it is necessary to replace the global thinking, but to supplement it. The joint use of the hybrid thinking and the thinking by characteristics is therefore also necessary. For example, in many activities, students can first classify the shape by global perception (we put together shapes perceptually close), and then only need to distinguish them through their characteristics.

Below is an example of strategy to find a shape in a collection of eight shapes  :

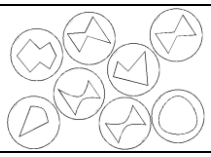
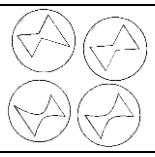
Starting Collection	Reduced collection	Identified shape in the reduced collection
		
I observe and manipulate the assortment of forms	I extract all "bow ties" (or "vases") of the assortment.	I then focus on the characteristics "it is the one that is not regular (=symmetrical) and with rounds (=curved edges)". I select then the corresponding shape.

Table 2

The choice of the selected shapes is essential because it can directly influences the way of thinking that the students will mobilize.

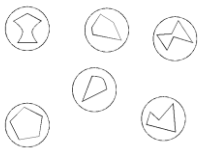
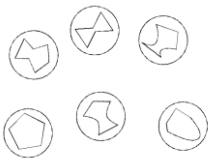
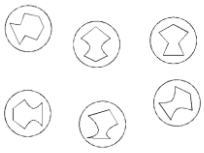
Perceptive thinking	Hybrid thinking	Thinking by characteristics
		

Table 3

Thus, if the selected shapes are perceptively distant, the overall vision is promoted. Conversely, if the shapes are perceptively close, an entry through the characteristics is necessary. These shapes are cut out from "translucent" Plexiglas. The choice of a circle does not favour any particular orientation. With this material, we can work on the shape using the edging by the exterior or the interior of the shape. Depending on the activity, we may choose to present one or both of these supports (or both). The interest of this material is that it can be embedded providing direct feedback for students.



The inside form



the empty circular part

With this collection of shapes we then create activities working on shape recognition with different kinds of tasks, classifications, associations and housing.

Methodology

For one school year we have worked with six classes including four classes in a downtown Geneva school, a class in the Geneva countryside and a class in neighbouring France. This diversity makes it possible to confront our material and our activities in different contexts (without pretending any generalization). In total there were 112 students aged from 4 to 6. At the beginning and the end of the year all the students took a test. We do not develop the results of the tests in this article. The students worked in small groups with the researchers from 2 to 4 periods in total. Each period lasts 45 minutes. In every class, except one, we realized, among the proposed periods, a session with big shapes in the “meso-space”.



Photo 1: Work with big shapes

In each class we have: 1) One individual pretest at the beginning of our research in which we use the activity « families to build ». This test has been passed by the teachers and was not filmed. 2) Many activities using the developed material. Each session was filmed by one or even two cameras.

3) One activity with big forms. 4) One « concluding activity » at the end of the year. 5) One post-test at the end of the school year (exactly the same as the pretest). 6) One pretest at the beginning of the following year (exactly the same as the previous ones). This test was filmed and students were systematically asked about their production.

We thus have many hours of observations that allow us to verify if students of this age can mobilize the characteristics of the shapes.

Presentation of an activity for 4 to 6 year old students: Families to build[5]

The activity is done in groups of 2 to 5 students. The teacher selects an assortment of 8 to 16 pieces from the collection (the inside form or the empty circular part). Below is an example of an assortment that has been frequently used in classrooms with students.

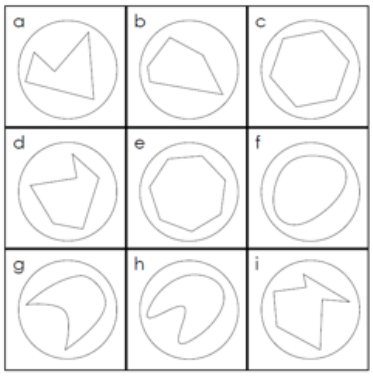


Figure 2: An assortment frequently used in classrooms

In this activity students must build families (with a number of families imposed or not). The pieces are scattered on a table. Students must create families by putting the pieces "that fit well together". Students must agree and be able to explain their choice, possibly giving a name to the families created. Various objectives can be identified for this activity: 1) classification of shapes based on characteristics 2) emergence of a common lexicon that can be reinvested in other activities 3) peer collaboration with the need to agree and to argue. Thus, according to the assortment of selected pieces, students can use global or hybrid thinking or thinking by characteristics. The choice of perceptively close shapes or not is therefore an important didactic variable for this activity as well as the number of families (imposed or free).

In the pooling phase the teacher can introduce new pieces to check the solidity of students' family choices. Either they manage to integrate the new pieces within the existing families, or they need to question their classification criteria and maybe modify them.

5 For more activities see Coutat & Vendeira (2015).



Photo 2: An example of three families built by a group of students: 1) "the mountains" 2) "the pebbles" 3) "the fish" (vision according to the resemblance of the shapes to well-known objects)

Some results

In this section we look at the productions of three students which reveal three different ways of thinking that the students mobilize about the shapes following our interventions in class. These are outcomes from the activity "families to build" carried out during the pretest (done at the beginning of our research) and the same test realized a year later.

Concerning the case of Luce, almost no change is noticable between the two productions at one year interval. During the first run, it is found that very perceptively similar objects are associated in order to create three families. A year later, the student explained that he had formed a family of *mountains*, *trumpets*, *teapots* and *lamps* and could not say more. It is possible to relate these objects to some of their characteristics as the sharp peaks for the mountains, the symmetry for the lamps or the asymmetrical spout for the teapot. However, this remains implicit and the primarily mobilized vision is, in this case, global.

	Luce			
Activity in 2016				
Activity in 2017	Mountains	Lamp	Teapot	Trumpets

Table 4: the two productions of Luce at one year interval

Lea's productions at the same task are identical to those of Luce for the first test. Her vision is essentially global. On the other hand, the two families created the following year are quite distinct. She chose only two families by mobilizing hybrid thinking. Indeed, the global perception is partially used with the second family where Lea recognizes *thunder* thanks to their "peaks in". As for the first family, it only possesses "rounded in" and does not belong to the family of the thunders.

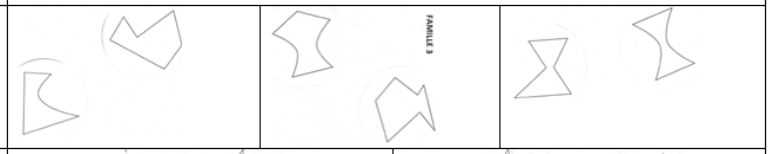
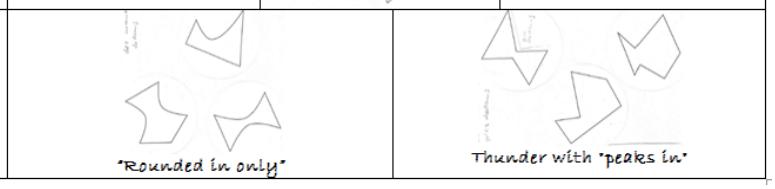
	Léa
Activity in 2016	
Activity in 2017	

Table 5: the two productions of Léa at one year interval

The first production of David is distinct from those of these two classmates. However, without a trace of his activity, it is difficult to understand how this student proceeded. It is conceivable (1) that devolution has not taken place; (2) that David mobilizes hybrid thinking but it is impossible for us to interpret. This is why we focus directly on the second production made the following year. The first family of David is justified according to two characteristics common to the three forms, namely "rounded and sharp". As for the second family, the forms are “all sharp”, but have no rounding. This student thus mobilizes some characteristics of the forms.

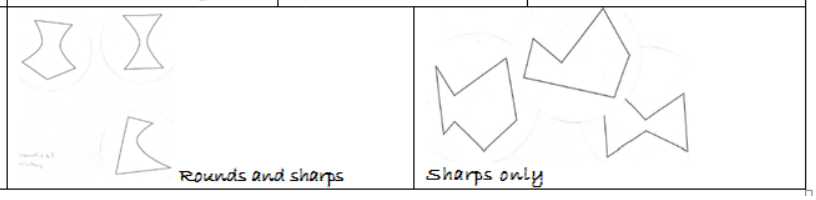
	David
Activity in 2016	
Activity in 2076	

Table 6: the two productions of David at one year interval

Conclusion

The task “build families” is very interesting for the researcher because it gives a lot of information about the student’s perception of the shape such as the visualization to build the families (global or not), the use of characteristics to build the families, the use of a pertinent language for oral interactions.

The various tasks created with the developed material and experimented in classrooms allow a progressive change of the visualization of geometrical shapes. It is important to note that students do not replace their perceptual way of thinking with a new way of thinking by the characteristics of the forms. Indeed, these ways of thinking must coexist and intertwine, sometimes giving rise to a hybrid way. It remains to be defined 1) whether the work undertaken allows all students to change their eyesight and 2) whether they are able to mobilize the appropriate thinking according to the situation.

Currently we are experimenting with new tasks with students from 6 to 8 years old and still analyzing the data collected with students from 4 to 6 years old. In addition, the developed material is currently tested in five schools. We look forward to the feedback from the teachers.

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The construction and study of the ellipse through Huzita-Hatori axioms: An investigative activity in analytic geometry classes

Fabiana Chagas de Andrade[1], Thais de Sales Ribeiro[2] and Bruna Mayara Batista Rodrigues[3]

[1] Centro Federal de Educação Tecnológica Celso Suckow da Fonseca, Brasil
bia.profmat@gmail.com

[2] University of Lisbon, Institute of Education, Portugal; thaissales.rib@gmail.com

[3]University of Lisbon, Institute of Education, Portugal; brunamayarabatistarodrigues@gmail.com

Keywords: Origami, geometric places, ellipse, investigation.

This work presents part of the thesis developed in the Professional Master's Degree in Mathematics, at the Pontifical Catholic University of Rio de Janeiro, entitled "The study of Conics through Origami." In the present work, we will focus on the construction and study of the Ellipse, highlighting the definition of its geometric place through the axiomatic geometry of the Origami.

Teaching geometric concepts may present many challenges, because its process requires the study of concepts and relations that are not learned by students in activities which use memorizing and exercise techniques. Therefore, it's necessary to have creativity and reasoning. In this context, it is important that the student is encouraged to see the Geometry study as a practice of investigative and exploratory nature.

According to Ponte, et al. (2006), "The geometric investigations contribute to realize the essential aspects of the mathematic activity, such as the formulation and conjecture tests, the search and demonstration of generalizations." Moreover, to investigate, explore and establish conjectures in activities of geometric nature are not easy tasks. Among various teaching methods that can be explored in order to foster the meaningful learning, we can be highlighted the use of manipulable materials. In this regard, we spotlighted the possibilities of geometric investigation offered by a simple and powerful teaching tool: the paper.

Ancient and modern at the same time, "the art of the paper folding", better known as "Origami", transcends the boundaries of a simple art with its ability of conducting to the mathematic learning, either by its harmonic visual or by the Axiomatic Geometry inherent in the folds. Its description is given by the Huzita-Haroti Axioms, which consists in the seven basic operations capable of align straight lines and pre-existing points in a paper sheet through a single fold.

Some researches have already chosen origami as a resource which improves the learning of maths, such as Monteiro (2008), who used the technique to solve some equations and to demonstrate theorems, as well as Asrlan & Isiksal (2014) who described an experience about preservice teachers training using this japanese tool.

Despite the different themes that can be broached through the axiomatic geometry of folds, allied to an investigative and exploratory practice, we will highlight an activity which the objective is to explore the definition of the geometric place of the Ellipse. In its teaching, it is verified an undue priority for the memorizing of equations, and in many cases, the students only dedicated themselves to the repetition of exercises which solely involved algebraic methods. In order to rescue the

geometric approach of the topic in question, we unite these two elements in the present work, conic curves and Origami, with the aim of developing concepts of the first one from the constructions of the second, emphasizing the investigative practice through the process.

In order to spread the geometry of folding as a resource for activities of geometric investigation, to rescue the approach of the geometric place of the Ellipse and to validate the presented research, a workshop was developed, applied, evaluated and enhanced in a group of 17 Mathematics Licenciate Degree students.

As a basis for the development of the activity, which consisted of the construction of the curve, we used four moments related to the investigation process in mathematics classes mentioned by Ponte, et al (2006): recognition of the situation, its preliminary exploration and the formulation of questions; conjecture formulation process; testing and eventual refinement of the conjectures; argumentation, demonstration and evaluation of the work done.

Thus, we will present in this poster the Huzita-Hatori axioms description, the method of the Ellipse construction, the mathematics concepts related to the folds and finally the results of the activity application with the future mathematics teachers.

Through the application of the construction activity and the study of the Ellipse, we realize that the geometric place concept is underexplored in Mathematics Licenciate Degree courses, which culminates in disfavouring this content in Basic Education. Thus, this work is expected to be a source of motivation to the teachers and professors to valorise the approach of the geometric concepts in an investigative way. Also, we look to find in the folds a possibility to mix various mathematical themes in a meaningful manner.

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A scale on cognitive configurations for tasks requiring visualization and spatial reasoning

Teresa F. Blanco[1], Pablo G. Sequeiros[2], Cristina Núñez[3], María Salgado[1]

[1]University of Santiago de Compostela, Faculty of Education, Spain; teref.blanco@usc.es, maria.salgado@usc.es

[2]University of Santiago de Compostela, Faculty of Teacher Training, Spain; pablo.gonzalez.sequeiros@usc.es

[3]University of Zaragoza, Faculty of Social and Human Sciences, Spain; cngarcia@unizar.es

Keywords: Visualization, spatial ability, preservice teacher, onto-semiotic approach.

Introduction

We present a preliminary study focused on a scale on cognitive configurations for tasks requiring visualization and spatial reasoning. The main notion used is that of cognitive configuration from the onto-semiotic approach (Godino, Batanero and Font, 2007). The results show that there may be several configurations at each level and these levels depend on both certain conditions of the task and the visualization skills involved.

Theoretical framework

From the onto-semiotic approach (OSA), the analysis of the mathematical activity, of objects and processes taking part in it, focuses on the practices done by people implied in the solution of certain mathematical problems (Godino, Batanero and Font, 2007). The enforcement of this approach to visualization leads to distinguish between visual practices and non-visual or symbolic/analytic ones (Godino, Cajaraville, Fernández and Gonzato, 2011). In order to make these principles operative, the OSA poses as one of its tools the 'onto-semiotic configuration', i.e., the network of objects and processes involved in the fulfillment of a mathematical practice (Font, Godino and Gallardo, 2013). These configurations can be socio-epistemic (networks of institutional objects) or cognitive (networks of personal objects). Here we will focus on the second ones.

Research problem and method

Can we set a scale on cognitive configurations associated to tasks requiring visualization and spatial reasoning (VSR)? To answer this question, three spatial tasks requiring counting, folding/unfolding and composing/decomposing have been selected. These tasks have been proposed to a total of 400 preservice teachers and the answers have been analyzed using the cognitive configurations (CC) proposed by the OSA (Fernández, Godino and Cajaraville, 2012).

Results

A variety of CC associated to each of the tasks has been found. This allows us to describe skill levels in VSR (Table 1) subject to certain conditions directly depending on the characteristics of the task, the visualization skills (Del Grande, 1990) required and the synergy between visual and analytical languages of each configuration (Godino, Blanco, Gonzato and Wilhelmi, 2013).

Task	Levels					Rank	%
Truncated cube	Level 1	Level 2	Level 3	Level 4	Level 5	High: levels 4 and 5	10,25
	CC2	CC1 CC4	CC3 CC8	CC6 CC7	CC5	Intermediate: level 3	11,50
						Low: levels 1 and 2	45
Folding/ unfolding	Level 1		Level 2		Level 3	High: level 3	8,5
	CC1		CC2 CC5		CC3 CC4	Intermediate: level 2	14
						Low: level 1	52,50
Perforated cube	Level 1	Level 2	Level 3	Level 4		High: levels 3 and 4	49,25
	CC2	CC4 CC5	CC3	CC1 CC6		Intermediate: level 2	0,75
						Low: level 1	24,25

Table 1: Scale of configuration levels

Conclusions

In general the ratio of students expressing high level CC is significantly below than of those exhibiting low level. The analysis shows that students mobilized variety and quantity of visual objects and processes. However, they do not reach the solution successfully. This fact might be due to students are not used to working with these objects and visual processes.

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Diagrams in students' proving activity in secondary school geometry

Keith Jones[1] and Kotaro Komatsu[2]

[1]University of Southampton, UK; d.k.jones@soton.ac.uk

[2]Shinshu University, Japan / University of Southampton, UK; kkomatsu@shinshu-u.ac.jp

Keywords: Geometry, proof, proving, diagrams, DGS.

Introduction

Diagrams appear in many different forms in mathematics and in its teaching and learning. While the use of diagrams is pervasive, research on learners' activity with diagrams is somewhat limited (for partial reviews, see Jones, 2013; Sinclair et al., 2016). In this short paper, our focus is on the question of how secondary school students work with diagrams during proving activity in secondary school geometry. We use data from a project investigating the design of dynamic geometry software (DGS) tasks that facilitate students' proving activity (see Komatsu & Jones, 2017).

Existing research and theoretical background

Samkoff et al (2012, p. 49) argue that "diagrams are viewed by mathematicians and mathematics educators alike as an integral component of doing and understanding mathematics". Even so, existing research indicates that, amongst other things, learners' beginning identification and interpretation of diagrams tends to be based on spatio-graphical properties represented in the diagram (Laborde, 2005) and that learners can have difficulties distinguishing within the configurations of a geometric diagram the visual characteristics that are relevant from those that are not (Gal & Linchevski, 2010). How secondary school students work with diagrams during proving activity in secondary school geometry is likely to vary depending on whether the geometrical diagram is 'static' (as in physical books and worksheets) or 'dynamic' (via digital technologies; for example Yerushalmy & Naftaliev, 2011).

The study

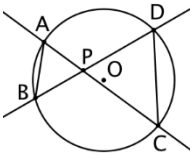
Data (in the form of transcribed student talk, student written work, and digital files) come from a task-based interview using the tasks in Figure 1 with a triad of 11th grade students (16-17 years old), Kakeru, Sakura, and Yuka (pseudonyms), from an upper secondary school in Japan. The students had previously learnt geometric proofs, including using the conditions for congruent and similar triangles. As such, they were familiar with the inscribed angle theorem, the inscribed quadrilateral theorem, and the alternate segment theorem. Prior to the task-based interview, they had four hours using DGS.

For Q1, and only using paper and pencil, the students conjectured that $\Delta PAB \sim \Delta PDC$ and wrote a suitable proof based on the inscribed angle theorem. For our analysis for this paper, we focus on what happened as the students worked on Q2 after they had used the DGS to construct the figure.

Findings

Our analysis found that during the time that the students worked on Q2, they moved ('dragged') points A, B, C, and D to various places on circle O. In working with this 'dynamic' diagram, we found that their discussion settled on various versions of the diagram that we could categorise into the six types of diagram shown in Figure 2.

Q1. (1) As shown in the diagram given, there are four points A, B, C, and D on circle O. Draw lines AC and BD, and let point P be the intersection point of the lines. What relationship holds between ΔPAB and ΔPDC? Write your conjecture. (2) Prove your conjecture.



Q2. Construct the diagram shown in Q1 with GeoGebra. Move points A, B, C, and D to various places on circle O to examine the following questions. (1) Is your conjecture in Q1 always true? (2) Is your proof in Q1 always valid?

Figure 1: Tasks used in the interview

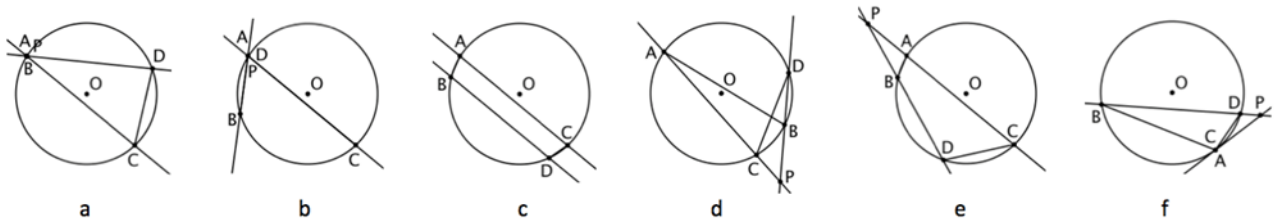


Figure 2: Types of diagrams the students produced

For Figure 2a, which the students created by dragging points so that points A and B (and consequently point P) coincided, they commented that "if (the points) overlap, (our conjecture is) impossible" because "(triangle PAB) disappears". For Figure 2c, where they considered that line AC was parallel to line BD, they concluded that "(our conjecture is) impossible" because "intersection point (P) disappears when (lines AC and BD are) parallel". Eventually, they concluded that their conjecture was not true for these two cases (for more on the other types of diagram, see Komatsu & Jones, 2017).

Concluding comment

The diagrams we found in the students' proving activity underlines the observation by Samkoff et al. (ibid) that the processes involved in using diagrams in mathematics are "surprisingly complex", especially the extent to which students are aware of the general result or a specific diagram.

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QED-Tutrix: Creating and expanding a problem database towards personalized problem itineraries for proof learning

Michele Tessier-Baillargeon[1], Nicolas Leduc[2], Ludovic Font[2], Philippe R. Richard[1] and Michel Gagnon[2]

[1]Université de Montréal, Canada; michele.tessier-baillargeon@umontreal.ca; philippe.r.richard@umontreal.ca

[2]École Polytechnique de Montréal, Canada; nicolas.leduc@polymtl.ca; ludovic.font@polymtl.ca; michel.gagnon@polymtl.ca

Keywords: Intelligent tutoring system, automated theorem proving, didactical contract, Design in use.

QED-Tutrix (QEDX) is an intelligent tutoring system which assists students in proof problem solving by providing hints while taking into account the student's cognitive state. QEDX stands out by the fact that it adapts to each user and class reality, not the opposite. However, this model implies recognizing, like a teacher would, proofs that do not necessarily conform to a formal logic. Hence, QEDX can't rely on an automated proof engine (Tessier-Baillargeon, 2016), raising the question of how to expand QEDX's problem database without manually implementing each valid proof. Therefore, our poster at CERME10 doesn't present a traditional research project with its research questions, it's methodology and conclusions. It rather aims at presenting the new research questions that stem from the challenges that arise with trying to broaden QEDX's problem bank while staying true to our main goal, which is to create a geometrical workspace (Kuzniak, 2006) according to witnessed student/teacher interactions through a design in use approach (Rabardel, 1995). Here we will focus our attention on the process of problem implementation, starting with how we currently generate a proof problem's solution graph.

Generating a proof problem's solution graph.

QEDX's **HPDIC** graph (Figure 1) is used to record all the valid proofs to a given problem. It includes **H**ypotheses, **P**roperties, **D**efinitions, **I**ntermediary results and a **C**onclusion. This graph is unique to each problem and is built from the inferences individually identified as true according to the problem to solve and the class context. The HPDIC graph for the *rectangle* problem in Figure 1 is fairly simple since it counts only 13 inferences. However, in the five problems implemented in the current QEDX version, one counts 214 inferences creating a much more complex HPDIC graph.

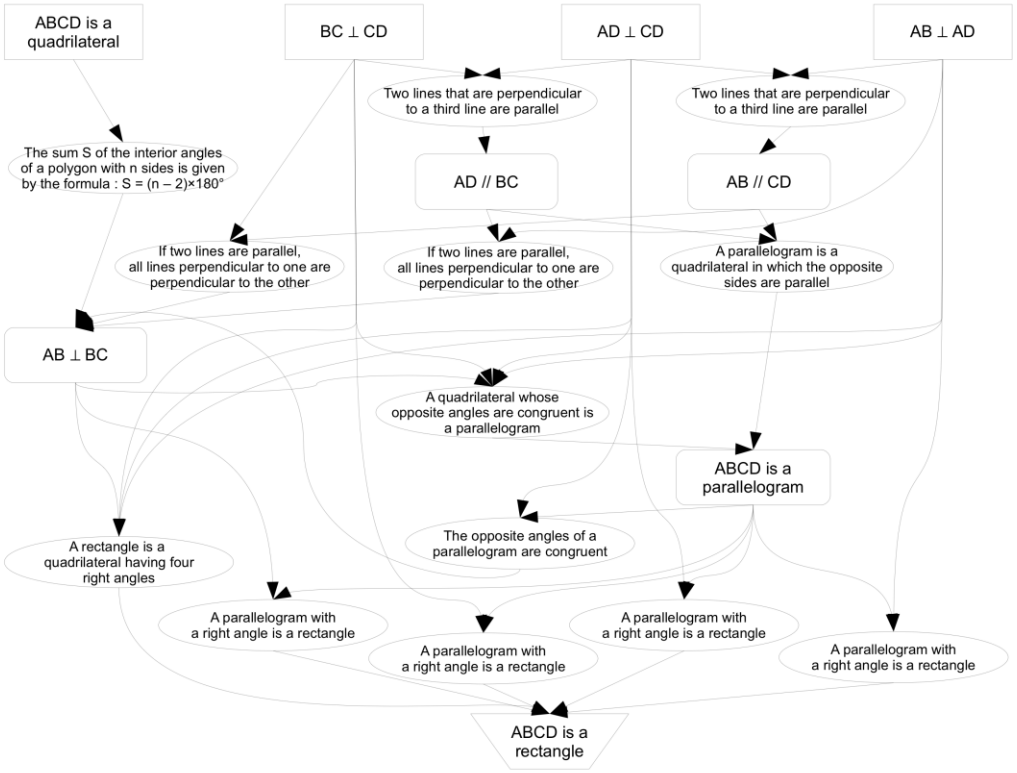

Figure 1: HPDIC graph, *rectangle* problem that asks to prove that a quadrilateral with three right angle is a rectangle

Needless to say, there is a great amount of prerequisite work to be done before a problem can be added to the system. Therefore, in order to expand QEDX's problem database, we need to, at least partially, automatically generate each problem's solution space. A rich problem database will allow the student to navigate a geometrical workspace made up of a sample of problems put together to help him or her overcome difficulties as well as exercise proving skills through personalized problem itineraries. However, since QEDX aims at adapting to every didactical contract (Brousseau, 1998) by expecting and recognizing proofs according to what the teacher of any given classroom would require, manually generating every valid solution becomes almost impossible. How can we take into account teaching traditions while maximizing our proof problem pool? This challenge will define the next steps in QEDX's design and development.

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Children's block-building: How do they express their knowledge of geometrical solids?

Susanne Wöller and Simone Reinhold

University of Leipzig, Germany, susanne.woeller@uni-leipzig.de; simone.reinhold@uni-leipzig.de

Keywords: Building of geometrical concepts, solids, construction activities, primary school.

Introduction

The main focus of the study is on primary students' concept knowledge about geometrical solids. In particular, we intend to detect the development of geometrical concept knowledge of Year 3, 4 and 5 students (aged 8 to 11). We investigate young children´s knowledge of geometrical solids by providing wooden blocks in construction tasks: 52 third-graders (German and Malaysian children), 30 fourth-graders and 9 fifth-graders were asked to construct cuboids and cubes according to their knowledge and visualization. Results are interpreted according to the Van Hiele framework. In addition, we have a closer look on the variety of cube and cuboid constructions and raise conclusions concerning the development of children´s conceptual knowledge.

Theoretical framework

The customary conception of a *concept* comprises the "ideal representation of a class of objects, based on their common features" (Fischbein, 1993, p. 139). In this sense, *geometrical concepts* refer to common features of a class of geometrical shapes or solids which can be visualized or perceived when encountering concrete representatives. Typical representatives (prototypes) depict specific features of the class of geometrical figures in particular (Mitchelmore & White, 2000). Based on this notion, students' conceptual knowledge of geometrical solids reaches beyond the capability of correctly naming concrete representatives or giving a verbal definition. It rather comprehends the perception, visualization and identification of distinctive properties which refers to individual mental images students have while thinking of a specific solid (Tall & Vinner, 1981). The development of geometrical concept knowledge from primary to secondary has been described by the well-known Van Hiele Model which defines five levels of development (Van Hiele, 1986). Yet, most research which refers to the Van Hiele framework has been concerned about children's geometrical concept knowledge of 2-D shapes, whereas little is known about children's concepts of 3-D solids. Based on this theoretical framework, we assume that analyses of similarities and differences in individual construction processes and products of Year 3, 4 and 5 children provide deeper insights into children's visualization of geometrical solids, regarded to be a core element of geometry and mathematics education in primary schools.

Research questions, methods and results

The results of the study are expected to contribute to a deeper understanding of the development of children's concept knowledge of geometrical solids at primary level.

- How do Year 3, 4 and 5 children (aged 8 to 11) articulate their conceptual knowledge of geometrical solids via construction activities with wooden blocks (cubes, cuboids, prisms and Froebel's Gift No 6)? Are these constructions in line with their verbal explanations?

- What kind of cuboids and cubes do they construct and which variations occur? Do they possess particular approaches in their activities?
- How can we interrelate these results to the Van Hiele framework and is there a necessity to enrich the *Model of Development of Conceptual Knowledge*?

We analyse the conceptual understanding, strategies and reasoning of Year 3, 4 and 5 children when observing and video-taping their construction activities of 3-D solids (cuboids and cubes) with wooden blocks. In a first step, German and Malaysian children were asked to explain their ideas and knowledge of geometrical solids in a short dialogue with the interviewer. Afterwards, a variety of tasks invited them to express their knowledge of cuboids and cubes via construction activities. During their constructions children were encouraged to describe their strategies. Data was coded with software support by *Atlas.ti*. A coding-guideline was developed mainly according to *Grounded Theory Methods* (Corbin & Strauss, 2015), trying to detect facets of articulating children's conceptual knowledge of geometrical solids and to relate our first results with the Van Hiele framework (Van Hiele, 1986).

Our results show an impressive variety of different types of constructed cuboids and cubes and of individual approaches, which indicate a wide variety in children's geometrical concept knowledge of the selected solids. Relating to cubes, most children focus on a square-base area during their constructions, some German and Malaysian third-graders only built one quadratic layer and name this a *cube*. Furthermore, we detected ambiguous mental images in children's concept knowledge concerning cubes. Relating to cuboids, our results illustrate the existence of prototypical representatives, e.g. convex constructions with various layers, followed by constructions only consisting of one layer. Furthermore, children's constructive activities can be (partly) interrelated to the Van Hiele framework, most children are at level of *Visualization* ("It's a cube because it looks like a cube."), resp. *Analysis* ("It's a cube because all surfaces are the same."). None of the children are in the phase of transition from *Analysis* to *Abstraction*, all children Year 3, 4 and 5 faced difficulties in realizing relationships between a cube and a cuboid.

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TWG05: Probability and statistics education

Introduction to the papers of TWG05: Probability and statistics education

Corinne Hahn, *ESCP Europe* and LDAR-Université Paris Diderot; Daniel Frischemeier, *University of Paderborn*; Sibel Kazak, *Pamukkale University*; Aisling Leavy, *MIC- University of Limerick*; Caterina Primi, *University of Firenze*

Overview

The working group gathered 30 participants from 15 countries. 22 papers and 4 posters were accepted.

The first session started with an ice-breaker activity. Firstly, the participants were asked to introduce themselves to a neighbor they had never met before. In a second time, each participant had to briefly present her/his colleague: her/his name, occupation, country and expectations about the working group. The cross-presentations made it possible to identify 3 main expectations. First researchers who attended TWG5 sought after constructive feedback on their work. Then they hoped to widen their point of view: gather new ideas, get informed of new trends, get to know better research works from other countries. The third main expectation was about networking: meet nice and smart people and develop possible collaborations.

The team of co-leaders had organized papers into 5 groups on 5 different topics. A session was devoted to each group and this session was managed and chaired by one of the co-leaders.

Teacher education (Aisling Leavy)

This session explored a broad spectrum of research in teacher education pertaining to statistics. The presentation by Artego, Diaz-Levicoy and Batanero reported on a study of the competence of 140 Chilean primary school students levels in reading pictograms. Frischemeier and Biehler reported on the development of statistical literacy and thinking in a statistics course for elementary preservice teachers. Negotiating the content and the teaching of statistics during teachers' professional development was the focus of the presentation by Bakogianni. The research reported by Gokce and Kazak examined middle school mathematics teachers' pedagogical content knowledge in relation to statistical reasoning. Afeltra, Mellone, Romano & Tortora reported on the results of a study designed to support teachers' interpretative abilities and use of errors as a didactic resource.

Discussion focused on the following guiding questions: What are the important understandings that teachers/students need to develop? What experiences can we provide to promote and develop teacher understanding? What considerations should we take into account when designing tasks to assess statistical understandings? In what ways might current knowledge frameworks limit/support the identification of teacher knowledge?

When discussing the important understandings and knowledge that teachers need to develop, there was acknowledgement of the limited experiences of pre-service teachers in statistics and the limited space it receives in teacher education courses. The greater recognition in teacher education of the dual role of content knowledge and pedagogical content knowledge was welcomed. In moving forward there was the recommendation of the need to support teachers in developing awareness of

the features of good tasks so they can design and modify their own tasks to ensure maximal learning benefit for children. When discussing the considerations that need to be taken into account when designing tasks to assess understandings/knowledge, there was agreement that it is difficult to assess teacher knowledge when we are not entirely sure what knowledge is important in order to teach statistics. Thus, there is the need for more research that elaborates what it is that teachers need to know in order to teach effectively in their classrooms. Finally, when discussing the ways in which we can we promote and develop teacher understanding, there was discussion around the importance of acknowledging what makes statistics unique (from mathematics). Consensus was also reached on the importance of task design is drawing attention to conceptual understanding, on the usefulness of focusing on children's responses (both errors/misconceptions and correct responses) and on the use of collaborative feedback is useful in revising understandings about teaching and learning statistics.

Informal statistical inference (Caterina Primi)

Recently several researches effort are made to understand the informal ideas relating to statistics inference (ISI) as it is seen as having a potential to help build fundamental concepts that underlie formal statistic inference. Indeed, students can show complex ideas regarding statistical concepts (such as distribution, sampling) that can be seen as "precursor notions" of regular statistical concepts.

Four papers were presented on this topic. Leavy explores the informal inferential reasoning of primary students (5-6 years old). In Büscher's paper students' patterns of thought and the processes of their conventionalization are reconstructed in a qualitative study with students of grade 7 (12-14 years old). De Vetten shows the importance of preparing primary school teachers for teaching ISI, identifies learning goals for the teachers, and reports on an intervention study focusing on these learning goals. McLean's paper reports on students' modelling activities involving resampling process of bootstrapping.

In the final discussion emerged the importance of informal inferential reasoning in particularly how it should be promoted at every level (from early years upwards) as it is the beginning of access to a statistical cultural. Additionally, the group agree that Informal inference has an important socio-political contribution and may represent the first exposure to prediction and data-based inference. For this reason, the group has identified the need for national curricula and policy bodies to promote an emphasis on informal inference at the school level. As future directions, the group has identified the critical role of task design, language and technology in accessing understandings of inference, and finally the research of evidence of whether and how informal statistical inference improves the transition to formal statistical inference.

Probability and sampling (Sibel Kazak)

Four papers were presented and discussed in the subtheme of probability and sampling. The paper by Paparistodemou, Meletiou-Mavrotheris and Vasou report on young students' ideas of randomness and expressions of probability of an event when designing their own games with the use of Scratch software. Silvestre and Sanchez examine high school students' reasoning while engaging with the idea of sampling distribution and the estimations of likelihood of outcomes in repeated random sampling by using Fathom software. Elicer and Carrasco explore the use of a

sequence of tasks designed based on the framework of didactical engineering to introduce conditional probability as a decision-making tool. Eichler, Vogel and Böcherer-Linder describe and compare the use of different visualization tools, such as unit square, tree diagram and 2x2 table for visualizing Bayesian situations that involve conditional probability and Bayes' rule. In the general discussion based on these papers the following issues were raised:

- Use of 'uncertainty' term instead of 'probability' to emphasis the link between probability and statistics and informal inferential reasoning.
- Current trend in teaching probability topics in higher grade levels in school mathematics curricula in different countries.
- Mismatch between emergence of probabilistic ideas in young students and how the school mathematics curricula are designed. Lack of focus on subjective probability in school.
- Role of representations in conditional probability situations and in decision-making. It was noted that there is no ultimate representation but variety of choices for different purposes.
- The need for attending to cognitive and non-cognitive components of learning both for teachers and for students of all ages (elementary through to college level).
- How to design tasks or activities in game-based environments like Scratch that will foster the development of statistical/probabilistic reasoning. By over stipulating the environment we may lose the affordances that the environment provides (i.e. openness, creativity).
- Use of technology in task-design and challenges in implementating such tasks in classrooms.
- Research (future directions): More research on ways of thinking about what children can do and more focus on philosophical perspectives in research are needed.

Technology (Daniel Frischemeier)

This session explored in what ways technology can promote and develop statistical reasoning. In detail we got to know about the use of different technology tools like educational software, spreadsheets, online platforms (for distance learning), response tools, and programming tools in statistics education. Overall we have five papers in the session on technology.

Parzysz reports on the use of spreadsheets to teach probability and introduces learning environments for French High school students to learn about the binomial distribution with spreadsheet software like Excel and discusses the potential and limitations of using spreadsheets when simulating random experiments and when shifting from discrete to continuous distributions. Serpe and Frassia present teaching examples to enhance the discussion on the meaning and interpretation of probability for higher secondary school students in Italy. In detail Serpe and Frassia show ways to implement programming for simulating chance experiments and introduce specific tasks (e.g., airplane task). In the article of Tacoma, Drijvers and Boon the reader gets to know about the potential of feedback devices and response tools in statistics education. Their research aims how students' models can be used to generate feedback in an online course on statistical sampling. The paper of van Dijke-Droogers, Drijvers and Tolboom describes a study which investigates ways to enhance grade 8 students´ statistical literacy through within-class differentiation. The study is framed in a design-based research project and interventions like Digital Mathematics Environments and digital tools are used. Finally, Meletiou-Mavrotheris,

Paparistodemou and Bayes describe an online course about statistical methods for post-graduate education majors and point out prospects and considerations of distance education.

Two fundamental issues have arisen in the discussion at the end of the session regarding the learning and teaching with technology in statistics education: design issues (task design and design of learning environments) and also the role of technology in teacher education. Regarding the first aspect "design issues" two fundamental questions have arisen in the discussion process of the whole TWG5: How to support learners in their use of technology? How to concentrate on the content rather than on technical issues of the tool? Regarding the second aspect "technology in teacher education", the group agreed that teachers need a solid technological knowledge (technological pedagogical content knowledge, TPCK) to be able to implement and orchestrate technology in statistics classrooms successfully. As future directions in the field of technology in statistics education the group has identified the potential of web-based applications, mobile devices and online learning systems for the learning and teaching of statistics.

Varia (Corinne Hahn)

Five papers were gathered in a "varia" group as they covered issues that were not directly related to other subthemes. Arteaga and colleagues describe a large study carried out in Chile with 6th and 7th grades students. The aim of this study was to explore students' competencies in reading pictograms. They report that the highest level they called "critical interpretation" is rarely reached by students. Gea and colleagues analyse correlation and regression problem situation in Spanish textbooks. Amazingly they report that there were very few problems with context. Trakulphadetkrai presents a qualitative study carried out with undergraduate students, with the aim of exploring how students learn statistical concepts through enactive story writing. Gonzales and Chitmun describe a socially open-ended problem in a sport related context and present the results of an exploratory study on the impact of one of these problems. Chiesi and Primi investigate how students' attitudes changes during an introductory statistics course and discuss educational implications.

The general discussion at the end of this session focussed on the question of task design. We raised many questions, among which the following three have been particularly prominent:

- Can a problem be "real" or simply realistic or meaningful?
- What could be the most appropriate task for the statistic classroom: didactical situation, project work, inquiry-based activity, open problem?
- What is the role and impact of tools?

Conclusion

In this working group, we challenged current frameworks and perspectives on statistics education research and some important issues emerged from the discussions. The participants agreed that we need to know more about:

- uncertainty in IIR,
- theoretical frameworks / philosophical frameworks for research purposes,
- childrens' naive conceptions and how to build on them.

Errors or didactic resources: A teacher education task in the context of probability

Laura Afeltra[1], Maria Mellone[2], Piera Romano[3] and Roberto Tortora[2]

[1]Liceo Scientifico "L. Einaudi", Italy; laura.afeltra1989@gmail.com

[2]University of Naples Federico II, Italy; maria.mellone@unina.it; rtortora@unina.it

[3]Liceo Scientifico "Mons. B. Mangino", Italy; pieraromano.72@gmail.com

In this paper we discuss a teacher education task centred on the request to interpret the reasoning of students addressing a 'throw of the dice' task. The task for teachers was designed after an analysis of class excerpts, carefully selected by our research group because of the interesting educational reflections to which they gave rise. The goal of the task is to support prospective teachers to develop interpretive ability in order to use errors as a didactic resource and to help students build their mathematical knowledge starting from their reasoning.

Keywords: Probability, teacher education, task design.

Introduction

In the framework of PISA 2015 the domain "Uncertainty and Data" is put at the core of the mathematical cultural experience. Items about this topic are, by now, present in all the national assessment tests and the bad students' results to these items, both in national and international tests, put in evidence the need to improve the teaching of Probability and Statistics in school. But this request, as is often the case, is not adequately satisfied, at least in Italy, by a specific plan of teacher education.

Many studies (see, e.g., Batanero, Godino & Roa, 2004) have highlighted that teachers do show interest for this topic but, at the same time, declare to feel a lack of knowledge and experience in it. Indeed, on one side probability was not included in school curricula until a few years ago, and, on the other side, many teachers, even with a Master Degree in Mathematics, did not take the exams of Probability or Statistics in their plan of studies, as in the past it was an optional exam. Nevertheless, it is well known that the success of any educational design aimed at developing students' knowledge and skills in Probability, as well as for any other topic, depends in great part on teachers' knowledge and attitude toward the topic itself.

In this study, we refer to Mathematical Knowledge for Teaching (MKT) framework that serves as a resource for specifically addressing the mathematical demands of teaching (Ball,Thames & Phelps, 2008). More and more researches all around the world are demonstrating the effectiveness of this frame for teacher education.

Several studies (Tversky & Kahneman, 1974; Jones, Langrall & Mooney, 2007), starting from the fact that situations of uncertainty are widely present in daily life, show that very often the common sense or the experience linked to gamble can conflict with the mathematical management of the situation. For this reason we think that it is crucial to design practices of teacher education in this topic which take into account students' answers and look at them as didactic resources (Borasi,

1994), even when these answers seem to be in conflict with the formalization goal of the educational design.

In this paper we discuss a particular teacher education task in teaching Probability. Its design and implementation is rooted in the "modus operandi" of our research team, in which university researchers, in-service school teachers and Master students are involved as a co-learning community throughout an inquire research experience (e.g., Jaworski & Goodchild, 2006). We will explain the rationale that guided us in the choices of some transcripts of classroom discussions, used to design a task for in-service teachers that, in our opinion, should be considered the final product of a long training path and an effective tool to develop their interpretive skills. Finally, we try to answer to the question whether this particular type of task could be used also with perspective teachers, in order to scale up the reflections and the outcomes developed in our co-learning community.

Theoretical framework

Teacher knowledge and, in a broader sense, teachers' attitudes and goals, play a key role in any effort of educational innovation. Among the different conceptualizations of teacher knowledge, the MKT effectively describes the particular features of mathematical knowledge needed for teaching (Ball, et al., 2008). In particular, many studies underline that, in order to improve the teaching of Probability, it is crucial to support teachers' training not only in the Common Content Knowledge (CCK), subdomain of MKT, i.e. the general knowledge of a math topic, but also in terms of the Specialized Content Knowledge (SCK), that is the knowledge about the topic specifically connected with its teaching. However, to develop SCK in the context of Probability, different aspects need to be taken into account (see, e.g., Ponte, 2008): i) epistemological reflections on the meaning of concepts (e.g.,,different meanings of probability); ii) awareness of the closed connections between probabilistic and statistical issues; iii) students' learning difficulties, errors, obstacles and counterintuitive ideas in this field; iv) necessity to plan assessment tests and instruments for interpreting students' responses.

In this study, we focus on the third point of the previous list, looking at errors and non-standard reasoning not as something to avoid but rather as a source to be capitalized, that really shapes the dynamics in mathematics educational process (Borasi, 1996). Indeed, in our perspective, one of the main task for a mathematics teacher is to grasp the "meaning" of students' answers, in order to develop their mathematical knowledge starting from students' reasoning.

This aspect is even more crucial in Probability than in other fields, considering that all the typical technical terms (probable, possible, random, event) are widespread common words. The epistemological consequent conflict can be faced only if teachers get aware that, according to Borasi (1996), ambiguities, in particular linguistic ones, can play a useful role in the development of mathematics learning. This peculiarity makes students, and, sometimes also teachers, more susceptible to counterintuitive ideas, which, in addition, arise in Probability already at an early level, more than in other branches of mathematics (Borovcnik & Peard, 1996).

We are firmly convinced that improving students' mathematical knowledge on the basis of their arguments, even when they seem naïve, requires that teachers activate a real process of interpretation, shifting from an *evaluative listening* to a more flexible *hermeneutic* listening (Davis,

1997). In this peculiar field, the wrong answers of students often arise from well-known misconceptions (Ang & Shahrill, 2014), the same ones that afflict teachers, even in service, as discussed in (Batanero, Godino & Cañizares, 2005). According with this vision, we have proposed a design of tasks to develop teachers' SCK, specifically addressed to support them in making didactic choices fit to develop students' mathematical knowledge, starting from their reasoning (Ribeiro, Mellone Jakobsen, 2016). The tasks we have designed address the difficulties encountered by students in coordinating the classic and frequentist approaches to probability, above all with respect to the representativeness of the sample, and in recognizing the notion of equiprobability as a basic aspect of any probabilistic judgement. These tasks come as a final result of a training path in our research group, with the aim to enhance teachers' professional development. But, after experiencing the task within our community, we also decided to propose a similar task for teachers' learning, because we agree with (Robutti et al., 2016) that it is necessary to scale up the significant experienced practices for teacher professional development.

Methodology

The particular task we are going to present was developed during the meetings of our research group. In the last two years, our group have been working as a co-learning community, where educators :

> (…) create opportunity to work with teachers, to ask questions and to see common purposes in using inquiry approaches that bring both groups closer in thinking about and improving mathematics teaching and learning. (Jaworski & Goodchild, 2006, p. 354)

This group is composed by five university researchers (two in mathematics education and two in physics education), fifteen in-service teachers with a long experience (one kindergarten teacher, five teachers from primary school, two from first order secondary school and 7 from second order secondary school), and some Master students in Mathematics or Education who have been working in their master thesis with us. Most of the teachers have been collaborating with the researchers since 2004, some were also involved in a three-year project (PDTR, Professional Development of Teacher-Researchers, www.pdtr.eu), financed by the European Community, aiming at "engaging classroom teachers of mathematics in the process of systematic, research-based transformation of their classroom practice" (Malara & Tortora, 2009 for the Italian contribution). It is worthwhile to say that we have found several similarities with the method of work of the mathematics learning communities in Norway described in (Robutti et al., 2016).

In the last two years, we decided to start a reflection about Probability and we used to meet once a month for three hours at the Department of Mathematics of the University of Naples. The training path was divided into two parts. Firstly, we organized several theoretic seminars on the epistemological bases of Probability, the connections between Probability and Statistics and on the necessity to enhance the study of Probability at school. Secondly, our meetings were devoted to plan didactic activities in Probability, from primary to secondary school. Each teacher had the possibility to adapt the didactic paths to his/her school level according with the national curricola. Successively, the teachers were asked to illustrate the outcomes of the didactic activities in their classrooms, sharing with the community audio and video recordings of class discussions, and samples of students' productions.

In this paper, we focus on some excerpts of the discussions occurred in Piera's (one of the author of this paper, and expert secondary teacher of the group) classroom of 10th grade students.
The class discussions were registered and transcribed thanks to the help of Laura (another author of the paper) who have been working to her Master Degree Thesis in Mathematics on the experimental educational activities on Probability, carried out in Piera's classes (Afeltra, 2015). Piera had already involved her two 10th level classes in inquiry based activities about descriptive statistics and her students were quite used to discuss each other, in order to deepen their understanding of what they were asked to study. The didactic path had been planned in the learning community with the aim to check if her students would have evidenced the predicted difficulties, also considering that most of them had studied the early elements of Probability in the previous years, often with a formula based approach. Among the others, we selected those class excerpts that gave rise to deep discussions during our meetings, about the possible interpretation of students' answers, as we will illustrate in the following. In order to mobilize teachers' SCK, we focused our attention on the following two frequent students' errors: to view different outcomes of an event as always equally likely (within a classic approach to Probability); and to consider a too small sample as representative to estimate the probability (within a frequentist approach).
These excerpts had a significant impact on the professional development of all in-service teachers, since most teachers, with a master degree in mathematics, seemed not able to act as students, because of a clear prevalence of their CCK on the SCK, necessary to predict students' learning difficulties. In our learning community, Piera's classroom excerpts were considered particularly interesting to be used for a pre service teachers' training activities, as they show, even on a small sample, that these misconceptions are a common way of thinking, so a teacher has to be ready to appreciate the opportunity to transform a wrong answer into a challenging one.

A teacher education task

In the following, we present the teacher education task based on excerpts of a class discussion on the game of tossing two dice, indeed, a very rich and complex context, in spite of its apparent simplicity. In fact, it is one of the typical contexts used to introduce Statistics and Probability, nevertheless several research studies show that both students and teachers can run into counterintuitive ideas about the representativeness of the sample, when rolling one die, and about the item of equiprobability, when playing with two dice (see, e.g. Batanero et al., 2005). Moreover, this context offers many interesting experiences involving the necessity to handle at the same time different approaches to Probability. It is necessary that students become aware of the intrinsic limits of the two approaches: in the classical case, it may be all but easy to identify all the possible equally probable cases, whilst in the frequentist approach it may be not possible to repeat a large number of trails at the same conditions. On the other hand, students who have already studied some elements of Probability in their previous scholar levels can get quite confused about the probability of an event, when they may calculate it in two different ways, for example when playing with dice, but with a small sample. Indeed, the game with dice, as all kinds of gamble, allows to introduce the Law of the Large Numbers in a significant way.

The task is organized as a questionnaire composed of three items, two of them centred on class discussion's excerpts. In the following, we comment on each item, and, in particular, on the class

excerpt, to illustrate the reasons why we consider them meaningful to be interpreted and interesting from the teacher education's point of view.

Item 1

Read and analyse the following class excerpt.

Teacher:	Playing with one die, how can we measure the probability of each side?
Daniele:	There is no favourite side, so for each of them the probability is 1/6.
Federica:	But rolling the die, we have got five 8 times over a total of 63. So the probability is 8 over 63, that is approximately 0,127.
Daniele:	One sixth is 0,167, they are not the same!
Federica:	Yes, we can use both definitions… but we have studied that in this case it is certainly one sixth!

a) For each students' answer discuss if, in your opinion, it is mathematically correct or not;
b) Plan a suitable didactic action to develop students' learning, starting from their answers.

We consider this excerpt as a good starting point to discuss the link between the classic approach and the frequentist one to Probability. The transcript shows that Federica is firmly convinced that the classical definition of Probability is prevalent over the frequentist approach. On the other hand, the play with one die is a very well known situation, and the "a priori" probability is so simple and evident that Federica's answer is quite "reasonable". In our opinion, the thought-provoking thing is her absolute indifference to the value of the relative frequency obtained as the result of rolling the die several times. Many speculations are possible: Federica's certainty can come from a previous formula-based approach to probability, experienced in previous years, or, instead, from the awareness that the frequentist definition needs a large number of data. In both cases, the item allows to reflect on the necessity that teachers clarify the question with their students. Moreover, it shows the importance to involve students in activities of data collection, which gives the chance to significantly introduce the Law of Large Numbers, also, maybe, with the aid of a digital simulator, to work with large samples.

Item 2

During a lesson on Probability, a teacher submits to his/her students the following question: "When two dice are simultaneously thrown, what are the chances of obtaining 7 as the sum of the two sides up?"

a) Answer the question, as you are able to do; b) Which possible answers, correct or not, in your opinion, will be given by students? c) Discuss about the possible difficulties that, in your opinion, a student will encounter when addressing this question.

The first item of the task aims to put teachers at the place of students. The whole task involves teachers in a brainstorming activity, mobilizing their CCK about the topic. In this way, teachers can recognize a didactic obstacle, and, consequently, realize the need to design an educational path to face it. Moreover, in the following item, the teachers can confront their own answers with those given by the students.

Item 3

In the following excerpt it is reported a discussion between a teacher and his/her students about the previous question.

Teacher: When two dice are simultaneously thrown, what are the chances of obtaining 7 as sum of the two numbers?

Gianluigi: There are 11 possible sums, from 2 to 12, so the probability is 1/11.

Ludovica: No, it would be true if we used a die with 11 sides. We have to consider all the possible couples, such as one and one, one and two and so on.

Teacher: Ok. And then, how many?

Students work in groups for a while until one of them affirms:

Massimo: There are 21 possible couples.

a) For each students' answer, discuss if, in your opinion, it is mathematically correct or not;
b) Identify a set of possible questions that you would pose to students to support them their learning process.

According to the classic definition of probability, Gianluigi's answer has to be considered wrong, since the 11 values of the sum are not equally likely. But Gianluigi's answer opens up the reflection about the difference between the sample space and the possible cases. The transcripts were enlightening for most teachers in our community. Even Piera, one of the author, witnessed that, in the previous years, on the basis of her CCK only, she would have rejected this answer, immediately underlining that the eleven possible outcomes are not equally possible. In the light of the reflections previously made in our community, this time she rather decided to give space to the classroom discussion. Moreover, Ludovica smartly refers to another context, where 11 would have been the correct answer, and this seems to us a significant process to support the construction of meaning, above all because it arose from students.

Even Massimo's answer, which is caused by the common belief that, throwing two dices, the order is not relevant (see, e.g., Batanero et al., 2005), is wrong but interesting. Indeed, we have selected this excerpt because, in our meetings, it triggered many discussions about the didactic choice, for example, to use or not dice of different colours, in order to avoid this error. Finally, we agreed on the opportunity to use dice of the same colours in order to open up the possibility that reasoning as Massimo's one emerge.. In this way, a teacher has the possibility to ask students to look for contexts different from the dice game, where the order turns to be relevant. In our meetings, many ideas were proposed: the most popular examples were contexts of sport competitions, while a more sophisticated context could be the Bose–Einstein statistics. Only on the basis of her learning's development, due to the participation in our co-learning community, the teacher was able to guide her students in finding a possible context in which Massimo's answer could be correct, activating in this way an hermeneutic listening (Davis, 1997) Her actions helped students not only to better understand the crucial role of equiprobability, but also to describe mathematical models and formulate hypotheses about them.

Conclusions

For several years researchers in probability education have been investigating the counterintuitive ideas arising when people formulate judgements in real situations involving uncertainty (Tversky & Kahneman, 1974). These ideas are often in conflict with the mathematical formalization of Probability, so they can cause, if not recognized, paradoxical results. It is widely known that a correct probabilistic reasoning needs a specific instruction (Fischbein, 1975), therefore, it is fundamental that a teacher is able to use students' intuitions, their counterintuitive answers or incorrect reasoning as resources for learning (Borasi, 1994). We think that this work is particularly useful for Probability, even more than for other mathematics topics, since, in this field, widespread counterintuitive ideas arise from many daily situations.

Our research group has been working as a co-learning community (Jaworski & Goodchild, 2006), where the professional development of all the participants was enhanced throughout inquiry activities, in order to improve learning from experience and reflection. In this paper we have focused on the process which led us to design a teacher education task, starting from the analysis of excerpts of classroom activities. We selected the excerpts most challenging to be converted in interpretative tasks for teacher training, with the aim to improve teachers' SCK about some crucial notions of Probability. The deep impact observed on the Math teachers of our community encourages us to redesign a similar path to be used for teacher training on a larger scale, as an attempt to face the issue of dissemination of the significant practices experienced in a co-learning community, as described in (Robutti et al., 2016). In this direction many questions are still open, and need further investigation. However, we are just now exploring the real effectiveness of this types of task with in-service and perspective teachers, their difficulties and limits.

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Negotiating the content and the teaching of statistics: Two complementary processes in teachers' professional development

Dionysia Bakogianni

National and Kapodistrian University of Athens, Greece; dbakogianni@math.uoa.gr

This study builds on Wenger's (1998) notion of meaning as the essence of a practice and the process of negotiation of meaning as determinant in the development of a practice. Following an emerging community of 11 secondary mathematics teachers, I aim at exploring the interactions between two dominant processes in teachers' work, the negotiation of the statistical content and the negotiation of the teaching of statistics. The results indicate that the two processes act complementarily in the formation of the practice of the community and consequently in the professional development of the teachers. Evidence of how the one provides feedback and meaning to the other is also illustrated and discussed.

Keywords: Community of practice, mathematics teachers, teaching and learning of statistics.

Introduction

The development of teacher education programs that aim to support teachers to promote statistical inquiry in their classroom, is gaining an increasing attention the last years in statistics education community. This current discourse emphasizes the importance of offering teachers opportunities to experience as learners statistical investigations (Makar & Fielding-Wells, 2011), and nominates the significance of teachers' engagement with inquiry in the teaching (Shaughnessy, 2014). A further approach suggests combining experiences in teacher education, namely as learners and as teachers, and provide evidence that this combination can help teachers not only to strengthen their statistical content knowledge but also to be able to transfer their understanding to the classroom (Heaton & Mickelson, 2002). Although the complementarity of teachers' experiences as both learners and teachers seems to be crucial in teacher's professional development, we still know very little on how these two types of experience interact and provide feedback to one another.

In this study I explore the interplay between the process of negotiating the content of statistics (NCS) and the process of negotiating the teaching of statistics (NTS) in the professional development of teachers. Particularly, I followed a group of 11 secondary mathematics teachers who worked collaboratively in an emerging Community of Practice (CoP) (Wenger, 1998) to develop the statistics teaching practice. During their work in this community, the teachers had the opportunity to engage in tasks that promote negotiation of meaning in both the content and the teaching of statistics, and so they participated as both teachers and learners. The research question that guides this exploration is the following:

> How do negotiation of the content of statistics and negotiation of the teaching of statistics interact and provide meaning to each other in the context of a Community of Practice?

Theoretical perspectives

In this study, I view the teaching of statistics from a *statistical thinking* point of view (Wild & Pfannkuch, 1999) which sets inquiry at the core of statistical teaching and learning and highlights both the specificities and the complementarity between statistics, probability and mathematics. I focus

on three dimensions related to statistics teaching practice. The first dimension is the *learning potentials* which refer to particular skills, abilities and knowledge that are connected to the statistical activity, e.g. that students are expected to understand and deepen in the fundamental statistical ideas (Burrill & Biehler, 2011) or that they need to be able to use and evaluate appropriate statistical tools and methods in order to analyze data (e.g. Franklin et.al., 2005). The second dimension is the *features*, namely instructional tools and strategies that seem to be crucial in supporting students to achieve the defined *learning potentials*. Examples of features are students' engagement in statistical investigations (MacGillivray & Pereira-Mendoza, 2011) or the use of dynamic software tools that support data explorations (Ben-Zvi, 2006). The third dimension is *resources*. The variety of the resources that are brought into the teaching of statistics constitutes a dynamic ground where teachers can build and form their practice. In my view of resources I adopt Adler's (2000) conceptualization which extends beyond material resources (e.g. software tools, physical objects, media extracts, real data sets) to include human resources (e.g. previous experiences, collaboration with colleagues, knowledge about concepts and procedures) and cultural resources (e.g. time, classroom habits).

In my view to practice, I use the lens of the social theory of learning (Wenger, 1998) which theorizes learning in practice through four components: *meaning* (the way we experience our life and the world as meaningful), *practice* (shared historical and social resources and actions), *community* (social configurations to which we belong) and *identity* (personal histories of becoming). In the social theory of learning, practice is about both action and interpretation of the action, and meaning is the essence of a practice and it is situated in the process of *negotiation of meaning*. In this study, I acknowledge two types of meaning that is negotiated in the CoP, the meaning related to the content of statistics where the teachers participate as learners in the negotiation, and the meaning related to the teaching of statistics where the teachers participate as teachers in the negotiation. The study of the interaction between the process of NCS and the process of NTS aims to get insight on how NCS can provide meaning for NTS and vice versa. Especially in the case of teaching statistics where the mathematics teachers are challenged with a content that they are not familiar with (Hannigan et.al., 2013) and which contains epistemological differences from the mathematics content they teach (Moore & Cobb, 2000), such insights could be rather helpful for the research in statistics teachers' professional development.

Methodology

To achieve my research goal I followed an exploratory case study methodology (Yin, 2003), where the case was a group of 11 secondary school mathematics teachers, 5 practicing (Akis, Dinos, Kimon, Lidea, Marcos) with 8-30 years of teaching experience and 6 prospective (Athina, Chloe, Eva, Lia, Ria, Sofi). All teachers were mathematics graduates and also graduates or senior students in a Master's program in Mathematics Education with no particular familiarity with the teaching and learning of statistics and with a varied background in statistics. This group was gathered in a voluntary basis and worked collectively for two years (2012-2013 and 2013-2014 academic years) in a regular base (about 2 meetings per month that lasted approximately two and a half hours each). Two researchers were also participants (the author-R1 and the supervisor of the study-R2) encouraging the active participation of the teachers, providing various resources, finalizing each meeting's agenda and challenging teachers to reflect on their experiences. The main agenda of the meetings was formed around a cyclic route of: (a) inquiring the content of statistics, (b) designing for their classroom (c)

implementing the designed tasks in their classroom and (d) reflecting on their practice. I considered this group as an emerging Community of Practice (Wenger, 1998) by encouraging the development of mutual engagement, joint enterprise and shared repertoire. This paper is based on the data of a full cycle (inquire the content, design teaching, teach the planning lesson and reflect on it) which covered 10 of the total of 12 meetings in the first academic year.

All meetings were audio and video recorded and the group discussions were fully transcribed. Semi-structured individual interviews at the beginning and at the end of the study were also conducted. Another source of data was teachers' journals regarding issues they considered as central in each meeting. Although these reports were not a complete source of data, they often constituted a useful source of triangulation in order to corroborate the study's findings.

The analysis of the data was based on a grounded theory perspective using the ATLAS.ti software. During the coding process I used as unit of analysis the task that the teachers were engaged in. In each task I distinguished NCS and NTS parts in which I assigned *features*, *learning potentials* and *resources* that were visible in teachers' discussions. In Table 1 I present an example of the process we followed. Last, in a second level I focused on identifying how the negotiation of each type of meaning was mobilized as well as exploring interactions among the negotiation of the two types.

Negotiating the content (extract from the 2nd meeting)	**Features**	**Learning Potentials**	**Resources**
Akis: Now, I **couldn't understand** this very well. As **Fischbein** mentioned in **his article** it is the concept of **proportionality** behind this. He said that we still use proportionality in order to make a conclusion instead of using the **Low of Large Numbers**. **Lidea:** An what is this Low of Large Numbers? **Ria:** It is when something is repeated many times, then it approaches the **theoretical probability**. **Marcos:** It is what we say to estimate the probability through the **relative frequency**. We then consider the probability as the number of the favorable cases to the total number of the cases, when we repeat this many times and the number of times tends to infinity etc. So let me explain what we have here. You have a small and a big hospital and you get a **sample** from each one, I think that this has to do with the **variability**, I mean practically it is like you have a small and a big sample, which one do you expect to be more accurate? More **representative** with regard to the theoretical probability. It is the small sample or the big sample? It is the big. This is the case with the big hospital, because in the big hospital the births will be closer to the **mean value**, while in the small hospital you will have large variability. **Lidea:** But I still can get why the small one is the right answer in this task? **Marcos:** Wait a moment. What is the question in this task? It asks in which hospital it is more likely to have more such days. **Akis:** Is it possible that I may didn't **translate** it correctly? **Lidea:** No Akis, it is absolutely correct. I translated it with exactly the same way. But I can't understand why. When he says 'such days'... **Dinos:** (talking to R1) Do you remember the **example** you had discussed with us (he means sometime **in the past**). The one that says that the half of the newborn children are boys. This is very similar and more comprehensible. **Maybe you could show us** this one. (R1 goes to the **computer** and **projects** the task mentioned by Dinos, they continued to discuss the **task** using the **projector** and the **blackboard**)	Engagement with an inquiring task Plenary discussions and negotiations (as experienced by the teachers)	Connect to probability ideas Deepen on sampling notions Appreciate variability	Educational research (Fischbein's article) Mathematical tools (portionality, Law of Large Numbers, theoretical probability) Statistical tools (relative frequency, sample, mean value, variability) Cultural tools (language) Human (collective experiences, difficulties) Material (computer, projector, blackboard)

Table 1: Example of the data analysis where the teachers were discussing the 5th problem presented on Fischbein & Schnarch's article (Fischbein & Schnarch, 1997)

Results

In Figure 1 I present the various tasks that the teachers were engaged in during the 10 meetings. As we can see, there were tasks oriented to encourage negotiations in the content of statistics (e.g. exploration of statistical tasks/situations), tasks that aimed at negotiating the teaching of statistics (e.g. design for the classroom, reflection on the teaching) and tasks that had the potentiality to immobilize negotiations in both types of meaning. However, as we can see in Figure 2 the realized negotiations indicate that the interaction between NCS and NTS was not only a function of the nature

of the task. This is especially obvious in the case of the design for the classroom tasks, where a quite large part of teachers' discussion was related to NCS.

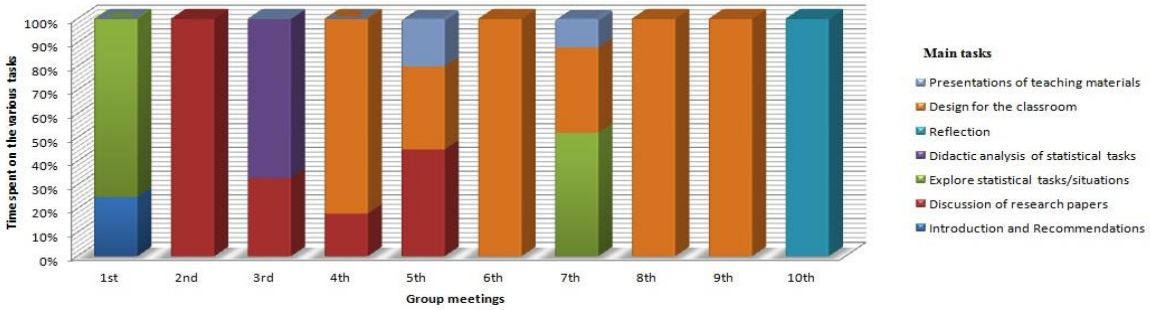

Figure 1: General description of the meetings agenda

How NCS provides meaning to NTS

The analysis of the data showed that NCS was mobilized by the teachers' need to understand better or deepen in a statistical concept or process. In the extract presented in Table 1 above, the teachers were discussing students' false intuitions with regard to the effect of sample size while Akis' intervention mobilized a negotiation of meaning related to sample notions and the Law of Large Numbers. This episode was expanded and lasted for about 10 minutes during which the teachers together with R1 exchanged arguments and utilized various resources that aimed to deepen their understandings around these notions. This discussion helped teachers not only to deepen their content knowledge but also to negotiate new difficulties that they had (making connections with particular knowledge and abilities required to gain meaning, namely the learning potentials) and means that helped them to overcome these difficulties (namely the features that can facilitate these learning potentials). In other words, they had the opportunity to incorporate their NCS experience into their teaching practice. Akis notes in his meeting report are indicative of how NCS provided meaning for the teaching practice: "What I observed is that our beliefs and our attitudes towards probability and statistics are very close to those of students, I mean we are guided by an intuitive way of thinking. If we want our students to adopt a more inquiring stance, we first need to give them appropriate tools to overcome their intuitions. I think that the way we discussed in the meeting, revealing our misconceptions and resolving them could be a good model for our teaching".

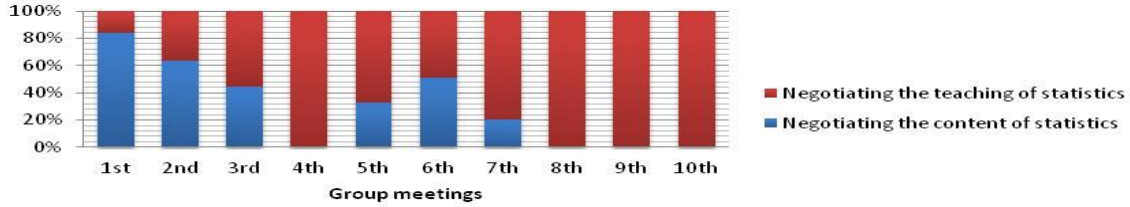

Figure 2: Alternations in the content of negotiation in the various meetings

Moreover, NCS was also mobilized by teachers' need to gain experiences with data. This was the case for example in the 6th meeting, where the group conducted a pilot study for an experiment that was designed for the students by a team of five teachers. Particularly, in this experiment the students were supposed to investigate if listening to music affects their ability to recall words. The teachers designed the experiment and before they implemented it in the classroom they collected data inside the group, explored the data, made conclusions and thought of possible modifications in the

experiment's design. In this sense the NCS helped teachers to acquire experience and confidence regarding the implementation of the designed activity. The words of Dinos in his final interview are characteristic: "this interaction gives you the impression that this (he means the task) will be considered by many couples of eyes, by many views. I mean especially in statistics where you can never acknowledge all the possible parameters,… it is not that you will learn something new, it is that it helps to illuminate other dimensions that you may have neglected at first".

Furthermore, in many cases the teachers transferred directly their experience as learners to their teaching practice. For instance, in the 6th meeting Dinos used an example to help his colleagues to understand the notion of the standard deviation in the estimation of a probability. Later in the discussion, Dinos suggested using the same example in the classroom and the others responded:

Sofi: So you suggest using the same example with the students. I like it. Actually it helped me to understand so it would be also helpful for students to understand.

Athina/Chloe: Yes, I agree too.

Last, as can be seen in the examples discussed above, it is also the general context of their enterprise, namely the community of statistics teaching practice, that mobilized them to analyze their NCS experience in terms of identifying learning potentials that could be supported in the classroom, features that could support these learning potentials and resources that could facilitate students in the learning process.

How NTS provides meaning to NCS

Apart from the nature of the task, NTS was mainly mobilized due to a question posed by one of the researchers, such as "Would you use such a task in your classroom and if yes how?" or "What do you think a student can gain from the experience of such a task?". Such questions encouraged teachers to extend their experience as learners to start a negotiation of meaning around the teaching and learning. The extract below is from the 3rd meeting when teachers discussed a statistical task and illustrates how such NTS was mobilized.

R1: What difficulties can someone face in the classroom with this task? Would you try to implement such a task? If yes, then how and with what goal?

Kimon: I could try it with 12th Grade students not with younger ones.

Dinos: I could do this with young students as well, with 8th Grade students for example.

Lia: I agree with Dinos. I think this would be useful in the formation of their attitudes towards probabilistic ideas. In 12th Grade level they have already formed quite formal conceptions.

Dinos: But the point is what modifications we can do.

This discussion for which the starting point was the question posed by R1, continued for about 30 minutes during which the teachers exchanged views and suggestions, referred to specific learning potentials, discussed potential features that could serve their goals and utilized or suggested resources (e.g. use of statistical tools / suggestion to include in their study all the students in the school instead of the students in the classroom) that could facilitate the learning process with regard to the defined learning potentials. In this way, teachers extended their NCS experience to consider aspects of

teaching and learning and to connect the statistical objects they negotiated with features, learning potentials and resources that broaden their view towards these objects.

Another example is the case where, during NTS, the teachers or the researchers asked for clarifications regarding particular teaching decisions or suggestions. In this way teachers reconsidered their choices and developed a deeper understanding on the underlying concepts or possible conceptual connections. The following example, from the 7th meeting, is indicative of this case. In this extract Marcos was trying to explain his choice to use mean values instead of median when students study the difference in our ability to recall words and no words.

R1: Marcos, why did you choose to use the mean values here?

Marcos: It actually depends on what you want to see.

Chloe: And what about you? What do you want to see?

Marcos: Look. It is true that with the median is easier to manage the results, I mean it could be easier to find subsets that can give a difference on median that is so big or bigger than the one we get. But on the other hand… What if the difference is small, like here? If I was to choose from a students' perspective, I would choose the median, but not with a very small difference. I mean maybe there are other parameters that result in such a small difference. I am not sure what I would say to students for a very small difference.

This episode continued and lasted for about 4 minutes during which Marcos developed arguments to explain his choice. In this way, Marcos got insight not only in his didactical choice but also in the role of the mean value and the median in the study of differences between two variables. Such an inquiry, although it refers to NTS, was also helpful in illustrating statistical concepts and thus provided meaning to NCS as well. Almost all the teachers of our study, in their final interviews, referred to the positive impact of their inquiry in teaching on their content knowledge. This was either because they were facilitated by their colleagues or the researchers' examples and explanations or because of their attempt to help or convince their colleagues, which guided them to develop appropriate examples or arguments and thus helped them to deepen in their own understandings. In both cases, the context of the CoP which encouraged the collaboration among them played a determinant role on this interaction. Below, we present two characteristic extracts from the final interviews.

I feel I gained a lot, especially as a learner. You see, the concepts we decided to work with in the classroom were blurred for me, too. It was mainly Dinos' examples and explanations that helped me to understand first these concepts and consequently what we were going to do with the students. (Chloe, final interview)

These discussions helped me to develop a deeper awareness of many issues. I mean, when you are trying to take a stand towards an issue or a particular decision, it helps you either to get a more clear position, by developing a deeper understanding or to consider new views or aspects that were out of your attention. (Marcos, final interview)

Conclusion

This study aimed at getting insight into the interaction of NCS and NTS in the context of a CoP. The results showed that both processes were mobilized either by the nature of the task itself or by reasons that are related to the teachers' needs and the context of their work inside the CoP. Moreover, we saw that NCS process, apart from teachers' content knowledge, helped them to enrich their teaching repertoire (appreciate learning potentials, explore features, get access to new resources) as well as to strengthen their confidence in handling particular statistical concepts inside the classroom. Similarly, NTS process, apart from a space for inquiring teaching and learning, was also a starting point for the teachers to develop deeper understanding on statistical content, to make conceptual connections and to acknowledge different dimensions of the underlying problem. These results go beyond the work of Heaton & Mickelson (2002), which shows the importance of the complementarity of the two processes in the professional development of teachers, to give empirical evidence of how the one process provide feedback and meaning to the other. Moreover, the collaborative context of a CoP acted supportively in the interactions between them. Thus, such a context, that fosters the co-existence of NCS and NTS, seem to help teachers not only to experience statistics as learners, but also to link this experience with classroom reality.

Last, this complementarity and feedback are especially important to the professional development in the case of statistics since the stochastic nature of the content on the one hand, and the unfamiliarity of teachers with the statistical tools on the other hand, constitute factors that, as we saw, reinforce the interactions between NCS and NTS.

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Common patterns of thought and statistics: Accessing variability through the typical

Christian Büscher

TU Dortmund University, Germany; christian.buescher@math.tu-dortmund.de

Allowing students to construct meanings of statistical concepts like variability requires building on their individual experiences. The notion of patterns of thought is utilized to conceptualize the difference between formal statistics and learners' initial thinking and to describe a pathway to bridge this gap between individual and mathematical thinking. Students' patterns of thought and the processes of their conventionalization are reconstructed in a qualitative study with n = 10 students of grade 7. An outlook is given on the possibility of connecting students' patterns of thought with formal statistics.

Keywords: Informal inferential reasoning, design research, concept development.

Introduction

One of the aims of statistics education is to foster informal inferential reasoning (IIR), the ability and disposition to use data in order to reason about some wider universe (Makar, Bakker, & Ben-Zvi, 2011). In IIR, statistical concepts are combined with contextual knowledge under certain statistical norms and habits, such as a "critical stance towards data" (see Makar et al., 2011 for a more thorough overview). While the framework of IIR can explain the *role* of statistical concepts in producing informal statistical inferences (ISI, see Makar & Rubin, 2009), it does not account for the *development* of statistical concepts. Accordingly, there exists a lack of research concerning statistical concept development of students with little experience in statistics. Tracking of the development of concepts, Bakker and Derry (2011) draw on the background theory of inferentialism, linking students' emerging concepts in inferential practices with IIR. Looking at the micro level of students' reasoning processes, Bakker and Derry argue that students can show complex ideas regarding statistical concepts such as center, variation, distribution, and sampling. These ideas are not formally articulated, but can rather be seen as "precursor notions" of regular statistical concepts (Bakker & Derry, 2011, p. 20).

These precursor notions could provide a promising basis for the development of more formal statistical reasoning. The challenge remains however to design tasks that draw on these precursor notions in order to foster students with little prior experience in formal statistics. These tasks need to draw on students' individual ways of reasoning as a resource in order to develop statistical concepts meaningful to the students. This requires paying careful attention to students' learning processes, and to uncover the links between formal statistical concepts and students' everyday thinking. This study aims at reconstructing students' individual ways of thinking on a micro level in order to find ways to connect students' everyday thinking to regular statistical concepts.

Patterns of thought in everyday thinking

The mathematician and philosopher Wille (1995) argues that in order for mathematics to become learnable and possibly meaningful to non-experts (i.e. the general public), the discipline of mathematics itself has to be restructured in a program he coined *Generalistic Mathematics* ("Allgemeine Mathematik"). Wille calls for mathematicians to reveal the reasons, the aims, the

common patterns of thought, and the boundaries and dangers of mathematical concepts and of whole mathematical theories. This however must not be done in the language of mathematical theory. Opening mathematics to the general public necessitates the use of common language, rather than specialized vocabulary, to describe those reasons, aims, patterns, and boundaries.

This approach to make mathematics open and meaningful resonates well with Lengnink and Peschek (2001), who see the challenge and goal of mathematics education in explicating the connection between *everyday thinking* and *mathematical thinking*. For them, mathematical thinking manifests as conventionalized everyday thinking. In this way, confidence intervals can be seen as a conventionalized form of the generalizations taking place in daily life: being reasonably sure that some observation of daily life can be taken as true, with a certain intuitive degree of variation.

This places a firm emphasis on the importance of common thought patterns and practices, so that a task of mathematics education becomes the identification of these patterns of thought. While Wille (1995) sees this task in the hands of (the philosophy of) mathematics, Prediger (2008) points out that empirical insights into mathematical learning processes form a valuable and even necessary basis for identifying connections between everyday thinking and mathematical thinking. This also means that thought patterns cannot be approached in a general way, but rather are tied to the specific mathematical content of the learning process under investigation. This study adopts the approach outlined by Prediger (2008), in order to reveal patterns of thought and their connection to the statistical concepts of center and spread when comparing distributions.

The typical as a common pattern of thought

One way students' thinking differs from formal statistics is in the use of measures. In statistics, measures such as median or standard deviation function as highly specialized tools for talking about statistical concepts like center or variability. In their intuitive approaches to statistics, students use strategies such utilizing "modal clumps" instead (Konold et al., 2002) to point out 'the majority' of the data. Since to the students the location and the width of the clump both are important, these modal clumps can simultaneously address the center of a distribution as well as a form of spread. Thus, in their everyday language, learners integrate many different statistical concepts that would formally be strictly distinguished through use of different specialized measures (Makar & Confrey, 2005).

This raises the question on what would be promising patterns of thought to build on. One such candidate would be the practice of identifying 'typical' values or ranges of values within datasets. Some research indicates that 'typical' might be a good way for students to think about the average (Makar, 2014), while other research finds 'typical' or 'normal' to be a term for talking about ideas combining center and spread (Büscher, 2016a; Büscher, 2016b). Thus, the pattern of thought of identifying the 'typical' of a distribution seems a promising candidate as a resource for concept development, although it remains unclear for which concepts exactly.

Patterns of thought and concept development

To address the question of how patterns of thought can support concept development, this study follows a conceptual change approach (Duit & Treagust, 2003). Learning is understood as a restructuring of prior conceptions, occurring when these conceptions no longer satisfactorily explain phenomena. The goal of statistics teaching is then to initiate the development of conceptions into statistical concepts. Since statistical concepts show a high degree of connection to each other, learning

trajectories in statistics however should not address concepts in an isolated way, but rather in a holistic way (Bakker & Derry, 2011). This calls for organizing structures that (a) connect to learners' prior conceptions, (b) holistically address statistical concepts, and (c) lead to regular statistical concepts. Patterns of thought can provide just such a structure, as they encompass different prior conceptions and thus enable the connection of everyday thinking to mathematical thinking.

Research questions

The theoretical background of this study suggests that designing learning trajectories towards statistical concepts should start from fruitful patterns of thought. It is thus an issue for empirical investigations to find those patterns of thought in students' thinking which can indeed serve this function as starting points in learning trajectories. Looking at what is 'typical' was identified as one potential pattern of thought that could result in such concept development. This study therefore aims to answer the following research question: *What concepts do students address and develop when conventionalizing the vague pattern of thought of 'identifying the typical'?*

Research design

This study adopts the methodological framework of topic-specific didactical design research (Prediger & Zwetzschler, 2013) with a focus on learning processes (Prediger, Gravemeijer, & Confrey, 2015). This approach aims at providing empirically grounded local theories on topic-specific learning processes as well as design principles and concrete teaching-learning arrangements for the topic. While the framework utilizes iterative cycles of design experiments, this study reports on the third cycle of design experiments.

Data gathering

Design experiments were conducted with five pairs of grade 7 students (aged 12 – 14) who had only little experience with statistics within the mathematics classroom. Experiments consisted of two sessions of 45 minutes each, with each session having its own arrangements of data and tasks. They were fully videotaped and partially transcribed. The process-focused analysis of the video data from the first session of the design experiments allows to reconstruct the development of students' patterns of thought, in their relation to task design.

Task Design

For designing the teaching-learning arrangements, various design principles have been implemented throughout the different cycles of design research, two of which play an important part in this study.

Drawing on 'typical' as pattern of thought. As outlined above, connecting to the pattern of thought of characterizing what is 'typical' of a certain distribution can potentially provide a starting point for processes of conventionalization that lead to meaningful use of formal statistics. Therefore, 'typical' has to be explicitly addressed in the task design, and the setting of the task has to provide a context in which this pattern of thought can naturally occur.

Criticizing conventionalizations. As Lengnink and Peschek (2001) point out, mathematical learning has to explicitly address the relation between everyday thinking and mathematical thinking. Following this, it is not enough for a task to just utilize students' patterns of thought and to encourage conventionalization of those patterns of thought. It is rather these conventionalizations that have to

become the object of investigation. Under a Generalistic Mathematics perspective this could mean addressing reasons, aims, and boundaries of these conventionalizations.

These design principles were realized in the design of the *Antarctic weather task*. The initiated activity puts the students into the role of consultants to researchers at the Norwegian Antarctic research station *Troll forskingsstasjon*. In the first phase of the task, the students investigate dot plots of daily temperatures for the month of July 2004 and predict the weather for next July by giving a distribution of ten days. In Phase 2, additional data for July 2002 and 2003 were included (Figure 1). Predicting the weather was chosen as activity because it is rooted in everyday thinking, combining experiences of short-term variability (one can never be too sure about the weather…) with long-term signals (… but there are typical temperatures after all).

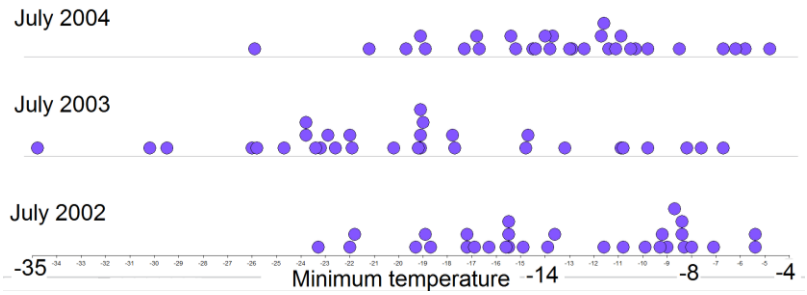


Figure 1: Distributions of the Antarctic weather task (translated from German)

In order to support conventionalization of patterns of thought, the third phase introduces a design element called *report sheets* (cf. Figure 2 and 3). The report sheets are introduced to serve as a brief summary of the Antarctic weather in July. They combine graphical representations and the use of measures with a brief inference. First, the students are asked to fill out their own report sheet. After that, in the fourth phase, the students receive three different filled-in report sheets by fictitious students (Figure 2). These filled-in report sheets differ in their interpretation of typical. This serves as a basis for discussing and criticizing the different conventionalizations of typical, as the students are asked to evaluate the correct use of typical.

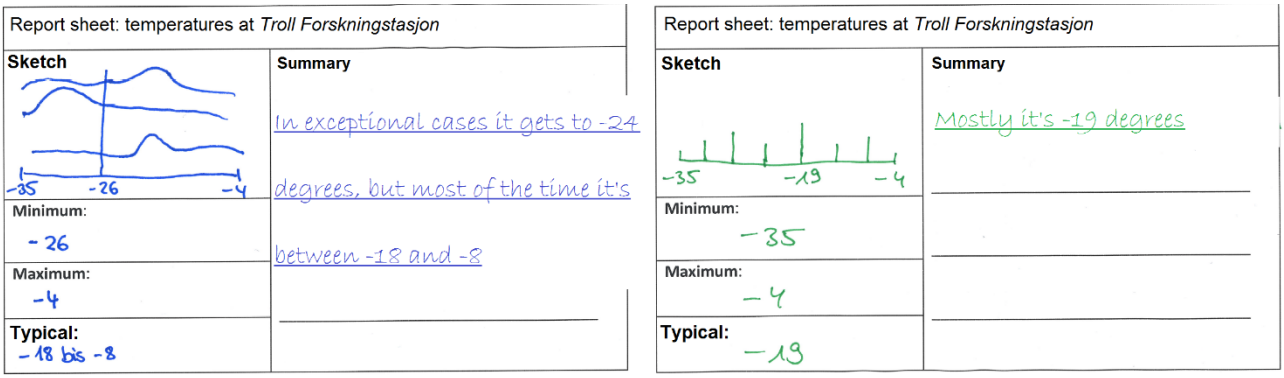


Figure 2: Fictitious students' filled-in report sheets (translated from German)

Data Analysis

The students' patterns of thought were reconstructed in an interpretative analysis using concepts-in-action and theorems-in-action from Vergnaud's (1996) theory of conceptual fields. Concepts-in-action are "categories (objects, properties, relationships, transformations, processes etc.) that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate

selection of information according to the situation and scheme involved" (Vergnaud, 1996, p. 225). Theorems-in-action are statements held to be true by the learner.

Which concepts-in-action and theorems-in-action are activated depends on the pattern of thought utilized by the learners. Thus, patterns of thought are conceptualized as groups of concepts-in-action concurrently occurring in the learners' activity. Concepts-in-action and theorems-in-action are reconstructed from the students' point of view, and do not necessarily correspond to regular statistical concepts. In the analysis, the reconstructed concepts-in-action are symbolized by ||…||, while theorems-in-action are denoted by <…>.

Empirical Insights: The case of Maria and Natalie

The first snapshot starts with Phase 2 of the Antarctic weather task. After getting the additional data of the years 2002 and 2003, Maria and Natalie, Grade 7, try to explicate their view on the data.

1	Maria:	We are pondering what the relationship, like, how to…
2	Natalie:	Yes, because we want to know what changes in each year. And we said that there *[2003]* it came apart.
[…]		
8	Maria	Yes, I think it *[2004] is* somehow similar to that *[2002]*, but that one *[2003]* is different.
9	Natalie:	Like here *[points to 2004, around -12 °C]* are, like, like the most dots, and here *[2002, -12 °C]* are almost none. And there *[2002, -8 °C]* are the most and here *[2004, -8 °C]* are almost none.

This excerpt serves as an illustration of the starting point in the students' reasoning. The students are trying to characterize the differences observed in the distributions. At first, the students formulate differences between 2003 and 2002/2004 in terms of ||*spread*||: in 2003, the temperatures "came apart" (#2). Another difference they notice is the difference of the location of the ||*center*|| between 2002 and 2004, indicated through modal clumps ("the majority", #9).

It is important to note that at this stage, the students face difficulties in trying to express their findings. The distributions of 2004 and 2002 are found to be "somewhat similar" (#8) though "different" (#8) to 2003, with further explanation supplied by Natalie through use of gestures and improvised vocabulary ("like, like the most dots", #9). Few minutes later, the students find a way to deal with the complexity.

21	Maria:	Well, we first should look at how many degrees it has risen or fallen. Generally. In two years.
[…]		
27	Natalie	You mean average, like…
28	Maria	The average, and then we look at how the average changed in two years.

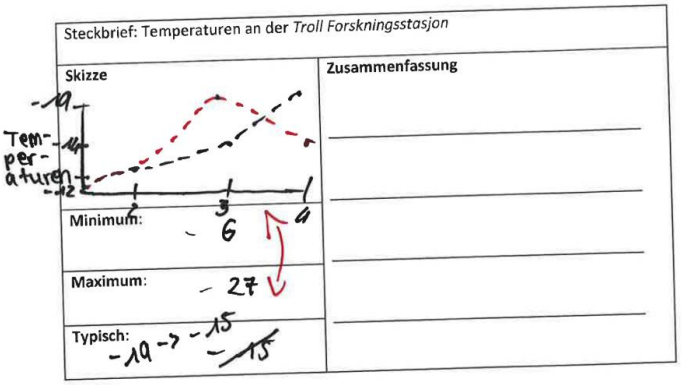
By introducing the notion of reducing the distributions to a ||*general value*|| ("Generally", #21), the students are able to handle the complexity of the differences between the distributions. For this general value, they appear to already know an adequate measure: the ||*average*||. To the students, <*the average represents the general value of a distribution*>. The average acts as a summary to be used in further procedures, as <*the differences between distributions can be described by differences in general values*>.

In the following exchange, after having estimated the averages to be -12 (2002) and -14 (2004), Maria and Natalie try to use their result for a linear extrapolation of the weather in 2015 to be predicted.

41	Natalie:	Wait. If it gets colder by 2 °C in two years, then it gets colder by 1 °C each year, so we have to…
42	Maria:	Nah, eh, yeah okay
43	Natalie:	13 °C colder average temperature. Right?
44	Maria:	Yes.
45	Natalie:	But that's too much, isn't it?

When reflecting on their result however, the students realize that a decline of the average temperature by 13 °C is not a realistic proposition (#45). While their knowledge of the real-world context helps them to identify this contradiction, they are not able to find another solution. In the minute following (not shown here), the students stay insistent in using the average and a linear extrapolation of the trend. It is important to note that, at this stage, their ideas concerning ||*spread*|| as expressed in the first excerpt seem to have disappeared, replaced by the stronger notion of ||*general value*|| expressed through the more conventionalized form of ||*average*||.

The design experiment progresses through the third phase, in which the students create their own report sheet (Figure 3). The analysis picks up at beginning of the fourth phase, with the students comparing the different interpretations of 'typical' in the filled-in report sheets.



Translations:

"Report sheet: temperatures at Troll Forskningsstasjon"

Skizze – Sketch

Typisch – Typical

Zusammenfassung – Summary

Temperaturen - Temperatures

(The black graph was drawn first, labeled a mistake, and immediately replaced by the red graph. Typical was first assigned as -15, then after the fourth phase changed to -19 to -15.)

Figure 3: Maria and Natalie's report sheet

Comparing the different interpretations of 'typical', Maria and Natalie are intrigued by the possibility to use an interval to characterize 'typical'. This leads them to reflect on their use of the average.

61	Natalie:	But the average temperature isn't really typical, is it?
62	Maria:	What, typical? Of course the average temperature is the typical.
[…]		
66	Maria:	Well, no. Typical is more like where the most… no…
67	Maria:	The average temperature isn't the typical after all. Because it's only the general, the whole. The typical would be for example for this *[2004]* here *[points to -14 on the 2004 dot plot]*.
68	Natalie:	Typical I think simply is what is the most or the most common.

The students utilize Typical to differentiate between two different ideas: The ||*general value*|| that is expressed through the ||*average*|| ("the general, the whole", #67), and the ||*most common*|| part of the

distribution, expressed through the ||*Typical*|| ("the most common", #68) – although at this point it is not yet clear if Typical consists of a number or an interval. With the ||*most common*|| corresponding to the notion of ||*center*|| expressed earlier ("the most dots", #9), Typical seems to help the students to express ideas that got swept aside by the more conventionalized average. Both, average and typical, start to act as conventionalized tools for talking about specific aspects of distributions.

Some minutes later, Natalie summarizes her view on the relation between 'typical' and average.

81 Natalie: And average is pretty imprecise, because it doesn't say anything about a single day. And with typical, I'd say, that it's a span between two numbers, because that way you can better overlook how it is most of the time.

In the end, typical and average provide two different applications. Whereas the average acts as a summary, 'typical' gives an overview into a distribution. The average can be used in comparing distribution in an efficient way, while 'typical' gives an insight into a range of 'normal' or 'expected' temperatures, to which any single day can be compared. In this way, 'typical' combines aspects of ||*center*|| and ||*spread*||.

Conclusion

The aim of this study was to examine the interaction of concept development and students' patterns of thought. The students showed two different patterns of thought: Characterizing data through a general value and through a range of typical values. These patterns differed in their degree of conventionalization. While the general-pattern was addressed through use of the average, the students lacked a conventionalization corresponding to the typical-pattern. This resulted in the typical-pattern to be suppressed in favor of the general-pattern, as the ideas addressing spread disappeared. Only when the students were supplied with different conventionalizations of Typical, they were able to reconnect to their typical-pattern. This then allowed them to express ideas combining center and spread.

This identification of thought patterns provides a promising starting point. Work still remains however utilizing these thought patterns to develop regular statistical concepts. This could be achieved by reconceptualizing formal statistics in terms of a typical-pattern. One possible connection could be interpreting the 'box' of a box plot as the typical area of a distribution. Additionally, more insight into processes of conventionalization is needed in order to be able to successfully guide students on their way to meaningful statistics (for one example see Büscher, 2016).

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Do attitudes toward statistics change during an introductory statistics course? A study on Italian psychology students

Francesca Chiesi and Caterina Primi

University of Florence, Italy; francesca.chiesi@unifi.it, caterina.primi@unifi.it

Mixed results have been reported about changes that might occur in students' attitudes as a consequence of attending introductory statistics courses and about male and female students' differences in attitudes. Thus, the aim of the current study was to shed light on attitudes changes in students attending introductory statistics courses taking gender into account. Overall, we observed changes in attitude that resulted in a more positive attitude from the beginning to the middle of the course. Nonetheless, along with a general positive trend, it was possible to highlight that some students get significantly worse attitudes and many of them do not substantially change their initial attitudes. Overall, no significant differences were found between male and female students. Finally, probabilistic competences along with statistics anxiety accounted for individual changes in attitudes toward statistics. Educational implications were discussed.

Keywords: Statistics education, attitudes toward statistics, attitude changes, gender differences.

Introduction

Attitude toward statistics is a disposition to respond favourably or unfavourably to objects, situations, or people related to statistics learning (Schau, Stevens, Dauphinee & del Vecchio, 1995). It is commonly described as a multi-dimensional concept that consists of affective (students' positive and negative feelings about statistics), cognitive (beliefs about the ability requested to learn statistics and about the discipline), and behavioral (interest and effort) components, which are deemed to have an effect on achievement. Emmioglu and Capa-Aydin (2012) provided a meta-analysis that addressed this relationship suggesting that there is a significant correlation between students' achievement and statistic-related beliefs, motivation, and feelings. Indeed, whereas the reviewed studies employed different research approaches and included different kinds of samples and courses, more positive attitudes were correlated - with different extent, and directly or indirectly- to a better course performance.

For this reason, a basic question of research refers to the changes that might occur in students' attitudes as a consequence of attending introductory statistics courses. Although some studies reported an increase in attitudes as a result of the courses (e.g., Chiesi & Primi, 2010), Schau and Emmioglu (2012) conducted a large-scale investigation reporting that the different attitude dimensions do not substantially change through the courses. Nonetheless, Millar and White (2014) highlighted that the mean changes were around zero but the variability in the individual changes was relatively large, i.e., whereas in some cases the attitudes actually did not changed, positive changes (i.e., shifts to a more positive attitude) and negative changes (i.e., shifts to a more negative attitude) were both observed.

Finally, literature on attitudes toward statistics addressed the issue of gender differences. Presumably due to the different sample and course characteristics (engineering students, economic students, psychology students, pre-service teachers), inconsistent results were reported. Some authors found

that men expressed more positive attitudes toward statistics than women (e.g., Chiesi & Primi, 2015; Tempelaar & Nijhuis, 2007), others studies found no gender differences (e.g., Martins, Nascimento & Estrada, 2011), and some others documented more positive attitudes for women (e.g., Rhoads & Hubele, 2000).

These mixed results suggest, in line with the recommendation made by Eichler and Zapata-Cardona (2016), to intensifying research on students' statistics-related attitudes. Thus, the aim of the current study was to shed light on attitudes changes in students attending introductory statistics courses taking into account gender differences. The specific aims can be detailed as follows.

a. To investigate the possible changes in attitudes as result of the course. Based on previous studies conducted on similar samples (e.g., Chiesi & Primi, 2010), we hypothesized that a positive overall change might occur from the beginning to the middle of the course. To take into account possible gender-related difference in attitudes, we observed the differences from pre- to post-test in male and female students separately. We expected that the course had an effect on both genders.

b. To provide a more fine-grained investigation we looked at the individual differences in attitude changes, i.e. if the student shifted to a better attitude, or if she/he got worse attitudes as a result of the course, or if the student's attitudes remained unchanged. In defining these typologies we referred to Schau and Emmioglu (2012) and Millar and White (2014). They suggested that, along with the statistical significance of the change in attitude scores, it is important to ascertain if students' attitudes change consistently, i.e., if there is a substantial increase/decrease in the observed scores. As such, following their indications to determine the relevance of the score change, we investigated individual differences in attitude changes controlling for gender. We expected that positive and negative shifts as well as no changes might be observed in both men and women.

c. Since we expected that students – all attending the same course - might change in positive or negative their attitudes towards statistics or maintain them stable, we explored if some specific factors could accounted for individual differences. In line with previous studies on cognitive and non-cognitive factors related to statistics education, we looked at mathematical and probabilistic competences along with test anxiety and statistics anxiety.

Method

Participants

Participants were 136 psychology students enrolled in an introductory statistics course at the University of Florence in Italy (mean age = 20.93 years, *SD* = 3.59; 70% female). They were first year students who did not have previous experience with the discipline at the university level but they might have encountered the discipline before in school-related contexts or in their out-of-school lives. All students participated on a voluntary basis after they were given information about the general aim of the investigation (i.e., collecting data for a research project on students' statistics achievement).

Description of the course

The course was compulsory. It covered the usual introductory topics of descriptive and inferential statistics (including basic concept of probability theory and calculus), and their application in psychological research. It was scheduled to take place over 10 weeks, and takes 6 hours per week (for

a total amount of 60 hours). During each class some theoretical issues were introduced followed by exercises using either paper-and-pencil procedure or a computer package (*R-commander*). Students were assigned homework for which they were allowed to work in groups. Consultation hours were also offered for one on one help with exercises. The instructor was one of the authors of the current paper.

Measures

Attitude toward statistics was measured administering the 28-item version of the *Survey of Attitudes toward Statistics* (SATS) (Schau et al., 1995; Italian version: Chiesi & Primi, 2009). The SATS contains Likert-type items using a 7-point scale ranging from *strongly disagree* to *strongly agree*. It assesses four attitudes components: Affect (6 items) measures positive and negative feelings concerning statistics (e.g. "I will feel insecure when I have to do statistics problems" or "I will like statistics"); Cognitive Competence (6 items) measures students' attitudes about their intellectual knowledge and skills when applied to statistics (e.g. "I can learn statistics" or "I will make a lot of math errors in statistics"); Value (9 items) measures attitudes about the usefulness, relevance, and worth of statistics in personal and professional life (e.g. "Statistics is worthless" or "Statistical skills will make me more employable"); Difficulty (7 items) measures students' attitudes about the difficulty of statistics as a subject (e.g. "Statistics formulas are easy to understand" or "Statistics is a complicated subject"). Two versions to use at the beginning (pre-SATS) and during or at the end (post-SATS) of the course were developed. For both the pre- and post- versions of the SATS responses to negatively scored items were reversed. Because the subscales were composed of a different number of items, scores were obtained by dividing each component score by the number of items that assess that component. As such all the scores ranged from 1 to 7 and higher scores indicated a more positive attitude. For Difficulty a positive attitude (i.e., high scores) means that students believe that statistics is easy whereas a negative attitude (i.e., low scores) means that it is harder.

The *Mathematics Prerequisites for Psychometrics* (MPP, Galli, Chiesi & Primi, 2011) was employed to measure the mathematical skills needed by students enrolling in introductory statistics courses. The MPP consists of 30 multiple-choice format questions (one correct out of four alternatives) from which a total score (range 0-30) was calculated. Additionally, the *Probabilistic Reasoning Questionnaire* (PRQ; Primi, Morsanyi & Chiesi, 2014), designed to measure proportional reasoning and basic probabilistic reasoning ability, was administered. The scale consisted of 16 multiple-choice questions from which a total score (range 0-16) was calculated.

The *Test Anxiety Inventory* (TAI; Spielberg, 1980) was administered to measure anxiety associated with test-taking situations. The TAI is self-report instrument consisting of 20 items. Respondents are asked to report how frequently they experience specific symptoms of anxiety from 1 (*almost never*) to 4 (*almost always*). A total score was calculated as the sum of all items, with higher scores corresponding to high test anxiety. Along with this general anxiety indicator, the specific anxiety toward statistics was assessed using the *Statistical Anxiety Scale* (SAS; Vigil-Colet, Lorenzo, & Condon, 2008; Italian version: Chiesi, Primi & Carmona, 2011). The SAS is a self-reported measure consisting of 24 items with a five-point rating scale ranging from 1 (*no anxiety*) to 5 (*very much anxiety*). The SAS includes Examination anxiety (8 items, e.g., "Studying for examination in a statistics course"), Asking for help anxiety (8 items, e.g., "Asking the teacher how to use a probability table"), and Interpretation anxiety (8 items, e.g., "Trying to understand a mathematical

demonstration"). A composite score was calculated with higher scores corresponding to high statistics anxiety.

Procedure

Students were administered the SATS-pre, the MPP, the PRQ, and the TAI at the beginning of the course. At the middle of the course (about four weeks later), the SATS-post was administered along with the SAS. The questionnaires were introduced briefly to the students and instructions for completion were given. Answers were collected in paper-and-pencil format and the time needed to complete them ranged from 20 to 40 minutes.

Results

To ascertain the possible changes in attitudes toward statistics and the gender related differences, we ran a 2×2 mixed ANOVAs with course (pre/post) as a within-subjects factor, and gender as between-subjects factors on each of the four attitude dimensions. It was found a main effect of course - that resulted in an overall increase - on Affect ($F(1, 134) = 17.34$, $p<.001$, $\eta_p^2 = .12$; pre: $M = 3.44$, $SD = 1.08$, post: $M = 3.74$, $SD = 1.25$), Difficulty ($F(1, 134) = 24.17$, $p < .001$, $\eta_p^2 = .15$; pre: $M = 3.23$, $SD = 0.64$, post: $M = 3.51$, $SD = 0.66$), Cognitive Competence ($F(1, 134) = 59.67$, $p < .001$, $\eta_p^2 = .31$; pre: $M = 4.19$, $SD = 1.01$, post: $M = 4.72$, $SD = 1.07$), and Value ($F(1, 134) = 4.89$, $p < .05$, $\eta_p^2 = .04$; pre: $M = 5.03$, $SD = 0.85$, post: $M = 5.20$, $SD = 0.94$). With the exception of Value ($F(1, 134) = 1.91$, $p = .17$), significant between-subject differences were found for the remaining attitude dimensions (Affect: $F(1, 134)=8.36$ $p < .01$, $\eta_p^2 = .06$; Difficulty: $F(1, 134) = 4.82$, $p < .05$, $\eta_p^2 = .04$; Cognitive Competence: $F(1, 134) = 6.17$, $p < .05$, $\eta^2 = .04$) with male holding more positive attitudes. Nonetheless, there were not significant course by gender interactions (Affect: $F(1, 134) = 2.08$, $p = .15$); Difficulty: $F(1, 134) = 0.26$, $p = .61$; Cognitive Competence: $F(1, 134) = 1.99$, $p = .16$; Value: $F(1, 134) = 0.89$, $p = .35$) indicating that attitudes improved regardless gender differences in the attitude degrees. In Figure 1 the descriptives by gender are reported for each attitude dimension.

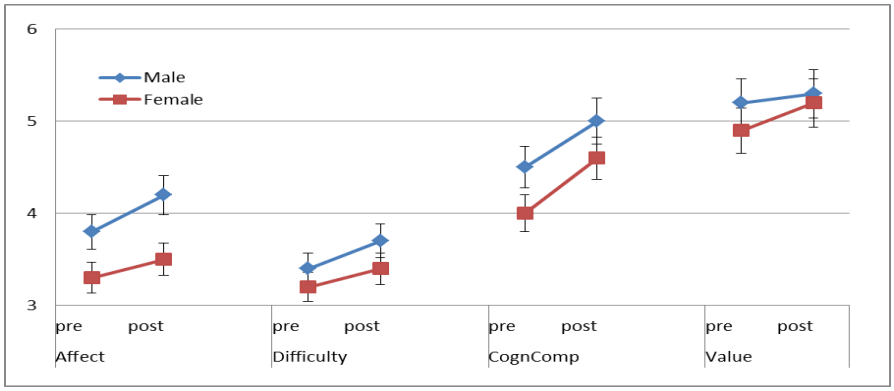

Figure 1. Mean scores of the four components of the *Survey of Attitudes toward Statistics* (SATS) at the beginning and at the middle of the course in male and female students.

Because the rating scale ranged from 1 to 7 and 4 is the midpoint, mean values revealed that male students were around the midpoint at the beginning of the course and later tended to be above it. Female students, whereas they get better across time, remained below it. On average, even taking into

account the positive shift from the beginning to the middle of the course, both men and women were below the midpoint for Difficulty, whereas scores over it were observed for Cognitive Competence and Value.

To look at the individual differences, i.e. if students get better, worse, or unchanged attitudes, we referred to Schau and Emmioglu (2012) and Millar and White (2014) to weigh the relevance of the change. Thus, we considered differences of about .5 point or more in absolute value as important. This means that students' scores would change consistently if they changed, for example, their Likert scale responses by 1 point on half of the items in the component. In the current study, to take into account the direction of change, we classified the score as follows: a negative difference of .5 point or less indicated a substantive decrease, a positive difference of .5 point or more indicated a substantive increase, all the other values indicated no substantive changes To take into account possible gender-related difference in attitudes, we observed the kind of pre-/post-test differences separately in male and female students (Figure 2).

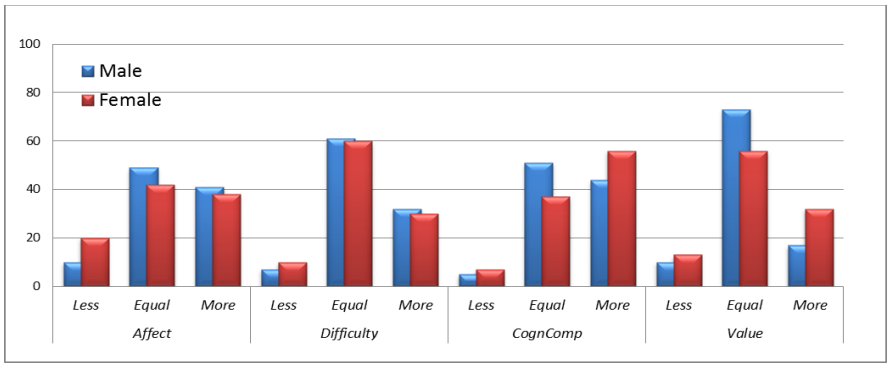

Figure 2. Percentages of negative, stable and positive pre-post difference scores of the four components of the *Survey of Attitudes toward Statistics* (SATS) in male and female students.

Chi-square tests indicated no significant differences between genders (Affect: $\chi^2(2) = 2.16, p = .34$; Difficulty: $\chi^2(2) = .34$, $p = .84$; Cognitive Competence: $\chi^2(2) = 2.48$, $p = .30$; Value: $\chi^2(2) = 3.83$, $p = .15$). Comparing the four components, the highest percentage of negative shifts (more than 15%) was found for the Affect component. A prevalence of stable scores (60% or more) was observed for the Difficulty and Value dimensions. Finally, we registered the highest percentage of positive shifts (about 50%) for the Cognitive Competence component.

To establish the relative impact of mathematical and probabilistic competences, test anxiety, and statistics anxiety on attitude changes, regression analyses were run (Table 1). In order to capture the variability in the changes occurred from the first to the second assessment, the criterion variable was the difference between the pre- and post-test scores. Given the overall absence of gender differences these analyses were conducted on the total sample. Results showed that none of these factors explained changes in Cognitive Competence ($F(4,130) = 0.71, p = .59$) and Value ($F(4,130) = 0.76$, $p = .55$). On the contrary, the regression models indicated that probabilistic competences and statistics anxiety contributed in explaining changes in Affect ($F(4,130) = 4.90$, $p < .01$, $R^2 = .13$) and Difficulty ($F(4,130) = 5.02$, $p < .01$, $R^2 = .14$). Specifically, higher competences were associated with higher

positive changes, whereas higher anxiety levels were associated with higher negative changes. Finally, test anxiety predicted changes in Difficulty in the same direction observed for statistics anxiety, i.e., the greater the degree of anxiety, the less the attitude increase (Table 1).

Criterion	Predictors	*β*	*t*	*p*
Affect change	*Mathematical competence (MMP)*	.045	0.41	.66
	Probabilistic competence (PRQ)	.29	2.87	.005
	Test anxiety (TAI)	.04	0.45	.65
	Statistics anxiety (SAS)	-.24	-2.33	.02
Difficulty change	*Mathematical competence (MMP)*	.15	1.46	.15
	Probabilistic competence (PRQ)	.25	2.47	.015
	Test anxiety (TAI)	-.20	-2.02	.045
	Statistics anxiety (SAS)	-.36	-3.60	<.001

Table 1. Regression analyses on statistics attitude changes (in brackets the scales employed to measure the predictor variables)

Discussion

The current study aimed at investigating in detail attitude changes in male and female Italian psychology students attending introductory statistics courses. In doing that, we took into account some cognitive and non-cognitive variables that might help in shed light on individual differences in attitude changes. Overall, we observed changes in attitude that resulted in a more positive attitude from the beginning to the middle of the course. Except for the Value component, men held more positive attitudes. However, there were not significant course by gender interactions indicating that attitudes improved in both male and female students. On average, and regardless the positive shift from the beginning to the middle of the course, both men and women believed that statistics was difficult, although they had confidence they would be able to learn it. Finally, all students valued statistics somewhat positively.

To provide a more fine-grained investigation we looked at the individual differences in changes as well as to the relevance of the change. Indeed, along with a general positive trend, it is possible to highlight that, although part of the students shifted markedly to better attitudes, some of them got significantly worse attitudes, and many of them did not substantially change their initial ones. Investigating gender-related difference in attitude changes, we observed no significant differences in male and female students. Thus, looking at the general patterns of change, it emerged that about half of the sample remained substantially stable across the four attitude dimensions (with the higher percentage for the Value component), more than one third of the sample shifted to a more positive attitude (with the higher percentage for the Cognitive Competence component), and a small percentage showed a negative shift (with the higher percentage for the Affect component).

When looking at the factors influencing the direction of the shift, we observed that probabilistic competences along with statistics anxiety accounted for changes in Affect and Difficulty components. That is, students with stronger competences were more likely to move to positive feelings about the

discipline and to consider it less hard. At the same time, more anxious students were more resistant to positive changes, i.e., they persistently dislike statistics and consider it hard.

Given these findings, it is interesting to note that the course *per se* promote positive changes in the students' attitudes. That is, arguably when interacting directly with the topics at an introductory level, some students tend to perceive it in a more favorable way. However, many students do not change or even get worst attitudes. Thus, it becomes important to identify methods for promoting better attitudes, for example arranging activities in which students could reinforce their basic competence in probability and providing them the adequate learning strategies to cope with anxiety. As such, they can perceive the subject easier and reduce negative feelings toward the discipline.

The present study has some limitations that we have to take into account when interpreting the results. First of all, it was conducted with Italian psychology students and this may limit their generalizability. Thus, future investigations should be conducted with different student populations to provide further evidence on the changes in attitudes and their determinants. Second, individual differences in changes of the value and cognitive competence components remain basically unexplained. As such, other factors (i.e., self-efficacy, motivation) should be taken into account to understand why some people do not change while others do. Finally, students valued statistics somewhat positively contradicting to some extent previous results on psychology students (e.g., Dempster & McCorry, 2009). This might be a result of social desirability effect that should be controlled in further studies.

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Informal statistical inference and pre-service primary school teachers: The development of their content knowledge and pedagogical content knowledge during a teacher college intervention

Arjen de Vetten[1], Judith Schoonenboom[2], Ronald Keijzer[3] and Bert van Oers[4]

[1]Vrije Universiteit Amsterdam, Amsterdam, The Netherlands & Hogeschool iPabo, Amsterdam, The Netherlands; a.j.de.vetten@vu.nl

[2]University of Vienna, Vienna, Austria; judith.schoonenboom@univie.ac.at

[3]Utrecht University, Utrecht, The Netherlands & Hogeschool iPabo, Amsterdam, The Netherlands; r.keijzer@uu.nl

[4]Vrije Universiteit Amsterdam, Amsterdam, The Netherlands; bert.van.oers@vu.nl

Teachers who engage primary school students in informal statistical inference (ISI) need to have a good content knowledge (CK) and pedagogical content knowledge (PCK) of ISI themselves. However, little is known of how teacher college education for pre-service teachers can contribute to the development of their ISI CK and PCK – a shortcoming we attempt to address in this paper. A class of 21 pre-service primary school teachers participated in an intervention consisting of five lessons. Design research methodology guided the design of the intervention. The preliminary results indicate that most pre-service teachers seemed to be aware of the inferential nature of research questions and the uncertainty inherent to ISI, but not all of them understood the fundamental idea that if a properly selected sample is representative for the population it can be used for an inference. Lacking this understanding might hinder them in teaching ISI.

Keywords: Informal statistical inference, informal inferential reasoning, primary education, statistics education, teacher education.

Introduction

In daily life, sample data is regularly used to make generalizations that go beyond the data collected; informal statistical inference (ISI) is a form of this phenomenon. The ability to critically evaluate such generalizations is increasingly useful in participation in present and future society. ISI is defined as "a generalized conclusion expressed with uncertainty and evidenced by, yet extending beyond, available data" (Ben-Zvi, Bakker, & Makar, 2015, p. 293). Unlike formal statistical inference, ISI does not make use of formal statistical tests that are based on probability theory (Harradine, Batanero, & Rossman, 2011). In recent years, statistics education researchers have turned their attention to how primary school students can be introduced to ISI. It is hypothesized that if students are familiarized with the concept in primary school, it will help them to understand the processes involved in its reasoning and in statistical reasoning in general (Bakker & Derry, 2011; Makar, Bakker, & Ben-Zvi, 2011). Evidence suggests that meaningful learning environments can render ISI accessible for primary school students (Ben-Zvi et al., 2015; Meletiou-Mavrotheris & Paparistodemou, 2015).

If students are to be introduced to ISI in primary school, future teachers need to be well prepared to conduct this introduction (Batanero & Díaz, 2010). This requires them to have an appropriate content knowledge (CK) of the subject and to have adequate pedagogical knowledge (PCK) (Burgess, 2009). Recent research shows that pre-service teachers have difficulty making generalizations and

understanding sampling representativeness and the logic of sampling (De Vetten, Schoonenboom, Keijzer, & Van Oers, 2016a; De Vetten, Schoonenboom, Keijzer, & Van Oers, 2016b). However, little is known how we can prepare pre-service teachers to teach ISI to primary school students. This paper reports on the design and preliminary results of an intervention at a teacher college for primary education that aimed to foster the development of the ISI content knowledge (CK) and PCK of pre-service primary school teachers.

Teachers' knowledge of ISI

In his seminal article, Shulman (1986) uses two broad categories to categorize the knowledge teachers need to have to teach a particular subject: pedagogical content knowledge (PCK) and content knowledge (CK) of the subject. Shulman's first category, pedagogical content knowledge, includes knowledge of how to present, illustrate and explain new material (Knowledge of Content and Teaching; KCT for short); and knowledge about the students' conceptions and misconceptions (Knowledge of Content and Students; KCS for short). Concerning content knowledge, teachers need to understand the subject at the level of student (Common Content Knowledge; CCK for short), but also need to have specialized knowledge of the subject that enables them to teach the subject and that is specific for the job of teaching (Specialized Content Knowledge; SCK for short) (Ball, Thames, and Phelps, 2008).

For ISI no categorization of CK and PCK is available. We combine the categorization of Shulman with the ISI framework of Makar & Rubin (2009) to conceptualize the CK and PCK needed for teaching ISI. Makar & Rubin (2009) argue that an ISI consists of the following three components:

1. 'Data as evidence': The inference is based on the data, and not on tradition, personal beliefs or experience.
2. 'Generalization beyond the data' (in short: 'Generalization'): The inference goes beyond a description of the sample data to make a claim about a situation beyond the sample data.
3. 'Uncertainty in inferences' (in short: 'Uncertainty'): The inference includes a discussion of the sample characteristics, such as sample size and sampling method, and a discourse on what these characteristics imply for the representativeness of the sample and the certainty of the inference. Moreover, it requires an understanding of basic logic sampling: the understanding that if a properly selected sample is representative for the population it can be used for an inference, because sample-to-sample variability is low. We have subdivided this component into three subcomponents: sampling method, sample size and uncertainty.

There are a limited number of studies that investigate (pre-service) primary school teachers' CK of ISI. The authors of this paper have investigated all three ISI components in two studies (De Vetten et al., 2016a; De Vetten et al., 2016b). In an exploratory design study, De Vetten et al. (2016a) show that regarding 'Data as evidence' most pre-service teachers indeed use data as evidence when comparing two samples to generalize to the population. In a large scale questionnaire study, De Vetten et al. (2016b) found less positive results when pre-service teachers were asked to evaluate whether data can be used as reliable evidence for a generalization. Concerning 'Generalization', De Vetten et al. (2016b) show that pre-service teachers are well able to discern that probabilistic generalizations are permissible, while deterministic generalizations are not. However, De Vetten et al. (2016a) report that pre-service teachers tend to only describe the samples, and do not generalize beyond the data.

The evidence on the 'Uncertainty' component suggests that many pre-service teachers show a limited understanding of sampling methods, sample size, representativeness and the logic of sampling and sampling variability (De Vetten et al, 2016a; De Vetten et al., 2016b; Meletiou-Mavrotheris et al., 2014; Mooney, Duni, VanMeenen, & Langrall, 2014; Watson, 2001).

Research on teachers' PCK of ISI and ways to prepare to teach ISI is even scarcer than research on teachers' CK of ISI. Leavy (2010) reports that pre-service teachers tended to focus excessively on procedures, spent too much time on descriptive analyses at the expense of discussion of inferences, and failed to stimulate data-based reasoning. Using the same data, Leavy (2015) shows that it is critical that pre-service teachers learn how to pose questions that invite students to reason about inference. Madden (2011) shows that tasks that are statistically, contextually and/or technologically provocative triggered high school mathematics teachers to reason about ISI.

Since little is known how to prepare pre-service primary school teachers to teach ISI, the aim of the present study is to investigate in what way teacher college education for pre-service teachers can contribute to the development of their ISI CK and PCK. The research question is: To what extent, and how, do the ISI CK and PCK of pre-service school teachers develop during, and as a result of, an intervention at primary education teacher college aiming at developing ISI CK and PCK?

Method

Context

In many countries, including the Netherlands, current statistics education curricula in primary and secondary education do not include ISI. Actual teaching practices focus primarily on statistical procedures and graphing skills, where concepts are learned without reference to the need to collect and analyze data (Ben-Zvi & Sharett-Amir, 2005; Friel, Curcio, & Bright, 2001; Meijerink, 2009). When statistical inference does form part of the secondary education curriculum, the ideas of sample and population are often only dealt with on a technical level. Consequently, many students enter tertiary education with a shallow and isolated understanding of the concepts underlying statistical inference (Chance, DelMas, & Garfield, 2004). In contrast to many other countries, where students can only opt for teacher education after completion of a bachelor's degree, in the Netherlands, initial teacher education starts immediately after secondary school and leads to the attainment of such a degree. For these students, mathematics teaching is usually not their main motive for becoming teachers.

The intervention was part of a course on mathematics education for grade 3 to 6. The course was the fourth course on mathematics education and the second for mathematics in grade 3 to 6. During the semester the pre-service teachers worked in a grade 3 to 6 class in a work placement school. Since in the Dutch mathematics curriculum for teacher college statistics gets only minor attention and since we wanted to have an intervention that would fit in the normal teacher college curriculum, we decided to restrict the length of the intervention to five lessons, out of the 16 lessons of the total course.

Design

We employed design research methodology to study the development in ISI CK and PCK of the pre-service teachers and to explain this development (Van den Akker, Gravemeijer, McKenney & Nieveen, 2006). Previous research (Ben-Zvi, 2006; De Vetten et al., 2016a&b; Paparistodemou &

Meletiou-Mavrotheris, 2008; Saldanha & Thompson, 2007) and our own ISI teaching experiences with primary school students informed us which learning goals are within reach of pre-service teachers and what PCK is necessary to teach ISI to primary school students (see Table 1). These learning goals were categorized into the three ISI components 'Data as evidence', 'Generalization' and 'Uncertainty'.

Table 1: ISI learning goals for the intervention at primary education teacher college

ISI component		Knowledge type[a]	Learning goals	Attained?[b]
General		KCT	To reason with students about ISI in a meaningful way, teachers can (1) have students conduct empirical investigations with an inferential research question about a meaningful topic or (2) have students evaluate research (for example as reported in the media) with an inferential research question about a meaningful topic.	0.75
Data as evidence		CCK	Use data as evidence, not other sources	1
		CCK	Sample provides information about likelihood of population parameters	0.5
		KCS	Many students do not use data as evidence	0.5
		KCS	Many students think that every sample distribution is evenly likely	0.25
		KCT	To teach students that data can be used as evidence for answers on inferential questions teachers can (1) have students conduct empirical investigations where sample and population are concretely visible, (2) regularly point at the sample, and (3) ask students on what arguments they or other researchers reached their conclusion.	0.25
Generalization beyond the data		CCK	It is possible to make claims about population, despite individual differences between subjects	0.75
		CCK	It is possible to use sample to make claims about population	1
		CCK	Awareness of inferential nature of research questions	1
		CCK	Claims about a population are often based on a sample	1
		CCK	Correctly articulate answers to inferential questions	0.5
		KCS	Many students answer inferential questions descriptively only	0.5
		KCT	To teach students that it is not necessary to investigate an entire population, but that a sample suffices for a reliable conclusion, teachers can (1) use examples from media where a sample is used, or (2) use resampling activities where different samples yield similar results.	0.25
		KCT	To make students aware of the inferential nature of research questions, teachers can (1) use real empirical investigations where sample and population are concretely visible, and (2) ask questions, such as 'Does our result hold for the sample only, or also for the population?'	0.25
		KCT	To help students to correctly articulate answers to inferential questions, teachers can reformulate students' responses.	0.25
Uncertainty inherent to inferences	*Sampling method*	CCK	Which of the following sampling methods are (in-)appropriate: convenience sampling, random sampling, quota sampling	0.75
		SCK	Understand why random sampling is appropriate	0.75
		KCS	Many students think random sampling is not an appropriate sampling method	0.5
		KCS	Many students tend to use incorrect matching techniques in sampling	0
		KCT	To teach students that a sample needs to be representativeness of the population, a teacher can (1) ask students to investigate how other researchers have selected their sample and what students think of the representativeness of the sample and (2) let students discuss how they would select a representative sample to answer a specific research question and let the children execute the sampling.	0.75
	Sample size	SCK	Why a larger sample leads to more certainty	0.5
		CCK	Sufficient sample size	0.25
		CCK	Sample size is independent of population size	0.5
		KCS	Many students do not take sample size into account in the certainty of their answers	0.5
		KCS	Many students make very certain inferences, even for small samples	0.5
		KCS	Many students think that sample size is dependent of population size	0.25

		KCT	To teach students the effect of sample size on the certainty of inferences, teachers can use real empirical investigations where resampling is used, select to two samples of different size and ask which sample provides more certainty.	0.5
	Uncertainty	CCK	Acknowledge uncertainty of inferences and impossibility of absolute certain inferences	1
		CCK	The larger the sample, the more certain the inference	1
		CCK	The better the sampling method, the more certain the inference	0.75
		CCK	When a sample is properly selected, the probability is small that another but likewise sample gives an entirely different result	0.5
		CCK	Correctly articulate uncertainty in inferences	0.5
		KCS	Many students express complete (un-)certainty in their inferences	0.5
		KCT	To make students aware of the uncertainty of inferences, teachers can use real empirical investigations where resampling is used, and confront the children that different samples lead to different conclusions.	0.25
		KCT	To help students articulate uncertainty in inferences, teachers can reformulate students´ responses or ask how much certainty the students have.	0.25

[a]CCK: common content knowledge; SCK: specialized content knowledge; KCS: knowledge of content and students; KCT: knowledge of content and teaching. [b]Each learning goal received one of the scores 0, ¼, ½, ¾ or 1 (0: not attainted; 1: attained by (almost) all pre-service teachers)

For each lesson, a hypothetical learning trajectory was designed, connecting activities with the learning goals for the lesson, while also explaining in what way the activities were hypothesized to help to attain the learning goals. Using example lessons was one type of activity used to foster the CK and PCK of the pre-service teachers. These lessons introduced many CK concepts and provided opportunities to discuss how children would deal with issues involved in these lessons. Each lesson was evaluated to inform the design of the next lesson. Table 2 provides an overview of the intervention. Part of the intervention was that pre-service teachers would give an ISI lesson in their work placement school. The analysis of these lessons is beyond the scope of this paper.

Table 2: Overview of the ISI intervention at the teacher college

Week	**Activity**
1	Informed consent, pre-test and homework instruction
Between 1 and 3	CK: Pre-service teachers make homework assignment: 1. Look up an article in the media that makes a claim about a population based on a sample and that somehow appeals to you. 2. Describe how the researchers came to their conclusions. 3. Write a critical evaluation about the quality of the research: to what extent is in your opinion the conclusion justified based on the research conducted?
3	**Lesson 1** CK & PCK: Small group and whole class discussion of homework assignment, attention for both CK and PCK aspects of the assignment CK: Explanation of random sampling and appropriate sample size using an ICT demonstration
5	**Lesson 2** CK & PCK: Teacher educator models an example lesson which the pre-service teachers could give themselves in their work placement schools PCK: Discussion of teacher educator's experiences with teaching the example lesson in primary school
5	**Lesson 3** CK & KCS: Discussion of equiprobability bias using a task CK Short recap of main ISI concepts PCK: Teacher educator presents learning goals of ISI lesson (KCT), discusses typical responses of students (KCS) and provides instructions for the ISI lesson (KCT)
Between 5 and 12	Half of the pre-service teachers teach an ISI lesson in their placement school.
12	**Lesson 4** PCK: Discussion of pre-service teachers' experiences with teaching with teaching the ISI lesson PCK: Pre-service teachers provide suggestions for alterations of the ISI lesson and tips for their fellow students.
Between 12 and 16	The other half of the pre-service teachers teach an ISI lesson in their placement school.
16	**Lesson 5** PCK: Discussion of pre-service teachers' experiences with teaching the ISI lesson
16	Post-test and evaluation of the intervention

Participants

One class of second year pre-service teachers participated in this study. They studied at a small teacher college for primary education in a large city in the Netherlands. The intervention took place in their second year of study, because statistics is part of the knowledge base that is tested in the third of study. This particular class was chosen, because the pre-service teachers had fewest credits open from their first year of all three second year classes. The class consisted of 23 pre-service teachers. They were asked to provide their informed consent. While all of them were required to participate in the activities and lessons, one pre-service teacher invoked the possibility to have her results excluded from the analysis. The results of another student were also excluded, because of absence during all but one of the lessons. This resulted in a sample of 21 pre-service teachers. The procedure was approved by the ethical board of the Faculty of Behavioural and Movement Sciences of Vrije Universiteit Amsterdam. The average age of the participants was 21 years (SD: 2.19); 6 were male; 10 had a background in secondary vocational education (students attending this type of course are typically aged between 16 and 20), 7 came from senior general secondary education, 2 had been enrolled in university preparatory education, and the educational background of the remaining 2 was either something else entirely or unknown. Their average score on the obligatory first-year mathematics exam for Dutch pre-service teachers was 135 out of 200 possible points (SD: 12.86). A score of 103 equals the 80th percentile of Grade 6 primary school students in the Netherlands. The first author was the teacher educator. He had four years of experience as a mathematics teacher educator, a master degree in economics and experience as a university statistics lecturer. He had taught most of the pre-service teachers during their first year of study.

Data collection

During the lessons, whole class interactions were recorded on video and audio, while small group interactions were recorded on audio. Furthermore, during most lessons one of the co-authors was present as observer, taking notes. Finally, all written work was collected. A pre-test and post-test were used to measure ISI CK and PCK at the start and at the end of the intervention. The results of these tests will be presented during the CERME presentation.

Data analysis

The goal of the analysis is to study the development of the ISI CK and PCK and to explain these developments. This paper reports preliminary analyses. After each lesson, the teacher educator's and observer's notes and reflections were used as for estimating to what extent the learning goals relevant for the particular lesson had been attained by the pre-service teachers as a whole. Each learning goal received a score ranging from 0 to 1 (0: not attainted; 1: by large attained by (almost) all pre-service teachers). These estimations per lesson were used to make an overall score for each learning goal. Next, average scores were calculated for the various components and types of knowledge. Based on these notes and reflections, these scores were related to the activities used during the intervention. The final results will show in detail the development of ISI CK and PCK at the level of the pre-service teacher and the role of the activities used in the intervention.

Table 3: Attainment of the learning goals, summarized by ISI component and knowledge type

Component	Average score[a]		Knowledge type	Average score[a]
General	0.75		Common content knowledge	0.73
Data as evidence	0.5		Specialized content knowledge	0.63
Generalization beyond the data	0.61		Knowledge of content and students	0.39
Sampling method	0.55		Knowledge of content and teaching	0.43
Sample size	0.43			
Uncertainty	0.59			

[a]Each goal received one of the scores 0, ¼, ½, ¾ or 1 (0: not attainted; 1: by large attained by (almost) all pre-service teachers)

Preliminary results

The preliminary results reveal to what extent the learning goals have been attained and shed some light on what activities helped to reach these goals. Table 1 shows to what extent the learning goals are attainted. Table 3 shows the average score for the learning goals summarized for components and knowledge type respectively. There are some notable results. First, most pre-service teachers seemed to be aware of the inferential nature of research questions and the uncertainty inherent to the results. The homework assignment at the start of the intervention seemed to have helped to foster this awareness. Second, the modeling of the example lesson seemed to be a good context to reason about sampling methods, because it naturally led to the use and discussion of random and quota sampling methods. Moreover, although average scores on PCK learning goals are lower than on CK learning goals, modeling helped the pre-service teachers to get a better idea how to teach ISI to primary school students. Thirdly, while the presentation of PCK issues by the teacher educator in lesson 8 did not seem to contribute much to the pre-service teachers understanding, the discussion of PCK based on the pre-service teachers' experiences in lesson 12 and 15 did. Finally, an understanding of the fundamental idea that if a properly selected sample is representative for the population it can be used for an inference, might be conditional for understanding other ISI concepts. While part of the pre-service teachers seemed to understanding this idea, part of them did not. Lacking this understanding might hinder them in teaching ISI.

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Chilean primary-school students' levels in reading pictograms

Danilo Díaz-Levicoy, Pedro Arteaga, and Carmen Batanero

University of Granada, Spain; parteaga@ugr.es

In this paper we analyse the competence in reading pictograms by 140 Chilean students (6th and 7th grades). The written responses to two activities taken from textbooks are used to describe the children's reading levels and strategies to interpret pictograms and translate them to a table. Results suggest that students do not find major difficulties in translating the pictogram to a table; however, few of them reach the upper reading level needed for a critical interpretation of the information displayed in the pictogram.

Keywords: Pictograms, understanding, primary education.

Introduction

A relevant part of the information we face every day is given in statistical graphs, whose interpretation is often needed to make different decisions; therefore there is a need for citizens to achieve enough graphical competence (Ridgway, Nicholson & McCusker, 2008). These reasons led countries like Chile to introduce statistical graphs in the primary education (Díaz-Levicoy, Batanero, Arteaga, & López-Martin, 2015). More specifically, in Chile children are requested to collect and record data to answer statistical questions about themselves and their environment, using bar charts, tables and pictograms, as well as to read and interpret these representations since the 1st grade (MINEDUC, 2012). There is however no available empirical studies providing evidence that Chilean children understand these graphs at the end of primary education. The aim of this paper is to analyse the reading level of the pictograms, reached by Chilean children in the last year of primary school (6th grade) and what they remember a year later (7th grade). Since statistical graphs are included in primary education in many countries, this information may be useful in other contexts. Below we describe the foundations, method and results.

Pictograms are statistical graphs that display information using icons, whose size is proportional to the frequency of each attribute. Each icon represent a fix value, and can be repeated, to achieve the attribute frequency; its iconic character has been considered relevant in conveying recommended modes of behavior (Tijus, Barcenilla, De Lavalette, & Meunier, 2007), such as helping sick children understand a treatment that they are undergoing (Hämeen-Anttila, Kemppainen, Enlund, Patricia, & Marja, 2004).

Understanding statistical graphs involve composing individual data values into an aggregate or distribution and perceiving this distribution as a whole; however, some students only perceive graphs as "collections of values" instead (Konold, Higgins, Russell, & Khalil, 2015). In our research we are interested in the children competence in interpreting a distribution represented in a pictogram and in the reading level shown in the children responses when reading pictograms. We use Curcio's (1989) categorization.

- *Level 1. Reading the data*. At this level the student can successfully perform a literal reading of the information presented in the graph, but does not succeed in more sophisticated reading, do not

provide an interpretation, or carry out additional calculations with the information displayed in the graph.

- *Level 2. Reading within the data*. In addition to making a literal reading, the student can obtain information that is not explicitly displayed in the graph, with simple mathematical processes, such as arithmetic operations or comparisons.
- *Level 3. Reading beyond the data*. The student is able to extrapolate or interpolate the data to predict values that are not shown in the graph. He or she is also able to make a critical reading to detect an incorrect interpretation of a graph.

We only found a few investigations related to our study. Cruz (2013) analysed the interpretation of pictograms by 21 children in 3rd year of Primary Education in Lisbon after a teaching process. The responses to a written questionnaire with different types of graphs were analysed: 82% of children properly completed those reading activities that required Level 1 in Curcio's (1989) classification, while 70.5% reached level 2 and 66.5% level 3. One of the activities consisted in reading a pictogram where each icon represents a unit. In his item the author obtained 95% of correct answers to Level 1 questions and 77.3% to the Level 2 questions. There were no questions of Level 3. Regarding other types of graphs, Evangelista (2013) proposed single and double bar graphs and line charts to a sample of 60 students in grade 5th in Brazil. The results show that children correctly answer 51% of the activities; on average, the students correctly answered 59% of the activities in bar graphs and 43% of line charts. Level 1 questions had an achievement of 60% and level 2 between 51% and 41%.

Method

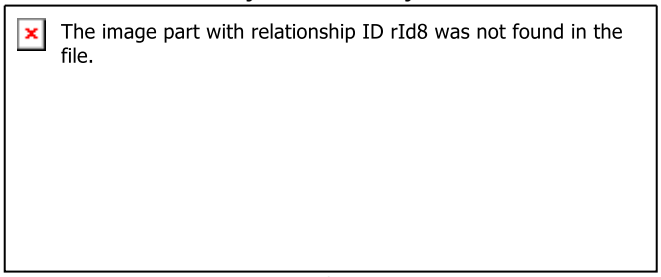
The sample consisted in 140 Chilean Primary Education students from 6th grade (69 students, 11-12 years old) and 7th grade (71 students, 12-13 years old). Two different schools in the city of Osorno took part with collaboration from the schools' principals and of the teachers responsible of these groups, to all of whom we sincerely thank. Although we used a convenience sample, the socio-economic setting and average mathematical ability of children is varied and represent the situation in Chile.

The questionnaire (Figure 1) included two items taken from Chilean mathematics primary education textbooks. In the pictograms included in the questionnaire each symbol represents a uniform and defined value; therefore, a priori, should be simple for children. In the first adapted from a 3th grade Primary Education textbook (Charles et al., 2014, p. 253) item the student should read the pictogram, where each icon represents 15 statistical units (books). The student has to read two sentences; the first one is false (since there are 30 science fiction books) and the second is true (there are 60 children books). To complete the task the student must recognize the row for each value of the variable *book type* and understand that its frequency is given by the number of icons multiplied 15 (the frequency represented by each icon). Therefore, the student first has to *read within the data* (Level 2 according to Curcio's, 1989 classification), as he or she has to perform calculations with the values in the graph. In addition, the student should reason that each statement is true or false; therefore the student has to make a critical reading of the graph and consequently work at Level 3 (reading beyond the data in Curcio's, 1989 classification).

In the second item, adapted from a 4th grade primary education textbook (Batarce, Cáceres & Kükenshöner, 2013, p. 343), the student has to translate information from a pictogram to a table.

Besides reading the number of icons corresponding to each value of the variable, the student has to perform calculations; in this case, two types of icons are used to represent either 10 or 5 hours of light. The student has to reach level 2, *reading within the data* and complete the table, calculating the total amount of hours the light was turned on.

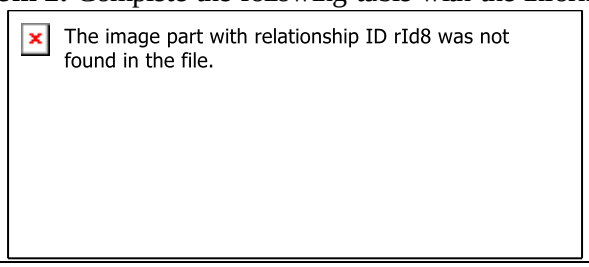
Item 1. The school librarian made an inventory of the library books.



Decide if each of the following statements is true or false. Explain your answer.

	True	False
1. There are only two science fiction books.		
2. There are 60 children books.		

Item 2. Complete the following table with the information displayed in the graph.



Number of hours per week that the light is turned on	
Place	**Number of hours**
Exercise room	
Dressing room	
Swimming pool	
Tennis court	
Total	

Figure 1: Questionnaire

Results in item 1

In item 1 we first analyse the percentage of correct responses and then the reading level that the students of our sample reach when providing a justification of their response.

Percentage of correct answers

In Table 1 we present the percentage of correct answers in the questions regarding the truth or falseness of the two statements given in this item. We observe that less than 50% of the students provided a correct response to these claims. The results are apparently worse than those obtained by Cruz (2013) in reading pictograms, but this author only made questions that involved reading within the data (Level 2) and each icon in his pictograms represented only one unit, while in our item each icon represents 15 units; therefore, our items are comparatively more difficult.

There was a higher level of success in the 6th graders than in the 7th graders, which may be due to the effect of forgetfulness, since pictograms are studied in Chile with more intensity in the first four years of primary education, and in 6th grade some activities related with pictograms are proposed in textbooks (Díaz-Levicoy et al., 2015). On the contrary, pictograms are rarely used in 7th grade. Anyway, the differences were not statistically significant difference in the t-test of difference of proportions (that test the hypothesis of having equal proportion of correct responses in both groups).

Statement	Total (n=140)	6th grade (n=69)	7th grade (n=71)	p-value*
1.It is not true that there are only two science fiction books	47.9	55.1	40.8	0.0904
2. There are 60 childhood books	49.3	55.1	43.7	0.1774
* Test of difference of proportions in independent samples				

Table 1: Percentage of correct answers according to statements of item 1

Reading level

We secondly analyse the reading level that the students achieved to decide the truth and falseness of the statements and to justify their response. Their arguments were classified according to Curcio's (1989) reading levels, which are interpreted as follows:

- *Level 0* is reached if the information requested in the question is not read or the graph reading is incorrect (students do not even read correctly the number of icons).
- The students' justifications are classified in *Level 1* if they simply read the number of icons for the variable values indicated in the question without performing any calculations. The student has identified the graph row corresponding to the variable value, and has counted the number of corresponding icons. However, he or he does not take into account that each icon represents 15 books, and does not perform the necessary calculations to determine the frequency corresponding to each category. Some examples are:

  It is true, because in the inventory two science fictions books appear (Student 55, first question)
  It is false, there ae four children books (Student 73, second question)
- *Level 2* is reached if the student correctly answers the question and apparently has carried out the calculations required to determine the frequency of a category in multiplying the number of icons by 15. In this case, the student is able to correctly interpret the pictogram but do not sufficiently argues the truth or falseness of the claim posed. We have also considered within Level 2 those responses in which students perform an incomplete argument, i.e. they do not explicit the arithmetic operations performed. For example:

  It is false; there are 30 science fiction books (Student 21, first question)
  It is true, one book is 15 and there are 4 books (Student 66, second question)
- We consider that a student response reaches *Level 3* if the student has performed the calculations required to determine the frequency of the category and interprets correctly the pictogram. In addition, the student reaches a critical reading, because he or she can give an argument that supports the correct or incorrect statement.

  It is false, because as you can see each icon represents 15 books and there are two icons for science fiction books; for this reason if we sum 15+15 the results is 30 books (Student 38, first question).

In Table 2 we present the distribution of the reading levels achieved by the whole sample, as inferred from the arguments that children provide to express their agreement or disagreement with both statements.

Statement	Reading Level			
	0	1	2	3
1. There are only two science fiction books	1.6	37.3	55.4	5.6
2. There are 60 childhood books	2.3	33.4	53.2	11.1

Table 2: Percentage of reading levels achieved by students in their responses to item 1

Overall the most common reading level in both questions was Level 2, which involves comparisons and data operations. The second most frequent was Level 1, where students read literally the information displayed, and very few students reached Level 3, which involves critical reading (a few more in the second statement).

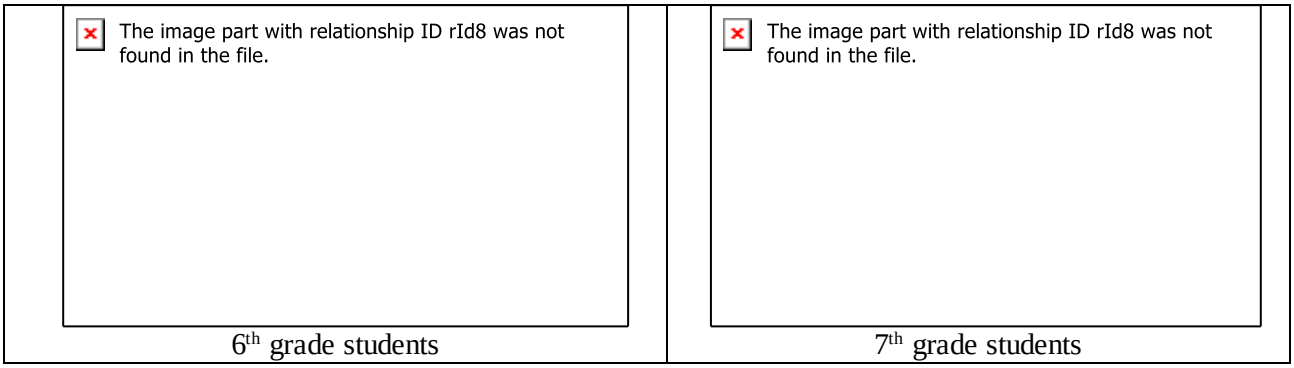


Figure 2: Percentage of Reading levels in both parts of the item1 in each group

When analysing the results by grade and statement (symbolized by S1 and S2) (Figure 2), 7th grade students more frequently provided Level 1 answers in both statements (63.4% in the first statement and 59, 2% in the second) that the 6th grade students (50.7% in the first statement and 43.5% in the second). These, also had higher percentage at Level 2 (42% in the first statement and 40.6% in the second), while the 7th grade student's percentages at level 2 were 26.8% and 21.1%. Level 3 answers were scarce in both grades; at this level, 7th grade students got better results than 6th grade students, due to their better level of reasoning, but the difference is small. Overall the 6th grade students have better results, since they achieved a higher percentage of responses in Levels 2 and 3.

Results in item 2

In this item we first describe the results achieved in the reading level and then those related to student competence to translate the graph to a table.

Percentage of correct answers

In this item, students should make a translation of the pictogram to a table. The answers given by students are classified according to the following categories:

- *Correct table*. When the student has successfully translated all data on the pictogram to the table. He or she has also correctly calculated the total of the table.

- *Partially correct table*. The student makes a partially correct translation of the information shown in the pictogram; the table is correct, with some mistake. These errors are: a) taking into account one icon more or less when calculating frequency (12 students); b) wrong calculation of total hours (7 students); c) considering that the icon that represents half bulb is equivalent to 15 hours of consumption, instead of five (1 student); d) considering that one of the icons represents one hour of consumption, while the rest has been translated well for 10 hours (1 student); e) producing some correct rows in the table, but not finishing the table (1 student); f) not calculating the total, although the table is correctly constructed (1 student); g) making two of the above errors (1 student).
- *Incorrect Table*. When all or most of the rows in the table are incorrect, what happens, in particular, to all of the students who only reached the Level 1 when reading the pictogram.
- *Do not complete the table*. When the student does not develop the activity or when students reached Level 1 partially.

Table 3 shows the distribution of the answers given by the students in translating to a table, where most students have completed the task successfully, with a percentage bigger than 76% in each course and globally. The differences were not statistically significant.

Type of answer	6th grade (n=69)	7th grade (n=71)	Total (n=140)
Correct	76,8	78,9	77,9
Partially correct	18,8	15,5	17,1
Incorrect	0	2,8	1,4
The task is not completed	4,3	2,8	3,6

Table 3: Percentage of students by table correctness

Reading level

In this item these levels are interpreted as follows:

- At *Level 0* are the students who provided no answer or who fail to read the graph.
- At *Level 1* students perform a literal reading of the data (either all or part of the data). For example, some students believe that each bulb represents a unit. An example is given in Figure 3 in which the student considers the bulb as an unit in all the categories, except one.
- At *Level 2* are those students who identify the number of icons corresponding to each variable value and multiply this number by 10 or 5 to obtain the corresponding frequency. An example is given in Figure 4.

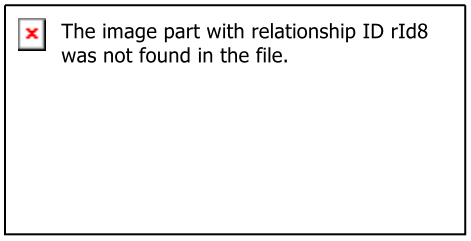


Figure 3: Level 1 table (Student 29)

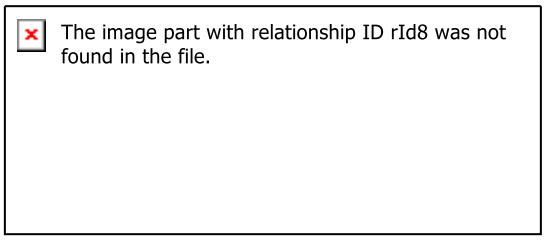


Figure 4: Level 2 table (Student 75)

In Table 4 we present the distribution of the reading levels reached by the students in this item, where the highest percentage of responses are located at Level 2 (reading the data) in both courses. Over 90% of students are able to correctly read the pictogram at level 2 and 7th grade students show better performance levels when the translation is done correctly.

Level	6th grade (n=69)	7th grade (n=71)	Total (n=140)
0	0	2.8	1.4
1	8.7	2.8	5.7
2	91.3	94.4	92.9

Table 4: Percentage of students reaching each reading level by course and in total

Discussion and conclusions

This study has provided information about the reading levels reached by the students in the sample when reading pictograms and when translating between pictogram and data table, complementing previous work. In relation to the work of Cruz (2013) we used more complex pictograms, as each icon represents several units, while in those proposed by the author each icon symbolized one unit. In relation to the work of Evangelista (2013) with lines and bar graphs, the results in levels 1 and 2 are somewhat lower than in our work but not too much.

In our first item students can reach up to Level 3 in Curcio's (1989) categorization, while Cruz only considered the first and second level. This explains why our results are apparently lower than those in Cruz. However, the second item proposed in our study has been very easy; both in reading the pictogram where our results were better, as in the translation to a table, a task that was not proposed by Cruz. Part of the errors found in the first item, where students are asked to refute or confirm a statement, are due to their lack of critical reading, and argumentations skills. As suggested by Freedman and Shah (2002), graph comprehension is influenced not only by the display characteristics of a graph, but also a viewer's domain knowledge, graphical literacy skills, and explanatory and other scientific reasoning skills that these children may lack. In the next step in our research we are expanding the sample of children and schools to confirm the results.

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Conditional probability as a decision-making tool: A didactic sequence

Raimundo Elicer[1] and Eduardo Carrasco[2]

[1]Roskilde University, Denmark; raimundo@ruc.dk

[2]Metropolitan University of Education Sciences, Santiago, Chile; ecarrasc@gmail.com

Conditional probability arises as a tool for analyzing a strategy for decision-making that molds to new conditions. From that point of view, an introductory sequence which utilizes diachronic games is designed and analyzed under the framework of didactical engineering, bringing conditional probability into play as a decision-making tool. It can be observed that students tend to base their decisions on heuristics and experiential considerations, and do not see the need for a proper calculation of theoretical probabilities. At most, they use experimentation as a tool, not computing probabilities based on relative frequencies, but comparing absolute frequencies.

Keywords: Decision-making, conditional probability, didactical engineering.

Introduction

UNESCO has dedicated a full chapter about confronting uncertainties on its "Seven complex lessons on education for the future". It describes uncertainties of reality and knowledge, and proposes ways of taking action despite the unavoidable uncertainty of the world (Morin, 1999). Ignoring or downplaying uncertainty could lead us to make fragile decisions, which can generate a negative impact as soon as the circumstances change (Taleb, 2012). In this matter, probability and statistics emerge as the core mathematical subjects for facing this challenge. Thus, these subjects should play an important role in modern mathematics curricula for general education.

However, in general students have low scores on these fields. PISA 2012 reveals that 76.9% of tested students do not pass the second level of accomplishment in the areas of uncertainty and data (Organisation for Economic Co-operation and Development [OECD], 2014). At the best they can apply suitable calculation basic procedures in familiar contexts such as coin tossing or dice rolling. However, they are not able to reason and make critical reflections in order to make valid contextual or general conclusions.

Probability and statistics have given shape to a field of economics studied since the 40s. It has been dominated essentially by "expected value theory" as a normative model of rational choice, proposing that rational individuals maximize the expected value of their utility functions (Friedman & Savage, 1948). This approach has been criticized lately by behavioral psychology and behavioral economics, pointing out many situations in which the axioms of the theory show themselves inadequate for modelling reality (Kahneman & Tversky, 2007). The authors propose a "prospect theory", taking into account individuals' biases towards the probability and impact of each choice.

The research related to this paper is embedded within a broader domain that embraces the relationship between didactics of probabilities and statistics, and decision-making under uncertainty. It involves facing the philosophical debate between the idea of probability and statistics as decision-making tools, against decision-making scenarios as resources for improving the institutionalized techniques within probability and statistics. Moreover, the research relates to critical mathematics education, which emphasizes the empowerment of students as citizens as an argument for mathematics in general education (Skovsmose, 1994).

In this frame, two research questions are addressed. (1) "What kind of probabilistic contents and related didactics could help students make better and more reflexive decisions in their lives?" and (2) "How do we assess the students' learning of probability theory and methods, and which role can decision-making scenarios play in such assessment?"

In particular, this paper reports the results of an exploratory application of didactical engineering that involves conditional probability in decision-making scenarios. The purpose is to illustrate challenges and difficulties involved in the teaching of probability under this point of view.

Theoretical framework

From an *enactivist* perspective (Brown, 2015), knowledge is only reflected by and detectible through action by those who know. One learns with an embodied mind, within a process called *enaction*. This notion breaks *representationist* ideas of the mind, considering an incarnated cognition, in which meanings arise as particular states of cooperation in neuronal networks. These states are put into action –enacted– via retroactive co-definitions between the subjects and the contexts they live in. This concept implies that, when faced upon a learning situation, students will enact the knowledge that had let them to act in similar situations, not only their school experience.

Within the scope of this research, students would enact what they have learned from situations of uncertainty and decisions they made before, their previous formal school knowledge, operational aspects such as heuristics and perceptual aspects like feelings that the context evokes in them.

According to mathematical philosophy literature (e. g. Leitgeb & Hartmann, 2014), two types of decision-making scenarios can be defined. Namely, a *situation under uncertainty* is a setting in which one does not know what the relevant probabilities are, and in decision-making *situation under risk*, the probabilities of the various outcomes are in principle. In both cases, decisions are made based on the best available information and building conjectures about likelihood of different results. We will consider the latter one in this paper. Here we will intend to use mathematical objects, such as probability theory and calculations, in the context of mathematics teaching.

Morin proposes "(…) two ways to confront the uncertainty of action. The first is full awareness of the wager involved in the decision. The second is recourse to strategy" (Morin, 1999, p. 47). Strategy must prevail over program, which gets stuck as soon as the outside conditions are modified. A strategy is meant to be adaptable to variations in the context. In this regard, as one acquires new information about the situation in which one requires to make decisions under uncertainty, the notion of conditional probability lets us incorporate changes in the degrees of belief about possible outcomes (Batanero & Díaz, 2007), improving decision-making based on predictions. As a consequence, the rank of experiments to consider in the classroom becomes wider.

The central mathematical object of study is, therefore, conditional probability. The learning goal is, as stated on the study program at 11th grade in Chile, to "solve problems that involve computation of conditional probabilities within simple situations" (Ministerio de Educación [MINEDUC], 2004). The choice of situations and means of representation (tree diagram, 2x2 tables) are left open for the teachers, but the given established notions involved are:

- Meaning of "probability of event A, given event B", using the notation P(A|B).
- If A and B are independent events, then P(A|B) = P(A).

- If A and B are not independent events, then P(A|B) = P(A and B)/P(B), with P(B) not equal to 0.

As an example, let's say that an experiment consists of tossing a fair coin twice. Let A be the event of obtaining two successive heads, and B be the event of obtaining a head in the first toss. According to previous contents, students should be able to calculate theoretically that P(B) = ½ and P(A and B) = ¼. Given the context, students should be able to interpret and calculate that P(A|B) = ½, because now that we know that B happened, then for A to happen, a second head should be obtained. Also, it may be obtained that P(A|B) = P(A and B)/P(B) = ¼ / ½ = ½.

Methodology

Didactic engineering is assumed as a research and design method, and includes four phases: preliminary analysis, *a priori* analysis, execution, and contrast and redesign (Artigue, 1988).

For the preliminary analysis, historic-epistemological elements are obtained by selected authors, in particular, Pascal and Huygens, and their significant work on the development of probability (Pascal, 1983; Basulto, Camuñez, Ortega, & Pérez, 2004). The analysis is made from a chronological construction of the concepts by the authors and their sociocultural contexts. Practices that build the necessity and give meaning to gambling and decision-making are documented. For the cognitive analysis (Elicer & Carrasco, 2014), exploratory tests are taken to gather productions of students about their probability notions and their strategies to make decisions in the Monty Hall game (Batanero, Fernandes, & Contreras, 2009). The didactic analysis is made from the 11th grade study program and textbook delivered by the Chilean government to public and subsidized schools, which represent 91.1% of the total enrolled students in 2015 (MINEDUC, 2015).

The didactic sequence is then designed taking into account key notions resulting from preliminary analysis and an increasing level of complexity. Students should start making specific calculations and end making justified decisions. For the *a priori* analysis, conjectures emerge from the authors, according to the cognitive analysis. It is fair to anticipate similar outcomes from the students and, therefore, to add questions that help them have a critical insight about them.

The execution stage consists in the application of the didactic sequence to a group of students and the contrasting between initial conjectures and the students' actions and productions. The experimental group is one upper secondary class of 19 students aged 15-17 years old. They have already been introduced to probability calculations using Laplace's law, tree diagrams and basic combinatorial techniques. Students have not yet studied the concept of conditional probability.

Finally, transcriptions of students' written outcomes are tabulated according to defined categories in the preliminary analysis (Elicer & Carrasco, 2014). Those which are unexpected and do not match these categories are highlighted and mentioned in the results. Suggestions for redesign resulting from the discussions with the teachers are mentioned for each activity.

Results and discussion

The designed sequence is fully exposed on the Appendix of this paper. It is meant to be executed as an introduction to the mathematical object of conditional probability. This means no new institutional contents would be presented, they should use their previous knowledge. The first session includes Activities 1 and 2, and the second session concludes with Activities 3 and 4.

For a full revision of relevant elements of the preliminary analysis see Elicer & Carrasco (2016). Those considered for the design are as follows, given in parentheses the questions implemented.

Historic-epistemological. In Pascal-Fermat correspondence, probability analysis arises from projective decision-making, in the effort of setting a fair share (1.1, 1.4, 2.1 and 3.4), in particular when a gambling game stops (2.5, 2.8 and 2.9). After this notion comes the idea of betting with some kind of advantage. Studying the ratio between favorable and all cases comes from getting to know every possible case. Huygens brothers association between forecasting situations, such as life expectancy, and gambling, allows us to give a new meaning to the idea of probability in a game, as an a posteriori calculated probability. From this point of view, possible outcomes of a game are described in statistical data (1.2, 1.4, 2.4, 2.7 and 3.3).

Cognitive. Students conceive that different realizations are all possible cases, without weighing them, and draw upon non-mathematical arguments to make a choice (3.1). They also recognize that they would make different decisions if they played the actual game, where they had to make a choice in situ and not a priori (4.1 and 4.2).

Didactic. The main activity proposed in the textbook is theoretical probability calculation (1.3, 2.2, 2.3, 2.6, 3.2 and 3.5), without decision-making or searching for an advantage in gambling. This is implemented on Activity 3, where the Monty Hall problem involves an actual decision.

Activity 1 goes as expected. Students unanimously recognize this is a fair game because both players have the same probability of winning (1.1), which is well calculated using the Laplace law (1.3). Usually, the distraction of doing eleven repetitions is recognized by them, saying there are too few repetitions, that it is an odd number (1.2), and that results depend on chance or luck (1.2 and 1.4). One particular comment we didn't expect was that "the game is not fair, because the results depend on chance and not on personal abilities" (1.1). Considering a future design, the meaning of fairness should not be trivialized. Might be defined or discussed.

Activity 2 throws similar responses about the basics (2.1 and 2.4), which is the intention. Some students still confuse the concept of "possibility" and "probability" (2.2) when giving their answer, which could be revised before. They do not use their previous combinatorial reasoning. Instead, they count different scenarios than come up to their minds (HTT, HHT, …), which not always lead to counting four possibilities for each player (2.2). For this reason, they might answer that each player has the same probability of winning (2.3), based on their intuition and the scheme made in question 2.2. The same schemes lead some of them to wrong answers.

For the second part of Activity 2, most students recognize that one player has an advantage after the first toss. Just a few could actually compute the theoretical probability (2.6), so most of them base their answers on interpreting experimentation results (2.7). Here the probability of success arises as an estimation of compared absolute or relative frequencies, giving use to fractions. Our observation is that an exact calculation of these probabilities does not seem to be necessary for answering the question, so the intention must be revised.

As for the repartition when the game is interrupted, (2.5, 2.8 and 2.9), there are two main types of reasoning among the students. One big group defends an equal repartition of 50% and 50%, arguing that "even when one of us has more possibilities of winning, randomness says that any of us could win", giving

randomness a mean for equality. Others recognize fair to split it according to probabilities, using fractions constructed on their ratio phase (Fandiño, 2015) as an operator to multiply the poll to be shared.

In Activity 3, most of students believe each of the remaining doors give an equal chance of winning (3.1), falling into an isolation effect (Tversky, 1972). Since there is a choice to make, they use personal experience-based explanations, repeating many of the answers obtained on the preliminary analysis. This is expected to change after the experimentation, finding the need for a proper probability calculation. Most of them change their position (3.3 and 3.4), based only on experience. This means the frequentist meaning of probability is stronger than a classic or theoretical one, as a decision-making tool. Questions 3.2 and 3.5 are too confusing for them and most of answers are left blank. We recognize there is no need for analyzing the sample space in an introductory session.

Conclusion

In the context of primary and secondary compulsory education, probability and statistics usually arise as mathematical concepts that represent tools for description of uncertainties. In order to move forward, the authors participate on the idea of having them as elements for decision and action. Didactical sequences should involve escalating decision-making scenarios and questions. According to the historical development of probabilities, it is convenient to ask if a game is fair or not (Hernández, Yumi & de Oliveira, 2010), followed by building a strategy to make it favorable.

Researchers and teachers should anticipate that heuristics and personal experiences are frequently more powerful considerations than calculations about probability and risk, when students are faced with decision-making scenarios. This has been documented not only in the didactics of mathematics research (e.g. Serrano, Batanero, Ortíz & Cañizares, 1998), but also (even previously) in psychology and economics literature. In particular, teaching and learning the conditional probability object could involve decisions within diachronic games. These are subjected to the isolation effect (Tversky, 1972; Kahneman & Tversky, 2007), among other difficulties, such as perceptions of independence and sample space, and interpretations of convergence (Batanero et al., 2009).

We recommend creating new sequences for other probabilistic concepts. Natural extensions are total probabilities and Bayes' theorem. Given information about medical research, students may decide whether approving or not a certain pharmaceutical product; or deciding about changes on their habits according to the relationship between cancer and processed meat or smoking.

Acknowledgments

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Appendix: Proposed sequence for execution

Activity 1: Single coin toss

Two players choose heads or tails. They toss a coin and whoever guesses wins.

1.1 Do you think this is a fair game? Why?

1.2 Repeat this game eleven times and register who wins each time. Do you keep your answer for question 1?

1.3 What's the probability of winning for each player?

1.4 [Teachers compile the results on the board]. Do you still keep your answer for question 1?

Activity 2: Best out of three

The game consists in two players choosing heads or tails, betting 60 each. They toss a coin successively three times and whoever obtains the most guesses wins.

2.1 Do you think this is a fair game? Why?

2.2 How many options does each player have of winning?

2.3 What's the probability of winning for each player?

2.4 Repeat this game ten times and register who wins each time. Do both players have the same amount of victories? Why do you think this happens?

Now suppose the first coin toss results on heads and the game is interrupted. You must decide what to do with the poll.

2.5 Would it be fair to split the poll by 60 each? Why?

2.6 Could you calculate the probability of winning for each player starting from that point?

2.7 Still assuming the first toss resulted in heads, simulate ten times the two remaining tosses, and register who wins the best out of three each time.

2.8 Given this scenario, would it be fair to split the poll giving 80 to the player who betted heads, and 40 for the one for tails?

2.9 Propose a repartition coherent with each one's probability of winning.

Activity 3: Monty Hall game

You are faced against three doors. Behind two of them there are goats and the other has a new car. Your goal is to guess the door where the car is hidden. The sequence is as follows: (1) The host offers you to pick a door. (2) After your choice, the host opens another door, different from the one you have chosen and shows there's a goat. (3) Now he offers a second chance: will you keep your first choice or change it to the other closed door?

3.1 What would you decide; would you keep your first choice or change it? Explain what is relevant for you to make this decision.

3.2 Which events have the same probability of occurring?

3.3 In pairs, play the game with your cups and car toy. One of the players will always change his or first choice, and the other will never change it. Repeat this ten times and compile the results for the whole class. Is there any difference between both types of players?

3.4 Is, therefore, any way of betting with an advantage?

3.5 Reconsider your answer from question 3.2. Given that the game has two stages of choice, which events have the same probability of occurring?

Activity 4: Plenary

Each pair of students responds the following questions in front of the class.

4.1 How would you face the Monty Hall game if you had to be there?

4.2 What recommendations would you give to someone who is about to play?

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Stepwise development of statistical literacy and thinking in a statistics course for elementary preservice teachers

Daniel Frischemeier & Rolf Biehler

Paderborn University, Institute of Mathematics, Germany; dafr@math.upb.de

In this paper we describe the design, realization and evaluation of a course for elementary preservice teachers, applying the PPDAC-cycle (Wild & Pfannkuch, 1999), using innovative methods and digital tools like TinkerPlots (Konold & Miller, 2011). We will refer to design principles of the course and show in which way a stepwise development of statistical literacy and thinking with TinkerPlots works in cooperative learning environments.

Keywords: Elementary preservice teachers, design based research, cooperative learning, statistical literacy and thinking, TinkerPlots.

Introduction

Since the implementation of the leading idea "Data, Frequency and Chance" (Hasemann & Mirwald, 2012) in mathematics classrooms in primary schools in Germany, statistics has become a central topic in primary school. This has set requirements not only for schools and teachers, but also for universities who have to educate preservice teachers in statistics for their upcoming school career. Requirements for teacher education in statistics can be found on German national level (e.g. AK Stochastik, 2012), and on international level (e.g. Batanero, Burrill & Reading, 2011). Two important aspects appear at both levels: applying a whole data analysis cycle (like PPDAC, see Wild & Pfannkuch, 1999) and analyzing data with digital tools (see Biehler, Ben-Zvi, Bakker & Makar, 2013). We decided to use TinkerPlots (Konold & Miller, 2011) for our purposes, since it is easy to learn, no formulas are needed, and it enables learners to create multiple representations of data. In our sense, TinkerPlots can serve as educational software for pupils from grade 4, as software for teachers for analyzing data, and as medium for demonstration purposes in classroom. This was our motivation to design, realize and evaluate a statistics course for elementary preservice teachers with TinkerPlots on the basis of the Design Based Research paradigm (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003).

Our course to develop statistical literacy and thinking with TinkerPlots

The main goal of this course is to develop statistical literacy and thinking components (for a definition see Garfield & Ben-Zvi, 2008, pp. 34) and the technological knowledge of our participants. At Paderborn University in Germany elementary preservice teachers for mathematics attend an obligatory course "Elementary Statistics", which is about data analysis, combinatorics and probability theory. Due to limits of time, there is no space for going through a whole data analysis cycle like PPDAC or to do further data explorations in multivariate datasets. For that reason our course was designed taking into account the principles of the "Statistical Reasoning Learning Environment" (Garfield & Ben-Zvi, 2008), to expand preservice teachers' knowledge in data analysis and to introduce a new tool to them, which better fits to primary and lower secondary school. Fundamental ideas realized in our new designed course are to "focus on developing central statistical ideas", to "use real and motivating data sets", to "use classroom activities to support the development of students´ reasoning", the integration of "appropriate technological tools", to "promote classroom discourse that includes statistical arguments and sustained exchanges that focus on

significant statistical ideas" and the "use of formative assessment" (see Garfield & Ben-Zvi, 2008, p. 48). On the paradigm of the PPDAC-Cycle (Wild & Pfannkuch, 1999), the course has the aim to encourage participants to define a statistical problem and to pose statistical questions (First "P" in PPDAC), to plan and to prepare a data collection (Second "P" in PPDAC), to collect data (with regard to data management and cleaning – "D" in PPDAC), to analyze data ("A" in PPDAC) and to make conclusions of the data explorations (interpretation – "C" in PPDAC). We implemented cooperative learning environments like the "Think-pair-share" method to develop the statistical literacy and thinking components of our participants (see Roseth, Garfield & Ben-Zvi, 2008) and to support peer-learning, peer-feedback and expert-feedback. In this respect, in a "Think-pair-share" setting, students first deal with the task on their own ("think phase"), discuss about their findings in peers ("pair phase") in a second step and finally discuss their findings in class with the teacher ("share phase"). All in all, the course consists of four modules: The first module deals with the generation of statistical questions, the preparation of data collection and the collection of data. Here the participants get to know the "PPD" elements of the PPDAC cycle with a special emphasis on the generation of adequate statistical questions (see Biehler, 2001). The second module has the intention to introduce the participants into data analysis with TinkerPlots. Here the participants can learn first steps in data analysis using data cards and hands-on-activities and then use TinkerPlots for first explorations in small datasets. Because the focus is on the "AC" (Analysis & Conclusion) elements of the PPDAC cycle, the datasets are given to the students in an already prepared form. Furthermore the students learn to describe and interpret distributions of categorical and numerical variables with special emphasis to the elements and characteristics of distributions like center, variation, etc. as it is proposed in Rossman, Chance & Lock (2001) and Biehler (2007a, 2007b). Module three builds on module two and covers advanced data analysis with TinkerPlots in large multivariate datasets. Here the learners are introduced into comparison of groups (Pfannkuch, 2007). A major aspect in module two and three is to enable the participants to explore datasets and make their own statistical investigations with their own statistical questions (for a typical task for a statistical investigation in this respect see Figure 1). Module four has the intention to introduce the participants into inferential statistics, especially into randomization tests with TinkerPlots (Frischemeier & Biehler, 2014). Further details on the course design and the lesson plans can be read in Frischemeier (2017).

Accompanying research of the course: Stepwise development of statistical literacy and thinking

The course was taught by the first and second author and consisted of 14 sessions, each session lasted 90 minutes. One major goal of the course was to develop the elementary preservice teachers' statistical literacy and thinking components with TinkerPlots. Since statistical investigations of complex datasets are new to our participants, we want to evaluate the statistical investigations in the introductory stage (module 2) and see in which way the quality will improve over time in cooperative learning environments. Two major research questions arise: How is the quality of the statistical investigations in the intermediate steps in module two? How does the quality of the statistical investigations develop in process of module two?

Participants, task and data collection

All in all 22 elementary preservice teachers participated in the course. All of them attended the course "Elementary Statistics" as described in the introduction. As a typical task a multivariate dataset with an exercise sheet consisting of four subtasks was given. As example you can see the "KinderUni"-task in Figure 1, where the dataset "KinderUni" had to be explored. The KinderUni dataset, is a (non-random

sampled) dataset with 28 variables containing information about leisure time and school activities of 39 pupils in the area of Kassel, Germany. In the introductory phase of module two the idea was, that learners at first explore small multivariate datasets to get used to data explorations with TinkerPlots and then to explore larger datasets in module three. So when working on the "KinderUni" task, the participants are at first (subtask (i)) asked for a short description of the dataset to get familiar with it. The second part (ii) of the task was to generate an appropriate statistical question. This statistical question of subtask (ii) is the starting point for subtasks (iii) and (iv). In subtask (iii), the participants are asked to create suitable graphs with TinkerPlots, which allow answering the statistical question arisen in part (ii). In subtask (iv) the participants are supposed to describe and interpret the TinkerPlots graphs of (iii) and finally to answer the statistical question posed in (ii).

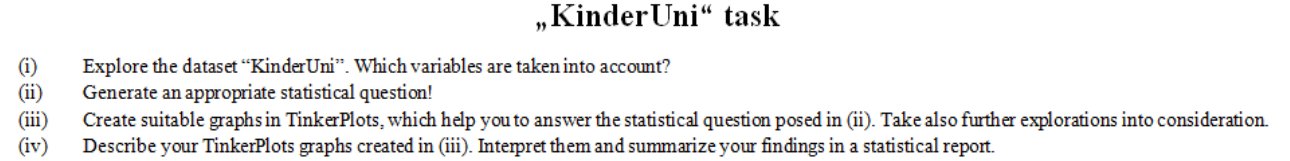

Figure 1: Task "KinderUni" as typical statistical investigation task in module two

The participants worked in pairs of two on the "KinderUni" task. So all in all, we had 11 pairs, who remained constant all over the course. When working on the task "KinderUni", the participants were asked to document the procedure of their statistical investigations in written form in Microsoft Word with the TinkerPlots graphs implemented. We collected all word documents from the "KinderUni" task. As mentioned above, one major idea of the course was to improve the quality of the statistical investigations by peer-feedback and expert-feedback. This happened with cooperative learning activities like "think-pair-share". First, in the "think" phase, all pairs worked on the task on their own and produced the preliminary version of the task (preliminary version: V1). Then two pairs came together and discussed the products of their statistical investigations (not necessarily with the same questions) in peers in the "pair" phase with the goal to find improvements for the TinkerPlots Graphs, for the descriptions of the TinkerPlots Graphs, etc.. Finally after revising the documents after the "pair" phase (version after peer feedback: V2), as a last step, the revised documents were discussed in plenum with the first and second author. After this phase the participants were again asked to revise their products for a final version (version after peer- and expert-feedback: V3). So for our data analysis we have the documentations on the statistical investigations of the participants as preliminary version (V1, n=11 documents), as version after peer feedback (V2, n=11 documents) and finally as version after expert-feedback (V3, n=10 documents).

Methodology for data analysis and coding

Our main goal was to rate the quality of the statistical investigations by points. Due to the huge amount of data, we used qualitative content analysis (Mayring, 2010) for rating the quality of the subtasks. We decided to weigh the subtasks (ii), (iii) and (iv) with equally two points maximum since these tasks are fundamental for the statistical investigation. In subtask (i) only one point is given, since this is an introductory task and easier than subtasks (ii), (iii) and (iv).

For subtask (i) we expected a description of the dataset (number of cases and variables, description of variables). In our course we have set the norm to begin every statistical investigation with an introduction. So two codes are given: "subtask (i) done correctly" and "subtask (i) not done correctly". If subtask (i) is done correctly, one point is given, if it is not done correctly no point is given for this subtask. Details given with examples can be found in Frischemeier (2017, p. 350).

To distinguish the quality of statistical questions posed in subtask (ii), we took into account the classification of Biehler (2001) in "one-variable-" and "two-variable-" questions from a deductive point of view. So we distinguished whether the questions take into account one variable (example: "What is the distribution of the variable height?" - variable: height) or two variables (example: "In which way do boys and girls differ in respect to the variable height?" - variables: gender and height). For questions taking into account only one variable one point is given as maximum, because the exploration coming out of questions containing one variable is easier than for questions taking into account two variables. For "two-variable-questions" a maximum of two points are given. In between we inductively identified different qualities of statistical questions: So there can be "one-variable-questions", which have just "yes" or "no" as answer (example: "Do 60% of the pupils have a mobile phone?") – rated with 0.5 points, whereas "one-variable-questions" in regard to a characteristic of a distribution (example: "How many pupils have a personal computer?") – rated with 1 point - are a little bit more sophisticated. Also in the set of "two-variable-questions" we find different types: There are questions leading just to a "yes"/"no" – answer (example: "Is there a difference between boys and girls in their time spending on computer use?"), whereas other types of questions lead to working out differences between the distributions (example: "In which regard does the computer use differ between boys and girls?"). Questions of the first type are rated with one point, questions of the second type are rated with two points. There is also another type of "two-variable questions", which we call "open and complex"-questions like "which differences exist between boys and girls in regard to their leisure time activities?" This type of "two-variable-question" is also rated with two points. In this course we have set the norm to try to pose statistical questions which aim at two variables. As an example for our rating in regard to subtask (ii) we take the question "How many kids have a way to school of 30 or more minutes?" of the pair Anne and Alice. We rated the question with one of two points, since it only covers one variable ("way to school") and it is aimed at one characteristic of a distribution ("how many …?"). Further details for the categorization of questions are given in Frischemeier (2017, p. 350).

For subtask (iii) an adequate TinkerPlots graph has to be created, which enables participants to answer the statistical question posed in (ii). Since we want our participants to focus on the distribution of the investigative variable and on the influence in regard to further variables, we have set the norm in our course that the icons should be stacked in TinkerPlots and further explorations (taking into account other variables) have to be made. If all three requirements (informative TinkerPlots graph, stacked dots and further explorations) are fulfilled, subtask (iii) is rated with the maximum of two points. Table 1 shows the several ratings for subtask (iii).

Informative TinkerPlots graph, stacked and further explorations	2 points
Informative TinkerPlots graph, stacked and no further explorations	1.5 points
Informative TinkerPlots graph, not stacked and further explorations	1.5 points
Informative TinkerPlots graph, not stacked and no further explorations	1 point
Non informative TinkerPlots graph/missing TinkerPlots graph	0 point

Table 1: Overview of ratings and their definitions of subtask (iii)

As an example for our rating with regard to subtask (iii) we take the pictogram (with stacked icons) of Anne and Alice in Figure 2. With this TinkerPlots graph they are able to answer their question ("How many kids have a way to school of 30 or more minutes?") posed in subtask (ii). Since icons are stacked, but no

further explorations are made, this graph is rated with 1.5 of 2 points. Further details and examples on the ratings of subtask (iii) can be read in Frischemeier (2017, pp. 354).

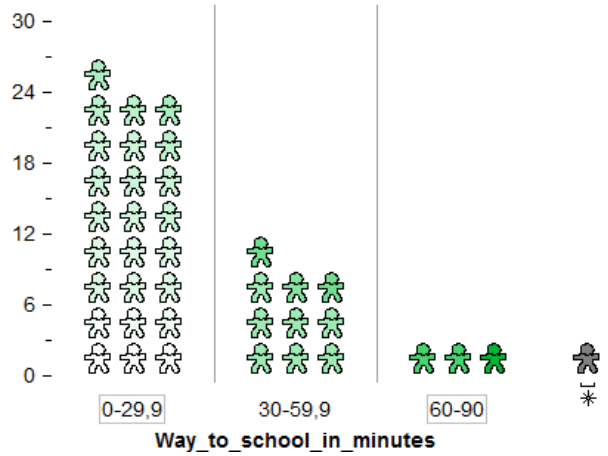

Figure 2: TinkerPlots graph for "KinderUni" task of Anne and Alice

In subtask (iv) the TinkerPlots graph (see Figure 2) has to be described adequately in at least one aspect and the question arisen in subtask (ii) has to be answered correctly. A maximum of two points are given, if both conditions are fulfilled. As adequate descriptions of the TinkerPlots graph we see elements like center, variation, shape, peaks, clusters and outliers (see Rossman et al. 2001, p. 48) but also absolute and relative frequencies of bins. For adequate elements to be carved out in group comparisons, see Frischemeier (2017, p. 42). In Table 2 we see the ratings for subtask (iv).

Component of TinkerPlots graph described and question (ii) answered correctly	2 points
Component of TinkerPlots graph described and question (ii) not answered correctly	1 point
Component of TinkerPlots graph not described and question (ii) answered correctly	1 point
Component of TinkerPlots graph not described and question (ii) not answered correctly	0 points

Table 2: Overview of ratings and their definitions of subtask (iv)

As an example for our rating in regard to subtask (iv) we have a look at the conclusion of Anne and Alice in subtask (iv): „We can see that 12+3 pupils have a way to school of 30 minutes or more." This was rated with the maximum of two points, since one component (absolute frequency of pupils in bins 30-59.9 and 60-90) of the graph is described and the question posed in (ii) is answered correctly. Further details on the ratings of subtask (iv) can be read in Frischemeier (2017, pp. 362).

Results

Let us have a look at the quality of the statistical investigations for the "KinderUni" task in module two in the different stages V1, V2, V3. For each team we rated the subtasks and calculated the success rate "points gained in all subtasks divided by the maximum points in all subtasks" for the "KinderUni" task in each stage (V1, V2, V3). In Figure 3 we see the distributions of the success rates in stages V1, V2 and V3.

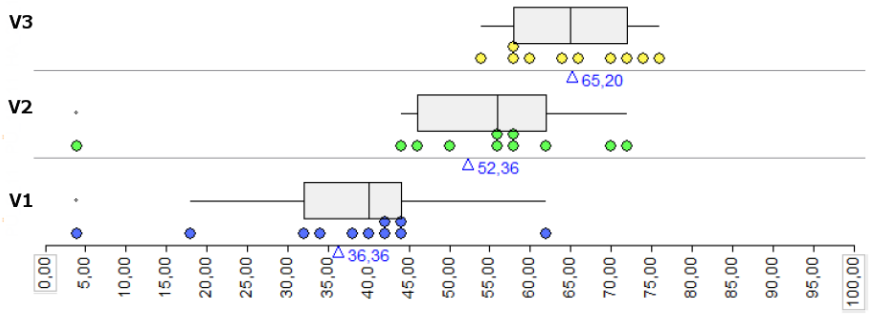

Figure 3: Success rates of the statistical investigations in stages V1, V2 and V3

Regarding to our research questions we can say that the median and also the mean (see blue triangles in Figure 3) of success rates of the different teams increase in the process of the several stages: In preliminary version (V1), where the pairs where on their own, 10 of 11 statistical investigations have a rate below 0.50, the median of the rates is 0.40, the mean of the rates is 0.3636. After the peer feedback phase ("pair"), there is a big positive shift in quality from V1 to V2. Exemplarily one peer feedback component which has often occurred was the advice to stack the dots in the plot to get a better view on the distribution of the data. The quality of the statistical investigations in V2 has increased a lot (mean=0.5236; median=0.56), since in this version only 3 of 11 statistical investigations are below the 0.50 rate. After the expert feedback ("share") in stage V3 all reports are over the 0.50 rate, the median of the rates is 0.65 and the mean of the rates is 0.6520. The expert feedback concentrated most notably on prompts which suggest a better description of the TinkerPlots graph and a more adequate answer to the statistical question posed in subtask (ii). Finally we can identify a positive development of the quality from V1 to V3. We can also see that the distributions in Figure 3 are heterogeneous at the beginning (stage V1) and become more homogeneous in V2 and V3. For a more detailed look we will have a look at Table 3, which identifies the changes within the development of quality in between the four subtasks (i), (ii), (iii) and (iv) in the stages V1, V2, V3. We see that there is an improvement of quality in all subtasks, but the amount of the improvements differ on the kind of subtask. In subtask (i) there is a high quality (0.73) even in the beginning at the preliminary version (V1). This quality improves over time in the different stages V2 (0.91) and V3 (1.00). In V3, the rate is 1.00, which means that every pair began their report at this stage with an introduction of the dataset. In subtask (ii) we see the smallest development of quality: for the questions in the preliminary stage (V1) the rate is 0.47 on average, there is no improvement in V2 (0.47) and only a small improvement in V3 (0.50). Even at the stage V3, in subtask (ii) all questions were only rated with one point, since none of the questions went beyond single characteristics of a distribution or beyond "yes"/"no" answers. One reason might be that the peer and also the expert feedback concentrated too much on the improvement of subtasks (iii) and (iv) but not enough on the development of the quality of the statistical questions.

	V1	V2	V3
Average success rate in subtask (i) of all pairs	0.73	0.91	1.00
Average success rate in subtask (ii) of all teams	0.47	0.47	0.50
Average success rate in subtask (iii) of all teams	0.39	0.51	0.68
Average success rate in subtask (iv) of all teams	0.19	0.55	0.74

Table 3: Development of quality (average of rates) between the subtasks

In subtask (iii) the tasks were rated 0.39 on average at the preliminary stage and improved over time (0.51 at V2 and 0.68 at V3). In subtask (iv) the performance was very poor at the beginning (0.19), but improved in progress: In V2 the rate was 0.55 on average and in V3 the rate was 0.74 on average. So in summary we can say that peer feedback and expert feedback in a think-pair-share environment enhances a stepwise development of statistical literacy and thinking components with TinkerPlots. Especially with regard to subtask (iii) und (iv) the creation of TinkerPlots graphs and their description seem to improve after peer and expert feedback. Only in subtask (ii) problems with the generation of statistical questions occur and there was no "big" improvement of quality.

Discussion and implications

The quality of statistical investigations depends on the statistical question rised for the investigation. As we could see, some questions only lead to a short exploration because the answer to that question is just "yes" or "no", wheares there can be also other questions which are aiming at carving out many differences between two or more variables. The analysis of the reports on the "KinderUni" task shows, that especially the creation of informative TinkerPlots graphs and also their description and interpretation with regard to the statistical question succeeds and the peer- and expert-feedback can improve the TinkerPlots graphs (subtask (iii)) and the descriptions and interpretations of the TinkerPlots graphs (subtask (iv)). The key point is the generation of adequate statistical questions aiming at more than only one variable. Although this was taught in our course, too many statistical questions lacked quality. For the re-design and the upcoming cycle of the course it would be important that there will be feedback on the statistical questions to improve their quality as well. Here it could be helpful to discuss adequate and non-adequate statistical questions in class to help learners to differentiate between adequate and non-adequate statistical questions.

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Variables characterizing correlation and regression problems in the Spanish high school textbooks

María M. Gea, Carmen Batanero, Pedro Arteaga and J. Miguel Contreras

University of Granada, Spain

mmgea@ugr.es, batanero@ugr.es, parteaga@ugr.es, jmcontreras@ugr.es

The aim of this paper is to analyse the problem situations within the topic of correlation and regression in the Spanish high school textbooks for Mathematics Applied to Social Sciences. In a sample of eight textbooks we firstly characterize the main problem situation used to contextualize correlation and regression, starting from the historical analysis of the topic. We then study the distribution of the following variables characterizing these problem situations: strength, sign, type of relationship and data contexts. Results show predominance of high and direct correlations, scarce examples of nonlinear regression and an excess of problem without context.

Keywords: Correlation and regression, textbooks, problem-situations.

Introduction

New curricular reforms emphasize statistical reasoning and its role in decision making and professional work (e.g., NCTM, 2000, CCSSI, 2010). Main content in these curricula for high school in Spain (MEC, 2007; MECD, 2015) and other countries are correlation and regression, which are fundamental statistical ideas that expand the previous knowledge about univariate distributions and mathematical functions. They also extend functional dependence to random situations and can be applied in a variety of other school subjects (Engel & Sedlmeier, 2011).

Previous research is mainly focused on students understanding of correlation (Estepa & Batanero 1995; Estepa, 2008; Zieffler & Garfield, 2009) with little attention to teaching materials, and, in particular, to the way the topic is taught or presented in the textbooks, in spite of their role as educational tools. From the official curricular guidelines until the teaching implemented in the classroom, an important step is the written curriculum reflected in the textbooks (Herbel, 2007). The selected textbook is an important part of teaching and learning mathematics, since it provides the main basis why the topic is taught (Shield & Dole, 2013). Moreover, mathematical textbooks receive increasing attention from the international community; see for example Fan and Zhu (2007).

The aim of this research was to analyse the tasks characterizing the problems used to present correlation and regression in high school Spanish textbooks directed to Social Sciences students. It is part of a wider project, where the way in which correlation and regression are presented in the textbooks in Spain is analysed. Complementary results were published in Gea et al. (2015).

Theoretical framework

We base on the Onto-semiotic approach to teaching and learning mathematics (Drijvers, Godino, Font & Trouche, 2013; Godino, Batanero & Font, 2007), where mathematical knowledge has a socio-epistemic dimension, since it is linked to the person's activity and depends on the institutional and social context in which it is embedded. In this framework, the meaning of mathematical objects

is linked to the mathematical practices carried out by somebody (a person or an institution) to solve specific mathematical problems. Around the mathematical practices linked to these specific problems, different rules (concepts, propositions, procedures) emerge (Godino et al., 2007) supported by mathematics language (terms and expressions, symbols, graphs, etc.), which, in turn, is regulated by the rules. All these objects are linked to arguments that serve to communicate the problem solutions, and to validate and generalize them to other contexts and problems.

The authors conceive different types of institutional meanings for a mathematical object (in this case correlation and regression): a) *reference meaning* (the system of practices used as reference in a particular research); b) *intended meaning* (the part of the meaning that is planned for teaching; for example, that proposed in the curricular guidelines); c) *implemented meaning* (what was finally taught to the students); and d) *assessed meaning* (content of assessment) (Godino et al., 2007). In our research we try to identify the implemented meaning of correlation and regression, as defined by the problems proposed in the textbooks, and to compare it with the intended institutional meaning for these students, as defined in the curricular guidelines (MEC, 2007; MECD, 2015).

Background

In spite of the relevance of these topics, previous research suggests poor results in people's understanding of correlation and regression. For example, Erlick and Mills (1967) found that negative correlation is commonly estimated as close to zero. Other authors studied the influence of previous theories about the context of the problem on the accuracy in estimating correlation. In this respect, Chapman and Chapman (1967, p. 194) described "illusory correlation" as "the report by an observer of a correlation between two classes of events which in reality (a) are not correlated, or (b) are correlated to a lesser extent than reported, or (c) are correlated in the opposite direction than that which is reported". The estimates are more accurate if people have no theories about the type of association in the data. If the subject's previous theories agree with the type of association reflected by the empirical data, there is a tendency to overestimate the association coefficient. But when the data do not reflect the results expected by these theories, the subjects are often guided by their theories, rather than by data (Jennings, Amabile & Ross, 1982).

According to Barbancho (1992), the correlation between variables may be explained by the existence of a unilateral cause - effect relationship (one variable produces the other), but also to interdependence (each variable affects the other), indirect dependence (there is a third variable affecting both variables), concordance (matching in preference by two judges in the same data set) and spurious correlation (or coincidental covariation). In addition to the estimate accuracy, understanding correlation involve the discrimination of these types of relationships between variables.

Estepa (1994) studied the understanding of correlation in a sample of 213 Spanish high school students. The author defined the *causal conception* according to which the subject only considers correlation between variables, when it can be explained by the presence of a cause - effect relationship. He also described the *unidirectional conception*, where the student does not accept an inverse association, considering the strength of the association, but not its sign, and assuming independence where there is an inverse association.

As regards research on textbooks, Sánchez Cobo (1999) classified the definitions of concepts

presented in 11 textbooks published in the period 1977-1990 as procedural, structural or a mixture of them. Lavalle, Micheli and Rubio (2006) analysed the concepts and procedures included in 7 high school textbooks from Argentina. The current paper complements these publications and our previous paper (Gea et al., 2015), where we analyse the presentation of concepts, properties and procedures in the same textbooks that we are analysing here.

Method and results

The sample was made of eight mathematics high school textbooks (H1 to H8), directed to Spanish Social Sciences students that are listed in the Appendix. They still are used in the schools, and were published by editorials of prestige and wide diffusion in Spain just after the past curricular guidelines were introduced (MEC, 2007). We performed a content analysis (Neuendorf, 2002) of the chapters devoted to correlation and regression with an inductive and cyclic procedure, and classified all the problems, exercises and examples used in the chapter according the variables described below. The total number of problems analysed were 2166, distributed according Table 1.

	Textbook							
	H1	H2	H3	H4	H5	H6	H7	H8
Number of problems analysed in the book	268	221	258	225	318	176	403	297

Table 1: Sample of problems analysed

Main types of problem fields

Anthony and Walshaw (2009) reported on the different types of tasks that have been analysed in mathematics education research, which include problems centred on specific mathematical content; problems that promote mathematical modelling; tasks requiring students to interpret and critique data and those that prompt sense making and justification of thinking. The problems analysed belong to the first category and were classified according the main types of problems identified by Gea, Batanero, Cañadas and Contreras (2013) in the study of correlation and regression:

P_0: *Organising and summarising bivariate data*, which include graphical representation and computation of summaries statistics.

P_1: *Determining the existence of a relationship between the variables*, which can be subdivided in four types of problems: (P_{11}) Defining the univariate variables that constitute the bivariate data; (P_{12}) Determining the type of dependence (functional, random or independence); (P_{13}) Determining the strength of the relationship; and (P_{14}) Determining the direction (direct, inverse or nonlinear).

P_2: *Predicting a variable from the other*, which can be subdivided in the following types: (P_{21}) Fitting a model to the data (usually, the linear model); and (P_{22}) Making estimations from the model, where we also include assessing the goodness of fit.

All these types of problems appear in the books, with different frequency, as shown in Table 2, with the following distribution: 11% P_1, 61% P_2 and 28% P_3. The most frequent field of problems consisted in identifying the strength of correlation (P_{13}), (with percentages ranging from 19% in H3 to 24% in H2). The textbooks paid less attention to the problem field P_0, with the exception of H3, H4 and H7, despite the organisation of data is an important step prior to analysing a relationship

between the variables. We also observe that H4 is more balanced as regards the different types of problems, although the percentage in P_{22} problems is still a little low.

Problem field		H1	H2	H3	H4	H5	H6	H7	H8
P_0		6.7	10.9	17.4	16.9	8.8	4.5	17.4	7.7
P_1	P_{11}	10.1	11.8	6.2	20.0	6.6	15.3	8.4	12.8
	P_{12}	14.6	14.0	19.8	11.6	13.8	13.6	11.2	14.8
	P_{13}	20.9	24.0	19.0	20.0	23.9	22.2	22.8	20.5
	P_{14}	19.4	9.5	14.0	12.0	16.0	9.1	13.4	13.5
P_2	P_{21}	16.4	11.8	10.1	11.6	15.7	15.9	14.6	20.2
	P_{22}	11.9	18.1	13.6	8.0	15.1	19.3	12.2	10.4

Table 2: Classification of activities in the textbooks by problem field

Strength, sign and shape of association

For each problem analysed, we computed the Pearson's correlation coefficient when the data suggested linear relationship and the square root of the determination coefficient when the dependence was non-linear. We then classified the problems according to the strength of association in the following way: a) independence, if the value of the coefficient was very close to zero; b) low dependence if these coefficients ranged in the interval [0.1; 0.5); c)medium for the interval [0.5; 0.8), d) high for the interval [0.8; 1), and e) functional where there was a perfect fit of the data to a model and $r = \pm 1$ or $D = 1$. In Table 3 we classify the problems according the strength of the dependence in the data suggested by these coefficients.

Strength of association	H1	H2	H3	H4	H5	H6	H7	H8
Independence	0.7	2.3	9.3	5.3	1.6	3.4	7.4	7.1
Low	9.3	8.6	7.8	23.1	9.7	8.5	22.1	9.1
Medium	19.8	10.4	13.2	20.0	17.3	11.9	16.1	16.8
High	47.4	76.0	48.1	39.1	53.1	60.8	28.3	49.2
Functional	5.6	1.8	8.6	6.2	2.2	1.7	8.7	8.1
No data provided	17.2	0.9	13.2	6.2	16.0	13.6	17.4	9.8

Table 3: Percentage of problems, according strength of relationship

Most commonly the data showed a high association or medium association; there were scarce problems with independent data or corresponding to functional relationships, in agreement with previous results from Sánchez Cobo (1999). We remark that most statistical studies in Social Sciences (the speciality that these students intend to follow) deal with moderate correlation, so that we recommend to include more problems with moderate association and a more balanced distribution of this variable in future textbooks.

We also studied the sign of correlation (direct or inverse) in case of linear relationship and found about 60% of problems that used direct correlation, as shown in Table 4. The scarce presence of inverse correlation problems (20% on average), also noticed by Sánchez Cobo (1999), may contribute to the unidirectional conception of correlation (Estepa, 1994) where students wrongly identify negative correlation with independence.

	H1	H2	H3	H4	H5	H6	H7	H8
Independence	0.7	2.3	9.3	5.3	1.6	3.4	7.4	7.1
Direct	59.7	85.1	47.7	68.0	46.2	61.9	51.6	56.9
Inverse	22.4	11.8	28.3	20.4	36.2	21.0	22.8	26.3
Others	0	0	1.6	0	0	0	0.7	0
No data provided	17.2	0.9	13.2	6.2	16.0	13.6	17.4	9.8

Table 4: Percentage of problems, according sign of correlation

A third variable analysed, not considered in previous research, was the type of function that fits the data. In order to determine this model, in each problem, we fitted different types of functions to the data and selected the function providing the best fit. Most situations corresponded to linear relationship, as shown Table 5, because this type of function is easier for the students for an introduction for the topic. However, we recommend incorporating some examples of functions well known by the students, for example, quadratic, polynomial or exponential functions to develop their statistical thinking, while avoiding the deterministic conception (Estepa, 1994). We remark that all the books include the least square line, as well as its use for prediction, while only one (H8) includes the Tukey line (2% of problems proposed in the book).

		H1	H2	H3	H4	H5	H6	H7	H8
Independence		0.7	2.3	9.3	5.3	1.6	3.4	7.4	7.1
Functional	Linear	4.5	1.8	6.6	6.2	1.9	1.7	7.7	6.1
	Non-linear	1.1	0	2.0	0	0.3	0	1.0	2.0
Random	Lineal	76.5	57.9	62.0	66.2	76.1	80.1	63.8	62.3
	Non-linear	0	37.1	7.0	16.0	4.1	1.1	2.7	10.8
No data provided		17.2	0.9	13.2	6.2	16.0	13.6	17.4	9.8

Table 5: Percentage of problems, according type of function defining the line of best fit

Data context

The relevance of context in the teaching of statistics has been extensively discussed by different researchers. We analysed the context of the situations proposed to the students in these textbooks and classified them (see Table 6) in the following categories: a) Biology (e.g., parents and children heights); b) Science (e.g., speed and distance); c) Sport (e.g., distance and time spent in a competition); d) Economy (e.g., energy consumption and gross national product); e) Education

(e.g., score in two exams in a group of students); f) Sociology (e.g., birth rate and percentage of women at work). Results show a high percentage of problems with no context (students cannot relate their results to a meaningful context) and a similar distribution of other contexts.

Context	H1	H2	H3	H4	H5	H6	H7	H8
Biology	8.2	10.9	7.0	4.4	13.8	17.6	6.9	12.5
Science	13.1	33.0	14.7	13.3	13.2	16.5	9.4	9.1
Sports	4.5	0	5.4	2.2	2.2	0	0	0
Economy	14.2	8.1	13.6	5.3	6.9	14.8	3.5	10.1
Education	10.4	4.5	1.9	14.2	17.0	15.3	5.7	11.8
Sociology	6.3	17.2	10.9	14.2	9.1	12.5	9.9	9.8
No context	43.3	26.2	46.5	46.2	37.7	23.3	64.5	46.8

Table 6: Percentage of problems according to context

Discussion and didactic implications

The study suggests important differences in the problems proposed by the different textbooks; for example, H2 and H6 include a lower proportion of problems with no context, although H2 has the highest proportion of high correlation problems and H6 the lowest percentage of problems P_0. We also found some biases in the distribution of the variables analysed, in particular there is a tendency towards direct, strong and linear relationship. It is important that teachers complement these types of problems with a wider variety of strength, sign and type of association, as well as with contexts that are interesting for the students, as it is suggested in the curricular guidelines (MECD, 2015): "The teaching of this subject should not be dissociated from its application to social phenomena".

Our results suggest that the institutional intended meanings for correlation and regression and the implemented meaning in the textbooks analysed do not fit appropriately according to the variables analysed. In this sense, these problems should also be complemented with statistical projects in which the students experiment a complete cycle of statistical enquiry and get experience in the different modes of statistical thinking in Wild and Pfannkuch (1999)'s model (need for data, transnumeration, variation, reasoning with models, integration of statistical and contextual knowledge). Today there are plenty of data available on Internet that can be used to introduce ideas of correlation and regression via projects, as suggested in Batanero, Gea, Díaz and Cañadas (2014).

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Appendix: Texbooks analyzed

[H1]. Colera, J., Oliveira, M.J., García, R. & Santaella, E. (2008). *Matemáticas aplicadas a las Ciencias Sociales I*. Madrid: Grupo Anaya.

[H2]. Arias, J. M. & Maza, I. (2011). *Matemáticas aplicadas a las Ciencias Sociales 1*. Madrid: Grupo Editorial Bruño.

[H3]. Anguera, J., Biosca, A., Espinet, M. J., Fandos, M.J., Gimeno, M. & Rey, J. (2008). *Matemáticas I aplicadas a las Ciencias Sociales*. Barcelona: Guadiel - Grupo Edebé.

[H4]. Monteagudo, M. F. & Paz, J. (2008). *1º Bachillerato. Matemáticas aplicadas a las Ciencias Sociales*. Zaragoza: Edelvives (Editorial Luis Vives).

[H5]. Martínez, J. M., Cuadra, R. & Heras, A. (2008). *Matemáticas aplicadas a las Ciencias Sociales. 1º Bachillerato*. Madrid: McGraw-Hill.

[H6]. Bescós, E. & Pena, Z. (2008). *Matemáticas aplicadas a las Ciencias Sociales. 1 Bachillerato*. Vizcaya: Oxford University Press España.

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Middle school mathematics teachers' pedagogical content knowledge in relation to statistical reasoning

Rukiye Gökce and Sibel Kazak

Pamukkale University, Faculty of Education, Denizli, Turkey; skazak@pau.edu.tr

To investigate middle school mathematics teachers' pedagogical content knowledge (PCK) with regard to statistical reasoning, an interview protocol was developed and used with nine teachers. This paper focuses on one of the problems in this interview protocol (Basketball problem) to illustrate teachers PCK in relation to four components: big ideas, student responses, student difficulties, and instructional intervention. Our analyses showed that levels of teachers' PCK varied in each component. Teachers had difficulties mostly in explaining student difficulties, developing instruction intervention strategies and distinguishing appropriate and inappropriate student reasoning.

Keywords: Statistical reasoning, pedagogical content knowledge, middle school teachers.

Introduction

The reliability and persuasiveness of advertisements or arguments that people encounter in daily life were ensured by the statistical information (Ben-Zvi & Garfield, 2004). For being informed citizens, it is crucial to analyze and interpret such statistical information and to make inferences from them. These problem solving, inquiry, analysis, justification and interpretation skills in statistics are all related to statistical reasoning. Garfield, delMas and Chance (2003) define statistical reasoning as "the way people reason with statistical ideas and make sense of statistical information" (p. 8). In statistics education, teachers have a critical role in helping learners develop deep conceptual understanding of statistical ideas. Shulman (1987) notes the importance of the capacity of a teacher to transform their content knowledge into pedagogically more powerful forms. Shulman (1986) states that the teacher content knowledge is not sufficient by itself for teaching a subject and points out the teacher pedagogical content knowledge (PCK) referring to the combination of content knowledge and pedagogical knowledge. More specifically, PCK is described as "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). In statistics education literature, the studies by Watson and her colleagues (Watson, Callingham & Donne, 2008; Callingham & Watson, 2011) on examining levels of teachers' statistical PCK suggest that teacher knowledge needs to be investigated more systematically and note the need for uncovering the current state to advance the statistics education. However, there is little research on teacher knowledge particularly with regard to statistical reasoning (e.g., Mickelson & Heaton, 2004; Makar & Confrey, 2004). Thus, the aim of this study is to investigate middle school mathematics teachers' PCK related to statistical reasoning. Our research question is: In a distribution comparison task, to what extent do mathematics teachers consider the big ideas in statistics in their reasoning, how students might reason and how they would intervene to promote appropriate statistical reasoning?

Theoretical framework

The notion of PCK, as noted by Shulman (1986), entails teachers' knowledge about both the understanding/misconceptions of students and pedagogical strategies for instruction. According to An, Kulm and Wu (2004) there is an interaction between PCK and content knowledge and between knowledge of curriculum and knowledge of teaching, and the knowledge of student thinking is in the center. Moreover, teachers' competence of making instructional interventions is considered as part of 'enacting mathematics for teaching and learning' which is one of the components in the model of PCK developed by the Teacher Education and Development Study in Mathematics (Tatto et al., 2008). Accordingly, responding to unexpected mathematical issues, evaluating student solutions, identifying student misconceptions, providing appropriate feedback, explaining and representing mathematical concepts are amongst the essential criteria for teacher competency to achieve the goals of learning-teaching process and increase its quality. This study focuses on the knowledge of student and knowledge of instructional interventions as the two components of PCK.

The study by Watson, Callingham and Nathan (2009) focused on teachers' PCK with regard to statistical knowledge at the middle school level. The researchers identified four non-hierarchical components of PCK. The first two components, "Recognizing Big Ideas" and "Anticipating Student Answers", reflect the link between teachers' knowledge of content and knowledge of understanding students. The other two components, "Employing Content-Specific Strategies" and "Constructing Shift to General", involve elements of pedagogical practices that the teachers used by foreseeing the progress of student understanding. In another study by Watson and Nathan (2010), teachers' PCK was investigated using three different problems. Teachers were asked about the big statistical ideas in each problem, possible appropriate and inappropriate student responses and opportunities provided by the problem for teaching. Teachers' responses were analyzed based on the four components of PCK developed previously by Watson et al. (2009). In our study we utilized these two studies for developing interview questions and coding teacher responses.

Method

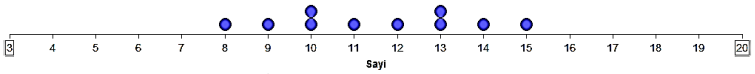
This exploratory study focuses on one-on-one interviews with nine middle school mathematics teachers (two females and seven males). The participants were selected on voluntary basis and from two different middle schools in Denizli. Their teaching experience changes from 2 to 30 years. The national curriculum of school mathematics includes data analysis strand since 2005 but according to our classroom observations, the participants tend to teach mainly computations and procedures in a traditional way rather than to focus on developing students' statistical reasoning.

Data collection and task

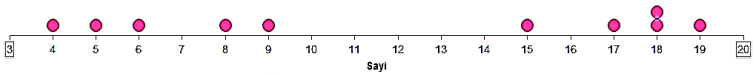
Interviews with each teacher were conducted by the first author and video-recorded. These video recordings were transcribed for analysis. An interview protocol was developed to study teachers' PCK with regard to statistical reasoning. In the larger study, the interview tasks involved three different scenarios, in which teachers were asked to determine possible student reasoning and difficulties. In order to develop teacher-student dialogues in these scenarios, these problems initially were administered to a classroom of 6th graders. Students' responses provided a basis for constructing the dialogues given to teachers during the interview (see Figure 3). Motivated by the design of statistical reasoning tasks used in Cobb and his colleagues' study (Cobb, 1999) each problem involves

comparing two or three data distributions from equal and small sample sizes to unequal and larger sample sizes. And four kinds of statistical reasoning were addressed in these problems: reasoning about distribution, reasoning about center, reasoning about spread, and informal statistical inferential reasoning. This paper focuses on one of these problems, called Basketball Problem. In the interview, each teacher was initially asked to decide and explain which of the players they would choose to the school basketball team looking at the given data in Figure 1.

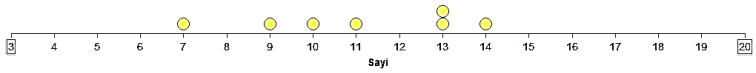
Players	**Players' scores**									
Arda	8	14	11	13	15	12	10	10	9	13
Baris	4	15	8	17	18	9	19	6	18	5
Cem	-	-	-	13	14	10	7	13	11	9



Graph 1: Arda's scores in the last 10 matches



Graph 2: Baris's scores in the last 10 matches



Graph 3: Cem's scores in the last 7 matches

Figure 1: Data given in the Basketball problem in the interview protocol

Next, in order to explore teachers' PCK in relation to students' statistical reasoning and instructional interventions, participants were asked to come up with examples of appropriate and inappropriate student reasoning in the context of given problem (see questions 1-5 in Figure 2).

Q1: In the context of this problem with appropriate reasoning, how might students answer?
Q2: Why do you think that reasoning is appropriate?
Q3: In the context of this problem with inappropriate reasoning, how might students answer?
Q4: Why do you think that reasoning is inappropriate?
Q5: How would you guide those students who reasoned inappropriately to correct reasoning?
Q6: In this page, there is a dialogue between students and teacher about basketball problem (see Figure 3). Firstly I want you to read this (dialogue sheet).

a) Here, could you identify which students make appropriate reasoning? Why?
b) Could you identify which students make inappropriate reasoning? Why?
c) What is the difficulty that prevents the student from reasoning in an appropriate way?
d) When you encounter such a situation in the class, what kind of questions would you ask to your students to make them reason appropriately? What kind of intervention would you consider?

Figure 2: Interview questions to elicit teachers' PCK in the context of Basketball problem

After these questions, teachers were given a dialogue between students and teacher about the problem (see Figure 3) to examine their PCK further by the questions 6 a-d in Figure 2.

Simge: Well, there is also the probability that Cem will score more or less than that you say he would make. So you also took risk in a way. In my opinion we can do this: Arda made 15 points the most, 8 the least score. So if I sum up these two numbers and divide by 2, I will find the number 11,5 as the middle point. Baris made 19 the most and 4 the least. The middle point of these two numbers is 11,5. Cem made 14 the most and 7 the least. The middle point of these two numbers is 10,5. So I would eliminate Cem.
Teacher: Well Simge, how would you choose between Arda and Baris?

Simge: I would look at the highest score for both of them. Since Arda made 15 points and Baris 18 points, I would choose Baris.
Kagan: As there is a difference between the numbers of matches played so far, we should look at the mean. As the mean of Arda's score is 11,5; Baris's is 11,9 and Cem's is 11, I would choose Baris.
Duygu: Baris's mean score can be high. But he performed very well in one match and very poor in another match. He has a varying performance in the range of 4 and 19 points. He is not consistent.
Teacher: What do you mean by 'not consistent'? How do you conclude that he is not consistent?
Duygu: When we look at the graphs we see that these scores that Baris made seem far and dispersed from each other. Arda's scores seem closer together. Arda has a varying performance between 8 and 15 points. He is more consistent than Baris. There is not much difference between Arda's average score and Baris's average score. I would choose Arda.

Figure 3: Part of a dialogue between students and teacher about basketball problem

Data analysis

Qualitative analysis of data was done by two researchers. Initially a number of codes were adapted from Watson and Nathan's (2010) study to examine interview transcripts for PCK in relation to statistical reasoning and codes were assigned. During the content analysis phase new codes were created from the data. The categorizations of teacher responses were discussed and agreed on by two researchers. As a result, four components of PCK were formed as seen in Table 1: Big ideas, student responses, student difficulties, and instructional intervention.

Aspects of PCK	Codes/Levels	f
Big Ideas	0- Response confused and /or incorrect	-
	1- Response implied and/or understanding revealed beyond initial question	4
	2- *Statistical reasoning*	5
Student Responses	0- Response irrelevant	1
	1- Appropriate or inappropriate but not both, or unclear	1
	2- Distinguishes both appropriate and inappropriate, *but no reason/explanation*	1
	3- Demonstrates understanding of students' reasoning *with reason/explanation*	6
Student Difficulties	0- *Response irrelevant*	5
	1- *Unclear (General statements, lack of knowledge)*	1
	2- *Correct reason/explanation*	3
Instructional Intervention	0- *Response irrelevant / personal view*	1
	1- *Noticing content without given data (common beliefs)*	5
	2- *Promoting the appropriate use of percents, numbers, measures of center and spread for statistical reasoning*	3
	3- *Promoting generalization beyond data*	-

Table 1: Codes for PCK (new codes were indicated in italics) and their occurrences

Differently from Watson and Nathan's study, a new component called "student difficulties" was considered in the current study. In addition, "instructional intervention" was used as a more general component that covers the two components in Watson and Nathan's study, namely "Employs Content-specific Strategies" and "Constructs Shift to General". "Big ideas" aspect of PCK was formed to uncover teachers' statistical reasoning by using big ideas in statistics with the addition of "code 2-statistical reasoning" to the codes in Watson and Nathan's study. Moreover, "no reason or explanation for the appropriateness of student reasoning" and "explaining the reasons for the appropriateness of student reasoning" criteria were added to code 2 and code 3, respectively, in "student responses" component. The codes in "student difficulties" component were formed as a

result of the content analysis. The codes in "instructional intervention" component were formulated as a result of the content analysis and aspects of statistical reasoning skills.

Findings

Teachers' PCK level with regard to big ideas

Five of the teachers were able to reason with the big ideas in statistics (code 2 level). These teachers took into account both variability and central tendency in their comparisons of distributions as it can be seen in Semih's reasoning: "First we should compute their means. Arda's is 11,5; Baris' is 11,9; Cem's is 11. Baris scored the most. I would choose Arda because of his stability. His scores are placed between 8 and 15 in the graphs. Baris' scores are placed between 4 and 19. Baris' average is not much higher than the others." Another teacher, Ebru, made her decision based on somehow a middle range in addition to the stability criteria. Her reasoning was: "I would transfer Arda. I chose a lower and an upper limit [*between 8 and 15*]. Even though Baris scored more points than this upper limit, from stability point of view Arda is more consistent." Other four teachers' responses were yet considered partially correct (code 1 level) because they focused on either the mean or the spread. The following two quotes illustrate code 1 level responses.

> I would transfer Baris. He scored in every game, no points less than 4. By just looking at the average, it does not look bad. (Eren)

> Their means are close. Arda's performance is better. These data aren't adequate. I would prefer Cem. His scores are closer. I am hesitant but I don't have any reason. I said that I would choose Cem, but I wouldn't choose Baris. He scored once 4 and the other time 19. There is no consistency. Arda scored 10 twice, 13 twice, these numbers are closer, so more stable. I can take Arda. (Suat)

Teachers' PCK level with regard to student responses

Responses of six teachers showed an understanding of students' reasoning with appropriate explanation (the code 3 level) as seen in the answer below:

> A student who reasons appropriately will firstly look at the mean. Arda's is 11,5; Baris' is 11,9; Cem's is 11. Student would look at the place where the cluster is more. Not only the cluster but also between which numbers the cluster is [*pointing to the range*]. His mean is high and the cluster is above a certain number, so he would choose Arda. Moreover their means are close to each other [*pointing to Arda and Baris*]. A student who reasons inappropriately would choose Baris scoring the highest number 19. Student reasoned inappropriately because there is only one match in which Baris scored 19, he scored 4 as well. Student should look at the whole. (Semih)

The other three teachers' responses were identified at the other levels. For example, Eren's response was considered at the code 0 level because his answer was related to the use of representations given in the problem rather than using statistical ideas/measures for his reasoning: "Student reasoning appropriately would look at both the graph and the table, evaluate them together and choose Baris. Student who doesn't reason appropriately would only look at the graph and choose Arda." In the following quotes, Suat was uncertain about the appropriate student reasoning due to the mistrust in

his own reasoning (the code 1 level) while Sema could distinguish appropriate and inappropriate student response but with inadequate explanation response (the code 2 level).

> Since I had a difficulty in answering the question in this problem, I am not able to guess how students could respond. I don't even think that my answer was sound, but a student who reasons appropriately might choose Arda because of the two modes and their being close to each other. A student who reasons inappropriately chooses Baris because he makes the high score. (Suat)
>
> Student reasoning appropriately would interpret the data looking at the table. By computation, he would choose Baris with the highest mean. Because we have only numbers, there is nothing else. Student who reason inappropriately would choose Baris since he scored the highest point. (Sema)

Teachers' PCK with regard to student difficulties

When asked to explain student difficulties in their reasoning, the teacher responses mostly were inadequate and had irrelevant details (the code 0 level) as seen in Ismail's response: "A student saying 'I'd choose Baris because he scores the most number' thinks in a shallow way." At the code 1 level Okan linked student difficulty to a wider context of school practice while at the code 2 level Suat made an evaluation based on what is expected statistically.

> The difficulty in a student who would choose Baris because of his highest mean could be the following. Students have computed the mean since primary school and so always mean stays in mind. The others [*other measures of center*] are introduced in the middle school. For students, the first thing they have learned seems more accurate. That is a student who chooses the extreme values maybe do not know measures of center and spread. (Okan)
>
> A student who chooses Baris according to highest score is looking at the players' best result. In fact the student should look at all of the results. (Suat)

Teachers' PCK with regard to instructional intervention

When teachers were asked about the kind of questions they would ask and interventions they would consider in the classroom to guide students to reason appropriately, most teachers responded at the code 0, code 1 and code 2 levels and none at the code 3 level. Since Suat's example of what he would ask to students was far from the context and irrelevant to the problem, his response was at the code 0 level. When Ismail gave the following answer: "I would ask the following question to the students who choose Baris scoring the highest: 'Guys, if you made a critique about the performances of Arda and Baris, which of them would you find successful? Why?'. If it is the final match, student may say Baris or for league matches Arda. This is important.", it was considered at the code 1 level because his response involved a use of context but not based on data. However, in the following response (the code 2 level) teacher's approach tended to involve having a discussion about the measures of center and spread within the problem context:

> I'd ask the student who chooses Baris due to his highest score: 'Who scored the lowest?'. Scoring the least is Baris as well. I would also ask Arda's highest and lowest scores. I would have the student find the range. Arda's range is 7, Baris' is 15. I would show that the scores Baris has made are in a wider interval. I would ask student: 'Would you choose the player who scores 4 in one match and 19 in the other or the player who scores 8 in one match and 15 in the other?' (Semih)

Identifying appropriate and inappropriate student reasoning in a given dialogue

In the given dialogue between teacher and students within the basketball problem context, only one of the teachers was able to distinguish appropriate and inappropriate student reasoning properly. This teacher (Göksel) argued that since consistency is important in this context, both mean and spread should be used in making the decision. The other eight teachers' responses were partially correct. For instance, one of these eight teachers (Suat) correctly identified student responses using both the mean and range with attention to consistency and yet he also considered the student responses using only the mean or the increases/decreases in the scores as appropriate reasoning.

Discussion and conclusion

In this study, teachers overall seem to do better in using big ideas in their reasoning and considering possible student responses on the Basketball problem. Yet, some teachers still had slight difficulties consistent with those that students might have in comparing distributions (Cobb, 1999) and in using big ideas for solving statistical problems (Watson & Nathan, 2010). Similar to the previous findings (Watson & Nathan, 2010; Watson, et al., 2008) some challenges also exist for these teachers when considering how students might reason in their decisions. In contribution to the previous research results, this study shows other challenges that most of the teachers tend to have: (1) somewhat incomplete understanding of potential student difficulties, (2) inadequacy for suggesting instructional intervention strategies that could help students make generalizations beyond data and (3) distinguishing appropriate and inappropriate student reasoning in the given classroom dialogue. These findings related to teachers' knowledge about student responses may not be unique to statistics teaching since they show consistencies with previous research with mathematics teachers in other mathematical topics, such as functions and undefined mathematical operations (e.g., Even & Trosh, 1995). The insufficiencies in teacher PCK both in our findings in relation to statistical reasoning and other studies by Watson and her colleagues suggest that there is a need for in-service teacher training on PCK in statistics. More specifically, teachers need to be able to anticipate possible student difficulties in statistical reasoning and develop ways to respond to them.

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Using sport-related socially open-ended problems to coordinate decision-making, variability modeling and value awareness

Orlando González [1] and Somchai Chitmun[2]

[1]Assumption University, Grad. School of Human Sciences, Bangkok, Thailand; ogonzalez@au.edu

[2]Tepleela School, Bangkok, Thailand; schitmun@hotmail.com

The purpose of this article is to describe the features, as well as the instructional potential and advantages, of socially open-ended problems set in a sport-related context, through which students could be challenged to model variability in order to assess performance and develop data-driven decision-making skills. These arguments will be illustrated by using as an example the "Darts game" task, a problem from a study conducted by González and Chitmun (2015) on Japanese and Thai secondary school mathematics teachers' professional knowledge of data-driven decision-making.

Keywords: Decision-making, socially open-ended problems, variability modeling, value awareness, statistical investigations.

Introduction

The importance of using, handling and interpreting data to inform decision-making is fundamental to participate competently in today's society. Such importance has been acknowledged by recent reforms to the mathematics curriculum in many countries, in which having the attitude and ability to purposely process and grasp features of daily-life data, and making informed decisions in real-life situations, are instructional goals explicitly stated in the statistics-related strands of the mathematics curriculum (e.g., MEXT, 2008, 2009; MOE, 2008). Thus, it is necessary for teachers to be able to choose the most appropriate tasks to take students further in the achievement of such goals.

Although important, being able to grasp features of daily-life data is not enough to make informed decisions. Today's society is a value-pluralistic one, in which it is natural for people to hold multiple values which, due to life experiences and interactions, will be prioritized, accommodated, negotiated, compromised and traded-off. Moreover, value awareness has been regarded as fundamental for engaging in effective decision-making, since values are principles used to evaluate actual or potential consequences of action or inaction, of proposed alternatives and of decisions (Keeney, 1992).

Under this scenario, socially open-ended problems (Shimada & Baba, 2015) emerge as an appealing and plausible way of providing students with the possibility of using, handling and interpreting data to inform decision-making, as well as with the opportunity to develop the skill of value awareness. The purpose of the present article is to clarify the instructional potential and advantages of socially open-ended problems set in a sport-related context by analyzing the features of a problem of this kind from a previous study (González & Chitmun, 2015). Moreover, in light of the aforementioned analysis, it is our purpose to propose this kind of problem as an instructional way to challenge students to structure variation among repeated observations of the "same" event, to model variability, and to make data-driven decisions. A lesson implementation of the aforementioned socially open-ended problem will be discussed and analyzed, and some conclusions will be given based on the results of the lesson implementation.

Decision-making process: Definition and related skills

A decision is the process within which a choice among specific options will be made, regarded here as having six phases: Definition, planning, data, evaluation, weighing impact, and making and justifying a decision (e.g., Arvai, Campbell, Baird, & Rivers, 2004; González, 2015). Some related skills are ability to identify, design, and choose optimal ways to make a decision; ability to seek, collect and process data relevant to a decision; and ability to give an informed justification for a decision made. Factors such as values hold by the decision-maker, communication and interpersonal interaction all influence the development of each phase of the decision-making process (Keeney, 1992).

Statistical investigation: Definition and type of problems to engage students in it

A statistical investigation is one of the four central aspects of statistical thinking used in statistical inquiry of authentic problems (Makar & Fielding-Wells, 2011). It is a process, comprised of the following five phases: Problem, Plan, Data, Analysis, and Conclusion (PPDAC). Since each phase of this process can be tied to a particular phase of the decision-making one, in the present article, the process of statistical investigation will be considered as being included under the most general process of decision-making (González, 2015; Keeney, 1992).

Teachers need to be able to plan and conduct statistical investigations that develop rich statistical understandings in their students, and one key issue to achieve this is using problems worthy of investigation, or to guide students to do so (Makar & Fielding-Wells, 2011). Such problems should have the following characteristics (Makar & Fielding-Wells, 2011, pp. 349-350):

Interesting, challenging, and relevant: Topics of interest include sport, weather, music charts, movies, and more serious topics of social issues relevant to teenagers (González & Chitmun, 2016). Challenging questions are those calling for a thorough analysis of the given data.

Statistical in nature: Problems posing questions calling for students to gather and interpret data and to justify their choices based on such interpretation. Appropriate interpretation of evidence requires simultaneous consideration both of one's knowledge about the domain and of the discernible patterns in the data (Lehrer & Schauble, 2004). The data related to the problem at hand must also offer enough complexity to generate interesting results.

Ill-structured and ambiguous: Questions such as "who is the best player?" or "which one is the best team?" raise the issue of what the meaning of "best" is, which enables negotiation and data-driven argumentation and discussion by students.

Socially open-ended problems

Socially open-ended problems (Shimada & Baba, 2015) are problems that are embedded in a real-life context, are familiar to the students and, by extending the traditional open-ended approach (Becker & Shimada, 1997), have been developed to elicit and address students' mathematical values (e.g., visual appeal, parsimony, efficiency, elegance and sophistication), social values (e.g., social responsibility, compliance with the law, human rights, fairness, compassion and equity), and personal values (e.g., persistence, integrity and friendliness) through modeling and argumentation. According to Shimada (2015, p. 11), this ability to address multiple values is one of the competencies expected to be developed in students by using socially open-ended problems in the mathematics classroom. Such ability requires the following skill-set: (1) The ability to build mathematical models—such as a formula, equation or system of equations describing how underlying factors are interrelated—based

on values, which is usually manifested in the first-half of the mathematics lesson; (2) the ability to appreciate the diversity of mathematical models based on values, which is usually manifested in the middle stage of the mathematics lesson; and (3) the ability to critically examine mathematical models based on values, which is usually manifested in the last-half of the mathematics lesson.

An example of socially open-ended problems: The "Hitting the target" task.

Shimada and Baba (2015) carried out a problem-solving lesson using the socially open-ended problem "Hitting the target" (see Figure 1).

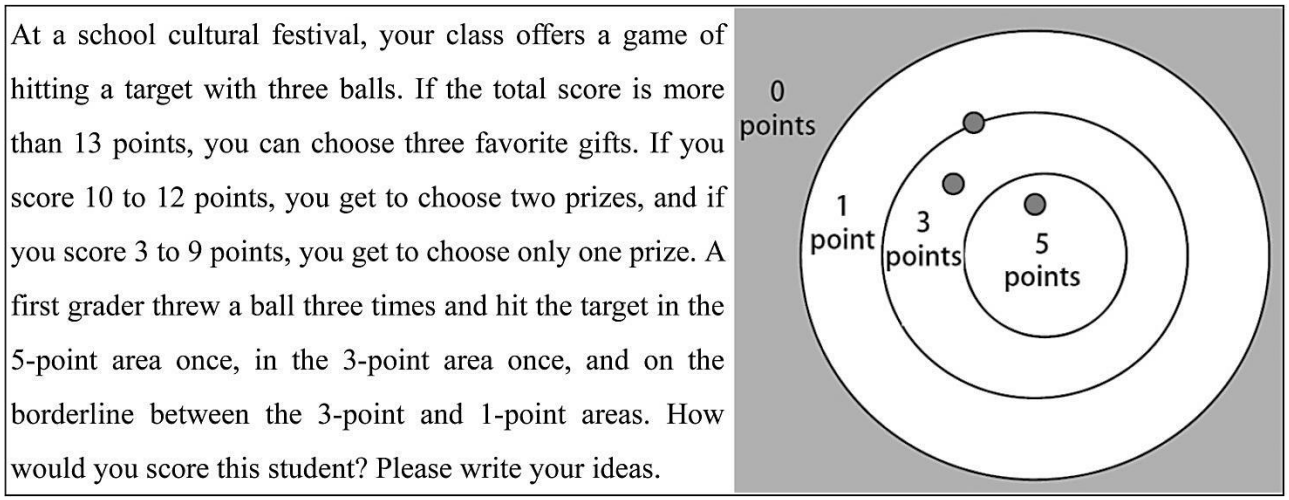


Figure 1: The "Hitting the target" task (Shimada & Baba, 2015)

"Hitting the target", which is a very popular game among children in Japan, is basically a darts game, but instead of throwing pointed darts at a concentric circles dartboard, small tennis-like balls covered with magnetized foam are used as a safety measure. Through engagement with the "Hitting the target" problem, it is expected from students to create rules in the form of mathematical models, in order to assign a score to the game performance of a participant after completing a set of three throws. The lesson planned by Shimada and Baba (2015) was deployed as follows: Provision of the problem; individual solutions by the students; whole-class presentation and discussion of students' mathematical models and reasons, and individual selection of one model with its reason at the end of the lesson. Each of these steps can be matched to a phase of a statistical investigation, and the problem itself seems to meet the criteria to be considered a problem worthy of investigation, or to guide students to do so (Makar & Fielding-Wells, 2011). Through the analysis of classroom interaction and students' data, Shimada and Baba (2015) identified four characteristics: (1) Diverse mathematical models with the same values; (2) Implicit values became apparent through comparison with other values; (3) Some students who transformed their initial values and those who did not; (4) Some students who changed their initial mathematical models under the same values.

Shimada (2015) and Shimada and Baba (2015) concluded that the decision of adopting a mathematical model for this particular problem was made based not just on the ball position on the target board, but also on the values (mathematical, social or personal) held by the students. This is in line with Keeney (1992), who stated that values are the primitive for considering any decision, since the first step in an effective decision-making process is for decision-makers to carefully consider their own values by clearly defining what it is they want to achieve in the decision context.

The "Darts game" task: Socially open-ended problems as potential tools for engaging students in decision-making and statistical investigations

Assessing sport outcomes (such as in the case of the "Hitting the target" task) has been discussed as potential way to build students' data-driven decision-making skills through statistical investigations (González & Chitmun, 2016). A similar (but more complex) problem to the one posed by the "Hitting the target" task is the "Darts game" task (González & Chitmun, 2015, 2016). This problem was the only one with a sport-related context posed by a Thai teacher—hereafter T1—from a sample of 15 secondary school mathematics teachers in the study conducted by González and Chitmun (2015) on professional knowledge of data-driven decision-making. In the present article, we will use the lesson plan designed by T1 (see Figure 2) as an example of a potential way of instructional design linking understanding of variability, modeling, statistical investigations and value awareness, all under the umbrella of the decision-making process.

The "Darts game" task—which can be considered a socially open-ended problem—represents an enhanced version of the "Hitting the target" task, with a more robust statistical nature. For example, the problem posed by the "Darts game" task will allow students to actively engage in the processes of generation, testing, and revision of real-life models of the world, which are at the very heart of what it means to think statistically (Lehrer & Schauble, 2004, p. 636). Furthermore, this problem focuses on the role of inventing measures of variability as a means for structuring variability. Inventing measures (statistics) of variability affords opportunities for coming to see differences among cases in new ways (Lehrer & Kim, 2009). As in the discipline of statistics, inventing statistics is not a solitary act: The meaning of a statistic for "team performance score" could be discussed and negotiated amongst students in a classroom community (Keeney, 1992; Lehrer & Kim, 2009; Petrosino, Lehrer & Schauble, 2003). Moreover, the formulation of a mathematical model—and scoring functions are mathematical models—can present an interesting challenge to the students.

In the "Darts game" task, unlike the "Hitting the target" problem, landing a dart on any region of the dartboard was not assigned a pre-established score (adding so ill-definition and ambiguity to the problem). So, by interpreting the patterns of the thrown darts to determine which team is the best one, students will engage in data modeling. Characteristics of distribution, like center and spread, could be made accessible and meaningful to students by displaying and structuring variation among observations of the "same" event, instead of considering such observations simply as a collection of differences among measurements (Lehrer & Schauble, 2004; Petrosino et al., 2003). In the "Darts Game" task, students will be required to structure variation among observations of the "same" event. The outcome of that process will be a mathematical model for scoring each team's performance, in order to make a decision regarding which team is the best one. Such models will be a source of information on students' structural behavior, observations, in situ measurements, and values.

According to Kazak (2006), engaging students in scoring a dart game will provide them with opportunities to engage in analysis concerning scores assigned around the target point, as well as in a discussion about what to do to get the highest score.

• **Intended grade :** Grade 9

• **Main topic :** Measures of central tendency (this class would be the final one on this topic)

• **Lesson objectives :**
1. Students will be able to bring out mathematical knowledge in order to make a decision.
2. Students will be able to understand and differentiate among the meanings of mean, median and mode.
3. Students will be able to explain their own ideas in terms of their mathematical knowledge.

• **Learning process :**

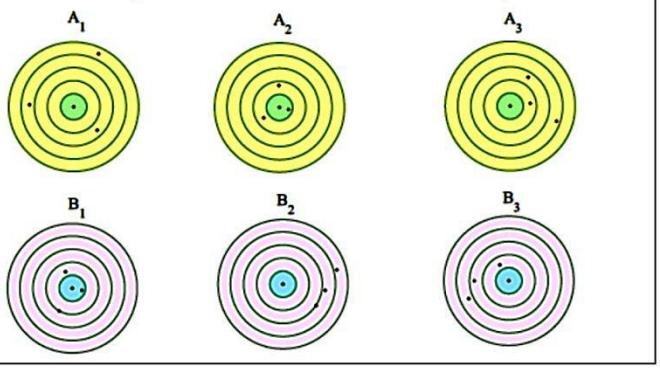
Phase	Instructional Activities	Instructional Remarks
Introduction	1. The teacher will pose the following problem to the students, and will give them time to understand the situation and the problem: From darts game: There are 2 teams (A and B) competing against each other, and each team have 3 persons (A_1, A_2, A_3 and B_1, B_2, B_3). Each person throws a dart 3 times. The positions where the darts landed are shown in the figure below. Which team is the winner? Please explain. 	1. The teacher will provide students with an exercise sheet with the problem. 2. The teacher will stick a larger version of the exercise sheet on the blackboard to display it for the whole class to see. 3. Students will be instructed to write individually as many answers as possible to the problem on the exercise sheet.
Development	1. Students will try to find as many answers as possible to the problem at hand, will justify their decisions, and will write down their answers and justifications on the exercise sheet. 2. After students finish answering the problem, they will be organized in groups, with each having 3-5 students. Every group will discuss the problem within its members. This discussion should lead to find the conclusion of the group regarding the problem at hand. 3. A representative of each group will show the group answers' conclusion in front of the whole class. • Some expected answers from the students are the following: ① Assign a score to each region on the dartboard (e.g., from 5 points for landing a dart in the innermost region of the dartboard, to 1 point for landing a dart in the outermost one), and consider team performance score as the total sum of points scored by team. ② Consider team performance score as the result of adding the total distance from the center to each of the 9 marks resulting from the team member's dart throwing trials (A lower score represents a better performance). ③ For each dart throwing trial within a team, identify the one with the shortest distance to the center, and consider team performance score as the result of adding the total distance from the center to each of these 3 marks (A lower score represents a better performance). ④ For each dart throwing trial within a team, identify the one with the largest distance to the center, and consider team performance score as the result of adding the total distance from the center to each of these 3 marks (A lower score represents a better performance). ⑤ Consider team performance score as the result of calculating the average distance to the bullseye of all the 9 marks resulting from the team member's dart throwing trials (A lower score represents a better performance). ⑥ Consider team performance score as the median distance to the bullseye of all the 9 marks resulting from the team member's dart throwing trials (A lower score represents a better performance). ⑦ Assign a label to each region on the dartboard (e.g., from "a" for the innermost region, to "e" for the outermost one); count, tally and compare the frequencies of darts landing on each region for each team; and consider team performance score as a function of this comparison of tallies obtained in each region by each team. ⑧ Group players to compete in singles matches (e.g., A_1 vs. B_1, A_2 vs. B_2, and A_3 vs. B_3), and consider the team with better performance the one with the largest number of singles matches won. 4. After all the presentations are complete, both teacher and students together will try to categorize the presented answers by mathematical knowledge.	1. Teacher will walk around the class while students work both individually and in groups, and will write notes, focusing on the strategies used in answering the problem. The teacher will keep these notes with herself. 2. The teacher will write down the answers while students present their group conclusions in front of the whole class.
Conclusion	1. Teacher and students together try to find conclusions of this lesson.	1. The students will write down the conclusions to their notebooks.

Figure 2: Lesson plan designed by T1 to engage students with the "Darts game" task

T1 did not mention explicitly the role of values in her lesson; however, as a socially open-ended problem, the "Darts game" provides an opportunity to develop students' ability to address multiple values through the implementation of the ability to address multiple values (Shimada, 2015).

Research methodology overview: Implementing the "Darts game" task

Study sample and research methodology: Overview

On November 16, 2016, the second author carried out a problem-solving lesson using the "Darts game" task with Grade 12 students in a public high school in Bangkok. The sample consisted of 34 students, with18 boys and 16 girls. The second author is a teacher specialized in mathematics education, with 13 years of teaching experience. In this study, almost the same lesson design depicted in Figure 2 was implemented, with two main modifications: Firstly, before lesson implementation, students were asked to find out as many answers as possible to the task as a homework; and secondly, during the presentation and discussion of the individual solutions and reasons, students were not organized in groups, following the lesson design presented by Shimada and Baba (2015).

Findings and discussion of results

In average, about two answers were provided per student, being the mode one solution (21 students, 61.8%), and the number of given solutions ranging from one to seven. Among those answers, 10 different decision models were identified (see Table 1). Since students were allowed to provide more than one decision model, the total will be more than 100 per cent.

	Model description	Frequency (%)
Decision model	M_1: Tallying the number of darts landing in each concentric circle. The winner team will be the one with more darts landed closer to the innermost circle.	9 (26.5)
	M_2: Tallying the number of darts landing on the outermost circle.	1 (2.9)
	M_3: Determining how clustered the three thrown darts are.	5 (14.7)
	M_4: Tallying the number of darts landing on a line.	10 (29.4)
	M_5: Measuring the distance of the farther shot to the bullseye.	3 (8.8)
	M_6: Developing a scoring function	21 (61.8)
	M_7: Tallying the number of times a dart hit the bullseye.	3 (8.8)
	M_8: Counting the times a player landed the darts inside the same concentric circle.	1 (2.9)
	M_9: Tallying the number of darts landing on the innermost circle.	7 (20.6)
	M_{10}: Measuring the total distance to the bullseye	1 (2.9)

Table 1: Different decision models posed by the students in this study

The reasons given by students during the phase of individual solutions revealed a diversity of considerations to decide the winner team: taking into account all the members of a team (21 students, 61.8%) vs. only considering a team representative (7 students, 20.6%), or posing decision models considering both perspectives (6 students, 17.6%). Among the reasons for choosing particular models in the phase of individual solutions are the following: difficulty in landing the shot in a particular area of the dartboard (e.g., on a line or on the innermost circle); precision (i.e., how tightly clustered, or spaced apart, the thrown darts are among each other in a particular trial, regardless of their position relative to the bullseye); accuracy (i.e., how close the thrown darts are to the bullseye); fairness and consideration to the whole team; and maximizing the likelihood of winning by selecting the best team player under a particular condition. Students mainly used tables while tallying, and calculated frequencies and modes. As expected, multiple scoring functions emerged from the students' answers. Although different scores were assigned to landing a dart on a particular circle region, most of the scoring functions assigned the highest score to the innermost circle, and the lowest score to the outermost one. Landing a dart on any line was also scored in multiple ways: averaging the score of

adjacent circle regions; assigning either the highest or the lowest score of adjacent circle regions; a fixed score (e.g., 10 points); no points; or invalidating the trial results for the player.

After engaging in the stage of whole-class presentation and discussion of the mathematical models and reasons, some students changed the decision model from the initial self-resolution stage in the final selection stage (7 students, 20.6%), while most of the students kept one decision model within those posed by them during the self-resolution stage (27 students, 79.4%). Five out of seven students who changed their initially-posed model somehow modified it, introducing aspects that emerged during the whole-class discussion (e.g., scoring by team instead of considering a team representative, and vice versa; considering landing a dart on the line as disqualification). The reasons to make a final decision were also varied: acknowledgement of the effort by all team members; fairness; establishing equal winning opportunities for both teams; easiness to either score or explain the final decision to the team members; and keeping the nature of the game (i.e., the shot closer to the bullseye wins).

Conclusions

All in all, students engagement with the socially open-ended problem "Darts game", set in a sport-related context and involving measurement of different observations of the "same" event, seemed to be an appealing and plausible way to, among other things, (1) make accessible and meaningful to students characteristics of distribution such as center and spread; (2) provide students with opportunities to structure variation and coordinate variability and chance by engaging actively in modeling challenges; (3) develop an aggregate view of data; (4) provide students with an opportunity to discuss, argue and negotiate the meaning of a statistic for "team performance score" as members of a classroom community; (5) actively engage students in the processes of generation, testing, and revision of real-life models of the world; and (6) develop students' value awareness in a climate of open discussion, by implementing the threefold ability to address multiple values. So, in a society in which the role of values, data and decision-making is fundamental for both education and active engagement in critical citizenship (Ernest, 2001), implementing sport-related socially open-ended problems in the mathematics classroom to address statistical contents seems to be a plausible way to help teachers achieve the aims of the mathematics curriculum regarding statistics education.

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Insights into the approaches of young children when making informal inferences about data

Aisling M. Leavy

Mary Immaculate College, University of Limerick, Ireland; Aisling.Leavy@mic.ul.ie

There is growing awareness of the statistical reasoning abilities of young children. In this study the informal inferential reasoning skills of a class of 5-6 year old children are examined as they reason about data in the context of a week-long data investigation unit. The strategies young children use to make predictions about data are identified. A discussion ensues around what these strategies communicate about early understandings of statistical inference. The findings suggest that making inferences from data can be challenging for younger students primarily due to the powerful influence of their developing understandings of number. However, there is evidence that children possess some of the building blocks of informal inference most notably in the approaches that point to a pre-aggregate view of data. Situating data investigations within interesting and relevant contexts, alongside good teacher questioning and opportunities to listen to the reasoning of their peers, contributes to the creation of statistical environments that support and develop early understandings of inference.

Keywords: Informal inferential reasoning (IIR), early childhood, data modelling.

Theoretical perspective

Informal ideas relating to inference are those understandings that are foundational to the development of inferential reasoning. While many different definitions of informal inference have been posited, a useful definition of informal inference is "the way in which students use their informal statistical knowledge to make arguments to support inferences about unknown populations based on observed samples" (Zieffler, Garfield, delMas & Reading, 2008, p. 44). Zieffler et al. (2008) identify three components of an IIR framework as: making judgments or predictions, using or integrating prior knowledge, and articulating evidence-based arguments. Arising from research with primary students, Makar & Rubin (2009, p. 85) propose three principles that are considered essential for informal statistical inference as '(1) *generalization*, including predictions, parameter estimates, and conclusions, that extend *beyond describing the given data*; (2) the use of data *as evidence* for those generalizations; and (3) employment of probabilistic language in describing the generalization, including informal reference to levels of certainty about the conclusions drawn.'

One statistical perspective identified as a necessary building block to form a basis for IIR is the ability to *view data as an aggregate* (Rubin, Hammerman & Konold, 2006). Statistical properties of aggregates such as their centers, variability and shapes emerge from attending to features of distributions rather than features of individuals. Thinking about aggregates, while possible, has been shown to be challenging for children (Cobb 1999; Hancock, Kaput & Goldsmith, 1982). Recent work by Konold, Higgins, Russel & Khalil (2015) has resulted in the identification of four perspectives that students use when working with data. The use of these perspectives as a way to analyse an individual's particular interpretation of data may provide valuable insights into their statistical reasoning and in turn the extent to which they possess the

necessary building blocks for informal inference. These perspectives include *data as pointer* (focus on the event rather than the data), *data as case value* (focus on individual data values or cases), *data as classifier* (identifying subsets of data values that may be the same or similar) and *data as aggregate* (view all the data values in aggregate as an "object" or a distribution).

A study of first-grade student's data modelling approaches carried out by English (2012) categorised children's predictions and approaches when working with data using the lenses identified by Konold et al. (2015). Using this framework to guide categorization, 6-year olds in English's study viewed data in a variety of different ways. Many children focused on the values of particular cases (*case value lens*) and others demonstrated the ability to consider the frequency of cases with a particular value (*classifier lens)*. There was also evidence of what English (2012) terms a *pre-aggregate lens* which included approaches that considered all the data, compared the frequencies and had some attention to overall trends. While not as sophisticated as an *aggregate lens,* which involves consideration of the entire distribution as an entity in itself, the presence of this pre-aggregate lens is a strong indicator of the nascent potential of young children to engage in informal inferential reasoning.

A number of studies have explored the reasoning abilities of young children when engaged in data modeling activities in environments supported by the use of picture books (English 2010, 2012; Kinnear 2013, 2016) and data-visualization tools and technologies (Ben-Zvi, 2006; Paparistodemou & Meletiou-Mavrotheris, 2008). This study continues this line of research by exploring the informal inferences young children make about the data presented in a data modelling environment and examining what these inference tell us about children's perspective on data.

Method

A multitiered teaching experiment (Lesh & Kelly, 2000) was carried out with twenty-five 5-6 year olds as they engaged in a weeklong data modeling activity. Statistical activity was motivated by a driving question (Hourigan & Leavy, 2016) and the context was the 'design of a zoo' as it was familiar to children and incorporated opportunities to work with data, encourage exploration of variation and make predictions about the data. The Ertle, Chokshi, & Fernandez (2001) lesson note format, developed for use as part of Japanese Lesson Study, guided lesson design considerations. This framework promoted a focus on expected student reactions, concomitant teacher responses and evaluation strategies. These foci supported examination of students' ability to engage in IIR. This study examines the final lesson which focused on making informal inferences about data.

The inquiry was stimulated by playing a video excerpt that we produced:

> Hi, I am James the zoo keeper. The elephant's home in the zoo is getting a little bit crowded. I think we need to make it a bit bigger. But, I don't know how many elephants will be in the zoo next year which makes it difficult to plan ahead. I was hoping you could look at the numbers of elephants in the zoo for the last four years, and predict how many will be there next year?

Children were then shown live video feed of the elephant enclosure in Dublin zoo and presented with a table of data illustrating the number of elephants born in the first year (3 elephants), second year (4 elephants), the third year (7 elephants) and fourth year (5 elephants) (see figure 1). Children worked in

groups of 5 to reason and make predictions about the number of animals born the following year (year 5). Following the predictions, other data relating to the birth rates of wolves [5, 6, 2, 3], giraffes [8, 8, 5, 5] and monkeys [3, 5, 0, 2] across four years were presented. Children worked in groups and predicted the number of respective animals born in the fifth year. The design of these tasks was informed by the Zieffler et al. (2008) framework to support IIR by challenging students to make predictions and judgments about data and by incorporating opportunities to capture students' informal inferential reasoning.

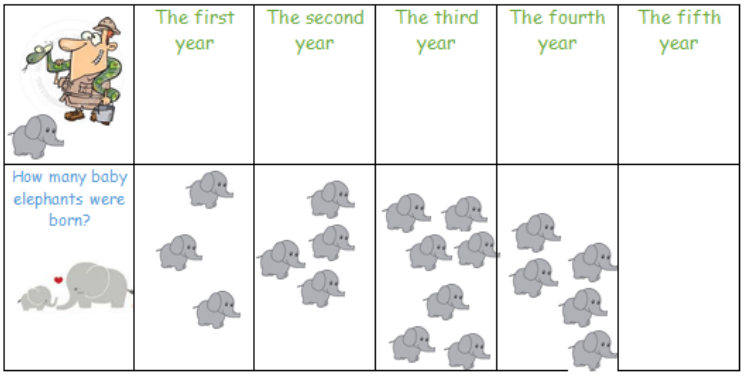

Figure 1: The 'elephant birth task'

Conversations in 4 of the groups were audio recorded and one group was video recorded. Our primary focus when analyzing the data was on identifying the ways in which young children make informal inferences in a context rich data modelling environment.

Recordings of group conversations were transcribed. Transcripts were coded according to whether they embodied Makar & Rubin's (2009) principles of IIR. Thus each transcript was coded at least three times in an effort to identify children's ability to generalize beyond the data, to use data as evidence and to use probabilistic language. All predictions were further categorized as representing data as pointer, case value, classifier or aggregate perspectives on data (Konold et al., 2015).

Findings

Children understood the task and were enthusiastic when making predictions about animal births in Year 5. However, making data informed predictions was challenging for some. Initially there was some evidence of idiosyncratic reasoning that was distanced from the context and from the data presented, however, this soon disappeared once the data and context were discussed further. Many children based their predictions on their knowledge of the context and modified their prediction based on discussion with peers. The findings are structured using the three principles of IIR that are considered essential for informal statistical inference (Makar & Rubin, 2009).

Principle 1: Generalizations beyond the data

While all children made predictions regarding the number of births, not all of the predicted values indicated an ability to generalize beyond the data. Rather, they reflected the influence and power of counting in the mathematical development of the young child. For these children, there was an awareness of frequencies and this was demonstrated in the tendency to list the numbers, order them and then compare the outcome

to the counting numbers. This focus on the frequencies resulted in two approaches to predicting births. The first approach was to *fill in the gaps*. Children compared the frequencies to the sequence of counting numbers usually leading to the identification of a 'gap' in the list of numbers. Children were eager to fill this gap i.e. identify a count/frequency that hadn't occurred in the presented data and avoid presenting a value that had already occurred. Thus, they believed that this missing number would likely be the number of animals born the next year (see discussion between Sheena and Ayesha below around wolf birth rate [5, 6, 2, 3]). The second approach was to *extend the number sequence*. In these situations children were not overly perturbed by an identified gap in the counting sequence and chose instead to extend the numbers beyond the range of the presented data (see Kate below). Generally, the next highest counting number above the upper value of the range was their prediction for the number of births in year 5. Both these strategies indicate a focus on pattern in the sense of ordinal counting numbers and thus the 'power of counting'. However, from a statistical sense the reasoning was located and justified within the world of counting numbers thus indicating a lack of focus on pattern and trends in the data.

Teacher: How many wolves will be born this year (pointing to year 5)?

Sheena: We say maybe 4 cause 5, 6, 2, 3. And there's no 4.

Ayesha: We've got our reason. 2, 3, 4, 5, 6. It's 4 cause 2, 3, 4, 5, 6.

Teacher: What would happen if there was a year 6? How many animals might be born then?

Kate: 1. Cause it would start 1, 2, 3, 4, 5, 6.

Principle II: Using data as evidence

Analysis of the transcripts revealed an abundance of situations when children *used data as evidence* to support their predictions and conclusions about data. The explanations provided by children were categorized as falling within one of the four perspectives on data posited by Konold et al. (2015).

Observation 1: The prevalence of a *case value lens*

The focus on individual data values indicated the presence of a *case value lens*. In particular, children were attuned to the appearance of zero births for year 3 in the monkey data [3, 5, 0, 2] and commented 'there were none that year' and 'there are zero there'. While this case value lens indicates a lack of focus on the aggregate, the individual data values were considered within the greater data context. For example, Eva drew on her knowledge of the context in her efforts to explain why no monkeys were born in year 3 when she stated 'because if they had too many babies there [pointing to the 5 born in year 2], the mommy babies would have to rest all day'.

Young children's approaches involving summing data values and calculating totals have been used as indicators of a case value lens (English, 2012). Similarly, in this study, several predictions of the births in year 5 also indicated a case value lens as they were based on summing all or some of the values and presenting this total as the prediction for year 5. Matthew predicted 'I decided there will be 10 monkeys altogether born' based on summing the births in years 1-4, and Kornelia predicted that 16 wolves would be born in year 5 'because I counted all of them'. This difficulty in attending to the variation in the data was also evident in another child's response that 10 giraffes [8, 8, 5, 5] would be born because '5+5 makes 10'.

However for many of these same children, while there was a focus on counting and the application of a *case value lens*, there was an awareness of pattern in magnitude of numbers. When large numbers were presented as predictions, children rejected these numbers as too big and drew on contextual information to justify their reasoning. In the following segment, children are predicting the number of giraffes [pattern: 8, 8, 5, 5] that will be born in year 5.

Thomas: I think 85. Because there is an 8 there and a 5 there.

Polina: They wouldn't fit into the box. They are definitely not going to fit into the zoo also.

Teacher: Really? Why do you think that?

Polina: Because they [giraffes] are very big.

Observation 2: Awareness of trends in the data and evidence of a *pre-aggregate lens*

The responses of 20% of children suggest an awareness of overall patterns and trends in the data. This was termed a *pre-aggregate lens* by English (2012) and may point to some emerging sense of distribution. The awareness of pattern was evident in Mia's response to the wolf births which she described as 'going up and going down'. Similarly the recognition and subsequent extension of a repeating pattern in the giraffe data set [8, 8, 5, 5] was evident when Melios predicted '8. It's 8, 8, 5, 5, 8. Cause it's a pattern: 8, 8, 5, 5, 8'. During a whole class discussion about the number of monkeys that would be born in year 5, awareness of patterns was evident in the comments from Otille and Kate below:

Teacher: How many monkeys did you think were born in year 5? [3, 5, 0, 2]

Otille: I think 1 because it goes down, up, down, up, down.

Kate: 5. Cause 5 here [points to 5] and then low [points to 0 and 2] so it would go back to high.

As can be seen from the transcripts above, children's justifications did not explicitly refer to the context of the data (in this case birth rates) and hence there is the possibility that this awareness of trends stemmed more from an algebraic rather than statistical perspective. However, the greatest indication of the presence of a *pre aggregate lens* was in the reasoning of those children who married an awareness of trends in the data with their understandings of the context in constructing their predictions. In the discussion of the trends in elephant births [3, 4, 7, 6] Polina imagined that animals born in year 1 would have grown up by year 5 and be giving birth to elephants in year 5.

Polina: We put 8 elephants (born in year 5)

Teacher: Why did you put 8 in?

Polina: Because I think these are going to grow up [pointing to the 3 elephants born in year 1] and these ones will be in their tummies [pointing to her prediction for year 5]. It is always going to get bigger.

Teacher: So do you think it will always get bigger?

Polina: Yes, I think so, I think there will be babies born from these ones. These ones are going to be all grown up, they will be adults.

Thus her understanding of the variation in the data influenced her predictions and ensured she always predicted beyond the upper range of the presented data. Another child, Anna, demonstrated her ability to view the trends across the years and used this trend to inform her initial prediction. However, she

subsequently used her knowledge of the context and adjusted her prediction downwards. Here, Anna is discussing her prediction for the number of giraffes born [8, 8, 5, 5] and her initial prediction of '3' may indicate some developing notion of center. However her attention to the context makes her mindful of how her prediction (if it were correct and acted upon) would affect the other animals in the zoo and she adjusts her predict to 'protect' other animals from the negative outcomes arising from her prediction.

Anna: It was different on different years sometimes 5 [pointing to the values for year 3 and 4] but here and here it was 8 [pointing to the year 1 and 2]. So I think 3.

Teacher: Why 3?

Anna: Cause it is like the others. Not too many (baby giraffes) but not none (baby giraffes).

No. No. I think 2. Because if there are too many, all of the branches and the leaves would be gone and there would be no place for a monkey.

Principle III: Use of probabilistic language

Makar & Rubin (2009) emphasize the importance of expressing uncertainty when making inferences – this can be identified in efforts to avoid deterministic claims and in the use of probabilistic language. Analysis of the transcripts reveal that children drew conclusions based on the data presented to them (birth rates over time) and used this data to make predictions beyond the data. All the while they were articulating uncertainty as demonstrated in their use of terms such as 'I think' (see Thomas, Otille, Polina and Anna above), 'probably' 'maybe' (see Sheena above) and 'I'm not too sure'. It is particularly interesting to note that children were comfortable with uncertainty and with the different predictions of others. This openness was evident when Eva pointed out 'we don't know' in relation to how many elephants the mother elephant would have. Her partner Paul continued the reasoning and stated that 'maybe there would be six elephants born because there are 6 elephants there and they could have 6 babies'.

Conclusions

Young children in this study demonstrated the seeds of informal inference in their ability to 'look beyond the data' and engage in data-based argumentation to support their predictions. However, making data-based predictions was a challenging task for some children. Case value perspectives were most prevalent. The lack of repeating data values in the presented data may account for the low incidence of classifier perspectives as compared to the study by English (2012). Similar to English's study there was evidence of the presence of a pre-aggregate lens in the approaches taken by children. A large proportion of children scanned the data for patterns, sought 'missing numbers' and many made predictions based on patterns in the ordered lists of data rather than thinking from a statistical perspective. This reliance on number and algebraic reasoning is not surprising given the curricular emphases in early years mathematics curricula. It is interesting to note the influence of zero on children's deliberations about data was also a factor in the work of Kinnear (2013) and Kinnear & Clarke (2016) when engaging young children in data modeling activities.

The success that some children experienced in making informal inferences was due to a number of factors. The role played by the data and task context is particularly evident. The use of an interesting and relevant context provided a 'crutch' for the children when making predictions. Their personal experiences and high task interest ensured that rather than reasoning about decontextualized data, children were reasoning about and making sense of the situation at hand – this supported their inferences. Secondly, the development of skills in making data-informed predictions was due in large part to the use of good questioning on the part

of teachers and due to their efforts in drawing children's attention to aspects of the data and clarifying misunderstandings as they occurred. Similarly, the work of Paparistodemou & Meletiou-Mavrotheris (2008) highlighted the important role that prompting by the researcher played in supporting children in speculating about larger data sets. The third factor was the importance of peer interactions. Children built on the ideas of others as they reasoned and made prediction within their groups thus providing evidence for the power of co-constructing meaning in small groups and demonstrated 'building on the ideas of others' (Whitin & Whitin, 2008, p.93). This importance of peer interaction in promoting inference and deriving conclusions from data was also evident in the work of third grade students when engaging in inference (Paparistodemou & Meletiou-Mavrotheris, 2008).

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Students' conception of a sample's relation to the population when constructing models of resampling

Jeffrey A. McLean[1] and Helen M. Doerr[2]

[1]St. Lawrence University, United States; jmclean@stlawu.edu

[2]Syracuse University, United States; hmdoerr@syr.edu

In this study, we examined the models of resampling that a group of students constructed in order to use one sample to make informal inferences about a population of data. Students participated in a model eliciting activity which aimed to elicit the resampling process of bootstrapping. We will discuss the model of resampling and inference constructed by one group of students, the factors which led to the group constructing a model of resampling with replacement, and how these students' conceptions of the sample related to their model. We suggest that the students' conception of the relationship between the sample and population was a key factor in constructing a model of resampling and inference similar to the method of bootstrapping.

Keywords: Statistics education, modeling, informal inferential reasoning.

Introduction

Bootstrapping is one method of simulating data through resampling that has become an important tool for statisticians, who suggest that it is intuitive to novice statistics students (Lock, Lock, Lock-Morgan, Lock, & Lock, 2013). The term bootstrap is used in the idiom to pull oneself up by one's bootstraps, which means to accomplish a goal with the resources on hand. Rather than take many samples to estimate a sampling distribution, the bootstrap method uses the one sample on hand to accomplish the same goal. Bootstrapping is a process of simulating data beginning by drawing one sample from a population. Resamples are constructed by choosing elements from the original sample, one at a time with replacement, until as many elements are drawn as in the original sample. This new sample is called the bootstrap sample. The process is repeated many times to create a collection of such bootstrap samples. A statistic from each of these bootstrap samples is aggregated to form an empirical bootstrap sampling distribution, which can then be used to make inferential claims about the population from which the original sample was drawn.

Bootstrapping was introduced by Efron (1979) as an alternative to earlier resampling methods. He asserted that the bootstrap was more widely applicable and dependable than earlier resampling methods, while also using a simpler procedure. New curricula for introductory statistics courses at the secondary and tertiary levels have been created which focus on simulation and resampling methods such as bootstrapping (Garfield, delMas, & Zieffler, 2012; Pfannkuch, Forbes, Harraway, Budgett, & Wild, 2013; Tintle, VanderStoep, Holmes, Quisenberry, & Swanson, 2011). Pfannkuch et al. (2013, p. 2) asserted that "the bootstrapping and randomisation methods using dynamic visualisations especially designed to enhance conceptual understanding have the potential to transform the learning of statistical inference." Lock et al. (2013) claimed that bootstrapping capitalizes on students' visual learning skills and helps to build students' conceptual understanding of key statistics ideas. While these curricula and subsequent research have examined students' understandings of the use of method of bootstrapping, studies have not examined how students construct and develop these understandings. We suggest that this lack of research on student

understandings of eliciting the concept of bootstrapping is a gap in the research literature. In this study we examine aspects of the model of resampling and bootstrapping that one group of students constructed in order to make informal inferences about a population of data and these students' conception of the sample's relation to the population. The research question driving this study was what factors led students to construct models of resampling with replacement.

Review of literature

The current research literature has viewed the bootstrapping method through the lens of formal inferential reasoning, such as using the method as a means to estimate standard errors and construct confidence intervals (Garfield, delMas, & Zieffler, 2012; Pfannkuch, Forbes, Harraway, Budgett, & Wild, 2013). In this study, rather than focus on students' understandings of how this method could be used for formal inference, we focused on how students constructed and developed methods similar to bootstrapping and made informal inferential claims from the resulting empirical bootstrap distributions. Key to eliciting students' models of resampling and inference in this study was understanding the use of students' informal inferential reasoning to make claims about a population. Informal inferential reasoning is the drawing of conclusions from data that extend beyond the data, from viewing, comparing, and reasoning with distributions of data (Makar & Rubin, 2009; Pfannkuch, 2007).

In order to make informal inferential claims in this study, groups of students first needed to determine a method of simulating data. Saldanha and Thompson (2002) explored students' conceptions of a sample in relation to the population while the students participated in an activity which constructed an empirical sampling distribution by collecting repeated samples from a population. Students' conceptions of a sample in relation to the population and sampling distributions were categorized as either additive or multiplicative. Those with an additive conception of the sample only viewed the part-whole relationship between the sample and the populations, with multiple samples representing multiple parts of this whole. The resemblance and relationship between the sample and population distributions was not a factor for those with this conception. Those with a multiplicative conception of the sample viewed the sample as a "quasi-proportional, mini version" (p. 266) of the population. The sample can be used to approximate the distribution of the population, with an understanding that various samples' distributions may bear more or less resemblance to the distribution of the population. We posit that this multiplicative view may be critical to understanding the use of bootstrapping for informal inferential reasoning, since the proportion of each element in a sample is assumed to represent the proportion of those elements in the population.

The focus of analysis for this study was the model of resampling and bootstrapping which participants constructed and developed while engaged in a model eliciting activity (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Models are "conceptual systems … that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)" (Lesh & Doerr, 2003, p. 10). Model-eliciting activities encourage students to generate descriptions, explanations, and constructions in order to reveal how they were interpreting situations. Model eliciting activities are designed in order for students to:

- Make sense of the situation drawing on both their school mathematics real-life sense-making abilities;

- Recognize the need to construct a model to complete the activity, rather than produce only an answer.
- Create documentation that shows solution paths, patterns, and irregularities that the students considered while constructing their model;
- Assess when their responses need to be improved, refined, or extended,
- Create models that can be extended to use in a broader range of situations (Lesh, Hoover, Hole, Kelly, & Post, 2000).

By using a modeling approach to examine student reasoning, we viewed reasoning as dynamic and developing over the course of instruction. Observing students as they participated in a model eliciting activity allowed us to view the construction and development of their thinking of resampling and inference.

Design and methodology

This study is a qualitative case study and part of a larger study (McLean & Doerr, 2016) that consisted of an eight-class-session teaching experiment that was enacted in four introductory statistics classes at the high school and community college levels (n=68) in the United States. This study focuses on the student reasoning that developed during one class-session as a group of four students from a community college participated in a model eliciting activity that aimed to elicit the method of bootstrapping. During the model eliciting activity, we collected written classwork from all participants and videotaped the group of students in order to document the group's model construction and development. We analyzed the videos and written classwork for evidence of informal inferential reasoning, quantities used and the relationships between these quantities, and the representations and explanations used in their arguments in order to reconstruct the development of the reasoning that were created by the participants.

The group of students participated in a model eliciting activity, using hands-on manipulatives, where they were given one sample from a population and constructed a model of resampling and inference in order to make claims about a population. The activity was designed to elicit a model that could be used to resample from a sample, with replacement, in order to construct a bootstrap sample of the same size as the original sample and use the distribution of these bootstrap samples to determine which outcomes are most likely to occur. The model eliciting activity asked groups of students to help a manager in a grocery store predict the percentage of peanuts in a certain brand of mixed nuts. The students were given a sample of mixed nuts in the form of seven craft sticks marked with a 'P' for peanut and 18 not marked to represent other kinds of nuts. The model eliciting activity prompt and manipulatives are shown below (Figure 1).

> [Grocery store] carries many types of nuts, dried fruits and candies in their bulk food section. The manager of bulk food is always interested in bringing new types of food for her customers to try. She recently ordered a sample of a new brand's mixed nuts. From past experiences, she has determined that customers prefer mixed nuts with fewer peanuts. She plans to order a large shipment of mixed nuts and is considering this new brand. Before she orders, the manager wants to know more information about the percentage of peanuts in this new brand. From this one sample of mixed nuts, the manager has asked that you determine a likely range for the percentage of peanuts in the entire brand of mixed nuts. She would also like to know the methods that you

develop to come to your conclusion. Your methods may of use for future bulk food purchases. The bag of sticks is the sample of mixed nuts. Sticks marked with a 'P' are peanuts. Those not marked are other types of nuts.



Figure 1: Model eliciting activity manipulatives

Prior to the model eliciting activity, the group of students participated in a model development sequence which elicited and developed models of repeatedly sampling from a population, constructing an empirical sampling distribution, and making informal inferential claims regarding the population. In this model eliciting activity, students could no longer repeatedly sample from a population in order to construct an empirical sampling distribution and needed to construct a new model of resampling and inference that could use only one sample from the population to make informal inferential claims.

Results

In the larger study from which this data was gathered (McLean & Doerr, 2016), groups of three or four students constructed two categories of models for making inferential claims of a population from one sample. The first category of model (n=16 groups of students) treated the sample of 25 nuts in a manner similar to a population and collected resamples without replacement, from the 25 nuts. The second category of model (n=4 groups of students) discussed or collected resamples from the sample of mixed nuts in a manner which preserved the makeup of the sample while resampling. We will report findings from one of the four groups who constructed this second category of model. This group constructed a model of resampling with replacement similar to the method of bootstrapping. We will discuss two findings of student understanding that were key to constructing these models: the representativeness of the sample to the population; and a method of resampling that preserved this representativeness. Findings like these were unique to the groups of students who constructed the second category of model of resampling and inference, which preserved the make-up of the sample while resampling.

Representativeness of the sample

Before discussing methods of resampling, the group of students first discussed how they believed that one sample could be used to make inferential claims and how their sample of mixed nuts related to other possible samples of mixed nuts taken from the sample population

Susan: I would say from this sample, that a little over a quarter of the peanuts, of the nuts are peanuts, from our random sample.

Randy: But it's only one sample.

Ted: But this is the only sample we have.

Brenda: So if you pick another random sample what's going to happen?

Ted: It's most likely going to change.

Susan: It's going to change, but I feel like it will probably be still about the same.

Susan made an assertion about the percentage of peanuts in the population under the condition that she was basing her assertion on the random sample, which Randy stressed is only one random sample. Susan emphasized that although other samples of nuts will be different from their one sample, they will probably be "still about the same". Susan demonstrated an understanding that the percentage of peanuts in the sample likely represents the percentage of peanuts in the population. This is a key aspect of inferential statistics. When making inferential claims you take for granted that the sample likely represents the population because as Ted stated, "this is the only sample we have" and as Susan asserted, "it will probably still be about the same" as other samples.

Preserving the representativeness of the sample when resampling

The group initially decided to take a resample of 14 mixed nuts by drawing one at a time from the bag of mixed nuts, without replacement. The group only took one sample with this method, which yielded five peanuts out of 14 nuts. The instructor then came back over to the group to discuss how the group had collected the sample of 14 nuts.

Instructor: So how are you choosing those?

Susan: He [Randy] randomly puts them together, and then I randomly without looking draw them out.

Instructor: Okay. So you're drawing out one at a time?

Susan: Uh-uh.

Instructor: Okay, and you're setting it on the table?

Ted: Yes.

Instructor: Then you're going back in and you're drawing another one?

Susan: Yes.

Instructor: Okay, so …

Ted: Ohh! Wait, you said that you're putting them on the table. Was that like, do you think that we should put them back in the bag after we draw it out? Like for probability simulators?

From this exchange Ted considered how the sampling would change if they resampled with replacement instead of resampling without replacement. By using the term "probability simulators" he was approaching the idea that if you do not replace the stick after choosing each one, the probabilities of choosing a peanut or another nut will change. This term was not used previously in the coursework for this class, but likely came from his earlier experience in a mathematics or statistics classroom. Ted was combining the idea of the representativeness of the sample to the population, that the group discussed earlier, with the idea of the probabilities staying constant for each choice of nut, the key concepts of bootstrapping. The group continued to discuss how this process of replacement was different than their initial approach without replacement.

Randy: What are we going to do now?

Susan: Now I'm going to hand you back the Popsicle stick and you're going to mix it back in.

Randy: So you're going to draw …

Susan: From the 25, not from, you know how before, like when I drawed [sic] and set it down, it went less and less and less and less and less?

Ted: So we're going to do it again, we're still going to draw 14, we're just going to put them in.

Susan: So each time we're drawing from 25, instead of a reduced …

Ted: 'Cause as we would draw, in this one we would draw and there were 10, that meant that there were only 15 left in the bag, which doesn't account for the sample, right? 'Cause you're reducing it.

They asserted that when not using replacement, the nuts that they were drawing from the bag no longer represented the sample after some nuts were drawn and not returned. The group followed this procedure of resampling with replacement to collect a sample of 14 nuts. This was the only sample that they had time to gather before the class was reconvened to discuss each group's approach to determine the likely range of peanuts in this new brand. The group did not use this one sample to draw a conclusion since time ran out for developing their model.

Discussion and conclusions

We assert that a key difference between this study and previous research addressing data simulation and bootstrapping (Garfield, delMas, & Zieffler, 2012; Pfannkuch, Forbes, Harraway, Budgett, & Wild, 2013), is the *elicitation* of bootstrapping methods by groups of students rather than the instruction of students on how to use the method. By eliciting the method, we were able to view students' statistical reasoning which led students to construct a process similar to bootstrapping: the representativeness of the sample to the population; and a method of resampling that preserved this representativeness. We assert that these students exhibited a multiplicative conception of the sample (Saldanha & Thompson, 2002). These students claimed that the distribution of the sample likely

represented the population and constructed a model of resampling with replacement that simulated resamples which upheld the quasi-proportional relationship of the sample to the population. These finding have implication for future curricula design by examining the key understandings that students may need before instruction on methods of resampling, such as bootstrapping. The finding also suggested that the use of model eliciting activities are useful design for instruction in introductory statistics classrooms in order to analyze students' developing thinking of resampling and inference.

Two limitations of this study were that: although this group of students constructed a model of resampling similar to bootstrapping, they did not collect resamples that were equal in size to the original sample of mixed nuts; and the time needed to sample by replacement with manipulatives did not allow the students to collect a distribution of samples from which to draw informal inferences. Collecting resamples of the same size is key in order to observe the variability in a statistic for samples like the original. While designing the model eliciting activity, we did not foresee issues with how the context of the activity encouraged students to make claims about the populations of mixed nuts, rather than the proportion of peanuts in future samples from the population of mixed nuts. Within this context, rather than use a bulk supply of mixed nuts in which samples of many sizes could be taken, the model eliciting activity may have been better designed to focus on the small packages of mixed nuts with consistent sample sizes. The time demanding nature of resampling could have been alleviating by using smaller original samples or with transitioning to the use of technology. A different context from mixed nuts where smaller samples felt natural, rather than merely a simplification may have allowed more resamples to be simulated. We made the choice to initially explore resampling with manipulatives and in later activities use technology to simplify the resampling process.

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At-distance college-level training in statistical methods: Prospects and considerations

Maria Meletiou-Mavrotheris[1], Efi Paparistodemou[2] and Ana Serrado Bayes[3]

[1]European University Cyprus; m.mavrotheris@euc.ac.cy

[2]Cyprus Pedagogical Institute, e.paparistodemou@cytanet.com.cy

[3]La Salle-Buen Consejo, Spain; ana.serrado@gm.uca.es

This article focuses on how information and communication tools made available online could be effectively exploited to help improve the quality and efficiency of college-level, at-distance statistics training. The paper first provides an overview of the content, structure, and pedagogical and didactical approach underlying Quantitative Educational Research Methods, an online course targeting post-graduate education majors that has been built based on contemporary visions of web-based statistics instruction and computer-mediated communication. It then presents some of the insights gained from a case study of a group of students (n=49) participating in a recent offering of the course. The article concludes with some instructional and research implications.

Keywords: Statistics education, distance education, statistical inference, model-eliciting activities.

Introduction

The affordances offered by modern Internet technologies provide new opportunities for statistics instruction, making it possible to overcome restrictions of shrinking resources and geographical locations, and to offer, in a cost-effective and non-disruptive way, high-quality learning experiences to geographically dispersed students. In recent years, we have witnessed an exponential growth of distance education worldwide. Online course delivery has become common in a wide variety of disciplines, including statistics. This expansion is likely to continue, given the expanding access to the Internet and the greater emphasis to lifelong learning. Several advantages associated with distance education have been identified in the literature. In addition to the flexibility and convenience it offers, the distance option may also allow students the opportunity to take courses from established experts in their field of study that might not be available locally. From the viewpoint of statistics, it creates some unique opportunities for enhancing instruction, including the provision of a vast array of technological tools and resources for better understanding of statistical methods and concepts (e.g. interactive applets, virtual laboratory experiments, etc.). Several successful examples of successful programs of teaching statistics via distance have been documented in the literature (e.g., Evans et al., 2007; Everson & Garfield, 2008; Meletiou-Mavrotheris, Mavrotheris, & Paparistodemou, 2011).

Despite its proliferation and the unique opportunities for enhancing statistics teaching and learning that it offers, online statistics course delivery also presents several unique challenges. There are several pedagogical and technical issues that need to be incorporated into the design of an online statistics course for it to provide an effective learning environment. Review of the existing research literature alerts us to the fact that the quality and effectiveness of online statistics training currently offered is variable and inconsistent (Evans et al., 2007). While most of the conducted studies indicate that students taking courses with an online component have similar achievement and satisfaction levels compared to students in traditional, face-to-face classrooms (Mathieson, 2010), there is

growing evidence of many web-based distance learning courses failing to meet the expectations raised. For example, while it is well-documented in statistics education research that the incorporation of discussion and active learning in the classroom can help learners to think and reason about statistical concepts, bringing these important learning approaches to an online course has proved very challenging (Gould, 2005; Meletiou-Mavrotheris & Serrado, 2012).

Early attempts at web-based instruction assumed that setting up an attractive website with interesting multimedia applications was adequate for learning to occur. However, it has now been recognized that the level of success of distance education is determined by multiple factors, including underlying theory, technologies, teaching strategies, and learner support. Elements in the design of an online course such as its content and structure, the tools and cognitive technologies employed, and the amount of interaction between instructors and learners as well among learners, are important factors affecting students' learning and attitudes (Tudor, 2006). A particularly important criterion for the level of success of at-distance statistical training is also the extent to which instruction allows learners to experience the practice of statistics and to apply statistical tools in order to tackle real-life problems.

This article provides an overview of a post-graduate quantitative research methods course that has adopted a non-conventional approach, which promotes online participation and collaboration of students using contemporary technological and educational tools and resources. After describing the course pedagogical approach, and content and structure, the paper presents some of the insights gained from a case study of a group of students participating in a recent offering of the course.

Nature of the Quantitative Educational Research Methods course

"Quantitative Educational Research Methods" is a graduate-level course targeting students enrolled in the M.A. in Educational Studies program offered at European University Cyprus. Although originally developed for a face-to-face setting, the course was in 2013 redesigned as an online course to make it accessible to students enrolled in this program through distance education. In designing the e-learning course, efforts were made to preserve the pedagogical approach, and content and structure of the classroom-based course.

Contemporary visions of web-based instruction and computer-mediated communication underpin the course design. Concurring with Roseth, Garfield, and Ben-Zvi (2008), the online learning environment has been built upon the premise that instruction of statistical methods ought to resemble statistical practice, an inherently cooperative problem-solving enterprise. Students enrolled in the course are provided with ample opportunities for interactive and collaborative learning. They are actively involved in constructing their own knowledge, through participation in authentic educational activities encouraging enculturation such as projects, experiments, computer explorations with real and simulated data, group work and discussions. Statistical thinking is presented as a synthesis of statistical knowledge, context knowledge, and the information in the data in order to produce implications and insights, and to test and refine conjectures. There is a focus on modeling and simulation—along with inference – which is being facilitated by having students use the dynamical statistical software package TinkerPlots2 (Konold & Miller, 2011) for all modeling and analysis. This software was selected because it is designed explicitly to support integration of exploratory data analysis approaches and probabilistic models, and to allow for generation of data (e.g., drawing samples from a model) and experimentation (e.g., improving models, conducting simulations).

The course lasts 15 weeks. It is made up of 7 modules, that are concept driven and focused on enriching students' knowledge of quantitative research methods (mainly inferential) by exposing them to innovative learning situations, technologies, and curricula. Each module involves a range of activities, readings, contributions to discussion, and the completion of group and/or individual assignments. The activities and assignments mirror those completed in the classroom-based course.

Throughout the course, students use TinkerPlots2 to work on a set of carefully designed open-ended Model-Eliciting Activities (MEAs) (Lesh et al., 2000) in which they create and test statistical models in order to solve real world problems of statistics (Garfield, delMas & Zieffler, 2012). The activities are carefully designed to support but, at the same time, also explore students' evolving understandings of fundamental ideas related to statistical inference. Some of the MEAs are completed individually, and others collaboratively in groups of 3-4 students. The MEA "How many tickets to sell?" (adapted from http://new.censusatschool.org.nz/resource/using-tinkerplots-for-probability-modelling/) is a typical example of these activities. It is based on the following fictitious scenario: *"Air Zland has found that on average 2.9% of the passengers that have booked tickets on its main domestic routes fail to show up for departure. It responds by overbooking flights. The Airbus A230, used on these routes, has 171 seats. How many extra tickets can Air Zland sell without upsetting passengers who do show up at the terminal too often?"* In this MEA, students use Tinkerplots2 to model the Air Zland flight (e.g. model the scenario in which AirZland sells five extra tickets, i.e. books 176 tickets). They repeat the experiment a large number of trials using the "Collect Statistic" feature of Tinkerplots2 to keep track of the number of passengers not showing up, and draw the resulting distribution of collected sample statistics. Students then decide whether their model should be adjusted or not and, based on that, make recommendations to the airline as to how many extra tickets it should issue. Finally, they use the properties of the binomial distribution to determine theoretical probabilities when booking a certain number of seats (e.g. 176 seats) and compare the results with those they get through the Tinkerplots simulation. (see Meletiou-Mavrotheris et al. 2015 for more details).

A progressive formalization approach is being employed in the course. The first part focuses on building a teaching pathway towards formal inference by helping students experience and develop the 'big ideas' of informal inference. Through their engagement with the open-ended MEA activities, students learn where these ideas apply and how. Later in the course, students are introduced to confirmatory or formal inference methods, and begin comparing empirical probabilities with the theoretical ones. They learn the formal procedures for building sampling distributions, constructing confidence intervals, and conducting hypothesis testing using different statistical tests. The similarities and differences between ideal, mathematical models of reality, and statistical models based on experimental data are being emphasized throughout the course. From informal uses of models early in the course to formal uses as part of significance tests at the later part, instruction encourages explicit discussion of how every model is essentially an oversimplification of reality which involves loss of information, and of how the success of probability models depends on their practically and potential to give useful answers to our research questions.

The course is delivered completely online using the instructional content and services of the project platform (on the LMS Blackboard system). In addition to the course content (video lectures, PowerPoint presentations, video tutorials, links to statistics resources available on the internet, etc.) the site offers access to various tools for professional dialogue and support (email, videoconferencing,

chat rooms, discussion forums, wikis, etc.). The course instructor acts as a facilitator of a deeper learning experience through guiding discussions, encouraging full, thoughtful involvement of all participants, and providing feedback, in both asynchronous and synchronous activities.

Methodology

The case studied was a group of students taking the online version of the Quantitative Research Methods course during the Fall 2014 semester. The first author was the course instructor. There were forty-nine (n=49) students enrolled in the course, residing in Greece (*n=38*) or Cyprus (*n=11*). Course participants were characterized by a wide diversity in a number of parameters including age, and professional and academic background. Their age ranged from 23 to 55. Some had an academic background in primary education (*n=18*), while the rest were secondary school teachers in different domains (languages, humanities, natural sciences, physical sciences etc.). While the majority were experienced educators with several years of teaching experience, a sizeable proportion were either unemployed or employed in non-education related occupations. Students also had a varied background in statistics. Most of the older participants had very limited prior exposure to statistics, while the younger ones had typically completed a statistics course while at college. Even students who had formally studied statistics had attended traditional lecture-based courses that made minimal use of technology. Thus, upon entering the course, almost all students had very weak statistical reasoning and/or a tendency to focus on the procedural aspects of statistics.

Documenting online student activity and collaborative knowledge construction is a multifaceted phenomenon that requires complementary methods of data collection and analysis in order to understand how learning is accomplished through interaction, how learners engage in knowledge building, and how designed media support this accomplishment (Hmelo-Silver, 2003). Consequently, to increase understanding of the research setting, the current study employed a variety of both qualitative and quantitative data collection techniques, including: (i) The contents of the online discussion boards, chats, and wikis, in which students had been participating during the course; (ii) Bi-weekly collaborative assignments, in the form of Model Eliciting Activities (MEAs); (iii) videotaped synchronous sessions taking place weekly throughout the semester using Blackboard Collaborate as a communication tool; (iv) final course examination administered to both students enrolled in the course under study, and students enrolled in a face-to-face version of the same course again taught by the same instructor; (v) an open-ended survey administered at the course completion, aimed at determining students' perceptions, opinions, and feelings regarding the course; (vi) Quantitative statistics automatically collected by the system (e.g. number of students participating in a discussion forum or successfully completing group assignments, etc.).

The text-based and video-based data collected during the course (MEAs, discussion forums, videotaped synchronous sessions, open-ended survey at course completion) were eventually analysed in order to examine how students' engagement with Tinkerplots2, with MEAs and with each other impacted their motivation and participation levels, and how it scaffolded and extended their understanding of the big statistical ideas encountered during the course. We did not use an analytical framework with predetermined categories. What we instead did was a content analysis aimed at identifying, though careful reviewing of the transcripts, the recurring themes or patterns in the data. Quantitative data (system statistics, performance on final examination) were analysed using descriptive and inferential statistics. Linking the depth of qualitative data with quantitative breadth

provided a more holistic picture of the course impact on students' attitudes and learning of statistics.

Results

Analysis of the data obtained during the case study, indicates that the online Quantitative Educational Research Methods course provided students with experiences parallel to those provided in its face-to-face version. The course was characterized by high levels of student engagement in online discussions and participation in videoconferencing sessions, and by successful collaborations for the completion of group assignments. Findings also suggest that the adoption of a pedagogical approach focused on modeling, using a dynamic statistics software like Tinkerplots2 for the conduct of statistical investigations, and of technological tools for facilitation of communication and collaboration among learners, is a viable option for online statistics instruction. The informal approach to statistical inference espoused by the course, using TinkerPlots2 as a tool for investigating authentic, open-ended model-eliciting activities (MEAs), fostered students' ability to reason about the stochastic, while also developing their appreciation for the practical value of statistics. Through their engagement in MEAs in which they collaboratively built models and used them to evaluate research claims and hypotheses, the graduate students in our study developed relatively coherent understandings of fundamental concepts related to statistical inference.

The affordances offered by Tinkerplots2 for building and experimenting with data models to make sense of the situation at hand, proved instrumental in supporting student understanding of both informal and formal inferential statistical ideas. Of course, similarly to other researchers we also witnessed a number of challenges associated with the adoption of a modelling approach (Konold, Harradine, & Kazak, 2007), and different levels of student reasoning and understanding of the role of models and modelling, and of the key assumptions underlying the models simulated by the computer (for more details, interested readers could refer to Meletiou-Mavrotheris, Paparistodemou & Serrado, 2015). Nonetheless, use of Tinkerplots2 enabled students to build and modify their own models of real world phenomena, and to use them to informally test hypotheses and draw inferences. Their engagement with data-driven inferences helped them to develop sound informal understanding of the logic of hypothesis testing and its related statistical ideas (significance level, p-value, null and alternative hypothesis etc.), and served as a foundation for the formal study of inferential statistics.

Student performance on assignments and assessments was comparable to what was observed in the face-to-face version of the course concurrently taught by the first author. When the end of the semester, both groups of students were administered an identical assessment instrument (as a final exam) with several open-ended tasks aimed at investigating their understanding of the main ideas and concepts related to statistical inference covered in the course, both groups obtained very similar results (*Mean Score: At distance=76.1, Face-to-face=77.06).* A two-sample unequal variance t-test (conducted after checking all assumptions) indicated that there was no significant difference in mean scores ($p=0.73$) between the two groups of students (see Table 1).

Course	**No. of students**	**Mean Score**	**Standard deviation**	**t test for equality of means**	
At-distance	49	76.10	14.68	t	Two-tail Sig.
Face-to-face	34	77.06	10.49	-0.3463	0.73

Table 1: Two-tailed t-test Comparison of Mean Differences in final exam scores

In the survey administered at the end of the course, students were asked to indicate what they liked

most about the course. The overall feedback regarding the course content, services, and didactical approaches was generally very positive. The flexibility and convenience associated with distance education was an aspect appreciated by all course participants, since it made it possible for them to determine their own place, pace and time of study. Another aspect also much appreciated by the majority of the participants was the fact that both the discussions and the assignments were carefully designed to be learner-centered, and to make explicit ties between theory and practice by utilizing students' own experiences as learning resources. The promotion of communication and collaboration was also considered to be an important strength of the course by most learners. Students, in general, enjoyed the interaction and the sharing of experiences and ideas, although there were a few who expressed a preference for individual assignments, arguing that *"group assignments are less flexible since you have to regularly meet online with your group",* or that *"in group tasks, some members do minimal work while the rest work very hard, but the end they all get the same score… this is unfair."*

In a previous study conducted by the authors in the context of a transnational online teacher professional development course in statistics education, the biggest difficulty experienced was the limited success in establishing a functional online community of practice (Meletiou-Mavrotheris and Serrado, 2012). Similarly to other statistics education researchers (e.g. Gould and Peck, 2005), there was a much lower than anticipated level of learner-to-learner interaction in the course. Although community building was a main objective, and while at the course outset there was big enthusiasm and very high participation in discussion forums, interaction dropped off over time. The vast majority of messages (around 80%) had been sent during the first half of the course, while in the second half only a handful of learners actively participated in the discussion forums, while the rest had made minimal or no contributions. In the current study, by contrast, students' level of engagement in the discussion forum was consistently high throughout the semester. All discussion forums created during the course were characterized by vibrant interaction and rich dialog.

We consider the active participation of students in the discussion forums witnessed throughout the semester to be an important success of this course since, as the literature indicates, leading a discussion of substance on a "discussion board" is much more challenging and difficult to achieve than in a real classroom (Gould & Peck, 2005). We believe that, in the current study, the adoption of the following strategies led to more successful community building compared to our prior research: (i) Making participation in group activities a compulsory element of the course that counts towards learners' grade; (ii) Establishing a clear set of criteria in the course syllabus to help learners better understand the academic expectations and increase the intellectual depth of their contributions; (iii) Providing sufficient time for group members to make meaningful interpersonal connections before the assignment of the first cognitive task; (iv) Increasing the duration of each discussion forum to allow adequate time for learners to formulate and articulate their contributions; (v) Providing more prompt and effective moderation of online interactions.

Despite the overall success of the course, analysis of the collected data has allowed us to identify a number of issues and student concerns that adversely affected the online participation of course participants. The biggest shortcoming identified was the course overload. When requested, in the end-of course survey, to indicate what they liked the least about the course, most participants mentioned the course workload that made it extremely difficult for them to keep up with the course requirements due to their overburdened schedules. Also, participation in videoconferencing and other activities that

required synchronous communication (e.g. chat sessions) proved very difficult to schedule, as it was almost impossible for all of the students to be available at the same time.

The Quantitative Educational Research Methods course team has adopted a continuous improvement iterative model. Insights from the current study informed the revision of the course, so as to further improve its quality and effectiveness. The heavy workload was corrected in subsequent offerings, and more realistic work expectations were set so as not to overburden students. There has also been more careful scheduling of course activities to offer students more flexibility.

Discussion

Teaching online courses is a new, unexplored territory for most statistics instructors. Distance education is similar yet different from classroom-based instruction, and requires new teaching skills and strategies. Several pedagogical and technical issues should be taken into account in the course design to provide an effective online learning environment. Using the case study of a distance-based approach to a quantitative educational research methods course as an example, the paper has provided some suggestions on how to best exploit the affordances offered by modern e-learning technologies to improve the quality and attractiveness of the online learning experience through the promotion of hands-on and collaborative knowledge construction. In accord with contemporary visions of web-based instruction that support collaborative and participatory models of online learning, the article has offered some insights on how to build an online learning environment in ways that resemble statistical practice, an inherently cooperative, problem-solving enterprise involving participation in projects, modelling and experimentation with real and simulated data, group work, and discussions.

Statistics education research in distance education settings is still at a developmental stage. More research is needed to advance our understanding of how to best take advantage of computer-mediated communication tools to support the development of effective virtual learning environments. By exploring the forms of collaboration and shared knowledge building undertaken by the group of students participating in our online course, the current case study has contributed some useful insights into the factors that may facilitate or impede the successful implementation of distance education. These insights have helped to further improve the quality and effectiveness of the course, and sketch a road map for our future research work, and for other similar endeavours.

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Insights from students' reasoning about probability when they design their own Scratch games

Efi Paparistodemou[1], Maria Meletiou-Mavrotheris[2] and Christina Vasou[2]

[1] Cyprus Pedagogical Institute, Cyprus, e.paparistodemou@cytanet.com.cy

[2]European University of Cyprus, Cyprus, m.mavortheris@euc.ac.cy, christina.vasou3@gmail.com

Making modeling, generalization and justification an explicit focus of instruction can help to make big ideas available to all students at all ages (Carpenter & Romberg, 2004). Because mathematical models focus on structural characteristics of phenomena (e.g. patterns, interactions, and relationships among elements) rather than surface features (e.g. biological, physical, or artistic attributes), they are powerful tools in predicting the behavior of complex systems (Lesh & Harel, 2003). Based on this assumption, we attempted to enhance students' (aged between 8 to 13 years old) reasoning about probability by asking them to design a computer game for modeling probabilistic ideas. Students were introduced to the block-based programming language Scratch 2.0, and used it to create their own games. The findings show that the idea of chance had an important role in their games and that they expressed many probabilistic ideas while they were designing and playing their game.

Keywords: Statistics education, randomness, educational game, game design, Scratch.

Background of the study

Although probability is increasingly being integrated into the school mathematics curriculum, students face difficulties in understanding a variety of probabilistic concepts. Probability is difficult to teach because of the gap between intuition and conceptual development, even as regards elementary concepts (Batanero & Diaz, 2012). The current article contributes to the emerging literature on game-enhanced statistics learning by exploring the capabilities of a learning environment that uses programming logic in a game setting, as a tool for facilitating the emergence of young learners' informal reasoning about randomness and other key probabilistic concepts. Based on a case study of a group of students (aged 8-13) who developed their own games through use of the visual block-based programming language Scratch 2.0 (Massachusetts Institute of Technology, 2013), the following question was explored: How do students use elements of reasoning about probability when they design their own games?

The research literature suggests that digital educational games have many potential benefits for mathematics and statistics teaching and learning. One of their foremost qualities is the capacity to motivate, engage, and immerse players. It has been shown that educational games captivate students' attention, contributing to their increased motivation and engagement with mathematics and statistics (Ke, 2008). Studies have also demonstrated that, in addition to providing an incentive for young people to engage in learning, games also have the potential to yield an increase in students' learning outcomes (Kolovou, van den Heuvel-Panhuizen, & Köller, 2013). Although much of the research on the effectiveness of gaming on learning is inconclusive at this point, there are strong indications in the literature that appropriately designed and constructively used games can support experimentation with mathematical and statistical ideas in authentic contexts, and can

be used as the machinery for engaging students in problem solving activities, and for promoting the attainment of important competencies essential in modern society (Lowrie & Jorgensen, 2015).

While digital educational games can provide a range of potential benefits for statistics teaching and learning, high quality, developmentally meaningful, digital games for students are less common than hoped. There is a wide variability in content, scope, design, and appropriateness of pedagogical features, with many educational games including mediocre or even inappropriate content, being drill-and-practice, and focusing on basic academic skills rather than on high-level thinking. Nonetheless, some exceptional exemplars that can help create constructive, meaningful, and valuable learning experiences do exist. One promising type is coding gaming software, which teaches students the concepts behind programming in a playful context. With an increasing focus on programming and coding finding its way onto the curriculum in many different countries across the world, some innovative, educationally sound game-based learning environments that support the development of computer programming skills from a young age have begun to appear. Several educational applications are currently available for helping students with no coding background or expertise to grasp the basics of programming through the exploration and/or creation of interactive games (e.g. Scratch, ScratchJr, HopScotch, Bee-Bot). Often, coding game applications enable students to share their games with others, and to play or edit games programmed by others.

Having taken their inspiration from Logo (Papert, 1980), educational programming environments promote a constructionist approach to technology use, with the emphasis being on students using technological tools to become creators instead of consumers of computer games. In addition to the provision of a highly motivational and practical approach for introducing students to computer programming and developing their computational thinking, coding software provide rich opportunities for the reinforcement of problem-solving, critical thinking, and logical thinking skills (e.g. sequencing, estimation, prediction, metacognition) that can be applied across domains.

Methodology

Participants and context

A total of four workshops were organized and each one lasted for 2 hours. Twenty-six students (N=26, 16 male, 10 female), aged between 8 and 13, participated in all four workshops during July 2016 (summer school vacations) on a volunteer basis. An invitation to parents was placed in social media and the students were selected from a priority list based on registration date. All participants had the right to pause or stop their participation entirely at any given moment. Additionally, all parents provided their written consent regarding the use and publication of their students' work for research purposes. In this paper, all names used are pseudonyms in order to preserve participants' anonymity.

To serve the role of the gaming platform, our research team chose Scratch, a visual programming language developed at the MIT Media Lab that consists of reusable pieces of code, which can easily be combined, shared, and adapted. Scratch can be used to program interactive stories, games, and animations, art and music and share all of these creations with others in an online community (http://scratch.mit.edu/). It was created to help students think more creatively, reason systematically, and work collaboratively, all of which are essential skills required for the 21st century (Resnick, 2007). The software was first released in 2007 while Scratch 2.0, which is its second current major version, came out in 2013. In this study, we deemed Scratch 2.0 as the most appropriate option to

adopt, due to the fact that there is very little research on how coding learning environments could be used as a tool for developing concepts related to the stochastic.

For each workshop, a different set of extra-curricular activities were closely designed based on constructionism (Papert, 1980), and each meeting was structured in such a way as to promote an unhurried and creative process. The first workshop aimed to a general introduction to the software and in the second workshop students worked on activities based on the movement of a sprite around the screen. In the second workshop the x and y axis were discussed based on the position of a sprite. In the third workshop, students worked on variables and the idea of randomness through experimentation with a flipping coin game, and ways to pick random block. Finally, students were asked to create their own game based on what they had learned. In the last session, students continued their games from their previous meeting, changed them if they wished, and asked a friend to play their game so to identify any bugs and fix them.

Data collection

For the purposes of collecting our data, we used a variety of methods, including live video recording of the workshop and screen capturing of the participants' interactions with the software. Other sources of data also included field notes and classroom observations. In six cases, we also conducted individual mini-interviews of selected students (interviewed while engaging in game design) that expressed some exceptional ideas regarding the element of randomness, in an attempt to study further their contributions to this project. For the purpose of analysis, we did not use an analytical framework with predetermined categories. What we instead did was, through careful reading of the transcripts and field notes and examining of the various interactions for similarities and differences, to identify recurring themes or patterns in the data. To increase the reliability of the findings, the activities were analyzed and categorized by all three researchers and any inter-rater discrepancies were resolved through discussion.

Insights from students' reasoning about probability in their Scratch games

In the following paragraphs, we present two main categories of students' reasoning about probability in the context of creating their Scratch games. First of all, we describe how students used the idea of chance and randomness in their games and secondly how they used spatial representations for expressing probabilistic ideas. The students' games we present here were from the last workshop.

The role of randomness in designing games

In our sessions, students experimented with different mathematical and statistical ideas while designing their games. One of the ideas brought up during the class discussion was that of randomness. The 'pick random' block, which allows users to bring randomness into Scratch projects, was casually explained to students, in a similar way to how the rest of the blocks were introduced. It was interesting to find out that the students ended up using randomness in their games.

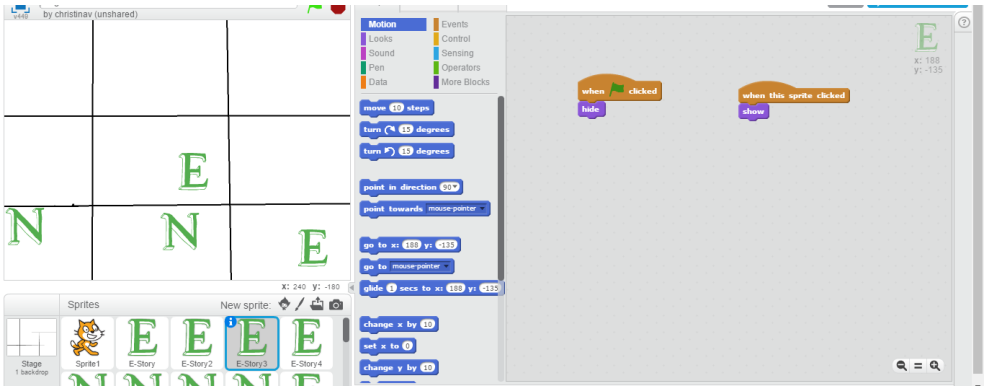

Figure 1: Eric's and Nicole's random game with letters

Eric (a ten-year-old boy) and Nicole (a twelve-year-old girl) designed a game where the first letter of their name appears randomly when you click on the board. It is like a tic-tac-toe, but the player is not sure where the letter goes.

> Eric: I like the fact that the letters appear in a random position. This makes our game more *interesting*.
>
> Researcher: Why is that?
>
> Nicole: You have to see the probability, where it might go [the letter], and then select the letter.
>
> Eric: You don't know at the beginning…You need to make a guess. If you don't look at the results and just play, then you are more likely to lose…but nothing is for sure.

Eric and Maria used the random rule in their game in order to make it more interesting. Randomness and uncertainty made their game to have action. Nicole referred to the concept of probability in order to make a correct guess based on the results of the game. So, students were playing the game and trying to guess where the next letter would appear based on the idea that the probability of each letter to appear somewhere is equal-according to their design.

Charis, a nine year old boy, also made a game by using randomness.

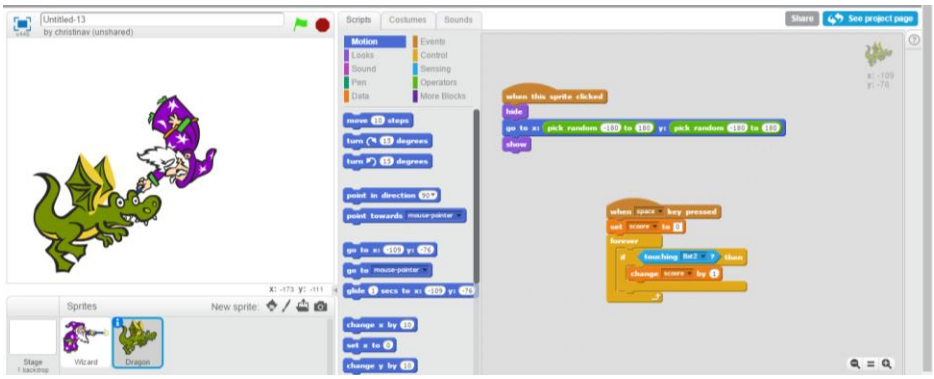

Figure 2: Charis' dragon random game

The aim of Charis' game is to click on the dragon. When the dragon is clicked, it appears in a random position. The magician then follows the dragon to its new position.

Charis: You know, I made it just for fun! It is nice to see the dragon moving around without *knowing*…But I will develop it. I made the dragon to move all over the place.

Researcher: So, will it appear again in this position we see now?

Charis: Of course! I will make something to count where it goes, so we will see which position it takes…May be to touch something…Let me see what I can do…

Charis realizes that randomness is something that you don't know in advance. It is interesting that he designed a dragon with a random move and then he tried to predict its movements by counting the dragon's position each time. He admits that this is how the game begins to have fun! The idea of using the x and y variables in a random way, and of trying to predict the next position prompted Charis to use the idea that the dragon will move on the pre-designed space and after a long time (law of large numbers) the dragon will pass from every point (based on x and y).

Spatial representations for expressing probability

Chris, a 13 year-old boy, was one of the students who really liked using randomness in his games. Chris designed a game of a dog crossing the street. The aim of the game is to help the dog to cross safely (without touching any of the cars).

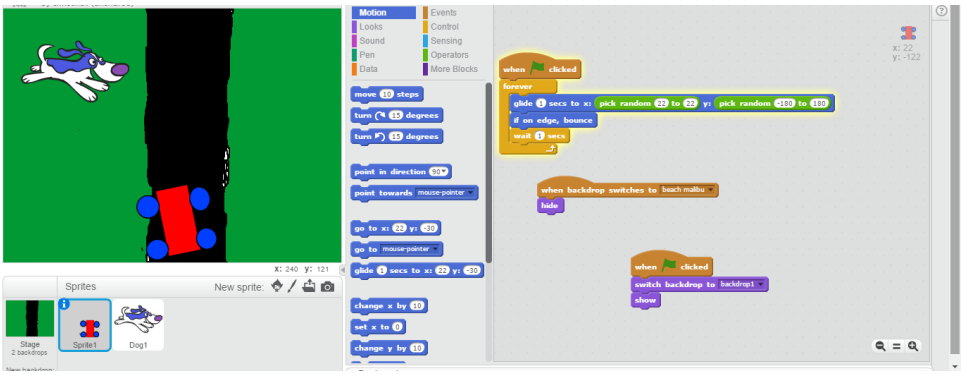

Figure 3: Chris' first version of random game

Researcher (R): So, what is the game here?

Chris (C): Try and see…

R: Interesting… [While R is playing the game.]

C: Yes, you don't know where the car goes. You should be careful!

R: Why? The car will move and cross the road.

C: Not only…It [the car] moves randomly on this road I design. That's the interesting part…So, you don't know where it goes. And when you touch it! You see! *The dog touched the car*. Do you like it?

We have also here the existence of randomness in games as a factor of making a game interesting. It is important how Chris refers to the dog's movement - the one that the player controls - and not to the car's movement. This also shows a realization that randomness in his game is something 'uncontrolled' and this was made on purpose for making the game interesting.

R: Why didn't you just make the car to move forward?

C: This is boring…just seeing the cars and move around. Now you don't know…Of course it is easy with one car. *…Chris is making different things on his game.*

We found it interesting that Chris' game was a non-deterministic model of crossing a road. His idea of moving the cars in the road randomly is what makes his game appealing. Chris designed a car that moved in a random way. Although a random movement of the car might have sufficed for the aim of the game, he also used the road as a spatial sample space and tried to increase the difficulty of the game by increasing the number of cars.

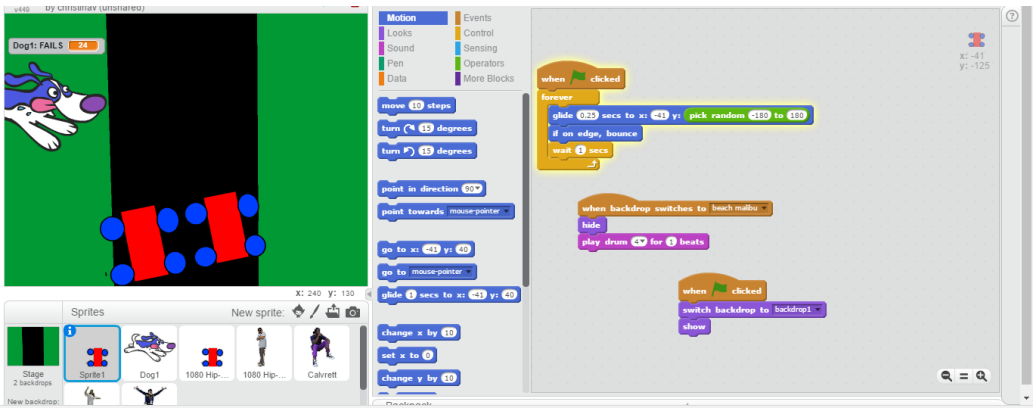

Figure 4: Chris' second version of random game

R: What have you done?

C: I just put two cars, a counter, made a bigger road and I changed the dog. I changed the code of the cars.

R: Why?

C: It is better this way. I made the road bigger and I asked the cars to move randomly all over the road. This makes it more difficult for the dog to cross.

[The researcher plays the game. The dog cannot cross the road. The counter keeps track of her failed attempts.]

R: It is very difficult that way.

C: Yes [he laughs]. This is something that reduces the probability of the dog safely crossing the road to less than fifty-fifty. Actually, it makes it go to zero.

R: Would you like to play it?

C: Actually, that way is not interesting…it's not fair. You know…I can make some change to the design. I will make the dog smaller. That will make it fair…Let's see.

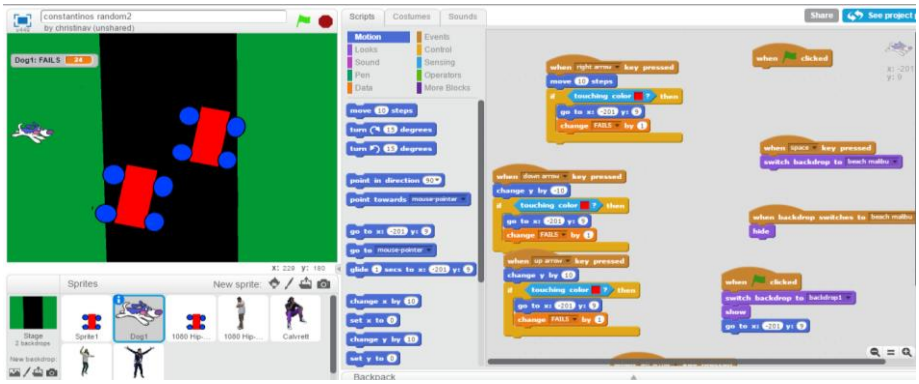

Figure 5: Chris' final version of random game

Chris uses the idea of fairness and the probability of 1/2 in his game while he is designing and re-designing his own game. It is interesting that although in the workshop we never referred to spatial probability, Chris in his game connects the concept of space with the concept of probability. We can see that he did not change the code in his game, although he could have done that in order to reduce the probability of the car crossing the road. What he did instead was to reduce the space in the road.

Discussion and conclusions

The aim of the paper was to explore how students use elements of reasoning about probability when they design their own games. The students in the study experienced statistics as an investigative, problem-solving process. Although we tried to separate the use of randomness from the spatial representation of probability, the reader might notice that this was difficult to do. Because of designing, students used simultaneously the idea of randomness in terms of the icons they had in their game. We were really surprised with how these ideas came out without even mentioning what sample space is, or how we calculate probability. The design, coding, revision, and debugging of computer commands, helps students develop higher order problem solving skills such as deductive reasoning, while at the same time improving their conceptual understanding of key mathematical and statistical ideas. Thus, it becomes crucial to incorporate computer programming into existing statistics curricula. Game coding learning environments provide an ideal opportunity for doing so in an engaging, non-threating, and child friendly manner (Resnick, 2007). Educators and others can ensure that coding gives opportunities for new expressions, even for reasoning about probability.

This increased popularity and proliferation of computer games has led to a widespread interest in their use as learning tools. Several statistics educators have, in recent years, been experimenting with digital games, investigating the ways in which this massively popular worldwide youth activity could be brought into the classroom in order to capture students' interest and facilitate their learning of statistical concepts (e.g. Pratt et al., 2008; Paparistodemou et al., 2008; Meletiou-Mavrotheris, 2013; Erickson, 2014). Our study shows evidence that randomness is an important factor in playing games and a software like Scratch can give opportunities to fill the gap between intuition and conceptual development of probabilistic ideas (Batanero & Diaz, 2012). When we reconsider prior work on randomness (for example, Pratt, 2000), we find resonance in the use of symmetry between apparent fairness and the tendency for children to consider the appearance of the dice (or coin, or spinner…), something that we also found in Chris' case. The present study showed some insights from students' reasoning about probability while they were designing Scratch games. We intend to

further analyse our collected data and to continue with further research on how students express probabilistic conceptions like the law of large numbers and how students use and elaborate their codes.

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Using spreadsheets to teach probability in French high school

Bernard Parzysz
Université d'Orléans & L.D.A.R. (université Paris-Diderot)
parzysz.bernard@wanadoo.fr

Including a 'frequentist' point of view has resulted in experimentation becoming an important issue in the teaching of probability in high school. Spreadsheets are now widely used, but the status of the results produced and how to use them are not always clear for the students, since two domains are at play in turn: statistics and probability. Through the French example – but this can also be applied to the teaching of probability in other countries − this paper reviews some questions about spreadsheets, namely simulating random experiments and shifting from discrete to continuous distributions.

Keywords. Probability, simulation, continuous distribution, spreadsheet.

In France as in many other countries, probability has become a prominent subject in the teaching of mathematics, in link with the growing importance of numerical data in everyday and professional life and the development of technologies allowing to process them. Conversely, the point of view on probability has evolved, taking into account a 'frequentist' point of view implying students making experiments. Among the technological tools spreadsheet has become much used in the teaching of probability, the main reason being that it makes getting a large number of tries of a random experiment very easy and fast. This paper is based on the French current situation, but it certainly can apply to many other countries all over the world. Its aim is to give some insights on how new didactic questions have occurred and have now to be tackled by teachers in their classes. As it has already been noticed, "it is not good enough to only consider which technology to use, but (…), in order for effective learning to take place, it is how the technology is integrated into the curriculum and learning process and how the teacher uses it that are vital" (Pratt, Davies & Connor, 2008, p. 98), the more so that "most teachers have little experience with probability and share with their students a variety of probabilistic misconceptions" (Batanero et al., 2005, p.1). I shall discuss some questions about the use of spreadsheet in high school, namely:

- the nature of simulation, implying using a model of the random experiment and making in turn intervene probabilistic and statistical paradigms;

- possible purposes of a simulation: visualize the law of large numbers, make conjectures, bring out the notion of stochastic model, help solving probability problems…

- the suitability of spreadsheet for introducing continuous distributions (exponential, Gaussian…).

For this, I shall use a theoretical framework including Kuzniak's Mathematical Working Space (MWS), Kuhn's paradigms and Duval's semiotic registers.

Theoretical framework

In order to get a holistic view of the work undertaken by somebody solving a mathematical problem, one has to take into account not only the domains at play but also the cognitive processes involved. The Mathematical Working Spaces, or MWS, framework (Kuzniak, 2011) considers two "planes" −epistemological and cognitive−, each one having three components:

- in the epistemological plane: a set of representations ('real space'), a set of artefacts (instruments) and a theoretical reference system;
- in the cognitive plane, three processes: visualization, construction and proof

An important feature of the model is the interaction between these two planes according three dimensions, semiotic, instrumental and discursive, linking each component of one plane to a corresponding component of the other (Figure 1). The model also assumes that efficient mathematic work results from involving the 3 dimensions together with interactions between them.

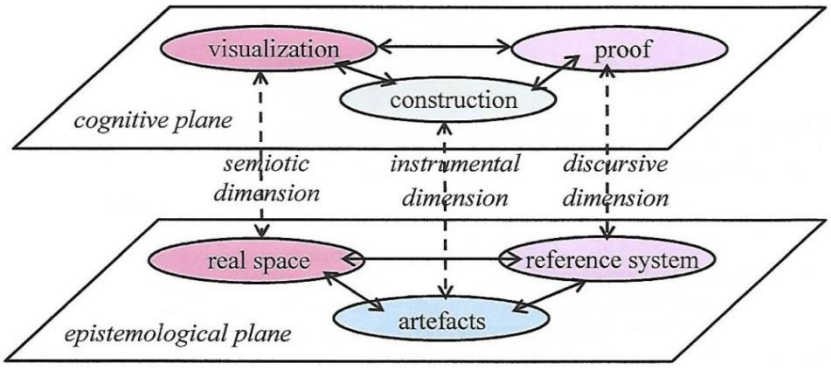

Figure 1: The MWS model (after Kuzniak, 2011)

Kuzniak distinguishes 3 main MWSs:

- reference MWS, defined by the syllabus,
- suitable MWS, planned by the teacher to be implemented in his/her class,
- personal MWS of the student.

Kuhn (Kuhn, 1962) defined scientific paradigms as "universally recognized scientific achievements that, for a time, provide model problems and solutions for a community of researchers," (page X of the 1996 edition). This notion was adapted by Kuzniak to taught mathematics, regarding the epistemological plane. In the case of probability several paradigms can be distinguished (Parzysz, 2011):

- a realistic paradigm (R), i.e. the real ('concrete') random experiment itself;
- a paradigm (P1) resulting from a first ("light") modelling of the real experiment by establishing a precise protocol, a list of issues and assigning a probability to each of them;
- a paradigm (P2), in which notions of random experiment and probability are defined, together with properties of probability which can be used for solving problems.
- a paradigm (P3) of the axiomatic type, taught in university.

N.B. In France, at secondary level, only P1 and P2 paradigms are considered, the latter being possibly extended with some elements of calculus at the end of high school (P2+) for the study of continuous distributions.

Regarding the semiotic dimension I shall refer to the notion of 'semiotic register', i.e. a coherent semiotic system allowing 3 cognitive activities: produce identifiable elements (representations), transform an element into another of the same register, convert an element into an element of another register (Duval, 1995). For Duval, a better knowledge is obtained through the use of several registers interacting one with another. He also indicates that the shift from one representation to another one

is more efficient when there is a 'semantic congruence' between them, i.e. when there is a one-to-one correspondence between signifying elements of the two representations.

Simulation

In the beginning of its being studied in high school, probability was taught as an application of combinatorics, through Laplace's formula (probability = number of favorable issues / number of total issues). This 'cardinalist' point of view implies that all the issues have the same chance to appear, and then other random phenomena had to be left aside. For instance this is the case for drawing pins: when tossed, they may come down in two ways, like coins, but no argument of symmetry can help and one cannot assign a plausible a priori probability to each of them. In such a case you have to observe the relative frequencies of the issues, assuming that they will 'converge' toward their probability when the number of tries grows 'to infinity'. This is the 'frequentist' point of view, theorized by the law of large numbers. This point of view was introduced in French high school 20 years ago. Anyway, whatever is the point of view on probability, one has to decide which probability will be allocated to each issue of the experiment, the difference being that the decision is made:

- either on *a priori* ground (e.g. 'symmetry' of the issues) in a cardinalist approach;
- or on *a posteriori* ground (frequencies of the issues) in a frequentist approach.

In past days, teachers were reluctant to let their students perform sequences of a random experiment, mostly because it was noisy and requested too much time, but the coming of computers in classrooms, namely spreadsheet including a so-called 'random' generator (see for instance Kroese et al., 2011), provided them with an alternative path (although starting with real experimentation remains necessary to materialize the link with reality). Spreadsheet is now widely used to simulate random experiments, with various purposes.

1) Spreadsheet can be used to visualize the compatibility of the cardinalist and frequentist points of view, and finally get the students confident in the generator. For this purpose one has to introduce into the machine a probability for each issue. Then the evolution of the relative frequencies on fairly large numbers of tries can be observed (Figure 2), this process being in fact a visualization of the 'law' of large number (belonging to the P3 paradigm).

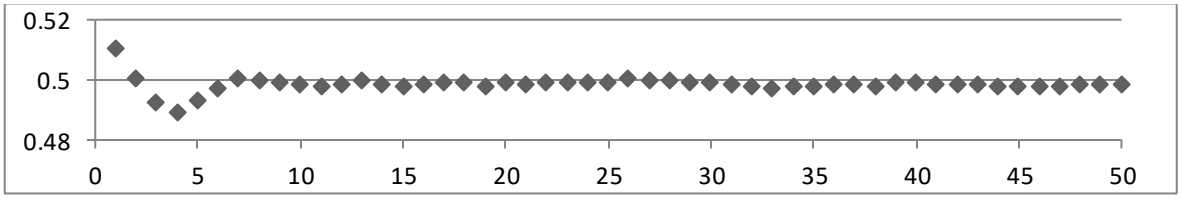

Figure 2: Relative frequencies of heads in 1000, 2000, …50000 tries of heads and tails

Thus performing a simulation implies constructing at least a simple probabilistic model (within the P1 paradigm), in order to implement it in the machine. Hence simulation is a ternary process: what is implemented in the software is *not* the real experiment but a theoretical *model* of it (Figure 3).

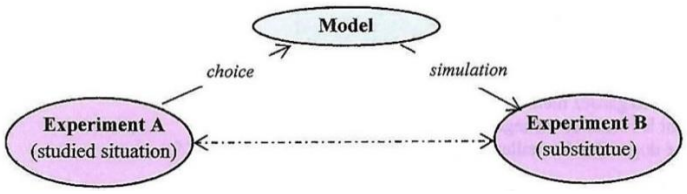

Figure 3: The ternary process of simulation

In such a task several registers are appealed to in turn: natural language, symbolic language (software) and Cartesian graphs. The semiotic-instrumental plane of the MWS is at play, involving the initial P1 paradigm (finding a model of the experiment), then shifting to another paradigm: descriptive statistic (DS) (results of the simulation). It is only when a conjecture about the experiment is asked that the discursive dimension appears (within P2 paradigm).

Both as teacher and teacher trainer, I could observe that some students find it difficult to distinguish between statistical and probabilistic paradigms (the more so than some notions are similar), somewhat analogous with geometrical paradigms. Here, like with GDS, the dynamic feature of spreadsheet, allowing an easy and fast observation of many samples −and consequently many different results (Figure 4)−, can help distinguishing the P2 paradigm (theoretical value) from SD (observed value). But French textbooks do not put the stress on the distinction between the two domains, in particular using the notions in a very loose way (e.g. confusion average / expectation), this probably reflecting actual teaching in classrooms. Similarly, about the instrumental dimension a tendency of textbooks to 'overguide' the students, in order to help them deal with the software, must also be noticed.

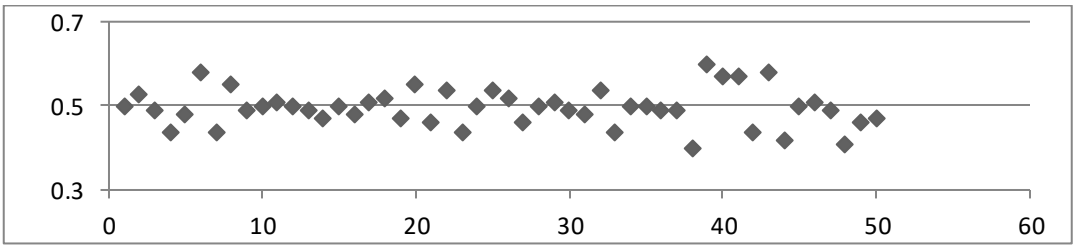

Figure 4: Relative frequencies of heads in 50 samples of 100 tries of heads and tails

2) Simulation can also be used to estimate the value of a probability. For instance, if various models of a same experiment give different probabilities for a given event, a simulation mimicking the concrete experiment can tell which model(s) can be discarded. This is the case with the following problem, which was the basis of an action research with high school students introducing the frequentist approach (Parzysz, 2007):

Toss a well-balanced coin; if head (H) happens you win and if tail (T) happens you toss the coin again; then if H happens you win and if T happens you lose. What is your probability of winning?

In 1754, Jean Le Rond d'Alembert thought that one has 2 chances against 1 to win, at variance with "all authors" claiming that one has 3 chances against 1; Mimicking the process with spreadsheet leads to some difficulty, since one has to distinguish between the two possible results of the first toss. For an easier implementation in the computer the teacher may decide that the coin would be tossed in every case. From a probabilistic point of view the two processes are equivalent but all students are not convinced. (Historically, a similar argument opposed Blaise Pascal with Gilles Roberval in 1654).

The reason for such a reluctance is that the second process is not semantically congruent with the real experiment.

Table 1 shows the corresponding spreadsheets.

Try n°	1st toss	Again?	2nd toss	Result
1	H	no		won
2	T	yes	H	won
3	T	yes	T	lost
4	T	yes	T	lost
5	T	yes	H	won
6	H	no		won
7	H	no		won

Try n°	A coin	B coin	Result
1	H	T	won
2	T	H	won
3	H	T	won
4	T	T	lost
5	H	T	won
6	T	T	lost
7	T	H	won

Table 1: Spreadsheets of the two simulations

When comparing the sheets, one can see that putting anything (H or T) in the empty boxes of the left sheet has no influence on the final result. After that one can forget the "Again?" column and have a second toss in all cases, i.e. replace the initial procedure by the second one without any inconvenience (Figure 5).

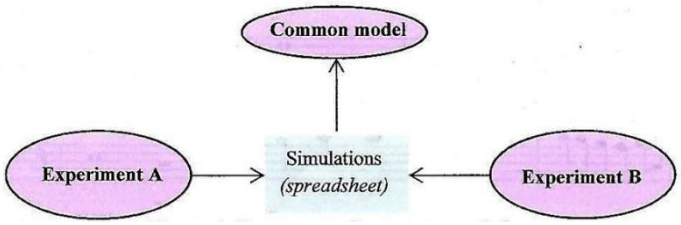

Figure 5: From experiments to model

Thus a visual comparison within the register of double entry tables, in the semiotic-discursive plane, can be a means for deciding if two models are equivalent. And for younger students this can be a possible path towards the bnotion of stochastic model.

3) A most widespread type of activity in French 10th and 11th grades describes a random experiment and then asks the student to simulate it a number of times with the spreadsheet, observe the results and formulate a conjecture about the probability of one of the issues or the possible value of a parameter. Then he/she is asked to solve the problem using the probability theory and compare the theoretical results with the initial conjecture.

In this process several paradigms are at play. As seen above, starting from reality (R), the student shifts to probability (generally P1), then moves to descriptive statistic (DS) to extract information from the spreadsheet (frequency, mean, etc.) and back to P1 to formulate a conjecture; solving the problem within P2 will imply the discursive dimension (Figure 6).

N.B. In such activities the spreadsheet is used as a multi-purpose tool: it intervenes in turn as logical tool (instructions for software), random generator (simulation), copying machine (results of tries), calculator (statistical parameters) and plotter (diagram).

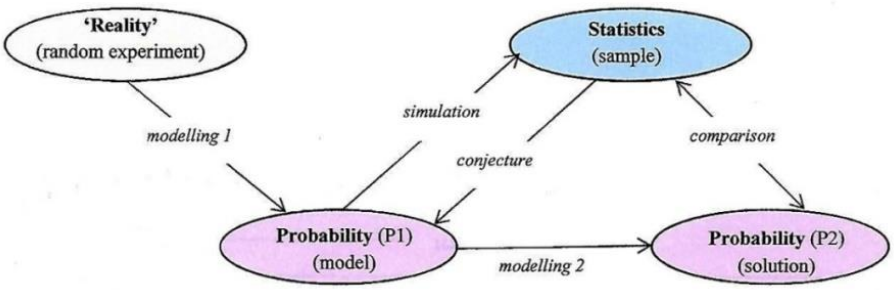

Figure 6: The paradigms and processes involved in the activity

Continuous distributions

N.B. This point is based on a recent research (Derouet & Parzysz, 2016).

The French syllabus for 12th grade includes an introduction to continuous probability laws, and the official resource document for that level suggests starting with a statistical continuous variable, in order to approximate the histogram of a sample by a continuous curve "which fits the histogram, the area under the curve being equal to 1". The general idea is to link a random variable X, not with a set of isolated probability values as was previously the case, but with a function f (density) verifying : $P(a \leq X \leq b) = \int_a^b f(x)dx$ for any a and b with $a \leq b$. This modelling process, which seems quite sensible, implies also a shift from statistics (DS paradigm) to probability (here P2+) and an essential, though transitional, point is histogram. This rises a difficulty, since spreadsheet *cannot* produce histograms, or rather what it calls histogram is in fact a bar chart. In order to overcome this problem one may widen the bars till they become contiguous (Figure 7), but this trick is restricted to cases in which all intervals have the same width. But the notion of density gets sense only with unequal intervals, since in a histogram the basic notion is area, not height. Thus in this case spreadsheet appears to be of no help if the software does not permit producing *real* histograms.



Figure 7: From bar chart to pseudo histogram

This same syllabus includes the study of normal law and recommends introducing the standard law $\mathcal{N}(0, 1)$ from the observation of the distribution of $Z_n = \frac{X_n - np}{\sqrt{np(1-p)}}$, where X_n follows the binomial law $\mathcal{B}(n, p)$. Then the bar chart for Z_n is approximated by the curve of a function of the $x \rightarrow \lambda.exp(-ax^2)$ type. As above a shift from discrete to continuous law occurs, but this time spreadsheet can help since, contrary to the general case, the values of Z_n are equidistant (the distance being $1/\sqrt{np(1-p)}$) and then a pseudo histogram is suitable.

All textbooks follow this scheme, in which three types of diagrams are at play in turn: first bar chart (for Z_n), then pseudo histogram, then bell curve. Spreadsheet is necessary at every stage of the process, first to get the values of $P(X_n = k)$ for $0 \leq k \leq n$) and various values of n and p, then the corresponding

bar charts of Z_n, then its pseudo-histogram and finally the (pseudo-)curve of the standard normal law (in fact a polygon). The main difficulty comes from the histogram and the curve looking proportional but *not equal* (Figure 8) because the distance between the values of Z_n is different from 1 (see above).



Figure 8: Pseudo-histogram and pseudo-curve

This problem of scale is tackled rather awkwardly in textbooks, as is the standardization of the binomial variable. However, the question appears when one wants to compare the shapes of the bar charts for several binomial distributions (Figure 9); one may then think of a 'calibration', i.e. changing the units on the axes, in order to get diagrams with the same average and height.

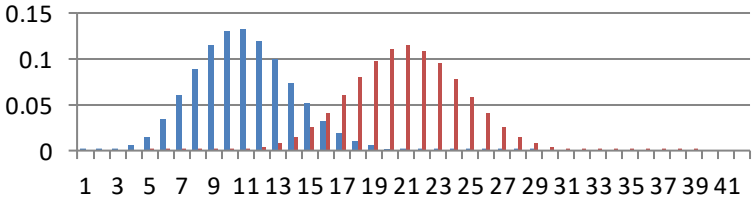

Figure 9: Comparison of binomial distributions ($\mathcal{B}(100, .1)$ and $\mathcal{B}(50, .4)$)

In the process the semiotic and instrumental dimensions of MWS are much appealed to, but the discursive dimension is not much present, due to the students' lack of knowledge.

Conclusion

The current French high school curriculum starts with descriptive statistics (from 6th grade on) and later goes on with probability (at 9th grade), introduced through a dual, frequentist and cardinalist, point of view involving several mathematical paradigms (DS, P1, P2, P2+). Experimentation has become a central issue in teaching probability and in this process spreadsheet extends real tries, for the reason that it is incomparably faster once its use (language, gestures) is mastered. It is now included in the semiotic-instrumental plane of the MWS and can play an important role in many ways and for multiple purposes. Some points are of importance for teaching with simulation, namely pay attention to the model subjacent to the 'real' random experiment (even when it does not clearly appear), help students distinguish between the statistical and probabilistic paradigms, bring out the idea of stochastic model… When coming to continuous probability a sensible way to introduce it consists of approximating a histogram by a continuous curve. Unfortunately usual spreadsheet cannot produce general histograms –i.e. with unequal classes– but only bar charts, becoming possibly 'pseudo histograms', and histograms should have to be produced with another software. However class experimentation showed that a pseudo histogram may prove useful as a transitory artefact in the particular case of shifting from binomial to normal law.

On the whole, although spreadsheet was not conceived for educational but for professional purposes, it has now become a quite appreciable, if not indispensable, tool for the teaching of probability in high school.

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Technology will solve student's probability misconceptions: Integrating simulation, algorithms and programming

Annarosa Serpe[1] and Maria Giovanna Frassia[1]

[1]University of Calabria, Department of Mathematics and Computer Science, Italy
annarosa.serpe@unical.it; frassia@mat.unical.it

The paper presents a proposal of a teaching practice aimed at higher secondary school students, which intends to enhance discussion on the meaning and interpretation of probability, a topic which is often neglected in Italian schools. The authors are convinced that turning the traditional teaching method upside down - that is, proceeding directly to a problem solving approach with the aid of computers as programming tools - better develops in the students the ability to analyse and address uncertain situations correctly, and consolidates a probabilistic mentality. The chosen method was a phenomenon-inductive approach with the help of simulations, which encouraged the formulation of hypotheses and speculation concerning random phenomena.

Keywords: Computer-based simulations, problem posing and solving, programming environment, random phenomena, secondary high school.

Introduction

Probability is a formidable tool for investigating the world around us, so it definitely ought to be part of the education of informed citizens. The importance of the calculus of probability in the higher secondary school curriculum in Italy had already been acknowledged by the ministerial experimental curricula[1]. That is because the topic of probability lends itself well to mathematical and formal elaboration processes of sensible reality, as well as numerous applications which project a dynamic vision of Mathematics open to the real world. Other European countries have also contributed to the debate on the meaning and interpretation of probability, alongside the development of mathematical theory, by including it in the school curriculum (Ahlgren & Garfield, 1991; Shaughnessy, 1992). However, efforts in this direction in most cases have not produced significant results, that is to say the topic has not received adequate attention for a number of reasons, among which the teachers' insufficient education and training stand out (Stohl, 2005). The fact that in Italy the teaching of probability in higher secondary schools has been relegated to a secondary and often marginal level has led with time to a number of misconceptions, which can have dangerous consequences, especially when one takes into consideration the strong growth of gambling. Learning about probability helps to understand the structure of gambling, but probability has much more important applications[2]. The difficulties in the classic approach have so far represented a real obstacle to classroom teaching and learning of the topic, not only in Italy but also in other countries (Batanero, et al., 2005). However, in recent years the growing interest for statistical methods and the use of information technology have contributed to the study of probability as a limit of stabilized frequency (Biehler, 1991). The

[1] PNI (Piano Nazionale Informatica) in the second half of the 80s and Brocca at the start of the 90s.

[2] One only has to think of modern Physics, Biology, Economics, telecommunications, etc.

modelling point of view was adopted in recent years, linking probability teaching with statistical thinking. The introduction of efficient computers in secondary education allows us to simulate models resulting from statistical observations and to introduce students to the large field of statistical inference. The advantage of using simulations is that we can overcome much of the difficulty encountered when using the formal rules. The present paper fits this framework, and proposes an example of teaching practice - aimed at higher secondary school students - which intends to enhance discussion on the meaning and interpretation of probability.

Theoretical framework

In the Italian Secondary school the approach to probability is either classic or frequentist. In particular, the classic approach has long been dominant; but in recent years the frequentist approach has also been used. The classic approach, based on combinatory calculus, is very hard for the majority of students because of the calculations involved in solving the formulae. This perspective introduces an *a priori* approach to probability in which probabilities can be calculated before any physical experiment is performed (Prodromou, 2014). The frequentist approach is based on observations of relative frequencies of an event associated with a random experiment that is repeated a sufficiently large number of times under the same conditions. In this view, experimental probability is estimated as a limit towards which the relative frequencies tend when stabilizing (Von Mises, 1928). The idea of stabilization is based on the empirical laws of large numbers. However, the frequentist approach does not provide the exact value of the probability for an event and we cannot find an estimate of the same when it is impossible to repeat an experience a very large number of times. It is also difficult to decide how many trials are needed to get a good estimation for the probability of an event (Batanero & Díaz, 2012). To adopt one or the other approach does not help the students to understand the meaning and conceptual scope of the calculus of probability; it is even detrimental because it widens the gap between formal education and actual practice (Pfannkuch & Ziedins, 2014). So we need to look for a synthesis of the two approaches focused on problem-based learning, which also stresses connections with inductive applications and reasoning. Today all this is possible thanks to technology, which both supports training activities effectively, and can itself be the object of the training, and deal with the related problems. The use of appropriate didactic software, for example, while drawing attention to conceptual aspects of the methods, also familiarises the students with the handling of data. The programming environments - already familiar to the learners - which are used as mathematic tools to simulate and analyse casual experiments, are very productive. Computer simulation in fact allows to reproduce a model of a real or imaginary system; from a learning point of view, it facilitates learning ‘through discovery' and ‘programmed learning'. Moreover, when teaching probability it is fundamental to make effective and efficient representations which help to clarify and explain why some resolving approaches work and aid generalisation. The practice of programming is an important resource in teaching because it is at all effects an infrastructure for the representation of mathematical objects; in this case, simulation through the realisation of algorithms and their implementation helps the student to understand the importance of using models to describe reality and at the same time to formulate clearly and in detail the theory at the basis of the phenomenon to be represented. This paper presents a teaching proposal which emphasises the value of information technology to support probabilistic reasoning, and at the same time moves beyond pre-packaged didactic software which imply the rigid application of ‘recipes' in limited contexts and cannot be easily applied to models in

real situations. Specifically, the proposal harmonically combines the classic and frequentist approach to probability into a pedagogic perspective which sees the computer as a programming tool in the MatCos[3] environment. The authors aim to study and experiment new curves in the field of teaching-learning of the calculus of probability taking full advantage of computers, whose potential has already been well illustrated by William Feller in his book (New York: Wiley, 1950).

Methodology

The psychology of conceptual learning has shown how concepts and judgment are formed along a lengthy pathway which starts from often confused or even distorted intuitions. These however become progressively clearer as new cases either confirm or disprove the initial assumptions, thanks to the reflections triggered by the new experiences. This is also true for probability, one of the concepts which is most likely to be misunderstood or distorted. Form the didactic point of view, computer programming plays a crucial role in the creation of social interactions which can assist students in the difficult transition from an empiric and objective interpretation of numbers to a relational and functional one. Based on these premises, the choice of methodology has privileged a phenomenological-inductive approach, which encourages the formulation of hypotheses and conjectures on random phenomena. In Figure 1 we can see the synoptic outline of the process:

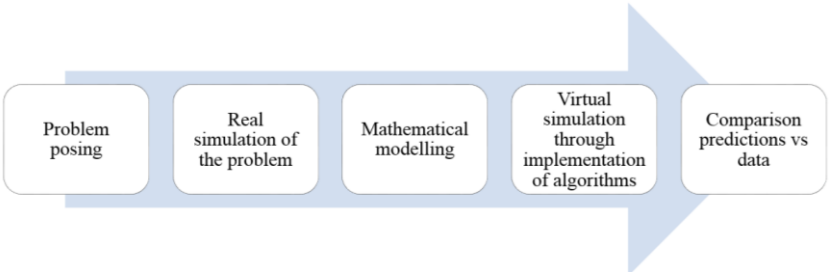

Figure 1: Model of the didactic procedure

The model in Figure 1 consists of five phases, with appropriate intermediate steps, which lead to the next phase. In practice, starting from reality, first we investigate it; then, after mathematisation, we derive rules of behaviour to be applied to reality itself. The process is carried out encouraging inductive reasoning, in an effort to blend the topics and solution methods while leaving plenty of opportunity for discovery. The simulations allow the students to take stock of the situation and start again from what they already know, to make appropriate considerations and understand why phenomena occur, and their implications. In particular, computer simulation through programming represents a constructive and cognitive activity because it enables the student to acquire skills, strategies and techniques for the solution of problems through the concepts of variable, procedure, repetition and reoccurrence; concepts which are also cross-curricular.

Design of the teaching procedure

Here we provide details of all the phases of the didactic model described previously.

[3] Designed and developed by the Interdepartmental Centre for Didactical Research of the University of Calabria, and widely tested as part of MIUR projects (Ministry for Education, University and Research).

Problem posing

The teacher proposes to the students the following problem situation:

> *For a tour in Sicily, a well-known airline company provides a small plane with 23 seats laid out in the following way: the first row has 2 seats while the others have 3. Those wishing to travel must book on-line through a dedicated platform, which allocates seats in a completely random way. Luisa is the first to access the booking platform; how probable is it that the system allocates her a seat in the first row? If Luisa also books for her husband, does the probability that the system allocates at least one seat in the first row remain the same?*

A careful reading of the text is followed by a discussion guided by the teacher. Such a process of verbalisation is important from the cognitive point of view, and represents a first step towards the formalisation of the problem; at the same time, from the constructive viewpoint it becomes a bridge towards the next phase, to be carried out in groups, which implies the real simulation of the problem (Frassia, 2016).

Real simulation

The teacher divides the class into groups giving each group a non-see through plastic urn containing 23 spheres, numbered from 1 to 23, identical in shape and material. The task for all the groups is to simulate the previous problem situation and register the results obtained on a chart. The following is the specific task set for the real simulation:

> "*From the urn containing 23 spheres* numbered from 1 to 23, one *is taken out. Calculate the probability of getting a number lower than 3*".

The students simulate for *n consecutive times* (for example *n=25*) the casual choice of seat by drawing a numbered sphere, recording the results on a two-way chart. When comparing the results obtained by the different groups, the learners realise they are different, and this motivates them to make a sufficiently high number of trials in a limited time. The need for a tool which can aid the learning experience, an instrument able to simulate a high number of repeated trials within a reasonable time, thus becomes evident.

Mathematical modelling

The actual simulation of the proposed problem constitutes an important occasion to highlight the ability to switch from the plane of reality to that of mathematics; but in order for the teaching to contribute to a real understanding of the concepts and a solid acquisition of them, the use of computers as programming tools acquires great importance because it is a method, "a mental place" where students have a real chance to explore mathematical concepts, to formulate conjectures to be validated or refuted, and then to continue the experience of problem solving (Frassia, 2015). Mathematical modelling requires the students to reproduce some aspects of sensible reality in order to analyse and study them. Moving from the experience to the construction of the meaning requires the construction of a simple algorithm representing the simulation of the event. Being able to work within a representation register and going from one to the other - that is what Duval (1993) calls 'competences of dealing with and converting' - is fundamental because the meaning of mathematical objects is accessible only through their representations. The probabilistic model is made explicit and reviewed in a logical sequence thanks to the algorithm; furthermore, the use of a programming environment

like MatCos adds value because it helps the students to reinforce their skills in handling mathematical language (Costabile & Serpe, 2009, 2013). The construction of the algorithm is an important and delicate phase because the students have to design the 'finite sequence of steps' that allows the computer to get to the solution. The steps of the algorithm are:

Step 0 Assignment: *n* (Simulation number).
Step 1 Initialization: *cf = 0* (counter for the number of favorable cases).
Step 2 Cycle (simulation of *n* prove consecutive)
- Creating one variable *a*;
- Control action: if (*a* = 1 o *a* = 2) then do
  - Increase counter *cf*.

Step 3 Calculation: *p=cf/n*.
Step 4 Print action: *cf*.
Step 5 Graphic representation: histogram of the absolute and relative frequencies.

Virtual simulation

The previous algorithm is easily implementable in MatCos.

Code MCS1

```
n=readnumber; cf=0;
for(i from 1 to n) do;
 a=int(random(1,23.99));
 if((a=1)o(a=2))then do;
  cf=cf+1;
 end;
end;
print("In ", n , " extractions, a number lower
3 is taken out ", cf , " times ");
v=Array(2); v(1)=cf; v(2)=n-cf; histogram(v);
w=array(2);w(1)=cf/n;w(2)=1-cf/n;
histogram(w);
```

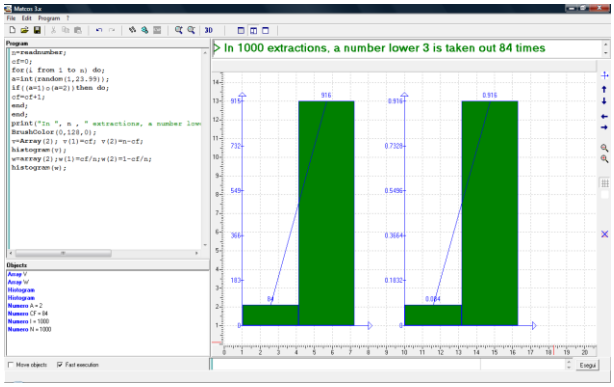

Figure 3: Output of simulation for n = 1000

The students now proceed to the algebraic solution of the set problem and compare the results with the numerical values of the relative frequencies obtained in *output*. Summing up, considering the event:

A = "*Draw a number lower than 3, drawing a sphere from a pool of 23 spheres numbered from 1 to 23*"

The number of favourable cases and the number of possible cases related to the event *A* are:

$$cf = 2 \qquad cp = 23$$

So, the probability of the event *A* is:

$$p(A) = \frac{2}{23} \approx 0{,}087.$$

From a comparison of the results obtained in *output* with the theoretical ones, the students realize that when the number of repeated trials increases, the value of the relative frequencies gets closer to the real value of the probability.

A question still standing…

The first question has now been answered, but the second is still standing:

If Luisa books also for her husband, does the probability that the system assigns at least one seat in the front row remain the same?

The corresponding question is made explicit in the following task:

"*From an urn containing 23 spheres numbered from 1 to 23 we pull out two spheres. Calculate the probability that at least one is lower than 3*".

Supported by the experience so far, the students decide to start directly from the virtual simulation and so make some changes to the previous algorithm. They are convinced that the two questions must have the same solution, and that the teacher just wants to trick them.

Revisiting the previous algorithm implies further steps.

Here we report the steps of the algorithm and one *output*.

Step 0 Assignment: *n* (Simulation number).

Step 1 Initialization: *cf = 0* (counter for the number of favorable cases).

Step 2 Cycle (simulation of *n* prove consecutive)

- Creating one variable *a*;
- Control action: if (*a* = 1 o *a* = 2) then do
  - Increase counter *cf*.

Else do

- Creating one variable *a;*
- Control action: if (b = 1 o b = 2) then do
  - Increase counter *cf*.

Step 3 Calculation: p=cf/n..

Step 4 Print action: *cf*.

Step 5 Graphic representation: histogram of the absolute and relative frequencies.

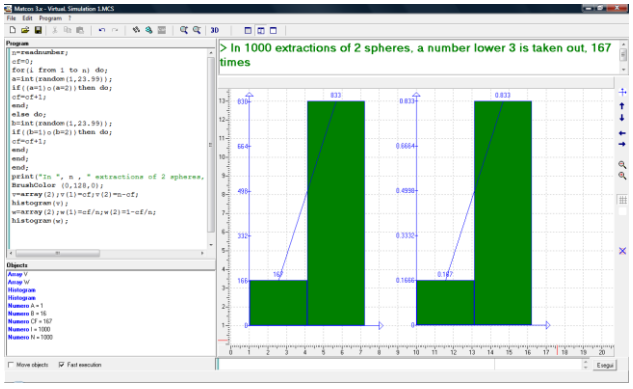

Figure 4: Output of simulation for n = 1000

The students now proceed to the algebraic solution of the set problem and compare the results with the numerical values of the relative frequencies obtained in *output*. In the drawing of 2 spheres from a pool of 23 spheres numbered from 1 to 23, the students consider the event:

E = "*Obtain at least a number lower than 3*"

The event complementing event *E*:

E^C= "*From the drawing of 2 spheres from a pool of 23 spheres numbered from 1 to 23, both are lowered than 3*"

So:

$$p(E) = 1 - p(E^C) = 1 - \frac{21}{23} \cdot \frac{20}{22} = \frac{19}{55} \approx 0{,}170$$

The students by now have all the necessary information to answer the second question in the problem, which requires a comparison of the probability of the two events: event *A* and event *E*. From such comparison it becomes apparent that the probability of the two events does not coincide. So the students' prior conviction that the two questions of the problem - albeit expressed in formally different terms – share the same solution is proven wrong. The probability of event *E* is indeed bigger than the probability of event *A*.

Conclusions

The teaching proposal increases students' confidence on the effectiveness of statistical methods, and at the same time raises awareness of random events. The difficulties encountered during the learning phases of the mathematics of uncertainty can thus be overcome. Virtual simulation, through the practice of programming, plays an essential role because it helps the student to develop good *problem solving* skills. In particular, simulation aids the understanding of the concept of probability of an event, assigning to it a 'degree of reliability' in the prediction of random phenomena. The novelty of this proposal is the setting up of an environment for the learning of probability, showing the close link between probability and statistics thanks to a very specific task (problem posing). The objective is to insist on the role of intuition because, in the majority of cases, students' probabilistic intuitions lead to erroneous convictions and answers. The use of computer programming enables the students to take explorative steps which can lead to the solution of a problem and trains their ability to 'anticipate' and "being able to see" in mathematics. The learner through the programme breaks down complex concepts into simple ones, thinks of and adopts new solution strategies, and compares with previous results. In so doing learners expand their mental processes and consolidate constructive knowledge. After all, education that promotes informed and solid learning cannot fail to redress wrong perceptions on the notions of probability and encourage reflection on its conceptual implications, but this had already been pointed out by Bruno De Finetti in 1967.

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High school students' first experiences with the sampling distribution: Toward a distributive perspective of sampling and inference

Eleazar Silvestre and Ernesto Sanchez

CINVESTAV-IPN, México; eleazar.silvestre@gmail.com, esanchez@cinvestav.mx

We report some initial results of an ongoing investigation about Mexican high school students' (17-18 years) reasoning when working with the sampling distribution (SD) for the first time in a regular classroom setting. Learning activities involve the usage of simulations and are part of a hypothetical learning trajectory (HLT) aimed at fostering a distributional and stochastic perspective of sampling and inference. We describe some appropriate strategies and limitations that students exhibited as evidenced by their a priori reasoning.

Keywords: Sampling distribution, statistical reasoning, high school, inference.

Introduction

Several errors and limitations have been reported on statistical and probability education in the past decades (Batanero, Godino, Vallecillos, Green & Holmes; 1994). In recent years, the development and availability of educational software in these areas have allowed some researchers to explore the idea that notions or informal versions of different objects and processes embedded in such disciplines, such as statistical inference, can be introduced before tertiary educational levels (Zieffler, Garfield, delMas & Reading; 2008). In particular, we point out that a number of proposals use empirical SD's and random simulations as key resources for developing appropriate conceptions and reasoning of fundamental statistical ideas, such as sampling, estimation and inference (e.g. Rossman, 2008; Batanero & Diaz, 2015; Garcia & Sanchez, 2015). Hence, we consider SD's to be a strong candidate and a starting point for instruction and teaching, with assistance of technology, when introducing statistical inference. We report on some preliminary results of an ongoing investigation that uses an HLT (Simon & Tzur, 2004) aimed to develop and foster a distributive and stochastic concept of sampling based on the SD. We focus on the initial learning activity that explores students' notions of probability, classification of most and least probable sample values, and the use of average as a measure for representation and summary of the SD. Related literature, method of inquiry, analysis of students' performance and some final reflections and discussions are described.

Some related literature

There are a variety of interrelated elements that intervene in the process of construction, applicability and interpretation of statistical inference such as sample and population, statistic and parameter, distribution and variability, probability and significance, etc. Some researchers suggest that several conflicts and limitations of students, teachers and even professionals regarding their capacity to draw inferences from sample data are strongly related to a lack of or poor formation of concepts such as variability, sampling variability and SD's. For example, a review by Herradine, Batanero & Rossman (2011) points out that people in general hold deep misconceptions about sampling and inference (e.g. representativeness heuristics, law of small numbers, incapacity to reason about many samples) that traditional teaching does not help to overcome, as it tends to focus on calculus and computations instead. Chance, delMas & Garfield (2004), and Liu & Thompson

(2007) also indicate that a deficient or limited understanding of concepts such as the SD and probability could inhibit an appropriate reasoning of statistical inference.

On the other hand, Fathom is a specialized software developed to assist teaching and learning in statistics and probability. It offers the possibility to, among other capabilities, generate random simulations that can cause the emergence of abstract and complex mathematical objects such as the SD. It also allows access to analyze how this concept behaves through the manipulation of its parameters (sample size, population parameters, number of samples); this opens the possibility for didactic treatment substantially different from the traditional one that parts from a theoretical posture. These are key resources that can potentially promote and foster students' statistical thinking (Chance, Ben-Zvi, Garfield & Medina, 2007; Biehler, Ben-Zvi, Bakker & Makar; 2013) and, particularly, can introduce ideas of the logic behind statistical inference (Rossman, 2008).

However, only a few of the aforementioned studies target high school students, with practically none having been conducted in Mexico. The actual curricula incorporate very few, if any at all, statistical and probabilistic content for elementary and secondary education; it is only for some high school programs that introductory courses of statistics and probability are included. In addition, traditional practices use no technology to assist teaching and focus heavily on computations. This makes us believe that students' lack of experience with sampling and statistical inference within schools' practices make them become highly dependent on wrong or inappropriate intuitions and misconceptions when making inferences. The presence of this conflictive situation, and the availability of research results that propose alternative ways to overcome such difficulties, have motived us to set our initial phase of this investigation: to explore and gain insight as to how these students reason when working with the SD, and with the assistance of Fathom (simulations of random samples), in order to better understand the extent of their reasoning and potential learning.

Method

A regular high school classroom of 44 students participated in the study (28 female and 16 male); they had previously taken an introductory course (*traditional* style) to Statistics and Probability containing descriptive graphics, measures of center and variability, introduction to classical and frequentist approach to probability, calculations of probabilities for simple events, probabilities for conditional and independent events. Students didn't have any previous experience with Fathom.

This particular learning activity is divided in two stages and was applied in two class sessions of two hours each (one stage per session). The problem was posed/introduced to students (described below) during the first stage and, in the second, they were allowed to work freely in pairs to respond some questions using the software. Data for analysis consist of students' responses (using digital worksheets), observations of two teachers/researchers who also attended the sessions and some brief interviews. We are using principles and initial techniques of the Grounded Theory methodology (Glaser & Strauss, 1967; Birks & Mills, 2011) to analyze students' responses; specifically, we are using: initial purposive sampling, initial (and open) coding, and concurrent data generation and analysis. These techniques have oriented our methodological approach by: (1) selecting appropriate participants to study the phenomena of interest (students' reasoning); (2) depart from a pre-established theoretical framework and apply an open codification and categorization of data (based on incidents-patterns in procedures and arguments students exhibit

when resolving the activities); (3) comparison between incidents-patterns and codes for a greater refinement. At this point, the initial codes provide a general description and evidence of students' reasoning; our ongoing analysis focuses on identifying core categories, something at this time beyond the scope of this communication.

The HLT and the unfolding of the learning activity

The main goal of the HLT is to encourage and provide students with an image of sampling and inference grounded on the study and exploration of the SD from a stochastic and distributional perspective; a conception set apart from the one that usually consists of an image that just reflects a "mini version" of the population. This HLT includes eight learning activities organized in three phases: 1. Introduction to the SD (to obtain estimations of probabilities for samples' outcomes; to generate an unambiguous method to identify usual or typical sample results and; to relate the average of the SD with the population's parameters); 2. Analysis of the SD when sample size is modified (changes in form, sampling variability, estimations of probabilities and typical sample results); 3. Estimation of an unknown population's parameters (assessing estimations based on sample size). In addition, after the three phases are completed, students must face a situation that requires the usage of (informal) hypothesis testing; a key aspect of interest in our research is to analyze students' performance in order to evaluate how much they draw on their knowledge about the SD and aforementioned concepts to do so.

The selected learning activity for this communication is the very initial one of the HLT and the core's first phase. It is based on Saldanha & Thompson's (2007) exploration of students' thinking of sampling and inference, with the distinction of being focused on specific sample outcomes. Due to space restrictions, we do not include explicitly all the components for this activity (learning objectives and hypothesis about the learning process) but briefly describe the unfolding of the mathematical task and its procedure of application.

The first stage of the activity follows a dynamic of an open class discussion guided by the teacher/researcher. It begins by presenting Mr. B to students: a small-medium size container composed of 7,000 beans, of which 50% are black beans and the rest 50% are pinto beans (Figure 1); Mr. B represents a physical dichotomous population whose composition is initially unknown to students. The main interest at first is to estimate the total percentage of pinto beans based on a small sample outcome. After one student draws by hand a random sample of 10 beans from the jar, the starting question that opens discussion is: "is this result (sample's %) enough evidence to assure that Mr. B has X% of pinto beans?" Some students are expected to intuitively detect sampling variability and ask to obtain more samples; the teacher encourage students to explore their argument by allowing them to draw as much samples as they consider necessary to propose a parameter.

We have observed that students don't pay much attention to high levels of variability and tend to hand-draw only a small collection of samples (less than 15), and then roughly conjecture a population percentage out of it using the frequencies, mode or average. After making this proposal (most likely different from 50%), the parameter is revealed to students and their "failed" estimation produces an engagement that is used to explore what parts of the process can be refined. Once the parameter is known to students, the exploration starts by the teacher proposing the classification of the obtained sample values as "favorable" (such as 40% - 60%) or "unfavorable" (replacing

Saldanha and Thompson's "usual/unusual" to highlight the tension between the presence of sampling variability and the expectation of obtaining sample outcomes that match or are very close to the parameter); the next step is to investigate how is it possible to quantify how expectable these results might be and how much for those considered the opposite. That is, we land and focus on obtaining probabilities of sample outcomes through the SD.

Then the teacher guides discussion to the proposal of generating many samples by posing the questions "What if we repeat this experiment a lot of times? Could the generation of many samples help us to calculate a probability or identify which are the most and least likely values?". The second stage of the task begins and Fathom is used to simulate a random collection of 300 sample outcomes and to generate a graph/distribution out of them (an empirical binomial SD emerges, with n = 10 and P = .1, that students can generate repeatedly instantly using the software). Then the students are organized in pairs to freely respond a series of questions using the software and digital worksheets, which aim to explore key aspects of the SD such as estimations for probabilities, most frequent/probabilistic values and the identification of center. We extract for this report the analysis of the following items:

a) Express a number from 0% to 100%, which represents how likely you consider it that you'll get a value of 70% of pinto beans in the sample #301 (n = 10). Write down your procedure and any calculations you made.

b) Which of all possible sample percentages do you think are the most likely to obtain in sample #301? Which are the ones you consider to be the least likely? Describe the method you used to select those values.

c) Express a particular value that summarizes and represents all 300 percentages obtained in a particular simulation. Explain your procedure and include any computations you made.

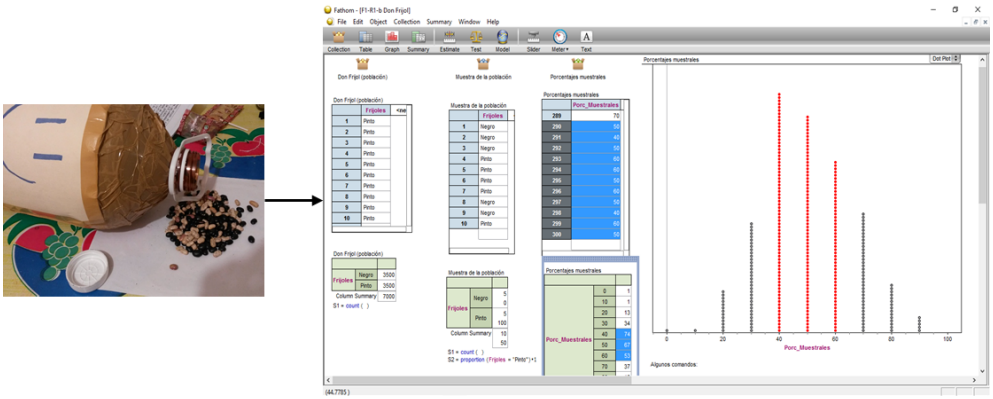

Figure 1: Mr. B (left) and display of an empirical SD using Fathom (right)

Results: Students' responses

Codes for summarizing students' responses in question (a) are:

	Codes: Procedures and arguments
Makes computations: 6 pairs	Computes correctly an estimation for the probability: provides intervals or particular values – (6)
Makes no computations: 16 pairs	Assigns a probability of 30% or 40% and points out one or two of the following features: the sample value (70%) is less likely to obtain due to its low frequency; considers that the values of 40%, 50% and 60% are the ones that present the highest frequencies (most likely values); specifies the sample's value frequencies – (6) Makes no numerical assignment and only points out one or two of the previous features – (3) Assigns a probability of 35% or 0% because they consider that sample values have those tendencies – (2) Others: Assumes the maximum frequency of the sample value as the probability; assigns 10% because it's one of 10 possible options (of the random variable); assigns 35% as the probability, incoherent argument – (5)

Table 1: Students' quantification of probability of a particular sample value (numbers in parenthesis indicate frequencies/number of pairs with the same type of answer)

Two main groups of responses appear; the first one consists of six pairs of students that made an appropriate estimation for the probability of obtaining a specific sample value; five of these gave a particular value for the probability regardless of the number of simulations they made of the sampling distribution, while only one of them expressed it in a form of an interval. For example, pair 16 (P16) mentioned:

> "We estimate that sample #301 has a probability of 10%-13% of obtaining 70% of pinto beans because we observe that, when generating different samples, the sample value of 70% appeared in a range of 30 to 40 times, which equals to 10% -13%"

It seems that the other group of students was not able to quantify correctly their expectation of obtaining the value of 70% and, instead, they assigned (six pairs) a value of 30% or 40% as the probability with no computations but most likely based on the frequency of the sample value. Nine of these 16 pairs also mentioned some features they observed in the SD, such as values of very low or high frequencies that they considered were related to a high or low expectation (probability) of obtaining the sample value. For example, P9 answered: "40% because when running the simulation, the value of 70% was not one that appeared constantly or continuously more than the previous values.". Codes for students' responses in question (b) are:

	Codes: Procedures and arguments
Arguments based on frequencies: 14 pairs	Sample values of 40%-60% because they present the highest frequencies (specifies a particular value for the frequency) – (1) Sample values of 40%-60% because they present the highest frequencies (points out some sample values or the interval) – (13)
Includes or based entirely	Sample values of 40%-60% because they present the highest frequencies and because the population's parameter equals 50% – (1)

on the proximity criteria: 6 pairs	Sample values of 40%-60% because they are the closest to the population's parameter (points out some sample values or the interval) – (4) Sample values of 40%-60% because the population's parameter equals 50% – (1)
Others: 2 pairs	Sample values of 40%-60%, redundant answer or incoherent argument – (2)

Table 2: Students' selection of the most likely/probable sample values (numbers in parenthesis indicate frequencies/number of pairs)

Fifteen pairs of students used the highest frequencies to determine which sample values they considered to be more likely, but only one of them specified a value for the frequency as a main reference to compare the rest. These two types of responses are shown by P2: "40%, 50% and 60% because their frequency is higher to 55"; and P22: "The values of 40%, 50% and 60% because the Fathom graph shows us that these percentages appeared the most". Six pairs argued that these percentages are the most likely since the population's parameter equals 50% (five pairs' responses are based entirely on this), which is what we call the "proximity criteria" (note that this type of thinking is devoid of reasoning with the main objects at hand, such as frequencies, image of distribution or sampling variability embedded in the SD). To exemplify this, P15 answered:

> "40%, 50% and 60% because the quantity of pinto and black beans is the same (50-50) but it's not likely that we will get 50% pinto beans in every sample, so the closest percentages are 40% and 60%."

Due to space restrictions, we do not include responses for the selection of the less likely values; however, we mention that the obtained responses match closely with those previously described. In this case 21 pairs referred to the values of 20% or less and 80% or more (special emphasis on 10% and 90%) as the least likely percentages; only one pair mentioned the values of 30% and 70%. Codes for students' responses in question (c) are:

	Codes: Procedures and arguments
Uses (or proposes to) the average: 10 pairs	Computes correctly the average of the sampling distribution – (3) Computes incorrectly the average (applies another method to high frequencies of sample values) – (2) Computes incorrectly the average (applies the "rule of three" based on a single sample value of high frequency) – (1) Computes incorrectly the average, does not specify method – (1) Only proposes to use the average, makes no computations – (3)
Uses the mode: 10 pairs	Uses the mode (50%) as a representative for the sampling distribution – (6) Uses a biased-mode (values greater than 50% that are not in the set of the random variable values) – (3) Uses a biased-mode (less than 50%) – (1)
Others: 2 pairs	Values of 50% or 60%, does not justify or explain the reasoning – (2)

Table 3: Students' numerical representation of the sampling distribution (numbers in parenthesis indicate frequencies/number of pairs)

The first 10 pairs of students decided to use the average to represent the sample values but only three did so correctly. Three out of four pairs that incorrectly computed the average mistook sample

values for their frequencies, didn't include all values and applied a different method; three more pairs only proposed to use the average but seemed incapable to compute it. For example, P11 mentioned (considers some high frequencies and calculates a percentage):

> "We made an average out of the percentages that repeated the most and we came to the conclusion that this percentage value is at least of 68% of the 300 percentages. The calculations were to obtain a percentage of the percentages that presented the highest frequencies: 40 with 60 times, 50 with 82 times and 60 with 62 times. Computations: 60 + 82 + 62 = 204 / 300 = 0.68 * 100 = 68%"

The next 10 pairs decided to use the mode but four of these failed to do so correctly; four of them used a biased mode, where three selected values greater than 50% that are not included in the random variable. As an example of this, P21 responded: "Of the 300 simulated percentages, we estimate that the value that repeats the most is 54% because no sample is equal to the previous one and most of the time we get something greater than 50%".

Conclusions and discussion

The procedures and arguments showed above are evidence of an a-priori reasoning since we consider our students have not been previously instructed on this type of activities (despite their introductory course). In this study, we identify some appropriate reasoning such as students relating high and low frequencies of sample values to high and low probabilities of obtaining them (even a few calculated correctly this estimation); and the use of the average and mode to summarize and represent the sample outcomes of the SD. Also, we identify limited or biased reasoning such as an incapability to calculate reasonable estimations for probabilities of sample outcomes; an ambiguous method to classify the most or least probable sample values (there's no definition for "the highest/lowest frequencies"; sample values of 30% and 70% are missing for classification); the use of the "proximity criteria" as a resource that avoids reasoning with the available data and is based in distance or proximity to the population's parameter; and limitations to calculate an average (despite "having all data at hand"). We believe these kinds of learning activities helped us significantly to expose students' conceptions about sampling and probability when working for the first time with the SD. The evidence suggests that students are capable of grasping and displaying important statistical-probabilistic notions, as well as resources, when placed in a rich-exploration and problem based environment. These constitute a starting point for researchers (and teachers) to assess and target more efficiently the different learning objectives of statistics education from a more constructivist point of view; it is the focus on the limitations founded in their strategies what can potentially and progressively re-orientate their reasoning to a more appropriate one. Overall, we strongly feel that more studies that can exhibit and foster students' reasoning are still needed in order to better understand and influence more efficiently students' learning of the complex (and necessary) discipline that is statistical inference.

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Using student models to generate feedback in a university course on statistical sampling

Sietske Tacoma, Paul Drijvers and Peter Boon

Utrecht University, Faculty of Science, Utrecht, The Netherlands; s.g.tacoma@uu.nl

Due to the complexity of the topic and a lack of individual guidance, introductory statistics courses at university are often challenging. Automated feedback might help to address this issue. In this study, we explore the use of student models to provide feedback. The research question is how student models can be used to generate feedback to university freshman in an online course on statistical sampling. An online activity was designed and delivered to 40 Biology freshmen. Instruments for generating student models were designed and student models were generated. Four students were interviewed about the generated models, and about the differences with their own estimation of their understanding. Results show that it is possible to generate individual feedback from student work in an online learning activity and suggest that discussing differences between own estimations and generated student models can be a fruitful teaching strategy.

Keywords: Statistics, feedback, educational technology, higher education, student model.

Introduction

Many bachelor programs offer introductory courses in statistics. Success rates for these courses are often low, which makes them an obstacle for students in obtaining a bachelor's degree (Murtonen & Lehtinen, 2003; Tishkovskaya & Lancaster, 2012). One challenge is that such courses are often taught to large groups of students, making it difficult for teachers to provide individual guidance. The main challenge concerns the complex topics in these courses. One of these topics is statistical sampling, which involves concepts such as sampling distributions and sampling variability. Understanding statistical sampling does not only require students to understand these concepts, but also the connections between them (Castro Sotos, Vanhoof, Van den Noortgate, & Onghena, 2007).

To address both the problem of individual guidance and the complexity of the subject matter, an approach using automated individual feedback can be promising. If appropriately designed and timed, feedback has the potential to increase student learning (Hattie & Timperley, 2007; van der Kleij, Feskens, & Eggen, 2015). Especially for large groups of students, automated feedback can add individual support that would otherwise be unattainable. For the case of statistical sampling, such automated feedback should aim at student understanding of the complex concepts involved. This asks for feedback on a global level, aggregated per concept, rather than on a local, task level. To gather input for such an overview, a specific digital assessment activity might be set up, but this would take valuable instruction time. A less time-consuming option may be to use student work in an online learning activity. Although such an activity is primarily designed for learning rather than assessing, it might still be possible to track the students' evolving understanding (VanLehn, 2008).

The question now is: how can we use students' solutions to tasks in an online learning activity to generate feedback on their knowledge of statistical sampling? This paper makes a start in answering this question by describing a prototypical approach through the use of student models.

Theoretical background

The theoretical background of this study includes two main elements: an analysis of the difficulties in the domain of statistical sampling, and the notion of student model.

Difficulties in the domain of statistical sampling

Samples are the key instruments to make inferences about a population. An important idea in making these inferences is that samples provide useful, but not complete, information about the population. This idea relates to two concepts: sample representativeness and variability. Sample representativeness means that for properly selected samples, sample characteristics will likely resemble those of the population. Sample variability means that not all samples are equal, and that sample characteristics do not necessarily meet population characteristics, and may not even be close. Making inferences from a sample involves a trade-off between these two concepts; a balance that is influenced by factors such as sample size and population variability (Batanero, Godino, Vallecillos, Green, & Holmes, 1994).

Castro Sotos et al. (2007) identify three main misconceptions that students may have about samples and sampling distributions. The first concerns the effect of sample size on the variance of the sample mean: as sample size increases, sample characteristics are likely to approach the population characteristics more and more. Many students misinterpret this so-called law of large numbers and use the sample representativeness heuristic to conclude that any sample's characteristics should be very similar to those of the population. The second misconception concerns the different distributions involved. Students often confuse the distribution of one sample of data with the distribution of sample means for several samples (Chance, del Mas, & Garfield, 2004). This can, for example, result in confusion between the standard deviation in a sample, the mean of the standard deviation over many samples, and the standard error of the sample mean. The third misconception concerns the central limit theorem, which states that for sufficiently large sample sizes, the sampling distribution of the sample mean can be approximated by a normal distribution. Students tend to wrongly extrapolate this theorem and believe that the larger the sample size, the closer the distribution of any statistic in the population will approximate a normal distribution (Bower, 2003).

Automated feedback through student models

In many online learning environments, automated feedback is offered at a task level or even at a step level. However, as we are interested in the students' conceptual understanding of sampling, it seems more relevant to provide the students with an overview of their knowledge of the entire domain, which, of course, is still based on the scores on single tasks. Such an overview for an entire domain is often called a student model (Brusilovsky & Millán, 2007; Bull, 2004). Student models can be used to adapt the educational intervention (i.e. the series of tasks) to the specific needs of the individual learner (Bull, 2004) and are in this role often invisible to the student. However, opening up the student model can promote learner reflection on his knowledge and understanding, and may help learners to monitor and plan their learning (Bull & Kay, 2007; Sosnovsky & Brusilovsky, 2015).

A student model contains a domain model and an overlay. The domain model consists of knowledge components (KC's) that each describe a piece of knowledge in the domain. All tasks in the learning activity are connected to one or more KC's in the domain model. The overlay contains a score for each KC, based on the student's performance on connected tasks, which describes the student's current understanding. The KC's in a domain model can be more or less coarse grained. An advantage of a fine-

grained domain model is that it enables a very sophisticated and precise diagnose of the student's current understanding. However, when using a course-grained domain model, connections between tasks and the domain model are much easier to manage, while still a reasonable diagnosis can be accomplished (Sosnovsky & Brusilovsky, 2015).

In the light of this theoretical framework, the research question addressed here is: How can student models be used to generate feedback to university freshman in an online course on statistical sampling?

Methods

To address the research question, a prototypical environment to generate feedback on the understanding of statistical sampling through the use of student models was set up. In this explorative design research, 40 freshmen Biology participated. The design included an online activity on statistical sampling, and a domain model and Q-matrix for generating student models. Data collection included digital student work, a questionnaire and interviews with students. Analysis aimed at choices in generating overlays and describing the students' reactions to their student model.

Design of an online activity on statistical sampling

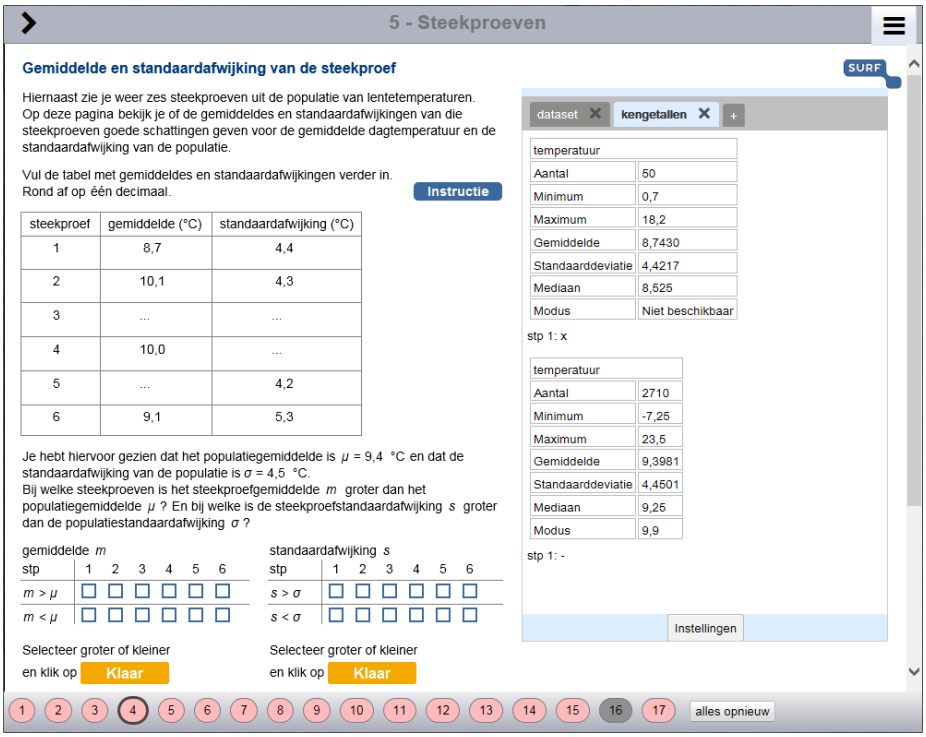

Figure 1: Example page of the DME-activity on statistical sampling

The online activity on statistical sampling was designed in the frame of the Utrecht University project "Innovative remedial digital learning modules for statistics", by an educational designer and the researcher (first author), in close collaboration with the teacher of the statistics course for biology students. For the design, the Freudenthal Institute's Digital Mathematics Environment (DME, see Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013) was used. Aim of the designed activities was to deepen the students' understanding of statistical sampling and sample variability. The activities contained theory, a simulation on sampling and questions about the students' intuitions, the simulation, and the theory. The difficulties described in the theory section were addressed extensively. Students were able to enter answers to all

questions and receive immediate feedback on the correctness of their response. For many tasks, hints and feedback on incorrect responses were designed. For an example page of the activity, see Figure 1.

Development of a domain model and Q-matrix

Through studying theory on statistical sampling[1], a set of knowledge components (KC's) for the domain of sampling was identified by the researcher. As the intended use of the domain model was to present it to students, a rather coarse-grained approach was chosen and too detailed KC's were avoided. Moreover, for a clear presentation to students, complete descriptions of the KC's were formulated, as opposed to one or two words per KC. Four main KC's were identified: Taking samples (procedure), Estimations based on a sample, Distribution of the sample mean, and Standard error. For each main KC, four detailed KC's were identified.

After the domain model was designed, all tasks in the module were linked to the corresponding KC's by the researcher. This resulted in a Q-matrix, in which entry (i, j) is 1 if task j is related to KC i, and 0 otherwise. The module contained 45 (sub-)tasks in total. For twelve tasks, the researcher judged that no KC's were relevant. In Figure 1, for example, the students are asked to read off values from a table. This activity helps students understand the table-tool they will be working with in this module, but how well students read off the values does not involve their knowledge of any of the KC's. For four subtasks, more than one KC was judged to be relevant. All other subtasks were connected to one KC.

Data collection: Student work, questionnaire and interviews

The participants in this study were 40 biology students, who participated in the first year introductory course Experiment & Statistics at Utrecht University. The students first attended a lecture on sampling and worked on the designed online module in the week following the lecture. Three sources of data were collected in this small-scale explorative study:

- Student work: the DME stores all student work, including all attempts that students do before reaching a final answer. Student work was collected for all 40 students;
- A questionnaire, in which the KC's from the domain model were presented to students. Students were asked to give their own estimation of their understanding of each KC. The questionnaire was completed by seven students.
- Interviews, in which students were questioned about the appropriateness of the domain model and generated overlay, and about differences between their estimated and generated overlay. Out of the seven students who completed the questionnaire, four were interviewed.

Data analysis

In the analysis, the students' attempts were extracted from the DME, exported to Excel and prepared for generating overlays. For each student, an overlay was generated. Next, the generated overlays for the four students who would be interviewed were studied and remarkably high and low scores were recorded.

[1] Sources used are two text books on introductory statistics: Whitlock, M., & Schluter, D. (2009). *The analysis of biological data*. Greenwood Village, Colo: Roberts and Co. Publishers, and Field, A. P. (2009). *Discovering statistics using SPSS: (and sex and drugs and rock 'n' roll)*. Los Angeles [i.e. Thousand Oaks], Calif.: SAGE Publications.

Explanations were sought by studying tasks connected to the remarkable KC's and by analysing student explanations in the interviews.

The interviews were transcribed and the students' answers were aggregated by topic. Next, summaries for each topic were written to create a general image of the students' reactions to the overview, and to identify issues in the current calculation of the overlays.

Results

In addition to the domain model and the Q-matrix, which are described in the Methods section, the results of this study include the generation of overlays, and the students' reactions to their student model.

Generating overlays

To generate overlays, the students' interactions with the DME had to be translated into scores for each KC. Student interactions with the DME are stored as attempts. Three attempt types are possible: correct attempts, half-correct attempts (for example when a student still needs to round off an answer) and incorrect attempts. To generate an overlay, correct attempts were counted as 1, half correct attempts as 0.5 and incorrect attempts as 0. For each task, the mean attempt score was calculated by dividing the sum of the attempts by the number of attempts. For each KC, the overlay score was calculated as the mean attempt score of all tasks that were connected to this KC in the Q-matrix. See Figure 2 for an example.

KC	P1	P2	P3	P4	E1	E2	E3	E4	D1	D2	D3	D4	S1	S2	S3	S4
Generated overlay	1,00	1,00	0,88	—	0,89	0,50	0,61	0,75	0,83	0,67	0,75	0,67	0,80	0,50	0,57	0,83
Student's estimation	0,80	0,80	0,70	0,70	0,70	0,80	0,80	0,70	0,70	0,80	0,70	0,60	0,70	0,70	0,80	0,80

Figure 2: One student's generated overlay and his own estimation

The method used is not the only possible calculation method. Another option that was considered is taking only the student's first attempt for each task into account. The student's subsequent attempts are guided by the DME's immediate feedback and therefore do not directly reflect the student's knowledge, but rather a combination of this knowledge and the student's reaction to the immediate feedback. However, this method neglects the fact that students are likely to learn from the immediate feedback, and therefore this approach is left out of the analysis.

The generated overlays are different for different students, which confirms that they provide individual feedback indeed. Moreover, the scores for each individual student were more or less spread out, so students did score different for different KC's. This shows that our way of calculating discriminates between KC's, and hence can inform students on their understanding of the different KC's.

Students' reactions to their student model

The domain model was first presented to students in the questionnaire, in which students were asked to estimate their own overlay. In the interviews, the domain model was presented again, this time with the generated overlay. All four students said that they understood the domain model well and thought it formed a useful summary of the domain of statistical sampling. One student explicitly mentioned that he would use the domain model in his exam preparation.

The comparison between the generated overlays and the students' own estimations resulted in fruitful exchanges. Most students seemed to adopt the generated overlay as a true representation of their knowledge. Three students seemed to adjust their own estimation to the generated overlay. For example,

when seeing the generated overview, one student concluded: “Apparently I can give myself higher grades than I did.” The fourth student, however, thought that the activities in the DME were easier than other activities in the course, and therefore thought her knowledge of the topic was not as good as her work on the DME-activity suggested.

When asked about differences between their generated and estimated overviews, students came up with some meaningful explanations:

- One student made some initial mistakes on a certain KC, because he did not understand it correctly yet. Therefore, his generated score was low. But with the help of the DME’s immediate feedback, he realized what he had understood wrongly, and therefore learned from these mistakes and the feedback. In his own overlay, he rated his knowledge in this KC as high.
- Some students were tempted to guess answers to see what immediate feedback the DME would provide. As in the previous explanation, such trial-and-error-behavior results in lower generated scores, but it is likely that students learn from the immediate feedback they obtain.
- Initial confusion about what students were required to do in the DME also resulted in low scores for some tasks. Here, a low score indicates difficulties with DME-interaction, rather than little knowledge or skills.

These explanations for difficulties show two things. First, these are exactly the difficulties that arise when learning material as opposed to assessment material is used for generating feedback. The tasks are above all designed to make students learn, and it is sometimes difficult to determine whether that learning has taken place before or after the student has answered a task. Second, the discussions with the students seemed to help them get a clearer picture of which KC’s they understood and which they did not. So using student work to generate an overlay, confronting that with the student’s own estimation and discussing differences seems to be a fruitful teaching strategy.

Conclusion and discussion

In this explorative study, we have shown an example of the use of student models for providing individual feedback in a university statistics course. We developed an online activity on statistical sampling, a domain model for the domain of statistical sampling and a Q-matrix connecting the tasks from the online activity to the domain model. Next, we used the students’ work to generate overlays and presented the generated student models to the students, to give them more insight in their understanding of the different concepts involved in statistical sampling.

The generated student models were different for different students, which indicates that they indeed provided individual feedback. Students regarded the domain model as a useful summary of the domain of statistical sampling. As such, the domain model seems a useful instrument to confront students once again with difficult aspects of the domain of statistical sampling. Moreover, students regarded the generated overlay as a more or less true representation of their knowledge of the domain.

Asking students to compare their generated student model with their own estimations resulted in fruitful exchanges and therefore seemed a promising teaching strategy. Students tended to adjust their own estimation according to the generated model, but were also able to explain remarkable differences between their estimation and the generated model. These explanations often involved the immediate feedback provided by the DME, or, more general, the fact that the feedback is based on student interaction with an

activity that is designed to learn from. Calculation methods that account more for this fact are available (VanLehn, 2008) and can be taken into account in future experiments.

Another lesson we have learned is that some tasks are important for the learning activity, but are not useful for the generation of overlays. This concerns, for example, tasks that serve to explain a tool or simulation to be used. Therefore, careful considerations should be made whether or not to include specific tasks in the Q-matrix.

In this study, with its explorative nature, we have shown that it is possible to generate useful student models, based on student work in an online learning activity. A next step is to investigate how these models can best be embedded in education to help students monitor and plan their learning.

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Undergraduate students' perceptions of learning introductory statistics through producing a statistical picture book

Natthapoj Vincent Trakulphadetkrai

University of Reading, Institute of Education, Reading, UK; n.trakulphadetkrai@reading.ac.uk

This qualitative exploratory study examined non-statistics specialist students' perceived benefits and limitations of learning statistical concepts through creative story writing. Stories can be a powerful tool as it provides an opportunity for statistics learners to refine their statistical understanding in different contexts – ones that are relevant to their personal experience and interest. The added benefit of learning through creating stories is how it can shift the focus from dealing with numerical data and formulae exclusively to the meaningful application of statistical concepts. Interview and observation data involving seven social sciences undergraduate students at an English university revealed a range of perceived cognitive and affective benefits as well as some limitations of this innovative statistics learning approach.

Keywords: Introductory statistics, statistical anxiety, creative story writing, picture book.

Introduction

A deficit in quantitative skills among UK university graduates has recently been highlighted as a major cause for concern. In its position statement titled Society Counts, the British Academy (2012) - the UK's national academy for the social sciences and the humanities - expresses its deep concern in the UK's weakness in the quantitative skills, particularly within the social sciences and humanities (SSH) disciplines, and highlights how such deficit can have serious implications not only for the future of the UK as a world leader in research and higher education, but also for its graduates' employability and its economy's competitiveness. While the focus of SSH disciplines is not mathematical per se, the recent drive for research-based teaching in higher education (e.g. Jenkins & Healey, 2005) implicitly requires SSH students to be confident in using their quantitative, particularly statistical, skills, to help critically interpret statistical data (as reported in some articles within their field of study) and, where applicable, to help them quantitatively analyse data for their own research-based dissertation project. However, statistics teaching is not often delivered to students in a relevant and exciting way (British Academy, 2012). It is hardly surprising then that statistical anxiety among university students, particularly those within the SSH disciplines, has been widely reported (Lalayants, 2012). The current study proposes an innovative statistics teaching whereby students produce a creative story where statistical knowledge and understanding are required to construct the storyline. More specifically, the study intends to examine the students' perceived benefits of the approach.

Literature review

Statistics education

The research field of statistics education is primarily concerned with studies that aim to explore different ways to make statistics learning and teaching more effective, and in some cases, even more enjoyable. In relation to the current study, two key strands of relevant studies include those that focus

on transferability, and those that focus on associating statistics learning with something lighthearted and enjoyable. Concerning the former strand, Groth and Bergner's (2005) study in the USA, for example, investigated the role of metaphors in providing insights into students' statistical thinking. Whilst the focus of Groth and Bergner's (2005) study is on using metaphors to reveal pre-service elementary school teachers' understanding of statistical sample (e.g. "a sample is one toy off a toy shelf" (p. 34)), it can be argued that the underlying principle of using metaphors is deep-rooted in the concept of transferability which can be applied to any group of students and any statistical topic. Concerning the second strand, Friedman, Friedman and Amoo (2002) argue for the use of humour in statistics teaching and learning. Specifically, they argue that humour can be used to build relationships and enhance communication between students and instructor, as well as can be used as a stress-reducing tool in statistics classes. In Neumann, Hood and Neumann's (2009) study, use of humour was evaluated. Through interviewing 38 students who were randomly selected from those enrolled on a first-year Psychology course in Australia, it was found that "humor aided teaching by providing amusement, breaking up content, bringing back attention, lightening the mood, increasing motivation, reducing monotony, and providing a mental break" (para. 1).

Whilst the aforementioned studies represent an attempt to make statistics learning more accessible and enjoyable, the current study would argue that the effectiveness of these attempts to improve statistics learning and teaching experience is limited due to their lack of emphasis on embedding statistics learning in a relevant and meaningful context. This study thus sets out to explore students' perceptions of using creative story writing whereby statistical knowledge is required to construct the story's narrative in order to help them develop their statistical understanding.

Creative stories as a learning tool

Egan and Judson (2016, p. 4) argue that "the old distinction of arts dealing with imagination and academic subjects dealing with reason has led to a neglect of engaging students' imaginations in learning academic subjects". This, they argued, acts as a key barrier to effective teaching and learning. The use of creative stories (whether as consumer or producer) thus has a great potential to bridge this gap. The current study would argue that two key features underpin the creative story writing approach, namely 1) transferability and knowledge application in meaningful context, and 2) self-motivation through relevance of and emotional engagement in the story. As highlighted by Groth and Bergner (2005) with metaphor and Martin (2003) with analogy, transferability one's (statistical) knowledge and understanding from, for example, statistics textbooks (context-free) to their own story (context-rich) encourages them to first think carefully about what the concepts are, and then engage in higher order thinking by applying the concepts in a meaningful context. This process is crucial as Tannen (1999, as cited in Haven, 2007, p. 64) argues that "Story merges abstract information with common sensory details to create context and relevance for the abstract". Both context and relevance, as Haven (2007) argues "trigger the conscious mind to pay attention and to remember" (p. 64). Additionally, when learners get to think of a context or storyline for their own story, they are more likely to be engaged and self-motivated in their own learning process and become more emotionally invested not only in the story (Egan & Judson, 2016), but also in the ownership of their knowledge construction.

In terms of research, to the best of the author's knowledge, only one small empirical study has been conducted to explore the potential benefits of using creative stories in statistics learning. The study, by D'Andrea and Waters (2002), set out to examine how short stories can be used to reduce statistical

anxiety among her 17 graduate Education students enrolling on an introductory statistics course in the USA. Using the Statistical Anxiety Rating Scale (STARS), the survey results showed that the students' anxiety towards the statistics course steadily declined when their ratings before and after the course were compared. However, one key limitation of this study is how the short stories were written by the researchers (i.e. the course instructors) themselves, as opposed to providing an opportunity for the students to create their own stories – a shortfall that the current study aims to address.

Theoretical perspectives

This study argues for a statistics teaching and learning strategy that is grounded in Papert's (1991) theory of constructionism. Unlike constructivism, constructionism places a great deal of emphasis not only on internationalization, but also the process of externalization. More specifically, constructionists argue that construction of knowledge takes place both in the head (internalization) and supported by "construction of a more public sort 'in the world'" (externalization), whereby learners creating a public artefact of what they know that can be "shown, discussed, examined, probed, and admired" (Papert, 1991, p. 142). In turn, this process helps to shape and sharpen the knowledge (Ackermann, 2001). In the context of the current study, such public artefact is the story created by the learners where knowledge and understanding of the assigned statistical concept is first required before applying such knowledge and understanding to construct their storyline.

The current study

The current study is exploratory in nature, and it sets out to investigate non-statistics-specialist undergraduate students' perceptions of using creative story writing to learn introductory statistics. More specifically, the key research question asks: *What are non-statistics-specialist undergraduate students' perceptions of key benefits of learning introductory statistics through creative story writing?*

Methodology

Research design

This study is predominantly qualitative, reflecting a recent call from the research field of statistics education and cognition to move beyond it being a purely quantitative field (Kalinowski, Lai, Fidler & Cumming, 2010). The data collection took place in May and June 2016, and it primarily involved semi-structured interviews with first-year undergraduate non-statistics-specialist students within the social sciences discipline. To allow the students to form well-developed perceptions of the approach, they were asked to attend a three-hour session where they mostly worked in pairs to independently research a given statistical concept for the first 30 minutes. This independent learning was supported by making a range of introductory statistics textbooks available to them during the session. They were also encouraged to watch tutorial videos available on Youtube on their electronic devices. For the remaining 2.5 hours, they were then asked to collaboratively produce a creative story to illustrate that concept. Before they started creating the story, the participants were asked to vote on the format of their story output, and everyone voted for the picture book format, over two other choices, namely the graphical novel and the story book formats.

Audio recordings were made of each team's discussion whilst they were working on separate tables. A week after the session, each team was interviewed separately and they were asked to reflect on their own experience of using creative story writing to learn introductory statistics at the interview. Together with the stories produced, these multiple sources of data were used as a form of triangulation to maximise the degree of reliability in the analysis.

Standard deviation, as a measure of variability, was chosen as an introductory statistical concept for the participants to base their story on, for its importance as a building block to more advanced statistical knowledge, such as sampling distributions, inference, and p-values (delMas & Liu, 2005).

It is beyond the scope of this paper to present in detail examples of the picture books created by the students in this study. Examples and brief discussions of these picture books were presented in the author's CERME10 presentation.

Sampling strategies and sample size

The participants were seven undergraduate Education students, who nominated themselves to be part of the study after a recruitment call. The students are English native speaking students of Caucasian origin, aged ranging from 18 to 19 years old. Non-random purposive sampling was used to ensure that none of the participants had a Mathematics or Statistics post-16 academic qualification (commonly referred to as *A Level* in England).

The students were split into two pairs and one triad: Team 1 (all male students) with Jim and Dylan (pseudonym), Team 2 (all female students) with Maria and Sarah, and Team 3 (mixed) with Rosie, Ryan and the additional student - Olivia. The reason for including both single-sex pairs and a triad of male and female students in the study was to minimise any impact certain type of pairing could have on learning. (One of the male students, Ryan, did not confirm his participation by the agreed deadline, prompting the researcher to recruit an additional student, Olivia, as no other male students were available. Thus, a total of 7 students attending the session. Ryan also did not turn up for the scheduled interview, resulting in having interview data from only 6 students)

Data analysis

Due to the study being exploratory in nature and not aiming to test any particular existing theory, an inductive thematically-coded approach to qualitative data analysis was adopted. Audio recording transcripts of both the group discussions during the session and of the interviews after the session were read and reread to identify emerging themes. The process was done manually without the use of any software. The researcher alone did the coding, and thus fully acknowledges the limitation this presents in terms of the reliability of the analysis.

Results and discussion

As previously mentioned, this study sets out to explore non-statistics-specialist undergraduate students' perceptions of key benefits of learning and teaching introductory statistics through creative story writing. The findings are presented and discussed below.

Perceived cognitive benefits

Four key cognitive skills were developed through the creative story writing approach, namely *understanding, application, visualization* and *communication.*

Concerning ***understanding***, in order for the participants to come up with a storyline, they first had to understand what the given statistical concept (standard deviation) was. For example, Rosie explained that "The process helped me because I had to concretely understand what the statistical concept was and what misconceptions there may be before we started writing the book". This resonates well with Dylan's view as he stated that "I think it helps because you have to completely get every single step of it … know how to do it … you will have to go over it all before you were able to even start thinking about how could we use this in the task [story]".

Application is another key cognitive skill put forward by the students. Using creative story writing to learn statistical concepts requires learners to think carefully of a meaningful and purposeful context in which the concepts can be applied. This encourages them to contextualise statistical concepts. Dylan, for example, explained that "figuring a story that would fit around it […] makes you think about how could I use it in real life – where it would be applicable. I think […] having to put it in the story helps you understand it quite a lot." Such view is also echoed by Maria who stated that: "having to work out a context for the story where there would be a need to use standard deviation further helped. […] If it doesn't have a context, it doesn't really make sense. It's harder to understand it". From the observation, students' conceptual understanding of standard deviation was evidently developing through their discussion about the context for and application of the concept. This highlights the role of what Donaldson (1987) referred to as embedded thinking.

As previously mentioned, all participants voted to present their story in the picture book format. Whilst originally not central to the key research question, several participants cited the benefits of the format, particularly ***visualization***, as contributing to the development of their understanding of standard deviation. More specifically, they highlighted how the format encourages them to think about how statistical concepts can also be represented visually through illustrations. Jim explained that "I think it gives you different ways to learn because you might be a visual learner. The pictures will help", highlighting how producing creative story in the picture book format could cater for diverse learning styles.

Finally, ***communication*** – another key cognitive skill that came up several times in the interview with the students. This is primarily concerned with how the creative story writing approach explicitly requires authors to break down the concept and communicate it to their readers. Jim, for example, explained that "It's also about breaking it down into a narrative that other people can understand because you write it for other people, so you have to … not dumb it down … but you would certainly break it down, and I think it helps you understand it that way". For Maria, she linked this aspect of the approach to teaching: "You're teaching it and you remember it better when you're trying to teach something rather than when you're just listening. We're writing a story to teach other people what it was". The participants who were not at all familiar of the concept prior to the session were later able to articulate that standard deviation measures the spread of the different data points in relation to their mean average. While the definitions offered by Sarah and Ryan did not make clear the relationship between the data points and their mean average, it can be argued that their understanding is still

emergent. Even for Maria and Rosie who already had some understanding of the concept prior to the session, the definition of the concept that they offered after the session was more detailed. In line with Haven (2007), this demonstrates that creative story writing can be a powerful learning strategy for a wide range of learners.

Perceived affective benefits

Different aspects of the story writing approach were highlighted by the participants as helping to make statistics learning more engaging. Rosie, for example, pointed out that "Personally, maths has always been my nemesis, so for me [the story writing] puts maths and statistics in a lighter viewpoint rather than being factual and quite off putting". Similarly, Olivia – echoing Egan and Judson's (2016) – found herself fully engaged in the process as she explained that:

> Before the session, the thought of statistics was fairly scary to me and seemed like something I would struggle to engage with. […] But, as the story writing process began, I was able to view standard deviation from a less scary lens. Before I realised, I was fully engaged with the story writing activity, rather than focusing on how scary the topic was initially.

Through its hands-on approach, Sarah and Jim commented about how the story writing approach helped to make them more engaged in statistics learning than they would otherwise be in traditional lectures. Sarah, for example, highlighted that:

> The activity was enjoyable and therefore more engaging than if I'd just read about standard deviation in a text book or been told about it in a lecture. Being able to have fun with it and produce something creative helped me to really get into it and actually want to pay attention to getting it right.

Jim also highlighted how learning statistics through story writing is more enjoyable and less scary when "you've got a picture involved for a start and you're making a narrative". This resonates well with Maria's view. Not only did she attribute the positive statistics learning experience to the story writing, Maria also attributed it to the picture book format specifically, as she explained "Picture books are also associated with happiness and adventure and putting statistics into a picture book can make statistics seem more exciting". Her view emphasises how the combination of story writing and the picture book format seems to compliment each other quite well.

The way these students described statistics and its learning in itself is revealing: 'nemesis', 'scary', 'factual', 'struggle', 'complicating', 'mundane' and 'off putting', highlighting how disengaged these students would have carried on feeling towards the subject had it not been because of the story writing approach to learning statistics. When examining how the same students described statistics and its learning using the story writing approach, the positive attitudes towards statistics learning became apparent: 'fun', 'fully engaged', 'more exciting', 'creative', 'more engaging', 'more accessible', 'immersed', 'more switched on', 'lighter viewpoint' and 'your own knowledge'.

Conclusions

Key findings and discussions

Overall, the approach, according to the students, appears to make use of four key cognitive skills, namely *understanding*, *application*, *visualization* and *communication*, and these are labelled as

perceived cognitive benefits. Concerning ***understanding***, in order for the participants to come up with a storyline, they first had to understand what the given statistical concept (standard deviation) was. ***Application*** is another cognitive skill that is put forward by the students. Using creative story writing to learn statistical concepts requires learners to think carefully of a meaningful and purposeful context in which the concepts can be applied. This encourages them to contextualise statistical concepts, and highlights the role of what Donaldson (1987) referred to as *embedded thinking*. The students themselves highlighted that the picturebook format encouraged them to think about how standard deviation could be represented visually through illustrations. This is particularly relevant when ***visualization*** is often seen to be a key mode of representation that can help learners develop their mathematical and statistical understanding (e.g. Bruner, 1966; Haylock, 1982; Haylock, 1984; Haylock & Cockburn 2013). Finally, ***communication*** is another key cognitive skill that came up several times in the interview with the students. This is primarily concerned with how the creative story writing approach explicitly requires authors to break down the concept and communicate it to their readers. Equally important, the students reported that the creative story writing approach motivated them to engage in learning introductory statistical concepts, as this is labelled as *perceived affective benefits*. The students particularly enjoyed incorporating humour in their storyline and page illustrations. The fact that laughter could be heard throughout what was essentially a statistics lesson was very encouraging as it demonstrates that it is entirely possible to have an enjoyable statistic teaching strategy that, according to the students, also helped them learn an introductory statistical concept. This is in line with the findings of Neumann et al.'s (2009) study that found students to be more motivated in their statistics learning process when they were able to include humour in their statistics learning.

Implications

The study highlights the potential benefits of using creative story writing, particularly in the picture book format, as an effective introductory statistics learning tool for non-statistics-specialist students. Such benefits are both cognitive and affective in nature. Additionally, whilst the students in this study were undergraduate students, this study would argue that both high school and postgraduate students would too find the approach beneficial.

Limitations of the study

The participants in the current study created their story in an arguably clinical setting, as opposed to their authentic learning experience. Additionally, it is important to remember that these participants volunteered to be part of the study. Taken together, it can be argued that the views and attitudes of this group of participants might be potentially different from those who are required to engage in creative writing as part of their course. Thus, any findings emerge from this study must be treated with caution, and this highlights the need for this study to be replicated in an authentic learning environment.

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Enhancing statistical literacy

Marianne van Dijke-Droogers[1], Paul Drijvers[2] and Jos Tolboom[3]

[1]Csg Prins Maurits, the Netherlands; m.j.s.vandijke-droogers@uu.nl

[2]Utrecht University, the Netherlands; p.drijvers@uu.nl

[3]SLO, the Netherlands; j.tolboom@slo.nl

Current secondary school statistics curricula focus on procedural knowledge and pay too little attention to statistical reasoning. As a result, students are not able to apply their knowledge to practice. In addition, education often targets the average student, which may lead to gifted students missing challenge. This study explored ways to enhance grade 8 (Pre-University level) students' statistical literacy through within-class differentiation. The developed course materials consisted of a differentiated module in the Digital Mathematics Environment (DME), combined with investigation activities during classroom sessions. The material focused on statistical reasoning using visual representations made with TinkerPlots We concluded that this teaching arrangement indeed increased students' statistical literacy.

Keywords: Statistical literacy, descriptive statistics, Digital Mathematics Environment, level differentiation, TinkerPlots©.

Introduction

Statistical literacy has become important for all of us, and statistics will only continue to become more critical in the future (Shaughnessy, 2010). Despite the global effort to innovate the ways in which statistics is acquired, current statistics education is still viewed as a field with a need for significant improvement (Garfield & Ben-Zvi, 2008). Strong educational foci on methodological skills, procedures and computations result in the limited ability to reason statistically and to apply statistics in practice (Allen et al., 2010; Gal, 2002).

The Netherlands are no exception to this. In grade 8 of the Dutch pre-university stream, for example, the statistics curriculum stresses the calculation of mean, modus and median. Statistical investigation and use of technology hardly occur in the current approach. The emphasis on calculating statistical measures contributes insufficiently to interpreting, critically evaluating and reasoning with data (Van Streun & Van der Giessen, 2007).

As a second concern, the current educational approach pays too little attention to gifted students. PISA research shows that the best quartile of Dutch students performs relatively poorly (Kordes, Bolsinova, Limpens & Stolwijk, 2013). It is plausible to assume that the education received is insufficient for these students. This is endorsed by the KNAW (2003), which calls for more differentiation between students and for offering enrichment material.

To address the abovementioned issues this study focuses on the following key question: Does a differentiated learning trajectory that focuses on statistical reasoning with visual representations increase students' statistical literacy in grade 8? The hypothesis is that an educational approach in which differentiated online tasks are combined with investigation activities in class will increase statistical literacy.

Theoretical framework

The theoretical framework we used integrates notions of statistical literacy and level differentiation.

Statistical literacy

Gal (2002) defines statistical literacy as interpreting, critically evaluating and reasoning with statistical information. This requires, in addition to procedural statistical skills, reasoning with and about data (Tolboom, 2012). Students should be taught the necessary skills to interpret and reason with statistical concepts. Research indicates that students in an early stage can reason meaningfully about distributions (Bakker & Gravemeijer, 2002). According to Piaget and Inhelder (1951), students have an intuitive sense of statistical reasoning. This intuitive concept can be used to develop statistical literacy. Moreover, research literature suggests that students become statistically literate by conducting their own research projects (Abel & Poling, 2015).

Web-based tools like TinkerPlots (Konold & Miller, 2011), which focus on the use of dynamic visualizations, may support statistical reasoning and literacy. The use of such software, in addition to manual data processing experiments, has the advantage that problems are taken care of, so there is more room for reasoning. There is evidence that the use of ICT in statistics can improve learning results (Morris, Joiner & Scanlon, 2002) and, especially if embedded in classroom discussions, can lead to increased statistical literacy (Bakker, 2004). Based on these findings, we focus in this study on literacy and reasoning using digital tools.

Level differentiation

The Dutch education system[1] is based on homogeneous streaming. Nevertheless, within a class of a specific achievement level, major differences between students in intelligence and performance may exist. Students' learning progress may suffer from neglecting these differences. In differentiated teaching, teachers provide individual learning paths to students, adapted to their levels, to learn as much as possible (Tomlinson, 1999). That level differentiation leads to better academic performance in primary education (age 4-12) has been shown by several researchers (e.g., Vernooy, 2009). With respect to differentiation in secondary education (age 12-17), less is known, even if Terwel (1988) and Van Dijk (2014) suggest that differentiation within mathematics lessons may lead to improved performance.

Differentiation assumes the classification of students. The RTTI model (Drost & Verra, 2015) can be used to identify the cognitive level of a learner. Dutch secondary schools and textbook editors increasingly use this model. It is based on four learning levels: Reproduction (R), Training (T1), Transfer (T2), and Insight (I). Based on RTTI test scores, the students can be clustered into level groups (Berben & Teeseling, 2014). In terms of the RTTI model, statistical literacy relates to T2 and I levels. Based on the aforementioned findings, in this study we opted for a differentiated educational approach based on the RTTI learning levels.

Methods

[1] For an overview of the Dutch educational system see https://www.epnuffic.nl/en/study-and-work-in-holland/dutch-education-system.

We successively describe the design, intervention, participants, data collection, and data analysis of the study.

Design research

Since teaching materials that aim at increasing the statistical literacy by offering differentiated teaching arrangements and using digital tools for this group hardly existed, we used a design research method (Bakker & van Eerde, 2015; Plomp & Nieveen, 2013). Because the learning objectives differ from the current curriculum in grade 8, no control groups were used; we compared the students' learning gains during the intervention through pre- and post-tests. This research can be characterized as a "proof of concept" of an intervention that focuses on statistical literacy and reasoning through a technology-rich, differentiated approach based on RTTI.

Intervention

The intervention consisted of statistics modules within the Freudenthal Institute's Digital Mathematics Environment (DME, see www.dwo.nl/en), combined with investigation activities during the classroom sessions using Tinkerplots. The DME is a digital environment in which students work on mathematical activities. It includes opportunities for differentiated education by offering several learning routes. The work of students is saved in the DME and teachers can monitor the results (Bokhove & Drijvers, 2012). In this study the procedural skills, e.g., calculating central tendency and variation measures and values of various graphs including boxplots, were offered within the DME. The DME modules were individually run and consisted of two learning routes: the basic route and the plus route. The students were assigned to these conditions according to their RTTI achievements during the past schoolyear. Students with average score T2 and I less than 65% followed the basic route, and others the plus route. Within the designed DME modules, students could check their work and correct it when necessary. Adjacent to each classroom session, students worked at home on the DME module. The hypothesis was that the procedural skills of students will strengthen through the DME-modules, so that they can use them in reasoning with statistical information.

Statistical reasoning in the frame of investigation tasks was central to the eight 60-minute classroom sessions offered in parallel to the DME modules. During this classroom sessions students worked in homogeneous teams (clustered according to the RTTI learning levels). The investigation activities were based on the stages of the statistical investigation cycle (Franklin et al., 2005). The students analysed their data manually and by using the software TinkerPlots© (Konold & Miller, 2011). This software provides rich visualization opportunities, flexible and investigative functions, and is user-friendly. Figure 1 shows some examples of the visual possibilities in TinkerPlots. The hypothesis was that clustering of students while working on investigation activities and using visual representations in TinkerPlots, sharpens and reinforces the

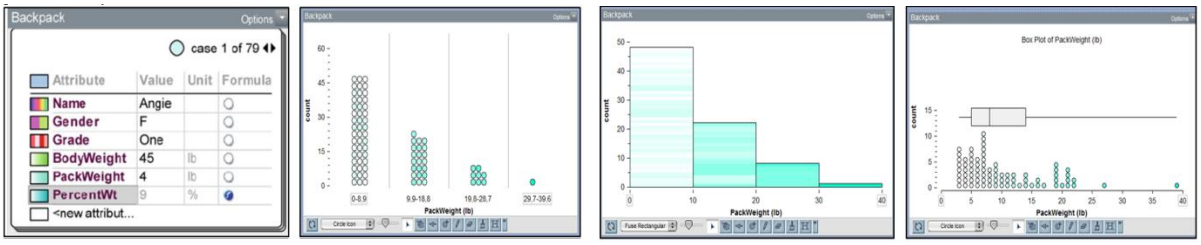

Figure 1: Examples of visual representations in TinkerPlots©

statistical literacy of students at the different levels.

Participants

In the pilot the designed material was tested in a classroom at the school of the researcher, the Csg Prins Maurits in a rural area in the Netherlands. The pilot class consisted of 25 pre-university grade 8 students (14-15 year olds), with sixteen students turning out to be basic students and the other nine plus students. The students had no previous experience with statistic education.

Data collection and data analysis

To verify whether the intervention improved statistical literacy, we examined students' DME progress, results on two statistical tests, logbook data from the teacher-researcher and students' final investigation task. The data of basic and plus students were analysed separately. To analyse the DME work we used data on score and time investment. Further to the DME modules two individual tests were taken. One of these tests was a RTTI standardized test conducted with 45% of questions at learning level R and T1 and 55% of questions at level T2 and I, the latter corresponding to statistical literacy. This ratio is in line with the standard approach in the research class, so the results can be compared with previous RTTI scores on math tests. The additional test consisted of questions at learning level T2 and I with a higher difficulty compared to the RTTI standardized test, so as to obtain additional information about the level attained.

The logbook of the teacher-researcher contained information about students' progress in interpreting, critically evaluating and reasoning with statistical information during the investigation activities in class. To find out whether the students in the end applied the statistical methods in practice, a final investigation task was administered in homogeneous level groups of 3-4 students. A rubric was developed for the analysis of this task to assess performance at learning level T2 and I.

	DME data	**RTTI test**	**Test at T2 and I**	**Log**	**Research task**
Statistical literacy in solving concrete problems	X	x	x	x	x
Statistical literacy in investigation activities	x	x	x	X	X

Table 1: Table of triangulation of research instruments

To ensure the quality of the research data triangulation is used. Table 1 shows how the statistical literacy has been measured with multiple instruments. The font sizes for x indicate the degree to which each instrument measures statistical literacy.

Results

We now present the results of the learning process using the DME, the test results and the development during the investigation tasks.

Learning process using the DME

Average time investment per DME module in minutes (sd)							
	Mod. 1	**Mod. 2**	**Mod. 3**	**Mod. 4**	**Mod. 5**	**Mod. 6**	**Total**
Basic students (n = 16)	36(21)	30(13)	36(19)	39(21)	27(14)	33(29)	34(20)

Plus students (n = 9)	42(28)	25(16)	30(27)	37(23)	19(16)	21(22)	28(22)
Average score per DME module in percent (sd)							
Basic students (n = 16)	87(9)	81(23)	80(13)	79(16)	70(30)	53(31)	75(24)
Plus students (n = 9)	79(24)	91(8)	72(27)	61(33)	65(37)	43(35)	67(32)

Table 2: Time investment and score using the DME

The investment of time and scores of the DME work for each level group and per module are summarized in Table 2. The students used the DME with an average time investment of more than half an hour per session. During the learning trajectory the time investment, in particular for the plus students decreased. Initially, the students needed extra time for getting acquainted with the material. Moreover, the students indicated that over time they thoughtfully chose their way through the module by skipping known problems.

The students' scores on the DME show an average of about 70%. The basic and plus students respectively achieved an average score of 75%(24) and 67%(32) per module. The exercises in the plus route were more difficult. The scores decreased during the learning curve when difficulty increased. In the last module there is a substantial decline. This module contains no new material, but includes joint exercises from the completed chapters. The plus students show more variability in score. The considered choices in learning exercises by these students might have strengthened this trend.

Test results

Table 3 provides an overview of the pre-test score (RTTI average score on nine math tests over the past schoolyear) and post-test score (RTTI score on the final statistics test). The RTTI pre-test scores indicate the achieved learning level of the students at the end of each chapter. This means that based on the presented pre-test scores the expected RTTI post-test scores should be on average 55% at the

	Whole class (n=25)		**Basic students (n=16)**		**Plus students (n=9)**	
	Pre av.	Post	Pre av.	Post	Pre av.	Post
Score on R and T1 in percent (sd)	79 (16)	85 (10)	79 (13)	85 (10)	80 (20)	86 (10)
Score on T2 and I in percent (sd)	55 (18)	64 (21)	47 (16)	59 (24)	69 (10)	73 (12)

Table 3: RTTI scores before and after the intervention

levels T2 and I. However, the RTTI post-test scores reached on T2 and I, the parts that measure statistical literacy, show a 9% higher score of 64%. The dissimilarity in progress between the basic students (12%) and plus students (4%)
at the levels T2 and I may be caused by a ceiling effect or maybe the exercises in the RTTI standardized test gave too little space to plus students to exhibit their knowledge.

On the additional test both level groups exhibit a high score on T2 level, in spite of the increased difficulty. A smaller increase appears on learning level I. The results cannot be compared with previous tests because the difficulty of the conducted questions was considerably higher.

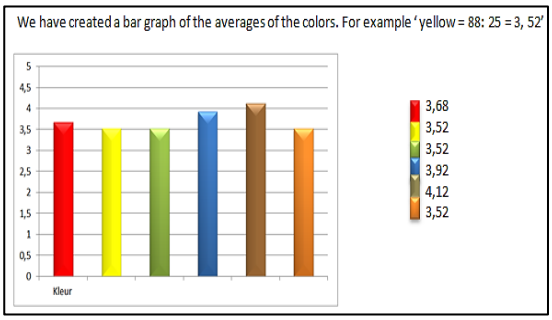

Figure 2: Students' work on the first investigation task

Learning process during investigation activities

The usual statements of students' written work at the start of this intervention can be characterized as short answers with a calculation of the mean. The used visual representations were limited to bar and pie charts. Figure 2 shows students' work on the first investigation task: Investigate the colour composition of a bag of M & M's. In the final investigation task the students' work contained detailed descriptions and rich visualizations with a wide diversity of graphs. Attention was paid to the interpretation of the data. Learning progress was visible in terms of interpreting, critically evaluating and reasoning with statistical information. Figure 3 shows a small part of students' work on the final investigation task in which they formulated and investigated their own research questions using datasets within TinkerPlots. The results on the final investigation task with respect to learning level T2 (65% of the total score) and I (35% of the score) are shown in Table 4.

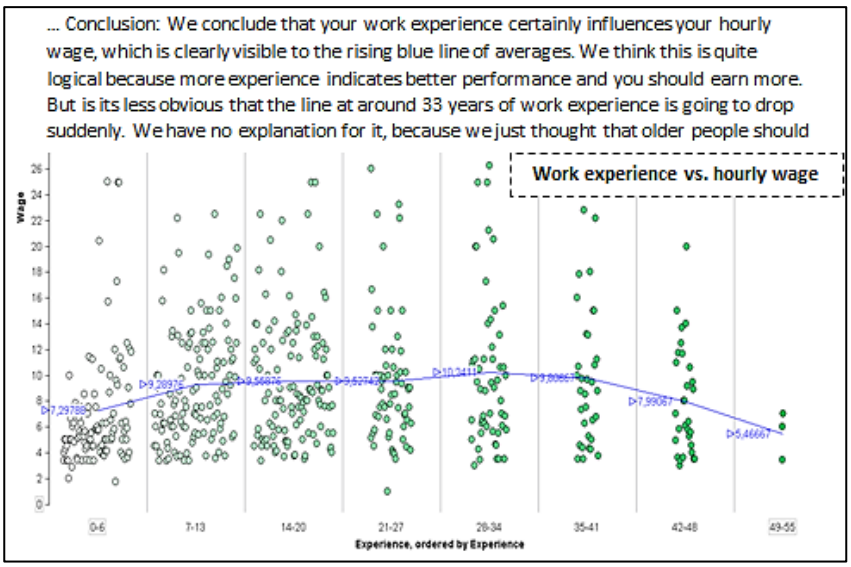

Figure 3: Students' work on the first investigation task

	Whole class (n=8)	Basic groups (n=5)	Plus groups (n=3)
Score on T2 in percent	89 (7)	85 (6)	95 (4)
Score on I in percent	51 (24)	35 (6)	79 (10)
Total score in percent	70 (26)	60 (27)	87 (11)

Table 4: Scores on T2 and I at the final investigation task

On level T2, for example, we evaluated in the assessment-rubric correctly representing and summarizing the data and on level I we examined the choice of an appropriate visual representation and the critical interpretation of the results. In comparison to the basic groups, the plus groups show a higher score on learning level I. This investigation task probably provided more room to gifted students to exhibit their statistical reasoning.

Conclusion and discussion

The main question in this research was: Does a differentiated learning trajectory that focuses on statistical reasoning with visual representations increases students' statistical literacy in grade 8? The results suggest it does. The RTTI scores reached on T2 and I, the parts that relate to statistical literacy, were much higher than would be expected according to the pre-test scores. Moreover, the final investigation task showed strong progress on interpreting, critically evaluating and reasoning with statistical information according to the start of this trajectory. This is consistent with the theories by Bakker & Gravenmeijer (2002) and Abel & Poling (2015) on developing statistical literacy. Both basic and plus students showed considerable improvement during the learning trajectory which suggests that both groups were challenged in this differentiated approach as suggested by Terwel (1988) and Van Dijk (2014). The use of all kinds of visual representations within TinkerPlots helped the students to explore their data. In summary, the results suggest that the designed educational intervention, which consisted of differentiated online modules within the DME combined with investigation activities using TinkerPlots during classroom sessions, led to increased statistical literacy. However, this study has its limitations: no control with other groups was possible; we cannot indicate whether the differentiated approach or the focus on statistical literacy and reasoning caused the increased level; the pilot took place in just one class, taught by the researcher. Therefore, the results cannot be generalized and further research is needed.

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Implementing the statistical investigative process with real data: Effects on education in secondary schools

Martin Andre

Pädagogische Hochschule Tirol, Innsbruck, Austria; martin.andre@ph-tirol.ac.at

Keywords: Statistics teaching, learner engagement, technology integration, problem-based learning

Theoretical background

In the Guidelines for Assessment and Instruction in Statistics Education (GAISE) College Report (2016) endorsed by the American Statistical Association following the GAISE College Report of 2005, the ASA revision committee recommends in addition to the teaching of statistical thinking and focusing on conceptual understanding, to use software combined with real data and to foster active learning. The main goal of the given recommendations for students is to develop statistical literacy and achieve the ability of thinking statistically. Therefore, students should understand the nature of data and all parts of the statistical process from obtaining and generating data to the communication and interpretation of the results after the analysis. In an international meta-analysis of 70 studies from the past 40 years Larwin & Larwin (2011) show that students in postsecondary statistics education strongly benefit from computer-assisted instruction (CAI) under certain circumstances, for example, when CAI is continuously and supplementary applied in lessons and homework. They also warn, however, of an overestimation of CAI. Several studies show the advantage of student-centered (e.g., Kuiper, Carver, Posner & Everson, 2015) and problem-based learning (e.g., Cantürk-Günhan, Bukrova-Güzel & Özgür, 2011) supported by technology (e.g., Koparan, 2015). Neumann, Hood and Neumann (2013) explored the benefits of using real data in statistical education. Gil & Ben-Zvi (2011) underline the importance of context in the emergence of younger students' informal inferential reasoning in an inquiry-based, technology-rich learning environment.

Research questions

The focus of the presented research lies on the beneficial implementation of statistics software in classes, considering, for example, the characteristics of various software programs. Another field of this research is how the use of statistical software in an inquiry-based learning environment leads to the development of conceptual understanding and not to dependence on the software and the learning of tools and procedures. This leads to the following research questions. First: How is the situation about the context- and inquiry-based learning of statistics supported by technology in Austrian secondary schools? Second: Is there an evident connection between context and using real data on the one hand and CAI on the other hand to provide a meaningful learning of the overall statistical process. Third: How should software under a given context in classes of higher secondary schools be installed to support conceptual understanding of the statistical investigative process and which characteristics of statistical software are especially beneficial to this purpose?

Research design

Various learning sequences for the 10th grade of Austrian higher secondary school are being created and will be inserted in different Austrian secondary schools. According to Strauss and Corbin's approach of Grounded Theory, diagnostic interviews will be carried out before and after the implementation of the

learning sequences. Some students' work on the sequences will be filmed and worksheets will be analyzed. The goal is, to develop provable hypotheses relating to the research questions.
The poster gives an overview of the theoretical framework, the research questions and design and focuses on a created worksheet that will be applied for this research. First results show that software in Austrian schools is commonly used in other mathematical fields, e.g. for plotting graphs, but not even in 30% of the cases for calculating statistical key figures.

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Metaphorical representations of multifaced statistical concepts: An access to implicit dominant interpretation by teachers?

Corinne Hahn

ESCP Europe and LDAR Université Paris Diderot, France, hahn@escpeurope.eu

Keywords: Metaphors, image, concepts.

Statistical concepts often have multiple faces and deep understanding of these concepts requires combining these different perspectives. For example, mean is a complex concept (Mokros & Russel, 1995) that can be thought as a typical value, fair trade or center of gravity (Gattuso, 1994). The teachers who teach statistics in social sciences curricula have very different backgrounds (Hahn, 2015). The question is whether this variety of backgrounds leads them to implicitly favor one of the dimensions of the concepts when they teach.

To answer this question, I chose to use metaphoric representations. Lakoff and Johnson (1980) have shown the central role of metaphors in the process of construction of scientific concepts. This question has been widely explored in mathematics education, a working group was specifically devoted to this topic at Cerme 2 to 5 (Parzysz, 2015). Soto-Andrade (2006) explains that a metaphor connects a concept already built in a familiar area with a concept to build in an area that is not familiar. These representations are usually "image like" (Sfard, 1994). That is why I chose to use depictive representations of these metaphors. In this research, these representations are external to the subject as they were not drawn by them.

The protocol for this research involves six steps:
1. Choice of statistical concepts
2. Didactic analysis of these concepts and identification of the different perspectives from the literature
3. Identification of metaphors who are used to teach these concepts and selection of the metaphors that can be associated with the different faces of each concept
4. Drawing of representations of these metaphors
5. Pretest with teachers from different backgrounds
6. Semi-structured interviews of professors who teach statistics in a business school.

The poster presents step 1 to 5 of this research. I chose the concepts of mean, sample and confidence interval. Following the review of the literature, different perspectives were identified and assumptions were made about the perspectives that should be preferred by teachers considering their background. From informal discussions with professors and the study of textbooks, I identified a few metaphors and selected those that seemed directly linked to the different faces of the selected concepts. I then draw for each concept, three pictorial representations of these metaphors. Through the pre-test, it seems that teachers do not at first always identify the concept behind the drawings. They associate the pictorial representations with concepts directly related to the target concept and linked to the perspective chosen. For example, regarding the mean, interviewed teachers "saw" the concept of distribution when they were shown the image referring to the "fair share" and the variance in the image referring to the "balance point".

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Non-mathematics majors doing statistics: Factors behind performance

Theognosia Michailidou

University of Leeds, UK; edtgm@leeds.ac.uk

Keywords: Statistics education, cognitive and non-cognitive factors, achievement.

Main objectives and theoretical foundation of the research study

Nowadays, a large proportion of undergraduate university students are required to undertake at least one statistics or statistics-related course. The main consideration guiding my research study is the exploration of several cognitive and non-cognitive factors, the "other" outcomes of statistics education (Schau, 2003), and their relationship with the students' academic performance in introductory statistics courses offered at tertiary institutions. Cognitive, affective and motivational factors under investigation include: students' feelings, attitudes and beliefs about statistics; students' anxiety feelings and perceived self-efficacy regarding statistics; motivational orientations, achievement goals, students' interest in and engagement with statistics (such as effort, persistence, learning strategies); resilient behaviour characteristics when learning and studying statistics; and prior mathematics or/and statistics background and performance. A proposed theoretical causal model, including direct and indirect effect of these variables on achievement in statistics, will be tested. Moreover, the study focuses on exploring and documenting non-mathematicians' perceptions, behaviors, challenges and experiences when completing a statistics course.

My theoretical foundation is supported by some key theories: Social-cognitive learning theory (Bandura, 1997) and Self-efficacy theory (Bandura, 1977), Expectancy-Value theory (Wigfield and Eccles, 2000) and Achievement Goal theory (Dweck, 1986). Some ideas behind these theories have been adapted to statistics learning experiences and they have been used as a guide for both the quantitative and qualitative strands of my research work.

Research design and methodology, and data collection and analysis procedures

In order to accomplish the research aims and goals, a mixed-methods research design (that is a combination of quantitative and qualitative data collection methods) has been employed. A self-reported questionnaire, which was designed and developed specifically for the purposes of the doctoral study, has been administered to a larger sample of students in university classroom settings. It comprised of open- and closed- ended questions, and Likert-type questions on a 5-point scale.

Individual (and pair) face-to-face semi-structured interviews were conducted with a sample of participants who had completed the questionnaire and consented to be interviewed.

The data collection procedure can be summarized in three many stages (not in chronological order): quantitative data-gathering (executed in two phases - at the beginning and at the end of the period of instruction of various statistics courses); qualitative data-gathering; and participants' final grades obtained at the end of a statistics course from each of the instructors. The data collection was carried out over a period spanning two academic semesters, the fall semester 2015 and the spring semester

2016. I collected data from six universities – all the recognized universities (both public and private) operate in Cyprus. I observed and gathered data from 35 statistics classes (23 courses) taught by 15 different instructors. I collected over 1000 questionnaires and I executed 60 face-to-face interviews. The participants, with a variety of mathematics background and experience, came from diverse academic departments and degree programmes.

The quantitative data will be analysed using a variety of methods: basic statistical techniques (descriptive statistics, reliability estimates, correlation coefficients, χ^2 tests, t-tests, analysis of variance and regression analysis); advanced statistical methods (multilevel analysis, factor analysis and structural equation modelling techniques). The qualitative data will be coded using thematic analysis approach.

Potential significance and contribution

This study may act as a springboard for further research in Cyprus and cross-cultural comparisons between Cyprus and other countries. The major findings and recommendations of the research work may constitute a helpful tool for the statistics instructors and stakeholders of statistics education in implementing interventions, innovations and instructional strategies/ practices to develop and improve the quality, the efficacy and the relevance of statistics courses that provided for non-mathematics majors.

The poster for the CERME-10 conference includes a brief introductory statement of the background, context and objectives of this doctoral study as well as information about the research design and methodology and data collection and analysis processes. Also, some preliminary findings (along with visual and graphical representations) are presented.

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Relating theoretical and empirical probability - Developing students´ conceptions of probability and chance in primary school

Kristina Penava

TU Dortmund University, Germany; kpenava@math.tu-dortmund.de

Keywords: Student conceptions, probability, chance.

Theoretical framework and research questions

Most of the recent studies emphasise the challenges when dealing with probability problems. As Horvath & Lehrer (1998) show, young children are able to see a relationship between theoretical probability and empirical outcomes, but „[t]he children never understood the role of the sample space without significant support and, hence, never completely understood the reasons why patterns were more predictable than simple outcomes“ (Horvath & Lehrer, 1998, p. 132).

The largest challenge for the development of conceptions of probability and chance seems to be coordinating empirical and theoretical perspectives on probability. This means to relate the (relative) frequency of outcomes (empirical law of large numbers) and theoretical assumptions considering the distinction between long-term and short-term on random processes (Schnell, 2014; Horvath & Lehrer, 1998). A core condition for emerging conceptions of probability seems to be the change of perspective from focussing on single outcomes to the long-term results (Prediger, 2005, p. 40) and changing from long-term to short-term perspective (Johnston-Wilder & Pratt, 2007).

This project aims at providing insights into the development of students´ conceptions of probability and chance during a board game. Towards this end, this study pursues the following research questions: Which elements of a teaching-learning arrangement can support the distinction between long-term and short-term? (RQ1) Which elements of a teaching-learning arrangement can support students in relating empirical and theoretical probability? (RQ2)

Method

In this project, students´ conceptions of probability and chance are identified within game interviews based on the board game „Who wins?“ (see Figure 1). Players take turns in throwing dice with asymmetrical colour distribution (green: 3 sides; red, blue and yellow: 1 side) and taking one step with the token of the matching colour. The goal of the game is to predict the winning token. During the interview the game is played several times, so that the learning environment provides stochastic experiences in a short-term context.



Figure 1: Who wins?

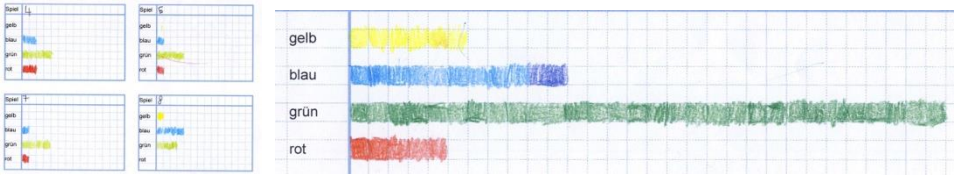

Figure 2: single game list and many games list

The students are asked to systematise the results with the help of different record types: a *winning list*

recording the winner of each game, a *result list* recording each individual thrown dice, a *single game list* providing histograms of colours thrown for individual games, and a *many games list* providing a histogram of the aggregated colours thrown over all games (see Figure 2). Simulated long-term documents provides opportunities to relate short-term to long-term results.

In total, 27 students were interviewed in groups of three. Each group met for three interview sessions. The 27 sessions were videotaped and fully transcribed (including oral statements, actions, and results of games and throws of dice). The students' conceptions on probability were identified in a qualitative interpretative analysis.

First conclusions

A first analysis of the empirical data shows that the learning environment can initiate a change of perspective from short-term to long-term. In particular, comparing long-term results with short term AND with other long term results revealed the difference between the contexts. Experiencing variability in a short-term context may support changing perspective from long-term to short-term. Different record types (e.g. single outcomes, cumulated outcomes) seem to support "seeing" the distinction between long-term and short-term on random processes.

Relating the outcomes and the colour distribution of the dice support the understanding of the relationship between empirical and theoretical probability. Reflection on mathematically unexpected outcomes may initiate explanations of the relationship in a more detailed way.

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TWG06: Applications and modelling

Introduction to the papers of TWG06: Applications and modelling

Berta Barquero[1], Susana Carreira[2] and Gabriele Kaiser[3]

[1]University of Barcelona, Spain;

bbarquero@ub.edu

[2]University of Algarve, & UIDEF, Institute of Education, University of Lisbon, Portugal;
scarrei@ualg.pt

[3]Universität Hamburg, Germany & Australian Catholic University, Australia;

Gabriele.Kaiser@uni-hamburg.de

Keywords: Mathematical modelling, empirical studies, theoretical approaches, modelling examples, usage of technology.

Introduction

Mathematical modelling and its teaching at various educational levels are widely accepted all over the world. There is consensus of the need to integrate mathematical modelling and applications in curricula and this has already taken place in many European countries. However, there is no unanimity on how to integrate mathematical modelling and applications into the processes of teaching-and-learning mathematics. In addition, there is little secure empirical knowledge available on how to implement efficiently the necessary new learning environments. In its discussions at CERME 10, the thematic working group TWG 6 on modelling and applications aimed to contribute answers to these open questions and hence further develop the work from previous ERME conferences. The contributions discussed at the congress are characterized by a strong and fruitful diversity in the research questions considered, the school levels addressed and the theoretical approaches taken. On the whole, the papers address theoretical, methodological, empirical or developmental research on the teaching and learning of applications and modelling. The group involved 35 participants from 16 countries – participants were from Europe, and also from South and Central America. A total of 20 papers and four posters were presented and discussed during the working group sessions.

In the following, we describe all presented papers, although not all have been submitted for the proceedings, grouped around eight comprehensive themes, which refer to current issues within the teaching and learning of mathematical modelling.

As a first important theme, we identify the *interplay between disciplines into modelling activities* referring to the specificities of interdisciplinary modelling activities, especially in engineering teaching. The second theme relates to the *connection of the problem solving perspective and mathematical modelling* and to the development of problem solving strategies and competences, when students work in groups and develop individual competences. The third theme covers *developing modelling strategies and competences*, for example theoretical and empirical work focused on the analysis and use of heuristic strategies adopted by teachers and/or students which illuminates strategies used to foster students' performance when solving modelling problems. The fourth theme refers to the *tools and methodologies used to analyse modelling processes*, namely

studies focusing on elaborating specific methodologies for analysing and evaluating modelling practices. The fifth theme focuses on *teachers' beliefs in relation to the teaching of mathematical modelling*, including, for example, research on the role of teachers to foster modelling practices. The sixth theme covers *teachers' interventions in mathematical modelling*. The seventh theme refers to *experimental materials and technology in modelling*, which covers two topics, a first primarily focused on the role of the auxiliary material and its impact on modelling and a second concerned with how to combine different resources with technology in concept development by means of real word contexts. The eighth theme refers to the *assessment of modelling practices.*

Overarching themes

Modelling and interdisciplinary teaching

This theme focuses on the interplay between disciplines in modelling activities. The first study by *Borromeo* and *Mousoulides* describes theoretical reflections about the differences and similarities between mathematical modelling and interdisciplinary mathematics education. Besides its focus on underlying similarities and differences, this paper provides some examples of projects that make connections explicit and how these may be useful for teachers while discussing modelling as a means for solving interdisciplinary problems. Within this thematic strand, other empirical works were presented based on particular case studies involving interdisciplinary projects with mathematical modelling as a central issue. In particular, *Brake* and *Lantau* described a pilot study of an interdisciplinary project used with experienced students of grade 12 based on modelling segways which supported them to build models based on linear systems of differential equations. Furthermore, *Sala*, *Font*, *Barquero* and *Giménez* reported on the design and analysis of an implemented interdisciplinary project, where mathematical modelling was embedded into an archaeological context. The study showed how the complementarity between two subjects (history and mathematics) can be an important tool in supporting modelling and inquiry. Besides the potential of designing interdisciplinary modelling situations, such as those presented in these papers, a couple of important question remain unresolved:

- What are the specificities of interdisciplinary mathematical modelling in relation to mathematical modelling more generally?
- How is it possible to manage the interplay between mathematics, mathematical modelling and non-mathematical knowledge to enrich teaching practices in the learning of mathematics?
- Can some analytical tools widespread in the research frame of modelling, such as the modelling cycle, be adapted to analyse mathematical modelling in interdisciplinary contexts? And, if yes, how?
- What relationships exist between the mathematical modelling cycle and the inquiry process?
- How do we best integrate different disciplines in developing modelling tasks?

Another important topic referred to modelling in the particular case of teaching of engineering. The papers from *Romo*, *Tolentino* and *Romo-Vázquez*, and *Siero*, *Romo* and *Abundez* reported intentions to design and analyse modelling activities for the mathematical education of teachers. Both studies were based on the Anthropological Theory of Didactics. They focused on analysing the roles and interplay of institutions in the educational programmes of engineers and on the institutional

conditions that the design of the study and the research activities expose in the teaching of modelling in such contexts. Some of the questions discussed were:

- How can different institutions become involved in mathematical modelling in engineering education?
- How can practical knowledge be grounded on mathematical knowledge?
- How can the steps of an engineering project (design, mathematical model, prototype) be described? How are these steps interrelated? How are they connected to the modelling cycle?

Connection of the problem-solving perspective and mathematical modelling

The second theme refers to the connection between problem-solving activities and mathematical modelling. First, *Clohessy* and *Johnson* examined the relationships between the problem-solving performance of small groups with that of individual students in order to identify the influence of group work as an effective instructional strategy when teaching problem solving. Second, *Karatas*, *Soyak* and *Alp* presented an investigation about mathematical non-routine problem solving processes of fifth grade students in small groups. Their study aimed to determine problem solving behaviours within different episodes of problem solving. As both papers focused on the description and measurement of problem solving competences in small groups and individually, some common questions appeared:

- How might individual competences improve when students are participating in a group?
- What instruments can we use to measure the improvement of individual competences before, during and after mathematical modelling processes? Which are the most valuable indicators we might use to measure these changes?
- What are the cultural aspects that have most impact on planning the implementation of problem-solving activities?

Developing modelling strategies and competences

This third theme concerns the use of heuristic strategies to support modelling practices. The paper from *Stender* and *Kaiser* presented a study on the usage of heuristic strategies by students in school within modelling activities and the promotion of strategic help provided by academic tutors, who guide the modelling activity of the school students. The paper of *Schmelzer* and *Schukajlow* focused on the relationship between reading comprehension and mathematical modelling. The study identified strategies to help learners comprehend a modelling problem and described ways these strategies might be implemented in the classroom. The following important questions emerged from discussions of the working group:

- How far can heuristic strategies developed in the frame of problem solving be transferred to mathematical modelling?
- Can their identification and characterisation be helpful in supporting teachers' strategic interventions?
- How, in different teacher education programmes, may these heuristics be made explicit? If we make them explicit, do we risk narrowing the radius of action of teachers when guiding modelling activities?

- What are the effects of text length and superfluous elements on reading comprehension in modelling problems?
- How can we prepare teachers to foster students' reading comprehension in modelling?

A closely related topic was that focused on the metacognition of modelling competencies as an essential part of developing competence in modelling. *Vorhölter, Krüger* and *Wendt* presented their results from a pilot study about the identification and measurement of metacognitive modelling competencies in small groups when working on modelling activities. The following relevant questions were discussed:

- How might we best define metacognitive competencies? How can their characterisation be used to evaluate students' development of competencies?
- How is it possible to separate the individual progress of metacognitive competences from that of the collective group?
- How might a detailed evaluation of students' teamwork enrich the understanding of metacognitive competences?

Analysis of modelling processes

This fourth theme addressed tools and methodologies used to analyse modelling processes. On the one hand, *Barquero, Monreal* and *Ruíz-Munzón* presented a study, proposed within the frame of the Anthropological Theory of Didactic, about how to forecast the increasing number of Facebook users. The analysis of the implemented research path was based on three dialectics essential for mathematical modelling: the questions-answers dialectic, that of the media-milieu and that of individual-collective dynamics. On the other hand, the paper from *Delgadillo, Viola* and *Vivier* presented an analysis of a modelling task in the context of pre-service teacher education based on the theory of the Mathematical Working Space. In this latter study, the modelling cycle was used as an essential tool to analyse the personal Mathematical Working Space of students solving a modelling task. Some questions appeared in the discussion of both papers:

- Which different dimensions, or levels, of analysis may be taken into account when analysing mathematical modelling practices?
- Which are the most valuable observables (depending on the focus of study)?
- Up to what point can tools for analysis be used as tools to help with designing mathematical activities? In which more general approaches to task-design do they appear?
- How do we carry out analysis that takes into account both individual activity and collective interactions in modelling processes?

Teachers' beliefs on teaching modelling

This theme refers to a study on teachers' beliefs on modelling tasks. The paper from *Ramirez* explored mathematics teachers' beliefs about teaching and learning mathematical modelling and about modelling itself. It presented an exploratory study of responses from teachers collected in an online questionnaire related to the characteristics of modelling practices. Several questions came up about the relation of teachers' beliefs with their experience and knowledge about modelling:

- While analysing beliefs many aspects have to be taken into account, which are difficult to separate. Thus, what is the relation between ideas, beliefs, previous experiences, and modelling competences?
- The role of teachers' beliefs in connection with teachers' knowledge is examined. Therefore, the question arises: How do we take into account teachers' knowledge about modelling?
- The role of teachers as individuals within institutions is studied. Thus, which cultural and school conditions influence their opinions?

Teachers' interventions in teaching modelling

This section refers to teachers' interventions in teaching mathematical modelling. *Ferrando, Donat, Diago* and *Puig* presented an analysis of the different kinds of interventions teachers made during a project in which students worked on a modelling task about the intensity of sound distribution throughout a classroom. The study aimed to identify the influence of such interventions on students' learning opportunities. The following questions arose:

- How do available resources and means influence the openness of the task and students' possible responses?
- Who validates the final answer in a project?

Experimental materials and technology in modelling

This theme focused on the role and use of experimental materials and of technology. Two topics were dealt with. The first focused on the role of auxiliary material and its impact on modelling and, the second concerned the combination of different resources with technology to assist concept development by means of real-word contexts. *Guerrero-Ortiz*, *Mena* and *Morales* discussed how the handling of auxiliary material can favour knowledge transfer between real world situations and mathematical models. The research, which was conducted within pre-service teacher education in Chile, presented insights into the design of modelling tasks and the affordances of auxiliary materials in supporting modelling. *Carreira* and *Baioa* base their research on an episode of a modelling activity with grade 9 students which aimed to reflect on the authenticity of the modelling task and to examine how students used experimental work to help them succeed in modelling activities. Some of the matters discussed were:

- How can auxiliary material support the learning process? Is it necessary to support students in its use or is it self-explanatory?
- Does the auxiliary material simulate the real processes taking place in the real setting?
- How far are problems authentic or meaningful to students?
- To what extent can we include the way things are really done in the real world in modelling tasks for students?

Regarding the second topic, *Karimianzade* and *Rafiepour* presented a study about the introduction of decimal numbers. The study showed how different resources were introduced and how experimental work helped fifth grade students to develop their knowledge and understanding of decimal numbers in the context of measurement. *Lieban* and *Lavicza* reported about students' work in using dynamic geometry systems in a geometric modelling situation. They suggested the use of some new manipulative resources together with digital applets that progressively enrich the development of the modelling process by students. Several ideas were debated around the questions:

- Which alternative ways exist to introduce decimal or rational numbers taking into account the introduction of standard units?
- How can we produce computer simulations of physical models?
- What are the goals for the different people (students, teachers and researchers) involved in creating GeoGebra models?

Assessment of mathematical modelling

This last theme refers to the assessment of mathematical modelling. The paper of *Greefrath, Siller* and *Ludwig* analysed the official school leaving examination in Germany allowing university entrance, the so-called Abitur examination, which is supposed to contain elements that examine mathematical modelling. For this purpose, they based their analysis on certain criteria: reference to reality, relevance, authenticity, openness and partial competence of modelling, to analyse the potential of problems included in the official examination. In addition to the difficulties of deciding how to evaluate modelling practices in these kinds of official examinations, several questions were discussed:

- Which criteria can be used to describe good examination questions for modelling?
- Can all the sub-competences of modelling be assessed within examination tasks?
- How can we design school examination tasks that cover various aspects and goals that include mathematical modelling?

Within this strand, *Ärlebäck* and *Albarracín* proposed an analysis of various definitions of Fermi problems from a modelling perspective. They focused on analysing how the definition and descriptions of Fermi problems in the literature align with different perspectives on modelling. They also discussed how far Fermi problems and modelling are connected and how strongly that connection is influenced by the definition of Fermi problems.

Concluding remarks and perspectives

The overarching themes tackled in TWG 6 show the variety of research questions the papers dealt with, for example concerning the theoretical frameworks used or the underlying perspectives on the teaching and learning of mathematical modelling (Kaiser & Sriraman, 2006, Kaiser et al., 2007). Furthermore, the educational levels involved span from primary to tertiary education.

In their analysis of the development of the teaching and learning of mathematical modelling over the last decades, Cai and colleagues (2014) introduce five perspectives, which can be helpful in order to identify the progress made during ERME conferences and potential future development.

The first perspective, the mathematical perspective, describes the differences between modelling at practitioner level and at school level. However, industrial examples can play a powerful role in education, because they are authentic and of varying complexity. There is a long tradition of discussing examples from engineering education in TWG 6 at various ERME conferences. In addition, the relation to other disciplines has always played an important role and needs to be mentioned under this perspective.

The second perspective, the cognitive perspective, focuses on students' cognitive processes when modelling, and cognitive barriers, when students work through the modelling cycle. Cognitively oriented analyses have played a prominent role in many sessions of the modelling group at previous

conferences. At CERME 10, this aspect can be found within two thematic strands, that on developing modelling strategies and competences and also that on the analysis of modelling problems. The discussions there broadened our focus on the cognitive perspective and helped us develop further our thinking.

The third perspective, the curricular perspective, refers to the inclusion of mathematical modelling in the curricula. This question was addressed in all meetings at previous conferences and is addressed in nearly all thematic strands of TWG 6 at CERME 10 and highlights the significance and urgency of this theme.

The fourth perspective, the instructional perspective, claims the necessity of high quality modelling education in order to promote effective learning. The question on how to implement effective modelling environments is a hot topic that has persisted for decades and was addressed at CERME 10 within various strands of TWG 6, especially within the themes that addressed experimental materials and technology in modelling and teacher interventions.

As a final and fifth perspective teacher education and teachers' activities in school are addressed, because of the obvious necessity of preparing pre-service teachers for the teaching of mathematical modelling, although the importance of this topic at the various sessions of the Applications and Modelling TWG at previous ERME conferences was less prominent than at this. At CERME 10, teachers and their role in teaching as well as teacher education played a prominent role within the work of the Applications and Modelling TWG. This was integrated into various themes such as modelling strategies, teacher interventions and teachers' beliefs. This shift shows a clear further development of the discussions and the work of TWG 6 and needs to be fostered and broadened. Teachers and their education are the key for the effective and efficient integration of mathematical modelling into mathematical education at various levels.

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Developing a classification scheme of definitions of Fermi problems in education from a modelling perspective

Jonas Bergman Ärlebäck[1] and Lluís Albarracín[2]

[1]Linköping University, Department of Mathematics, Sweden; jonas.bergman.arleback@liu.se

[2]Serra Húnter Fellow, Universitat Autònoma de Barcelona, Spain; lluis.albarracin@uab.cat

In this paper we use a modelling perspective to analyse three descriptions and definitions of so-called Fermi problems found in the literature. We discuss how the three definitions align with, and what they potentially have to offer to, realistic or applied modelling, contextual modelling, educational modelling (either a didactical or conceptual), socio-critical modelling, epistemological or theoretical modelling, and cognitive modelling. Our findings show that the definitions share some similarities, but for the most part are formulated in lose terms. From a modelling perspective, we found that the conceptualisation of Fermi problem we studied foremost and directly align with contextual modelling and both strands of educational modelling. We also discuss the seemly incompatibility between Fermi problems and the other modelling perspectives, and suggest new lines of research on Fermi problems in particular, and on conceptualizing modelling in general.

Keywords: Fermi problem, modelling, modelling perspectives.

Introduction

The notion *Fermi problem* is tributed to the Italian Enrico Fermi (1901-1954), the 1938 Nobel Prize winner in physics, who had a special liking for posing and solving problems like *How many shopping malls are there in the United States?* (Anderson & Sherman, 2010). Fermi's philosophy was that any thinking and reasonably educated person should be able to solve problems of this type by just combing one's capabilities of making quantitatively accurate realistic and intelligent order of magnitude estimates, reasoning, and doing simple calculations (Efthimiou & Llewellyn, 2007). The perhaps most famous and classic *Fermi problems* is *How many piano tuners are there in Chicago?* Allegedly Fermi repeatedly gave this problem to his physics students at the University of Chicago many times over the years, and illustrated of the power of such reasoning by quickly calculating an astoundingly accurate and reasonable answer based on just a few sensible assumptions and estimates. Besides going under the name *Fermi problems*, these types of problems are also called *back-of-envelope calculation problems* or *order of magnitude (estimation) problems*.

Much due to the influence of Fermi, *Fermi problems* have been widely used in physics and engineering college courses in the US. Indeed, one can find many "shout-out" advocating and claiming various beneficiary effects for using *Fermi problems* in teaching, often exemplifying the assumptions and calculations involved in an explicit example as well as listing *Fermi problems* to try out in the classroom (see for example Carlson (1997)). However, it seems that systematic science- and engineering education research focusing on *Fermi problem* is sparse or at best marginalized. In recent years however, a number of studies in mathematics education have focused on the use of *Fermi problems* in the teaching and learning of mathematical modelling. Peter-Koop (2004) used *Fermi problems* to investigate third and fourth graders' problem solving strategies and among other things found that students' solutions "revealed multi-cyclic modelling processes" (p.

461). At the upper secondary level Ärlebäck (2009) investigated the potential of using *Fermi problems* as 'miniature modelling problems' to introduce modelling. Using so-called MADs (Modelling Activity Diagrams) the result showed the complexity of the modelling process involved when students at the high school level engaged in solving *Fermi problems*, something which recently also have been documented for college students (Czocher, 2016). *Fermi problems* have also been used to study students' reasoning involved in solving so called *Big numbers estimation problems*, such as *How many persons can fit in the playground of our high school to attend a concert there?* Albarracín and Gorgorió (2013) showed that problems requiring equivalent mathematical solving approaches, but formulated using different context-specific wording, resulted in the students using differing solving strategies. Building and furthering this study, Albarracín and Gorgorió (2014) showed that some of the solving strategies the students used normally not would been considered valid as mathematics classrooms activities. For example, one such strategy found was *the exhaustive recounting of objects*, which requires excessive effort and/or time, or input from external sources which would eliminate the need to solve the problem altogether. However, it was concluded that 47% of the students' strategies were based on mathematical models.

Sriraman and Lesh (2006) have argued for the introduction of *Fermi problems* as interdisciplinary tasks which potentially bridge and connect mathematics and other school subjects. In addition, due to the directness aspect of *Fermi problem*, one can also easily incorporate different social issues of interest within the task, such as estimating the amount drinking water consumed, the consumption of gasoline or other fuels, the amount of discarded food or other ecological types of problems (Sriraman & Knott, 2009).

In this paper we present our on-going work aimed at doing an exhaustive and systematically review of the literature on *Fermi problems* from all educational fields. As part of this endeavour, we in this paper analyse three different definitions and descriptions of *Fermi problems* in the literature from a modelling perspective. We use the classification of perspectives on modelling by Kaiser and Sriraman (2006), and map the key features of *Fermi problems* in the definitions and description onto the different perspectives and discuss the potential of using *Fermi problems* in a modelling setting from different viewpoints. Our aim is that this preliminary analysis will point out areas and directions that are worth to further explore in the larger study.

The research question that guided our work in this paper was: *How does the definitions and descriptions of Fermi problems in the literature align with different perspective on modelling?*

Methodology and method

Three of our goals with doing a systematic review of *Fermi problem* is to i) elaborate a research grounded coherent definition that characterize *Fermi problems* as completely as possible; ii) find and describe the connection between *Fermi problems* and modelling in general and connections between modelling perspectives in particular; and iii) create a research agenda for future research (Ärlebäck & Albarracín, in preparation).

The literature for the exhaustive review was identified using a) search engines such as Academic Primer, ERIC, Google Scholar, and Scopus, and key word searches on *Fermi problem/question/ estimate, back-of-envelope problem, order of magnitude estimate, "how many piano tuners"*, b) snowballing (using literature already found and concluded relevant for the research to identify

further literature; cf. Petticrew & Roberts, 2006), and c) asking colleagues with other mother toughs than our own for papers in their native language. It should be noted that there are similar notions and concepts in chemistry and physics, and hence the searches will result in large numbers of hits. However, the majorities of these can be dismissed since they not are about education. The papers that did have and educational focus was skimmed and paper that only mentioned Fermi problems in the passing was excluded from the final selection. This resulted in a list of 59 papers from mathematics education and other educational subjects (such as science, economics and engineering), written in English, Spanish, German Japanese and Dutch. All 59 papers were read and three representative definitions and descriptions were selected. We then used the characterisation of perspectives on modelling by Kaiser and Sriraman (2006) as an analytic lens to compare the three definitions as well as contrast them relative the different modelling perspectives. We chose this high-level framework to structure the analysis rather than a more specialized and "derived" framework (such a framework classifying modelling tasks) for two reasons. Firstly we wanted do use the existing definitions and descriptions of Fermi problems in the literature as the point of departure for the analysis, and secondly we wanted to use a neural framework not based on too specific cultural or epistemological stances.

The three definitions and descriptions of Fermi problems

Although the number of papers related to *Fermi problem* found is numerous, many of them do not offer any explicit definitions of the notion, but are rather based on shared knowledge and often provide some elaborated examples to characterize how *Fermi problems* are conceptualized and understood.

For the analysis and discussion in this paper we have chosen to focus on the following three different definitions and characterisations of *Fermi problems* in the literature: Ärlebäck (2009), Goodchild and Fuglestad (2008), and Sriraman and Knott (2009). All three sources are selected from the mathematics education research literature and use and discuss characteristics of *Fermi problems* and how students work with these. Ärlebäck (2009) is included since the characterizing of *Fermi problem* in this paper is one of the most cited and used definition in the more recent literature (in 9 of the 59 papers in our list of research paper on *Fermi problem*). Goodchild and Fuglestad (2008) and Sriraman and Knott (2009) are both included since their papers are representative for much of the other papers in literature. One can discuss whether the expressed conceptualizations of *Fermi problems* in the three papers are definitions in strict sense or mere characterizations or descriptions, but to avoid ambiguity and awkward formulations in the paper we will from now on refer to the three simply as definitions.

The first quote, from now on referred to as (Ärlebäck), comes from Ärlebäck (2009) who suggested and adapted so-called *Realistic Fermi problems* defined by:

- their *accessibility*, meaning that they can be approached by all individual students or groups of students, and solved on both different educational levels and on different levels of complexity. A realistic Fermi problem does not necessarily demand any specific pre-mathematical knowledge;
- their clear real-world connection, to be *realistic*. As a consequence a Realistic Fermi problem is more than just an intellectual exercise, and I fully agree with Sriraman and Lesh (2006)

when they argue that “Fermi problems which are directly related to the daily environment are more meaningful and offer more pedagogical possibilities” (p. 248);

- the *specifying and structuring of the relevant information and relationships* needed to tackle the problem. This characteristic prescribes the problem formulation to be open, not immediately associated with a know strategy or procedure to solve the problem, and hence urging the problem solvers to invoke prior constructs, conceptions, experiences, strategies and other cognitive skills in approaching the problem;
- the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities. An implication of this characteristic is that the context of the problem must be familiar, relevant and interesting for the subject(s) working in it;
- (in connection with the last two points above) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities. (Ärlebäck, 2009, pp. 339-340, italics in original)

The second definition of *Fermi problem* is by Goodchild and Fuglestad (2008), who draw on (Swan & Ridgway, n.d.). Their definitions will be referenced as (Goodchild & Fuglestad):

> These [*Fermi problems*] are ‘plausible estimation’ tasks, which consist of one or two easily-stated questions which at first glance seem impossible to answer without reference material, but which can be reasonably estimated by following a series of simple steps that use only common sense and numbers that are generally known or amenable to estimation (Goodchild & Fuglestad, 2008, p. 52).

The third and last definition, from this point referred to as (Sriraman & Knott), is from Sriraman and Knott (2009):

> Fermi problems are estimation problems used with the pedagogical purpose of clearly identifying starting conditions or assumptions and making educated guesses about various quantities or variables which arise within a problem with the added requirement that the end computation be feasible or computable by hand. (p. 220)

Analysing and situating Fermi problems from different perspectives on modelling

We now briefly summarise the main characteristics of the different perspectives in Kaiser and Sriraman (2006) and discuss how the three definitions of *Fermi problems* above “fits” with the respective perspective and why. The brief characterization presented of *realistic or applied modelling*, *contextual modelling*, *educational modelling* (either a *didactical* or *conceptual*), *socio-critical modelling*, *epistemological or theoretical modelling*, and *cognitive modelling* are based on Kaiser and Sriraman (2006) and Blomhøj (2009).

The *realistic or applied perspective of modelling* stresses the importance of using authentic problems from science and industry as well as for the students to engage in the whole modelling process rather than fragmented parts thereof. Although none of the definitions explicitly excludes authentic contexts from science and industry, they all tend to suggest and promote more mundane

and everyday problem contexts: “the context of the problem must be familiar, relevant and interesting for the subject(s)” (Ärlebäck); “reasonably estimated by following a series of simple steps that use only common sense” (Goodchild & Fuglestad); “making educated guesses” (Sriraman & Knott). It could be noted that the use of the word ‘realistic’ in Ärlebäck’s definition might be misleading with respect to the realistic and applied perspective of modelling. This wording merely stresses that the Ferm*i problem* should have a meaningful real-world connection and not be purely intellectual in nature. However, in the sense that *Fermi problems* that focus on issues like the number of piano tuners in a city, or the number of grains of sand in a glass, are not normally relevant questions for students. On the other hand, problems that ask students to estimate the amount of trash produced, or the volume of fresh water consumption, connect with the students’ physical and social environment and have meanings by themselves. The meaning of ‘realistic’ in the realistic or applied perspective on modelling is much stronger. This suggests that Femi problems, at least as portrayed in the definitions discussed here, have little to offer to the realistic and applied perspective on modelling.

Contextual modelling, having its roots in the word problem solving tradition, is centred around the design of carefully structured and meaningful situations, were the students develop, refine, and extend their own mathematical constructs as well as apply these in different contexts. The emphasis on meaning-making in the contextual modelling perspective can be seen echoed in (Goodchild & Fuglestad) and (Ärlebäck) but not evidently in (Sriraman & Knott). In (Goodchild & Fuglestad) the students have to meaningfully understand and come to grips with the context of the *Fermi problem* at hand to overcome the “easily-stated questions which at first glance seem impossible to answer”, whereas (Ärlebäck) stresses the problem formulation to “be open, not immediately associated with a know strategy or procedure to solve the problem, and hence urging the problem solvers to invoke prior constructs, conceptions, experiences, strategies and other cognitive skills in approaching the problem“, which resonates with the ‘traditional’ problem solving tradition that historically has been strongly associated with the contextual perspective on modelling. (Sriraman & Knott) on the other hand describe *Fermi problems* as intentionally designed with the explicit “pedagogical purpose of clearly identifying starting conditions or assumptions and making educated guesses about various quantities or variables which arise within a problem”. This focuses more on solving (meta-) strategies than stressing meaning-making or for the students to develop, refine, and extend their own mathematical constructs.

Both ‘flavours’ of *educational modelling* (didactical and conceptual) are so-called integrative perspectives in that they seek to combine modelling as a learning goal in its own right as well as modelling as a vehicle for learning other content matter. The two strands within this perspective forefront pedagogical goals such as using modelling as a didactical tool for structure learning processes and modelling as a mean to introduce concepts and promote concept development. Within this perspective, the cyclic view of modelling (aka the modelling cycle) has a prominent role. Looking at the three definitions, we argue that (Ärlebäck) and (Sriraman & Knott) both put forward *Fermi problems* as vehicles for learning other curricula objectives as well as have explicit didactical considerations as central features. On the one hand the two characteristics of *accessibility* and *discussion promoting* in (Ärlebäck) address classroom dynamics and classroom norms as innate components of the *Fermi problems* themselves. (Sriraman & Knott) on the other hand explicitly

describe the use of *Fermi problems* as having a "pedagogical purpose". Looking at the definition in (Goodchild & Fuglestad) however, these educational aspects are not emphasised.

Central from the *socio-critical perspective on modelling* is critical reflection and critique of mathematics role and function in society as manifested in the use of mathematical models and modelling. Although it is an innate feature of *Fermi problems* to engage the problem solver in making reasonable, and arguable critically realistic, assumptions and estimates, these need not inherently nor explicitly focus on or be connected to the social dimensions involved in the context of the problem. Similarly as for the realistic and applied perspective on modelling, there are nothing in the definitions that explicitly stresses the fundamental core characteristics of the respective perspective. That is, with regards to the socio-critical perspective on modelling, neither of the definitions analysed forefronts the social aspects and implications of the use of models and modelling in society. However, it is worth noticing that (Goodchild & Fuglestad) use a formulation that indicates that *Fermi problem* can be used to get students to appreciate the potential and power of mathematics to address and make sense of real problems in the world, namely "questions which at first glance seem impossible to answer without reference materials".

Epistemological modelling focuses on theory building and uses modelling as a mean to re-construct topics and branches of mathematics as a discipline. Neither of the three definitions (Ärlebäck), (Goodchild & Fuglestad) and (Sriraman & Knott) express the ambition to draw on *Fermi problems* to derive theory in terms of re-building and constructing mathematical (sub-)topics or (sub-)areas. Indeed, as pointed out in Ärlebäck (2009), *Fermi problem* can be experienced as limited with respect to various mathematical content, and given a particular learning goal within mathematics, it might be very challenging to design and formulate a *Fermi problem* that focuses on eliciting this content in a natural way.

The *cognitive modelling perspective* is sometime described as meta-perspective in the sense that it focuses on fundamental research questions related to various aspects of modelling from a cognitive perspective. From the point of view of the cognitive modelling perspective being a meta-perspective that guides research into the practices of mathematical modelling and all that goes around and into the modelling process, we find it difficult to elaborate on what the different definitions might offer in this respect. We fear that such a discussion would be far too speculative to be constructive or productive and not inform our aim about how to classify definitions of *Fermi problems*.

Discussion, conclusions, and future research

The limited analysis we have presented in this paper points to some of the challenges in developing a classification scheme of definitions of *Fermi problems* from a modelling perspective. Having engaged in this exercise, we conclude that the level of interpretation needed to apply the different perspectives on modelling as analytical lens introduces uncertainty in the results. Partly we believe this has to do with the fact that the definitions and characterizations of *Fermi problems* in the papers found in literature are vague and ambiguous. However, we also contribute some of this difficulty to the used perspectives on modelling in Kaiser and Sriraman (2006), which describes the modelling debate from evolutionary viewpoint, connecting todays trends and approaches with their historical traditions and roots. This suggests on the one hand, that an overarching and general characterisation and definition of *Fermi problem* could make the research on *Fermi problem* more connected and

coherent, rather than scattered and compartmentalized. On the other hand, is also suggests that alternative ways of thinking about and characterize different aspects of the on-going modelling debate might provide new insight into the growing literature on the teaching and learning models and modelling – ranging from basic ontological and epistemological considerations to different aspects of both general and particular designs and practices involved in the teaching and learning of mathematical models and modelling.

In going through the papers in our list of research on *Fermi problem* and looking at the definitions, we found that most definitions adapted in the different papers are of a local and pragmatic nature in the sense that they are relevant and work fine in the particular setting and study described and reported on the paper. We also identified patterns of linkages between the work of some authors who draws and build on each other's work, whereas some pieces of research are more like isolated islands. To us this is a second indication motivating the need for a more coherent view and characterisation of *Fermi problems*, in order to coordinate the various research finding in the literature and advance our collective experiences and knowledge with respect to *Fermi problems*.

As we mentioned before, there is no consensus of what the characteristics of *Fermi problems* are in the research literature. This is perhaps not surprising since this type of problems have been part of everyday mathematics and science teaching in various degrees and in various forms for decades, but only in recent time been subject for more systematic investigations. Doing the analysis of the three definitions have pointed to some communalities and difference in general and from a modelling perspective in particular. We are of the opinion that *Fermi problems* have much to offer from a modelling perspective, both as a tool to promote modelling (cf. (Ärlebäck, 2009)) and as a research tool. Hence we would like to promote the use of *Fermi problems* in schools, and through our systematic literature review (Ärlebäck & Albarracín, in preparation) we hope to lay the foundation for finding a common ground for promoting these types of problems in education and research. Our next step is to build on the initial ideas and results presented in this paper to make a more carful analysis of our sought out literature, with the ambition, to among other things, come up with a tentative and coherent definition of *Fermi problems* together with a rationale for how, when and why to used then in connection to mathematical modelling.

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Levels of analysis of a mathematical modelling activity: Beyond the questions-answers dialectic

Berta Barquero[1], Nacho Monreal[2] and Noemí Ruíz-Munzón[2]

[1] University of Barcelona. Faculty of Education, Barcelona, Spain; bbarquero@ub.edu
[2] Pompeu Fabra University. Escola Superior de Ciències Socials i de l'Empresa – Tecnocampus, Mataró, Barcelona, Spain; jimonreal@tecnocampus.cat, nruiz@tecnocampus.cat

This paper presents the a posteriori analysis of an study and research path (SRP) on comparing reality versus forecasts of Facebook users, which appears as a teaching and learning proposal for mathematical modelling. We present the main elements of the SRP that have been designed through a virtual platform developed in the frame of the European project MCSquared and experienced in a first-year course at university in management sciences degrees. The three-layer analysis we present is based on three main study dialectics: the questions-answers, media-milieu and individual-collective dialectics, which are central for an SRP and for mathematical modelling. In particular, we focus our a posteriori analysis on how these three dialectics were fostered and which are the main weaknesses as well as strengths of the SRP experienced.

Keywords: Mathematical modelling, study and research paths, dialectics, university level.

Introduction: The SRP as teaching proposal for mathematical modelling

During the last decades, researchers and practitioners agree that teaching should not be focused only on the formal transmission of knowledge, but also should provide students of the tools for enquiring into the study of real phenomena and integrate mathematics as an essential modelling tool. It is thus important to foster a change of the school paradigm, with new functionalities to mathematical knowledge, novel responsibilities to teachers and students, different ways of questioning mathematical knowledge, that is, moving from a school paradigm which most of the time focused on introducing students to already built mathematical knowledge devoid of its rationale to a paradigm of questioning it, 'questioning the world' in the words of Chevallard (2015). In the particular case of the research on modelling and their applications and on inquiry-based approaches some big steps have been made showing how, under certain suitable conditions in different educational levels and curricular frames, modelling activities may be successfully put into practice (Artigue & Blomhøj, 2013; Burkhardt, 2008; among others). Hence, the dissemination and long-term survival of activities based on modelling, enquiring and other innovating proposals are one of the main challenges for Mathematical Education. Therefore, to support and analyse any kind of alternative teaching proposal, researchers need reference models that allow them to analyse and evaluate the impact that these innovative teaching practices have on the way mathematics is conceived, on the nature of the didactic systems and milieus emerged through these practices, and on the conditions and constraints that help or hinder the viability of these practices.

In our research, developed in the framework of the anthropological theory of the didactic, we bet on the use of the study and research paths (SRP) as epistemological and didactic model (Chevallard, 2015; Winslow et al., 2013) to face the problem of moving towards a functional teaching of mathematics and, particularly, where mathematics are conceived as a modelling tool for the study of problematic questions.

Traits and levels of analysis of an SRP

According to Barquero and Bosch (2015), the starting point of an SRP should be a 'lively' question of real interest for the community of study (students and teacher/s). The study of Q_0, called the *generating question*, evolves and opens many other *derived questions* Q_1, Q_2,…, Q_n. The continuous looking for answers to Q_0 (and to its derivative questions) is the main purpose of the study and an end in itself. As a result, the study of Q_0 and its derived questions Q_i leads to successive temporary answers A_i that can be helpful in elaborating a final response $R^{\heartsuit}$ to Q_0. These first characteristics can be associated to the first level of analysis of the SRP that we here consider, it consists in the dialectic establishing between the questions posed and the likely answers appearing (*questions-answers dialectic*) which also provide the basic structure of an SRP to be implemented and to be enriched after each implementation. This first layer can be linked to the *cronogenesis* dimension of the teaching and learning practices, referring to the evolution of questions to be faced and the necessary knowledge to be used.

Another central dimension for an SRP is the *media-milieu dialectic*, which constitutes the second level of analysis. As described in the aforementioned investigations, the implementation of an SRP can only be carried out if the students have some pre-established responses accessible through the different means of communication and diffusion (that is, the *media*), to elaborate the successive provisional answers A_i. These *media* are any source of information, such as: textbooks, treatises, research articles, class notes, or the teacher acting as main media. However, the answers provided are constructions that have been elaborated to provide answers to questions that are different to the ones that may be put forward throughout the mathematical modelling process. Thus they have to be de- and re-constructed according to the new needs. Other types of *milieus* will therefore be necessary to test the validity and appropriateness of these answers. This second level of analysis put attention to the *mesogenesis*, that is, the evolution of the experimental milieu. Finally, we may consider the collective dimension of the research and study of questions in an SRP. This third dimension focuses on the roles and responsibilities that, far from the traditional didactic contract, students and teachers may assume in experiencing an SRP and how the individual work is shared, transferred and agreed with the wider community, and vice versa. We will denote this third level as the *individual-collective dimension*, which refers to the *topogenesis*.

Didactic analysis of a mathematical modelling process: The case of an SRP about comparing reality against forecast

We focus on analysing the case of an SRP on *comparing forecasts against reality in the case of Facebook users' evolution*. This SRP was designed in the frame of the European project MCSquared (http://www.mc2-project.eu), the goal of which is the design of innovating teaching proposals (the so-called c-book units) to foster creative mathematical thinking. This c-unit in particular has been produced by a group of five multiple background designers: 2 Maths Education researchers, 2 university lecturers, who were then in charge of its implementation, and an expert on modelling in the field of Operations Research, which enrich a lot the way to structure the teaching sequence to prompt mathematical modelling. The teaching proposal was made in the virtual socio-technical environment developed along the project, the c-book that integrates some narratives with several applets of different factories of different educational technologies. Next we present the details about the design of the SRP experienced, combining the virtual environment offered by the c-unit with face-

to-face sessions during the winter term of the academic year 2015-16 with first-year students of Business Administration Degree and of Innovation Management (BAIM), all from the 'Escola Superior de Ciències Socials i de l'Empresa-Tecnocampus', Pompeu Fabra University.

The initial situation starts from a real news about a research developed by Princeton University in 2014, in which it was predicted that Facebook would lose the 80% of its users before 2017. Hence, the initial question Q_0 presented to students is about: *Can these forecasts be true? How can we model and fit real data about Facebook users' evolution to provide our forecast the short- and long-term evolution of the social network? How can we validate Princeton conclusions?* The experimentation was structured in three interconnected phases linked to the generating question Q_0, building up the a priori design of the SRP, then reflected in the design of the c-book unit. A first phase that focuses on the open research of real data about Facebook users, a second one focused on which mathematical models (mainly based on elementary functions) can provide a good fitting to real data, and a third one about the use of these models to provide the short-, medium- and long-term forecasts of Facebook users and about how to decide about best and most reliable model. The students, working in 'consultant teams' of 3-4 people, got the order from MS2 Consulting ('Mathematical Solutions Squared') previously described as Q_0 and they were asked to deliver a final report by the end of their work as an oral presentation as response to the order. The implementation combined face-to-face sessions in the teaching device called 'Math modelling workshop' (in a total of six 90-minuts weekly sessions) for the miss-in-common of the junior consultant teams' partial reports, with work out of the classroom. The guiding c-unit that gives the workshop support was ready before starting in order to let the different teams attach their answers, pose new questions by providing an interacting device (as chats or shared spread-sheets) and use some applets designed especially to get through the different phases of the SRP. Next we sketch how different dialectics were prompted by both: (a) the c-unit design (by its initial design but also by the different changes that were introduced according to students' requirements: new questions and answers not envisioned, new media required, etc.) and (b) the didactic gestures and devices to manage its implementation.

The questions-answers dialectics

We can visualize the first level of description of the SRP designed and experienced as an arborescence of the *questions* that were proposed and faced and the *answers* foreseen and appeared. This questions-answers structure constitutes a first layer of analysis of the process designed and of the trajectories followed in the implementation of the SRP. As introduced above the generating questions Q_0 of the SRP on comparing forecasts against reality in the case of Facebook users' evolution was broken into three main derived questions (see Figure 1 for the SRP questions' organization), which guided the successive phases of its implementation. A first phase that focuses on the open research of real data about Facebook users, a second one focused on which mathematical models (mainly based on elementary functions) can provide a good fitting to real data, and a third and last one about the use of these models to provide the short-, medium- and long-term forecasts of Facebook users and about how to decide about best and most reliable model.

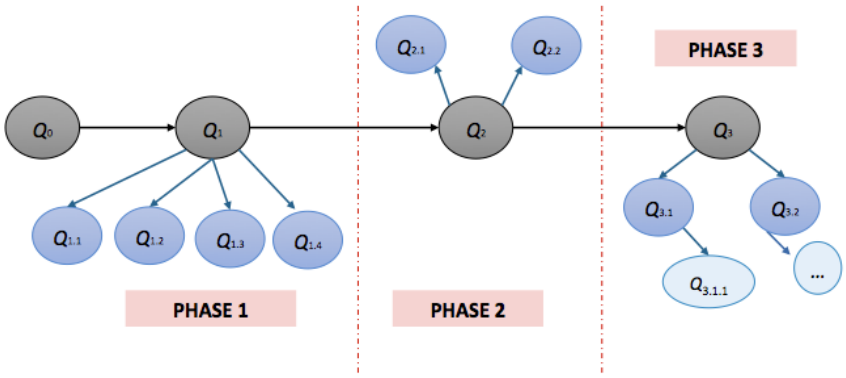

Fig 1: Tree of questions and answers of the different phases of the SRP

Q_1: Which data sets about Facebook users are better to consider in our research? → A_1: Each group look for the data set to be used and shared; the whole community agree on the terminology (year, period, units, etc.) and on the dependent and independent variables to consider.

$Q_{1.1}$: Which time intervals may be considered? $Q_{1.2}$: How can data be well-organized? $Q_{1.3}$: How to organise and visualise data? $Q_{1.4}$: What can we say about the growth tendency of the data analysed?

Q_2: Which mathematical models provide the best fitting of real data about FB users? → A_2: Each consultant group is asked to propose and justify three mathematical models fitting real data.

$Q_{2.1}$: Which models (based on elementary functions: linear, parabolic, exponential, etc.) may fit the data?
$Q_{2.2}$: How can the coefficients of the model be determined?

Q_3: How can we decide about the 'best' models fitting data? Can we use this model to predict the future evolution of FB users? → A_3: Need to create tools to justify why a mathematical model/s is/are the 'best' with respect to: (a) fitting data and (b) forecasting the evolution of FB users.

$Q_{3.1}$: How can we compare the error committed between reality and forecasts provided by models?
$Q_{3.2}$: Can be the same model used for the short- and long-term forecasts?

Let us comment the main aspects of the *a posteriori* analysis of the experimentation referring, in particular, to the questions-answers dialectic level. About the first phase, we should remark the ease with which the students found real data about the evolution of the social net. The most format they found the information was by means of a graphical representation (for example, a bar chart). This fact strongly determined their analysis, since they focused mainly in the graphical analysis of the data growth tendency, but not in their numerical versant (variation tax, that appeared in a tangential way). Besides, the fact that many groups found the same data triggered an intense debate and interchange of ideas among them, which took us to consider a brainstorming session about the previous hypothesis in the classroom, and as a consequence, the duration of the first phase was extended from 3 to 4 sessions. Due to the wealth of answers collected by the teams during the brainstorming session we decided to ask the students to deliver a first report in a poster format, so that each team could synthesize their findings and share their conclusions at that moment. About the second phase of the SRP, since many groups worked finally with very similar data on the worldwide evolution of FB users, we made two new decisions: (a) give each team a second set of different data, corresponding to different geographical areas, in order to contrast their hypothesis and extend their study; and (b) ask for more than one fitting model for each data set. The analysis of the teams proposals made arise

a non-expected aspect: many of them proposed using piecewise functions, so that the expected answers to Q_2 about the consideration of models based elementary functions (linear, quadratic, exponential, etc.) was extended. Otherwise, during the brainstorming in the first phase the teams started enquiring in the history of FB and about the possible reasons of the changes in the tendency of the data or number of users (IPO, new rival social nets, purchases of the company, new developments, etc.), and also the moments of change of tendency. New questions and answers appeared at this stage about changing the fitting model in accordance to a particular action or decisions of FB. Concerning the third and last phase, we only dedicated two face-to-face sessions of the workshop (one with the whole group and the other for the consultant teams' doubts) and were not enough for a rich development of Q_3. Although this time constraint, there were some applets designed and integrated in the c-unit to help on the simulation of models and its contrast to real data, as we explain in the following section about the media-milieu.

The media-milieu dialectic

Since we have the first layer of analysis of the SRP in terms of the arborescence of the questions-answers, it is important to ask when, where and how questions can arise and answers can be developed. It is at this new level when there may appear the different elements taking part of the milieu, composed of varied elements: questions, temporary answers, pre-existing answers in or out school, means to validate answers, experimental data, etc., accessible through different kind of media (textbook, lectures, website resources, etc.). The relation among these elements can be analysed through the *media-milieu dialectic*. In our SRP it has been central the constant dialectic between the search for data (for instance, real data about Facebook users, or about the company changes) and of pre-existing answers (ways to organise data, common models to fit population evaluation, elementary functions, tools to control error, etc.) that exist in different media that were available to students, such as web resources, contents of Mathematics course, answers from lecturers from other courses; and the creation of the appropriate means (milieu) to integrate (or refuse) them it their SRP path study. Let us stress the importance of some of them.

In the first phase of the SRP, it was important to count on the help of the teacher of another course called 'Introduction to digital communities' (running in parallel to the workshop) who helped on providing a general sense and functionality to Q_0 and to show how the students could look for real data about FB and some techniques to organise them. Also, having open access to the news and papers published by Princeton University about FB, as well as FB answer to Princeton or their monthly report about users growth was very useful for students. All these elements took part of the *media* accessible to students, at the time it enriches students' *milieu* mainly composed at this stage of the data sets that each team chose to work with, shared and debated with the whole class sessions. All these elements helped them to prepare a first report with the first temporary answer A_1, in a poster format given in the c-book platform, and then shared and debated in the face-to-face session (see Figure 1, left side). With respect to the second phase, the a priori design of the c-unit contained some applets (designed with Geogebra) proposed to help students to explore different models based on elementary functions (Q_2). These applets provided the main *media* for students to visualize data jointly with model simulation, and also took part of their *milieu* as main tools for contrasting, comparing and deciding on the 'best' models to choose. As aforementioned, in the SRP experienced students suggested using piecewise functions, which pushed designers to make changes in the

questions and applets as these necessities were coming up. Moreover, some groups started to present new hypothesis about model to use with non-elementary functions (such as Gaussian function), most of which were part of their milieu because they had been introduced in previous courses. So that, designers had to quickly cover this demand by designing a new applet with Geogebra to let them manipulate also these types of mathematical models.

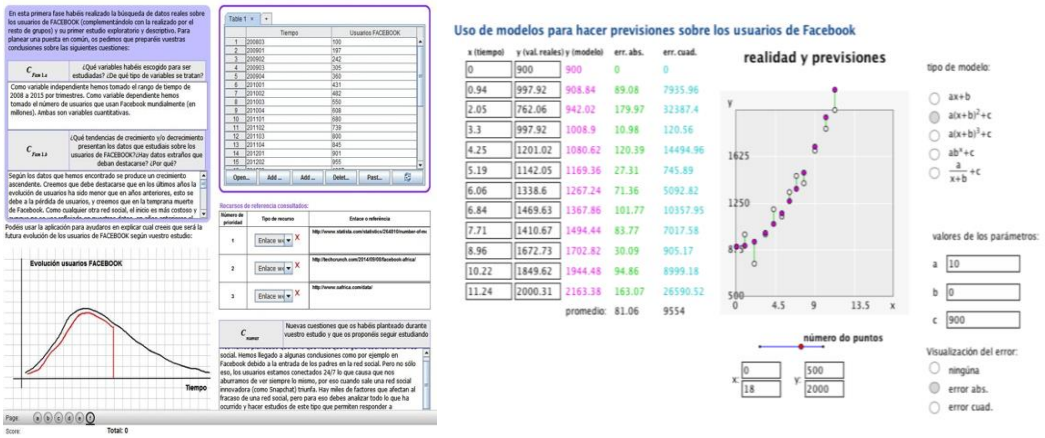

Fig 2: Example of page format and applets of the c-unit about Facebook users

With respect to the last phase, despite of the lack of time, the study of Q_3 that the consultant teams developed was in general very rich. Concerning $Q_{3.1}$ about comparing the error committed by models' forecasts against reality, we decided to integrate a new applet from Cinderella (see Figure 2, on the right) to provide students with the main media, also milieu, to simulate models they had bet for and to be able to compare graphically and numerically the error between data and forecasts coming from models' simulation. Through this tool, one could obtain the numerical calculation and graphical representation of the punctual and the averaged error (absolute and quadratic error) at the time one can changed the parameters that define the model to obtain a better fitting. Although the several advantages that this applet provided to students, students assumed and used uncritically the tools proposed by the applet. The lack of time and these designers' decisions made that the media-milieu dialectic at this stage was not so rich as it could be.

The individual-collective dialectic

In this level of analysis we focus on the relation between the teaching devices habilitated for the implementation of the SRP and the changes on the traditional didactic contract that are necessary, that is, changes on the roles and responsibilities that both students and lecturers involved in this experience had to assume. This layer, closely related to the two previously introduced, provides a finer detail about how individuals and the group developed their work in the SRP. And, how all the actions and objects of study and research (looking for questions and answers, proposing new media, adopting external answers, enlarging the milieu, etc.) are shared, agreed and transformed from the individuals to the community. Although we will be not able to go very deep in this description, due to space restriction, let us stress some important features of the SRP about its collective dimension.

First, the *devolution of* Q_0 was presented as an external order coming from MS2 Consulting ('Mathematical Solutions Squared') and students were asked to spend more than a month to prepare and deliver a final report by the end of their work as an oral presentation. The lecturer of the (official)

Maths course now changed their role to become only a guide of the study and research process, transferring most of the responsibilities to students on elaborating their answer. Moreover at the end of the implementation, two external persons carried out the *validation* and *evaluation* of the final consultant team's answers, in coherence with what was asked at the beginning and trying to make students assume these responsibilities. Second, students were organised all the time in *consultant teams* with an autonomous functioning, who were asked since the beginning to jointly deliver a collective report, although in its evaluation each of them had to be responsible of explaining responses to one of the phases and all could be asked about any phase. To help on assuming this autonomy, lecturers took several decisions: (a) distribute different data sets to help them not to be very influenced by other teams work and rhythms; (b) use the workshop sessions for the common debate of groups work and to share the main advances in finding answers (although different) and new questions emerged, and (c) constantly introduce changes in the design of the c-unit design, which work as the shared support, with the new questions that had emerged and with the media that could help them at each step. For instance, at the last of phase 1, designers decided to integrate a poster format to fix the things students had to share with the rest of the groups and dedicate a workshop session when consultant's teams could explain their answers to Q_1; or, as explained in the previous section, in phase 2 and 3 designers decided to create new applets to deal with new models proposal and to facilitate them the media and milieu to make the SRP progress.

Final remarks and conclusions

In this paper we focus on the case of an SRP on comparing forecasts against reality in the case of Facebook users' evolution to show the use of three dialectics: the one of the questions-answers, of the media-milieu and of the individual-collective, corresponding to the three complementary level of didactic analysis of teaching and learning processes (Chevallard, 2008). Besides their analytic use, they suppose a productive framework to enrich teaching and learning practices, in particular, on modelling.

In what concerns to the *questions-answers dialectic*, since the beginning of the workshop, the generating question Q_0 about the controversy of the article by Princeton was adopted by the students with a great interest and, up to the end of the process, was kept alive. From its implementation we can underline very important conditions that were created. First, the flexibility of the lecturers and designers team that were opened to readjust the schedule according to students' team work, that is why we devoted more sessions to the first phase and consequently reducing the ones to the last phase. Furthermore, they were very attentive to integrate in the c-book unit all new questions and means that students asked for. Second, students were very active on workshop session to share their proposals from which many derived questions appeared, some of them planned in the a priori design, some others that extended the initial proposal. About the *media-milieu dialectic*, in the case of this SRP and with the support of the c-unit infrastructure, we took several decisions along the implementation of transforming the media offered to students to help them in the modelling process and also to observe the impact new media had on students' milieu. We may again insist on the importance of very important contributions, such as: collaboration with other subjects (as the one of 'Introduction to digital communities'), focusing some workshop sessions on the discussion external answers that students brought, or by the creation of widgets to foster students' experimental work, etc. Last but not least, about the *individual-collective dimension*, we may underline that students easily accept the

request of presenting their final response in front of an external committee. The last session that we dedicated to these presentations brought to the light the richness and variety of answers given to the stated initial question, as well as the complementarities of consultant teams' answers. But, we are aware of the weakness and insufficiency of mechanism to collect individual and teams internal work, as most of the workshop sessions were dedicated to the common debate. This is one aspect to be improved in the following experimentations and to integrate means to get access to the dynamics established between the individual and collective work.

Acknowledgments

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Mathematical modelling as a prototype for interdisciplinary mathematics education? – Theoretical reflections

Rita Borromeo Ferri [1] and Nicholas Mousoulides [2]

[1]University of Kassel, Germany; borromeo@mathematik.uni-kassel.de

[2]University of Nicosia, Cyprus; n.mousoulides@ucy.ac.cy

In the last years the discussion for promoting Science, Technology, Engineering, and Mathematics (STEM) education became a central goal of educational policy in many countries worldwide, in an attempt to prepare students for a scientific and technological society. However, interdisciplinary mathematics teaching and learning is not limited to the "STE" and should include other disciplines across the curriculum. Mathematical modelling, as a mathematical practice and key competence within mathematics education standards could be interpreted as an excellent example for promoting not only modelling competencies, but also interdisciplinary mathematics education (IdME) in school. In this paper we focus theoretically on the question, 'Which core similarities and differences can be stated between the two fields along three perspectives?', by presenting a piece of theory describing the interplay between IdME and mathematical modelling.

Keywords: Mathematical modelling, interdisciplinary mathematics education, theory development.

Introduction

The purpose of the present study is to examine, from a theoretical point of view, the interplay between mathematical modelling and interdisciplinary mathematics teaching and learning, and to propose how mathematical modelling can promote interdisciplinary mathematics education. Following our theoretical approach, we present an example of an activity, based on our previous work, which exemplifies the key features and components of such a modelling activity that can promote interdisciplinary mathematics teaching and learning.

In the following sections, we present mathematical modelling as a means for teaching and learning mathematics through an interdisciplinary lens, by referring to the characteristics of modelling that make this approach feasible. We later present the teacher perspective on using modelling in promoting an interdisciplinary approach, and an example of an interdisciplinary modelling activity. We finally present and discuss a model of the interplay between mathematical modelling and Interdisciplinary Mathematics Education (IdME).

Mathematical modelling as means for interdisciplinary teaching and learning

In this section we firstly give a brief general view on mathematical modelling and interdisciplinary mathematics teaching and learning by defining the two fields and by posing some initial thoughts for consideration. Although these definitions show strong overlaps, one has to look deeper concerning their differences, to understand both fields as exclusive as well. To make this more transparent we discuss mathematical modelling as a means for interdisciplinary teaching and learning, along the following perspectives: (a) the modelling cycle perspective and the individual modelling routes, (b) the teachers' perspective and the cross-link approach, and (c) the

interdisciplinary activities and the students' work. These perspectives are presented in the following sections.

General view and definitions

Whereas there is a strong consensus in the international discussion that mathematical modelling can be described as an activity that involves transitioning back and forth between reality and mathematics, the definition of interdisciplinary mathematics education is very vague. Recently several researchers from different disciplines, including both authors of this paper, published a monograph entitled "Interdisciplinary Mathematics Education – State of the Art" (Williams et al., 2016). Without going into detail here, it became clear that describing a 'discipline' is much easier than to think about, if more disciplines could be multi-, inter-, trans- or meta –disciplinary. So what does interdisciplinary mean? An interesting paper by Nikitina (2006) described three core approaches to the teaching of science and mathematics in integrative ways, that differ from one another in form and purpose. These three strategies, namely conceptualizing, contextualizing, and problem-centering, ask different questions of mathematics and science, and serve different learning goals. The authors discussed these strategies based on their empirical study and furthermore they claim that understanding the strengths and weaknesses of each strategy can help educators choose the optimal way to present their interdisciplinary material. In contrast, a brief pragmatic definition is formulated by Roth (2014), who stated that: "Interdisciplinarity denotes the fact, quality, or condition of two or more academic fields or branches of learning. Interdisciplinary projects tend to cross the traditional boundaries between academic disciplines" (p. 317). In the following we use this definition of IdME as a basis for our theoretical reflections in this paper. Following Roth's (2014) definition, some thoughts arise by contrasting it with mathematical modelling: Without having a real life problem, mathematical modelling activities are not possible. At first, real life questions come out of another '(scientific) discipline' than mathematics. This makes sense and so we have the solution and found the overlap between these fields by arguing that mathematical modelling is the same as interdisciplinary mathematics and vice versa! – Stop, this would be too easy! Can we easily speak about 'disciplines' or is it better to say that real life questions of modelling problems come out of other 'realities'? Do modelling problems always include or promote further disciplines/ realities and what is the importance when having one or more of them? If there are other disciplines implicitly and explicitly distinguishable in a modelling problem, is it the teacher's goal to connect them and make them understandable for the students? Are modelling problems per se a prototype for interdisciplinary mathematics education? In the next section the cycle perspective gives some answers to the raised questions and again new thoughts are presented.

Cycle perspective and individual modelling routes

Recently the importance of modelling cycles, independent of type (see Borromeo Ferri, 2006), became clear for the teaching and learning of mathematical modelling in the classroom. In addition to promoting general modelling sub-competencies, fostering the meta-cognitive modelling competency can be seen as a strong goal in the learning process as well. Research on students' cognitive processes while modelling also showed that the individual's process of modelling is far from linear. So, individual modelling routes (Borromeo Ferri, 2007) better describe students' jumps backward and forward within the cycle. These jumps could be empirically reconstructed between phases, not only inside "reality" or "mathematics", but mostly between "reality" and "mathematics".

Looking from the cycle perspective on modelling and interdisciplinary mathematics we would like to formulate two main thoughts/ideas, where (b) is dependent on (a).

(a) The modelling cycles which were developed in the field of applied mathematics and mathematics education during the last decades have a strong focus on the mathematics itself of course (Pollak, 1979; Blum, 1985). The interdisciplinary view is not explicitly visible. The often used terms in the modelling cycles, like "simplifying" or "working mathematically" do not imply that other disciplines are involved. This shows exclusiveness and no overlap when just focusing on the cycles. The same phenomenon can be observed when looking at modelling cycles in physics or chemistry (e.g. Goldhausen & DiFuccia, 2014), because they indeed focus on their discipline, but applying mathematics, if necessary, is only a side product.

(b) Considering that the interdisciplinary view is not explicitly visible in the known cycles, it is clear that it can only happen in connection with appropriate modelling problems. The individual has to interpret by reading the problem the other disciplines/realities that are included in the problem. This means that the extra-mathematical knowledge is not only limited to one's own experiences, but to the knowledge of other disciplines, like physics or ICT. One could argue that the stronger the "discipline knowledge" and the "mathematical knowledge" the better a student's modelling process will be. Speaking on an abstract level, the individual modelling routes are on a multi-dimensional level, when the other discipline(s) included in the modelling problems is understood by the students. From this point of view, we see strong overlaps between mathematical modelling and interdisciplinary education.

Teachers' perspective and the cross-link approach

With respect to the previous section we focus now on the teachers' perspective. If we want teachers to be qualified in interdisciplinary mathematics, is it expected from them to become experts, for example, in all of the four STEM (Science, Technology, Engineering, and Mathematics)-fields? This question was the starting point for our theoretical conceptualization of STEM from a teaching and educational perspective, which is also based on the existing debate in STEM-education. Borromeo Ferri and her colleagues (2016) distinguish between the single-field teaching approach and the cross-link field teaching approach. The first approach describes promoting a single discipline in school very deeply, like for example an engineering learning environment (e.g. English & King, 2015). The other disciplines are not fundamentally included within this environment. The second approach means to promote multiple disciplines; at least two disciplines are promoted in *one* learning environment in order to cross-link these disciplines (see Star et al., 2014 for an example). Within the empirical classroom study of the "Leonardo-da-Vinci Project" (Borromeo Ferri et al., 2016) mathematics, physics, engineering and art were explicitly included in one learning environment. Grade 9 students (14 year olds) built and modelled the Leonardo bridge in an Inquiry-based Learning environment. The main goal of the lesson-unit was the permanent reflection of cross-linking the disciplines. On the basis of the empirical data and theoretical thoughts "cross-link" could be characterized as follows: One can speak from cross-linking, if at least two (scientific) disciplines are combined during one lesson or within the whole lesson-unit and are reflected with students on a metacognitive level (Borromeo Ferri et al., 2016). Again, the main aspect of making several disciplines explicit is at the foreground. If the teacher decided to look at the Leonardo bridge from only a strong mathematical perspective by neglecting the other disciplines, it is also possible.

The students had the opportunity to understand and to model the bridge by also using and naming the other disciplines.

Interdisciplinary activities and students' work

A great number of research studies has focused on the development of activities and learning materials, following an interdisciplinary approach. In this paper, we focus on the development (and the characteristics) of learning activities that have adopted a modelling perspective (e.g. English & King, 2015; English & Mousoulides, 2015; Mousoulides, 2016). Such activities are set within authentic contexts, and allow for students' multiple interpretations. With regards to mathematics, such activities provide students with opportunities to be engaged in important mathematical processes, such as describing, analysing, constructing, and reasoning (Lesh & Doerr, 2003).

Research in the field listed six design principles for developing such learning activities, following a modelling perspective. These design principles are based on the work of teachers and researchers and that have subsequently been refined by Lesh and Doerr (2003). The 'Model Construction Principle' ensures that the solution requires the construction of an explicit description, explanation, procedure, or justified prediction for a given mathematically significant situation. The 'Reality Principle' requires that students can interpret the activity meaningfully from their different levels of mathematical ability and prior knowledge. The 'Self-Assessment Principle' ensures the inclusion of criteria that the students themselves can identify, and use to test and revise their ways of thinking. Specifically, the modelling activity should include information that students can use for assessing the usefulness of their solutions, for judging when and how their solutions need to be improved, and for knowing when they are finished. The 'Model Documentation Principle' ensures that while completing the modelling activity, the students are required to create some form of documentation that will reveal explicitly how they are thinking about the problem situation and their solutions. The fifth principle is the 'Construct Share-Ability and Re-Usability Principle', which requires students to produce share-able and re-usable solutions that can be used by others, beyond the immediate situation. The 'Effective Prototype Principle' ensures that the modelling activity is as simple as possible yet still mathematically significant. The goal is for students to develop solutions that will provide useful prototypes for interpreting other structurally similar situations.

By adopting the principles mentioned above, Mousoulides and colleagues (e.g. English & Mousoulides, 2015; Mousoulides, 2016; Williams et al., 2016) have developed a number of interdisciplinary modelling activities for students. These activities have been piloted and mainstream tested in various schools in a number of countries. Such an example, the '*How can I lose weight*' activity is presented in the monograph by Williams and colleagues (2016). The activity, which targeted 11-12 year olds, focused on the balance between nutrition and physical activity for a healthy life. The activity required students to actively participate in the collection, presentation and interpretation of data regarding their nutrition and exercise habits. Based on an analysis of their own data, students had opportunities to explore the variables (and their dependencies) that may affect the amount of energy intake on a daily basis (e.g. height, mass, age) and suggested specific diet and exercise plans, always taking into consideration the need of balancing the two.

The activity consisted of three parts. In the first part, the case of Mary, a 14-year-old girl who cannot fit into her favourite clothes, was presented. The students then considered the general question, "Is

not eating the best approach to losing weight?" Students, with teachers' support, quickly realised that the question needed to be refined in order to be answered meaningfully. On refining the question in their own ways, students acknowledged that real (actual) data on nutrition, and also on physical activity are needed. Students were then encouraged (by teachers) to work with their parents to collect the required data, through an anonymously completed questionnaire. Using their own data, students worked in groups to summarise their results, by categorising data into the different food categories (e.g. protein, carbohydrates, dairy products, fruits, vegetables, sweets, etc.), and by discussing the advantages and disadvantages of each category (Reality Principle). Students also explored trends and relationships in their data, by using a spreadsheet software (Self-Assessment Principle). An example of their work is presented in Figure 1.

During the second part of the activity teachers guided a student-centred exploration for identifying the factors that determine a person's daily calorie intake (age, gender, height and body mass). Students worked on analysing tables and graphs by using an applet software, designed to support the interdisciplinary activity.

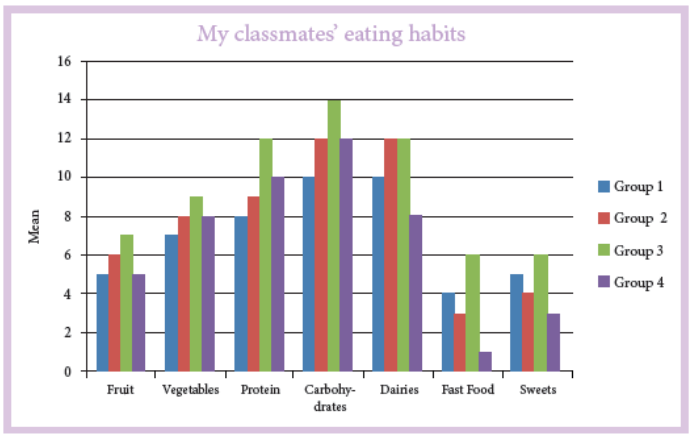

Figure 1: Student eating habits

In the third part of the activity students worked on suggesting a balanced diet plan (for a single day), taking into consideration the daily amount of energy a person needs (Model Construction Principle). Students could use the provided 'food database' for creating the person's diet for a day (Model Documentation Principle) and then explore the appropriateness of the diet with regards to the calories taken and the food categories (Figures 2 and 3). After completing the tasks and sharing their results in a whole class discussion, students then moved to the last part of the task, in which they designed their own balanced nutrition and physical activity case (Construct Share-Ability and Re-Usability Principle).

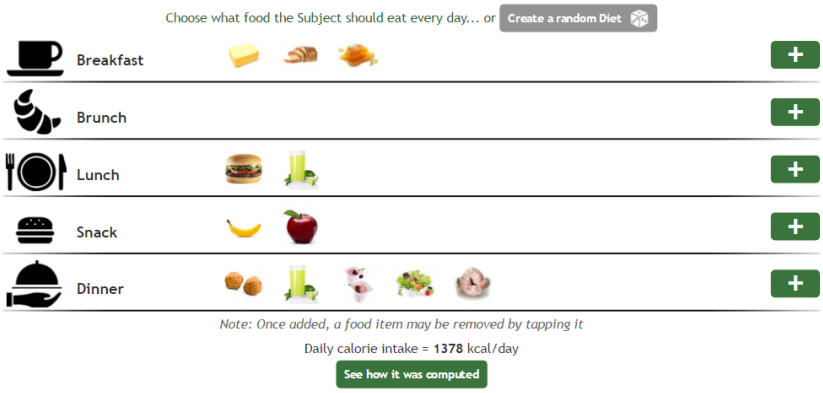

Figure 2: Creating a person's diet

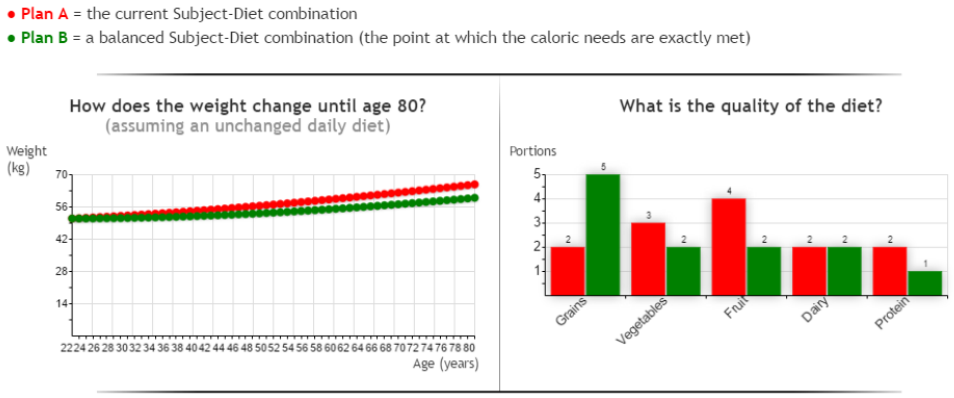

Figure 3: Examining the appropriateness of a diet in terms of calories and food quality

The interdisciplinary nature of the activity focused on the role of the bridging concept (balance between calories intake and exercise) (Effective Prototype Principle). The quite complex activity setting provided opportunities for students to explore important concepts from mathematics and biology. The implementation of the activity revealed that both teachers and parents found the interdisciplinary nature of the activity challenging; for teachers, it provided a new way of thinking and working, while for students it provided a real world problem framework, in which they could explore and connect concepts from different, yet connected, school subjects.

Theorizing the interplay between mathematical modelling and interdisciplinary mathematics education

When summarizing the presented theoretical analysis, it becomes clear that there are strong overlaps between mathematical modelling and IdME. The real context of modelling problems, like the case presented in the previous section, in fact evokes interdisciplinary activities, but the teacher is at first the person who should make them more explicit to students, and finally actively connect the different fields, through her/his teaching. Although in some ways mathematical modelling could serve as a prototype for interdisciplinary mathematics education, mathematical modelling has its own conditions. Theorizing the interplay between mathematical modelling and IdME is a challenge, which we like to think about furthermore and also find an appropriate visualization. At this point we argue that mathematical modelling has its own "theory(-ies)", because the characterization of mathematical modelling, and further terms/ concepts/ processes like "problem understanding", and "validation", are part of the theory(-ies) of modelling. This can be seen as the theoretical part of

mathematical modelling, based on the theoretical and empirical research in this field in the past decades. It is rather difficult to separate mathematical modelling from IdME, because on one hand mathematical modelling as itself is (can be) a part of IdME, but also on the other hand we can view mathematical modelling as a comprehensive research field. By adopting this modelling oriented approach in the "nutrition-exercise" case study, students could work in finding/proposing a model for balancing the intake-consumption of calories. For instance, students could be asked to propose models for different people (e.g. peers, professional athletes, teachers, parents), which balance their daily diets and their exercise habits. In doing so, the emphasis of the activity would not be within mathematics or biology, but rather on modelling.

IdME can be situated in its own field, if the (interdisciplinary) task does not fulfil the criteria of the modelling problems, e.g. when you have some kind of a "word problem". Not every interdisciplinary task, which has (some) mathematics in it, is a modelling problem per se. IdME can be done differently and cannot be connected with a modelling problem, so one can cross the disciplines of mathematics, and for example biology in a task, but only focusing on mathematics when dealing with the problem at last. This is what we mean with pure crossing disciplines. The interplay of modelling and IdME is clearly observed, when a real life question is embedded in a real modelling problem, in which students understand the context, recognise all the disciplines involved, and use or get to know about the extra-mathematical (other disciplines) knowledge. By adopting this perspective, in the Nutrition-Exercise case study students could work in solving a problem related to finding algebraic formulae for calculating the number of calories in various types of food and/or sport activities. In doing so, students have to work with both mathematical and scientific concepts, in solving the required problem, but the emphasis would be on the mathematical concepts or the biology ones (e.g. different types of food and relation to calories per gram, etc.).

There is a number of possible implications, especially for the teachers, in promoting both modelling competencies and IdME-competencies. By using modelling problems, teachers can work with their students in crossing the boundaries between disciplines. In doing so, interdisciplinary modelling activities can provide unique opportunities for teachers to collaborate, synthesize and integrate more interdisciplinary pedagogies and teaching methods in their teaching (Berlin, & White, 1995), and for students to develop better and more coherent solutions for complex, yet interesting and real world based problems. Such approaches raise teachers and students' expectations and confidence in working in a more interdisciplinary way, and lessened their focus on the difficulties in using and working with interdisciplinary modelling activities.

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Mathematical modelling of dynamical systems and implementation at school

Martin Bracke and Jean-Marie Lantau

University of Kaiserslautern, Germany; {bracke} or {lantau}@mathematik.uni-kl.de

Dynamical systems like a segway can be described by linear systems of ordinary differential equations of the form $\dot{x} = Ax + Bu$. *Obviously, modelling of dynamical systems at school is a huge mathematical challenge for students and teachers. Therefore, in a pilot study with a math course consisting of 12 students of grade 12, it was analysed if, and up to which depth, modelling projects of dynamical systems (using the example of a segway) can be implemented at school. The interdisciplinary project was based on the modelling cycle of Blum and Leiss (2007). First of all, we describe the implementation and results of the pilot study. Then, we outline the design of a follow up study that will be carried out during school year 2016/2017. In this part we formulate our research questions, mainly the design of teacher training with a special focus on teachers' attitudes, which are driven by the findings from the pilot.*

Keywords: Dynamical systems, interdisciplinary project, mathematical modelling.

Introduction

In a pilot project the modelling of dynamical systems at school (with a math course of twelve grade 12 students) was carried out by the example of a segway. The project was designed as an interdisciplinary project and started with the students independent modelling that lead to a physical description of acting forces of a segway. After a review on the appropriate linearisation of nonlinear terms, a linear time-invariant system for the segway was proposed. This introductory phase took five lessons. Next, we had three full project days where the students worked in groups of up to four students to concentrate on one of the several interdisciplinary aspects of the project. Namely they worked on 1. How to construct and control a segway and the basics of the Lego Mindstorms® EV3-brick, 2. How to simulate a segway by the help of suitable software and 3. The mathematical model of a segway, including finding of a mathematical solution for the theoretical stabilisation. The project was completed by a final presentation of the students for a mixed audience of teachers as well as university staff members. Obviously, the theoretical and practical issues are highly challenging for students and teachers and therefore the research issue of the pilot study was if an interdisciplinary modelling project of this complexity can be realised at school. The results of the pilot project were impressive, because the students managed to construct and control a Lego Mindstorms® segway as well as acquiring mathematical knowledge about systems of ordinary differential equations and their solutions and eigenvalues to assess the stability of solutions of a system of linear ordinary differential equations. Furthermore, they applied a theoretical and a practical control of a segway by using a proportional derivative control (PD-control). Based on the outstanding results of the pilot study, but with concern that the data set of 12 students is not representative for a quantitative analysis, the following main research questions are raised for a follow up study: Which form of modelling supervision (*open* = research-based, *fine-structured* = content-based, or *semi-structured* = connecting the two approaches) promotes the best development of mathematical competencies for the students during this kind of interdisciplinary projects? Secondly, it will be discussed how teacher training with respect to modelling projects for dynamical systems has to be designed. Altogether there will be a short introductory meeting at the beginning followed by a two-day teacher training focussing on possible implementations in class (with a focus on different aspects of the whole STEM project). The third part is going to be the implementation of the project by the teachers participating in our study. Accompanying the whole study, the attitude of teachers (regarding self-efficacy and prior experience) as well as self-assessment (regarding

opportunities and difficulties for themselves and their students towards interdisciplinary modelling projects of dynamical systems) will be evaluated by a series of pre and post surveys (and interviews). Furthermore, it will be researched, which competencies can be promoted through an interdisciplinary modelling project. The study shall be conducted with 10 to 15 math teachers and 100 to 150 students from grade 10 to 12.

The pilot project

The pilot project (cf. Lantau, 2016) will be described in detail since due to the complexity this is necessary to understand the generation of our research questions as well as the design of the upcoming study.

A short mathematical background of modelling a dynamical system, such as a Segway

The mathematical analysis of a dynamical system is based on the comprehension of system and control theory. A real dynamical object, in this case a segway, is at first described by physical equations to model the acting radial- and horizontal forces. After using suitable simplifications for nonlinear terms in the equations of forces (by using linear Taylor approximations) the physical model is transformed in a mathematical model by the introduction of variables like angle and position and their derivatives into a state-space vector x. Considering the force u of the regulating motor, after linearisation of nonlinear terms one can model a dynamical system by a linear time-invariant system $\dot{x} = Ax + Bu$ of ordinary differential equations. In the case of a segway, the state-space vector *x* includes the variables angle, angular velocity, (horizontal) position and (horizontal) velocity. This linear time-invariant system can be shifted into a linear system of ordinary differential equations by feedback-regulation $u = Fx$, to get $\dot{x} = (A + BF)x$. Now, the theorem of Wonham (Sontag, 1985) yields that *if and only if the linear time-invariant system is controllable (which means that every possible state* $\tilde{x}$ *can be reached due to a regulation of the motorforce u; that holds true for the segway), for every real monic polynomial P there exists a matrix F, such that P is the characteristic polynomial of the matrix A+BF*. Since the eigenvalues of the matrix *A+BF* are the roots of the polynomial *P*, the goal is now, to place all eigenvalues of the matrix *A+BF* into the open left half complex plane $\mathbb{C}_-$, such that the segway stabilizes itself to the rest position $x = 0$. Another approach to control a segway can be realised by a proportional-derivative control. The basic idea of a proportional-derivative control, e.g. for the control of the angle φ, is to choose the motor force $u = -\alpha \cdot \varphi - \beta \cdot \dot{\varphi}$, to stabilise the segway to its rest position. To implement this type of control, some conditions on the parameters α and β have to be fulfilled.

Realisation of the pilot project

The pilot study was inspired by the modelling cycle of Blum and Leiss (2007) consisting of the seven steps Constructing (1), Simplifying/Structuring (2), Mathematising (3), Working mathematically (4), Interpreting (5), Validating (6) and Exposing (7). In what follows, the seven steps of the modelling cycle will be connected to corresponding parts of the pilot study. During the project the teacher played different roles (research guide/advisor, leader of phases with questions and development) that will be explained in detail in what follows.

1. Constructing:

The introduction into the project was given by means of a video showing the popular German entertainer Stefan Raab unsuccessfully trying to control a segway (URL: https://www.youtube.com/watch?v=_m3YBSQYGuw). After this motivating start, the students were asked to create their own model of a segway. This task, carried out through a group work of four students per group, lasted 30 minutes and was done as an independent work. The construction of three fitting models for a segway has successfully been realised by the students (cf. Figure 1) who were experienced in modelling real-life situations, since many modelling tasks have been

established in this class. But in general it is not necessary to be experienced in mathematical modelling to participate in teachers trainings or as a student in the project.

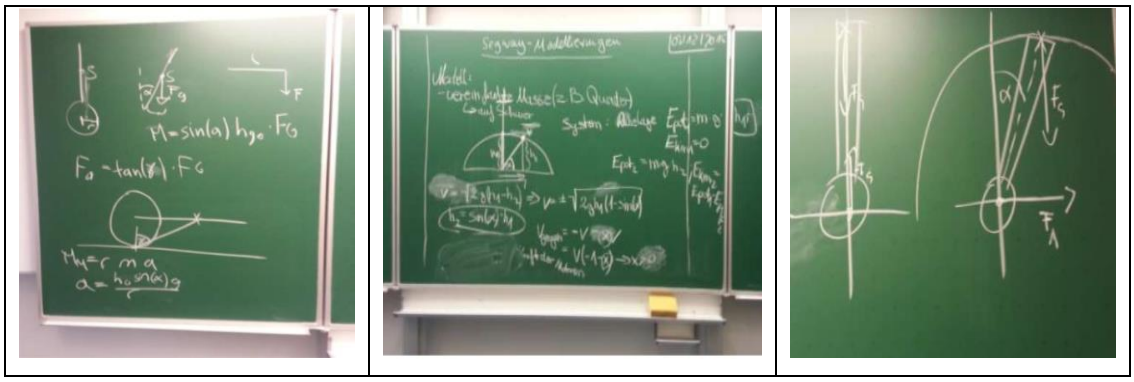

Figure 1: The independent modelling of a segway – Three different approaches

2. Simplifying/Structuring

Figure 1 shows, that the students recognized that the description of acting forces is a necessary part to obtain a mathematical model of a segway. In the next step the three different approaches to model a segway were summarised in one model, to describe the acting forces of a segway.

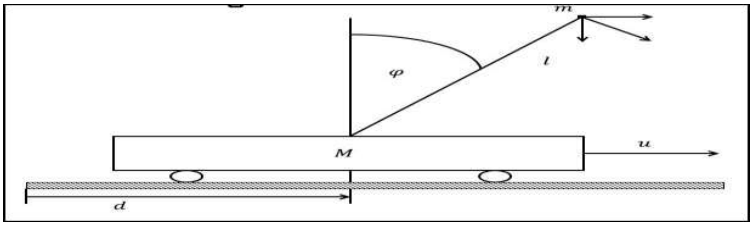

Figure 2: Model of a segway based on acting forces

The part of simplifying and structuring was supported by a matching task, where the students should match acting forces, such as radial forces and horizontal forces, to equivalent terms. In the sequel the different forces were collected to describe the horizontal forces and the radial forces in two equations:

$u = (M + m) \cdot \ddot{d} + m \cdot l \cdot cos(\varphi) \cdot \ddot{\varphi} - m \cdot l \cdot \dot{\varphi}^2 \cdot sin(\varphi)$ (horizontal forces)

$0 = m \cdot l \cdot \ddot{\varphi} + m \cdot cos(\varphi) \cdot \ddot{d} - m \cdot g \cdot sin\,(\varphi)$ (radial forces)

At this point it has to be mentioned, that during the project three different approaches to model a segway were proposed to the students. The first one, presented in this paper, leads to a control of a segway using the variables *angle* and *position* of the segway as well as their derivatives. Another possibility to model a segway, is by modelling acting horizontal forces for the variables *position of the wheels and the centre of gravity* of the segway (and their derivatives). The third approach is to model the segway as an inverted pendulum by describing radial forces. This leads to a proportional-derivative control for the *angle* and the *angular velocity* of a segway. Due to the fact, that in the modelling approaches 2 and 3, only the variables position (of wheels and centre of gravity) and angle, respectively, are analysed, the number of physical terms to describe the segway is less in comparison to the first approach. However, it has to be clarified, that the physical comprehension of acting forces is essential to promote a mathematical model. And this motivates some deep mathematical concepts: (linear) ordinary differential equations and their solutions as well as a stability analysis through the concept of eigenvalues/-vectors.

3. Mathematising

The transfer from the physical model to a mathematical model is achieved by a linear Taylor-approximation around the rest position taking into account the nonlinear terms in the equations of

forces. The linearisation for the functions sine, cosine and quadratic function were discussed geometrically on the blackboard but the general concept of Taylor-approximation has not been introduced to the students. After linearisation, the two equations can be summarised into a linear time-invariant system introducing a so-called state-space vector. This phase of the project was designed by lessons with question and development. The introductory phase of the study preparing the three project days was concluded by the formulation of the linear time-invariant system that was jointly developed on the blackboard.

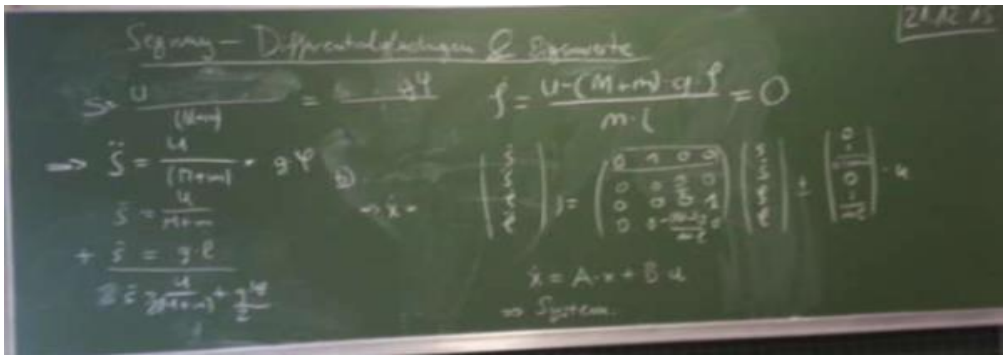

Figure 3: Development of a linear time-invariant system on the blackboard

4. Working mathematically

To emphasize the interdisciplinary character of the project the three project days (each lasting from 8 am to 2 pm) started by the formation of three groups. The task of the first group was to construct a segway by using a Lego Mindstorms® set. Afterwards, the aim was to control the segway, specifically regulating the segway into the rest position, using the Lego Mindstorms® Software or using the Java-based software lejosEV3 – in each case with the help of a gyro-sensor. The students were highly experienced in working with Lego Mindstorms®, since they have managed several projects in the past, where the use of Lego Mindstorms® was necessary. For the second group four exercise sheets were designed in which the students were introduced to new mathematical concepts, namely: One-dimensional ordinary differential equations and their solutions, linear systems of ordinary differential equations, the matrix exponential function as a solution for linear systems of ordinary differential equations, stability theory for solutions of linear ordinary differential equations, eigenvalues in a non-geometric concept, proportional-derivative control and at last, feedback-regulation for linear time-invariant systems. The task of the third group was to carry out computer simulations according to the mathematical concepts worked on by the second group. To this end, a single exercise sheet including six tasks was designed to guide these students. All exercise sheets can be found as an appendix to the master's thesis of Lantau (2016). During the three project days the students worked nearly autonomously in class on their tasks and the teacher supervised the work of the different groups following the concept of minimal help.

5. Interpreting and 6. Validating

The interpretation and validation of the mathematical model also took place during the three project days and was motivated by the interdisciplinary character in a very natural way. In particular, this can be seen in the application of the proportional-derivative control: While groups 2 and 3 used the inverted pendulum to model the segway in order to use the proportional-derivative control for the angle and angular velocity, the first group used the information about the theoretical restrictions for the proportional- and derivative parameters α and β to practically stabilise the Lego segway for about 10 seconds. While two groups established a theoretical concept to model and control a dynamical system, including the development of fitting constraints for the control (*interpretation*), the third group used the results of the other groups to practically stabilise the segway (*validation*).

Considering steps 2, 5 and 6 of the modelling cycle, we observe that a successful modelling of dynamical systems, such as a segway, includes several disciplines like physical comprehension of acting forces, engineers' competencies to construct and control a Lego Mindstorms® segway, scientific programming for the control and finally, mathematical competencies to acquire a theoretical comprehension for the control of dynamical systems. The students also observed that several disciplines must be considered in a highly connected sense to promote the success of the modelling task.

7. Exposing

In order to collect and structure the theoretical and practical results of the project the students were asked to create a final presentation. This promoted the mathematical learning success in a holistic sense because the students who worked more practically on the project got a deeper insight into theoretical results through the explanations of their classmates and the connection of the theoretical results to practical results. The same also holds vice versa. During the creating of a final presentation the students needed to prepare their newly acquired knowledge properly to present it for a mixed audience of math teachers, students, schoolmates, parents and math professors. The exploration of the pilot study shows that the phase of presenting (*exposing*) the results of the modelling project promotes many mathematical competencies. Hence, in our view, this step is essential when modelling dynamical systems at school.

Main results of the pilot study

Considering the project´s realisation it can be observed that a modelling project of a dynamical system sets a high demand on physical-, engineering-, computer science- and mathematical competencies for both students and teachers. Therefore, the main question of the pilot study was if this kind of interdisciplinary projects can be realised at school. The pilot gives a positive answer, and it also shows that fundamental mathematical competencies, as proposed by the German Education Minister Conference (KMK, 2012), are highly promoted by the students during the project. Next, the six fundamental mathematical competencies are listed and connected to processes that promoted the corresponding competency during the interdisciplinary project.

1. *K1: to argue mathematically*: This competency was promoted at three different stages of the project. At first it was promoted during the discussion of linearisation for nonlinear terms, secondly during the development of stability criteria for linear systems of ordinary differential equations and thirdly during the discussion of constraints for the parameters of the PD-control.
2. *K2: to solve problems mathematically*: The research issue for the students was how a segway can be theoretically and practically stabilised. This problem has been solved practically by a PD-control based on the Lego Mindstorms® Software and theoretically, by learning the concepts of feedback- and PD-control.
3. *K3: to model mathematically*: The competency of creating mathematical models has been the centre of the pilot and was highly promoted. The students got the insight that mathematical modelling is very useful to solve real-life problems.
4. *K4: to use mathematical forms of representations*: By the preparation of a final presentation the students were requested to illustrate their main results in a mathematically correct way.
5. *K5: to work with technical, symbolic and formal elements of mathematics*: During the development of a linear time-invariant system as a model of a dynamical system the students used technical, formal and symbolic elements. This competency was also highly requested for the two groups that worked out the concepts of eigenvalues, PD-control, feedback-regulation and ODE-theory.
6. *K6: to communicate*: During their creation of a model for a segway and especially during preparation of the final presentation the students communicated frequently to discuss models, mathematical solutions and open questions. Furthermore, this competency was promoted through the final presentation itself.

Corresponding to the development of mathematical competencies during the project, the mathematical learning success of the students has been detected. To investigate the learning success a multiple-choice test was developed containing 10 questions regarding the mathematical concepts that are introduced during the project. The test has been carried out one week after the final presentation and was analysed by counting the correct answers of the content-related questions. More than half of the class reached at least 83 % of the maximum score and two students answered every question completely correct. This shows that the concept of ordinary differential equations, their solution and the stability of solutions, the concept of eigenvalues considering the stability of solutions for a linear system of ordinary differential equations, modelling a segway as a linear time-invariant system and controlling it by means of feedback-control or PD-control can be taught through an interdisciplinary modelling project. The concepts of eigenvalues (embedded in a geometric context of linear mappings and its characteristics) and (one-dimensional) ordinary differential equations and their solutions are part of the curricular standards in mathematic for secondary level in Rhineland-Palatinate (MBWWK, 2015). Both of them can not only be taught through an interdisciplinary modelling project, but also the connection between a classical algebraic concept (eigenvalues) and a concept of analysis (ordinary differential equations) can be established. But not only these two fundamental mathematical concepts for secondary level courses could be transmitted; even concepts that are usually taught in master degree courses for mathematics teachers can be included. From our point of view, the combination of promoted mathematical-, physical-, engineering-, and computer scientific competencies and a huge learning success in advanced mathematical concepts legitimates the implementation of interdisciplinary modelling projects in secondary level courses. Beside the description of the pilot project, the paper focuses on research issues concerning a follow-up study, explained in the next section.

Research issues for a follow-up study

Regarding the results of the pilot study two main research questions are aimed to be analysed within the framework of *Grounded Theory*:

1. How shall teacher trainings for interdisciplinary modelling projects be designed?
2. Which competencies can students acquire, depending on the implementation and the type of supervision of an interdisciplinary modelling project?

As explained in Kaiser (2013) the most advanced approach of including mathematical modelling in school is by an interdisciplinary project which requires each discipline – in our case of modelling dynamical systems: physics, mathematics, engineering and computer science – to share its concepts and information in order to guarantee success of the modelling project. This task is very challenging for math teachers, considering the fact that some teachers neither have the specific mathematical knowledge to understand real problems (here: theory of dynamical systems) and its didactic transposition (Chevallard 1985) nor they are experts in other required fields like physics or computer science. Therefore, one aim is to design appropriate trainings that help teachers to convey the interdisciplinary character of modelling dynamical systems. In our example this includes the adequate physical-didactical preparation of detecting acting forces of a segway as well as an introduction to basic concepts for the control and programming of a Lego Mindstorms® segway. However, in order to promote mathematical competencies and a mathematical learning success, it is also necessary to design trainings for the inner mathematical concepts required by this project. For this purpose it is planned to design learning material for complex mathematical concepts like *stability theory for ordinary differential equations* as well as an *introduction to linear time-invariant systems and their control.* To this end the material designed for the pilot study will be analysed and modified based on the students´ and teachers' comments.

During the follow-up study the attitude of teachers towards mathematical modelling of dynamical systems, represented by the example of a segway, will be explored and evaluated through a series of

surveys. The attitude of teachers will be measured by analysing three connected questions. Since the mathematical foundation of a dynamical system is located within the scope of *systems and control theory* and the project requires knowledge from several disciplines, one question is which inner and extra mathematical comprehension issues do maths teachers have or expect when they think of modelling a dynamical system. According to this question it will also be analysed how far the teachers are engaged to fill their content-related gaps. Additionally, teachers will be asked for possible implementations of each step of the modelling cycle in secondary level math courses. During the first teacher training (end of 2016) there will be pre and post surveys to answer these questions. Then, the results will be considered to prepare the second teacher training in spring 2017. Moreover, the teachers´ expectations on possible obstacles will be assessed before, in between an after a series of trainings regarding the design and supervision of the modelling of a segway. Regarding these questions it is planned to design a survey, based on the theory of planned behaviour (Ajzen, 1991). The survey shall point out to what extend the teachers´ attitude to modelling projects, their subjective norms and their perceived behavioural control influence their intention and, later on, the execution of their modelling project of dynamical systems. Additionally, teachers shall comment on the possibilities of developing each of the six mathematical competencies K1–K6 by modelling a dynamical system. Furthermore, they shall assess the potential learning success regarding the two fundamental mathematical concepts of *ordinary differential equations* and *eigenvalues*/-vectors that can be addressed by this modelling project. Finally, we want to analyse the correlation between the teachers´ attitudes and the students learning success.

Beside the teacher focussed research, our second research question is *Which mathematical (physical, engineering- and computer scientific-) competencies of the students can be strengthened significantly through an interdisciplinary modelling project of dynamical systems*. It is planned to use video recordings to detect the enhancement of students´ mathematical competencies during the project. The video material will be transcribed based on the work of Mayring (2015). To evaluate the enhancement of mathematical knowledge there will be pre-and posttests. Furthermore, the mathematical working techniques preferred by the students will be evaluated. For the analysis of the students´ prerequisites surveys will be designed and it is planned to use video recordings to analyse the mathematical working techniques and to detect the phases of the project in which competencies of the students are promoted.

For the teacher trainings, options for different ways to conduct and supervise the project will be designed: The first approach is an open modelling process in which the initial problem of a segway to be stabilised is given to the students without further hints or work sheets. The main idea is to develop the students' independence in modelling activities including the competencies of „developing productive dispositions, flexible strategies, and foster student persistence and independent thinking" (Common Core State Initiative 2010; National Research Council, 2001 as cited in Doerr & Ärlebäck 2015, p.1). As described in the paper of Doerr and Ärlebäck (Brodie 2011; Lobato & Ellis, 2005; Magiera & Zawojeski, 2011 as cited in Doerr & Ärlebäck, 2015, p.1), this type of modelling challenges the teacher „to tackle classroom discussions, to structure group interactions and to provide effective feedback to students." Regarding the task of modelling a segway we would like to find out how far students can model a segway independently in a way that develops mathematical techniques (e.g. PD-control) to solve the real problem. This question refers to the framework of Wake, Foster and Swan, who proposed that students´ competencies of a simplification of the reality and the development of a mathematical structure that represents and simplifies the reality are under-emphasised in school mathematics (Wake et al. 2015, p. 8). Following the theory of Wake et al. it is also planned to create material that promotes a pre-structured and more teacher-controlled modelling of a dynamical system as an alternative. Our aim is to check which approach teachers prefer for their class and to analyse which competencies are

promoted in a modelling project depending on the conduction and supervision in one of the two specified ways.

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Creating a color palette: The model, the concept, and the mathematics

Susana Carreira[1] and Ana Margarida Baioa[2]

[1]University of Algarve and UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal
scarrei@ualg.pt

[2]Group of Schools D. Manuel I – Tavira and UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; ambaioa@gmail.com

Based on a brief episode of a modeling activity involving experimentation, performed by 9th graders, the goal is to reflect on criteria of authenticity for school modeling tasks. The study shows that students adhered to the experimental work and materials; still their understanding of the real event and their proposed solution to the problem seems to have missed the real problem setting.

Keywords: Concept development, experiential learning, mathematical models, proportion.

Introduction

Worldwide school mathematics curricula embrace the goal of ensuring the knowledge, skills, and competences that will enable individuals to understand today's highly mathematized and technological world and to allow them to critically and consciously engage in processes that involve mathematics and its applications. This strategic goal of mathematics education is clearly echoed by the rhetoric of international tests such as PISA and TIMSS. However, mathematical modeling is admittedly a demanding activity for students, not only because each step of the modeling cycle can become a potential blockage (Blum, 2015) but also because, for example, the focus for many students seems to be to arrive at a single solution instead of developing a suitable mathematical structure for a real world problem situation.

Recently, Blum (2015) has reaffirmed that the aims of teaching applications and modelling can be seen as having a dual function. On the one hand, the knowledge of mathematics and its use is vital to the real world and its progress, mainly in solving real problems and carrying out complex projects; on the other hand, the real world and the way it incorporates mathematical knowledge is extraordinarily important as a vehicle to provide meaning to the learning of mathematical concepts and generally to mathematics as a discipline.

Bonotto and Basso (2001) also discussed another dual nature: that of cultural artifacts, which besides belonging to the world of everyday life are also part of the world of symbols. If artifacts can be the basis for understanding the mathematical structures underlying reality, they can also be used for vertical mathematization, where the concepts embedded in real-world objects and processes become material for reflection and problem solving.

In this article we want to reflect on this duality of goals of modelling real situations and, in parallel, on the duality of roles that a particular type of task, here labeled as an experience-based task, can play in the application/understanding of the concept of direct proportionality. Experiential mathematics is a way of students dealing with living and practical knowledge of a problem or situation in the context of mathematical modeling at school. It intends to develop students' ability to collect and interpret data in addressing real world situations as well as the ability to develop mathematical thinking and communication, while reporting their ideas and findings to others. The experiential mathematics puts

the real situation as an integral part of the students' work, allowing them to understand how it works in reality and how they can "handle" it mathematically (Palm, 2008, 2009; Vos, 2011, Galbraith, 2006).

Theoretical perspective

Authenticity of a modelling situation

The question of the authenticity of the examples proposed for the implementation of mathematical modeling has been dissected by several researchers and gains particular emphasis on educational environments informed by a socio-critical view of mathematical modeling. For example, Rosa and Orey (2016) advocate the use of ethno-modeling as a tool for teaching purposes where rather than distorting the cultural and ethnic reality, the aim is to help students learn how to find and work on authentic situations and real-life problems emerging from cultural and historical artifacts, embracing mathematical ideas, thoughts and practices as developed by cultures across time and space. In any case, the perspective of the actors who actually participate in the actual settings has long been emphasized by Niss (1992), in pointing out that genuine problems, issues and situations must be recognized as such by the people who actually work in them. Despite such concerns, there is still considerable lack of authenticity in many situations that school mathematics presents to the students, which often end up just mimicking real-world scenarios, prioritizing educational goals and curriculum alignment over facing the ill-defined reality of many problems (Vos, 2011; Palm 2008, 2009; Eames, Brady & Lesh, 2015).

Blum (2015) explicitly notices that the motivating or the marketing role often attributed to mathematical modeling tasks in textbooks and curriculum materials results in the creation of problems that are only a disguise of a real situation. But other authors have strongly criticized the pseudo authenticity of modeling problems, particularly in the PISA tests. This does not invalidate the accepted notion that an out-of-school scenario cannot match completely a proposed scenario in a school mathematics assignment. And that seems to be even more unquestionable when it comes to formulating problems in assessment tests.

> A school task can, of course, never completely simulate an out-of-school task situation. Nevertheless, sometimes the school situation can be organized and the assignment formulated in such a way that many of the aspects of a real life task situation may be simulated fairly well following that the students' task solving can take place under conditions fairly close to those in the simulated situation. Other times, for example in largescale high stakes testing, the conditions under which the task solving takes place put severe restrictions on the possibilities to simulate many of the aspects with high fidelity (Palm, 2008, p. 40).

According to Palm (2008), students have been predominantly faced with unrealistic mathematical tasks and have given in many cases unrealistic answers to them. At the same time, it is not totally obvious to what extent the realism of the tasks would produce greater caution and attention from students to the realism of their answers. Furthermore, Palm notes that there is a lack of consensus in the mathematics education community on the concept of realism of a task in terms of adjustment between a school assignment and a real life situation. And the evidence of that may be found in the variety of different terms for tasks that somehow try to emulate real situations (authentic tasks,

realistic tasks, real life tasks), together with the many different meanings that have been attached to each of them.

Vos (2011, 2015) has been systematically addressing the issue of authenticity and suggesting criteria that allow us to judge the more or less authentic trend of a task. Her first concern is the use of the term authentic tasks because the educational environment will require adaptations in the same way as with a flight simulator: the criteria of the *effectiveness* of the learning environment and the *realism* of the proposed situations should therefore prevail. The leading proposal is that the concept of authenticity of a task is first and foremost a social construct, so never universally definable. Therefore, a task may have some authentic aspects but it may also have others that are deliberately introduced and designed to respond to educational purposes. In short, it must be acknowledged that not all aspects of a task need to be authentic while accepting that the tasks are more engaging if a number of issues that have their origins in the situation are respected.

Once put into perspective the question of authenticity of modeling tasks, the consequent challenge concerns the search for principles to guide the construction of interesting modeling tasks, that are both effective and intellectually honest (Blum, 2015; Galbraith, 2006; Palm, 2009). To a great extent the challenge is to convert a fruitful idea rooted in a real-world situation in a problem that is fairly well framed by the details of the situation and that is accessible to a resolution that also respects the authentic school mathematics (Vos, 2011).

There are at present some proposed formulations of basic principles to inform the authentic character of modeling tasks in school mathematics. Here we will consider the work of Palm (2009), Galbraith (2006) and Vos (2015).

Palm (2009) considers a number of variables: Event; Question; Information; Presentation; Solution Strategies; Circumstances; Solution Requirements; and Purpose. Galbraith (2006) proposes a number of principles: Link to the real world; Tractable question; Feasible solution process; Existing knowledge; Evaluation available; Structure. Vos (2015) offers a set of criteria within a pragmatic definition of authenticity where the purposes and methods of the modelling researchers and task designers converge: authentic aspects in the field of mathematics (symbols, research questions, research experience) and authentic non-mathematical aspects (apparatus, professionals, mathematics applicability, problem settings).

In this study, we will concentrate on the aspects which directly relate to experiencing the real-world situation and developing a mathematical solution. We will thus select the following parameters: the *event* and the *question*, the *link to the real world*, the *apparatus* and the *problem setting*.

Experiential learning

The use of cultural artefacts and, in particular, objects that are part of children and adults' everyday experience is suggested by Bonotto and Basso (2001) as a strategy to make connections between the mathematics involved in real-world situations and the mathematics that is targeted in the classroom. They found that for students to bring mathematics into reality, it is helpful to introduce mathematical facts that are embedded and encoded in artefacts. According to their reports, in carrying out mathematical experiences from the interpretation of artefacts, the students make the transition from the real world to the world of symbols (horizontal mathematization) but in addition the use of artefacts also give them the opportunity to advance the construction of mathematical concepts (vertical

mathematization). The artefacts (concrete materials) may also be used as tools for the application of previous knowledge in new contexts and for consolidating the existing mathematical knowledge, pushing it to a higher level (Bonotto & Basso, 2001).

Bonotto (2007) advocates the need for change if we want to create realistic situations in mathematical modeling activities, i.e., recommends less stereotypical and more realistic situations, namely with the use of concrete materials; those are relevant to the students as part of their life experience, offering meaningful references related to concrete situations.

Other theorists also argue that fundamental knowledge and skills may be more easily accessible if students are directly involved in practical and experimental activities, as those can promote not only the perception of the usefulness of the materials in question but also a better understanding of the concepts explored. The discovery learning and the learning in practice are ideas inherent to the experiential education model. The model of learning by doing, developed by Kolb (1984), consists of a four-process cycle. The four processes are: experience (perform an activity); reflect (ask questions and talk about what happened in the experiment, analyzing possible inconsistencies between the observed and the predicted); abstract (generate a new idea or modify a previous idea); and apply (use what was learned in a similar or different situation, which can in turn create the need for new experiences). Experimental activities lead students to interact, analyze, question, reflect and transfer. The activity comes first; learning comes from the thoughts and ideas that arise as a result of the process of learning by doing. Accordingly, the drive for the development of new concepts is provided by new experiences. "Learning is the process whereby knowledge is created through the transformation of experience" (Kolb, 1984, p. 38). "Learning by doing" therefore appears as a natural learning perspective if modeling is seen as an activity that is similar to the methods in the experimental sciences or to the applied mathematics research.

The perspective of experimental modeling environments described by Halverscheid (2008) focuses on activities supported by experiments that give the opportunity to build mathematical models and produce mathematical knowledge around the questions investigated during experimentation. The role of experience is to lead the search for a suitable mathematical model that can explain the real data and results. The practical experience, as advocated by the author, becomes the rest of the world inside the classroom or school laboratory. The models are therefore produced to explain and interpret that intended authentic world.

Our theoretical approach, intends to combine the matter of authenticity with the idea of experimental modeling environments. We therefore consider that the authenticity of the school modelling task requires a clear account of an *event* that takes place in an out-of-school situation and the formulation of a *question* that is relevant and pertinent in that real world. But we add that the search for a solution entails a *link to the real world* that is made by an experimental *apparatus*, and a *problem setting* which is experientially emulated in the classroom.

Methodological approach

This study adopts a qualitative methodology of action research as the pillar of the methodological approach is a teaching experiment on the implementation of a new pedagogical approach involving modeling and applications in mathematics classes (Loughran, 2007; Latorre, 2003). The pedagogical motives of the teaching experiment corresponded to the need of providing meaning and practical

sense to mathematics, to contribute to change the negative view that many students held about mathematics, and to develop their appreciation for mathematics in their daily lives.

Two classes of 9th graders participated in this study. The ages of the students ranged between 14 and 17 years old, with an average of 14 years old. They were usually collaborative and committed students. Most of the lower achievers worked hard to improve their performance but showed many difficulties in applying mathematical concepts and in problem solving. None of the students had come across modeling activities in previous school years.

The modelling activities included 4 sections. The first is the introduction to the topic under study. The second consists of a practical activity, using manipulatives and everyday objects. The third comprises the analysis of the data obtained in the experimental stage, with the purpose of creating a model that might be used in similar situations. Finally, a written report should entail the following points: explanation of the experimental situation, assumptions made, strategies used, results, evaluation of the work and the difficulties found.

To solve the tasks students worked in groups of three or four elements organized at their will. The time allocated to each task was between 90 minutes (1 lesson) and 180 minutes (2 lessons). During the activities, the students had the opportunity to move around the room and discussing with colleagues the ideas from another group.

The data collection included participant observation and field notes from the teacher. The classes were recorded on video and audio with a mobile camera and the activity of a target group was also recorded on video and audio with a still camera. For each activity, a target group was randomly chosen. The written reports delivered by all the groups at the end of each activity were also collected. Here we will focus on the activity and work of a single group of four students. We will consider in particular what they developed in the experimentation phase and on the model proposed.

The task involved color dispensing and mixing paint solutions. It aimed to simulate the procedure used in industrial machinery called tintometric systems. In a tintometric system once the white base has been dispensed the container automatically positions itself under the volumetric tinting system, which dispenses all the coloring pastes and then mix them. The task was presented as follows:

Introduction. An indoor and outdoor paint shop uses a mixing paint machine in which it is only necessary to place the can with the neutral paint and introduce the code of the chosen color in the catalog. But a problem arose in the machine and it partially crashed. It no longer dispenses the required pigments, it can only mix them. Now the shop assistants must manually place the pigments to obtain the color chosen by the client. This means a new problem. They do not have information about the quantities of pigment to be used for each color of the catalog. So, your mission is to create a palette of colors and provide the amount of pigments to be used for each color.

Practical activity. You have at your disposal white liquid (milk) and colored liquid pigments (food coloring), a gauge and 1 ml syringes. Your mission is to produce a color palette that involves two primary colors and a reference table with the precise amounts to be used by the assistants.

Analysis. Choose two primary colors and prepare a palette of colors with various shades and make a table with the amounts of pigments for tins of 1L, 5L, 10L and 20L.

Report. Record all your procedures and results and submit your final report.

A case of proportional reasoning

The process held with the experimental apparatus was the same in most of the groups. It was clear that the students knew the idea of a paint catalog and knew that colors are labeled with names. Therefore, in general, they were creating a sequence of shades and were making labels for each shade. They used a small amount of pigment at a time and always added that amount to the previous mixture. At the beginning they only had milk (equivalent to the neutral paint) and then added color by the addition of two pigments. Throughout the experiments all groups recorded the values in the tables and subsequently calculated the amounts of pigment for different paint tins. Some of the groups did not present a formula to generate any amount of paint (unknown value) and only made the calculation of the amounts necessary to make the quantities asked in the problem.

The target group used yellow and green pigments. In their report the group described how they created a palette of four shades:

> Using the syringe we started with 0.1 mL of yellow pigment in the cup with milk (60 mL) and this color was **Vanilla**. Then we added another 0.2 mL of yellow pigment to the glass that already had 0.1 mL of yellow and the color was **Cream**. After we added another 0.2 mL of yellow pigment and 0.1 mL of blue pigment to the glass which already had some amount of pigment and the color was **Mint**. Lastly we added to the glass another 0.4 mL of yellow pigment and another 0.5 mL of blue pigment and the color was **Amazon Green**. Now, using the results of our table, we will determine the amount of pigment to be used with 1L, 2L, 5L and 10L of paint. To obtain those values, we use the rule of three. For the case of Vanilla we get:

$$\begin{array}{l} 60 \text{ -----} 1000 \\ 0.1 \text{ -----} \text{x} \end{array} \qquad x = \frac{1000 \times 0.1}{60} = 1.(6)\ \text{mL}$$

To complete our description of the results of the group we note that they created a table for each tin size and the four shades, showing the quantities of *paint*, yellow pigment and blue pigment. For the Amazon Green, the table indicated: 20000 mL of paint, 300 mL of yellow, and 200 mL of blue.

Discussion and conclusion

In this teaching episode and from the empirical data presented we want to emphasize that the students apparently attributed significant authenticity to the task, namely: to the *event* of producing a color catalog of paint; to the *question* of finding the quantities of pigments for manual mixing; to the *link with the real situation*, that is, helping the assistant to retrieve the quantities of pigments for each shade; and to the use of the experimental *apparatus* which was essential for the ongoing mathematization and emulated the real process.

However it seems that some aspect of authenticity has lacked. At a certain point the mathematical model is clearly formulated by the students and it shows their awareness of the fact that there is direct proportionality between the quantities of white paint in the experimental model an in the real sized tin, as well as direct proportionality between the quantities of a pigment in the model and in the real sized tin. That is obviously a sound conclusion. However, they went on assuming that the amount of white paint in the real tin was equal to the capacity of the tin, which is equivalent to say that the tin would be completely full with white paint and thus with no capacity available for an additional amount of color paste.

We are suggesting that the missing aspect of the modelling situation concerns the authenticity of the *problem setting*. The students had actually visited a paint supplying store in the neighborhood and could learn about a machine used to automatically dispense and mix the paints and produce the desired color. But one detail is relevant on the machine operation: it begins by asking the operator the size of the tin and then retrieves the amounts of neutral and of colored pastes. The total of the mixture has to be equal to that volume (or slightly below). Instead, the setting in which they worked was a scenario involving much lower quantities than the actual tins. Therefore, no absolute urge for correctness of the calculations was at stake, as Vos (2011) points out. Indeed, the initial volume of milk was little and imperceptibly changed by adding small doses of pigments. In other words, there was almost no distinction between the volume of white and the volume of the mixture in the glass of the experiment (Figure 1). But that is no longer valid when we want to produce a can of 20 liters of Amazon Green color. In this case, the value obtained for the combined quantities of yellow and blue pigments was 500 mL and that would be a quantity which, if added to the neutral paint, greatly exceeds the capacity of the tin.



Figure 1: Two shades and not distinguishable volumes

Surely, other factors can be considered as responsible for the inaccurate mathematisation carried out. One of them refers to the difficulty students have with the notion of ratio, which is well documented in research. In fact, the rule of three was a ready strategy for students. And the ratio was never really considered. In any case, what happened is that students did not differentiate the amount of colored paint from the amount of white paint, as it is evident from their table that shows 20000 mL of "paint" instead of "white paint". Still, that value was the correspondent to the 60 mL of milk in their rule of three. In conclusion, the concept of ratio was never activated and instead proportionality was used as a process of enlarging the "size" of each ingredient in the mixture (enlarge a small cup to a large tin). Therefore, questions of authenticity that are recognized by those who work with them in practice (Niss, 1992) seemed to have been absent from the process of translation between students' real model of the mixture produced in the classroom and the actual process of paint production.

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Examining the role of group work as an effective instructional strategy when teaching problem solving

Eamonn Clohessy and Patrick Johnson

University of Limerick, School of Education, Ireland; eamonc1@hotmail.com; patrick.johnson@ul.ie

This paper reports on a post-primary classroom intervention conducted to investigate the effect that carrying out problem solving in small groups as an instructional strategy has on the problem-solving performance of individual students. Over the course of the 6-week intervention students were introduced to an explicit problem-solving framework and challenged to solve weekly problems both in small mixed-ability groups and also individually during their traditional mathematics classes. It was found that there was a strong correlation between the problem-solving performance of the small groups and that of the individual students which suggests that group work could be utilised as an effective instructional strategy when teaching problem solving.

Keywords: Group activities, problem solving, secondary school students, secondary school mathematics.

Introduction

In 2008 in Ireland there was a change in the mathematics syllabus in secondary education in response to a number of studies and publications (e.g. Conway & Sloane, 2005). All of these studies identified that there were major deficiencies in the mathematical competency of students in secondary education and those commencing third level education. These concerns, along with others, fuelled the introduction of a new secondary mathematics syllabus in Ireland in 2008 named *Project Maths*. Project Maths identified five key skills that they saw as being central to effective teaching and learning across the new curriculum - *information processing*, *being personally effective*, *communicating*, *critical and creative thinking* and *working with others* (DES, 2015).

This new secondary mathematics curriculum also places an increased emphasis on group work and the development of problem solving skills within the classroom. This syllabus change should lead to an increase in the amount of group work occurring within Irish classrooms, but this beg the question as to how effective group work actually is? Can we measure what effect group work will have on the individual student, particularly when dealing with activities such as problem solving?

In terms of problem solving it has long been accepted that increasing the problem solving skill set of students is one of the primary goal of mathematical instruction (Travers, 1977). In the early nineties in America, the National Council of Teachers of Mathematics (NCTM) set out the goals for promoting problem solving as a curricular focus by declaring it as one of the three main goals of mathematical instruction in a second level school (Szetela & Nicol, 1992). In Ireland this focus on problem solving has only taken place in recent years due to the change in the syllabus brought about by the new Project Maths course. Many research papers focus on the individual problem solver but others have focused on the idea of problem solving in small groups (Artzt & Armour-Thomas, 1992). In addition to the obvious benefits of improving the problem solving skills of the students research highlights the additional benefits that working in small groups yields e.g. developing personal and social skills (McGlinn, 1991), enhancing self-esteem (Slavin, 1991) and reducing the dependency of

the students on the teacher (Sandberg, 1990). In light of these benefits and the emphasis that the new syllabus places on working with others and problem solving, this current research project decided to examine whether working in small groups to complete mathematics problems would improve the general problem solving ability, and overall mathematical ability, of individual students. This research aimed to address this hypothesis by answering the following questions:

1. Is there a relationship between an individual student's problem solving achievement scores and their achievement scores when solving problems as part of a group?
2. Do students believe that working in small groups to solve problems is beneficial in the development of their individual problem solving ability or overall mathematical ability?

Framework for problem solving

With the increased emphasis placed on problem solving in the new syllabus, and the relative newness and unfamiliarity of both teachers and students with problem solving, it was deemed necessary to provide students with guidelines to assist them during their initial problem solving exploits. The instructional framework selected by the authors to assist in the problem solving activities in the classroom was developed by Artzt & Armour-Thomas (1992) and was specifically tailored for problem solving in small groups. This framework was based on an earlier framework developed by Schoenfeld in 1985 but expanded and added additional episodes. Schoenfeld (1985, p. 292) defined an episode as "a period of time during which an individual or a problem-solving group is engaged in one large task". Artzt & Armour-Thomas (1992) finally settled on eight episodes when looking at problem solving in small groups – *read, understand, analyse, plan, explore, implement, verify,* and *watch and listen*.

Methodology

Participants

34 students from a medium sized rural secondary school in the west of Ireland participated in the research project. 22 of the students (12 male and 10 female) were from a mixed ability first year class (average age 13 years) whereas the remaining 12 students (7 males and 5 females) were from an ordinary level[1] third year class (average age 15 years). The first year cohort only had two classes per week with their teacher as part of this intervention whereas the third year group had four classes per week with their teacher over the course of the six week intervention. A typical mathematics class lasts for 40 minutes.

Selection of content and questions

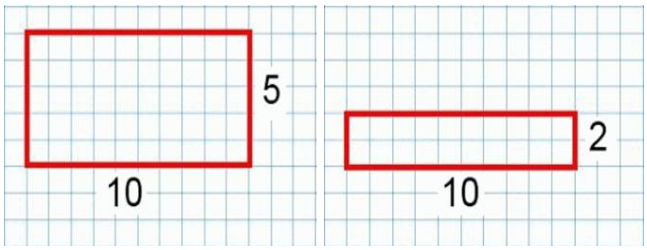
The selection of content for this study was primarily based on the Project Maths Junior Cycle[2] syllabus. The topic covered by the first year students during the six week intervention was 'area and perimeter', whereas the third year students covered the topics of 'circles and cylinders' and 'area, perimeter, and volume'. Note that the third year cohort covered more material due to the extra contact

[1] After first year all classes in Irish secondary schools are streamed into two groups, Ordinary and Higher, with higher being the more challenging stream.

[2] The Junior Cycle caters for students aged from 12 to 15 years and covers the first 3 years of post-primary education.

time with their teacher during the intervention. An example of a question given to the first year students whilst working in groups is provided here:

The first rectangle has a perimeter of 30 units and an area of 50 square units. The second rectangle has a perimeter of 24 units and an area of 20 square units. Charlie wondered if he could find a rectangle, with a side of length 10 units, whose perimeter and area have the same numerical value.



Each week during the intervention the students were challenged to solve some combination of either purely mathematical or worded problems based on the previous week's mathematical content. Verschaffel, Greer, and De Corte (2000) use the term 'word problem' to refer to any mathematical task where significant background information on the problem is presented as text rather than in mathematical notation and this is also the meaning that the authors have adopted as part of this study. Problems were selected from past examination papers, books and online websites and were deemed appropriate for the age and ability levels of the students.

Intervention and assessment

The intervention took place in the students' traditional 40 minute classes over a period of six weeks. Normal teaching operations took place during the intervention with approximately 10 minutes of certain classes being assigned to the testing of the students' problem solving abilities. In the first week of the intervention the students were given an individual assessment to gauge their initial problem solving skills prior to working in groups to solve problems. During the following five weeks students were regularly placed in small mixed ability groups of 3 or 4 and asked to solve 4 problems together during that week. At the end of each week students were asked to individually solve 2 problems so that their individual progress could be monitored. Individual students and all the members of a group were awarded a single correct mark if the problem was answered correctly and awarded no mark if they failed to correctly solve the problem. A focus group with 6 students randomly selected from both the first and third year groups was conducted following the intervention.

The role of the teacher during the intervention

Throughout the intervention the primary role of the teacher was that of a facilitator or a time-keeper. The teacher answered any questions that the students had with regards to the use of the problem solving framework or specific questions such as issues regarding units of measurements (i.e. is cm^3 associated with volume). When asked a question related to the solving of the problem the teacher declined to answer directly and instead used probing questions to try and guide the students towards a solution. The frequency of questions directed at the teacher lessened after the initial weeks of the study as students began to rely on the other members of their group for assessing their ideas and potential solution strategies and not the teacher. Fewer questions were directed at the teacher during

the individual problem solving sessions as the students viewed these more as ‘tests’ and therefore assumed that they were not supposed to ask questions of the teacher.

Findings

First-year students

Looking at the assessment trend among the individual students it can be seen from Figure 1 that over the course of the intervention there was an upward trend in the number of questions answered correctly. The pre-test resulted in only 5 correct answers (11.36%) out of the possible 44 (22 students times 2 problems). This number of correct answers increased to 16 (36.36%) in week 1, 28 (63.63%) in week 2, 30 (68.18%) in week 3, 31 (70.45%) in week 4, and finally 38 (86.38%) in week 5. From Figure 1 it is clear that there is an almost linear increase in the number of correct responses between the pre-test and the week 2 assessment but this increase is then followed by 2 weeks of a much slower advancement by the students before increasing more significantly in the final week.

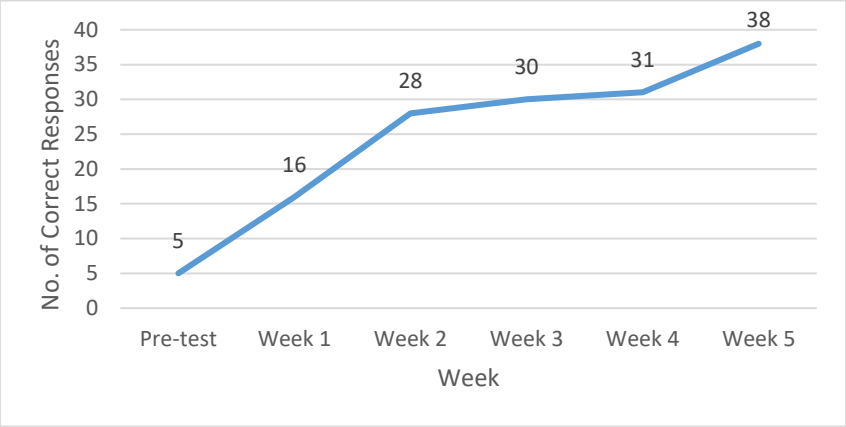


Figure 1: Individual assessment scores for first year students

In terms of the group assessment, Figure 2, there was evidence of a positive increase in the number of correct group responses as the weeks progressed. Out of a total score of 24 (6 groups times 4 problems) possible correct responses each week 8 groups (33.33%) answered the problems correctly in the first week. This increased to 17 (70.83%) in week 2, remained at 17 for week 3, increased marginally to 18 (75%) in week 4 and increased again to 19 (79.16%) in week 5.

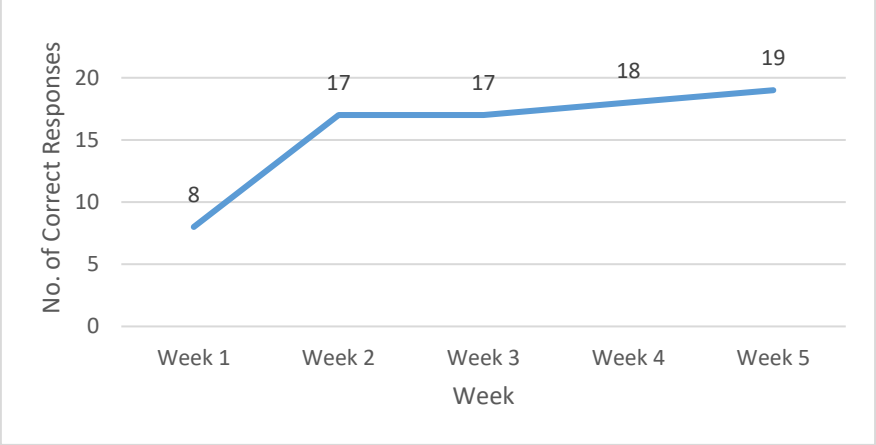


Figure 2: Group assessment scores for first year students

Comparing the number of correct individual solutions against the number of correct group solutions each week it was found that there was a strong correlation between the number of correct answers among individual students and among the groups (r = 0.95). This suggests that as the first year students became more efficient at solving problems within groups so too did they become more efficient at solving problems individually.

Third-year students

Similar trends are noticeable among the third year group in both the individual and group problem solving assessments, although the scale of the improvement is not as significant as with the first year students. The pre-test of the individual students problem solving ability resulted in 2 (8.33%) correct responses out of a possible 24 (12 students times 2 problems). At the end of week 1 the number of correct responses had decreased to 1 (4.16%) before increasing to 14 (58.33%) at the end of week 2, 15 (62.5%) at the end of week 3, 22 (91.66%) in week 4 and then dropping slightly again to 19 (79.16%) in week 5 as seen in Figure 3.

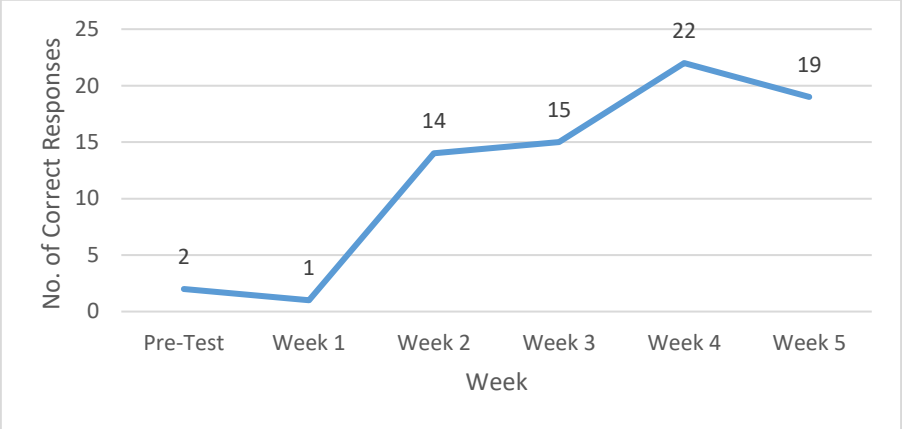


Figure 3: Individual assessment scores for third year students

In terms of the number of correct responses from the groups of third year students there appears to be less fluctuation in the results. Out of a total score of 12 (3 groups times 4 problems) 4 groups (33.33%) answered the problems correctly in the first week. This increased to 6 (50%) in week 2, 7 (58.33%) in week 3, 10 (83.33%) in week 4 and 11 (91.66%) in week 5 as seen in Figure 4.

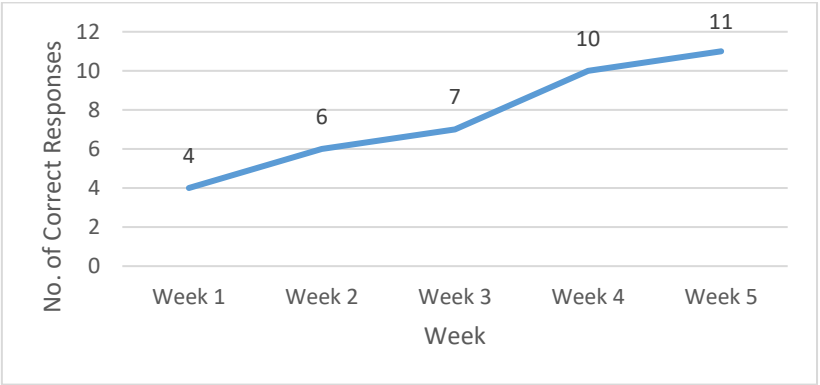


Figure 4: Group assessment scores for third year students

Comparing the number of correct individual solutions against the number of correct group solutions each week for the third year class again found that there was a strong correlation between the number of correct answers by individuals and among the groups ($r = 0.889$).

The second research questions focused on whether the students felt that working in small groups to solve problems had been beneficial in enhancing their individual problem solving skills or their overall mathematical skills. Overall the students were positive in their responses to the focus group question relating to whether they felt that their individual problem solving abilities had improved as a consequence of the intervention.

Interviewer: Do you feel that your problem solving ability has improved? Why do you think this?

Student4: I think it has because I have been finding it easier to figure out my homework, so I think it has.

Most students responded in a similar manner although two of the group did confuse the question slightly and make reference to working in groups.

Student2: Ah, yes because I now know that I can ask others for help and use their opinions to build on to get my answer.

As already alluded to by some of the students in the previous question, all of the students responded that they found working in small groups enjoyable and some even stated that it increased their confidence in the mathematics classroom. Worryingly some of the students appeared to suggest that this type of active is not common place in their traditional classroom which is in opposition to the overall aims of the new syllabus.

Interviewer: How did you find working in groups as part of your mathematics class?

Student1: I enjoyed doing maths more because I got a fair share of trying to work it out and it wasn't as boring as a normal maths class. I felt like my opinion mattered which is different to other classes. I found it weird that the teacher encouraged us to talk while in class, usually they are trying to keep us quiet.

Student5: I feel more confident because maths seems a bit more fun when you can talk to your friends. Also it improved my ability to say I could do things when I though I couldn't. I now try a different way of answering the question if I get stuck when I try it the first time around.

When asked about whether working in groups was beneficial in helping them solve the problems all the students agreed.

Student5: Am, yes because it helps it go faster and if you're stuck you'd have another person's opinion to help work it out. It was really fun working with your friends in class like that.

The final question focused on whether or not the students felt that, as a result of their participation in the intervention, their overall mathematical knowledge had improved. The responses to this question were positive, but varied. Some students focused on the idea of being able to approach questions differently now because they were able to ask other students their opinions and then solve the problem

themselves, based on the insight from the other student. Another student spoke about being able to analyse the ideas of the other members of their group and how it forced them to look at the problems from different perspectives.

Interviewer: Have the classes improved your overall mathematical knowledge? In what ways?

Student2: I think that I have new ways of solving problems. Before the group work, I used to look at the question and if I couldn't understand I used to leave it because I didn't know what to do. Now I would ask someone else if they could do it and see if I could use their idea to answer it.

Student3: I found it improved my knowledge because I think I had to think more about the question.

Interviewer: What do you mean by 'think more'?

Student3: Am, well because if someone in the group had a different opinion, I would try to see where they got that idea from and try and see if that would work. I also tried to see if it was the same as my idea but said in a different way.

Conclusion and discussion

The primary aim of this study was to investigate whether problem solving in small groups had any effect on a student's problem solving achievement when working on their own. Solving problems in small groups affords students the opportunities to ask questions, challenge assumptions, discuss opinions and share work among colleagues. The results of this study found that the there was a strong positive correlation between the weekly number of correct responses to the problems solving tasks in small groups and the problems solved by individual students in both of the student groups. This would suggest that working in small groups to solve problems has had a positive impact on the individual problem solving skills of the students. Reading too much into this results could be misleading though as the unfamiliarity of the students with problem solving, or problem solving approaches, meant that the improvement in overall problem solving skills shown by the students could be a consequence of being introduced to a problem solving framework rather than from working in small groups, or some combination of both.

This been said, all of the students commented positively when asked about whether they felt that working in small groups had been beneficial to them. The students highlighted how they liked the ability to talk to this classmates and discuss the problem which was not something that was common in their previous mathematics classes. This seems to contradict the aims of Project Maths which stresses the importance of developing the key skills of *communicating* and *working with others* (DES, 2015). In line with the finding of Slavin (1991) students commented positively about several qualities which they felt that working in groups had helped to develop, such as feeling like this opinions mattered and feeling more confident towards mathematics. Additionally Sandberg (1990) found results that coincide with the findings of this study in that students are willing to persevere when faced with a problem that they cannot solve straight away and overall become less dependent on the teacher. These are all key skills that need to be developed in students and this would suggest that the teaching of a problem solving framework in conjunction with working in small groups to solve problems appears to be an effective instructional strategy.

Finally it is worth noting that the students did highlight some issues with the intervention in its current form. Two students commented that they felt that there wasn't enough time allocated to the group work activities at the end of the classes and that they always felt rushed. Another two students commented that in one instance one student in their group had taken over the activity and proceeded to solve the problem by themselves without consulting or involving the other members of the group. Obviously there are limitations to every study but going forward it is important that more focus be placed on the roles and monitoring of individual students within the groups.

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Modelling problems in German grammar school leaving examinations (Abitur) – Theory and practice

Gilbert Greefrath[1], Hans-Stefan Siller[2], Matthias Ludwig[3]

[1]University of Muenster, [2]University of Wuerzburg, [3]University of Frankfurt

greefrath@uni-muenster.de; siller@dmuw.de; ludwig@math.uni-frankfurt.de

Examples of mathematical problems in German school leaving examinations are examined in detail as to the extent to which they verify the competence of mathematical modelling. For this purpose, the relevance, the authenticity of the context, the openness of the task and the partial competencies as criteria are examined in more detail. It is shown that authentic contexts but no relevant contexts are contained in the problems. As a rule, the use of mathematics is not authentic and problems are not open. The partial competencies in modelling are quite unequally divided and are not covered completely. It is recommended to use all the criteria for the design of future examination questions.

Key words: Modelling, examinations, Abitur, authenticity.

Introduction

In Germany, an overall policy of the Conference of Education Ministers has been in place with respect to "Educational Monitoring" since 2006. The intent is to enhance competence orientation within the educational system. In mathematics, the focus also lies on the general competence in the field of modelling. From the year 2017 onwards, a task pool of Abitur examinations will be available out of which all Federal States can take problems for Abitur examinations. This is an important step towards improving the quality of examination problems and to align levels of requirements between the Federal States step by step. Such examination problems have been developed on the basis of educational standards and also contain, in particular, elements examining mathematical modelling. Studies regarding problems in written Abitur examinations are rare. Still today, it is obvious that there are very few theoretically and methodically profound and empirically substantiated approaches in respect of examination problems in the framework of the written Abitur examinations, which at times just focus on individual standard-related aspects (Kühn, 2016, p. 75). Therefore, this applies in particular to problems regarding mathematical modelling. At the core of mathematical modelling lie the translation of a problem from reality into mathematics, the work with mathematical methods and the application of mathematical solutions to the real problem. Usually, the entire modelling process is represented as a cycle. Idealisation here means that the representation itself is, again, a model. In Borromeo Ferri et al. (2006) the creation of a mathematical model is contemplated in detail, representing the process of the individual, creating the model, from the starting situation to the mental representation as a first step.

Modelling competence

Blum et al. (2007) describe modelling competence as the capacity to execute the relevant process steps required when switching between reality and mathematical problem adequately as well as to analyse models or comparatively evaluate such models. This can be discussed in more detail on the basis of modelling cycles by Blum and Leiss (2005, p. 19). Thus, it is possible to analyse the various partial processes of modelling activities considering various details and other aspects. It is

also possible to consider the capacity to implement such a partial process as a *partial competence* of modelling (Kaiser, 2007; Maaß, 2004). In consideration of the modelling cycle one could characterise such partial competences as shown by Greefrath and Vorhölter (2016, p. 19). We do not mention inner-mathematical working between the mathematical model and the mathematical results as a partial modelling competence since this is not specific for modelling processes.

Characteristics of modelling problems

It is not always easy to select or develop the right modelling task. As an indication, characteristics may be specified that a modelling task should fulfil. In her comprehensive classification scheme for modelling problems, Maaß (2010) focuses in particular on the type of relation to reality, to the openness and the modelling activity as criteria for modelling problems. With respect to the focus on the modelling activity, partial competences of modelling will be observed more closely. As regards the relation to reality, the relevance and authenticity of the context will be examined more closely (cf. Greefrath et al. 2013).

Relevance

Initially, we deal with the classification of reality-related problems handed down in German discussions. In connection with the factual context of problems, we traditionally talk about *"embedded problems"*, *problems formulated in text form* and *"factual issues*". These types of problems provide facts about the relevance of the factual context used for students.

Embedded problems have no real relation to reality. The factual context is of no importance regarding the solution of problems and can be exchanged arbitrarily. The intention of embedded problems is the application and practice of numeracy skills. An embedded task as a modelling task is suitable only to a limited extent since the mathematical model is already implicitly contained in the problem. *Word problems* are problems formulated in text form – sometimes complemented by figures. Similar to the embedded problems, the issue is basically exchangeable as reality is often represented in a very simplified manner. The intention is to promote mathematical skills. It is not really about an autonomous development of a mathematical model because of the lack of reality and the given simplifications. Nevertheless, one major problem for students is the translation of the text into the corresponding mathematical symbolism such as terms or equations. For this reason, the known term "mathematisation" in the field of modelling is normally used in this context (Schütte, 1994, p. 79). In the case of problems formulated in text form the mathematical problem dominates the embedding of it. Another focus then lies - depending on the specific problem - on the interpretation of the mathematical results in the factual situation and in the wording of a corresponding answer. The substantial and - particularly in connection with factual problems - frequently exclusive treatment of problems formulated in text form in mathematics classes has been strongly criticised. One reason is the lack of genuine reference to reality. Another reason is the process of practising similar mathematical issues, formulated in text form, as a result of which a real reflection about the context used becomes redundant. In the case of *factual issues* a factual environmental problem is of primary importance. In this context, the function of "Factual Calculation" is described by Winter (2003) as "Environmental Development". In this context, real data are frequently given in respect to which authentic questions are then asked. Since the processed issue plays a real role, information about the respective matters must be gathered and processed.

Therefore, the processing of factual issues is also to be considered as interdisciplinary or - ideally - even as discipline-linking. In this context, the factual issues can be viewed as modelling problems (Franke & Ruwisch, 2010).

Authenticity

Authenticity is one central characteristic of modelling problems. Authenticity means both the authenticity of the extra-mathematical context and of the application of mathematics in this particular situation. The extra-mathematical context must be real and not be specially designed for a certain arithmetical problem. The application of mathematics in this situation must also be reasonable and realistic and should not just be used in mathematics lessons. Authentic modelling problems are problems that genuinely belong to an existing subject or problem area where they are accepted by people working in those areas (Niss, 1992). Thus, an authentic task becomes credible and at the same time a realistic task for students from an environmental point of view. Authenticity helps students take problems seriously and avoid superficial substitute processing strategies as in the case of embedded problems (Palm, 2007). In the case of authentic problems, students can assume that the things they are dealing with really exist and that the task they are presented with is a real task that finds its justification outside mathematics lessons as well (the task have an out-of-school origin and a certification (Vos, 2015)). Authenticity of problems helps students take such problems seriously. However, authenticity of problems does not mean that the problems are actually important for the students‘ present or future lives.

Openness

Open problems are those problems that - for instance - allow for more than one approach or solution. Openness enables students to choose their own approaches or solutions regarding the problems. There are various classifications of open problems. We limit ourselves here to the examination of openness by initial state, transformation and target state (cf. e.g. Wiegand & Blum 1999). Said classifications use the known problem-solving psychological description of a problem through its initial state, target state and a transformation that transfers the initial state into the target state and are not limited to modelling problems (Klix, 1971). Open problems are divided by the clarity of their initial and target statuses as well as by transformation. Maaß (2010) suggests another classification of open modelling problems based on Bruder (2003) which includes seven different types and which distinguishes between overdetermined and underdetermined problems.

Criticism concerning Abitur examination problems with modelling elements

The use of modelling in examination problems, however, is not unreservedly viewed positively. The fact that in many cases the relevance of the factual context used is not at the focus of examination problems, gave rise to (fundamental) criticism on the part of some expert representatives with regard to modelling in examinations. On the one hand, the criticism is directed against the fact that “modelling competence“ is not at all examined by the problems (Kühnel 2015, p. 76). On the other hand other authors show the categorical refusal of modelling problems (cf. Bandelt & Weidl, 2015, p. 4). Strong criticism is also directed against the greater part of texts in examination problems (cf. (Jahnke et al. 2014, p. 120): “Instead of dealing with mathematical problems, A-level students have to tackle wording problems“.

Criticism regarding examination problems can be absolutely justified, in particular if it is about embedded problems and not about factual problems in examinations (cf. Fig. 1). Henn and Müller (2013, p. 205) comment: “Unfortunately, most of the so-called “modelling problems“ at school - and in particular in the Abitur exam are not at all modellings according to our way of thinking. Almost always one starts out from a more or less complex function equation, allegedly describing a ski jump, a tower, a playground or another construction. Now, with this function, a common functional examination is to be made. However, the whole thing is not a modelling task, but entirely a mathematical problem.“

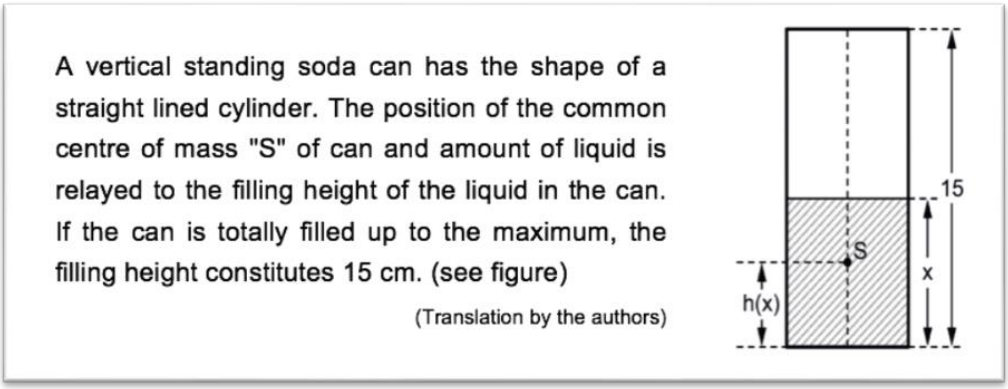

Figure 1: Stimulus text of an examination task (www.iqb.hu-berlin.de)

This brief insight shows that the use of the term “modelling problem“ - in particular in examinations - does not at all guarantee the clear characterisation of a certain type of problem. It should be considered, however, that the conditions given in examination or in normal lesson situations are different from each other (cf. Siller et al., 2016, p. 381/382). Ultimately, in examination situations the focus is on the term of “measuring“ the students‘ performance as outlined by Siller et al. (2016, p. 384). Thus, less attention may be paid to a creative phase in a test situation. In addition, psychometric findings (cf. Rost, 2004) reveal that - for the measurement of performance - such performance must be addressed explicitly and structured into small units in order to obtain a valid statement.

Research issue

Against the background of criticism directed towards current Abitur examination tasks and in consideration of the importance of competence of mathematical modelling for German levels of education and of international discussions as well as of the theoretically clarified criteria for modelling problems, we have to ask ourselves the question, how to evaluate the quality of existing Abitur examination problems with modelling elements in Germany. We limit ourselves here to the calculus subject area, which accounts for the largest part of the Abitur examination in mathematics in Germany. With regard to the criteria discussed above, the question is: How can German Abitur examination problems in calculus be assessed with regard to the partial competences of modelling, to relevance, to authenticity and to openness?

Research method

For preparing for the pool of Abitur examination problems which will be available from the year 2017 and from which all Federal States can easily draw Abitur examination problems, the Institute for Educational Quality Improvement in Berlin will provide a collection of examples (cf.

www.iqb.hu-berlin.de/bista/abi). Such samples were chosen as a basis for the examination because they best fit with the Abitur examination problems of 2017. Among the sample problems there is a total of four examination problems for calculus for general grammar schools. Two of the problems are of an increased level and a further two are intended to be used in Computer-Algebra-Systems (CAS). All in all, the four examination problems include 50 items.

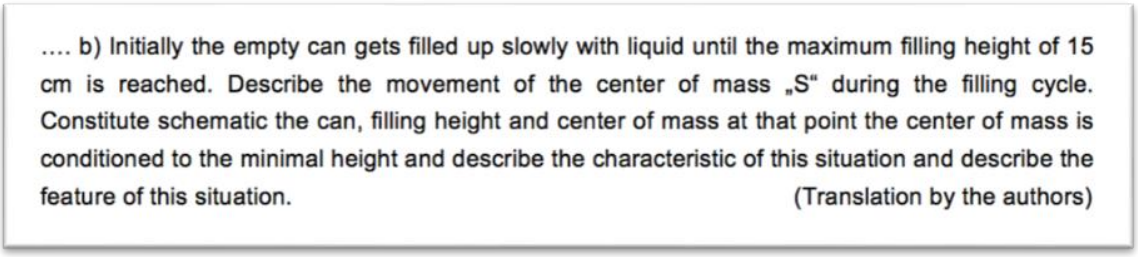
.... b) Initially the empty can gets filled up slowly with liquid until the maximum filling height of 15 cm is reached. Describe the movement of the center of mass „S" during the filling cycle. Constitute schematic the can, filling height and center of mass at that point the center of mass is conditioned to the minimal height and describe the characteristic of this situation and describe the feature of this situation. (Translation by the authors)

Figure 2: Part b) of an examination task (www.iqb.hu-berlin.de)

Problems were evaluated by items according to different criteria. The selection of criteria follows the frequently quoted central characteristics of modelling problems (cf. Bruder 1988, Greefrath & Vorhölter 2016, p. 17, Maaß 2004, p. 22). Problems were discussed and evaluated per item and on the basis of Tab. 1 in the framework of a qualitative research process including three evaluators. Finally, all items could clearly be allocated the corresponding characteristics (cf. Bortz & Döring 2006) – by using 0 (does not match – e.g. Relevance 0 in Fig. 2) or 1 (matches the criteria – e.g. Authenticity in Fig. 2). In this way, a detailed compilation of the examined items was created and an allocation to the aforementioned criteria became possible.

Reference to reality	Does the problem refer to an extra-mathematical factual context?
Relevance	Is the factual context relevant to the students (factual problem)?
Authenticity	Is the factual context authentically related to the actual situation?
Authenticity	Is the factual context authentical with regard to the use of mathematics?
Openness	Is there more than one possibility to solve the problem (solution variety)?
Partial competences of modelling	Which partial competences of modelling are required for dealing with the problem (simplifying / structuring, mathematising, interpreting, validating)?

Table 1: Criteria used for the Assessment of Items

Results

In an initial assessment of the existing data material, we focus on the criteria reference to reality, authenticity, openness, relevance and modelling competence. Due to the small data basis, it was not possible to make valid quantitative statements. However, certain trends are visible. 20 items out of a total of 50 contain a reference to real life situations. The problems, though, included in the basic level more items with reference to reality (52 %) vs. items without reference to reality than in the higher level group (31 %). Correspondingly, on average, problems with the use of CAS contain more items with reference to reality (46 %) than in the group without the use of CAS (33 %).

Authenticity: If one looks at the problems considering the requirement that the respective items should contain realistic situations, our analysis reveals that such items appear both at the increased

level and at the level where CAS are used. More than one half of the items with reference to reality also contain realistic situation at both levels. Regarding a realistic use of mathematics which from our point of view represents a further aspect of authenticity (cf. table 1), this has been identified in just three items. Students were required to explain in the factual context of a parabolic modelled trajectory that only certain parameter values are possible. It should be noted here that quadratic functions are used even by experts for modelling trajectories.

Openness and Relevance: Open problems have not been identified in the items examined. Also with respect to the relevance of the problem, no item can be identified in the present examination problems that would fulfil this requirement.

Modelling competences: The analysis concerning the partial modelling competences show that the items normally address one of the partial competences. However, the partial problems are not designed in such a way that the sequence of an idealised modelling cycle reflects the appearance of the partial competences of simplifying, mathematising, interpreting and validating one after the other. Most items focus on mathematising, followed by required interpretations. Simplifying or structuring or the validation within one item is not required by the current examination problems. One item requires both mathematising and interpreting. This item requires the determination of an intersection point on the basis of a model given by a function equation with an unknown parameter with self-determined conditions interpreted in the framework of the factual context.

Discussion

Generally, it can be said that around 40% of the items examined include reference to reality. However, this does not mean that such items also contain modelling problems. The application of common criteria for modelling problems shows that we can talk about modelling problems only to a limited extent. Not a few items include an authentic factual situation, but a realistic use of mathematics is given just in exceptional cases. None of the problems is open or relevant. This is why criticism towards Abitur examination problems must be taken seriously if one must conclude – as already concluded by Henn and Müller (2013).

It is comprehensible that in individual items only a partial modelling competence is dealt with due to the examination situation or to performance measurement, respectively. This can also be observed essentially in respect to the problems examined. It would be desirable though, that all partial competences of modelling were contained in a test problem and that this problem was tested in the typical sequence as in the modelling cycle. This is not the case with regard to the problems examined. There are even partial competences, which are not tested at all, such as simplification.

Numerous general conditions must be observed when preparing examination problems. In order to actually test the modelling competence, it is necessary not just to use problems with reference to reality which describe an authentic situation only partially and to test just a few competences of partial modelling competencies, but to start with an intensive and criteria-led development of modelling problems for an examination on the basis of the current educational standards.

In order to improve the competency modeling and the development of modelling problems for examinations, the criteria used here should also be used for the development of examination problems and for training problem developers. In order to increase the share of suitable modellings

in examinations, it seems that a substantially greater proportion of reality-related problems or a substantially targeted development of suitable items is needed.

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The investigation of mathematical problem-solving processes of fifth grade students in small groups

Firdevs İclal Karatas, Öznur Soyak and Ayşenur Alp

Middle East Technical University, Faculty of Education, Ankara, Turkey;

iclal.karatas@metu.edu.tr; osoyak@metu.edu.tr; alp.aysenur@gmail.com

The purpose of the study is to investigate mathematical non-routine problem solving processes of students in small groups. The study was conducted with nine fifth grade students in three small groups. A framework developed by Artzt and Armour-Thomas (1992) for protocol analysis of problem solving in mathematics is used for this study to determine problem-solving behaviors observed within different episodes of problem solving. The findings revealed that although the understanding episode was coded as the greatest percentage, the analyzing episode was coded as the lowest percentage within the three groups.

Keywords: Problem-solving process, small groups, fifth grade students.

Introduction

Problem solving has become an integral part of learning mathematics since it helps students to understand mathematical contents. It also leads students to understand how to apply their knowledge into their daily lives when solving problems. The NCTM (2000) also emphasizes the mathematical problem solving activities from pre-kindergarten to grade 12 in all mathematics classrooms. Moreover, problem solving is strongly emphasized in recent Turkish elementary mathematics curriculum. It is considered as a basic skill that should be developed in each content area (MoNE, 2005). Besides, students should be able to develop their own strategies and apply them to solve their real-life problems when solving problems (MoNE, 2005).

Moreover, problem solving entails engaging in a task for which the solution process is not identified beforehand (NCTM, 2000). Mayer (1992) defines problem solving as a cognitive process in which one figures out how to solve a problem of which the solution is not already known. Most definitions of problem solving emphasize problems that require problem solvers to use information and procedures in unfamiliar ways. Problem solving is an extremely complicated human endeavor. It is considerably more than the implementation of well-learned procedures or the simple recall of facts. Problem solving involves the construction of sequential procedures that build strategies in addition to the application of the structure (Hammouri, 2003). Problem solving also entails arranging several cognitive and metacognitive processes, deciding and performing suitable methods, and regulating behavior for varying demands of problems (Montague, 1991).

A variety of models are proposed that describe the processes that problem solvers use from the beginning until they finish their tasks (Garofalo & Lester, 1985; Mayer, 2002; Montague & Applegate, 1993; Polya, 1957). For example, Polya's model comprises of four stages; namely, "understand the problem, make a plan, carry out the plan, and look backwards" (Polya, 1957). Later, Garofalo and Lester (1985) revised the model proposed by Polya and include cognitive and metacognitive components. Their model is described in four stages as orientation, organization, execution, and verification. Montague and Applegate (1993) presented a model focused on seven

cognitive processes “reading, paraphrasing, visualizing, hypothesizing, estimating, computing, and checking” and three metacognitive processes “self-instruction, self-questioning, and self-monitoring”. Mayer (2003) proposed another cognitive process model that included translating, integrating, planning, and executing processes. In particular, Artzt and Armour-Thomas (1992) developed a framework to examine the problem-solving processes of individuals as they work in small groups.

Furthermore, non-routine problems are problems where how to solve the problem is not obvious immediately, or they have not been encountered before in the curriculum. Non-routine problems require critical thinking and an extension of prior knowledge that may include concepts and techniques which will be explicitly taught at a later stage, and may include finding connections among mathematical concepts (Schoenfeld et al., 1999). The findings of this study support that challenging problems are likely to enable metacognitive process so that students consciously adjust and regulate their cognitive processes (Montague & Applegate, 1993).

All over the world the importance of exploring elementary school students’ problem solving abilities is highlighted. There are some international examinations such as Program for International Student Assessment (PISA) and National Assessment of Education Progress (NAEP) to determine the performance on non-routine problems and in problem solving. In the 2012 version, students’ PISA ranking scores in problem solving show that Turkey is among one of the worst in the world (42nd out of 65 countries, 2012 PISA). Therefore, researchers need to find out why this is so. As a country students’ problem solving abilities can be improved if how students think, and their awareness of their actions while solving non-routine problems are determined. In addition, exploring cognitive and metacognitive abilities is difficult in problem solving. Especially, elementary school students may not be aware of what, and why, they are doing. As a result, the purpose of this study is to investigate problem solving processes of fifth grade students when they solve non-routine mathematical problems.

Significance of the study and research question

Many research studies and projects have pointed out the importance of learning problem solving in school mathematics courses (Higgins, 1997; NCTM, 2000; Verschaffel et al., 1999). One of the major goals of mathematics education is the acquisition of the skill of learning how to solve problems. However, there are conflicting views about the attainability of these goals (Verschaffel et al., 1999). Despite long years of instruction many research studies show that children are insufficient and not confident in having the aptitudes required for approaching mathematical problems in a successful way (Higgins, 1997; Doorman et al., 2007). The reasons for these deficiencies, particularly in elementary students, can be attributed to two factors. The first of them is the lack of specific domain knowledge and skills (concepts, formulas, algorithms, problem solving). The second factor is shortcomings in the heuristic, metacognitive and affective aspects of mathematical competence. When confronted with unfamiliar complex problem situations, students mostly do not spontaneously apply heuristic strategies such as drawing a suitable schema or making a table. They usually only glance at the problem and try to decide what calculations to perform with the numbers. In addition to this, many students have inadequate beliefs and attitudes towards mathematics itself, learning mathematics, and problem solving. These beliefs exert a strong negative influence on their willingness to engage in a mathematical problem. Some examples of such beliefs and attitudes are

that there is only one correct way to solve a problem; that a mathematical problem has only one right answer; and that ordinary students cannot solve problems which requires higher order thinking. These insufficiencies in students' beliefs are related to the nature of the problems given in the lessons and the classroom culture. Hence, problem solving activities should give opportunities to students for investigation, reasoning and deciding on the solution process and improve their problem-solving skills. Small groups in problem solving may provide natural setting for interpersonal monitoring and regulating of students' goal directed behaviors. In this study, the problem-solving processes that occur as individuals engage in mathematical problem solving in small-group settings are examined.

The findings of the study may contribute the studies on the process of fifth grade students' thinking. The determination of these students' thought processes will helps teachers to design and adjust problem solving instruction and better support the development of students. The findings of this study may be applicable for developing teaching methods and materials to enable the development of fifth grade students' problem solving skills in future non-routine problem solving classrooms. In addition, this study can be significant for the design of curriculum in that the results support the design of educational or special programs that can be more effective and supportive of elementary students. Thus, the research question of the study could be stated as follows:

Which are the most dominant problem solving processes of fifth grade students when solving non-routine problems?

Theoretical framework

It is necessary to appraise the information about problem solving processes to develop a framework that can explain how students figure out mathematical problems. Mathematics educators and psychologists have suggested various problem solving process models. Polya (1957) proposed four phases called *"heuristics"* to understand problem solving processes. The phases are known as "understanding the problem, devising a plan, carrying out the plan and looking back". Polya also proposes several strategies that can be used when students solve problems. His strategies include using diagrams, looking for patterns, trying special cases, working backward, intelligent guessing and checking, creating an equivalent problem and creating a simpler problem. Considering the problem-solving processes, an appropriate strategy can be essential to reach the solution of the problem.

After that, Schoenfeld (1982) developed a model for mathematical problem solving based on the Polya's model. The model includes five episodes; namely, "reading, analysis, exploration, planning/implementation and verification". Adding cognitive and metacognitive aspects of problem solving to Polya's and Schoenfeld's model, Garofalo and Lester (1985) proposed a framework with orientation, organization, execution and verification phases. Montague and Applegate (1993) also proposed cognitive-metacognitive aspects of mathematical problem solving. This model focused on seven cognitive processes "reading, paraphrasing, visualizing, hypothesizing, estimating, computing, and checking" and three metacognitive processes "self-instruction, self-questioning, and self-monitoring". These various models have been used to investigate problem solving processes, but only two models by Garofalo and Lester's model as well as Montague and Applegate's model have been used with gifted students as a framework to describe problem solving processes in the literature (Garofalo, 1993; Montague, 1991; Montague & Applegate, 1993; Sriraman, 2003). Several metacognitive actions during problem solving were described in each phase of those models by

cognitive theorists, in mathematical problem solving. To examine the problem-solving behaviors and cognitive processes of individuals as they work in small groups, Artzt and Armour-Thomas (1992) developed a framework based on Schoenfeld's (1982) framework. Schoenfeld (1985) defined an episode as "a period of time during which an individual or a problem-solving group is engaged in one large task" (p.292). The framework for the protocol analysis of problem solving in mathematics is used for this study to differentiate between cognitive and metacognitive problem-solving behaviors observed within the eight episodes (read, understand, analyze, plan, explore, implement, verify, and watch and listen) of problem solving. The framework synthesizes the problem-solving phases identified in mathematical research by Garofalo and Lester, Polya and Schoenfeld, and of cognitive and metacognitive levels of problem solving behaviors studied within cognitive psychology, in particular, by Flavell (1981). This framework used in this study is to examine the interactions between two levels of cognitive processes (cognitive and metacognitive) observed in the problem-solving behaviors of students working in small groups on mathematics problems.

Methodology

This project consists of qualitative research in which case studies are employed. A qualitative design is appropriate for this study because the study focuses on gaining in-depth information about what actually occurs during the problem-solving process. The study conducted with nine fifth-grade students in a private school in the capital of Turkey. Purposeful sampling was used to select the participants as the researcher wanted to obtain more knowledgeable information about the problem-solving processes within the groups. Voluntary participants were involved in the study. The students who have high self-expression skills were selected by two mathematics teachers.

This study used multiple methods including a think aloud procedure when the students are engaged in solving problems, researcher's field notes of observation, and analysis of students' solution papers to collect data. Prior to the data collection, the participants practiced the think aloud technique with a sample problem. The procedure provides participants with important practice for understanding and developing confidence prior to utilizing the technique with the research problems. Over the one day 3-hours period of data collection, three mathematical problems which were selected from PISA problem solving sample questions (going to the cinema, transportations system and holiday) from decision making units (OECD, 2005), were given to participants to solve in small groups by using think aloud method. All three students in small groups had their own paper and problem sheet to follow the process. They continuously spoke aloud while they work on the problems explaining their thoughts. Also, they had unlimited time to solve each problem. Since misinterpretations of the data might have resulted with only a single researcher as the data collector, the researcher maintains a record of field notes explaining her reflections about the activities. The field notes included the explanations of questions, reactions, and behaviors that occurred during data collection.

In the study, group members were chosen from different classes, and they had never studied together before. Data collection was in a one-to-one setting between the participants and the researcher to have some field notes. One researcher observed exactly one group and took field notes. The researcher videotaped all the processes to record the participants' behaviors, how they responded to the problems, and what mathematical language they used. All data from the think aloud session, participants' solution papers, and researchers' field notes were transcribed for analysis by the

researchers. To generate the categories, the researchers read through all transcribed data sentence-by-sentence and identify words or phrases that described the participants' responses. For example, in group 1, student 1 says for cinema question: "First we will read the question then we will discuss". This sentence is coded as *Read*. Again in group 1, student 1 says for cinema question: "Until now, what did you understand?" and student1 suggests:" Let's underline the important sentences" These two sentences are coded as *Understand*. Also, student 2 says: "They cannot go to that film because it is for above 18 years old". This sentence is coded as Analyze. For the second question, student 1 says: "Let's try the other way" and coded as *Explore.*. Student 2 says: "They cannot go to Children in Deep, Carnaval and Pokemon" coded as Implement. Student 2 says: "Let's look at it carefully. They can all go to Mystery. Let's check" coded as *Verify*.

After each interview is transcribed, participants check the accuracy of the described experiences and themes. Then, the codes were applied based on a review of the data and the concepts emerging from the data. The responses of one student were compared with those of other students in the same problem, as well as the same student across other problems. Multiple data sources were used to triangulate and confirm patterns that emerged. Each response was compared with other responses with the same idea, regarding the source of the responses. The codes were grouped into categories. At this point, preliminary categories were developed. Responses were compared across categories in terms of similarities and differences. Next, the researchers revised categories with transcribed data again and again until the final categories are confirmed. The final categories were also reviewed against the transcribed data for the last time.

Findings

The coding for each of the three groups was done and the behavior of each group was categorized by episode. As it was suggested by the study, the three groups' episodes or problem solving behaviors were recorded and ranged. The audio records of the groups were coded in 1-min intervals based on the emergent behaviors through sentence-by-sentence and identify words or phrases that described the participants' responses. Groups were differentiated from one to another by giving numbers such as Group 1, Group 2 and Group 3.

Table 1 lists the number and percentage of behaviors across all groups, of the 519 behaviors that were coded, 25% belonged to Group 1, 21.65% belonged to Group 2 and 53.4% were demonstrated by Group 3.

According to the results students in Group 1, out of 130 items (sentences, phrases, words), show "understanding" behavior (e.g. Student 1 says: "Stanley cannot come to the cinema on Sunday also he cannot watch Pokemon" which represents rephrasing the questions in different ways) 44 times, 33.8%, which is the most frequent behavior observed, "exploring" behavior 25 times, 19.2% which is the second most frequent behavior observed and "implementing" behavior, 22 times, 16.9% which is the third most frequent behavior. As it was mentioned, the greatest percentage of existing behaviors was in understanding followed by exploring.

For Group 2, out of 112 items, students show "understanding" behavior 53 times, 47.3%, which is the most frequent behavior observed, "exploring" behavior 19 times, 16.9%, and "implementing" behavior, 15 times, 13.3%, which is the third most frequent behavior observed.

For Group 3, out of 178 items, students show "understanding" behavior 75 times, 42.1%, which is the most frequent behavior again, "exploring" behavior 27 times, 15.1%, which is the second most frequent behavior observed and "planning" behavior 19 times, 10.7%, which is the third most frequent behavior observed among the other behaviors. Differently from Group 1 and Group 2, this group shows more "planning" behavior than other groups.

Table 1: Percent distribution of behavior categories (episodes) by problem solving group

	Groups					
Behavior Category	Group 1		Group 2		Group 3	
Read	12	(9.2%)	6	(5.3%)	17	(9.5%)
Understand	44	(33.8%)	53	(47.3%)	75	(42.1%)
Analyze	1	(0.7%)	2	(1.7%)	1	(0.6%)
Explore	25	(19.2%)	19	(16.9%)	27	(15.1%)
Plan	7	(5.3%)	9	(8.0%)	19	(10.7%)
Implement	22	(16.9%)	15	(13.3%)	15	(8.4%)
Verify	11	(8.4%)	4	(3.5%)	10	(5.6%)
Watch & Listen	8	(6.1%)	4	(3.5%)	14	(7.8%)

Of all the episodes coded, the understanding episode was the coded as the greatest percentage within three groups while analyzing behavior was very rare. Among 419 items, 172 items represent "understanding" episode, 71 items represent "exploring" episode, 52 items represent "implementing" episode. The percentages of each episode are given in Figure 1.

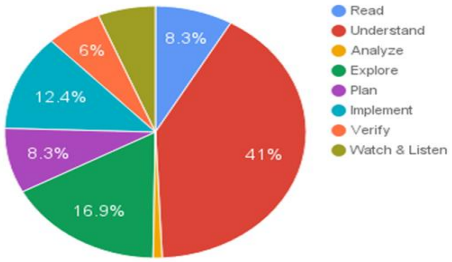

Figure 1: The percentage of the problem-solving behaviors for each episode in all groups

Discussion

According to Artzt and Armour-Thomas (1992) the greatest percentage of time was exploring (60.4%). In contrast to their study, we found that understanding was the most frequent observed behavior (33.1%). Watching and listening can play an important role in addressing the issue of communication between group individuals. These low and high percentages of watching and listening behaviors may be as a result of different reasons. For example, in Group 1, one student interrupted the other two students and also this student was writing the majority of the solution. The records and observations indicated that in each group some students assumed a leadership role. Therefore, it was possible that this kind of act would discourage the other two students in the group. However, during the problem-solving procedures, some productive interactions occurred while Student 1 (in Group 2) was not only supporting and guiding others but also got benefits from group members' ideas. These

results are in agreement with Artzt and Armour-Thomas' (1992) findings which showed different patterns interactions between group members and show the significance of intergroup relations for active and productive contribution.

The framework contributed the observation of individuals while working in small group settings. As it can be realized from the records and observations, group composition affected the group life. As group members were chosen from different classes, it is interesting to note that in all three groups students reflected a pragmatic desire in order to achieve the common goal by working together productively. With the exception of one student in Group 3, the small group study enabled researchers to observe peer to peer communication in a small group environment.

'Understanding' was the behavior that was coded the greatest percentage by students in this study. It would be expected that after this phase, students could decompose the problem into basic components and examine the relations between given elements and common goals at the analysis level, and then explore the problem by guessing and testing. In our study, understanding led group members to the exploration without making visible analysis.

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Geometric modelling inspired by Da Vinci: Shaping and adding movement using technology and physical resources

Diego Lieban[1] and Zsolt Lavicza[2]

[1]Federal Institute of Education, Science and Technology, Bento Gonçalves, Brazil; diego.lieban@bento.ifrs.edu.br

[2]Johannes Kepler University Linz, Austria; lavicza@gmail.com

We report on the experience gathered in a study using a geometric modelling approach based on dynamic geometry systems complemented by physical modelling. Our strategy intends to strengthen the interconnections between the current trends in Science, Technology, Engineering, Art and Mathematics Education. To help students to better understand how certain physical mechanisms work, some da Vinci machine prototypes were reconstructed and used as a starting point for this study. Building upon previous experience, our work currently concentrates on the analysis of connecting physical and digital resources and on how they contribute to students' creative thinking and problem-solving. We discuss the concept of geometric modelling, focusing on spatial thinking, joints and their movements. Further, we present some new manipulatives that are being tested together with digital applets and discussions from this practice.

Keywords: geometric modelling, GeoGebra, problem solving, creative thinking.

Introduction

Geometric modelling has been used in a wide range of contexts, but continues to deserve more attention in mathematics education. The term "geometric modelling" usually refers to different digital techniques for representing specific objects or surfaces. This concept seems to be associated mostly with computer-aided design, the tasks of which are usually related to shape and to improving models in order to approximate real structures by means of sophisticated algorithms and software. This work intends to bring geometric modelling, supported by dynamic geometry systems (DGSs) and combined with physical resources, to the classroom. While an analogue model provides "hands-on experience" to students so they can comprehend certain mechanical movements (and their restrictions), a digital model forces them to develop suitable strategies for transcribing such actions. Students must therefore think about how to apply mathematical concepts properly in order to use them successfully in their digital models. In our case, creating a digital representation is not only a mathematical exercise, but also an opportunity for the students to refine and review their comprehension processes and to improve their physical models. In this modelling approach, students are faced with problem-solving in Science, Technology, Engineering, Art and Mathematics (STEAM) contexts.

Modelling is characterized as the branch of mathematics that deals with the translation of a real-world situation into mathematical language. In publications discussing mathematical modelling (e.g., Brinkmann & Brinkmann, 2008; Lingefjärd & Holmquist, 2001), data are used to optimize particular processes or to develop algorithms for analysis or prediction. Geometric modelling approaches also concentrate predominantly on processes rather than on actual goals and concepts (e.g., Mason, 2001; Henning & Keune, 2008; Siller, 2008). We use geometric modelling for both the virtual (on the computer screen) and physical (concrete manipulative) representation of objects, which allows both the objects' functionalities and the interactions between them to be analyzed. Exploring linked

mechanisms by using either concrete manipulative or digital tools such as GeoGebra, students can examine mathematical ideas in order to improve their constructions. We focus on the process of using modelling to support geometrical relationships and vice versa. Our goal is that students gain a better understanding of how certain mechanisms work, beyond connect such STEAM areas. As a starting point, we used some mechanisms introduced by Leonardo da Vinci and shared them in an interactive GeoGebra book (see https://ggbm.at/AnHK7nCX). Although da Vinci invented them more than 500 years ago, their basic physical principles are still used in engineering and in education. Reconstructing these models does not only mean redoing what has already been done; it means connecting ideas and strategies for learning and using new resources based on some classical ideas, and maybe even improving them along the way.

In this paper, we present an activity developed within a vocational course in Brazil. Building upon this preliminary experience, we intend to investigate (in the form of a PhD project) how the combined use of physical and digital tools is of mutual benefit and promotes learning in science and mathematics.

Theoretical framework

Enhancing geometric modelling through Dynamic Geometry Systems

More than a "mathematical playground", dynamic geometry systems (e.g., GeoGebra) must be considered as a proper space for transforming students into explorers – a platform where mathematical understanding takes on another dimension and goes beyond merely applying formulas. Numerous studies have advocated this (e.g. Schumann, 2004; Gawlick, 2005; Bu & Hohenwarter, 2015) and stress the importance of transcending traditional geometry courses before the increasingly sophisticated and widespread application of geometry in science and daily life. In particular, they highlight how DGSs have supported changes in mathematics teaching and learning as well as in professional mathematical practice. Among the perspectives associated with the DGS approach, we outline those that are closer to geometric modelling: problem-solving, concept formation, construction, measurement, visualization, exploring, and variation/animation. DGSs often grant access to information that would otherwise be inaccessible. For instance, in spatial geometry, GeoGebra enables users to see an object from different points of view and its cross sections. Such features provide an important link between 2D and 3D representation. Investigating whether some ideas established in 2D also work in 3D or how they should be adapted is a promising strategy for promoting spatial thinking. Consider an example where the reasoning based on circles in a plane can be extended to a sphere in space: We have a point given in a plane and a line at a certain distance d from the point. In order to mark those points on the line that are located at the same distance $e > d$ from the point, one could use a circle in 2D, but this would not suffice in 3D space (a more appropriate approach would be to use a sphere in this case). The attempts to find a solution using DGS software enhance the students' conjecturing processes – another benefit of DGSs.

Teaching approach using analogues and DGS

In the didactic domain, Alsina (2007) proposed that students gain insights from the functionality of objects and thus engage themselves in a creative process because they can identify the potential or restrictions of a particular phenomenon. Gravina (1996) suggested that dynamic geometry can foster an approach to geometric learning in which assumptions are made from experimentation and the

creation of geometric objects. In the same vein, Swan et al. (2007) stated that students can refine their own thinking by interacting with different representations of problem situations. In our case, the aligned and parallel modelling process led students to a practice where they had to hone their ideas in every construction step. Facing the need to convey their ideas, students tapped into their previously acquired understanding (an example is discussed in more detail further below).

Recent studies (e.g., Sinclair, Bussi, de Villiers, et al., 2016; Camou, 2012; Lesh and Sriraman, 2010) support the positive effect of the design and implementation of a multi-representational approach to exploring 3D objects using crafts, computer technology, and paper-and-pencil methods. In this context, we seek to provide an integration of geometry with algebra and trigonometry (using the example of joints with circular movements) that goes beyond technical instrumentation.

In our case in particular, the use of mechanical principles provides the background for the modelling process, as can be seen from the diagram in Figure 1, which was adapted from De Sapio and De Sapio (2010): they considered the relevance of applying an approach to problem-solving at an elementary stage through constructing mechanical analogues to geometric problems. In this case, mechanical reasoning supports geometric reasoning. Note that we added the arrow in the opposite direction, since the reverse case (geometric reasoning supports mechanical reasoning) is equally possible, as shown in our study: On the one hand, mechanical reasoning was essential to discussing the proper ratio for a pulley system in one case. On the other hand, with the help of rotational simulations (i.e., geometric reasoning) by means of digital modelling, the students figured out how to build a functioning physical prototype in another case.

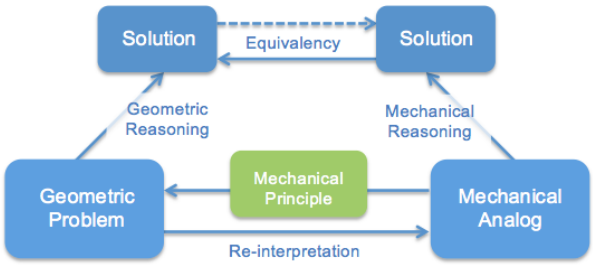

Figure 1: the solution's correspondence come out in both directions

Concerning the activity's driving, the activity is consistent with DeHaan (2009), who stated that some strategies can transform the lecture hall into a workshop or studio classroom (even partially), and stressed the use of computer-based interactive simulations as a promoter of creativity instruction.

In fact, researchers investigating creativity generally argue that projects tend to be more creative when the solution is redefined, revisited, and questioned numerous times during the process (Lee & Carpenter, 2015). Furthermore, there are many different ways of developing prototypes. The process of refining ideas and designs puts students on an unrestricted path. Siswono & Novibasari suggested that problem-posing activities using the "What's another way?" strategy could improve students' abilities in creative thinking (as cited in Siswono, 2010). In our study, the students discussed their different ideas, especially in digital modelling. In this phase, they considered various points of view and also tried to gain some insights to check whether their ideas were feasible or not in order to do make the model as simple as possible.

Methodology

Our experiment started in September 2015 at the Federal Institute of Education, Science and Technology in Brazil. It took approximately four months and had the form of a partial extra-class activity with two weekly meetings to follow the progress. The students (most of them were 16 years old) participating in the vocational (informatics) course were supported by two additional teachers (physics and mathematics). Although supervised by teachers, they chose their own topics to investigate. We first present the Da Vinci Rotatory Bridge Project (Figure 2) developed by 4 students. It was agreed that they should develop both physical and digital models in order to try to improve the joints of the existing mechanisms. Our intention was that, by comparing similarities and differences between the models, the students should be able to use one to support the other. No particular order was prescribed, but parallel development was suggested. Use of GeoGebra was also optional but recommended, since we were exploring it during class. In fact, da Vinci projects have been promoted since 2011, but this was the first time GeoGebra materials and GeoGebra 3D features were integrated. Particularly in the digital modelling process, the students concentrated on principles of rotation, translation, and spatial geometry.

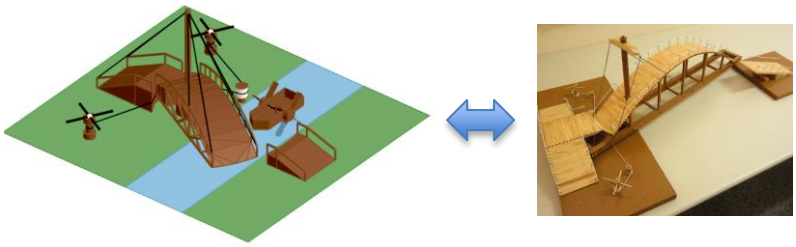

Figure 2: The digital prototype developed using the GeoGebra 3D feature (left) and the physical model made of wood (right) were developed in parallel

As a further development we are combining this activity with a new resource, 4Dframe[1], which is a flexible material and easy to manipulate. In the next example (Figure 3), we follow the development of a digital catapult in two versions. The second one is based upon the 4Dframe model.

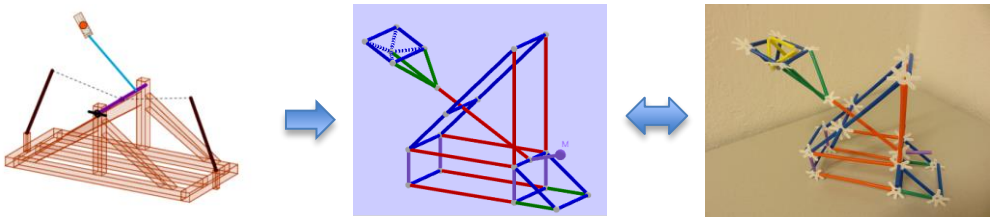

Figure 3: Catapult evolution and becoming easy to represent

Since mechanical principles are highlighted in a simple way, it becomes easier for students to represent them in GeoGebra. Furthermore, elementary models with straws and connectors become part of the digital modelling more easily when only segments and points are used to represent the structures. The functional principle of the joints, however, is preserved. In addition, various colours are used to represent the corresponding elements and to contribute to the visualization and facilitate spatial comprehension.

[1] For some examples, see https://www.geogebra.org/m/xCxJUyyx.

Finally, another important benefit is that the digital model allows a wide range of representations to be created by simple cursor movements (in the example in Figure 4 by dragging the blue points). Using 4Dframe, the students can think freely about different possible solutions and the constraints arising for an eventual construction, and are engaged in a learning process involving critical and creative thinking.

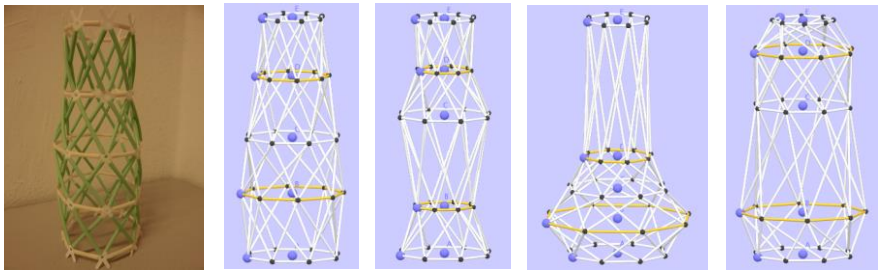

Figure 4: Multiple representations promote creative thinking. By dragging the blue points in the digital model, students can create a range of possible solutions.

Construction's ideas and discussion

In order to illustrate some basis used by students to support such models, we share some parts of them. The students were free to construct according to their own previous knowledge and were not required to use any specific content. However, if the need arose in a particular task, the students received proper support. In such cases we enjoyed discussing the problem at hand with them and, together, introduced new concepts or strategies, as presented below.

Given a circle with its centre in A (located on the x-axis) and an arbitrary radius[2] we started the task (Figure 5). First students investigated the relative position of a point B on the circle in relation to its center while sliding along a line. Since the goal here was to simulate the movement of a wheel, they needed to implement rotational movement. To figure this out, they changed the definition of point B to (x(A) + sin(x(A)), y(A) + cos(x(A))). Naturally, this result was obtained after several attempts and discussions. Note that, in this case, the point B is a function of point A.

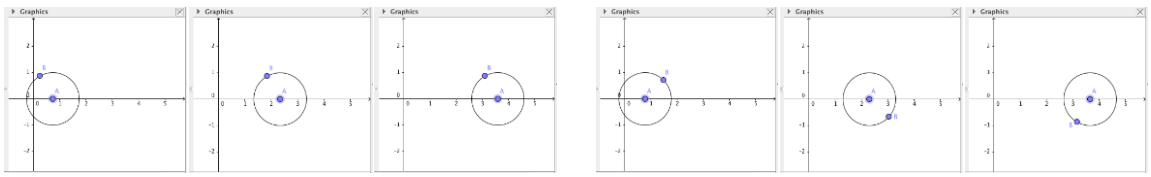

Figure 5: Three-frame cycles represent the transition from a "dragging circle" to a "rolling circle"

In a 3D representation, the same logic is preserved, but this concept was totally new for the students: Each point then has three inputs. If a circle is perpendicular to one of the standard axes, then all points belonging to this circle have a constant input regarding to such axis. The other two coordinates repeat the idea from the previous 2D example. For instance, in the case below (Figure 6), B is given by (x(A) + sin(x(A)), 0, z(A) + cos(x(A))). The connected elements can be completed by rotation or symmetry. Additionally, some principles applying to spatial coordinates were used to define proper points as a

[2] The radius could also be controlled by a slider. In this case, the scale had to be in accordance with the remaining construction.

basis for such constructions. In the boat example (Figure 6), a first reference segment was built to guide the following marks through translations and reflections.

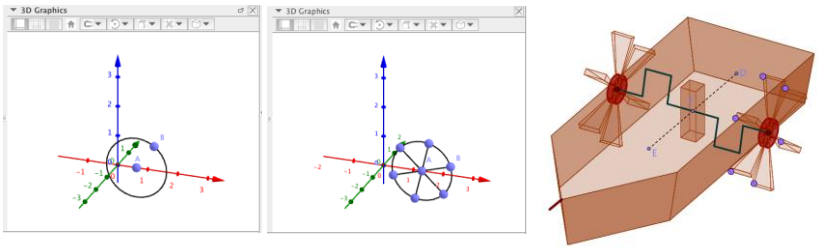

Figure 6: Developing a 3D representation in 3 steps

In this activity, the students should preserve the coherence between analogue and digital model as much as possible. To illustrate this, we refer to the bridge example, where the relation between the turns of the driver pulley and the bridge was to be determined; this is an issue easily identified in the physical prototype but not as readily in the digital model. When the students realized that the models were not in agreement, they concentrated on the geometrical problem and concluded which ratio between the number of turns of the bridge and of the pulley should be appropriate. In contrast, when students modelled only the digital boat (in a previous experiment), they recognized the following misconception before building the physical model: if the paddle wheels spun together at the same speed, the boat would move only forward and backward. They then fixed this problem in the physical model. This suggests that it does not matter which model they build first as long as the whole model contributes to their experience and improves their learning process. When digital geometric modelling was the goal, students needed to use their knowledge of trigonometry and parameters (functional thinking) in order to establish links between elements to obtain the desired representation. Questions such as *"If you want to change the direction of the rotation, what do you have to do?"* often initiated their investigative process and sometimes became a challenge.

Conclusion

"Inviting" students to reconstruct historical models is one of many possible ways to teach mathematical concepts and to promote students' creative thinking processes. Students must decide how they can use their previous knowledge in order to solve a given task. In this way, students increase their autonomy and become more involved in their own learning processes. An important step of Pólya's heuristic strategies can be outlined: "If you cannot solve the proposed problem, try to solve first some related problem" (as cited in Schoenfeld, 2016).

While we promote STEAM on the one hand, we introduce different dimensions of learning to students and enable deep learning driven by their own interests on the other. Problem-solving and geometric modelling can also become a basis for the integration of mathematical learning into trans-disciplinary educational frameworks, currently referred to as STEAM.

In the course of this study, students left some testimonials on Moodle. One such testimonial referred to some physics aspects: *"I enjoyed two videos posted by the teacher that showed how da Vinci's bridge worked. They enabled a better understanding of the functioning of the rotatory bridge. I understood better how the system of ropes and pulleys works in order to reduce the workforce."*

Feedback such as this and examples such as those reported above illustrate the students' interaction with different resources that supported them. They indicate that the geometric modelling approach can motivate and contribute to their learning process. We therefore seek to evaluate and promote this activity among teachers as part of our current research. We are now working on developing new resources that connect the physical and the digital world and will report on these at a further occasion.

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Choosing a Mathematical Working Space in a modelling task: The influence of teaching

Elizabeth Montoya Delgadillo[1], Fernanda Viola[2] and Laurent Vivier[3]

[1]Pontificia Universitaria Católica de Valparaíso, Valparaíso, Chile; elizabeth.montoya@pucv.cl

[2]Universidad Nacional de Córdoba, Córdoba, Argentina; fviola@famaf.unc.edu.ar

[3]Université Paris Diderot, Paris, France; laurent.vivier@univ-paris-diderot.fr

In this paper we report on students' work in Argentina and in France when performing a modelling task. The problem given, named the "gutter", is quite a classic in university calculus courses. Analyses have been realized by using the Mathematic Working Space (MWS) research model in relation to mathematization of a modelling process. We are mainly concerned with the influence of the teaching in students' productions. Since the modelling process is an important issue of teaching, we have chosen the population of the study among pre-service teachers.

Keywords: Mathematical Working Space, paradigms, modelling task.

Introduction

Nowadays, modelling is fostered in secondary school and at the university level, and one expects teachers and engineers to be trained to achieve the modelling competence. For instance, modelling constitutes a specific competence that has to be developed by civil engineers in Latin America, according to Tuning project (Guerrero, 2013). But, what modelling problems have to be considered in upper secondary school and in university? Are the modelling problems used in high school, and in the university, standardized by the institutions? Do they refer to real situations? We think that mathematical objects, different representations used, and properties involved in the modelling process give a richer sense for students in term of the knowledge they can construct. Teachers have of course an important role to play that highlights the importance of the teachers training. In that way, the population of this study was mainly choosen in secondary teacher training.

In this study we present a relation between the Mathematical Working Space research model (Kuzniak & Richard, 2014; Kuzniak, Tanguay & Elia, 2016) and the modelling cycle (Blum & Borromeo-Ferri, 2009) that could be understood as a *first cycle* for the resolution of the modelling task. Mathematical Working Space (MWS) is a model that is used in research in mathematics education, first developed in the field of geometry. When a student starts with the given situation, one assumes that he/she begins a horizontal mathematization process which is a foundation for bringing the situation problem into a mathematical domain. Then a vertical mathematization process takes place where the MWS framework and the modelling cycle can interact with each other. That is our approach in this study.

We first present how to use the MWS model to analyse a modelling activity. Then, we study a modelling task given to pre-service teachers in Argentina and in France. Our aim is not to make a comparative study, since both populations are different, but rather to identify the personal MWS of the students when solving this modelling task. The focus is mainly on three levels of teachers training in Argentina. We also looked at primary teachers training in France in order to have an idea of no scientific students' answers.

MWS and Modelling

We consider a MWS that depends on a specific mathematical field (Kuzniak, Tanguay & Elia, 2016) such as, among others, analysis, geometry, algebra or statistics. Paradigms in an MWS, depending of the domain, serve to characterize the work according to a community or an institution. We develop the paradigms for analysis (Montoya Delgadillo & Vivier, 2016).

The MWS model

Three types of MWS may be distinguished: (i) **MWS of reference**, which is defined according to the relation to knowledge, ideally under mathematical criteria; (ii) **suitable MWS**, which depends on the institution involved, and is defined according to the way that this knowledge is supposed to be taught in the institution; (iii) **personal MWS**, which depends on the individual and is defined by the way in which the individual handles a mathematical problem with his or her own knowledge and cognitive capacities.

MWS is an environment in which reflection results from the interaction between an individual and a problem in a mathematical domain. It is an environment organized for an expert of this domain, by means of two interconnecting planes: the epistemological and the cognitive planes (Kuzniak & Richard, 2014; Kuzniak, Tanguay & Elia, 2016).

The *epistemological plane* is composed of three poles (Figure 1a), namely *referential* (properties, theorems, definitions…), *representamen* (semiotic signs), and *artefacts* (material or symbolic). The *cognitive plane* consists of the processes of *visualization*, *construction* and *proof*. The functioning of a MWS must not be understood as a union of single components lying on the epistemological and cognitive planes, but rather as links activated by two or three geneses, *semiotic*, *instrumental* and *discursive genesis*, that articulate the two planes.

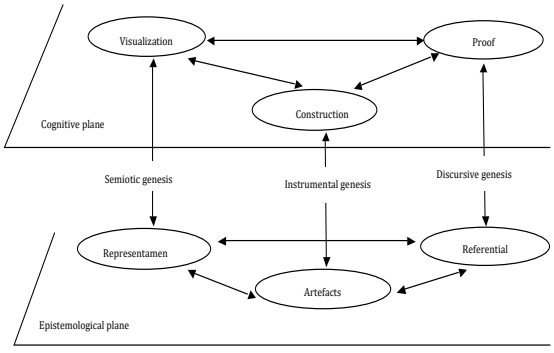


Figure 1a: The Mathematics Working Space, geneses and vertical planes (Kuzniak & Richard, 2014)

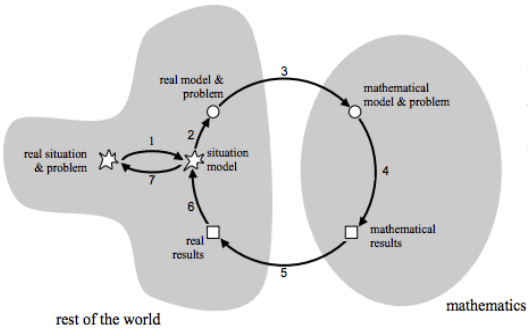


Figure 1b: The modelling cycle (Bloom & Borromeo-Ferri, 2009)

Paradigms of analysis

The situation we propose (see following section) is an optimization task. Hence, using (mathematical) analysis is quite natural, even if the problem can be solved in various ways. In order to identify the paradigms of analysis in the answers of the students, we present the three working paradigms of analysis identified by Montoya Delgadillo and Vivier (2016):

- **Arithmetic/Geometric Analysis** (**GA**): it enables interpretations with implicit assumptions based on geometry, arithmetic calculations or the real world.

- **Calculation Analysis** (**CA**): the rules of calculation are defined more or less explicitly and are applied independently of reflection on the existence and nature of the objects.
- **Real Analysis** (**RA**): it is characterized by work involving approximation and neighbourhoods; definition and properties are set theoretically; an "ε work".

Modelling

The development of the modelling skills (Blum & Borromeo Ferri, 2009) mobilizes notions and mathematical objects of different mathematical domains, where the knowledge that the students can learn is grounded on arguments that belong to different domains (analysis, probabilities, geometry, etc.). This gives rise to different MWS and paradigms.

Recently, two doctoral theses in probability in Paris Diderot university proposed a use of MWS in modelling processes[1]. The whole modeling cycle (Figure 1b) is not taken into its totality: the focus is on phases 3 to 5 of the cycle and our aim is to analyse, with the MWS model, the mathematization process when students solve a modelling task. We expect to identify, in students' work, the solving mathematical domain and mathematical objects, representations or signs, artefacts, mathematical knowledge, and the working paradigm. We show below how the two frameworks may be used together and their complementarity.

In the following, we do not consider all the modelling process with the lens of the MWS model, but rather how is it possible to analyse mathematical activity. In particular, we set the following questions: Given a specific task, what domain and (personal) MWS will a student choose during the modelling process? Is it possible to see an influence of a suitable MWS?

Experimental study

The "gutter" situation (see below) was given to three groups of future secondary mathematics teachers in Argentina: two groups of a private training institute, namely 1PC of 1st year and 2PC of 2nd year, and one group of 4th university year, 4PF, of the Universidad Nacional de Córdoba – unfortunately, a 3rd level group was not available. The study focuses on these pre-service teachers, but the task was also posed in France, in an examination of 3rd year future primary teachers, at the Université Paris Diderot. This extra population helps to understand what kind of solving processes one can expect from non-scientific students. Students of the study are named L1 to L28, 1PC1 to 1PC24, 2PC1 to 2PC15, and 4PF1 to 4PF12 accordingly.

As announced above, it is an optimization task, quite classic except for modelling aspects. Here, we make the hypothesis that the majority of students who followed a calculus course that provided a method for solving a class of optimization problems (derivation or optimum of quadratic functions) will work in paradigm CA, and that they will resort to functions.

The aforementioned was the case for 4PF students trained at the university, since these students had courses on mathematical analysis. But CA paradigm was not expected for 1PC students and 2PC students, since in Argentina the first calculus course takes place during the second year and the experimentation was at the very beginning of the academic year. We then expected more GA methods (see below).

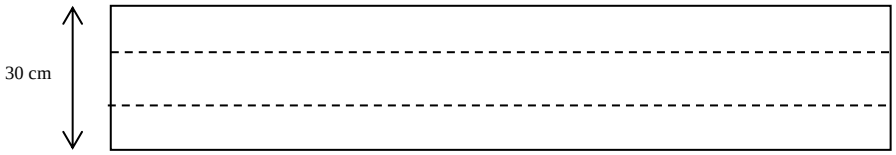
[1] See also the poster "Modelling tasks and mathematical work" in TSG6 of CERME 10.

On the other hand, French students were in a multidisciplinary third university year after the validation of two disciplinary university years (mathematics, or biology, or history, or English…). Students' profiles were very diverse and a few of them studied mathematics, and specifically calculus. We make the hypothesis that, spontaneously, most of L-students work in GA paradigm, whether or not using functions.

The gutter situation: a priori analysis

First, we give the statement of "the gutter", then we make an a priori analysis.

We have a rectangular metal sheet of 30 cm width and of big length. We fold up perpendicularly edges on each side to make a gutter (see dotted lines on the figure below). For obvious reasons, both side edges of the gutter will have the same size.



How should we fold up the metal sheet in order to obtain a gutter with a maximum flow?

This is a partially modelled task, since many parameters are fixed and, moreover, the geometrical mathematization is given. The choice to fold perpendicularly allows simplifying the task by eliminating other cases. There is a lack of parameters to study the flow, among them, the slope of the gutter (that we shall not discuss) and the length of the sheet.

The latter is not necessary because we can replace the study of the flow by the study of the area of a cross-section of the gutter supposing that the flow through this section is constant. Nevertheless, it is simpler to work with the volume than with the flow which is a quotient magnitude. Thus, some students may choose a length to make a calculation of volume. Is this length considered as a parameter or either as a numerical value added to the statement? This length is used in calculations or is it only a useful intermediary to think of the situation?

Let us note that there may be problems of understanding of the proposed geometrical modelling. An inadequate understanding of the situation corresponds to the addition of an idea or a belief, which is not correct regarding the situation. We expect in particular to have equitable answers with 10 cm for each of the three edges, as well as the assertion that the area, and thus the flow, does not vary because of the constant 30 cm.

On the other hand, there is no indication on the mathematization allowing to make calculations for solving. We focus our analyses on this phase of mathematization, which we interpret as the choice of a MWS by students, and the phase of solving. We then look at students' answers with an adequate understanding of the modelling situation: good geometrical shape (specified in the statement), and considering the area of a cross-section, or a volume, to study the various values (the variations) of the flow. We also look at a schema of the gutter: cross-section, in 2D or in 3D.

In the following, we present the types of answers expected for solving the modelling task. Before that, let us notice that the task can be solved in the geometry field of the statement, with magnitude and a knowledge on the areas of rectangles of given perimeter: By taking two "gutters", one forms a rectangular pipe the section of which has a perimeter of 60 cm. Since the area is maximal for a square,

the solution is obtained for a 60 cm/4 side, that is, 15 cm. So, the basis has 15 cm of length and one has to fold in 15 cm/2 = 7.5 cm.

The resolution can be made in a numerical MWS, that is, by doing several calculations of the area or the volume. Several levels are possible: choice of some values of the length of the edge, calculations, and then decision-making. Numbers to be considered are essentially whole or decimal numbers. The solution being 7.5 cm, considering only integers multiples of cm does not allow to find the optimal value (unless changing the unit of length). These values can be grouped in a table or not, obtained in an organized way or not (for example, with all integers from 0 to 15, or by an oriented search), with a sign (as the letter x) to denote the length of the edge or not. A formula allows to automate calculations and, possibly, the implementation in an instrument of calculation. Obtained values can be also put in a graph.

The production of a formula can allow to change of MWS to a functional MWS, in calculus. One expects the introduction of a quadratic function[2]:

- Using only algebra, with the expression of a global variational principle: $f(x) \leq f(a)$ where a is the candidate value, that has to be find, to be the optimal value;
- With recognition of a quadratic function, its properties, vertices or symmetry axis of a parabola, allow to solve the problem. This is a work in CA paradigm.
- Using the derivative of the function. Is the theorem quoted? Is the change of sign of the derivative evoked or forgotten? This is mainly a work in CA paradigm.
- Setting a values table or a graphic, recognizing or not a parabola, in GA paradigm. The difference with numerical MWS mentioned above may be difficult to identify.

Results

Let us note that 12 L-students do not answer, and utterances from 5 students of 1PC are not classified. Three of these 1PC-students gave the solution 7.5 cm without any justification: possibly an exchange of information between students or, for at most two students, the divisions of 30 cm by 4 like for a square[3]. There is also the atypical 1PC24's production, which will be mentioned later on.

In Table 1 there is a summary of the results with four groups of indicators:

1. Representamen: cross-section of the gutter (CS), letter for the edges (Ledg), letter (L) or value (V) for the fixed length of the gutter (Par), table (T) and graph (G);
2. Objects used: function (Fu) or formula (Fo);
3. Knowledge: derivation (Der), vertex of a parabola or quadratic function (Ver) in a MWS of function, CA paradigm; numerical calculations (Num) in a numerical MWS;
4. Non adequate modelling: 10 cm for each edge (10), no variation (noV) and also extreme folding (ExtFold) for "the edge must be the lower" or "the greater".

[2] A quadratic function limits the technical difficulties. The task was also chosen for this reason.

[3] That method is not expected here: since $(a,b) \rightarrow ab$ is bilinear, optimizing the area of rectangles with fixed perimeter p is the same than optimizing the area of rectangles of sizes a and b with $a + b + a = p$.

Table 1: Students' answers to the "gutter task"

	CS	Ledg	Par	T/G	Fu/Fo	Der	Ver	Num	10	noV	ExtFold
4PF (12)	8	12	6L,0V	0T, 3G	12Fu, 0Fo	7	5	0	0	0	0
2PC (15)	3	2	0L,1V	2T, 0G	0Fu, 1Fo	0	1	5	4	5	2
1PC (24)	7	2	1L,2V	0T, 0G	0Fu, 0Fo	0	1	11	5	2	0
L (16)	0	7	2L,3V	2T, 2G	4Fu,2Fo	2	0	5	2	0	4

There is a lot of 2D schema, close to the statement schema, but only four 3D schema. On the other hand, a cross-cutting (see Table 1) seems characteristic of 4PF-students, more advanced mathematically. It is the same with formalization, introduction of letters (variables and parameters) or using functions: the 12 4PF-students used a modelling with functions. It is also the same with the use of knowledge: 4PF-students work in CA paradigm, either by using derivation or the formula giving the vertex of a parabola (rarely justifying that it is a maximum; only two students justified it, by calculating the second derivative). There were a few students that used this knowledge in the three other groups, or in an empirical way by working in GA paradigm (by numerical calculations, more or less organized, or graphs).

Although the work seems largely guided by the knowledge the 4PF-students have, with few variations, modelling was adequate, what is not the case in other groups. For students who do not well understand the situation, either no MWS rises from modelling, or a very poor MWS appears, not allowing a substantial mathematical work. Answers of inadequate modelling are:

- Equitable length of the three sides: 10 cm each (5 students);
- There is no variation, all is constant (4 students);
- An extreme folding, the littlest lateral side possible (5 students), but sometimes with the (real) constraint that 0 is forbidden, so that the water remains in the gutter;
- L27: a pyramid (to have a pipe?) without understanding the geometry of the statement;
- 1PC24 drew a roof explaining: "if there is a few water that does not go strongly", and "if there is a lot of water and with great strength".

Although the productions of the group 1PC are very poor, with not much formalization, in the groups 2PC and L we find a various types of rich productions: trying values with calculations, sometimes leading to the accurate solution when the student thinks out of the integers domain (some students remain in the field of the arithmetic of integers, sometimes with answers with two solutions, 7 and 8, and even the constraints to have a basis of even length in order to be able to divide by two), graphs and tabulations which can lead to the awareness of a symmetry (L6, L28 below).

L14 proposes a formula with a letter x for the length of the edge and l=10 for the length of the gutter, to calculate what he calls Dmax (figure 2). He uses this formula to find a values table for a whole number between 1 and 15. He writes: "we notice that the maximal flow is achieved for an edge between 7 and 8 cm". Other students stop at this stage, while L14 continues with another table for the values of x between 7 and 8 with a step of 0,1. This allows him to conclude correctly, in GA paradigm, that "to have a maximal flow, edges have to be 7.5 cm each".

Student L6, after the introduction of the function $f(x) = (30-2x)x$, establishes a valuable table for integers and draws the graph of f (figure 3). She concludes with visualization on the graph to determine the solution by drawing what seems to be the axis of symmetry of the curve.

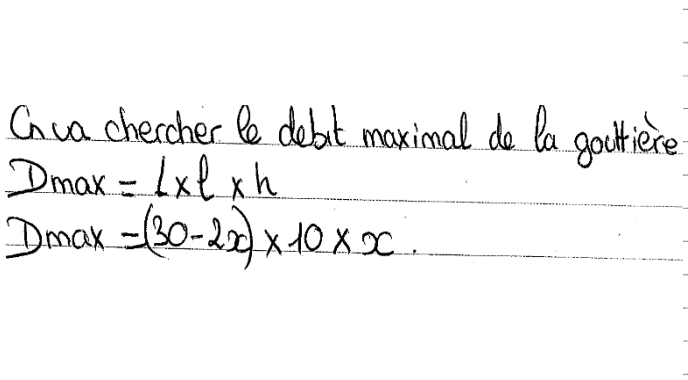


Figure 2 : Answer by student L14

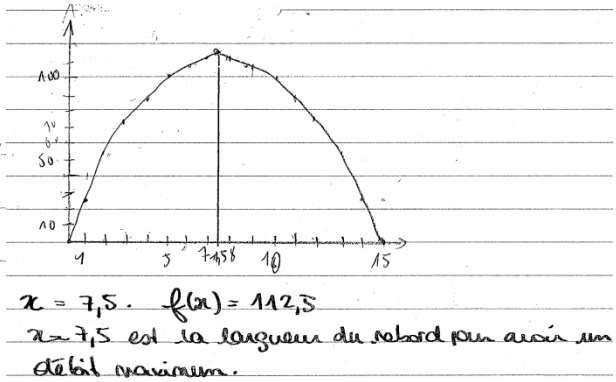


Figure 3 : L6's graph

L28 declares the same function but in a more formal way: "$\forall x \in (0,15)$ $f(x)=(30–2x)x$". He then calculates the values for the whole values of x that he associates two by two, which is an organization adapted to the function at stake. "We notice a symmetry in the values of $f(x)$ when x varies between 1 and 15. We deduct that the maximum is in the middle of the values, that is: $f_{max}(x)=f(7,5)=112,5$ cm^2."

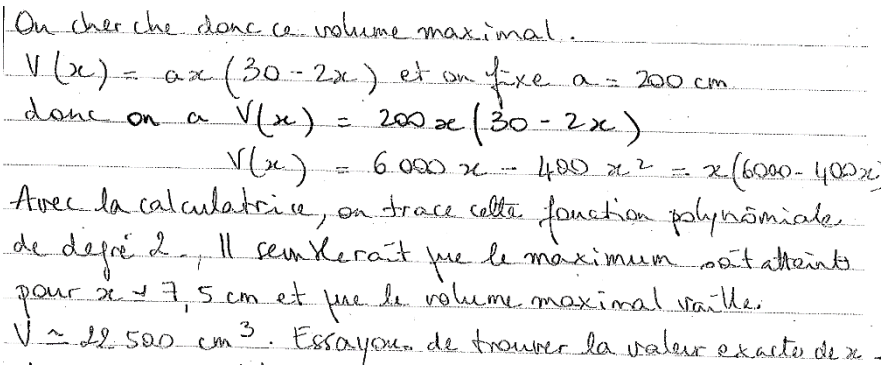


Figure 4: Production of L10

L10's work is very complete (figure 4). She uses a formalism (she fixes the parameter a to 200 cm), a function, and a graphic calculator to propose a conjecture for the candidate maximum value in a visualization work in GA paradigm. She strengthens her conjecture by considering the middle of the function roots. She continues by calculating, algebraically, in paradigm CA, $V(x)–V(7.5)= –(20x – 150)^2$, what allows her to justify her conjecture and to conclude.

Finally, 2PC4 has an atypical production with a modelling by means of a paper model: he takes two paper strips, which he folds to form a pipe and gives, by a reasoning close to the solution in the $MWS_{Geometry}$ described above, the right solution.

Conclusion

The recognition of a function in the situation, like the introduction of a letter, is an important modelling activity, because a concept is introduced, which a priori has no relation to the situation. But more than a letter – by using it the work can remain in a numerical MWS –, the introduction of a

function switches the work to a MWS of functions with specific techniques (e.g. derivation) and representations (graphic, table value). However, the work done can be very different and the personal MWS involved can also be different from a student to another.

The understanding of the situation and the asked question made each student choose a MWS, that is to say, mathematical objects, knowledge, theorems, signs (letters, graphics,…), artefacts (calculator, spreadsheet,…) and the processes of visualization, construction and proving in various domains – in other words, the MWS... – . Those are complex issues for a teacher to control the various knowledge involved.

Then we wonder whether the modelling process for some students is a routine that has been "normalized" by the institution, mostly for a specific class of problems, or a mathematical activity which has been taught in their institution. It seems that for 4PF group, students' answers are standardized with only a *choice* between two knowledge on functions. This is a first influence of the suitable MWS that we point out. The diversity of the L-students productions reinforce this standardization and we set the problem of the possibility for teachers to be aware and open to alternative answers that their students may have.

MWS model allows us to analyse in depth the concepts and mathematical objects coming from mathematization identified in the modelling process. However, the possibility that different MWS from this process are generated makes an attractive but complex task to control for the teacher, overall for a *second modelling cycle*. Indeed, this study focus on a *first cycle* where one asks to students to solve a modelling task. Then, teachers may use a *second cycle* in order to reorganize and to point out knowledge that arose during the first cycle.

Acknowledgments

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Fifth-grade students construct decimal number through measurement activities

Abolfazl Rafiepour[1] & Azam Karimianzade[2]

[1]Department of Mathematics Education- Faculty of Mathematics and Computer- Shahid Bahonar University of Kerman, Iran; Rafiepour@uk.ac.ir or drafiepour@gmail.com

[2]Kerman Math House, Kerman, Iran

In this study, a teaching experiment was used in which fifth grade students developed their own knowledge about decimal number based on their prior knowledge and real life actual experiments. A Realistic Mathematics Education approach was used for designing this research. Participants in this study were 27 students from a primary school and they did not have formal instruction at school on decimal number until participating in the current study. These students engaged in some measuring activities during four sessions in two weekends. They discovered the idea of calibration of a measurement unit and tried to use this idea for measuring the length of objects. This study shows how real life experiments of students help them to calibration a measurement unit.

Keywords: Decimal number, measurement activities, realistic mathematics education, primary students.

Introduction

Understanding of decimal numbers is very important for people who live in twenty first century because they use computer, calculator, digital monitor and other measurement activities in their real life experiments. Usually, people encounter with decimal number in their real life. For example nutrition facts written on foods, factors for buying products, vaccination card of children and … have a lot of things related to decimal number. All children encounter with decimal number before formal education at school, although its meaning is not understood. According to Bonotto (2009) connection between outside of school experiments of children and formal education can support students' conceptual understanding.

In traditional teaching, decimal numbers developed through place value. Usually students use tens and hundreds blocks for consolidating decimal numbers concept. Operations and procedures explain by teachers abstractly and then ask students to do some similar exercises. Indeed, students hadn't active role in developing their own knowledge and there is no connection between out of school students experiment and their math classroom activities.

It seems to we need to develop new approaches for teaching decimal number. One of the useful ideas in this regard is realistic mathematics education approach. In current study, we try to use this approach for finding some real contexts which are authentic for students and these contexts used for starting point for constructing mathematics by students themselves. Indeed, this paper will be introducing a teaching experiment in decimal number domain. During this teaching experiment, students do sequence of measuring activities to develop a measurement unit and calibrate it for measuring different lengths. Students also discover different representation for decimal number through these activities. The main purpose of this research was that show how students develop their

own knowledge about decimal number in the context of real world with using their common sense and prior knowledge.

Literature review and theoretical framework

Decimal number is one of important topics in school mathematics which has plenty of application in students' real world experiments. But several studies show that students and even adults haven't good understanding about decimal numbers (Moloney & Stacey 1997; Steinle, 2004; Lai & Tsang, 2009; Sengul & Guldbagci, 2012). Lai and Tsang (2009) show that procedural knowledge of students in decimal number was very good, but their conceptual knowledge in decimal number and decimal notation were so weak.

One of the important concerns of Lai and Tsang (2009) is that do the mathematics teachers know how to deliver decimal conceptual knowledge to the students? Bonotto (2001) believe that students' difficulties in decimal number rooted in teachers teaching strategies which have no connection to real life of students. Indeed, students usually encounter with decimal number in format of some stereotype word problems. Niss, Blum, and Galbraith (2007) said word problems exist several centuries in school math curricula and used as application of mathematics, but these types of problems in fact are a pure math problem in cover of words. Greer (1997) believes that word problems are artificial. Verschaffel (2002) states that emphasis on word problem cause to suspend common sense during mathematical problem solving.

Bonotto (2004) in the line with realistic mathematics education, believe that engaging students into the contextual activities that related to their own personal life, help them to enhance their conceptual knowledge in mathematics and having positive attitude toward math. She mentions two below factors that separation between school mathematics and real life facts: Stereotype problems of mathematics textbooks and Classroom environment.

If teachers of mathematics wish to establish situations of realistic mathematical modeling in problem-solving activities, Bonotto (2005) proposes below suggestions:

- The type of activity to which teachers delegate the process of creating interplay between math classroom activities and everyday-life experience must be replaced with more realistic and less stereotyped problem situations;
- Teachers must endeavour to change students' conceptions, beliefs, and attitudes toward mathematics;
- A sustained effort to change classroom culture is needed.

Bonotto (2001) maintains that children's understanding of decimal numbers can be fostered by classroom activities where learners can transfer their out-of-school knowledge and utilize familiar tools, such as the ruler, that they also use in out-of-school contexts. She believes it is possible to attempt an innovative teaching trajectory in which decimal numbers are introduced through contextualized measuring activities. Indeed, measuring activities requiring vast use of a ruler can offer children good opportunities to move toward the construction of a comprehensive understanding of decimal number and notation. For example, a study of Astuti (2014) shows that, if students use paper strip and calibrate it by themselves, then their understanding about decimal numbers and notation will be more developed.

Method

In this study, a teaching experiment about decimal number will be introduced. Based on the research aim, the type of the research is categorized as design research. This research contributes to develop a Local Instruction Theory (LIT) to support students develop the understanding of decimal number and notation. LIT has cyclic spirit (Gravemeijer, 2004) that in this study is prototyped by a Hypothetical Learning Trajectory (HLT) (Simon & Tzur, 2004) which is elaborated and refined when conducting the design. The initial step of HLT in this study is developed based on the analysis of key areas of decimals from literature review, the analysis of Iran mathematics curriculum, and the analysis of the potential use of contexts and model based on the framework of RME.

Current study is part of larger study that investigates conceptual understanding of fifth grade students in several aspects such as constructing a measure, calibration a measure unite, familiarize with decimal and notation, comparison of decimal number, density of decimal number, and submission and multiplication of decimal number. This study concentrates only on constructing a measure, calibration a measure unite, familiarize with decimal and notation. Main purpose of this study was familiarizing students with concept of decimal number. In this regard five activities designed which related to students real life facts (see figure 1).

1. Select a measuring unit arbitrary and measure length and width of classroom board, approximately. Represent length and width of classroom board with mathematical symbol.
2. How we can get better approximation? Write your proposed method completely, then record measured length in a mathematical form.
3. Select an object that smaller than your measuring unit. How you measure the length of this object? Explain your method and write measured length.
4. In this week a one meter non-graded tape give to each group of students and ask them where do you hear about "half" concept? What is the meaning of "half"? What is mathematical symbol for that?
5. Divide non-graded tape in 10 parts. Then try to measure a selective object approximately, and then show it with new mathematical symbol (decimal number).

Figure 1: Activities of first week

These activities implemented in four 80 minutes sessions in two continuous weekends' day. First three activities implemented in session 1 and 2 in the first weekend. Fourth and fifth activities implemented in session 3 and 4 in the second weekend.

This study conducted in a primary school in the beginning of school year (Fall 2014). Participants of this study were 27 fifth grade (10-11 years old) female students. These students had no formal program in these two days and all of them participate in this extracurricular class voluntarily. All students work on activities in group. Each group contains three students. Two types of complementary communications occur in this study: group discussion and whole class discussion.

During these teaching experiment sessions, second author and two other math educators record all communication of students in group. In the end of each session, students' group works collected also. Below considerations navigate activities designing process, data collection and data analysis.

- Using of non standard measuring units for measuring objects in the classroom;
- Measuring with high accuracy;
- Calibrating a measuring unit;
- Numerical representation of length of object with using calibrated unit;
- Importance of decimal division and decimal representation of length of object.

After each session, video record of session and students' group works and researchers note analyzed and use them for leading teaching experiment in next sessions.

Results

Results of first day (Sessions one and two)

The students used different tools for measuring the length and width of classroom board, such as notebook, math book and A4 paper size. As it was asked them to measure the length and width approximately, so they write these sizes as follow and in term of a complete unit. In fact, the extra parts were neglected. A group of students who chose their notebook size as measuring unit (module) had stated that the classroom is 23 and a half notebook size length. Using the term "half" showed that they know the decimals informally.

In the second activity, it was asked them to measure the length more carefully. In all the groups it was seen that they divided the measuring unit (module) to small sections. The difference between group operations is in the numbers of divisions and choosing denominator. The mathematical symbols which were used in this activity are as the length of a natural number plus a fraction or a Mixed number.

To guide students to a more accurate calibration, they were asked to measure the length of an object that is smaller than the length of their module and write its mathematical symbol in the third activity. In this activity, student should choose smaller objects than unit. They chose the length of pencil lead packet, pen, notebook and etc to measure. The students' performance was divided in three classifications.

- Four groups of students neglect the previous division and creating the new one for their measuring. They changed the number of their part and explained that the length of chosen object is smaller than their unit, so they change the numbers of divisions. Indeed, they divided the length of measuring unit into larger equal parts.
- Three groups of students keeping their previous division. They just divide their previous division again.
- Two groups solve this problem in different way. They divide the unit into the five parts and then divide each part into the five parts again.

Maryam: we divided our set unit into 2 parts. We divided each into 3

Researcher: can you represent the length of measurement in math symbols?

Aida: one second (one a half) and three of this part.

(i.e. 1/2 and 2/3of this part).

Researcher: would you please represent it in math symbols?

Aida: we should calculate it.

Maryam: we should add 1/2 and 2/3.

Marjan: no, it's not the 2/3 of the notebook length. It is 2/3 of one half of its length… It means its 5/6. (She shows it on the picture to her partners).

In fact, this group is faced with a challenge in calculating the length with mathematical symbols. They needed to be able to add the fractions in unequal denominator. Dividing the previous parts into same part numbers, the operation of one group was different. First, they divided the unit into 5 divisions and do it again for each part. They stated:

> ”Although we can re-divide each part into 5 sections and we repeat it again and again for smaller parts. But if we want to notate the length, it will be difficult. For example, we have to write one fifth plus two twenty fifth plus … so, probably the next denominator is 125. Again 125 times 5 … then set the common denominator…”

The fourth group operation was, writing the numbers in base 5. They stated interestingly that they can continue this trend. They expressed that for setting the denominator, it's necessary to multiply the denominator to 5, so it was obvious that they found a regular algorithm for approximating the length of object.

In the classroom discussion, which is performed at the end of the first session, students expressed that writing the mathematical symbol for the length is very important. They have some problems in irregular dividing of parts, so they choose the third method as the best way to calibrate the units of measuring.

Results of second day (Sessions Three and Four)

In the first activity in the second session, the students were asked to express that where in real life have seen the word half and their symbol "half" and its mathematical symbol. Some of the students' expressions were as the following ones.

- We usually say in the grocery store: I need 2.5 kg lentils or in my mother's shopping list it was written 1.5 kg beans.
- The house is 15.5 m length.
- The jar contains 2.5 litter of water.
- The volume of the coke bottle was specified 1.5 liters of water.
- We need 1.5 meters fabrics for making this shirt.

All these written expressions showed that the students have seen and heard of these decimal symbols during shopping, in parents' shopping lists, food labels such as cokes etc. and they understand this concept. The following classroom discussion was conducted:

Researcher: what does one half mean?

Student: it means a half.

Researcher: what does number 5 mean based on 0.5?

Fatemeh: half a kilogram means a half of one kilogram, it is 500 grams. Half a meter means a half of a meter, so it is 50 centimetres. You can find number 5 in both of them.

Zahra: in my opinion, one second is equal to one half, so ½=0.5.

Rezvan: ....so it means that there are many fractions which are equal to 1/2.... For example, ½=5/10=50/100 ...that all of them are half or a half.

Maryam: eureka, eureka. 1/2 is one out of two, so 0.5 is 5 out of 10.

Parisa: that's right teacher. 0.5 is 5 out of ten. It is the same as 50 out of 100 or 500 out of 1000. All are the same and mean a half.

Teacher: could you tell me that what 4 out of 10 looks like? Please, write down your answers on the paper (all the students wrote 0.4).

In this discussion, the teacher tries to help the students using their own knowledge of "half" concept in real world to find an equivalent phrase for ½ in mathematical world. Using the concept of one half and 1/2 and in regards to their knowledge about equal fractions, they can interpret 0.5 by their own and this process shows the horizontal mathematizing. In fact, these students are moving back and forward between the real and mathematical worlds for exploring this concept.

In the second activity in which a one meter ribbon was given to all the groups, all the students divided the ribbon into 10 parts. In fact, it was asked them to specify the decimetres on the ribbon and measure the object in decimetre or one tenth accurately. In other words, they realize the relationship between number 10 in denominator and decimal numbers go out one decimal place.

According to the class discussion in the previous session, the students divided decimetres into 10 parts for more accurate measuring and made the centimetres. They made a calibrated ribbon in centimetre accuracy. In fact, it was asked them to specify the decimetres on the ribbon and measure the object in decimetre or one tenth accurately. In other words, they realize the relationship between number 10 in denominator and decimal numbers go out one decimal place (figure 2).

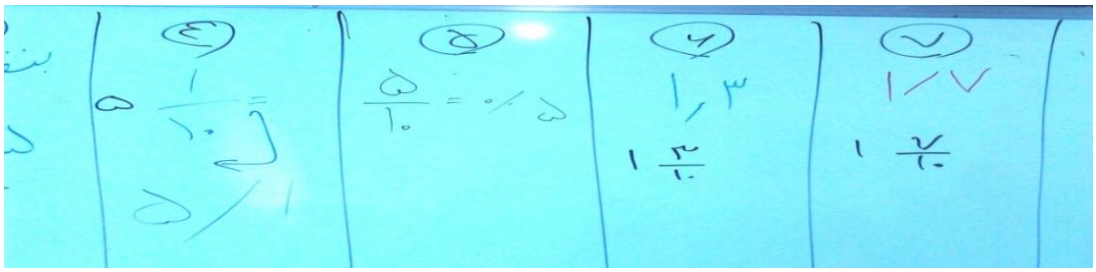

Figure 2: related to measuring the length of objects by calibrated ribbon in decimetre accuracy

When the teacher pointed that the ribbon is one meter length while one of the students said:

“That’s interesting, we divided the ribbon into 10 parts, and re-divided it into 10 so it means we have one meter into 100 divisions. These are centimetres. We knew one meter is 100 centimetres but I perceive it now”

The process of making a calibrated tape measure helps the students to realize the relationship between meter, centimetre and decimetre.

Discussion and conclusion

In the first session of teaching test, the students are allowed to calibrate a measuring unit freely but they had no idea about how to calibrate the unit. In the results of this study, it was seen that they faced a challenge with writing the mathematical symbol for the length, in the other words, these challenges caused to create a regular algorithm for calibrating the unit of measuring. The results of this study showed that choosing a division in base 10 is not natural. In the experience of Astuti (2014), the idea of direct division in based 10 was provided by researcher but here it was tried to conduct the students to dividing the unit in base 10.

This study confirmed the Van de Wall (2001) quote as said "students' mind is not like a whiteboard as entering the class". As it was mentioned, although the students had not learnt the decimal numbers formally and before the research but they used them in their real life frequently or found them on some objects covers, their parents' notes and etc. also they had seen the decimal numbers and separator mark. As Freudenthal (1991) stated, using students' background experience and information could help them in learning decimal notation and its concept. Measuring the length of an object for many times and in more accurate ways help them to realize the meaning of the digits after the decimal point. In fact, when they divided one meter into 10 sections and re-divided it again and again it means that they realize the position and concept of decimal and centesimal scales. One of them expressed, for more accurate scales; we can divide the centimetres into 10 and make it smaller and smaller. In fact, he noted the millesimal position and could guess the decimal demonstration correctly.

This study represented that if the classroom environment changes and better information in regards to the students’ real life experience are provided they can find new unknown mathematical structures by discussing and talking to each other and develop their mathematical knowledge.

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Teachers' beliefs about mathematical modelling: An exploratory study

Paola Ramirez

University of Bristol, Graduate School of Education, England; paola.ramirez@bristol.ac.uk

Drawing upon a theoretical framework based on beliefs, learning and teaching of mathematical modelling as well mathematical modelling itself, this paper explores mathematics teachers´ beliefs about these themes. Based on responses to an online questionnaire, teachers' beliefs, experiences, and mathematical modelling lessons were shared from the teachers' perspective; several similarities with the cyclical process of modelling emerged, as well as sharing a new point of view of the aspects that teachers consider as mathematical modelling for example, real-life situations and processes behind mathematics problems.

Keywords: Teacher beliefs, mathematical modelling, mathematical model, learning and teaching mathematical modelling.

Introduction

Mathematical modelling involves the development of models to explain real-world situations. Such models allow for making predictions, explaining phenomena, making decisions, and disseminating knowledge (Schichl, 2004). In addition, learning mathematical modelling is a cyclical process in which pupils' study a problem derived from the real world and create a mathematical model to explore, predict and explain in order to provide a solution to that problem (Mason & Davis, 1991).

Mathematical modelling may also be seen as an approach to learning that uses elements of reality to create models with mathematics. In this approach, students work together in a cyclical process that involves different stages, such as formulating a mathematics problem based on a situation in real life, setting up a mathematical model that explains the problem, attempting to find a mathematical solution for the problem, explaining the model and interpreting the solution, and comparing the solution with the original problem in real life (Mason et al., 1991; Blum & Borromeo, 2009; Lawson & Marion, 2008). It is important to note that while it is true that the students are engaged in the modelling process in a lesson, it is the teachers who initially implement the strategy, for example, by choosing the modelling task.

In this context, previous studies have suggested that, when applying mathematical modelling, teachers should consider, for example, 'teachers have to know ways how to support adequate student strategies for solving modelling task' (Blum et al., 2009, p. 54). Complementary to this idea, Tekin Dede and Bukova Güzel (2016, p. 1) suggest that 'some researchers indicate the teacher are not be sure about how they should act in this implementation process. Especially the teacher who are novice or have not enough experience in modelling can have difficulties in this process'.

Considering that the role of the teacher appears to be a crucial part of the development of the cycle of mathematical modelling; the question then arises of what beliefs teachers have about mathematical modelling that they could share with their students. As Kaiser (2006, p. 399) notes, 'teachers and their beliefs concerning mathematics must be regarded as essential reasons for the low realization of applications and modelling in mathematics teaching'.

Bearing these factors in mind, this study seeks to examine mathematics teachers' beliefs about mathematical modelling through an online questionnaire about teachers' backgrounds as well as their opinions about mathematics classes, mathematics models, and modelling. This study is the first part of an ongoing process of a doctoral research project; the aim of this paper is thus to examine teachers' beliefs and practices related to mathematical modelling in order to discuss future implications when conducting mathematical modelling.

Theoretical framework

Thompson (1992) discussed the idea that teacher beliefs about the nature of mathematics should be considered – for example, concepts, meanings, and rules, among others – as well as teachers' beliefs about teaching and learning mathematics. In addition, as Stipek, Givvin, Salmon, and MacGyvers noted in their study on evaluating teacher beliefs, we should consider (2001, p. 213):

(1) the nature of mathematics (i.e., procedures to solve problems versus a tool for thought), (2) mathematics learning (i.e., focusing on getting correct solutions versus understanding mathematical concepts), (3) who should control students' mathematical activity, (4) the nature of mathematical ability (i.e., fixed versus malleable), and (5) the value of extrinsic rewards for getting students to engage in mathematics activities.(6) Teachers self-confidence and enjoyment of mathematics and mathematics teaching.

Handal (2003, p. 47) states that teacher beliefs can be related to 'what mathematics is, how mathematics teaching and learning actually occurs, and how mathematics teaching and learning should occur ideally'. It would appear that beliefs about mathematics share common themes with the field of mathematics itself, as well as with teaching and learning mathematics. Indeed, many researchers currently study beliefs about mathematics, practices, and teaching, so it is natural that many researchers would study beliefs within specific areas of mathematics (for example, mathematical modelling), since mathematical modelling is part of mathematics.

When examining beliefs about mathematical modelling, we should consider that 'beliefs in the context of mathematics education can be classified as beliefs about mathematics (as a science), beliefs about the learning and teaching of mathematics [,] and teacher self-efficacy beliefs' (Mischo & Maaβ, 2013, p. 22). According to those authors, the first aspect – mathematics as a science – refers to the formal aspect of mathematics as theorems, rules, problem-solving, and applications as subjects to be learnt. In this context, mathematical modelling is part of this frame, since it is a process that involves a task that takes elements from reality to be explained mathematically. The second aspect – learning and teaching mathematics – refers to constructivist and socio-constructivist views as a way of teaching modelling. Finally, teacher self-efficacy refers to teachers' beliefs that should be carried out and activities that should be implemented in order to reach the teacher's goals within a lesson.

Ärlebäck J. (2009, p.2100) posits that in order to understand the beliefs from teachers about mathematical modelling and models it is necessary to take account of 'beliefs about the nature of mathematics, real world (reality), problem solving, school of mathematics and beliefs about applying, and applications of, mathematics'.

Previous studies from the literature review, however, do not seem to have taken into account the 'perceptions and beliefs about mathematics [that] originate from past experiences' (Mutodi &

Ngirande, 2014, p. 432); perhaps these studies would have had more to offer if they had included relations with past experiences, because if teachers have a history with mathematics (in particular mathematical modelling, because it is part of mathematics), then their beliefs about the field can be related to the background and experiences they have lived: for example, when they teach or learn mathematics.

According to the literature, we may observe that teachers' mathematics beliefs may be classified into different topics; these topics stem from the experiences that teachers have had with mathematics, either through teaching or learning or when they themselves have studied the subject. In this study, considering the idea that mathematical modelling is part of mathematics, teachers' beliefs about mathematical modelling will be classified into three dimensions on the basis of the literature review: (1) mathematics in itself, in particular considering mathematical models and modelling; (2) beliefs about learning and teaching mathematics modelling, considering students, behaviours, lesson planning, and task design; and (3) real-life experience with mathematical modelling, which means that the history described by each participant will be taken into account in relation with mathematical modelling and models, since our beliefs about mathematics stem from our past experiences. It is important to note that this third dimension cannot be separated from the first two, because the experience gained from any context will be enriched by the mathematics itself as well as by the experiences of teaching and learning mathematics.

Methods

The aim of this study was to explore teachers' beliefs about mathematical modelling, therefore an exploratory research was adequate because this attempts to 'seek new insights' (Robson, 2002, p.59) on teaching mathematical modelling in light of my considerations in the introduction, consequently gaining familiarity with the beliefs of mathematics teachers.

Bearing in mind that beliefs are related with mathematics on different dimensions, the selection of the participant was through an activity that is related to mathematics, such as teaching mathematics, conducting research in mathematics education, or studying mathematics itself. Consequently, three of the participants were Chileans who worked at a university in Chile in the mathematics faculty where they are training future mathematics teachers; six of the participants were from the United States, where they worked at high-need schools[1]. The richness of these participants, helps me to have an international overview of the beliefs in this area.

In order to research the beliefs of mathematics modelling, an online questionnaire was designed, comprising ten structured and open-ended questions based on the literature review about mathematical modelling with a focus on the teachers' relationships with the field of mathematics education, mathematical modelling and models. In addition, there are similarities between online questionnaires and structured interviews, in that the researcher has the same direct pre-established questions for each participant without giving interruptions among questions. Those similarities,

[1] 'The school is located in an area in which the percentage of students from families with incomes below the poverty line is 30 percent or more; or in an area with a high percentage of out-of-field teachers; is in an area in which there is a high teacher turnover rate; or is in an area in which there is a high percentage of teachers who are not certified or licensed; is within the top quartile of elementary schools and secondary schools statewide, as ranked by the number of unfilled, available teacher positions at the schools' (No Child Left Behind Act of 2001, 2002, p.115, STAT.1656).

made me consider and exploring the online questionnaire as a way to approach at teacher beliefs. Furthermore, taking into account the limitations of distances between countries and accessibility, the online questionnaire was adequate.

The online questionnaires were distributed between December 2015 and January 2016. The answers were transcribed and analysed based on categories described above; the way that I used to analyse the answers was considering all of them, highlighting common factors that emerged and after that observing how these related to the theoretical framework described previously. It would have been preferable to include the questions, but this was not possible due to space limitations.

Because this is an initial study on mathematics teachers' beliefs about mathematical modelling, it is impossible to generalise from this point of view, but the study can present an opportunity to explore what occurs in beliefs of teachers about this subject on persons related with mathematics but not necessarily those currently working with mathematical modelling and thus provide insights within this large and expanding field, for example, when an implementation is carried out by teachers.

Results

A few representative teacher examples are provided below to illustrate their responses; the responses were transcribed whole, to prevent loss of fluency. In the transcripts, certain parts of the texts have been put in italics font below for emphasis, showing the common factor highlighted on the analysis process. In addition, 'TUS' refers to teachers from the United States, while 'TC' refers to teachers from Chile. The number next to the initials indicates the person who has answered the question.

Mathematical modelling

Real-life situations and processes behind mathematics problems

According to teachers' responses about mathematical modelling, for example, what comes to mind when you think of mathematical modelling, teachers' beliefs may be categorised in two ways: (1) real-life situations or real-life problems and (2) the process involved in solving mathematics problems. Both considerations also take mathematics itself into account: more specifically, the real world and the application of mathematics, as Ärlebäck (2009) has noted.

TC1: *Real situations*, math representation, solving, and interpretation.

TC8: It came to my mind to think of applied math, i.e., to *relate math with nature* or a *daily situation.*

TUS2: I explain to the students that we use mathematics to *model situations in real life* to be able to understand them better and, if possible, to find a solution to the situation … I think of it as visuals that will simplify the situation I am reading. I think of equations or systems of linear equations that will allow me to find solutions.

TUS4: I think mathematical modelling means *showing the students the thought process that is involved in solving a math problem.* It's the problem-solving techniques in regard to a given (abstract or *real-world*) math situation – explaining which

method, formula, etc., is going to work and why. That's what I think mathematical modelling is.

TUS5: The teacher needs *to model the way a problem should be processed* and thought through in order to come up with a valid solution.

As the reader may have noted, there is a link between the beliefs about mathematical modelling and the 'real situation' which can be interpreted, for example, with real life and is related to the modelling cycle mentioned previously. However, there is also a distinction between these beliefs because to others 'modelling' means modelling through a 'step by step' approach showing the process behind a mathematics problem.

Mathematical models

In response to the question 'what do you think about mathematical models?', some of the teacher beliefs about modelling are related with the use of models within mathematics; in some of the teachers' responses, the teachers' beliefs were related to the nature of mathematics itself (Ärlebäck, 2009; Thompson, 1992; Handal, 2003; Stipek et al., 2001; Mischo & Maaβ, 2013) as well as with mathematics learning (Handal, 2003; Stipek et al., 2001; Mischo & Maaβ, 2013). In accordance with Mutodi and Ngirande's work (2014) about experience and beliefs, some of the teachers mentioned their past experience with models, when they were responding to 'what is your opinion about the statement: 'Modelling is everywhere'' (Mason & Davis, 1991, p. 9).

TUS7: Taking real-world data and using graphing tools and computers to assist in *obtaining models*.

TUS6: I think a mathematical model is a key factor in *describing measurement that is located in space*. [Models are] essential for describing the composition of matter in our universe.

TUS2: During college, we used mathematical models in *calculus and geometry* as well as *numerical calculus and number theory* … I agree with 'modelling is everywhere'. I believe some of the models are more elaborate than others, but in general, *I can see everything being a model of concepts*, including numbers and graphs.

TC9: *Visuals* are necessary; hands-on [teaching] is almost vital for understanding.

From the transcript, it is possible to infer that the teachers viewed mathematical models as ways to understand mathematics with a factor of utility of model, using words such as assist, essential or necessary.

Learning and teaching mathematical modelling

Experience with mathematical modelling

In terms of experience in modelling, some of the teachers mentioned that they had studied modelling at their universities (i.e., dynamic systems and geometry, among others); they also discussed their experience in teaching (or not having experience in mathematical modelling). This situation relates to what Mutodi and Ngirande (2014) discuss in their work: our beliefs about

mathematics come from our experience with mathematics. In addition, some of the teachers recognized that they had limited experience with mathematical modelling, or they lacked the time to do it; one of the teachers, however, also mentioned her intention to learn more about modelling.

These ideas could help to explain why teachers are often unsure of how to act when they work with mathematical modelling, which results in necessary (and time-consuming) planning during the implementation stage, as mentioned by Tekin Dede and Bukova Güzel (see the introduction to the current paper).

TUS2: As a school teacher, I would love to be able to *have the time to read and understand more about mathematical modelling*.

TUS3: *I have not spent nearly enough time* doing mathematical modelling.

TC8: My experience with *mathematical modelling is with dynamical systems*. In this area, you can find many examples of different situations where modelling is present. In the classroom, you can use simple examples and particular cases of this area to show the students.

Mathematical modelling class

After the question, 'how do you imagine a mathematical modelling class?', several teacher beliefs have presented several similarities when using elements from reality yet only one teacher mentioned focussing on models, as has been mentioned by several authors in this paper (Mason & Davis, 1991; Blum, 2009; Lawson & Marion, 2008); In addition, teachers' beliefs as to how mathematics teaching and learning can occur as Handal (2003) described previously were present. For example, some of the teachers' responses in the present study suggested that mathematical modelling lessons could be very creative, active, and didactic, where the students could take part in the learning process.

TUS2: I imagine [that mathematical modelling class] *focusses on models* more than a specific area of mathematics.

TUS6: Let's take, for example, the concept of geometry. *The setting within a classroom could be used as mathematical modelling*. Without going outside the four walls [of a classroom], one could introduce to students the concept of angles, lines, planes, perpendicular and parallel lines, congruent angles and similar figures, and so on … I think of [mathematical modelling class] *as very creative*; it is an atmosphere where *students are launched into using their minds* so that they will become creative, inquisitive, and analytical… Mathematical modelling is one of the most *intriguing, creative, and thought-provoking subjects that one can teach*. It blends into other subjects, such as art, physics, and chemistry.

TC8: I imagine a mathematical modelling class as being very *active and didactic – students working in groups and discussing the problem that has been assigned*.

Discussion

Through this study, we can observe that the beliefs are quite similar in both countries. Teachers link mathematical modelling with situations to be modelled; this situation stems from real-life

experiences. This belief shows a similarity with mathematical modelling as a subject that can involve the development of models to explain real-world situations (Schichl, 2004). One teacher, however, said that mathematical modelling was how 'a problem should be processed'; in this case, the teacher's beliefs were not related to real-life situations but to the process involved in solving mathematics problems. Perhaps this particular belief about mathematical modelling could be understood in light of some of the teachers' limited experience with mathematical modelling or, as Tekin Dede and Bukova Güzel (2016) have stated, because the teachers were unsure how to act when working in a class that involves mathematical modelling.

On the other hand, teachers' beliefs about mathematics modelling classes in general showed that they felt that classes could be very creative, intriguing, and thought provoking; as one teacher said, they could be taught with the participation of the students. One teacher did recognise, however, that it can be difficult to break down some students' beliefs. Even so, other teachers stated their intention to learn more about mathematical modelling.

In terms of mathematical models, sometimes teacher's beliefs referred to the utility of using a model as a way to demonstrate mathematics concepts. In this sense, mathematical model beliefs are related to the nature of mathematics (Mischo & Maaβ, 2013; Stipek et al., 2001; Thompson, 1992; Handal, 2003; Ärlebäck, 2009) as well as with the experiences that they have had in the past (Mutodi & Ngirande, 2014). Then the questions that naturally arise include, In which ways can the utility beliefs of a model influence the implementation of mathematical modelling? What kind of decision does the teacher take during an implementation? What types of feedback can the teacher give to the students?

Finally, through this particular study and in consideration of teachers' beliefs about mathematical modelling, more questions and insights have emerged; for example, how to lead an implementation of the mathematical modelling cycle that would bear teachers' beliefs in mind, and how can the usefulness and reliability of online questionnaires be linked to carry out an exploratory study on beliefs.

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Didactical activities to engineering training: Methodological proposal

Avenilde Romo[1], Santiago Tolentino[2] and Rebeca Romo-Vázquez[3]

[1,2]National Polytechnic Institute, CICATA, Ciudad de México, México; aromov@ipn.mx; santiago.tolentino1@outlook.com

[3] CUCEI, University of Guadalajara, Guadalajara, México; rebeca.romovazquez@gmail.com

In this paper we present a methodology to design didactical activities for training engineers. One phase of this methodology is selecting an extra-mathematical context, allowing identification and analysis of mathematical models used by engineers. We selected an industry beer context and we identified the Pareto chart as a tool to solve different problems, for example faults in production lines. This work uses elements from Anthropological Theory of Didactics. We present a praxeological analysis as basis of didactical activities.

Keywords: Beer industry, Pareto chart, training engineer, didactical activity.

Introduction

Over the past 30 years, mathematical modelling and applications have been a subject of study and development in the field of Educational Mathematics, as in the ICMI 14 study, edited by Blum, Galbraith, Henn and Niss (2007). Mathematical modelling and applications are activities in which math is used to solve problems in various contexts (math, engineering, economy, medicine, etc.). They are based on developing competencies associated with the application of mathematics and constructing mathematical models in extra-mathematical contexts (Niss, Werner and Galbraith, 2007). However, mathematical modelling and applications are not widely used in primary and secondary education, which give more importance to concepts and procedures. At the university level, modelling has emerged as a new educational paradigm (Bissell & Dillon, 2000). In the future, teaching professionals will be required not only to create but also adapt mathematical models to solve practical problems (Bissell & Dillon, 2000) related to interpreting solutions proposed by other professionals and to employ technology to perform mathematical tasks. This demands teaching math in such a way that it includes the use, adaptation, handling, and interpretation of mathematical models in order to adequately deal with tasks in extra-mathematical contexts. One of perspectives that comes closest to this demand is known as the realistic perspective, which is described in the proceedings of the Topic Study Group 21 (TSG21) of the 11th International Congress on Mathematical Education in 2008, as follows: "In this perspective, mathematical modelling is viewed as applied problem solving and a strong emphasis is put on the real life situation to be modelled and on the interdisciplinary approaches" (Blomhoj & Carreira, 2009). In this work, we consider the work of Kadijevich (2009), which illustrates this perspective, is a very interesting approach to the mathematical modelling due to the type of problems chosen to address it and to the use of information technology that is proposed. However, we believe that is not clear, from a theoretical point of view: how do you choose the actual context? How is a modelling activity generated in the context chosen? How is analyzed their relevance to the classroom? In general, it seems that the question of choosing the extra- mathematical context in order to propose the modelling activity has been theoretically little worked, it appears as a little cloudy element, leaving the emphasis on the characteristics of didactic activities (Galbraith & Stillman 2006) and on the modelling cycles that

allow to describe and analyse it (Blum & Borromeo Ferri, 2009). In an effort to attend to this lack of theoretical framing of the choice of additional mathematical context for the design of didactic activities, we have proposed a methodology based on the extended praxeological model (Castela & Romo, 2011) for designing didactic activities based on modelling that incorporate elements of the use of models in real contexts.

Elements of the Anthropological Theory of Didactic

Since the extended praxeological model (Castela & Romo, 2011) is based on the Anthropological Theory of Didactics (ATD), we present in this section some of its elements. The ATD is an epistemological model that allows the study of human activity in its institutional dimension. An institution is a stable social organization that defines the human activities generating resources that make them possible. These materials or intellectual resources, which are made available to the subjects, have been produced by communities along the confrontation of problematic situations with the objective of solve them in a regularly and effectively way (Castela & Romo, 2011). The classic praxeological model, proposed by Chevallard (1999), recognizes the praxeology [T, τ, θ, Θ] as a minimal unit of analysis of human activity. Its four components are: the task type (T), the technique (τ), the technology (θ); and the theory (Θ). The 'task' refers to what is to be done; the 'technique' is how it is to be done; the 'technology' is a discourse that produces, justifies and explains the 'technique'; while the 'theory' produces, justifies and explains the 'technology'. Mathematical praxeologies or mathematical organizations can be of different level and they serve to a hierarchy of levels of determination proposed by (Chevallard, 2002). Mathematical institution imposes a model of subjection to the mathematical praxologies: it rests on a structure that organizes praxeologies in different interlocked levels that are in increasing order of size as follows: specific, local, regional and global. The most basic level of a mathematical organization is the punctual [T, τ, θ, Θ] and it has only one technique for performing such tasks. The next level is the local, which groups all punctual mathematical organizations associated with the same θ technology. The regional level regroups all punctual organizations associated with same theory Θ, global or domain regroups certain regional mathematical organizations; discipline is the top level and combines all domains.

Codetermination of the mathematical and didactic

Chevallard (2002) develops the model presented below, in order to take into account the subjections that weigh on the didactic organization of the study of praxeologies. In this work, the author notes that didactic organizations cannot be developed if they are found far from higher levels, domain and discipline; reciprocally these levels cannot be imposed without considering the conditions of the educational institution. In that sense yields a co-determination of mathematical and didactic organizations.

> […] each level imposes, at a given moment during life of educational system a set of constraints and support points: ecology that results is determined both by what restrictions prohibit or drive, and the exploitation that actors make to the support points that different levels offers. (Chevallard, 2002, p.49)

As you can see the fact that Chevallard be interested in teaching leads to extend the range of levels. He introduces three higher levels: society, school and pedagogy, noting that levels of domain and discipline are also subject to restrictions imposed by these three levels that complement the scale

upward: Society → School → Pedagogy → Discipline → Domain →Sector → Local → Specific. This hierarchy makes us consider that the study of mathematical praxeology or modelling praxeology in an institution that is subject to various restrictions imposed by institutions of higher levels.

Moments of the study

In the frame of ATD, the study is seen as the construction or reconstruction of elements of a mathematical praxeology, in order to perform a troublesome task (a task type for which a mathematical praxeology does not exist or is not available). In order to finely describe this process of construction or reconstruction, the ATD proposes a model of study of a punctual mathematical praxeology. This model distinguishes six moments, which are also associated with groups of activities. A moment is a dimension of the activity, a phase in the process of study, which may appear several times but following an internal global dynamics. Chevallard (2002) presents the model as follows:

Group I (Study and Research Activities [SRA])

1. Moment of *the (first) meeting with T*;
2. Moment of *the exploration of T and technical emergency τ*;
3. Moment of construction of *the technological-theoretical block.*

Group II (Synthesis)

4. Moment of *institutionalization.*

Group III (Exercises and problems)

5. Moment of *work of* the mathematical organization (specifically of *the technique*).

Group IV (Controls)

6. Moment of *the evaluation.*

These moments are not detailed in this paper, but they are presented with the aim of showing how the ATD in the process of construction and reconstruction of a praxeology is conceived.

In particular, we are interested in considering group I and generating an SRA: a didactic device for students to construct, in this case, a modeling praxeology involving the Pareto Diagram. One of our questions is, what legitimizes the chosen mathematical modeling praxeology? Why is it important for students to build this praxeology? And in particular, future engineers. For us it is important that the mathematical modeling of the classroom is related to the mathematical modeling of its professional practice, which is seen as a relationship between institutions as shown below.

Training in mathematics and the professional world seen as institutions

In the framework of the ATD, analysis of mathematical activity is considered in its institutional dimension. Given that our proposal is to generate a methodology for designing activities based on mathematical modelling that links mathematical knowledge teach at the training institution and the one used in the professional field, we must identify institutions that are adequate to participate in this process, and their interrelations. Romo-Vázquez (2009) argued that training engineers involves three types of institutions: Production (*P*), where praxeologies are produced, Teaching (*E*) responsible for transmitting the praxeologies. Use or users *Ip*, where the praxeologies are employed. By producing institutions we refer to mathematics *P*(*M*) seen as a discipline, together with its

intermediate disciplines *P*(*DI*), which we will also call Engineering Sciences (e.g. signal processing, control theory, electrical circuits, etc.). Teaching institutions are represented by mathematics *E*(*M*), and the intermediate disciplines *E(DI*); while the practical institution is *Ip*. The latter is examined at two levels: the professional practice of engineers, and the devices that, upon approaching practice, are developed in schools; for example, a project to innovate a product or service. Also taken into account were three inter-institutional tours that can be followed by a mathematical praxeology by going from *P*(*M*) to *Ip*. These can be represented schematically as follows (Figure 2):

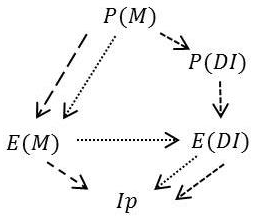

Figure 1: Institutional tours of a mathematical praxeology to go from P(M) to Ip

The transpositive effects (changes that occur upon moving from one institution to another) can be so large that a mathematical praxeology in *Ip* may not be recognized as mathematical. Research by Hoyles, Noss and Pozzi (2000) shows that some professionals utilize techniques and strategies in their practice that are based on mathematical models, but that when automated are no longer recognized as such. This begs the question: what sorts of didactic activities can be generated so that the mathematical models used in *Ip* practice or *E*(*DI*) find a place in methods of teaching mathematics *E*(*M*) (see Figure 2)? To answer these questions we propose the following methodology.

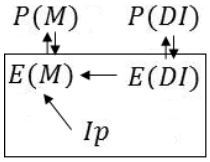

Figure 2: News institutional relations between *Ip*- *E*(*M*) and *E*(*DI*)-*E*(*M*)

Methodology for designing didactic activities (SRA) based on mathematical modelling

This methodology for designing didactic activities based on mathematical modelling emerged from research by Macias (2012). Here, modelling activities are seen as praxeologies (mathematical and/or modelling) to be performed in *E*(*M*), but in relation to praxeologies of *E*(*DI*) and/or *Ip*. It consists of four phases: 1) Selecting an extra-mathematical context, 2) Praxeological analysis and identification of a mathematical model. 3) Analysis of the mathematical model identified and its relation to *E*(*M*) and 4) Design of the didactic activity (SRA) for *E*(*M*).

1) Selecting an extra-mathematical context. First, we must consider the educational level at which teaching will take place, then the contexts where the mathematical applications will be put to use. For example, if we consider teaching programs for engineers, the natural contexts of use are specialty training *E*(*DI*) and professional practice Ip. After that, one must identify some of the elements (resistance of materials, control theory, data structure, among others) that are of macro scale and may include various sub-institutions for the analysis of the modelling activity that occurs there. Selecting this context must be based on an approach to the institution or sub-institution

through interviews with one or more of the subjects involved (e.g. professors, expert users, researchers), a review of relevant documentation (suggested by the aforementioned subjects, and/or one's own search), and visits aimed at identifying the type of mathematical and modelling activity that is used. Specifically, it is important to analyse whether the mathematical models identified as being in use correspond to those that are actually taught E(M); examples could include functions, vectors, matrixes, mathematical optimization, or differential equations, among others. In this way, one can determine whether the context chosen provides a suitable analytical basis for designing the didactic activity.

2) Praxeological analysis and identification of a mathematical model. In this phase mathematical modelling activity is analysed through praxeologies. Modelling activities in an extra-mathematical context may consist of mathematical praxeologies and/or mixed praxeologies.

3) Analysis of the mathematical model identified and its relation to *E*(*M*). A mathematical model that is in use but that is also taught in *E*(*M*) is identified and then analysed through the functions of the technology practice; i.e., describe, validate, explain, facilitate, motivate and evaluate. Describing the model in use allows us to elucidate the reasons relative to context on the basis of which that particular model was chosen to resolve tasks in the extra-mathematical context. Identifying the elements that validate the use of the model, and under what conditions, makes it possible to understand what contextual elements must be considered in designing didactic activities. For example, many mathematical models are used in "ideal" conditions such that they make it possible to resolve certain tasks more easily, though the solutions obtained will later need to be adapted to reality. This adaptation is conducted on the basis of certain elements that *validate* it. Recognizing the *explanations* of use allows us to understand what each element of the model represents and to what degree the model used allows us to model the context (or part of it). Analyzing the elements that *facilitate* the use of the model reveals the process of mathematical modelling, which entails not only assuring that the mathematical model chosen will make it possible to resolve a problem in an extra-mathematical context but also that the resolution reached will be the least complex one. Identifying precisely what it is that motivates the use of the chosen model is a medullar phase in designing didactic activities, but this analysis of use must be complemented by a didactic analysis of the model in the context in which it is taught.

4) Designing didactic activities (SRA). Designing a didactic activity must be based on both the praxeological analysis of use (praxeologies present in an *Iu*) and the mathematical model identified; *i.e.*, one must recognize the praxeologies of both use and teaching in order to perceive the relations between them. One must choose the types of tasks that, because they emerge from use, can be adapted for a type of school task; for example, studying the behavior of a continuous signal, determining the total cost of an inventory, or calculating the inverse of a mixed matrix, etc. These types of tasks require mathematical techniques that may be school-related (being part of curriculum) but are also used in the professional field, so mathematical and non-mathematical technologies will have to be built by the students (3rd moment) in order to validate, explain and justify techniques that emerges when confronting the problematic tasks. Specifically, technologies of use (part of practical praxeologies) must be adapted in such way the students can build them in the first college courses. The objective of the didactic activities must be oriented towards the type of praxeologies that figure

in the activity, whether this be constructing, mobilizing, or searching, for knowledge. It is intended that this activity may be an SRA which allows building a praxeology of modelling.

The context proposed: Pareto chart in beer industry

To make a praxeological analysis we are chosen a Beer industry that is constituted by ten production lines and produce about one million hectoliters of beer per month. Apart from domestic beer this industry produce lots of beer for export to countries in all continents. To meet domestic demand and shipments abroad, it requires each of the ten production lines, meet high levels of efficiency. However, in each production line problems requiring immediate attention in order to achieve the planned goals they are presented. Thus various problems for the maintenance of thousands of machines, components and parts that make up each of the production lines are also presented. And logistics to control the flow of materials required in the production and control of shipments. In Tolentino (2015), these questions are studied: What Mathematics used in this industrial environment? Are there some common mathematical tools to manage the wide range of problems in the brewing? Tolentino, was both a master's student of mathematics education program and he was working in the industry as an engineer, he found that a Pareto chart is used to solve problems in the industry. This chart is based on the principle that if 20% of the causes of problems is attended a solution of 80% is obtained in effect.

Praxeology faults problems

Type of task. Identify the most important causes of faults in production lines of an industry.**Task.** To solve faults (time) that arise in different production lines of beer production. **Technique. Step 1. Data collection.** It is recorded in a table stop time of production lines due to operational faults, faults in machinery or material defects. **Step 2**. Is ordered from highest to lowest the time column of this table, the percentages of stop time are obtained for each line, in relation to the accumulated from 157.62 hours. A column for cumulative percentage is added.

Line	Wasting time (h)	Percentage	Cumulative Percentage
Line 2	41.02	26.02	26.02
Line 8	25.65	16.27	42.30
Line 12	24.34	15.44	57.74
Line 6	14.73	9.35	67.09
Line 10	14.17	8.99	76.08
Line 3	13.8	8.76	84.83
Line 11	11.03	7.00	91.83
Line 5	8.33	5.28	97.11
Line 4	4.55	2.89	100.00
cumulative	157.62		

Table 1: Wasting time of production lines

Step 3. Two graphs are performed in Excel: in the bar graph the columns for each line indicate stop time. Graph of the Lorenz curve or cumulative percentage. On the right vertical axis measures the time from 0 to 41.02 h. For the Lorenz curve on the right vertical axis the percentage of 0 is measured at 100 percent (See figure 3). **Step 4.** Identify the diagram. Finally add a line from the 80 percent that intersects the graph of the cumulative percentage, and descends to the x axis, to separate lines that are to the left of this line it is the line 2, 8, 12, 6 and 10 (see figure 4). These are the lines that generate 80 percent of the total time of overall production stoppages during this time period.

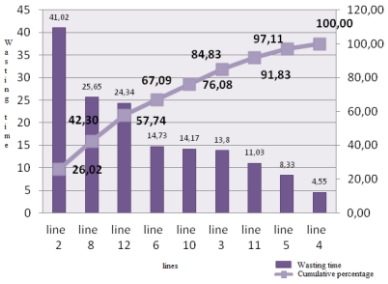

Figure 3: Bars and Lorenz curve

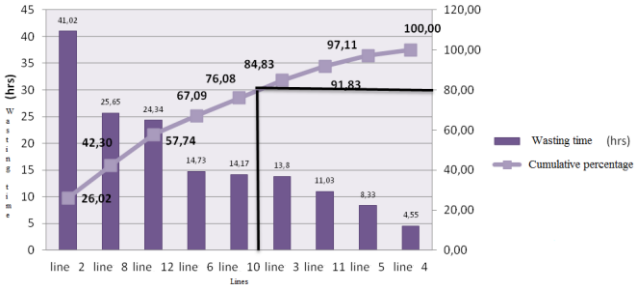

Figure 4: Analysis of the Pareto Diagram

Step 5. Analyze the Pareto chart. As a result of the above analysis, attention is directed to lines that generate 80 percent of stop time, *ie* lines 2, 8, 12, 6 and 10. Therefore proceeds to Pareto analysis of each one of these 5 lines, starting at line 2. Leaving as trivial the 4 lines: 3, 4, 5 and 11. That is although the line 11 was stopped 11 hours during this week, is considered out of importance according to the analysis Pareto realized. In this case, the Pareto Diagram is used again to analyze lines 2, 8, 12 and 6, to make decisions about the elements that must be repaired urgently, in each of the lines. The Pareto Diagram is made by engineers and allows them to identify the main causes of the problem, however a deeper analysis of the industry is necessary to recognize how the practical and theoretical knowledge allows solving the identified causes. This analysis is repeated in the brewing industry several times to have elements to act and address problems optimally, using fewer resources and obtaining the greatest benefits.

Conclusion

We consider that this praxeological analysis (phase 2 of methodology), briefest presented, allow us to see the importance of Pareto chart on the beer industry. To design the SRA, it is necessary to analyze university courses likely to identify a local Praxeology the Pareto chart. However, analysis of the Pareto chart of the beer industry gives elements for SRA: from data of different problems ask propose a model that allows the company to identify the major causes of the problems. Considering the problem of faults, you can ask students a model to identify the line that causes the greatest wasting time or the lines that cause 50% of the strikes, then 75% and then 80%. The interesting thing about this proposal may lie not only that students can work with data from businesses, but they reach handle the Pareto Principle, the 80-20 ratio. The Pareto Diagram also involves mathematical elements that support them as the Lorenz Curve, which has hardly been mentioned here. In Tolentino (2015) the mathematical analysis of the origin and evolution of the Pareto Diagram is presented and this must also be considered in the design of the SRA, involving three institutions $P(M)$, $E(M)$ and Ip.

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Mathematical modelling in an archaeological context: Their complementarity as essential tool for inquiry

Gemma Sala[1], Vicenç Font[2], Berta Barquero[3] and Joaquim Giménez[4]

University of Barcelona, Faculty of Education, Spain; gsala@ub.edu[1]; vfont@ub.edu[2]; bbarquero@ub.edu[3]; quimgimenez@ub.edu[4]

This paper focuses on analyzing the potentialities of using a codisciplinary inquiry context to enhance the appearance of mathematical modelling in teaching and learning practices. We focus on an archaeological context where modelling becomes an essential tool to enquire into and progress on the study of questions emerged from the interplay between two main disciplines: Mathematics and History. We present the design, implementation and analysis of a teaching sequence based upon the ruins of a Roman theatre discovered in Badalona (Catalonia) to promote inquiry and modelling students' competences. The sequence was implemented in 2015 with 12 and 14-year-old students. The central question introduced (what kind of building could have been the discovered ruins) involved students in a reflective inquiry process that facilitate them to progress on modelling, at the time modelling facilitate that the inquiry process could follow.

Keywords: Inquiry competence, mathematical modelling, co-disciplinary context, geometrical model, task design.

Introduction

The main aims of our research are to (1) design codisciplinary teaching sequences to promote inquiry students' competence, (2) implement them and (3) analyze their affordances to enhance inquiry and modelling students' competences. In particular, this paper focuses on the design, implementation and analysis of a teaching sequence based upon an archaeological context—the ruins of a Roman theatre discovered in Badalona (Catalonia, Spain) and how modelling became an essential tool to enquire into several questions that emerged from this extra-mathematical context, in order to progressively build up the necessary subject knowledge to provide and validate answers.

As aforementioned, the sequence started with an initial situation that leads to the formulation of the specific initial question and became the core of the students' research. The starting situation which is presented to students is the discovery of some Roman ruins in a Badalona's suburb by the archaeologists' team of the city Museum. The archaeological report that is provided explained that these ruins could belong to an ancient Roman building, but *what kind of building could it have been? Could be a theatre? A circus? An amphitheatre?* In order to face these questions, it is essential to make students unfold their inquiry and modelling competences and help teachers to guide student's reflections and to organize inquiry activities so that (mathematical and historical) knowledge could emerge and be used.

In the research we present, the codisciplinary context here considered for the design of the sequence —an archaeological context adapted from a real research— presents several advantages, which will be further developed in the following sections. On the one hand, requires that Mathematics and History work together to face most of the questions that appear in the teaching and learning

sequence. This no so common complementarity between both disciplines allows us to provide authenticity (Vos, 2011) to the questions faced and to promote the blend of inquiry and modelling in different steps of the study, as also some previous investigations have shown (Sala, Giménez & Font, 2013; Sala, Barquero, Font & Giménez, 2015). On the other hand, this codisciplinary context is enough close to students (as it is located in the same city that the school is) that facilitates that students can have access to the real ruins and to experts' answers about their real investigations, providing legitimacy to their inquiry process and responses. After describing the main theoretical frameworks upon which the design of the teaching sequence was proposed and analysed, we present the main research questions about how the use of codisciplinary inquiry contexts can help to break disciplinary boundaries (in particular between Mathematics and History) and about how their interaction leads to a rich complementarity between inquiry and modelling.

Theoretical framework and research questions

The research we present in this paper considers different theoretical aspects. First, it is assumed the notion of *basic competence* considered in Catalan curriculum. More concretely, we are interested in the notion of *inquiry competence* that, following Sala (2016) and framed according to the Catalan curricula of Primary and Secondary school education, it is defined as:

> The ability to mobilise the suitable knowledge and the appropriate resources that facilitate the development and application of a logical and critical methods —under the teachers' guidance— in order to look for and find answers to problematic questions or situations in some school and/or out-school context. (Op. Cit. 2016, p. 64)

In the same line than Artigue & Blomhoj (2013) underlines, an important step in legitimising inquiry-based approaches was the publication of the National Science Education Standards (NSES; NRC, 1996), which called for students to do and know about scientific inquiry, and that teachers should foster the development of inquiry skills. When the inquiry-based approaches migrate towards mathematics, it seems important to consider approaches paying attention to establishing connections among mathematical and extra-mathematical world, such as modelling approaches. In this sense, some of the inquiry requirements (such as making observations, posing questions, examining sources of information, identification of assumptions, considering alternative explanations; NRC 1996, p.23) are closely related to some essential steps of the cycling process through which mathematical modelling is developed. So that, sometimes, it is difficult to find the differences between both processes (see for instance, the modelling and inquiry cycles presented in Blomhoj, 2004) and the linked competences. In particular, as pointed in Niss, Blum & Galbraith. (2007, pp.3-8), modelling constitutes a competence in its own right, which needs to be developed through appropriate modelling activities. From our viewpoint, to promote inquiry mathematical modelling should be placed at the core of the mathematical (and scientific) teaching and learning practices to ensure the right and rich development of important enquire abilities. Also, in the other way round, to promote and ensure mathematical modelling practices, inquiry should nourish some essential steps that may make modelling successfully progress.

Second, we considered different theoretical elements to ensure that the sequence design could offer a rich and functional teaching and learning of mathematics. With this aim, the mathematical and

didactic design quality is justified based on the three criteria of didactic 'suitability' —*emotional suitability*, *ecological suitability* and *epistemic suitability*— proposed by the onto-semiotic approach (Godino, Batanero & Font, 2007). Different aspects of its design considered justify the emotional suitability of the teaching sequence: the students could work with real data evidence —data from the research report of the discovery of real Roman ruins—; they could interact with the archaeologists' team of the city Museum and could share their results with them, etc. In turn, *ecological suitability* was justified by the curricula of these secondary-school students having a competency-based approach. Likewise the implementation allowed students to unleash relevant processes of mathematical activity, in particular processes of mathematical modelling that justifies the *epistemic suitability* or mathematical quality.

Finally, we also use the notion of *research and study path* (SRP) proposed in the framework of the anthropological theory of the didactic (Chevallard, 2015) and their main characteristics to design the sequence of tasks to achieve a high epistemic suitability (see also Sala et al., 2015). The SRP proposal emphasises the necessary dialectics between 'research' or 'inquiry' (facing open-problems, problem posing, making observations, examining different media or source of information, etc.) and 'study' (attending teachers' explanations to provide mathematical knowledge, building up models, testing the validity of mathematical tools, etc.). As underlined in Barquero, Serrano & Serrano (2013), mathematical modelling cannot be considered only as an aspect or modality of mathematical activity but has to be placed at the core of it. Modelling is not only as a way to make the functionality of mathematics visible, but also as a key tool for the functional construction and connection of mathematical and extra-mathematical knowledge. In this sense, inquiry and modelling have to be mutually enriched. On the one hand, modelling activity needs from more inquiry moments (posing questions, looking for external resources, looking alternative proposals, etc.). For instance, in the SRP the role of the questions and of question-posing is essential (as the starting point of an SRP with a generating question, powered enough to pose many derived questions) so that it is essential to help students develop some reflective inquiry gestures such as posing new questions along their particular study or using extra-mathematical context to validate mathematical models and make new questions emerge and thereby starting new loops in the modelling cycle. And, on the other hand, inquiry needs from modelling processes (when mathematical tools and models are build, used analysed, validate, etc.) to make the study progress.

Research questions

The aim of our research is to analyse if the implementation of the sequence we had designed, with features as the ones detailed above, could promote the inquiry students' competences and how modelling could become an essential tool to enquire into the inquiry questions that were emerging in order to provide and to validate answers.. Therefore, our main research questions that we focus on are the following:

> Can a teaching sequence based on the study of an archaeological problem in a codisciplinary context help students to face an inquiry that needs modelling processes as an instrumental tool to find, test and validate answers? Which is the relationship between modelling and inquiry in this kind of teaching sequence?

Design and implementation of the teaching sequence

As it has been introduced, the starting situation is based on the discovery of certain Roman ruins, some years ago, in the centre of Badalona (a city next to Barcelona, Catalonia) by the archaeologists' team of the Badalona's Museum. They concluded that these ruins could have been an antique public building belonging to the classical *Baetulo* (the Roman old name of Badalona), surely a theatre, and they explained their research in an article (Padrós & Moranta, 2001).

The sequence was named «*What are these ruins hiding? Investigating the Roman ruins of Baetulo*» and all the sources, devices designed, links recommended and the worksheets that they had to follow were available in the blog designed by the authors: http://ruinesdebaetulo.blogspot.com/

We implemented the teaching sequence with a group of a 30 students (12-13 year-old) of a Secondary school education in Badalona during two weeks of June in 2015. The selection of the participants was intentionally chosen due to the facility that the first author of the paper had to the school, and the age and syllabus adequacy of the group class. Students worked during all sessions in 'inquiry teams' of three or four members, set up in the first session. They worked as real research teams, formulating hypothesis and doing tasks to validate them, discussing the partial results obtained during the process, finding points of agreement and writing an inquiry report to gather all the ideas, proofs and work done. Each team had on member in charge of explaining and defending their temporary report. Moreover, during all the study process, students got an inquiry guide to help them to progress in such a new activity.

From the archaeological reality to the emergence of questions

The didactic sequence was inspirited in a real archaeological investigation, a situation very close to a real extra-mathematical context, which was introduced to the students by the teacher and the first researcher paper signatory. The aim of the inquiry proposed to the students it is the same of the original aim research: discovering with which kind of building the ruins could be identified. In this sense, the context used to design the didactic sequence was an *authentic* situation (Vos, 2011) because the situation introduced to the students is clearly not created for educational purposes even though some elements are included for educational purposes.

Therefore the students had to investigate —from real data, archaeological reports, and canons of Roman architects— what type of building the discovered Roman ruins could have been and its features. In the first session, they could explore the map of the zone where the ruins were located by the link in the blog. They also visited later the place of the discovery accompany by one of the archaeologists of the Museum of Badalona and asked questions, took photographs, and measurements. The current constructions in this zone, houses and streets, followed a curious curved shape—easily perceptible in the map. It indicated, surely, that all these constructions were built on top of the ancient structures.

The role of the context in the students' process of enquiring

Students looked up and investigated information about the Roman architecture to find what type of buildings had a curved part of their perimeter. They found few buildings that showed this feature at least. For instance, theatres were circular, amphitheatres were elliptic or circuses had a part circular

and other part quadrangular. This fact generated some other questions that promoted the developing of the inquiry process because the students could formulate their first hypothesis about that kind of building the ruins could below: *Which Roman building (theatre, circus, amphitheatre, etc.) shape would concur with the shape of the part of the Roman wall found? What do the geometrical shape of the partial Roman wall discovered determine?*

In the first stage of the process, the problematic situation was introduced to the students by the History teacher and the first author. Then, first questions emerged and the students started to look up information about the public Roman buildings in the links, maps and books. At the end of this stage most of the students should have understood the context of problem. So, they could formulate their preliminary hypothesis and conjectures about which kind of public building the ruins could have been only based on the historical information found. So that, at this stage when they had only took into account this historical information, their assumptions included more than one building (all the buildings that could have a curved part). Each inquiry team had to write in their report the agreed hypotheses they reached.

Actually, the inquiry was based on a mainly discovery: a part of a Roman curved wall, a metre and half high. This partial wall belonged to a building that was the centre of the inquiry. It was the external wall of the public building and determining it shape could mean to know which type of Roman building it was. The students had to follow their inquiry from the study of this element and the context where it was find. The number of types of Roman buildings existing limited the quantity of different type of shapes the students had to consider; but other information deduced from de context directly as the orography and the dimensions of the place where the ruins were discovered, or the dimensions of the curved Roman wall discovered, also limited the options that the students could choice in order to formulate their hypothesis about the building the ruins would fit with. In this case, the context influence on the inquiry helps students to deal with a problem that would be very difficult to resolve from an only mathematical point of view and completely unachievable to their level.

Improving the first hypothesis: Building up models to systematise and mathematize the archaeological system

The second stage started with an important session that had the objective of discovering the geometrical shape that the Roman wall described (a circumference, an ellipsis, etc.) and test their first hypothesis. From this stage the teachers of Mathematics, Technology and Catalan Literature had involved in the sequence management. The results obtained from this central task provided enough data that allowed formulate sufficient plausible hypothesis about the type of public building.

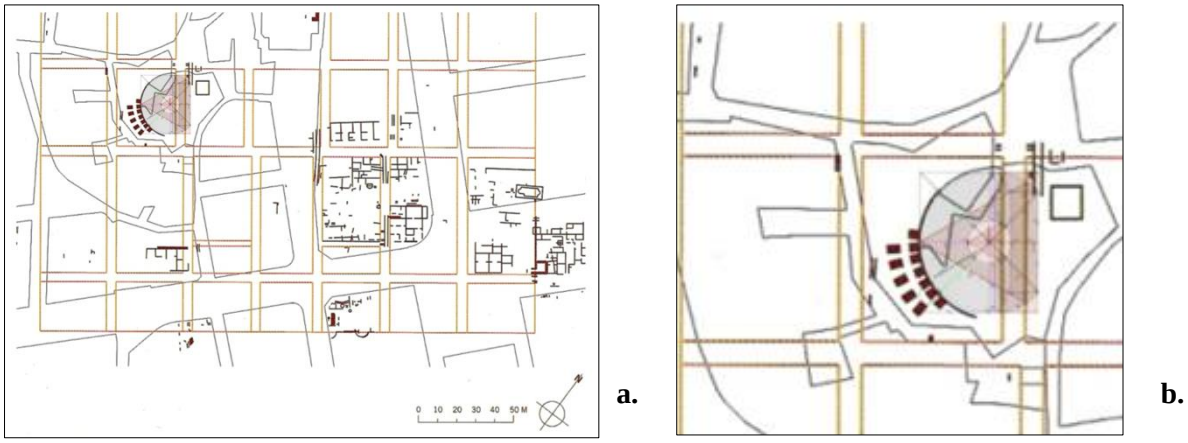
The session in where the problematic situation began to be systematised and mathematized were placed on the public square next to the school, and was recorded and afterwards analysed. The students work with an exact representation of the part of the roman wall drawing on the ground. They could experiment with different ways to proof what was the geometrical shape that fit with the Roman wall. After few tasks in which students had to construct and drawing different curves on the ground, all the teams could check that the perimeter of the curve Roman wall fits with a circumference and so, they could conjecture about the building would have been a theatre. Also,

this evidence lead students to think that it was necessary to find the radius of the circumference in order to know how large the building was. The students find the radius of the circumference (16 m.) with graphical methods, always on the ground of the square. Then they spread out on the square, drawing with their bodies the perimeter, to notice the likely real dimensions of the Roman theatre.

After this session they returned to the classroom and tried to explain the experimentation in their reports but it emerge the necessity to get a tool to draw the theatre properly. Moreover, the teacher introduced the book by *Vitruvius* (available in the blog), classical Roman architect who wrote several canons that Roman people followed to construct each type of public building. In this stage the technology had an essential role in order to allow students drawing their geometrical model of the theatre following the *Vitruvius'* instructions. The students constructed their model of the theatre using the software *Geogebra*.

Interpreting and validating mathematical models within the archaeological context

When each team had their model of the theatre finished could export the file as an image and pasted it on the map of the area studied, fitting it properly in the exactly site where the ruins was discovered (Figure 1). This task allowed students interpret their model considering the specific context and verify if their construction and hypothesis were suitable.



a. **b.**

Figure 1. (a) Application the theatre model, which is construct with Geogebra, on the map studied from one of the inquiry teams report; (b) Detail enlarged

To do the tasks in this second stage the students had to deal with information from the Mathematics —data related of geometrical shape of the wall, extracted from the activity on the square—and from the History —considering the features of the context in where the ruins were discovered—. Modelling, in that moment, became an essential tool that facilitated find answers.

Then, when the students were became competent researchers yet, it started the third stage of the sequence. They arrived at that point with an important collection of questions and doubts but in the following session they had the opportunity, firstly, to share it with other groups and, thereupon, preparing an interview to ask the archaeologist of the Museum. During this interview students could contrast and validate their results about the model selected. It was another especial time you could notice that in the real world different disciplines interacts in a natural way to find solutions of real

problems. After the session outdoors talking to the archaeologist, they could apply their geometrical model construct with *Geogebra* on maps with other ruins of theatres in Europe and thus could check how the same model fits there, too. The model also was useful counting how many people would fit in the theatre, the last task of this stage. Finally, the students finished writing the final report of their inquiry describing their process, the mathematical tools used, the result of verifying their hypothesis, new opened questions, etc.

Conclusions

We would like to stress the importance that the context and the initial question studied had to deeply involved students in the inquiry activity. Moreover, the possibility to work with real data, the facilities to get access to the ruins and to other real archaeological investigation facilitated that students faced the inquiry questions from a wider approach. What the design and implementation of the activity shows is that the context easily offered different kind of information, such as the one coming from the history, from orography, from geometrical models used to measure and drew plans. It facilitated that students considered all these kind of different nature information and tried to systematise it and to work with it in order to provide answers.

On the one hand, we have shown how the enquiring process carried out in the implementation nests a sub-process of modelling that appears as a tool to contemplate information emerged from the context that could become mathematized (such as: the measure of the curved wall, of the ratio, of the building perimeter or the possible use of the Vitruvius canon) in order to be able to progress in looking for specific answers. Thus, at some stages, mathematics appears as an essential modelling tool to look into systems, build up mathematical models, simulate and test them.

More concretely, in the task done on the square next to the school, students wondering how could know the geometrical shape of the whole Roman wall from the partial wall in the map. But this is a difficult mathematical problem to resolve—to find the geometrical curve from a part of it— because there could be a lot of solutions. Besides, the methods to finds these solutions only from the mathematical point of view, are totally beyond the powers of secondary school students'.

On the other hand, the dialogue with History limits the possible answers because the Roman buildings only had three relevant shapes: ellipsis (amphitheatre), circle (circus) or semi-circle (theatre). Due to the contribution of the historical information the problem became achievable at the students' level of mathematical knowledge and allows the beginning of modelling.

Last but not least, it was also very important the possibility that students know about the experts' work (the real research of archaeologists) and to realise that the process they follow are quiet similar to the process of enquiring and modelling that they were following. It also had an important impact on their motivation and on the perception they had about the usefulness of Mathematics and of modelling. The interaction with the archaeologists allowed students to validate their whole process of enquiring and their results.

Acknowledgment

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Strategies for fostering students' reading comprehension while they solve modelling problems

Madlin Schmelzer and Stanislaw Schukajlow

University of Münster, Germany;

madlin.schmelzer@uni-muenster.de, schukajlow@uni-muenster.de

A profound level of reading comprehension is essential for solving modelling problems, as a problem solver has to understand the real-life situation presented in the task in order to construct an adequate situation model, real model, and mathematical model. The aim of this paper is to present the theoretical grounds and a sample learning environment for fostering reading comprehension. In the first part of this article, we summarise research on reading comprehension while solving modelling problems, report on research on two strategies (text highlighting and self-generated drawing) that can help learners comprehend a modelling problem, and describe ways to implement both strategies in the classroom. In the second part, we present a learning environment that can be implemented to foster reading comprehension in the lower secondary classroom.

Keywords: Modelling, strategies, reading comprehension.

Introduction

Reading comprehension is important for learning in school and for students' everyday lives. As each school subject offers students different kinds of texts, fostering subject-specific reading comprehension strategies is necessary for learning and problem solving. In math classes, students have to read different kinds of texts such as proofs or word problems. We address word problems, which, in order to be solved, require a demanding transition from the real world to mathematics as *modelling problems* (Blum, Galbraith, Henn, & Niss, 2007). To solve a modelling problem, students need a profound understanding of the task, as a superficial combination of the numbers given in the task is not sufficient for finding a solution to a modelling problem. In this paper, we characterise the role of mathematical reading competency for the solution of modelling problems, summarise research on two strategies (highlighting and self-generated drawings) that might improve reading comprehension, and present a learning environment for fostering these strategies in the classroom.

Theoretical background for reading comprehension and modelling

Reading comprehension while solving modelling problems

Analyses of the cognitive processes that people engage in during mathematical modelling have often distinguished the following activities in the solution process: (1) The problem solver constructs a *situation model* on the basis of the information presented in the task and on his or her prior knowledge. The situation model reflects the learner's mental representation of the given situation. A profound level of reading comprehension is essential for constructing the situation model if a problem is partly or completely represented as text. (2) The situation model is then simplified and structured to obtain a *real model* that contains only the information necessary to solve the task. The problem solver needs a deep understanding of the problem to separate relevant from redundant information. (3) Mathematising the real model leads to the *mathematical model*. (4) Working

mathematically then serves to produce a *mathematical result*, which is then used to refer back to the real situation given in the task by interpreting and validating the *real result*. In the following, we concentrate on understanding and simplifying/structuring, for which reading comprehension is critically important.

Understanding and simplifying/structuring are expected to be essential for solving the entire modelling problem. These expectations were supported by the results of the study by Leiss et al. (2010). In this study, the strategies that students used in the construction of an adequate situation model and real model were found to have a positive influence on their modelling competency. Therefore, we conclude that it is important to support learners' reading comprehension by the use of strategies. Two strategies suitable for fostering reading comprehension in modelling problems are presented in the following chapter.

Highlighting and self-generated drawings for reading comprehension

Highlighting. Highlighting is a cognitive strategy that aims at directing the learner's attention to the specific words or sentences in a text. To highlight important information, the learner has to identify and select the relevant information.

The prompt "Use the highlighting strategy" is not sufficient by itself to support the learner's reading comprehension, as the quality of strategy use has been found to be important for effects on students' performance. The use of highlighting has to be regulated by students in order to be efficient, otherwise the learner might highlight too many words in the text or might forget to highlight some important information. In a study of college students, Leutner et al. (2007) examined the effects of training the highlighting strategy with expository texts. They compared three groups: The first group did not receive any training and served as a control group, the second was given a strategy training in highlighting, and the third was given a combination of strategy and self-regulation training. The findings revealed that the third group outperformed the others in text comprehension. Thus, self-regulation seems to be an important factor for strengthening the appropriate and purposeful use of cognitive strategies such as highlighting. The combination of strategy and self-regulation training included the following elements: First, the learning goal was presented. Second, the participants observed how a fictive person applied the highlighting strategy. Third, they were given a strategy training in which the steps of the highlighting strategy were presented and then applied. Fourth, a self-regulation training followed, in which the steps of the highlighting strategy were recalled, and then the steps of the metacognitive strategy (monitoring, self-evaluation, and reaction) were introduced and applied.

Highlighting strategies can also be expected to be useful in the domain of mathematical modelling, as modelling problems with reference to real life often contain redundant information. In order to construct a situation model and a real model, students have to identify relevant information. In an exploratory case study by Leiss et al. (2010), some difficulties in the use of the highlighting strategy while solving modelling problems were observed. An analysis of students' solutions showed that some students highlighted all numbers written in numeric form given in the task, including numbers that were not needed to solve the modelling problem. Further, some students did not highlight numbers written in word form, even when these numbers were essential for the solution. These

observations demonstrate the limited quality of the highlighting strategy by students and emphasise the importance of improving the quality of the use of this strategy.

Self-generated drawings. Another important cognitive strategy that can be applied to support a learner's reading comprehension and problem solving is the creation of self-generated drawings. Whereas highlighting is aimed at selecting the most important information, drawings are aimed at organising and visually representing information given in the text.

A study by Leopold and Leutner (2012) revealed the advantages of drawing activities for the comprehension of science texts. Students in grade 10 were instructed to read text paragraphs and then make a drawing that represented the main ideas of the paragraphs. To train the students to use the strategy, they worked on an example that demonstrated how to process the text with a related drawing. The results showed positive effects of the drawing instructions on students' science text comprehension. Drawing activities encourage students to construct a mental model and seem to offer a useful strategy for facilitating students' deeper understanding. In the domain of mathematics, positive effects of drawing activities were found on 3rd-grade students' word problem solutions (Csíkos, Szitányi, & Kelemen, 2012).

Drawing activities might also support the construction of a situation model in the context of mathematical modelling (Rellensmann, Schukajlow, & Leopold, 2017). Strategic knowledge about drawing was found to have a positive effect on modelling performance. This effect was mediated by the accuracy of the *situational* and *mathematical drawings* and emphasised the importance of the quality of the strategy for solving modelling problems. In addition, the study revealed that the accuracy of mathematical drawings is a strong predictor of modelling performance, whereas the situational drawing had only indirect influences on performance by facilitating the construction of a mathematical drawing. These findings suggest that self-generated drawings offer a strategy that is useful for fostering modelling. The most promising was found to be the generation of accurate mathematical drawings. Thus, when instructing students how to generate a drawing, teachers should pay special attention to the accuracy of the mathematical drawing (namely that it contains correct relations and all relevant numbers). Learners should be encouraged to generate a situational drawing if they do not succeed in drawing a more abstract mathematical model in their first attempt.

Even though highlighting and drawing seem to be useful strategies for fostering reading comprehension during mathematical modelling, they need to be taught in rich learning environments. In the following chapter, we present some learning environments that are appropriate for teaching these strategies.

Highlighting and self-generated drawings in learning environments for improving modelling

Several studies have investigated the effects of different learning environments on students' modelling competency. In the following, we present two studies that integrated highlighting and drawing strategies (among other elements) in their learning environments to foster modelling competency.

Verschaffel et al. (1999) revealed the positive effects of a certain learning environment on 5th graders' modelling and problem-solving competency. The learning environment contained the acquisition of an overall metacognitive strategy that involved five stages in the planning of the whole solution process. Eight strategies (e.g., "Distinguish relevant from irrelevant data" or "Draw a

picture") were embedded in the first two stages (Verschaffel et al., 1999, p. 202). The distinction between relevant and irrelevant data is related to the highlighting strategy, as appropriate highlighting aims to make this distinction. Another condition under which the results were acquired was the instructional technique used in this study. It consisted of systematic changes between whole-class discussions and small group work. In both phases, the teacher encouraged the use of strategies and encouraged the students to reflect on their purposeful use in order to stimulate the regulation of strategy use.

A learning environment for modelling that included strategic elements was examined by Schukajlow et al. (2015). A scaffolding instrument called the *solution plan* with four steps was used in this study to support students' modelling activities. Strategic prompts were assigned to each step. As a whole, the solution plan served as a metacognitive planning strategy that was designed to guide students through the process of solving a modelling problem. Fostering reading comprehension was not the sole focus of the solution plan, but it included cognitive strategies that were aimed at improving reading comprehension (e.g., strategies such as "Look for the data you need and, if necessary, make assumptions!" or the strategy "Make a sketch!"; Schukajlow et al., 2015, p. 1244). Although the highlighting strategy was not explicitly mentioned in the solution plan, it is closely connected to the strategy of looking for relevant data.

The student-centred *operative-strategic* learning environment used in this study is characterised by a systematic change between individual work in groups and whole-class discussions. The whole-class discussions included presenting solutions and reflecting on the solution processes (Schukajlow et al., 2015, p. 1243). The study found that an experimental group that was taught the solution plan outperformed a control group that was not taught the solution plan in solving modelling problems. Furthermore, students in the experimental group reported more frequently using self-reported strategies than the control group.

On the basis of the theoretical and empirical findings on the effects of highlighting and self-generated drawing, we developed a learning environment for fostering reading comprehension. We describe this learning environment which will be approved in the next step of the project in the following section.

Learning environment for fostering students' reading comprehension while they solve modelling problems

Based on the theoretical grounds presented in the first part of this paper, the following learning environment was developed to foster 9th graders' modelling competence with special regard to the beginning of the modelling process, namely understanding, structuring, and simplifying. The aim of the learning environment is to improve students' performance in these sub-competencies by fostering their reading comprehension via trainings in highlighting and the use of self-generated drawings. Similar to some other studies that implemented strategy trainings (see e.g., Leutner et al.,

2007, or Schukajlow et al., 2015), the duration of the teaching unit will be five lessons with a total of approximately 225 minutes[1].

The modelling problems that will be used in the present learning environment include text and can be solved by applying the Pythagorean Theorem as a mathematical procedure. The Pythagorean Theorem was chosen because of the importance of this mathematical procedure for national and international curricula. Before the beginning of the teaching unit that was designed to foster reading comprehension, students are expected to know the Pythagorean Theorem and to practise it on intra-mathematical problems. A sample problem *Reaction time* is shown in Figure 1.

Reaction time

During the 2016 European Championship, Germany played against Slovakia in the round of the last sixteen. With goals by Boateng (minute 8), Gomez (minute 43), and Draxler (minute 63) the German team won with a score of 3 to 0.

In the 14th minute, Germany was allowed a penalty kick after a foul by Slovakia. A penalty kick is shot from a distance of eleven metres from the goal, which has standard measures of 2.44 m in height and 7.32 m in width. The German penalty taker was Mesut Özil, and Matus Kozacik was in the Slovakian goal.

Unfortunately, the penalty kick was stopped by Kozacik so that Özil missed the chance to have an early score of 2 to 0 for Germany. His penalty kick was shot a bit too feebly and flew just over the ground to the lower right corner where Kozacik was able to deflect it away from the goal. Although the penalty kick was not shot very hard, the goal-keeper didn't have much time to react, as the football flew at a speed of about 80 km/h towards the goal.

Calculate how much time the goal-keeper had after Özil's kick to reach the position where he stopped the ball just before the corner of the goal.

Figure 1: Sample modelling problem *Reaction time*

In line with the solution plan study by Schukajlow et al. (2015) and the study by Verschaffel et al. (1999), the learning environment that we chose for our teaching unit includes systematically changing between individual work, group work, presenting solutions, and reflecting on the solution process as a class (Schukajlow et al., 2015, p. 1243).

In the first lesson, both of the strategies of highlighting and using self-generated drawings are introduced. The students are given the modelling problem *Reaction time* (cf. Figure 1) and are requested to highlight important information and to generate a drawing while doing their own individual work in groups, but they are asked not to solve the problem. The task requires the application of both the highlighting and drawing strategies.

[1] Schukajlow, Kolter, and Blum (2015) measured effects after 205 minutes of total treatment. Leutner, Leopold, and den Elzen-Rump (2007) used a time of 150 minutes. Verschaffel et al. (1999) used 20 lessons to teach eight strategies, so five lessons for teaching two strategies seemed appropriate for our teaching unit.

After the individual work in groups, students present their highlighted texts and their drawings and describe how they proceeded in applying both strategies. To encourage a discussion about potential difficulties in the use of these strategies, the group that the teacher chooses to make the first presentation should be one that had difficulties with the generation of the highlighting or drawing. During the presentation and the subsequent reflection on the presented solution, the teacher should direct students' attention to typical problems that result from the misapplication of strategies. The teacher should then present the learning goal of the teaching unit to the students, namely to improve reading comprehension and the ability to solve modelling problems.

After that, the first two steps of the solution scheme (cf. Figure 2) are introduced. They save the results of the class discussion in written form and might also provide some advice that was not mentioned by the students. The solution plan by Schukajlow et al. (2015) was adapted to better fit the aim of fostering reading comprehension and to guide the application of both of the strategies of highlighting and producing self-generated drawings. The solution scheme with the reading strategies of highlighting (integrated in step 1) and the creation of self-generated drawings (integrated in step 2) is shown in Figure 2.

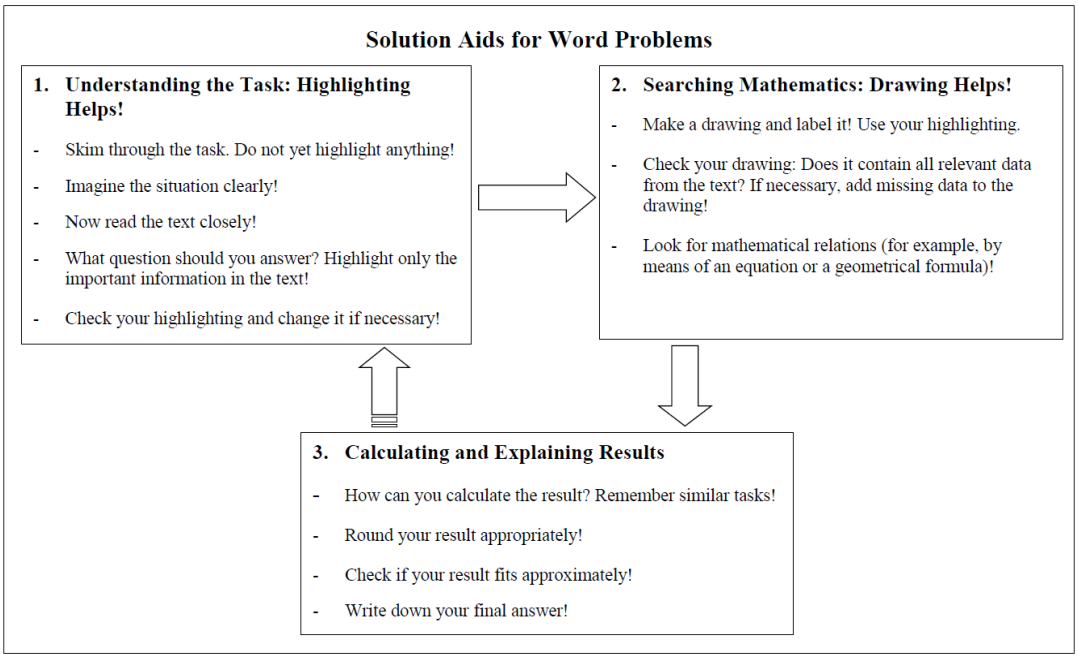

Figure 2: The solution scheme used in the learning environment

The entire solution scheme serves as a planning strategy for the whole solution process. In step 1, the highlighting strategy is presented. First, students are told to skim the task. This means they should get an overview of the task, which might contain a title, text, questions, pictures, tables, or diagrams. While skimming the text, students do not need to understand each word in detail. After skimming the text, the students are prompted to imagine the situation presented in the task. This might help to activate prior knowledge about the topic and facilitate reading the text in detail in the next step. These activities stimulate the *understanding* of the real situation. The highlighting is prompted after they read the question again, as the selection of relevant data depends on the question posed in the problem. After they finish the highlighting, students evaluate whether they highlighted only the most important information in the task and revise their highlighting if needed. The highlighting is aimed at helping students to *simplify* the given information.

In step 2 of the solution scheme, the students are asked to make a drawing and label it with relevant data from the text. These activities help them simplify and structure the information given in the text. During the following monitoring activities, students check whether their drawings contain all relevant data from the text and whether all mathematical relations are represented correctly. If students do not succeed in constructing an accurate mathematical drawing, they can first generate a less abstract situational drawing. At the end of step 2, students *mathematise* the information given in the drawing. In step 3, the students calculate a solution and obtain the mathematical result. They interpret, validate, and present the result in a final answer in step 3. The arrow pointing back towards step 1 indicates that the solution process might be restarted if the result does not fit.

In the first two lessons of the teaching unit, the students are familiarised with only the first two steps of the solution scheme in order to train their reading comprehension strategies and the sub-competencies of understanding, simplifying, structuring, and mathematising as part of the modelling process. In lessons 3, 4, and 5, students practise the entire modelling process by applying the entire solution scheme.

In line with Leutner et al.'s (2007) study, the current learning environment contains the same main elements to stimulate self-regulation. First, goal setting is realised in the first lesson. Students are confronted with a modelling problem that requires a profound level of reading comprehension. The teacher explains that the aim of the teaching unit is to learn strategies that support reading comprehension and to solve reality-related modelling problems. Second, instead of observing the application of the strategy by a fictive person, students analyse their classmates' highlighted texts and drawings in both the work done in small groups and the presentations involving the whole class. These practises are implemented in order to stimulate students' activities in the classroom. Third, the strategy training begins with a presentation of the steps that are necessary for highlighting and drawing and is followed by an application of the strategies while solving modelling problems. In contrast to Leutner et al.'s (2007) study, the self-regulation training is integrated in the strategy training. If requested, the teacher gives strategic advice by referring to the relevant steps of the solution scheme and stimulates reflection on the use of strategies during the individual work in groups and during the whole-class discussion when solutions are presented and reflected on. This process helps to encourage the use of strategies and to establish the solution scheme as a scaffold for solving modelling problems. In order to stimulate the self-regulation of strategy use, we included the prompts "Check your marks and change them if necessary" in step 1 and "Check your drawing" in step 2 of the solution scheme. Further, the validation of the results of solution problems is stimulated by the prompt "Check if your result fits approximately" in step 3 (cf. Figure 2).

Summary and future steps

In the first part of this paper, we discussed the theoretical background for reading comprehension and modelling. Based on the theory, we presented in the second part of the paper a learning environment to foster students' reading comprehension while solving modelling problems. This learning environment will be evaluated in a project for pre-service teachers. The pre-service teachers will obtain the material and instructions to implement the learning environment in their classrooms to gain practical experience in fostering reading comprehension in mathematical

education. They will attend a seminar to prepare for the project and to reflect on the experiences they made while implementing the project.

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Methodology for designing didactical activities as an engineering project of a tactile sensorial therapeutic ramp

Luis Siero[1], Avenilde Romo[2] and Arturo Abundez[3]

[1] National Polytechnic Institute, CICATA, Ciudad de México, México; lsiero@uabc.edu.mx

National Polytechnic Institute, CICATA, Ciudad de México, México, aromov@ipn.mx

[3] National Center for Research and Technological Development, Cuernavaca, México,

The objective of this investigation is designing an activity research study in mathematical modeling for the training of future engineers in the subjects of linear algebra, and structural mechanics of composite materials, in an application of stress and strain calculation. Considering elements of the Anthropological Theory of Didactics (ATD) implementing a methodology within the mentioned theory. This allows the analysis of mathematical models in use, as basis for designing didactical activities, in order to create a link between these two subjects, showing the future engineers that mathematics can be used to solve problems in an extra-mathematical context. This took place in the Research Center for Applied Science and Advance Technology of the National Polytechnic Institute (CICATA-IPN) and in the School of Science of Engineering and Technology of the Autonomous University of Baja California (ECITEC-UABC).

Keywords: Mathematical models, linear algebra, engineering training, ATD.

Introduction

The objective of this work is to design modeling activities for a mathematical training of engineers. The design is based on mathematical modeling analysis on specialty training courses, focusing on laminated composite materials. This collaborative work involves aerospace, mechanical engineers and mathematicians that teach in the carriers of Aerospace engineering and mechanical engineering, having an opportunity to analyze a real context of modeling; namely Calculation of stress and strain of composite materials. This project was proposed on a structural mechanics of composite materials course; because the students wanted to know where they could use the mathematics they were learning. For this work we have considered elements of the Anthropological Theory of Didactics proposing a methodology associated to this theory that permits the analysis of mathematical models in use based on the design of didactic activities.

Elements of the Anthropological Theory of Didactic

The ATD is an epistemological model that allows the study of human activity in its institutional dimension. An institution is a stable social organization that defines the human activities generating resources that make them possible. These materials or intellectual resources, which are made available to the subjects, have been produced by communities along the confrontation of problematic situations with the objective of solve them in a regularly and effectively way (Castela and Romo, 2011). The classic praxeological model, proposed by Chevallard (1999), recognizes the praxeology $[T, \tau, \theta, \Theta]$ as a minimal unit of analysis of human activity. Its four components are: the task type (T), the technique (τ), the technology (θ); and the theory (Θ). The 'task' refers to what is to be done; the 'technique' is how it is to be done; the 'technology' is a discourse that produces, justifies and explains the 'technique'; while the 'theory' produces, justifies and explains the 'technology'.

The training of engineers through institutions

The training of engineers can be seen through institutions, Romo (2009) distinguish three types: production of knowledge $P(S)$, teaching of knowledge $E(S)$ and use of knowledge or practices Ip. This distinction seeks to recognize the first vocation of the institutions and the production of knowledge correspond to scientific disciplines, such as mathematics or engineering sciences, are validated in these the existence of knowledge and the relations between them. The teaching of knowledge $E(S)$ are responsible for displaying and disseminating praxeologies, meanwhile in the institutions of usage Ip, the praxeologies are used to solve problems of practice. This does not mean that within the institutions $P(S)$ there is no teaching or usage praxeologies nor in teaching $E(S)$ and practice Ip there is no production of knowledge. In this investigation linear algebra is considered an institution of teaching mathematics $E(AL)$, and in teaching engineering, structural mechanics of composite materials $E(MC)$ and connect them through a Study and Research Activity (SRA). The SRA constitute didactic devices for the construction of a praxeology through three didactic moments that are: first encounter with T, exploration of T and the emergence of the τ technique and construction of the technological-theoretical block, in this case coming from the structural mechanics of composite materials as outlined below

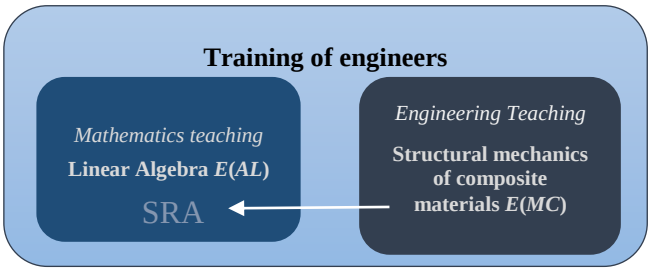

Figure 1: The SRA as an element linking educatinal institutions

In order to initiate the transition from the traditional paradigm in mathematics ”visitinng works” (to the teaching of pre-existing mathematical objects) to questioning the world according to inquiry-based mathematics education (IBE).

Didactical design methodology of an SRA

The methodology initially proposed in Macias (2012) allows the design of SRA that involves non mathematical elements. Its four stages are: 1) Election of an extra-mathematical context; 2) Praxeological analysis and identification of a mathematical model; 3) Analysis of the identified mathematical model and their relationship with $E(M)$ and 4) design of the SRA.

Election of an extra-mathematical context

To choose an appropriate extra-mathematical context for the design of an SRA, the following elements were considered.

1) Generation of surveys aimed at teachers and students about mathematical needs of engineers in training:

Students. Which of these subjects you thought more important, and why? Have you used or adapted a mathematical model? Which model and for what?

Teachers. In any of your subjects taught in an engineering career you use mathematics? Have you used a mathematical model or adapted one in any of your courses? Which model and for what?
67 students from 3rd to the 7th semester were surveyed, finding that the most useful subjects are calculus and linear algebra. 54% recognizes the work with mathematical models. All teachers who teach subjects of MC, integral calculus CI and AL were surveyed, 92% of them recognize the use of matrices in their courses

2) Interviews with the coordinator of the career of mechanical engineering, who noted that the subject of MC required early in the course of a review of linear algebra.

3) Approach with MC teachers, who pointed out that in general, students do not recall procedures or operations previous courses, AL for example.

4) Joint work with engineers-researchers in the area of materials.

Praxealogical analysis and identification of a mathematical model

The praxeology that is identified and analyzed is the calculation of stress, strain and elastic modulus in laminated materials. The type of task is to calculate the stress or strain of a laminated material, the technique is associated with the use of the matrix *S* (strain) or matrix *Q* (stiffness), the technology is Hooke's law and the theory is the mechanics of materials. The analysis is based on a technical report of basic mechanics of laminated composite plates (Nettles, 1994), suggested by an engineer-researcher, who indicated it, as a heavily used reference material. In this section the theoretical technological-block is shown, which displays how the stiffness matrix associated with the technique of calculating the stress and strain is presented. This will provide the basis to show in the next section full praxeology from the analysis of an exercise presented in a class of structural mechanics of composite materials.

Technology: Generalized Hooe´s law for anisotropic materials[1]
Nettles explains that the relationship between stress and strain is independent of the direction of the force, and is provided by the constant of elasticity (Young's modulus), this is for isotropic[2] materials. In nonisotropic materials it should use two elastic constants at least. The relationship stress / strain for isotropic materials appears as follows: $\sigma = E\varepsilon$ (1)

Where σ: is the stress, *E:* Denotes the Youngs modulus and ε: is the strain.
For orthotropic materials [3], the direction must be specified in the stress/strain relationship:

$$\sigma_1 = E_1\varepsilon_1;\ \sigma_2 = E_2\varepsilon_2 \qquad (2)$$

where
σ_1: Denotes the stress in the longitudinal direction
E_1: Denotes the stiffness in the longitudinal direction (Young´s modulus)
ε_1: Denotes the strain in the longitudinal direction
σ_2: Denotes the stress in the transversal direction

[1] Anisotropic materials: is the material that its mechanical properties differ according to the load direction

[2] Isotropic materials: It is the material that has identical mechanical properties in all directions regardless of the direction of the load

[3] Orthotropic materials: is the material in which mechanical properties are different in three perpendicular directions

E_2: Denotes the stiffness in the transversal direction (Young´s modulus)
ε_2: Denotes the strain in the transversal direction
$E_1 = E_L$ Defines the stiffness in the longitudinal direction and $E_2 = E_T$ is the stiffness in the transversal direction. This law produces different techniques, Nettles initiated by the special orthotropic plates and is why we analyze them below..

$$S_{11} = \frac{1}{E_1} \qquad S_{22} = \frac{1}{E_2}$$
$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \qquad S_{66} = \frac{1}{G_{12}}$$

Stress and strain for special orthotropic plates
The author begins by explaining that on a plate, stress can be given in more than one direction. Immediately he defines Poisson's ratio as the strain perpendicular to a given loading direction, showing the relationship for different loads.

For loading along the fibers:

$$Poisson´s\ ratio = \nu_{12} = \frac{\varepsilon_T}{\varepsilon_L} = \frac{\varepsilon_2}{\varepsilon_1} \quad (3a)$$

For loading perpendicular to the fibers

$$Poisson´s\ ratio = \nu_{21} = \frac{\varepsilon_L}{\varepsilon_T} = \frac{\varepsilon_1}{\varepsilon_2} \quad (3b)$$

The strain is equal to the difference between the stretched component deformation due to an applied force and contraction of the Poisson´s effect due to other forces perpendicular to the applied force, thus:

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \nu_{21}\varepsilon_2 \quad y \quad \varepsilon_2 = \frac{\sigma_2}{E_2} - \nu_{12}\varepsilon_1 \qquad (4a)$$

aplicando la ecuación (2)

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \nu_{21}\frac{\sigma_2}{E_2} \quad y \quad \varepsilon_2 = \frac{\sigma_2}{E_2} - \nu_{12}\frac{\sigma_1}{E_1} \qquad (4b)$$

Subsequently, the author considers the presence of shear forces. The shear stress and shear strain are related by a constant called shear modulus, denoted by G.

$$\tau_{12} = \gamma_{12}G_{12} \qquad (5)$$

Where: τ_{12}: Shear stress, γ_{12}: Shear strain y G_{12}: Shear modulus

Equation (5) is similar to equation (1) it only considers shear stress and strain, where the indices 1-2 indicate shear in the 1-2 plane. The author mentions that a relationship exists between the Poisson constant and the Young's modulus in both directions, in the longitudinal direction and the transverse direction to the material, and then it holds that:

$$\nu_{21}E_1 = \nu_{12}E_2 \qquad (6)$$

Equations (4b) and (5) can be written in their matrix form obtaining

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \quad (7) \qquad \text{where,}$$

Here it is where we can see a relationship between the subjects of linear algebra and structural mechanics of composite materials as a matrix model for calculating stress or strain of laminated

materials whether they are isotropic or orthotropic. Calculating the inverse stress matrix S, we obtain the stiffness matrix Q turning out to be:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} \quad (8)$$

where,

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}}$$

$$Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}=\frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{66}=G_{12}$$

We can appreciate in equation 7 to 8, the calculation of the inverse matrix was made, to switch the stress matrix to the stiffness matrix, with basic operations taught in linear algebra. The author explains, broadly, the mathematical model based on the algebraic work, having left the reader the task of verifying the connections between the different equations, until reaching the mathematical model that relates the stress, strain and Young's modulus (which are the elastic properties of the material). All this is the technological component of the praxeology. To illustrate the types of tasks that can be solved and the associated techniques to the praxeology, we analyze below a classroom exercise from the subject of structural mechanics of composite materials.

Analysis of the identified mathematical model and its relationship with *E*(*M*)

In the class of structural mechanics of composite materials (MC), where we analyzed and identified the praxeology of the stress calculation of a laminate material, with a task type *T*, calculate the modulus of elasticity of a laminated material in a particular direction, with two tasks, t_1 y t_2:

t₁: Calculate the modulus of elasticity of a laminated material in the *X* direction.

t₂: Calculate the modulus of elasticity of a laminated material in the *Y* direction.

For laminate fiberglass polyester matrix that is laid up in a $[45_2/-45_2\ /0]_s$ stacking sequence.

τ₁: Find the stiffness matrix of a laminate material, with the following information

$$\boldsymbol{E_1=40\ GPa} \qquad \boldsymbol{G_{12}=2.8\ GPa}$$

$$\boldsymbol{E_2=9.8\ GPa} \qquad \boldsymbol{\nu_{21}=0.3}$$

$$[Q]=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{SS}\end{bmatrix}$$

$$[Q]=\begin{bmatrix}40.90 & 3.01 & 0\\ 3.01 & 10.02 & 0\\ 0 & 0 & 2.80\end{bmatrix}GPa$$

Determine the stiffness matrix of the lairs in 45°.

Using the expressions for the calculation of the stiffness matrix, for any fiber orientation it can be written by:

$$[Q]=\begin{bmatrix}\bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xS}\\ \bar{Q}_{yx} & \bar{Q}_{yy} & \bar{Q}_{yS}\\ \bar{Q}_{Sx} & \bar{Q}_{Sy} & \bar{Q}_{SS}\end{bmatrix}$$

xy axes: global axes; *1,2 axes* : material axes. Thus the stiffness matrix turns to be: $[Q]=\begin{bmatrix}17.03 & 11.43 & 7.72\\ 11.43 & 17.03 & 7.72\\ 7.72 & 7.72 & 7.01\end{bmatrix}GPa$

For the $-45°$ the stiffness matrix is $[Q] = \begin{bmatrix} 17.03 & 11.43 & -7.72 \\ 11.43 & 17.03 & -7.72 \\ -7.72 & -7.72 & -7.01 \end{bmatrix} GPa$

Stiffness matrix in the flat tension of the laminate material

$[A] = \sum_i [Q]_i \cdot h_i$where h is the thickness of the material $[A] = \begin{bmatrix} 218.04 & 97.46 & 0 \\ 97.46 & 156.28 & 0 \\ 0 & 0 & 61.68 \end{bmatrix} 10^9\, h \frac{N}{m}$

Normalized stiffness matrix in the plane stress of the laminate is written as $[A^*] = \frac{[A]}{10^9\, h} = \begin{bmatrix} 21.8 & 9.75 & 0 \\ 9.75 & 15.63 & 0 \\ 0 & 0 & 6.17 \end{bmatrix} GPa$

τ_1: Applying a tensile stress in the *X* direction $\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix}$. As mean stress acting on the laminate material $\begin{Bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix}$; The relation between the average stress and the strain of the material is given by: $\{\sigma\} = [A^*] \cdot \{\varepsilon\}$; Calculating the strain state for the loading state. $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 0.06362 \\ -0.03969 \\ 0 \end{Bmatrix} 10^9\, \sigma_x$

In the direction of the *X* axes the relation between the strain and stress is given by: $\varepsilon_x = 0.06362\ (10^9)\ \sigma_x$, on the other hand the apparent elasticity modulus in the *X* direction is presented as $E_x = 1.572\ GPa$

τ_2: Applying a tensile stress in the *Y* direction

Analogously the mean stress acting on the laminate material is

$\begin{Bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \sigma_y \\ 0 \end{Bmatrix}$ Therefore the strain state is $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.03969 \\ 0.08874 \\ 0 \end{Bmatrix} 10^9\, \sigma_y$

Particularly in the Y axis direction the relationship between the stress and strain is: $\varepsilon_y = 0.08874\ (10^9)\ \sigma_y$. Furthermore the apparent elasticity modulus in the *X* direction is given by $E_y = 11.27\ GPa$. The praxeological analysis of the technical report and the classroom exercise presented here very briefly, allowed to identify the matrix as a mathematical model for calculating strain of a laminate (isotropic, anisotropic and orthotropic) material. Also, the matrix operations and calculating the inverse of the matrix are used to determine the modulus of elasticity of a composite material, specifically laminated materials.

SRA design proposed for a tactile sensorial therapeutic ramp (RTEST)

For the SRA design, we considered the construction of a product that requested the use of a laminated material and calculating the stress and strain of the material to ensure the usefulness of the product. Therefore, it was intended that the use of the mathematical model appear the way it happens in engineering projects. Determining that for students in the early college years, more than a structure was necessary to think of a product consisting of plates, which also would be useful for the

community. Thus, it came to the proposal of a tactile sensory therapeutic ramp (RTEST) with laminate material to help children from three to ten years to correct gait problems.

Moment of the first meeting with T. To design the ramp students must develop three basic tasks, t_1) design the ramp, t_2) choose the laminate and stress calculations, t_3) determine the type of material with which the ramp will stimulate the sensory part. *Moment of the exploration of T and technical emergency τ.* Students should find the technique or techniques to solve t_1, t_2 y t_3. For t_1, they should investigate the types of ramps, analyze and choose one. For t_2, they should investigate laminated materials, choose one and make the stress calculation (the technic, p. (3-5)). And for t_3, investigate the materials that promote sensory stimulation and choose one, justifying the reasons for their choice. *Moment of construction of the technological-theoretical block.* The third moment intersects with the second because here students must build the ramp, using drawings and stress calculations previously made, as well as the preparation of the composite material. In t_2 students should know and apply Hooke's law, for the problem they are solving, they have to know how to build the stress matrix and determine the stiffness matrix. To perform these calculations students can use computer programs such as MathLab, Scilab and SolidWorks.

First implementation of the SRA

Presenting a first implementation of the evolution of the SRA of one team:

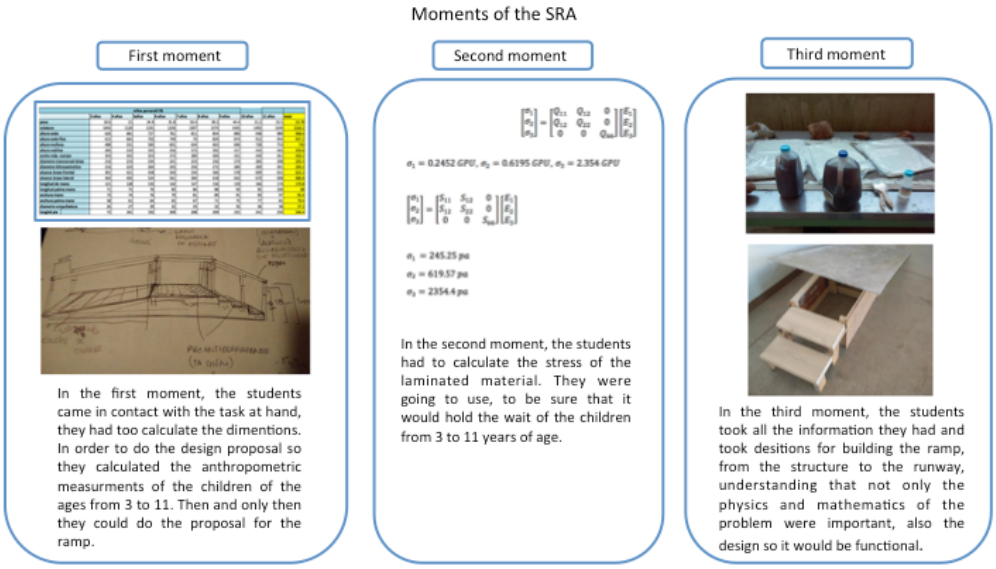

Figure 2: Momentum schematics of the SRA

The implementation with students of the core curriculum of engineering, designers and aerospace engineering; Of three different semesters (2, 6 and 7), trying to mimic the form of work of the industry: as different specialties as well as novices and experts engineers. The SRA was proposed to each teacher of the course to see if the assignment was pertinent to their subject and if they could make it part of their class and grade it. The development of the SRA was parallel to the classes of the teachers that agreed to work on the project assigning a certain time in each course for doubts they might have. A report was requested for each of the three phases. Phase 1: Proposal of a design for a RTEST ramp (3 weeks), phase 2: Strain calculation of the laminate material (3 weeks) and phase 3: Elaborate and choose materials for the RTEST ramp: Laminate materials and for the tactile and

sensorial part of the project (4 weeks). The SRA had three phases associated with the first three moments described above.

Conclusion

The SRA is proposed within the framework of the paradigm of questioning the world in the training of engineers. In this SRA unlike the commonly proposed projects of engineering the mathematical topics are highlighted. In addition the engineering topics are shown in a more important roll in the mathematics subjects. The SRA involved students and teachers from different specializations. The design of the material and the RTEST requires the calculation of stress -matrix model-, knowledge of materials and design. To do this the students must investigate and study elements from different disciplines as well as practical knowledge, students learn to do research, model, use available knowledge to create new, teamwork, communicate their ideas and justify the practice with theoretical elements of different levels. The analysis of the development of the SRA would allow us to understand the institutional necessary conditions for designing SRA in a more complex environment.

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The use of heuristic strategies in modelling activities

Peter Stender and Gabriele Kaiser

University of Hamburg, Germany; gabriele.kaiser@uni-hamburg.de

For students working on realistic modelling problems as autonomously as possible the support by a tutor is indispensable. However, how this support can be realised is still not a sufficiently answered question. In the paper we describe a study in which students worked on complex, realistic, authentic modelling problems over three days supported by tutors. The tutors participated in a teacher training prior to the modelling activity. The focus of the study is the usage of heuristic strategies by students within modelling activities and the promotion of strategic help provided by the tutors. Based on videotaping of ten groups while they were working on the modelling problem of the optimal placement of a bus stop, the study could show that heuristic strategies are an indispensable basis for adequate decisions in the modelling process. Their promotion by the tutors seems to be highly adequate in order to foster modelling competencies under a broader perspective.

Keywords: Mathematical modelling activities, heuristic strategies, modelling example.

Introduction

Modelling and applications are receiving increasing attention all over the world, modelling competencies are required internationally by many curricula. However, the complexity of real world examples and the according modelling process to tackle the problem leads to a strong discrepancy between the high relevance of these kinds of activities in curricula and their factual relevance in school. In the following we will present a study, which examines how tutors can foster the tackling of complex and authentic modelling problems by students in special learning environments, the so-called modelling days. We will in particular focus on heuristic strategies from the problem solving discussion and their possible usage in modelling classrooms.

Theoretical framework

Mathematical modelling and modelling cycle

In our research modelling is understood as a process where a ‘real situation’ from the ‘Rest of the World’ (Pollak, 1979) comes up and needs to be understood and simplified and transferred into a realworld model. The real world model is transformed into the world of mathematics, i.e. the formulated mathematical problem is (partly) solved and the solution is validated according to the real world situation. Often the first results do not answer the primary problem adequately, so the modelling cycle is run through again with an adjusted real world model. This process is repeated until a solution is produced which is adequate for the real situation from the standpoint of the modeller. The according process can be visualized with the following modelling cycle (figure 1).

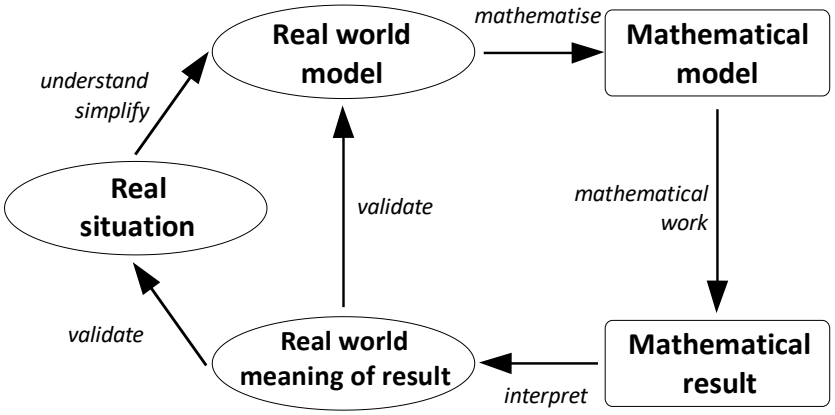

Figure 1: Modelling cycle (Kaiser & Stender, 2013, p. 279)

Fostering student's independent modelling activities in cooperative learning environments

If it is intended that students work as independent as possible on the modelling task in cooperative learning environments self-directed learning environments as required by many scholars in the modelling discussion (e.g. Kaiser and Stender, 2013, for an overview see Blum et al., 2007), the support by a tutor has to be adaptive. We use the definition given by Leiss (2007) of an *adaptive intervention*:

> Adaptive teacher interventions are defined as those kinds of assistance by the teacher to the student, which supports the individual learning and problem solving process of students minimally, so that students can continue to work at a maximal independent level. (Leiss, 2007, p. 65, own translation)

As guidelines for teachers supporting students who need help in their work, we refer to a framework developed Zech (1996), who suggests a five step approach to realise adaptivity: (1) *motivate*, e.g. 'You will make it'; (2) *feedback*, e.g.: 'Go on like this!'; (3) *strategic help* based on strategy what to do next, e.g. 'Simplify the situation by making it as symmetric as possible!"; (4) *content-related strategic help* gives a strategic help with additional information to the problem, e.g. which aspect should be described as symmetric; (5) *content-related help* shows the students aspects of concrete steps to work on.

If motivational and feedback support are not sufficient to enable the students to continue their work, strategic help is according to this framework the next step to support the students, as the students only get a possible way how to go on, but the students themselves still have to realise the work by their own. Important strategic helps are based on references to the steps of the modelling cycle: 'Simplify the situation!', 'Try to transfer this into a formula!', 'What does the mathematical result mean in the real world?', 'Does the result answers the real world situation meaningfully?' More specific support is offered by the use of heuristic strategies.

Heuristic strategies

The usage of heuristic strategies is a well-known approach in mathematical problem solving (e.g. Pólya, 1973), that can be used while tackling modelling problems as well. Based on the work of Pólya and others we distinguish the following heuristic strategies, which were formulated in the frame of ongoing empirical research in mathematical modelling (for details see Stender and Kaiser, 2015):

- *organise your material / understand the problem*: change the representation of the situation if useful, try out systematically, (Pólya, 1973) use simulations with or without computers, discretize situations,
- *use the working memory effective*: combine complex items to supersigns, which represent the concept of 'chunks' (Miller, 1956), use symmetry, break down your problem into sub-problems),
- *think big*: do not think inside dispensable borders, generalise the situation (Pólya, 1973),
- *use what you know*: use analogies from other problems, trace back new problems to familiar ones, combine partial solutions to get a global solution, use algorithms where possible (Pólya, 1973),
- *functional aspects*: analyse special cases or borderline cases (Pólya, 1973), in order to optimise you have to vary the input quantity,
- *organise the work*: work backwards and forwards, keep your approach – change your approach – both at the right moment (Pólya, 1973).

Although these heuristic strategies are well-known in the problem solving discussion, there is only little empirical research known, how these heuristic strategies can be implemented in classroom teaching and how far these heuristic strategies can be transferred to other mathematical activities such as mathematical modelling.

Research Question

In our research study we aim to evaluate, how far heuristic strategies developed in the problem solving discussion can be transferred into the teaching and learning of mathematical modelling. Furthermore, we examine how far these heuristic strategies are appropriate strategic interventions for the tutoring of students who are working on complex, realistic and authentic modelling problems.

Design of the Empirical Study

Modelling Days as learning environment

As this kind of work is not very common in usual classes we established modelling projects in schools as learning environment, called modelling days and offer these to schools as special, project-oriented activity organised in their school. The problems, on which the students work, are developed by the university research group. The tutoring of the students, who work in groups of four to six, is organised either by the teachers or future teachers within their master studies. Both groups receive a special training, in which they become acquainted with complex modelling examples and how to support students during their modelling activities. During the modelling days students of grade 9 (15 years old at the end of lower secondary level) work on one modelling problem for three full days in school. The students can choose the problem out of three problems presented by the research group.

In the following we describe one example and describe exemplarily one possible solution.

Modelling problem: The Bus Stop Problem – a possible solution

We used the bus stop problem in two different versions within various modelling days: the more complex one asks for the best positions of the bus stops for the entire public transport system of the

city of Hamburg. The simplified (but still complex) version only asks for the bus stops of one single bus line.

A solution for the more complex version is based on the idea of covering the city with circles of the same diameter in a regular pattern where the centres of the circles are the bus stops. In a second step a rule is developed based on the adjustment of the bus stops to the requirements of the city-map. The diameter of the circles has to be calculated, which leads to the distance of two bus stops by a certain bus line. For this problem there are a lot of possible aspects that can be considered. One possible approach is to reduce the bus line to a straight line, where the bus stops all have the same distance to the next stop.



Figure 2: Bus line

In this solution the following aspects were taken into consideration: the average walking time from and to the bus station (velocity v_F), the time the bus drives (velocity v_B) over the distance s and the extra time (T_H) each stop causes in between. An optimal bus stop distance shall minimize the total travel time $T(x)$.

This leads to the function $T(x) = \frac{2r+x}{2v_F} + \frac{s}{v_B} + \frac{s}{x} \cdot T_H$ and setting the derivation as zero yields the following solution $x = \sqrt{2v_F \cdot s \cdot T_H}$. These formulae now can be interpreted according to the situation, e.g. in respect of the influence of the distance s or the walking velocity. Students usually will not receive this general result using variables, but with set numbers and they often produce a graph like figure 3. For a more detailed version of this solution see Stender (2016).

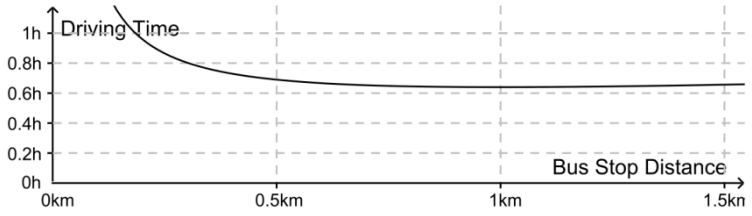

Figure 3: $T(x)$: Travel time depending on the bus stop distance

Data collection and data evaluation

Within our study we videotaped ten groups of students who were working in five rooms at higher track school in Hamburg (so-called Gymnasium), overall about 40 students participated in the study. Over three days the students worked around 15 hours on this modelling problem. We transcribed the phases during which the tutor communicated with the students, including a short time before, so that we could identify the causes leading to the contact and a few minutes after the communication so it was possible to analyse the effect of the tutors' intervention. In total 238 contacts between tutors and individual groups were transcribed and coded using qualitative content analysis (Mayring, 2015). Based on the analysis of the codes for the phases before, during and after the intervention the success of the interventions could be determined. Detailed findings were presented in Stender & Kaiser

(2015). Interventions that were not successful or gave too strong content-related help were subject to a more detailed examination. In these cases we tried to formulate alternative strategic interventions for use in further teacher training. The solution processes of the students were reconstructed based on the work of different groups and hereby an idealised modelling process could be reconstructed (for more details see Stender, 2016).

Results of the study

The reconstructed and idealised modelling process is in the first part of the results section used in order to identify, which heuristic strategies students used either intuitively or by referring explicitly to the modelling cycle, to which they had been introduced explicitly using the example of the length of traffic light phases. All students had worked on this example as introduction. The second part of the results section identifies possible interventions by tutors, introducing the students to the usage of heuristic strategies or by using these heuristic strategies by themselves in order to support the students.

Reconstruction of heuristic strategies in the solution process

In the following we analyse this reconstructed idealised solution regarding the use of heuristic strategies, which are highlighted in *italic*.

The first step of every modelling process is the exploration of the situation, that means as heuristic strategy *organise your material / understand the problem*. The students explored public transport maps and collected important places like schools or hospitals. It took a longer time to *change* this point of view to a more abstract *representation* of the situation, where the bus line is a straight line and special places do not matter. In this situation the more abstract representation is less complex as a lot of details from reality (traffic lights, curves, crossings, hills, …) are missing. So, here a heuristic strategy derived from the modelling cycle is applicable: *simplify* the situation as much as possible at the beginning! Describing the representation of a bus line as a straight line needs another heuristic strategy, namely to construct the situation *symmetrically*. Using this strategy leads to the assumption that the distance between two contiguous bus stops should be all the same. Figure 2 shows that even more aspects are symmetrical in this model. The transfer from the complete public transport system to one single bus line, that is used later to reconstruct the whole transport-system, uses as heuristic strategy to *break down your problem into sub-problems*! This is another way to *simplify* the situation.

To understand the problem of the simple straight bus line *two extreme cases* should be *analysed*, a powerful heuristic strategy: If there are few bus stops, the bus can drive fast without being interrupted by time consuming boarding, but the walk to the next bus stop will be very long for many passengers, which leads to a high total travel time. The other extreme situation has many bus stops, e.g. every 50 m. Now the walking time to the bus stop will be short for all passengers, but the bus will need a long time for a certain distance, because it is stopping every 50 m. So it becomes clear that between these two extremes there is an optimal distance between two bus stops that minimises the total travel time.

For the students the situation was still too complex and they were not able to formulate a functional based approach as they did not have enough experience with these kinds of problems. Now several heuristic strategies were employed: *use analogies, break down your problem into sub-problems, try out systematically or work on special cases.* As already mentioned all groups had worked on the length of traffic light phases as introductory example, they therefore knew the formulae $s(t) = \frac{1}{2}at^2$

and $v(t) = at$ and how to calculate acceleration processes. They reduced complexity again by the heuristic strategy *simplify* in the modelling cycle setting the distance between two bus stops a x = 500 m. This heuristic strategy is related to the heuristic strategy of *working backwards* as x should be the result of the calculation but is used here as if the result is already achieved. Then the students calculated the driving time between the two bus stops using certain values for the acceleration and the velocity of the bus and using *analogies* from the traffic light problem as heuristic strategy. This can be described again as using the heuristic strategy of *break down into sub-problems* as the calculation was not done for the whole bus line, but only for the part between two bus stops. As the choice of x = 500 m was made as ad hoc decision it can be described as heuristic strategy of *trying out* a *special case*. The calculation itself that led to a certain driving time includes several steps and is mathematically challenging, but was achieved due to the use of the *analogy* from the traffic light problem. The result of this approach was the calculation of a certain traveling time that unfortunately was not the answer to the question of the best distance between two bus stops. So, in a second loop through the modelling cycle the students calculated the time for a longer distance of 12 km with a bus stop every 500m *combining the partial results from above*. Again this shows an interesting result but no answer to the initial question.

The next step was to vary the number of bus stops on the 12 km journey using a heuristic strategy *for optimisation you have to vary the input quantity*. The travel time was calculated in the same way as before so an *analogy* was used. The calculation was realised using a spreadsheet and used the heuristic strategy *trying out systematically* different distances. This approach yields the result that with increasing number of bus stops the traveling time increases proportionally. This was expectable as one of the crucial aspects – the walk to the bus stop – was not considered up to now.

Based on this insight an average walking time to the bus stop was included in the calculation while the rest of the calculation was *analogue* to the previous one. This led to a result similar to figure 3, but the students still used as variable the number of bus stops, not the distance between two bus stops.

With the heuristic strategy of *a change of representation* the students switched to the more meaningful variable, again the new calculation was *analogue* to the previous one. Still everything was calculated with a spreadsheet so it had the character of a *simulation* or just *trying* out special cases. Only few students were able to realise the next *change of representation* and combined the single steps of the calculation into one formula. They developed one function $T(x)$ that included several steps of the calculation in one single mathematical term and $T(x)$ works as a *supersign*, which is another, less discussed heuristic strategy. This approach opened the way to use the derivate and calculate a solution like it is shown above. This was done with concrete numbers instead of parameters (v_F, v_B, T_H, s, r) by one group of students, but in another group there were students able to use parameters instead of concrete numbers, which again means the use of *supersigns* as each character stands for an infinite amount of numbers.

The results from different traveling distances s were compared and it became clear as meaningful result that on shorter trips the bus stops may be closer together. This result was *validated*, a heuristic strategy from the modelling cycle, by analysing the map of the public transport in Hamburg. Near the city, where people often use the bus only for the short distance to the next metro station, the bus stops were much closer than in the outer parts of Hamburg. The calculated distances matched very well to the distances in the map.

The subsequent step was to go back to the public transport network and cover the city with circles of a certain diameter. Now the results from the single bus line were used, which uses the heuristic strategy to *combine partial solutions to get a global solution* in order to choose meaningful diameters. These diameters were not the same over the whole city according to the previous results and in opposition to the initial idea.

To summarise, the analyses of the modelling activities by the students showed an intensive usage of heuristic strategies, partly referring to the various phases of the modelling cycle and partly as intuitive usage.

Heuristic strategies as strategic interventions

The heuristic strategies that were used quite often intuitively by the students in the modelling process can be transferred into strategic interventions by tutors, if the students are not able to continue their work on their own. In the following examples for these activities are described, which were shortly included in the teacher training beforehand and which were used by the tutors, but not as intensively as wished, probably due to their low importance in teacher training.

In each situation, where an analogue acting to previous work occurs, the following hints are possible: "This work is analogue to something you have done bevor!" or "Calculate this in the same way you did in the traffic light problem".

While constructing the real model the simplification of the situation is essential. "In your first approach build the real model as simple as possible – for this, it's a good idea to describe the model as symmetric as possible!"

The idea to break down the problem into sub-problems can be initiated by "For this problem you have to work on several steps – try to solve only one simple part at the beginning and then try to use these result for the next steps!"

The idea of using special cases can be implemented as follows: "If you have no idea how to go on, select specific numbers and work with them! Just work on special cases in the beginning!" As shown above this strategic intervention is a powerful mean for modelling activities.

The heuristic strategy *For optimisation you have to vary the input quantity* can be helpful for students who are not familiar with functional thinking. The following hints can be given: "You calculated with 23 bus stops. What happens if you use more or less bus stops?" "Vary the number of bus stops!" "If you look for an optimal solution you have to make sure that a nearby situation is less good!"

To summarise, these examples show how a heuristic strategy can be used to create a strategic intervention. Depending on the work of the students, more or less information on the concrete modelling problem can be included in the intervention in order to give a less abstract input to the students if necessary.

Summary and conclusions

The empirical study displays a great variety of heuristic strategies used by the students within their modelling activities, a few were developed intuitively, a few derived from the description of the modelling cycle introduced beforehand.

An important result of the study is that strategic interventions often were successful when a tutor supported students working on complex modelling problems, because the usage of these heuristic

strategies is not self-evident. Because the usage of adequate strategic interventions by tutors is very hard, it often only will be possible if prepared beforehand. In order to react on the students in class in an adaptive way the tutor needs a deep insight into the modelling process, the modelling problem, possible solutions as well as heuristic strategies and strategic interventions.

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Metacognitive modelling competencies in small groups

Katrin Vorhölter, Alexandra Krüger, and Lisa Wendt

University of Hamburg, Germany; Katrin.vorhoelter@uni-hamburg.de

Metacognitive modelling competencies are an essential part of modelling competence. When working on a modelling problem in small groups, the metacognitive modelling competencies of an individual may be less important, but in particular those shared in the group are of major importance. In this paper, results of a pilot study are presented which clearly indicate that measuring metacognitive group competencies is challenging. Furthermore, it is shown that the measurement of metacognitive competencies of individuals is not sufficient to get insight in the students' metacognitive behavior.

Keywords: Modelling competencies, metacognition, group work.

Introduction

Solving complex modelling tasks in mathematics education in school in Germany is usually done in small groups, at least in tandems. The reason for this is not the promotion of skills for cooperative collaboration - basic skills in this regard are taken for granted. Rather, there is the conviction that such complex problems can only be solved by students, if the ideas and skills of many are shared. This does not only refer to the so called sub-competencies of modelling competencies, which are necessary for getting from one phase of a modelling cycle to the next, but also concerning overarching skills such as metacognitive modelling competencies. However, in recent years, research on modelling competencies has merely focused on individual students. Thus, group dynamics were often neglected. This paper presents first results of a pilot study, in which students' perception and attitude towards metacognitive strategies used by themselves as well as by group members were measured by using different instruments for data collection.

Theoretical background

Modelling competence

Working on modelling problems successfully and goal-oriented requires modelling competence. Although there is no in general accepted concept about which competencies are comprised, the definition of Maaß (2006) is widely accepted. According to her definition, modelling competencies include "abilities and skills to conduct modelling processes adequately and in a goal-oriented way; as well as the willingness to put these abilities and skills into practice." (Maaß, 2006) Here it becomes obvious that those competencies, necessary for getting from one step of a modelling cycle to another, are surely part of modelling competence (Kaiser, 2007). Furthermore, the definition given above, indicates that appropriate beliefs and insights as well as comprehensive competencies such as working cooperatively in groups, communicating with each other and metacognitive competencies are necessary as well.

Metacognitive competencies

The concept of 'metacognition' is a fuzzy one. Schneider and Artelt (2010) define metacognition as "people's knowledge of their own information-processing skills, as well as knowledge about the nature of cognitive tasks, and of strategies for coping with such tasks. Moreover, it also includes

executive skills related to monitoring and self-regulation of one's own cognitive activities." This definition provides the most common distinction of metacognition into metacognitive knowledge from metacognitive skills (often called metacognitive strategies). Thus, metacognition comprises metacognitive knowledge about the specifics of modelling tasks, the knowledge about appropriate strategies for working on modelling tasks successfully and knowledge about person's own skills and competencies and as well as those of other people involved the modelling activity. Furthermore, the procedural aspect of metacognition contains the use of strategies for planning, monitoring, regulating and evaluating the whole modelling process (see Vorhölter & Kaiser, 2016)

For solving a modelling problem successfully and goal-oriented, both aspects of metacognition mentioned above are necessary: A complete lack or only a very low level of meta-knowledge about modelling processes and problems can result in considerable problems when working on such tasks. For transitioning between the stages of a modelling process and for dissolving cognitive barriers while working on them, meta-knowledge as well as metacognitive strategies are needed (Maaß, 2006). Regarding problem solving processes, for example Schoenfeld (1992) points out the importance of planning the solution process. Furthermore, monitoring each other by reciprocal asking and answering metacognitive questions while working on a complex task can improve mathematical performance as well as metacognitive competencies at the same time (project IMPROVE, Mevarech & Kramarski, 1997). This finding is confirmed by the conclusion of Goos (1998): collaborative interactions deliver metacognitive benefits. However, not only metacognitive strategies referring to planning, monitoring and regulating the modelling process are of great importance for solving modelling problems: Blum (2015) points out that reflecting one`s own activities is crucial for transferring knowledge and skills from one task to another.

Important metacognitive strategies for working on modelling problems in small groups

The influence of metacognition on learning results was investigated in many studies, but the conclusions are ambiguous, as mentioned above. A reason for the ambiguity may be the fact, that metacognition is normally measured regarding a single person and correlates with her/his own mathematical performance. Solving modelling processes, however, is usually done in small groups. Therefore, one has to distinguish between the performance and metacognitive competencies of single team members and those of the group as a whole. But research on metacognition in the past has merely focused on individual processes. „By focusing on the individual student, researchers have failed to address the dynamics required for progressive knowledge building by collaborative learning groups" (Chalmers, 2009). However, "team cognition emerges from the interplay of the individual cognition of each team member and team process behaviors." (Cooke, 2004) So to solve a modelling problem successfully, not the individual, but the group competencies are crucial: Students have to share their knowledge and their competencies (Artzt & Armour-Thomas, 1992).

Thus, for working on modelling problems successfully in small groups, metacognitive strategies are of great importance. In previous studies at the University of Hamburg the following strategies were identified as those, that were used by students as well as classified as useful or even necessary:

- Strategies for planning:
    - P1: Subdivide the solution process in several steps,
    - P2: Allocate parts of work to different team members,

  - P3: Structure the solution process according to the time available,
  - P4: Choose useful solution strategies
- Strategies for monitoring and, if necessary, for regulating the working process
  - M1: Identify different kinds of red-flag-situations
  - M2: Notice incomprehension
  - M3: Keep track of the time available
  - M4: Check the work habits
  - M5: Reconsider solution strategies
- Strategies for evaluating the modelling process to improve it
  - E1: Evaluate the strategies used
  - E2: Reflect on the working habit
  - E3: Validate on the solution (cf. Schroeder, 2013)

The identified strategies were used for developing instruments for measuring students' use of metacognitive strategies while modelling, as shown in the next paragraph.

Measuring metacognitive strategies

In general, for measuring procedural metacognitive modelling competencies, two possibilities exist: Online-methods such as thinking aloud, observations, eye-movement or logfile registration enable process diagnostics concurrent to task performance. Thus, a deeper look into the metacognitive behaviour of students is possible without disturbing and influencing them too much. However, these methods cost a lot of time and money. Therefore, they can only be used for small samples. While using offline methods like (prospective or retrospective) interviews or questionnaires, the results rely on the students' self-reports. These methods bear the risk that strategies may be used unconsciously or their use may be forgotten by the students. Furthermore, the item formulation may remind the students on the usefulness of certain strategies. Consequently, they will answer according to their metacognitive knowledge and not on basis of their behaviour. However, in contrast to observations and thinking-aloud-protocols, processes which were not verbalized because of different reasons can be measured with the help of questionnaires or interviews. In addition, questionnaires can be used for bigger samples. (Veenman, 2011).

For obvious reasons, the development of a questionnaire is desirable. For doing so, the identified metacognitive strategies mentioned above were used as a basis. The questionnaire used in this study consists of 40 items divided into the sub-processes of planning, monitoring, regulating and evaluating, 27 of them concern individual metacognitive strategies, 13 items regard group strategies. Students are asked to judge their use on a five-point-scale. Furthermore, students are asked to judge their motivation to work on the task, the task difficulty and their satisfaction with their small group. To give an impression of the questionnaire, selected items and the relation to the coding guideline presented above are shown in Table 1.

Item	Relation to coding guideline
1.1 I have thought about how to solve the Problem best on my own.	P1
1.2 We tried to recognize possible steps together.	P1
2.1 I questioned my own ideas.	M5
2.2 I questioned the others' ideas.	M5
3.1 When we found a solution, I reconsidered the whole solving process.	E1
3.2 When we found a solution, we were wondering what we can do better next time.	E2

Table 1: selected items of the questionnaire

Research questions

As presented above, metacognitive competencies seem to be necessary for working on modelling processes successfully. Therefore, teaching units for fostering these kinds of competencies are desirable. To evaluate these teaching units, instruments for measuring metacognitive competencies are needed. Thus, a questionnaire for measuring students' individual metacognitive modelling competencies as well as those of a small group was developed. In the study presented in this paper, the students' self-reports in the questionnaire were compared to experts' ratings on the students' use of metacognitive strategies while working on a modelling task and to students' self-reports in an interview afterwards. Hence, the research questions of this study are:

- *For which metacognitive strategies – at an individual as well as at group level - do the students' statements in the questionnaire correspond with experts' ratings as well as with students' statements in interviews?*
    - *Which metacognitive strategies can be measured more reliable by students' self-reports?*
    - *Is any further information required to interpret students' self-reports?*
    - *Which metacognitive strategies can be measured more reliable by experts' ratings?*

Design and methods of the study

For answering these questions, students of grade nine of three different classes were introduced to a modelling cycle and then worked in groups of four on a modelling problem. The working process was videotaped. After working on the problem, the students were asked to fill in the questionnaire presented above. While doing so, they were not allowed to speak to each other and discuss the items. In the afternoon, students were interviewed using a stimulated recall-interview (Gass & Mackey, 2000). For this, selected scenes from the video were shown to them and they were asked to comment them. Afterwards, some questions about their attitude towards the importance of metacognitive strategies were posed.

To answer the research questions, the videos as well as the interviews were analyzed using the items of the questionnaire as coding guideline, following qualitative content analysis (Mayring,

2010). Those codings were compared with the answers in the questionnaires. In the next section, first results from the study of one of the small groups are presented.

First results of the study

The group consists of four girls, which are named Anna, Julia, Olivia and Lea in this paper; three of them were interviewed afterwards. Anna, Julia and Olivia all mentioned in their interviews that they were used to work together in this group; Julia indicates in the interview, that Lea is a new student and is not familiar with the other students. She assumes that this might be the reason for Lea not taking part during group work.

In the following, special attention is paid to items that were answered very differently by the students within the small group or items for which the different sets of data provide different information. Thus, students' statements in the questionnaire concerning selected items of the sub-processes of planning, monitoring and evaluation will be compared with the respective statements in the interviews as well as with outcomes of the analysis of the videos. An overview of the students' statements in the questionnaire is shown in Figure 1.

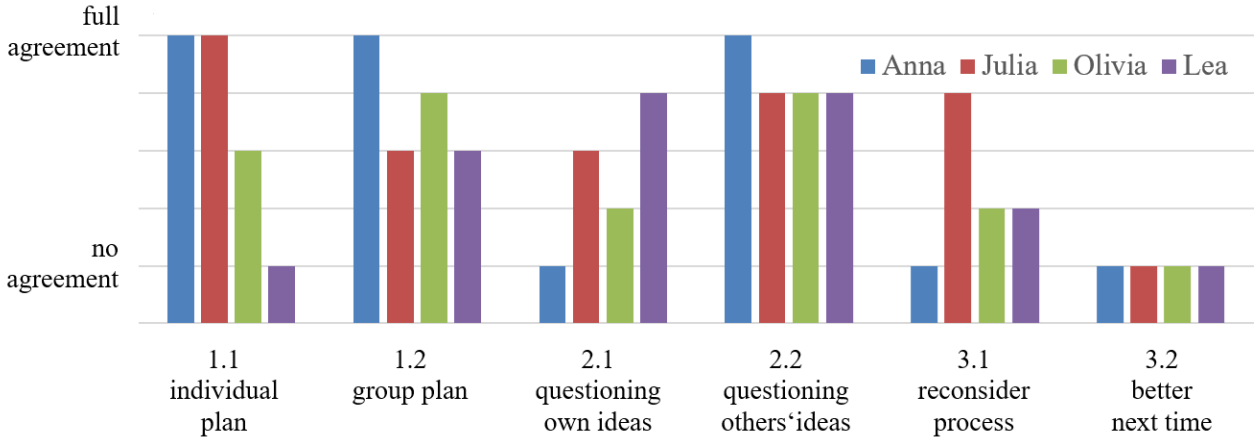

Figure 1: selected students' judgements in the questionnaire

In the questionnaire, Julia and Anna indicate that they worked out a plan own their own before planning the solving process together, whereas Olivia only did this partly and Lea did not plan on her own at all. Their perception of developing a plan in the whole group differs (see Figure 1). Thus, questions on the causes of these differences arise. When analyzing the video, one can clearly identify a scene in the beginning, in which the group is discussing how to proceed. You can see that Anna is the one, who develops a plan, whereas Julia and Olivia are not convinced and ask several questions. Before their questions are answered satisfactorily and before they are convinced, Anna starts to work. This scene was shown to the girls during the interview. When asked to comment on the scene, Olivia did not say much about the planning process:

Interviewer: How did you proceed in this situation?

Olivia: I don't really know. Actually, Anna said we should use a scale and then we knew what to do.

Only when asked about the necessity of having a plan, she talked about the importance of planning:

Olivia: Most times, planning is better, because you then know this is the next step, and then that step.

Anna on the other hand spontaneously commented her behavior:

Interviewer: In this scene, you have decided how to solve the problem. How did you decide?

Anna: I said how to, I don't know. We have. I had the idea of scale and then, Julia wanted to calculate the volume. But then we decided for the scale.

Summing up, the statements in the questionnaire express the level of conviction concerning Anna's plan. This suggests that the students' perspective on group planning was measured correctly, although the statements regarding group planning differ.

Regarding the sub-process of monitoring, both Anna and Olivia state in the questionnaire that they have not or only to a very small extent questioned their own ideas, but to a higher extend others' ideas (Table 1, 2.1 and 2.2). Comparing this data with those from expert ratings and from the girls' statements in the interview, different reasons for these statements are revealed. By analyzing the video of the girls' working process, Anna can be described as the one, who brought in the most ideas and managed the group in some ways. Although she wasn't aware of doing so during group work, she recognizes her behavior in the interview when asked to comment the scene:

Anna: I said that doesn't matter, I took over power and blocked other's suggestions and explanations. Seeing my behavior frightens me, I did not realize I was doing this.

Thus, her statement in the questionnaire indicates that she was a group leader that did not approve of others' ideas, because she was very convinced of her own. In contrast, Olivia did not participate with her own ideas or took over any other active responsibilities. However, it becomes clear from several statements that she was monitoring the whole process and questioned the process if necessary. But based on her statements in the interview, it becomes clear that she is not aware of doing so herself:

Interviewer: And what about looking about one's own shoulder? […]

Olivia: I don't think so.

Interviewer: Why not?

Olivia: I don't know. We are a group that simply work. And then, ready.

Thus, Olivia uses metacognitive strategies of monitoring unconsciously. This makes clear that it is sometimes necessary to have further information about the group processes and the different roles of the students. One possibility are ratings by experts. However, those are not sufficient solely, as one can see in regards to Lea. She did not say a word while working on the problem, but states to have monitored the whole process. If her judgement is right, it cannot be proved. But as the answers from the questionnaire do not count regarding marks and it was clear to the students that their math teacher will not get their judgements, you can state that Leas statements are correspondent with her perception of her own behavior.

The students' statements about evaluating the whole modelling process correspond and match with the researchers' analysis completely: The group did not evaluate their working process significantly (except of Julia).

Conclusion and outlook

The selected results of the pilot study presented above illustrate in a considerable way the importance of sharing metacognitive competencies in a group: Presumably, none of the girls would have solved the task on their own. Even Anna, the "group leader", needed Julia and Olivia for monitoring and validating the modelling process. However, it also becomes clear that measuring students' metacognitive competencies is challenging. Measuring metacognitive group competencies is even more challenging. In this study, different methods for measuring the use of metacognitive strategies while working on a modelling problem in small groups were used: students had to fill in a questionnaire and were interviewed. Furthermore, their behavior was judged by researchers.

The presented results clearly show that some answers in the questionnaire are not consistent with statements in the interviews or with the analysis of the students' working process. In addition, students' judgements about incidents during group work differ.

As presented above, almost all differences could be explained by consulting not only one, but different items or by using all three datasets. However, not all students, who take part in the main study (about 600), can be interviewed nor can their group work be analyzed. Therefore, it should be analyzed next, if there are any key or filter-items in the questionnaire that give information about how to judge other items. In accordance, the items have to be identified that can be rated by experts better than by students themselves and it has to be analyzed whether this is a question of special items in general or a question of students and the role of the students in small groups.

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Product orientation in modeling tasks

Wolfgang Bock, Martin Bracke and Patrick Capraro

KOMMS, University of Kaiserslautern; {bock, bracke, capraro}@mathematik.uni-kl.de

Keywords: Real-world problems, modelling tasks, client action.

Introduction

Beginning with the first Pisa study in 2000 (OECD, 2001) there is a worldwide discussion of an appropriate education in the so-called STEM subjects. One cause was the alarming fact that in many countries the majority of secondary school students fail to reach proficiency in math and science (Kuenzi, 2008). One significant reaction to the above mentioned debate was to strengthen the role of mathematical modeling in teacher training and the curricula. In Germany, this can be seen from the fact that mathematical modeling is mandatory in most of the recently introduced master programs for mathematics teachers. Moreover, there is a strong increase in the number of publications on mathematical modeling of real-world, realistic or authentic problems over the past decade. The following definition is due to Bock & Bracke (2013):

Definition 1: *An authentic problem is a problem posed by a client, who wants to obtain a solution, which is applicable in the issues of the client. The problem is not filtered or reduced and has the full generality without any manipulations, i.e. it is posed as it is seen. A real-world or realistic problem, is an authentic problem, which involves ingredients, which can be accessed by the students in real life.*

Real-world problems and authentic problems are used and studied in many different modeling activities such as the TheoPrax method (TheoPrax), modeling weeks and modeling days, the Junior-Engineering Academy (Bock & Bracke, 2013), Fraunhofer-MINTeC[1] Talents (Bracke et. al., 2015). Especially in TheoPrax and also the modeling activities in vocational education (e.g. Wake, 2014) the activities aim towards a specified product from the beginning. In product management a product is a deliverable or set of deliverables that contribute to a business solution. In (Kotler et.al. 2006) it is defined as "anything that can be offered to a market that might satisfy a want or need" (p. 230). In the client-provider situation the product thus is the good sold to the client by the provider. It, therefore, is directly related to the needs of the client meaning the task the client gives to the provider. In the real world, e.g. in industry, an actual client is not really interested in mathematical models – very unfortunate to most of mathematicians - but mostly in a tool or a strategy he can directly use for his purposes. In our opinion for modeling activities having real-world or authentic problems this is a dimension which is to be included into the existing modeling process. For authentic and real-world problems we therefore define the notion of a product.

Definition 2: *In a mathematical modeling situation (with an authentic and real-world problem) a product is a deliverable in the language of the client which satisfies the needs incorporated in the*

[1] See also: Fraunhofer MINT-EC Talents programme. http://www.fraunhofer.de/de/jobs-karriere/nachwuchsfoerderung/mint-ec-talents.html and MINT-EC e.V. http://www.mint-ec.de

task given by the client to the provider in such a way, that the client can use it directly for his purposes.

Research questions

At the University of Kaiserslautern modeling activities including products have been performed for years. Even in the first modeling week in 1993 one can find in 3 of 8 projects concrete products such as computer programs and strategies. It is planned to study how modeling tasks and the development of products changed during this time based on the reports of past modeling activities. It is to expect that both changed with the introduction of computers and the increase of programming skills. For this purpose we intend to analyze existing material from the past 23 years on how strong the focus on project orientation in the modeling tasks have been. The use of computers during the modeling weeks is well documented, however the data is not enough for empirical studies. Thus for a more involved study we plan to gather material from other universities involved in modeling weeks. Furthermore we want to investigate how the client action during the modeling process changes the whole process of the modeling cycle and the product itself. A product can in reality be produced without a modeling cycle; on the other hand the modeling cycle can be performed without producing a product. However the quality of the product can be improved by an iterated mathematical modeling process in most cases. The client in this case plays the role of an external control in the sense that he/she will accept or not the product. He/she can also ask for new features or ask for specific extensions. To study this we plan to apply different approaches (with client/without client) to different but similar groups in modeling activities. The performance of the group should be investigated via video analysis, such that we can estimate how often a modeling cycle was performed.

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Modeling tasks and mathematical work

Charlotte Derouet[1], Alain Kuzniak[1], Dominique Laval[1], Elizabeth Montoya Delgadillo[2], Laurent Moutet[1], Assia Nechache[1], Bernard Parzysz[1], Laurent Vivier[1]

[1]Université Paris-Diderot, LDAR (EA 4434), Paris, France; laurent.vivier@univ-paris-diderot.fr

[2]Pontificia Universitaria Católica de Valparaíso, Valparaíso, Chile, elizabeth.montoya@pucv.cl

Keywords: Mathematical Working Space (MWS), modeling tasks, modeling cycle.

In the Mathematical Working Space (MWS) model, an epistemological plane and a cognitive plane are introduced with a focus on their interactions related to semiotic, instrumental and discursive dimensions (Figure 1a). The model is devoted to the analysis of mathematical work with, specifically, paradigms guiding and orienting the work. Numerous researches are based on the MWS model and the reader may refer to special issues in journals such as Bolema 30(54) and ZDM-Mathematics Education 48(6) in which an introduction to the model is given in the survey paper. Nevertheless, until now, few studies on modeling tasks have been based on the MWS model and we want to highlight recent researches, in particular coming from PhD studies within our team.

Based on the modeling cycle (figure 1b) proposed by Blum and Leiss (2005), we suggest, in the poster, some adaptations which help to understand how MWSs can be used. The whole modeling process is not taken in its whole, and we focus more on how the analysis can be refined, mostly between phases 3 and 5 of the cycle, in relation to activity in different mathematical domains.

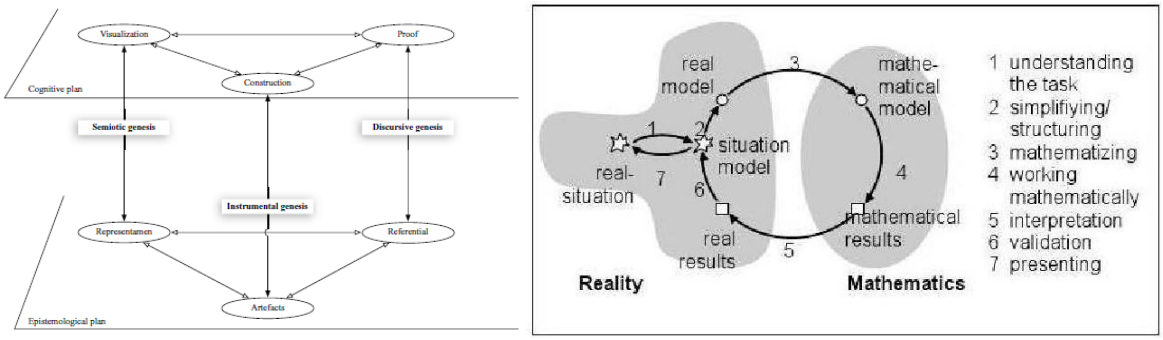

Figure 1a: MWS diagram; Figure 1b: modeling cycle (Blum & Leiss, 2005)

Nechache (2016) suggests describing the modeling work in probability situations, with the MWS framework. She identifies the importance of the theoretical referential of the MWS_{Proba} in the constitution of the real model. Then, for the analysis of the mathematical part, she fully uses the MWS_{Proba}. In the same way, the MWS model can be used for studying other mathematical domains.

Derouet (2016) proposes a similar type of use for the mathematical part, but she associates sub-phases to the stages of the cycle in order to investigate the progress of the modeling process. She isolates a part of the cycle containing "real model" and "real results" that she names pseudo-concrete. It allows her to identify, in a modeling situation related to continuous probability, a work within the MWS_{Proba} in various working paradigms.

In these studies, the MWS model allows to refine the analysis of the mathematical part by taking into account a first horizontal mathematization followed by a second vertical mathematization allowing to strengthen the mathematical model. Other types of change, or transition, are possible,

like the change of MWS or mathematical domains. In his study in relativist kinematics, Moutet (2016) suggests an extension of the MWS model to take into account a change of matters. He considers a second epistemological plane for physics, and he studies the interactions between these two planes and the cognitive level.

In these studies, simulation associated with digital models can also be considered as an important stage of the modeling process. It plays two different roles. The first one is in relation to the development of the real model with a simulation close to the initial situation (urn model or a calculator which proposes rolls of dice or coin, in probability). The second role presupposes a stronger mathematical expertise in the MWS of the domain at stake as, for example, the implementation of an algorithm of dichotomy in analysis.

Hence, the use of the MWS framework can enrich and strengthen the analysis of the modeling process based on the study of a cycle (figure 1b) in connection with a first resolution of the problem. It constitutes a first interaction between MWS and the modeling cycle, as a *first cycle*[1]: The modeling problem has been mathematized and it is possible to identify the epistemological and cognitive components of the MWS in relation to the student's activity and realization in the different domains and paradigms. But we can also, in a more didactic way, think of a *second cycle* aiming at a better understanding of the model and of the mathematical objects introduced to solve the problem by students. In that case, the modeling task proposed by a teacher aims not only at solving a real problem but more deeply at exploring and understanding the numerous uses of a mathematical notion, enriching the MWS, in particular the theoretical referential. This is what we are developing in a work on progress on the exponential function.

Acknowledgment

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[1] Even if, of course, one should not understand the cycle as requiring to go through every stage, in a linear way.

Mathematical modeling and competencies for biology students

Yannis Liakos

University of Agder, Kristiansand, Norway; ioannis.liakos@uia.no

The research aims at introducing modeling tasks with an ultimate goal to engage students more actively into learning mathematics through tasks that are biologically 'colored'. My focus is on the individual progression (if any) of students' mathematical competencies during a sequence of modeling sessions as part of their first year mathematics. My goal is to explore the nature of progression within the competency profile of participating student, the relation between modeling processes and this progression and what transformations are applied in these sets of competencies.

Keywords: Mathematical modeling, mathematical competencies, tasks, progression.

Short description of the research topic

The literature provides documentation of learning benefits by engaging students in mathematical modeling: educational benefits (Kaiser et al., 2006); students engaged in modeling may develop a deep understanding of the content and an ability to solve novel problems and that it can bring students into alignment with the epistemic aims of science and help them develop more sophisticated ideas within the area of study they are focused on. I adopt a design based research approach in which an iterative process of design, implementation and analysis takes place.

The research questions

RQ_1) What is the nature of progression in a student's competency profile through the course of the mathematical modeling unit?

RQ_2) Considering a set of competencies (competency profile) as a body of knowledge, what transformations are applied in the course of the modeling sessions?

Theoretical and methodological framework

My theoretical approach is informed by the anthropological theory of the didactic (Bosch & Gascón 2014) and, in particular, by the distinction between four bodies of knowledge: scholarly knowledge; knowledge to be taught; knowledge taught; and knowledge learnt. A significant work on mathematical and modeling competencies has been done by many researchers (e.g. Maaß 2006; Blum & Kaiser 1997) while Niss (2003) created an 8-fold (KOM project) system of mathematical competencies. I decided to proceed to an adaptation of the KOM model adjusted to the above mentioned literature and the specific context I am working on: students of a Biology Department. Part of my analysis will be based a 3-D (radius of action, degree of coverage and technical level) model of progression for each competency from Niss and Højgaard (2011). I have also constructed a coding system in order to "attach" specific students' expressions to a certain mathematical competency. These constructs form a first set of tools for data collection and analysis.

The context and methods for data generation and analysis

The context (or arena) of this study is the Biology Department of a Norwegian university and a mathematics course for first year Biology students. My main study took place with first year Biology students. Modeling sessions occur weekly during the first semester. These sessions (for groups of 3 to 4 students) are 50 minute in length and supplement lectures to the whole cohort of students. Tasks in the sessions are designed in five 2-week blocks focused on a subset of mathematical competencies that students should bring into action in order to complete the task.

To address RQ_1 I will, as above, explore selected students' small group discourse activity. Data were collected through audio-visual recordings of students working on tasks synchronized with their writing (using Smartpens which records audio and visual data accompanying written data). At RQ_2

I am addressing the question: what is the "offered body of competencies" (an a-priori analysis). For the next three steps I will use my data to provide answers or useful directions. A task-design analysis, for example, can provide what the existing literature provides on population dynamics and exponential growth (scholarly knowledge) but also which task was finally decided to be presented (knowledge to be taught) and this will happen for every different modeling block.

Potential significance and contribution

My research will contribute to the following areas (1) dynamic competencies profiles for students, (2) a critique of Niss' (2003) 8-fold system of mathematical competencies, and (3) a description, in the context of mathematical modeling in the Biological Science, of ATD's four bodies of knowledge.

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Phenomenological analysis of the line: A study based on the Didactical Analysis of mathematical knowledge

Guillermo Ramírez-Montes[1] and Fabián Gutiérrez-Fallas[2]

[1]Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal; grm1905@gmail.com

[2]Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal; fgutierrez92@gmail.com

Keywords: *Phenomenological analysis, line, Didactical Analysis.*

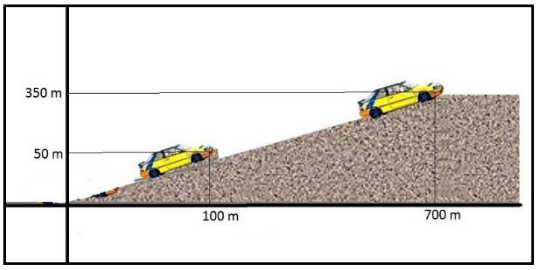
Improving the learning of students has been a concern in research in Mathematical Education. Consequently, didactical aspects and tools for teaching of a successful mathematic, have received a growing attention in different studies. The Didactical Analysis (DA) is considered by Rico, Lupiañez & Molina (2013), as (a) a tool for teaching training; b) an instrument for curricular innovation; and (c) a methodological and research tool, this latter considered in this study. The DA is a cyclic process which implies four sub analysis: content analysis, cognitive analysis, instruction analysis and performance analysis (Gómez, 2002). In the content analysis, we find the Phenomenological Analysis (PA) of a concept, that consist of "describing what are the phenomena for which (the concept) is the means of organization and what relationship has the concept or structure with these phenomena" (Puig, 1997, p.62). Freudenthal (1983) names didactical phenomenology to the phenomenology that is characterized by consider the concept as a cognitive process, as a subject of teaching and to be learned by students, that beside organizes phenomena in the student`s world and proposed in the teaching of concept. For this author PA's aims is to serve as the basis for the organization of mathematics teaching.

The intention of this poster is to present results of a PA of line concept on the Cartesian plane, based on the DA of this concept, to answer: what is the knowledge related with the line used for? The data were collected between 2013 and 2014 through a books selection of mathematical and didactics texts of secondary level. The PA of the line was carried out by documental analysis of these texts and its construction was realized in three phases: (i) define mathematical substructures of line concept on the Cartesian plane (mathematical-world), (ii) define the phenomena that each mathematical substructure organizes (real-world), (iii) establish the relationship between substructures and phenomena.

Our proposal consists to show that from this PA, mathematics teachers can organize the teaching of the line on the Cartesian plane promoting the mathematical modelling process of Borromeo Ferri (2006), where phenomena and mathematical substructures are part of different sets, but they are related each other according to the proposed steps in the modelling process. In our study, we identified four contexts that lead to this modeling process, each one referring to phenomena and substructures of how organize the contents of line: (C1) the slope of a linear trajectory with respect to a fixed reference line, (C2) the linear relationship that occurs between two magnitudes, (C3) the behavior between two or more linear relationships, (C4) the distance between two or more objects.

For instance, in C1, a possible real-world model related to the mathematical substructure of the *slope of the line* is as follows:

The picture presents the information of a car when climbing p a road. In this one is showed the position of the car in two different moments, when it has moved 50 m and 350 m respectively.



What is the slope and angle of inclination of the road?

What is the relationship between the slope and the tangent of the angle of inclination?

In relation to C2, a possible real-world model related to the mathematical substructure of *the linear equation of two variables* (or *the equation of two variables of first degree*) is the following:

Carlos has a bank account where he earns 0,5% interest from the initial savings each month. If Carlos opened the account with 2000€ ten months ago, how much money will Carlos currently have? How many months are needed to have 8000€?

In the poster, we would present examples of each context, highlighted the details to consider them like starting points to promote the modelling process in the learning of the line in the Cartesian plane.

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TWG08: Affect and mathematical thinking

Introduction to the papers of TWG08: Affect and mathematical thinking

Pietro Di Martino (Chair), University of Pisa, Italy

Inés Mª Gómez-Chacón, Universidad Complutense de Madrid, Spain

Peter Liljedahl, Simon Fraser University, Canada

Marilena Pantziara, Cyprus Pedagogical institute, Cyprus

Stanislaw Schukajlow, University of Münster, Germany

Introduction

This chapter introduces the contributions discussed during the working sessions of the TWG8 "Affect and mathematical thinking" in CERME 10.

In this edition our TWG has been enhanced by the inclusion in the group and the contribution in the discussion of the researchers from the TWG7 "Mathematical Potential, Creativity and Talent".

The quantitative data about the participation to our group confirms the interest toward affective issues in the field of Mathematics Education: 41 manuscripts were initially submitted to the groups, 26 were accepted for the discussion, and finally in these proceedings 24 papers and 2 posters are included.

Moreover, our group confirms its spirit of inclusion: 12 different countries were represented and 11 newcomers welcomed.

The papers (24) and the posters (2) were presented and collectively discussed in the first six sessions. Presenters had 10 minutes (5 minutes for posters) to introduce the key-ideas of their papers, then we developed a 10 minutes discussion. At the end of each session, 20 minutes were used to discuss the main aspects emerged in the section.

In the fourth section, one hour has been devoted to the presentation of the ERME Chapter about affect: the draft of the chapter was been sent to all participants before the conference, and each participants had to share two questions, comments or criticisms about this draft. The collected comments are been the thread of the discussion related to the chapter.

In the seventh and last session, we discussed the main themes emerged during our previous work, developing the structure of the report for the final day.

The issues emerged in TWG8 at CERME10 and the related discussion

The analysis of the affective focus of the papers discussed in our TWG reflects the current diversity of interests and approaches inherent in the field of affect research. Many different affective constructs emerged: beliefs, motivation, values, emotions, needs, relaxed, memory, aesthetic, confidence, meta-affect, identity, self-efficacy, meaning, motivation, values, images, views, flow, perseverance, tolerance, interest.

It is important to underline as many of these constructs are clearly related, sometimes – and this is a well-known critical aspect in the affective research – different labels are used to indicate the same constructs, and vice versa. Sometimes the same term is used in a very different meaning by different

researchers. As we will underline later, this communicative issue is particularly critical for the emotions.

In our discussions, we underlined the lack of conceptual clarity again, and called researchers to use a clear definition in their studies and to label the constructs appropriate.

Despite this variety, we recognized five recurrent and crucial dimensions involved in the discussion of our group: students, problem solving, self-concept, emotions and context.

In the following, we briefly discuss some aspects related to these five dimensions.

Students

Enriched by the contribution of researchers from the TWG7, we have long debated about the so defined "achievement problem", discussing around the following clearly related questions:

- Looking at school transitions: what are the effects of these transitions on school achievement? How and why do the parameters of the school achievement change dramatically from a school level to the next?
- What is the relationship between school achievement and mathematical talent?
- What is the distinction between low and high achievers in mathematics apart from the grade?

We also argued as it would be interesting to develop research around the above themes, looking at students who are studying mathematics in different contexts (for example modelling, IT environment, etc.).

Problem solving

There is a known and long tradition of research about affective construct and problem solving. This kind of research it is particularly important for our field, since it shows two crucial aspects: the strict relationship between affect and cognition (problem solving activities surely involve cognitive aspects but also strong emotional reactions); the relevance and peculiarity of the research on affect in the specific context of mathematics: indeed problem solving is one of the main activity for mathematicians.

As usual, one of the main issues is how teachers can create the context in order to develop the appropriate mathematical activities and environment for positive affect, increasing motivation and also performance. In several discussions within our group, it emerges as problem solving is not only an essential activity for developing the mathematical competence, but it has also the potential to draw attention and to motivate students, because – in some sense – problem solving is one of the beautiful side of math (the eminent mathematician Ennio De Giorgi used to say: a nice problem, even if you do not solve it, accompanies you).

Many papers presented in our group stressed the relevance of two aspects in order to take advantage of the *affective potential* of problem solving.

The first one is related to the setting: here setting is understood in a broad meaning, on the one hand we mentioned the classroom climate – a positive climate is needed to develop significant problem solving activities, in particular it appears crucial to not identify problem solving activities and

assessment – and the context (for example the spatial organization of the classroom). On the other hand, we mentioned the organization of the problem solving activities with the promotion of collaboration and discussion among students.

The second one is related to the choice of the mathematical problem. It is confirmed as only cognitive demanding (and not routine) problems can foster students' engagement, but, above all, can challenge students shifting their attention from products to processes. In particular, it is also crucial the monitoring of the possible imbalance between skill and challenge that student can experience during problem solving. In this setting, problem solving can be a tool to involve high-achievers, but also low-achievers, in mathematical activities and increase positive affects towards mathematics.

Self-concept

The fact that students/teachers self-concept strongly affects their choices, the effort they devoted and their perseverance in doing some specific activity is one of the fundamental points in our field. Therefore, it is not a surprise that many discussions in our group focus around self-concept and related constructs, such as: self-efficacy, self-perception, self-regulation, identity, personal meaning.

In particular, three aspects related to the self-concept have been analysed and discussed in this edition of the TWG8:

- How students' self-concept influences their interpretation of the *mathematical environment*;
- How the context affects the self-concept, in particular it emerges the idea that context provides available identities;
- The connection between the self and the emotions (in the context of mathematics education). Self has seen as a filter for interpreting experiences.

Emotions

Obviously the study and the discussion around emotions is a must in the group of affect.

In particular, we discussed some crucial dichotomy related to the concept of emotion:

- Emotions sometimes are a cause for some didactical outcomes, but sometimes are a symptom. In the first case, we see positive emotions as an educational goal, in the latter emotions are an indicator;
- Emotions have a double nature: more rapidly changing state-aspect and more stable trait-aspect. The study of state, as opposed to trait, is necessary to give a more detailed description of emotional experiences in the mathematical teaching-learning process;
- Emotions can be the cause for opposite pathways during problem solving, or more in general mathematical activities (go on vs. give up).

As usual, we also discussed how deal with two classical critical issues in the research about emotions: one related to the observation and the other related to the communication.

Emotions, in truth as many other constructs, are not directly observable, therefore we never observe emotions, but we infer them from some indicators. Sometimes we collect information about emotions through self-reports, and – already at this stage – communication issues intervene; studies

developed in different countries stress the level of *emotional illiteracy* of a large part of the population: it is difficult to reflect about emotions, and it is more difficult to have the dictionary to communicate them.

The *communication problem* involves the field of the research not only for the difficulties related to the emotional illiteracy, but also for the *internal* communication (the communication between researchers): as a matter of facts, the labels for emotions have different shades in every language, and sometimes they are associated to a different meaning.

Context

Affective issues are mainly social for their nature: as we have seen, context plays a crucial role in the development of affective reactions. For this reason, context is always been one of the *leading actors* in discussion within TWG8.

A still open problem is the exact definition of context: what is the context? We are convinced that the context is dynamic in nature (it depends on the group/individuals), and this dynamicity makes more complex to circumscribe it.

In the group discussion was observed that in the context of emotion, the term ‘culture’ needs to be ‘unpacked’ and broached not from the assumption that cultures are uniform, but rather from more dynamic conceits such as put forward by identity theories.

Some studies in this edition focus on the difficulties related to the transition from a context to another one (for example in the school transition), analyzing how and how much these *context-transitions* change affect.

Positioning and emotions in learning algebra: The case of middle-achieving students

Wajeeh Daher[1,2] Osama Swidan[1,3] Amani Masarwa[1]
[1]Al-Qasemi Academic College of Education, Baqa, Israel wajeehdaher@gmail.com
[2]An-Najah National University, Nablus, Palestine
[3]Kay Academic College of Education, Beer Sheva, Israel

Mathematics education researchers utilize different theoretical frameworks to study the role of affect in learning mathematics. This paper utilizes a discursive framework (Evans, Morgan, & Tsatsaroni, 2006) to study middle-achieving students' emotions in learning, utilizing technology, the topic of 'the quadratic function as a product of two linear functions'. The students' learning was videoed and then transcribed to be analyzed according to the discursive framework. The research results indicate that members in the middle-achieving groups claimed the collaborator positioning in order to learn the mathematical topic using mainly behavioral, social and cognitive processes. Leaders claimed their positioning through carrying out processes related to the different aspects of learning, mainly the cognitive, metacognitive, meta-emotional, social and linguistic aspects. Dominant emotions in the groups' learning were frustration, enjoyment and content.

Keywords: Discursive framework, positioning, emotions, middle-achieving students.

Introduction

The affective aspect is a growing area in educational research due to its relationships with other aspects of students' learning, especially the cognitive one. In the present paper, we study middle-achieving students' emotions in relation to their positionings when they learned the quadratic function as a product of two linear functions using dynamic software. In a previous research, we examined students' positioning and emotions in one group learning geometry (Daher, Swidan & Shahbari, 2015), where the group consisted of two high-achieving students and one middle-achieving student. Following that study, we wondered how emotions and positioning are experienced by groups consisting of only middle achieving students. This paper intends to examine this issue, using the discursive framework (Evans, Morgan, & Tsatsaroni, 2006).

Emotions in mathematics education

Polya (1957) addressed the necessity to consider emotions as they influence the problem solving process. Later, especially in the 1980's, researchers considered affect as a significant component of students' mathematical problem solving (e.g., McLeod, 1988; Schöenfeld, 1985). Emotion is one of the fundamental elements of the affective aspect (Hannula, 2004). Hannula (2004) describes emotions as connected to personal goals. Furthermore, emotions, when managed appropriately, become a potential tool for effective thinking rather than a disturbance to this thinking (Antognazza, Di Martino, Pellandini, & Sbaragli, 2015; Salovey & Mayer, 1990). In the present research, we intend to study, by using the discursive emotions and positioning framework, middle-achieving students' emotions when they utilize technology to study Algebra.

The discursive positioning framework for studying students' emotions

Positioning is defined as "the discursive process whereby people are located in conversations as observably and subjectively coherent participants in jointly produced storylines" (Davies & Harré, 1999, p. 37). Evans, et al. (2006) suggest this discursive positioning framework for studying students' emotions. This framework assumes that meaning making occurs in social practices, using semiotic resources. Social practices have an emotional dimension that helps maintain social identity. Moreover, empirical data in this framework is seen as text, the analysis of which demands attention to its context(s). This analysis entails a combination of structural and textual phases that each informs the other. The structural analysis considers the positions available to or claimed by the participants. Positions are associated with power in relation to others, as well as with differing values within the discourse, which creates spaces within which emotion may arise. Usually, there is more than one available position for a participant, either within a single discourse or several competing discourses. Evans, et al. (2006) describe the positionings taken care of in the structural analysis: Helper and seeker of help, collaborator and solitary worker, director of activity and follower of directions, evaluator and evaluated, insider and outsider.

The textual analysis considers the exchange of meanings. This phase has two functions (Evans, 2006): (a) showing how positionings in social interactions are actually taken up by the participants, and (b) providing indicators of emotional experience. The textual analysis has two stages. In the first stage, the focus is to identify the interpersonal aspects of the text that establish the positions of the participants. Indicators at this stage include reference to self and others, reference to valued statuses (e.g. claiming understanding or correctness), modality (indicating degrees of un/certainty), hidden agency (e.g., passive voice) or repetition. For example, leadership is indicated by demonstrating knowledge or meta-emotional behavior (e.g., trying to change the negative emotions of the group members). The collaborator position is indicated by the activity of a group member, as answering questions or doing actions in response to events occurring during the group learning.

The second stage of the textual analysis attends to (a) indicators of emotional experience that include: direct verbal expression (e.g., 'I feel anxious'), use of particular metaphors (e.g. claiming to be 'coasting'), emphasis by words, gesture, intonation, or repetition, body language (e.g., facial expression); (b) indicators suggested by psychoanalytic theory, as indicators of defenses against strong emotions like anxiety, or conflicts between positionings (as 'Freudian slips'), surprising error in problem solving, behaving strangely (as laughing nervously), denial (e.g., of anxiety).

Research rationale and goals

In spite of mathematics education researchers' acknowledgement of the role of affective aspects in mathematical education in general and mathematical problem solving in particular, research related to this aspect is still not widespread (Antognazza et al., 2015). We intend to study emotions in problem solving using the discursive framework developed by Evans, et al. (2006). In more detail, we intend to analyze the positionings taken by ninth grade students and their related emotions when learning in groups, with the help of GeoGebra, the quadratic function as a product of two linear functions. Doing so, we introduce to the discursive framework the different aspects of learning; as the meta-cognitive and meta-emotional aspects. This will shed more light on the factors that influence middle-achieving students' experiencing of positioning and emotions.

Research question

- How are positionings taken up by middle-achieving ninth grade students, working in a group to learn the quadratic function as a product of two linear functions, in the presence of technology?
- How are middle-achieving ninth grade students' emotions associated with the positionings that they claim, when learning with technology the quadratic function as a product of two linear functions?

Methodology

Research setting and participants

In a previous research (Daher et al., 2015); we analyzed the affective aspect of one group's learning of mathematics, where the members were both high and middle achieving students. We wondered how the affective aspect would be affected in just middle achieving or high achieving groups of students. In the present research, we analyze this aspect in three groups of grade 9 middle-achieving students (ages between 14 and 15 years). One group consisted of three female students (Sana, Amal, Asil), and two consisted of two female students and one male student each (Fairouz, Noura, Salim) and (Alaa, Siham, Amin). All the participating students had not worked with GeoGebra before, and they were introduced to it in two hours' session before learning the quadratic function topic. Furthermore, the students had learned some issues in the topic of the quadratic function (the function's maximum or minimum, the vertex of the function and the domain of increasing/decreasing), but not the quadratic function as a product of two linear functions. The third author taught the three groups in a middle school in Israel.

Data collecting and analyzing tools

We collected our data using observations of the learning of the three groups. We also conducted interviews with their members. Every group's learning was videoed and at the end of each lesson, the three students in each group were interviewed individually regarding their positionings and emotions during learning. We analyzed the two types of collected data using the discursive analysis framework presented above. Moreover, we combined the analyses of the data collected by the two tools (observations and interviews). The findings section in this paper sheds light on this method.

Learning material

The three groups of ninth grade students worked with a sequence of activities; all related to the quadratic function as a product of two linear functions. Following is an example of these activities.

In the same coordinate system, we want to draw the three functions: $y=x$, $y=x+2$ and $y=x(x+2)$.

- What are the algebraic characteristics of the linear function: $y=x$?
- What are the graphical characteristics of the linear function: $y=x$?
- What are the algebraic characteristics of the linear function: $y=x+2$?
- What are the graphical characteristics of the linear function: $y=x+2$?
- What are the algebraic characteristics of the linear function: $y=x(x+2)$?
- What are the graphical characteristics of the linear function: $y=x(x+2)$?
- What are the similarities and the differences between the characteristics of the two above linear functions and the characteristics of the quadratic function?

Note: Algebraic characteristics are related to the parameters of an equation, while the graphical characteristics are related to the intersection points with the axes, increasing or decreasing of a function, etc.

Findings

The present research aimed at characterizing middle-achievement students' positioning and emotions when learning algebra with technology. Doing so, we found that mainly the students had the leader and collaborator positionings during a lesson. We will describe how the students in the middle-achieving groups claimed each of the positionings and experienced their emotions and/or reported them in each positioning. Doing so, we will address the following aspects of learning that the positioning is related to: behavioral, cognitive, meta-cognitive, social and linguistic. The emotional aspect of learning will be considered in light of the taken positioning.

Collaborator's functioning

The middle-achieving groups utilized collaboration to learn the new mathematical ideas. This is exhibited in that generally the members of each of the participating groups claimed the collaborator positioning to pursue, with the help of the mathematical software, their learning of the quadratic function. This claiming resulted in making the group's members learn enthusiastically to understand the appropriate mathematical relations. This resulted in the group's members enjoying the activity and being content when arriving at its solution. Thus the collaborator's positioning helped make the students' emotions concerning their learning experiences positive ones.

To claim the collaborator positioning, the group members were involved with behavioral processes (working with GeoGebra), social processes (group discussions), as well as cognitive processes (mathematical reasoning). These three types of processes, not only helped the group members claim the collaborator positioning, but at the same time, supported their attempts, as described above, to perceive the new mathematical ideas. In the interview, the students associated their behavioral processes with positive and negative emotions: enjoyment of their work when the software helped them solve the mathematical problem, and frustration when finding difficulty to operate the software. Excerpt 1 shows this claiming the collaborator's positioning.

A1	Sana	We need to find the intersection with x for the three functions.
A2	Amal	What's the first function?
A3	Asil	3x-2
A4	Sana	[drew the first function in GeoGebra]
A5	Amal	What's the second function?
A6	Asil	2x+3
A7	Sana	[drew the second function in GeoGebra] Let's find the intersection points with x.

Excerpt 1: claiming the collaborator positioning

Excerpt 1 shows the claiming of the collaborator's positioning as connected with the behavioral aspect of the group's learning. This aspect is expressed by the students' action with the Geogebra software (A4, A7). However, this positioning also involves the meta-cognitive aspect. The utterances of Sana (A1, A7) are concerned with regulating the processes of the problem solution.

Leader's functioning

The leader's positioning in the middle-achieving groups was claimed by directing the learning of the group, as well as to advance this learning towards the solution of the mathematical problems and the sharing of the new mathematical ideas. Moreover, leaders in the middle-achieving groups claimed their positioning through carrying out different types of processes, mainly cognitive, metacognitive, meta-emotional, social and linguistic processes. Below, we elaborate on these processes.

The group leader's cognitive functioning was actualized through demonstrating knowledge during carrying out the mathematical activity. For example, Fairouz, a leader in one middle achieving group, argued that they only needed to know the intersection points of the two linear functions with the x axis in order to draw the resulting quadratic function.

The group leader's metacognitive functioning was actualized through asking questions during the group learning as means to decide upon the method of solving a problem. Moreover, the group leader's meta-emotional functioning was actualized through trying to change the negative mood of the group when encountering a difficulty. For example, Alaa, a leader in a middle-achieving group, tried to lessen the anxiety of group members by saying: "Don't worry. It's O.K. Sure we made a mistake. Let's read again our solution to find it".

The group leader's social functioning was actualized through answering other members' questions, asking questions and requesting actions from the group members to keep the group learning going. Regarding the linguistic aspect of the leader's functioning, the leaders in the middle-achieving groups used the first person plural pronoun to talk about the mathematical actions that they needed to perform, which showed them as collaborators with the other members of the group. This indeed happened in the middle-achieving groups but not numerously (See for example excerpt 2).

The leader's functioning resulted in different emotions, but generally speaking this functioning resulted in frustration, when unable to find a way for solving a mathematical problem, enjoyment during the successful solution process, and content when finally solving the activity.

Difficulties in claiming the positions of leaders and collaborators

The members of the middle-achieving groups, due to the lack of appropriate previous knowledge in the subject matter and sometimes in GeoGebra manipulation, encountered difficulties in claiming the positions of leaders and collaborators during the process of the mathematical problem solving. This led to their experiencing some negative emotions. Moreover, the members of the middle-achieving groups experienced calmness, anxiety and confusion in accordance with their leader.

Working with GeoGebra, the members of the middle-achieving groups encountered at the beginning difficulties related to working with a new technological tool, which could be related to the behavioral aspect of the group's learning. Excerpt 2 describes such a difficulty, where Salim, Noura and Fairouz wanted to draw the function (2x-9)(3x-4) in GeoGebra [B1], but found difficulty doing that due to not writing correctly the appropriate number of brackets [B2-B6].

B1	Fairouz	We should write 3x-4 multiplied by 2x-9.
		[Noura started to write the expressions]
B2	Salim	Perhaps the brackets can be put afterwards, wait Noura, wait, it keeps moving.
B3	Fairouz	Write it from the beginning.

B4	Salim	No, no [He takes the mouse from Noura who seemed annoyed by the act of Salim. Salim works on GeoGebra] O.K. Now write it again.
B5	Noura	[Noura puts her hand on her mouth with boredom]
B6	Fairouz	[Fairouz worked on GeoGebra, then she said with annoyance] something wrong with the brackets.
B7	Teacher	Don't get anxious. At the beginning, brackets are tricky. Everything will get O.K.

Excerpt 2: Students' difficulties in working with GeoGebra and related emotions

Excerpt 2 shows some of the difficulties encountered by the middle-achieving groups, as a result of their behavioral functioning; specifically when working with a technological tool. Fairouz, in the interview, said they felt out-of-control and thus frustrated not being able to draw from the beginning the graph of the function f(x)= (2x-9)(3x-4) in GeoGebra. Salim pointed at the teacher's interference as supporting them in getting back control over their work with GeoGebra, which made them satisfied with their work on the mathematical problem.

Encountering difficulties in learning the new topic, not only influenced students' positioning and emotions, but also colored the linguistic aspect of their learning, especially their use of pronouns. This is the case in excerpt 3, where the difficulty is related to simplifying an algebraic expression.

C1	Siham	We want to draw the quadratic function y=(2x-9)(-x-4).
C2	Amin	Multiply first the brackets.
C3	Alaa	(-x-4)(2x-9)= $-2x^2$+9x-8x+36, Now we compute +9-8.
C4	Amin	-17
C5	Siham	Minus 8 plus 9.
C6	Amin	Minus 17, plus .. [He seemed anxious, not sure of his computation]
C7	Alaa	What?
C8	Amin	Minus one or plus one
C9	Alaa	What?
C10	Siham	Plus 1.
C11	Alaa	Write $-2x^2$+x-36.

Excerpt 3: Having difficulty in simplifying an algebraic expression

Excerpt 3 shows that confronting difficulty constrained the group's sense of control and produced anxiety. In this situation, singular pronouns or no pronouns were used.

Discussion and conclusions

Research of students' emotions in mathematics learning is growing (e.g., Antognazza et al., 2015; Daher, 2011; Hannula, 2004). The present research aimed at characterizing grade 9 students' positioning and emotions when learning algebra with technology. The research findings indicate that to claim the collaborator positioning, members of the middle-achieving group were involved with behavioral processes (working with GeoGebra), social processes (class discussions), as well as cognitive processes (reasoning). These processes helped them reach their learning goal, thus resulting in positive emotions. It could be said that collaboration was associated mainly with positive emotions as enjoyment and content, though negative emotions as anxiety were experienced when having difficulty in solving the mathematical problem; i.e. in arriving at the learning goal.

To claim the leader positioning in a middle-achieving group, the member was involved with different learning processes, as demonstrating knowledge, which was also reported in Evans, et al. (2006), but their functioning was distinguished from the other group members by performing metacognitive and meta-emotional processes, as reported in Daher et al. (2015). These processes helped plan, monitor, evaluate and take decisions regarding the group learning, especially in time of difficulty in arriving at the learning goals. Thus, these processes helped maintain the leader positioning (Black, Soto & Spurlin, 2016), as they supported the leader in advancing the group learning.

In addition, the leader metacognitive functioning was actualized by asking questions as means to decide upon the method of solving a problem. This decision making could be looked at as a social process (Vroom & Jago, 1974) with the goal to advance the group learning. Moreover, it seems that critical thinking skills, actualized in decision making, were needed to claim the leader's positioning. Furthermore, the goal of the leader meta-emotional functioning was to change the negative mood of the group when encountering a difficulty, which motivated the members' work (Leithwood, Louis, Anderson & Wahlstrom, 2004). So, we argue that the leader positioning was claimed by paying attention to different aspects of the group learning, especially the metacognitive and meta-emotional aspects. This leader's functioning resulted in different emotions related to the difficulty and success in performing the mathematical task, which could be associated with Goldin's (2000) emotional pathway, where generally frustration preceded enjoyment and enjoyment preceded content. This emotional pathway included the three dominant emotions in the groups' learning, i.e. no singular emotion was dominant but the emotional pathway was thus. Furthermore, the group members' emotional experience was influenced by that of the leader, which could be related to the emotional contagion suggested by Hatfield, Cacioppo and Rapson (1993), where there is tendency to converge emotionally with others. We say this is especially true in group learning when the other is the leader.

Students encountered sometimes difficulties in learning the new topic. This encounter, not only influenced students' positioning and emotions, but also their linguistic use of pronouns. This was expressed in their use of singular pronouns or no pronoun at all when getting anxious for not being able to proceed with the carrying out of the activity.

Future studies are needed to compare the positioning and emotions of different achievement-groups in solving mathematical problems. Furthermore, research is needed to verify the effect of prior positions of the group members on their current positioning, which the present research did not target.

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The role of affect in failure in mathematics at the university level: *The tertiary crisis*

Pietro Di Martino[1] and Francesca Gregorio[2]

[1]University of Pisa, Mathematics Department, Pisa, Italy; pietro.di.martino@unipi.it

[2]University of Pisa, Mathematics Department, Pisa, Italy; gregorio.francesca@gmail.com

The tertiary transition between secondary school and university appears to be an insurmountable struggle for many students. This is also the case, surprisingly, in a certain sense, of students enrolled in Mathematics degree courses, and therefore students considered "gifted" with respect to mathematics. This case seems particularly interesting from an affective point of view: these students often live failure in mathematics as a tragedy, and – above all – initially they are not able to interpret their failure. For these reasons, it appears crucial to investigate which role is played by emotions in the emergence and management of this crisis, and how the students' view of mathematics and their self-perception develop in the tertiary crisis period.

Keywords: Mathematics failure, affect, gifted student, theories of success in mathematics.

Introduction and theoretical background

As Nardi (2008) underlines, the teaching and learning of undergraduate mathematics is a relatively new field of mathematics education research.

The most part of the research about this topic focuses on cognitive aspects, highlighting the difficulties related to the learning of advanced mathematics (Artigue, 2001). In particular, Tall (1991) discusses the students' difficulties in conceptualizing some specific mathematical constructs (for example the notion of limit of a function), and in using the formal definitions of these constructs.

Other scholars focus on the specific difficulties related to the tertiary transition, discussing the enormous gap between secondary and tertiary mathematics (De Guzmàn et al., 1998; Wood, 2001) in terms of cognitive, metacognitive, linguistic and also practical demands.

Alcock and Simpson (2002, p. 33) underline how "certain reasoning strategies are inadequate when applied to university mathematics, although they might be efficient and sufficient in non-technical contexts and in the kind of reasoning with specific objects required by school mathematics". Schoenfeld (1985) analyzes the undergraduate students' difficulties in managing with non-routine tasks. Ferrari (2004) discusses the linguistic difficulties related to the shift from an informal approach to mathematics to a formal one. De Guzmàn and colleagues (ibidem, p. 756-757) underline how "many students arriving at University do not know how to take notes during a lecture, how to read a textbook, how to plan for the study of a topic, which questions to ask themselves".

All the scholars describe the mathematical tertiary transition as a very challenging moment for students: Tall (1991, p. 25), points out that it "involves a struggle (...) and a direct confrontation with inevitable conflicts, which require resolution and reconstruction".

Nevertheless, affect appears to be the 'great absent' in this overview about the factors playing a role in this transition: for example, in the previous editions of CERME, there are no reports of the TWG about affect related to the undergraduate level (with the exception of the study related to the undergraduate course needed to become a primary teacher).

In particular, Clark and Lovric (2008), drawing on anthropological theories, describe the tertiary transition in mathematics as a three-stages rite of passage that includes: separation (from secondary school), liminal (from secondary school to University) and incorporation (into University). This rite of passage is characterized by a real crisis in which the consolidated routines are suddenly interrupted, changed and distorted.

As Bardelle and Di Martino (2012) underline, this crisis appears to be particularly challenging for mathematical high-achievers in the secondary school level: it can be difficult for these freshmen to understand why the reasoning strategies that worked in their previous mathematical experiences suddenly stop working at university level.

Therefore, it becomes interesting to investigate which role emotions play in the arise and management of this crisis, and how the high-achiever students' view of mathematics and their self-perception develop during this crisis period.

In other words, considering the TMA-model for attitude developed by Di Martino and Zan (2010), it is interesting to study the development of students' attitude towards mathematics during the *tertiary crisis*.

With this aim, in the AY 2014/15 we developed a narrative study involving different categories of current and ex-students of the Bachelor in Mathematics in Pisa:

1) *freshman*: to collect the voice of the subjects during the crisis period;
2) *expert students*, i.e. students enrolled in the third year of the Bachelor: to understand what they remember about their transition difficulties, their idea about the causes of such difficulties and about how they overcame them;
3) *dropout students*, i.e. students that have left the Bachelor in Mathematics without obtain the degree in Math: to collect their memories about the crisis period, in order to reconstruct their emotions, the motives of their resignation, their theories of success (Nicholls et al., 1990) and causal attributions (Weiner, 1986).

Context and methodology

Context. The Bachelor in Mathematics in Pisa is one of the most prestigious in Italy, the majority of its students is considered excellent in math during secondary school. This fact is confirmed by data collected for our research. We analysed data from AY 2009/10 to 2012/13: Table 1 shows the percentage of the high-rated students in final exam of secondary school (we define high-rated a mark between 90/100 and 100/100). It compares the situation in Pisa with the average from all Italian Bachelors in Mathematics.

	2009/10	2010/11	2011/12	2012/13
Italy	45%	42.4%	44.5%	40.1%
Pisa	73.6%	58.6%	65.7%	60%

Table 1: Percentage of high-rated students in the Bachelor in Mathematics, in Pisa and nationalwide.

The percentage of high-rated students in Pisa is much higher than the global one. However, the dropout rates of the Bachelor in Pisa are within the national average range.

The high concentration of above the average students and the presence of difficulties, as witnessed by the failure rates, make the Bachelor in Mathematics in Pisa the ideal contest for our research.

Procedure. The study was conducted in two different phases.

In the first phase, we developed and administered three online questionnaires (one for each category of the involved students) including open and close questions about the mathematical experience at the University and, in particular, the difficulties encountered. Students were requested to answers in an anonymous way: respondents were invited to share an e-mail address in order to participate in the second, non-anonymous, phase of the research. The participation to this study was voluntary. At the end of this first phase we had collected: 26 answers by freshmen; 75 by students enrolled in the third year of the Bachelor; 52 by students that had left the Bachelor.

In the second phase, 40 students (3 freshmen, 27 *expert students*, 10 *dropout students*) were interviewed by the second author of this report. The time for the interviews varied in a range from 5 to 90 minutes. The interviews were audio-recorded and then fully transcribed.

We will quote the students' answers using an alphanumeric code: F, E or D (which mean freshman, expert student or dropout student, respectively); a serial number (it indicates the order in which the student completed the questionnaire); Q and I (which mean questionnaire and interview, respectively).

Rationale. We developed a narrative approach because we wanted students to feel free to express what they consider important, using the words that they consider more appropriate. In particular, we considered the open-ended questionnaire and the interview to be two complementary narrative instruments: according to Cohen et al. (2007), an open-ended question can catch the authenticity, richness, depth of response, honesty and candor which are the hallmarks of qualitative data. On the other hand, questionnaires have their limitations: they are one-way compared with interviews.

Discussion

In the students' stories the difficulties are often linked to strong and negative emotions, persisting over time. Both questionnaire and interview had specific questions about emotions; for example, a question in the questionnaire was: "*Write a feeling that is linked to your experience at the Bachelor in Mathematics*". E69Q, despite he was able to overcome the initial difficulties, reports: "*unfortunately, now* [after having dropped out of the Bachelor in Mathematics] *I like math a lot less, or rather, it still fascinates me but it is now linked to very negative emotions that ruin it all*".

The percentage of students who indicate negative emotions changes drastically depending on the category of interviewed students. Only 32% of the freshmen report having bad feelings relating to their experience at the Bachelor in Mathematics and to their difficulties and failures. This fact may be connected with some characteristics of the sample: the questionnaire was published at the end of the academic year, when the students with serious difficulties have already left the Bachelor. The percentage of students that report having bad feelings increases among the *expert students*: as many as 52% write about bad feelings and emotions. Predictably, the percentage increases among the *dropout students*: 75% of these students link difficulties and the experience at the Bachelor in Mathematics with very negative feelings.

Different types of bad feelings are reported, we have identified some categories: anxiety/ distress/ anguish; frustration/ despondency/ hopelessness; fear/ apprehension; sadness/ sorrow/ depression; inadequacy/ insecurity.

The majority of bad feelings are related to the anxiety caused by the Bachelor in Mathematics: "*for the first two years I was in a permanent state of anxiety and distress*" (E6Q), "[my experience in Pisa has been] *angsty*" (E8I). This topic is reported by all three categories of students.

The frustration and despondency category appears despondency among the *expert students*, but also among the dropout students. Many students report they are not been able to reach their goals via techniques and mechanisms that had been successful in the recent past. The persistence of such a situation forces many students in a state of frustration and it brings them to reevaluate their skills: "*I realized that I would go to class and not understand a word of what was being explained. Therefore, I felt some frustration and I thought that I was not intelligent enough, that I was inadequate*" (D49Q).

The category of sadness and sorrow characterizes particularly the students who left the Bachelor in Mathematics. The decision of leaving the Bachelor in Mathematics seems to be linked to a strong sadness: "*I remember* [of that period] *just a lot of tears*" (D39Q).

Also the category of inadequacy and insecurity is strongly linked to the experience of difficulties, and in particular to their lasting and to the failure in overcoming them. These feelings are new for the students, they never felt them before because they have always been good at math. The consequences, in these cases, can affect the students' self-esteem and his or her learning abilities: "*I think that my experience at the Bachelor in mathematics left me with less confidence in my ability to study*" (D10Q), "*as far as I'm concerned, low self-esteem kills any productive drive*" (E8Q).

The emotions reported by the interviewed are often felt as negative because they are unexpected: the student shifts suddenly and in unexpected ways from a *mathematical welfare* to a *mathematical malaise*, and he doesn't understand the causes of the shifts. The persistence of the difficulties causes the growth of bad feelings and the sense of helplessness ("*there were a lot of difficulties and disappointment… I felt like there was no way out*", E29Q), contributing to foster a downward spiral.

In our study a special attention has been given to the students' attributions for their difficulties. The students both spontaneously or answering specific questions, have made causal attributions. The narrative data collected have permitted us to identify the more frequent causal attributions, and to organize them in categories:

- *Transition aspects*: differences between secondary school and University (contents, organization, teaching styles, assessment, …);
- *Low preparation*: insufficient secondary school prerequisites;
- *Low ability*: lower math ability than they thought, inadequate mindset (these factors are often attributed to a faulty way to assess in secondary school)*;*
- *Comparison aspects*: many of these students were considered (and perhaps they really were) the best math-students in their school, and for the first time in their life they are "one of many". This impacts with their self-perception in math.

The students blame an important part of their failures to the great differences (related to math) between secondary school and University. These differences and the subsequent difficulties often cause significant changes in the students' view of math, and in particular of what it means to be good in math. In particular, most of the students point out that they got good grades in secondary school without significant efforts: "[in secondary school I considered myself good at math] *because I could avoid studying it and still get the highest grades*" (F14Q) and "*I realized that the high learning speed was due to the easiness of the topics we studied in high school, rather than to an above the average skill*" (D51Q). The secondary school's math is regarded as a simplified and procedural math, surely not as the *math* studied at the University: they seem two completely different subjects! ("*Math you do in high school is not the one you do in your first year in University*", E52Q). Students recognize that in University more formalism, abstraction and proofs are required: math switches from numbers and figures (a practical mathematics) to structures, like vector spaces or groups (a theoretical mathematics), and it involves a radical and hard cognitive shift. Despite the connected difficulties, the discovery of this new math usually is welcomed ("*I think I like the subject even more than I did in high school. I've found topics that I find fascinating*", E65Q), but sometimes it isn't ("*I've changed Bachelor since I couldn't find the practical math I was expecting*", D6Q).

Anyway, the crucial point seems to be that this *discontinuity* in the subject is typically unexpected by the students: they choose Mathematics with a clear idea of what it is and of how much they are good in math, and suddenly they have to compare with a new reality.

Among the transition aspects, teachers and style of teaching have a predominant role. There is a shared perception amongst students that at the university level there is not a particular attention to the students' difficulties: it is interesting to underline that this perception is often shared also at secondary school level by students with difficulties in math (Di Martino & Zan, 2010).

Students also underline the fact they are left alone from the beginning: "*in University they gave for granted many notions, or they didn't focus enough on topics they deemed to be easy, creating enormous doubts and flaws*" (D41Q). So students feel abandoned and powerless against apparently insurmountable difficulties, unable to find successful strategies.

From our data, it emerges that students blame responsibility to secondary and university teachers for their transition difficulties. In their view, secondary school teachers did not teach them what math really is and how it needs to be studied, and university teachers do not pay attention to the natural difficulties in the transition.

In this framework, math is seen (often for the first time for these students) as intrinsically complicated, and the transition aspects seem to add up further difficulties. Many students thought, and continue to think, a particular mindset is required to succeed in math, and this "math mindset" is innate ("*from birth you are not cut out for it, as you would need to be*", E45I).

The great amount of difficulties in the transition and this belief represent an explosive mix: according to the students' narrations, it is one of the main causes of resignation (to be good in math you need to have an innate talent; now, with the real math, I'm not good in math; so I don't have the innate talent and I can't do anything to improve, because I'm not talented).

The above explosive mix is also strongly affected by the comparison with peers. Most of the interviewed were the best of their class, or even of their school, during secondary school. At University, the context is completely different: you are one of many, and – above all – there is a natural reluctance in sharing personal difficulties (this reluctance appears to be linked to the emotional reactions to the difficulties we have commented before). The consequence is the spread of a feeling of loneliness, a lot of students stated that they thought to be the only ones in that context with difficulties: they believed that most of their peers understood all without difficulties. This (wrong) perception affects and quickens the change in the math related self-perception of the students, creating doubts about their own brightness. This has strong effects on the emotional side: "*I have really downsized the opinion of myself I had by seeing that there were way more capable people than myself*" (D33Q), "*I had begun to think that maybe I wasn't so good as I had thought and that it had all been an illusion. Moreover, I saw geniuses that new everything and understood everything right away and so I felt like an idiot*" (D44Q).

Despite a lot of common themes, there are also some significant differences between the causal attributions for difficulties of the expert students and those of the dropout students.

In particular, most of the dropout students claimed that, despite a hard and extensive study, or even despite the supposed sufficient comprehension of math, they failed the exams. In their opinion, the reasons for the lack of success is therefore linked to their natural inability or even stupidity: they seem to think that a kind of innate ability is needed to succeed in mathematics. Other respondents said that one of the reasons for failure was that some professors seemed to teach only for the excellent students, without taking care of the bulk of the average students. So, the exceptional students' presence is seen as a problem, as much as the exams' scale of evaluation.

In the final analysis, the dropout students used especially external and uncontrollable causal attributions.

Also the freshmen and the expert students used external causal attributions but, they reported that after of an initial period in which the difficulties were perceived as uncontrollable, they found a way to turn them in to something controllable. In particular, they refer to a shift in their theories of success or to a change in the strategy they adopted to deal with their pre-existing theories of success.

The students report of some strategies or changes that have led them to the overcoming of their difficulties; the most frequent reasons are relative to the quantity and quality of their study and relative to their study habits. A lot of the interviewed spoke about the cooperation with peers as of being of great help: "*obviously a relevant part of my success is due to the people that have supported me*" (E44Q) and "*Personally it was group study that allowed me to go on*" (E7Q). Peers,

especially better students, also helped to find the right study habits and to create the necessary mindset to succeed in math: "*the older students helped me by convincing me that it was all about getting settled with new ways of reasoning*" (E44Q).

So, as a student said, "*challenging one's study habits*" (S63I) is important; the first step is to understand that the study habits are amendable and this happens especially after failures and through the comparison with peers or teachers. From this quote, as from many others, it appears that the personal awareness of what is going wrong and what can be improved is a necessary step towards overcoming one's difficulties. "*Seeing the teachers in action has been fundamental for me, in the sense that it helped me adopt the right mindset. By just studying on the books, I would have never obtained the same results*" (E52Q). Teachers are also fundamental for their emotional support: "*some teachers were fundamental in the process of overcoming my difficulties! In my opinion it is important that the professor lets you know that he believes in you, that he is aware you spent months preparing for the exam, that he is sorry if he fails you and that you are not just a number!*" (E28Q). Moreover, lots of students have found the meetings with peers or teachers very useful, overcoming the fear of the professor's judgment, which is instead very common in secondary school. Finally, great study and effort are necessary: "*I overcame my difficulties by endeavor and maximum commitment*" (E44Q).

Conclusions

From our study there thus emerges a path that seems to characterize the experience in the Bachelor in Mathematics. A student which was a high-achiever in high school enrolls in the Bachelor in Mathematics; almost always, in an unexpected and abrupt manner, he faces difficulties; these difficulties are linked to strong negative emotions (such as anxiety, frustration, sadness…) and are combined with a reevaluation of the previous scholastic experience and of one's skill in math; math is seen under a new light: it is, in some sense, new and it is taught differently; the student produces theories of success and causal attributions: these can be internal or external, but the difficulties are initially almost always perceived as uncontrollable.

Up to this point in the student's path, most of the stories we have heard agree, regardless of the interviewed student's category. But from now on there is a definite distinction between the experience of who has abandoned the Bachelor in Mathematics and who has succeeded in continuing his or her studies. The comparison of the experiences of the subjects from different categories has in fact provided us with precious information: those who, for possibly emotional reasons, persevere in producing uncontrollable causal attributions or in implementing the same strategies to reach success, will eventually drop out of the Bachelor in Mathematics; on the other hand, changing one's theories of success or one's causal attributions or just identifying them as controllable allows one to overcome difficulties and failure. Our study seems to suggest that what makes the difference between dropping out and overcoming the difficulties are one's success theories and causal attributions, and in particular the ability to modify them and identify controllable factors.

It thus seems that the processes that leed to changes in the students' success theories and causal attributions, which bring to light the controllable aspects of one's difficulties, is worthy of a deep investigation.

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Primary future teachers' expectations towards development courses on mathematics teaching

Silvia Funghi

University of Modena and Reggio Emilia, Faculty of Education, Italy; silvia.funghi@unimore.it

The issue of what teachers need to teach mathematics effectively has been widely dealt with in the past decades; nevertheless, there are still few studies focusing on what future teachers think they need in order to be effective. In order to get a deeper understanding of future teachers' viewpoints about what they need to learn, a narrative study concerning primary future teachers' expectations on a course on Mathematics Teaching was developed. Issues about a lack of connections between theoretical notions provided by university courses and teaching practice were raised, as well as problems about affective dimension in mathematics learning and about mathematics itself.

Keywords: Future teachers, expectations, mathematics teaching, professional development.

Introduction and theoretical framework

What do teachers really need to teach mathematics effectively? In Mathematics Education this question has been mainly addressed by studying the relationship between teachers' knowledge, behavior and attitude, and students' achievements (Ball et al., 2008). Other studies highlight several aspects perceived by different figures (students, administrators, teachers) as necessary for the teacher to be effective, e.g. competence in the subject, teaching style, enthusiasm for teaching, care for students' difficulties and interest in students' lives outside of the classroom (Stronge, 2007).

It emerges that people have different 'ideals' of good mathematics teachers, and this is true in particular in the case of future teachers. This case is interesting because this 'ideal' strongly affects how a future teacher approaches learning opportunities (Liljedahl et al., 2015). From this viewpoint, teachers' development can be seen as an ongoing process nurtured by teachers' desire to "*fill the gap*" between their current developmental stage as teachers and the "ideal" mathematics teacher they want to become (Sfard & Prusack, 2005). In particular, future teachers' development depends not only on the training planned by teacher educators, but also on individual needs and aims: "teachers do not come to professional learning opportunities as blank slates. Instead, they come to these settings with a complex collection of wants and needs" (Liljedahl, 2014, p. 1). This is crucial for primary level teaching, in which involved people are not specialized in mathematics.

In this framework, it appears relevant to collect future teachers' opinions about what they want to learn in their professional development, and, above all, what expectations they have about the mathematics courses they have to attend. In fact, in the field of mathematics education, *expectations* are quite an unusual focus even for research concerned with affective issues: more often, research regarding the implementation of educational courses or workshops takes into account future teachers' *a posteriori* evaluation of whether their expectations have been fulfilled or not (Bartolini Bussi, 2011).

For these reasons, this paper focuses on a study on narrative data collected from future teachers *at the beginning* of a course dealing with mathematics teaching, in order to bring out future teachers'

expectations and hopes about their education in mathematics teaching before they participate in a specific educational course.

Methodology

Population and procedure. This study concerns the analysis of the answers to one of the questions of a questionnaire that was administered in February 2016 to a class of future teachers during the first day of the course of the third year in Mathematics Teaching (Bachelor's Degree in Primary Teacher Education, University of Modena and Reggio Emilia, Italy; the duration of the whole degree course is 5 years). The answers were collected from a class made up of 49 future teachers.

The choice to consider a class attending the third year of degree course is not irrelevant, because it was strictly related to one of the purposes of the inquiry, which was to study the story of future teachers' relationship with mathematics, paying attention to their perception of the value of their university education. From this viewpoint, a third year class was interesting because future teachers in this university must attend two courses in Mathematics in the first two years and a course in Mathematics Teaching in the third year: in particular, they cannot take the exam for this last course until they take both Mathematics exams. Furthermore, the course in Mathematics Teaching is also the last compulsory course they must attend regarding mathematics. So, it was interesting to investigate what these future teachers expected about this course *after two compulsory courses on Mathematics.*

In order to pursue this goal, an open questionnaire was specifically developed. It was composed of three questions, where the first two aimed to investigate future teachers' relationship with mathematics and its changes during the years of university and pre-university education; the last question on the other hand – *"What are your expectations about the course on Mathematics Teaching?"* – was aimed to grasp also future teachers' viewpoints about their own wants and needs regarding their education as teachers. This last question is the focus of this paper.

The questionnaire was presented and administered at the beginning of the first lesson of the course; the time available to fill it in was about 30 minutes. Since the questionnaire is part of a bigger project aimed to study the development of future teachers' beliefs throughout the years, future teachers were asked to write their I.D. numbers, but it was explained them that it was impossible for the researcher to match these numbers with their personal details.

Rationale. Results obtained by previous narrative studies show that the analysis of narratives is a very powerful tool, especially for the purpose of research on future teachers' beliefs, identity and attitudes (Kaasila, 2007). A narrative approach, in fact, permits respondents to focus on the aspects *they* consider most important, and to explain their own thinking with the words *they* consider the most appropriate. This way, the variety of answers produced provides the researcher an amount of details and information that has an enormous value for the purpose of qualitative studies (Cohen et al., 2007).

Regarding more specifically the method chosen to collect narrative data, in this case it was essential to make future teachers not frightened of being penalized for their real opinions. So, the most suitable inquiry tool was undoubtedly an open questionnaire, because it allows respondents to

remain anonymous – a possibility that other inquiry tools, such as oral interviews, do not permit. This is also the reason why future teachers were assured of the confidentiality of their answers.

The data were analyzed adopting a content-categorical approach – e.g. the analysis was focused on the content of collected answers, not on their form, and the unit of analysis was single utterances isolated from the rest of the discourse, and not the whole narrative (Lieblich et al., 1998). This approach, in fact, has been proven to be very appropriate to study phenomena common to a group of people (Kaasila, ibidem).

In the discussion, we will use the acronym FT (Future Teacher), followed by a progressive number from 1 to 49, to quote excerpts of protocols.

Results and discussion

The first interesting feature of the collected data is that there were no answers such as "I don't know", and no answers left blank: future teachers had *something to declare* about their expectations on the course on Mathematics Teaching. The overall feeling emerging from the answers is a compelling need for courses focused on teaching practices, rather than lessons about theories of teaching (*"[…] I hope that we will deal with things that are more inherent to our future profession, because some courses of our degree course deal with many notions that are difficult to employ in practice",* FT16). This feeling is well reflected in the high rate of respondents – 40 on 49, around 82% – asking for teaching methods (*"[…] I hope that methods to employ once in class will be illustrated as much as possible",* FT44), or more generally asking for a course to learn "how to teach mathematics at primary school" (*"I expect a course to understand what and how to teach when I will be faced with children at primary school",* FT3).

However, after a closer examination of the answers, it emerges that expressed expectations involved mainly two dimensions, the pedagogical one and the affective one, so that it was possible to distinguish with enough clarity two kinds of expectations: those related to *pedagogy and pedagogical content knowledge* (according to Shulman's definition, 1986) and those about *affective issues*. Since most of the expectations fell in one of this two categories, the few ones that were not directly related to pedagogical or affective issues were grouped into a third category that we called *other expectations concerning mathematics or mathematics teaching*. In order to give to the reader a clearer explanation of the features of each category, they will be analyzed one by one in the following subsections.

Expectations related to pedagogy and pedagogical content knowledge

As one could expect, this is the most recurrent category, since every answer involves at least one expectation related to pedagogical knowledge or pedagogical content knowledge. In most cases, as it was anticipated earlier, the respondent makes a generic wish to expand his/her knowledge about methods and means for teaching mathematics. In particular, our future teachers mainly seek methods that are *efficient* (*"I hope that […] alternative methods will be proposed to us […] [that are] significant and efficient for children's learning",* FT46) and possibly *fun* (*"I hope to learn practical and fun methods to teach mathematics in primary school",* FT2). In many cases, respondents seem to wish to learn a sort of *recipe for good teaching*, e.g. which method works best according to the situation (*"I expect to learn several methods for teaching mathematics that are*

suitable for the various difficulties that a child can encounter in learning mathematics", FT31). Many answers even talk about learning "*the best* method to teach mathematics" (*"I hope to face more topics, for example the best way to teach mathematics in order to make children appreciate it",* FT34). Sometimes the list of notions the respondent expects to learn is so extensive that it can even be hard to believe that a single course would be enough to fulfill all these expectations (*"I expect to learn to understand children's learning processes, strategies to adopt to present this subject, which methods should be used, how to organize the lessons and the topics",* FT4).

In some cases, as previous excerpts show, future teachers' main concern is about the *real* teaching practice: this is also confirmed by the rate of answers (17 answers on 49 – around 35%) stressing the practical and concrete nature of the notions that future teachers would like to learn (*"I hope that in the course we will talk not only exclusively about theory, but, on the contrary, mainly about PRACTICE. Practice and advice about how to facilitate [children's learning] and make our way of teaching more efficient"*, FT41, capital letters in the original). Some of them ask more specifically for some practical examples from real experiences of in-service teachers (*"I hope that […] [the course] will be focused on efficient methods for teaching this subject, perhaps also those based on real experiences of teachers",* FT32). These answers perhaps point out a lack of connections to didactics of mathematics in previous mathematics courses.

A slightly different kind of expectation regards the enhancement of respondent' *competencies* as a teacher. In some cases, in fact, future teachers seem to be more concerned with personal improvement, rather than with learning a set of "ready-made" methods (*"I hope to join my enthusiasm for the subject to the competencies that are necessary to teach it step by step"*, FT11). However, what competencies they refer to is not clear from these answers. There's just one case where the competencies that the respondent would like to gain are sufficiently clear to be described as organizational competencies, competencies in explaining, and diagnostic competencies (*"[…] I expect that [the course] will be useful from a practical viewpoint, so that […] I will know how to organize the lessons, how to explain and how to help those who have more trouble",* FT4). The last competence in particular seems to be quite an important one for our future teachers, since many of them underline their need to learn more about children's learning processes and difficulties, and about ways to make children overcome such difficulties – as can be seen from some of the previous excerpts. It is noteworthy that none of the respondents talks about issues regarding creation of tests and assessment of tests' results – suggesting that, in our sample, future teachers are more concerned with making children *understand* mathematics, rather than *assessing* their learning.

Expectations about affective issues

This category of expectations includes those ones referring to emotional aspects related to the relationship with mathematics. We can distinguish in particular two kind of expectations, according to the subject of this relationship: in some case this subject is *the future teacher itself,* whereas in other cases the subject is *the class,* e.g. the pupils that the future teacher imagines to have.

The first kind of expectations refers to those answers that express the desire to reconcile with mathematics and the hope that the course on Mathematics Teaching could facilitate such reconciliation: such answers correspond, in fact, to narrations of difficult or fluctuating relationships with mathematics in the other answers of the questionnaire. A clear example is given by FT46, who

affirms: *"I hope to stop being stuck with this subject, and to be able then to learn and employ useful strategies to teach this discipline at best",* and in the other answers tells about her fluctuating relationship with mathematics, and in particular about her rejection for solving problems when she was a young student. In these cases, mathematics is described as far from future teachers' interests, and a course focused on mathematics teaching, not on the subject itself, seems to be an opportunity to come closer to mathematics (*"I hope that this course will let me come closer to a discipline that has always been too distant from me, but I hope above all that I will be able to look at it and perceive it in a different way",* FT37), as highlighted also in the investigation by Coppola et al. (2013) on the "math-redemption" phenomenon. It is possible that, beyond the sense of utility of studying teaching rather than the subject itself, for these future teachers plays a role their hope that studying how to teach topics could be also helpful to clarify *their* doubts from primary school. Only in a couple of cases did future teachers claim to expect to continue improving their relationship with mathematics, even if it is already a very good one (*"My expectations are varied. First of all, I hope to renew my interest in the subject",* FT16).

The second kind of expectations, on the other hand, regards future teachers' need to learn how to support the growth of a good relationship with mathematics among their future students. The request to learn how to support pupils' emotional involvement in doing mathematics is quite widespread: it is detectable in 17 answers out of 49 (around 35%). In particular, there are four main emotional responses to mathematics that future teachers wish to elicit into their future students – listed in order of increasing intensity:

- No *hate* for mathematics (*"I hope that it will teach me how to teach mathematics in order to make sure that children don't hate it",* FT17);
- Having *fun* when doing maths (*"I expect to acquire competencies to teach mathematics […] in a way that my children learn it as adequately as possible, perhaps also having fun",* FT8);
- Feeling a real *interest* in the subject (*"I hope to discover new methods to revive children's interest in mathematics [...]",* FT35);
- Being *passionate* about mathematics (*"I hope to learn, or at least to get hints and advice […] about how to make children passionate about mathematics, and to engage those who have more troubles or feel aversion for this discipline",* FT5).

As we can see from the above quotations, even if these emotional responses could all be interpreted as ways to make pupils have a good relationship with mathematics, they do not seem to have the same importance in future teachers' eyes. For example, in the answers that talk about *having fun*, it is not clear if future teachers are interested in an amusing emotional climate in the classroom *per se* or if they are just looking for fun methods for teaching mathematics – i.e. if establishing a good emotional climate for them is an *aim* to reach when teaching, or a *means* to facilitate the understanding of some mathematical content (*"Through this course I hope to learn instruments that are necessary to explain mathematics to children. It is not an easy discipline, and I would like to discover methods to make it fun, easy and efficient at the same time",* FT22). In fact, the importance of the influence of pupils' emotions on their learning seems to be underestimated in the answers that talk about avoiding making children hate mathematics: the feeling the reader gets is that the quality

of the climate to create in the class stops at a sort of "peaceful coexistence" between the pupils and the mathematics to be learnt – as we can see in the excerpt by F17 above. If so, a "peaceful" emotional climate could be seen not as an aim or as a means for teaching, but rather as a *constraint* for the learning to occur.

Other expectations concerning mathematics or mathematics teaching

Regarding mathematical contents of the course. Since the course is expected to address teaching practice, most of the answers do not mention expectations about its mathematical contents. "Less mathematics and more practice" could be a good motto to summarize some claims (*"I hope that there will be less numbers and less formulae with respect to the previous courses, and that it will be focused on efficient methods of teaching this subject",* FT32). However, when a respondent refers to some mathematical content, generally he/she expresses the need of a course focused more specifically on mathematical contents to be treated in primary school, rather than on new topics (*"I expect that the focus will be on the topics that we will have to teach to children, rather than on topics that actually we will never deal with at primary school",* FT39). Just in one answer we can find a demand for the explanation of new mathematics topics, but also in this case new knowledge is hoped to be useful in practice (*"I hope to come into contact with topics, materials and teaching methods we have never dealt with, that will constitute a new part of knowledge that I could employ in my future profession",* FT16).

Regarding the view of mathematics to communicate. Another group of answer express the wish to learn to teach in a way that communicates a particular view of mathematics (since the focus is not on teaching methods *per se*, this answers were not included among expectations regarding pedagogy and PCK). It could be possible to identify three main kinds of such demands, which can be summarized as: *I hope I will learn how to teach mathematics in order to make it…*

- *…"reality-based".* Some people ask for teaching methods that will make them able to make pupils recognize and use mathematics in real-life situations (*"In this course I'd like to discover new methods and ways of teaching that start to talk about mathematics from the real world, from everyday life",* FT19).
- *…"cross-cutting".* This feature is emphasized in particular in FT1's answer, where she describes her will to set her future lessons in a way to link mathematics to other scholastic subjects: *"[...] But above all I hope that [this course] will help me to understand how I can develop a course on mathematics that crosses other disciplines".*
- *…neither mnemonic nor mechanical.* A couple of answers express the will to learn to teach mathematics discouraging a view of it as a subject merely based on memory (*"[...] I'd like it if children could receive an approach to the learning of mathematics which is not mnemonic",* FT33) or on mechanical techniques (*"[...] I hope to receive hints to dispel the myth of mathematics as a strict and mechanical subject",* FT46).

Regarding the reasons for teaching mathematics. To conclude, I would like to report one answer which is particularly original. In this case the respondent expresses her hope to improve her mathematics education, because she seems to seek for a more complete and interrelated overview of mathematical topics, and for an explanation of the reasons why they are taught at primary school:

[…] I hope to reach a more organized and coherent way of thinking about mathematical contents. But above all I hope to understand the reasons why we study mathematic, in order to return it to the children of my future classes. […] Obviously acquiring competencies in the management of tools and communication techniques is important but is secondary; the main goal remains to know why one is studying a certain discipline. [FT10]

Conclusions

In the introduction we underlined how long researches in Mathematics Education has been pursuing the goal of establishing what do teachers need to teach mathematics effectively. Initially the efforts were directed to defining aspects related to knowledge – as in the work by Ball et al. (2008). More recently, research has widened its perspective to other aspects as well, as affective aspects of teachers' education (Coppola et al., 2013). Another aspect investigated just in recent years is what future teachers think they need to teach mathematics effectively: as Liljedahl (2014) underlines, it does matter "what the results [of the research] say about teacher autonomy and the role that workshops play in the professional growth of teachers".

These considerations also suggest another direction of inquiry: future teachers' expectations about courses for professional development. This aspect was the focus of the present study. The results obtained are obviously context-bound, since we are concerned with one class of a specific Italian university; nevertheless, there are some observations rising from our findings that seem to be more apt to be generalized and discussed here.

The first one concerns the shared judgment about the compulsory courses on Mathematics: courses focused only on mathematical contents are generally considered to be too theoretical – sometimes even beyond future teachers' capabilities – and scarcely connected with teaching practice. On one hand, as suggested by Boero and Guala, it could be advisable to incorporate the study of mathematical content with tasks "clearly related to crucial educational issues" (Boero and Guala, 2008, p. 232) aimed at stimulating a deep reflection on mathematics as a social and historical product; on the other hand, we believe that poor preparation in mathematics cannot adequately support the development of knowledge for teaching, e.g. a certain level of content knowledge is absolutely necessary for the development of pedagogical content knowledge. The challenge for teacher educators is to make future teachers aware of the importance of a strong preparation regarding content about this sense.

Another general aspect concerns future teachers' expectation about learning mathematics teaching methods that take into account also affective aspects. This consideration highlights two other issues for teacher educators: one is to develop workshops aimed to restore future teachers' relationship with mathematics; the other is to develop courses providing some space for reflections about affective issues, without losing the connection to mathematical content. The main risk, in fact, is to draw future teachers' attention to children's amusement, and thereby neglecting the activities' mathematical relevance. In our viewpoint, mathematical significance has to come first, and moreover, the focus should be not on fun activities, but rather on the ideal emotional conditions to make the activities work, making children feel free to express their thoughts and make mistakes. The analysis of future teachers' expectations raises serious challenges for us, both as researchers in

mathematics education and as teacher educators: accepting them is the further direction of this paper.

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Regression trees and decision rules applied to the study of perplexity: Issues and methods

Inés Mª Gómez-Chacón, J. Tinguaro Rodríguez

Universidad Complutense de Madrid, igomezchacon@mat.ucm.es,

This paper explores the use of the CRT (Classification and Regression Tree) methodology to analyse data from a fuzzy rating scale-based questionnaire. Based upon a questionnaire to assess the state of perplexity in mathematics undergraduate students, the rule structure obtained from the CRT analysis is reported. We anticipate these findings may be of interest both to evaluate the interplay between cognition and affect as well as to researchers in the Fuzzy Logic field.

Keywords: Emotions, heuristics, fuzzy set, cognition and emotion, mathematics, regression trees.

Introduction

In this paper, we will focus on perplexity in mathematics. In the studies concerning determination of affective pathways during the solving of problems (Goldin 2000; Gómez-Chacón, 2000, in press), the state of perplexity or puzzlement is considered to be one of the interesting emotional states into which the individual can drift along positive or negative pathways when solving the problem. Perplexity does not in itself have unpleasant overtones—but bewilderment, can include disorientation and a sense of having "lost the thread". If problem solving continues, a lack of perceived progress may generate frustration, where the negative affect becomes more powerful and more intrusive, but where there is still the possibility that a new approach will move the solver back to the sequence of a predominantly positive affect. The studies mentioned above show the need to understand and to know, in depth, the benefits that this state can achieve for the teaching and learning of mathematics.

A big challenge today is to improve the methodological tools for evaluating affect detection systems. This need is motivating studies in trying to explain this gradualness in the processing of affective mechanisms. Further research is necessary to reference the identification, discrimination, and the unclear boundary between the cognitive and affective processes. This paper explores the use of decision trees to analyse data from a fuzzy rating scale-based questionnaire. Thus, here we will report our results in the form of a tree structure providing rules to assess the state of perplexity in mathematics undergraduate students, built upon the basis of the previous questionnaire.

The present research is primarily exploratory for two reasons: 1) perplexity has been scantly analyzed in mathematics and educational psychology; and 2) the use of the CRT methodology to analyse data from a fuzzy rating scale-based questionnaire is a new development. The theoretical background and empirical studies related to perplexity need to be developed.

Theorists of science and mathematics (Lakatos 1976) claim that mathematical reasoning and complex problem solving are typical cognitive tasks in which perplexity is directly involved. However, an analysis of the psychological (cognitive and affective) processes involved in it lacking, in order to clarify the definition. (.Studies about confusion (Silva 2010) indicate an appraisal structure of this emotion of novelty-complexity that is reflected in a state of uncertainty and comprehensibility, reflecting an inability to understand. Smith and Ellsworth's (1985) appraisal

model maintains that in order to differentiate emotional experiences some dimensions are key (pleasantness, attentional activity, control, certainty, goal-path obstacle and anticipated effort). Taken together in this study, these dimensions and mathematical cognitive processes could provide information about perplexity, a knowledge emotion that is rarely studied, and illustrate the relationships between these cognitive variables and emotion in order to deepen understanding of its nature.

The following hypotheses guided the work: (1) the emotions associated with the perplexity state could be positive, negative, or neutral; (2) those participants with more positive ability to cope with the situation (control), better ability to predict, or with a wider understanding, are those who will better handle their state of perplexity in the reasoning.

Fuzzy rating method for questionnaires

The method of fuzzy rating scale applied in this research was introduced by Hesketh, Pryor, and Hesketh (1988) and subsequently developed in various studies by Gil et. al., (2015). The fuzzy approach is based on the idea that, in some cases, it is not reasonable to say that an object has to either verify a property or not verify it (Zadeh, 1975). Objects or people may exhibit some properties only partially—i.e. up to a certain extent or degree. In many of the conducted researches the evaluation of the emotion parameters is qualitatively performed through reports, interviews, recording observations or, if it is quantitative, through Likert scales. In the case of Likert scales (based on an implicit subjacent numerical continuum), such kind of imprecise information is lost, since finally just a single category has to be chosen (which excludes the representation of a potential hesitation between categories).

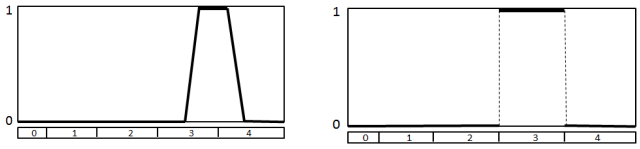

Fig. 1. Examples of Fuzzy sets valuating emotion

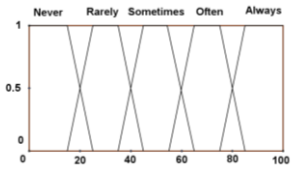

Fig. 2. Fuzzy sets modelling linguistic variable

In this sense, fuzzy logic allows relaxing this constraint by admitting valuations to be given in the form of fuzzy numbers over the subjacent numerical scale. That is, in this setting each possible numerical evaluation is assigned a degree of membership or verification, between 0 and 1, representing the validity of such number as a measure of the observed emotional phenomenon (Fig.1).

The idea of using these fuzzy sets to describe imprecise terms is closely linked to the concept of linguistic variable introduced by Zadeh (see Zadeh, 1975). A linguistic variable is considered one that takes linguistic values, which is less accurate than the use of numbers. For example, a linguistic variable used to evaluate ‘confidence’ may take the (linguistic) values: never, rarely, sometimes, often and always. Each of the linguistic terms that can take the variable is modeled by a fuzzy set (see Fig. 2, and notice the subjacent numerical scale that accompanies the linguistic descriptions). There are values of the variable that can be assigned up to a degree to two of these fuzzy sets (e.g. through disjunction) and, therefore, the boundaries between two consecutive linguistic values can be made flexible.

In this study, trapezoidal fuzzy sets (or fuzzy numbers) were used to perform evaluation on a continuum, assigning a membership degree between 0 and 1 to each point of the interval [0,100] (see Fig. 2 again). Notice that trapezoidal numbers allow representing a continuum of prototypes (i.e. elements that are assigned membership degree 1) with a linear decay.

Regression trees

The CRT methodology is a data mining approach widely employed to develop 'IF-THEN' rule models in order to explain the behaviour of a variable of interest (the dependent variable) in terms of logical conditions over a set of explanatory or independent variables (see Breiman et al., 1984). As such, classification and regression trees have been successfully applied to different data-analysis tasks, such as segmentation, stratification, forecasting, data reduction, variable selection, etc., in wide variety of practical contexts (see Strobl et al., 2009).

Particularly, the CRT methodology allows determining a subset of the available independent variables as well as a set of conditions over these variable's values that separate the data into groups as homogeneous as possible in terms of the values of the response or dependent variable. To this aim, the CRT method performs successive dichotomous splits of the data by identifying both the independent variable and its cut-point that provide the greatest variability (i.e. variance) reduction at the split data groups verifying either condition (i.e. being greater or lower than such cut-point). This process starts at the root node containing all the available data, and is iterated at the resulting nodes or groups until some stopping criterion (usually concerning the depth or number of splits or the sample size in the undivided nodes) is reached. The nodes that are left undivided at the end of this process, usually known as leaves, provide conditional response-variable distributions (assumed to be as homogeneous as possible) given the conditions or premises formed by the conjunction of the different branches (i.e. splits) that separate each leaf from the root node.

Notice that CRT is a data analysis methodology with almost no assumptions, and particularly that it is a non-parametric and distribution-free model-building method (e.g. no normality or independence assumptions are made). For these reasons, CRT is especially useful as an exploratory tool allowing to uncover some relationships and patterns in the data that may be expressed in logical form.

In this work we apply this regression tree methodology to develop a rule model capturing the relationships between a numerical dependent variable, measuring either the intensity of perplexity or pleasure experienced by students while solving a mathematical problem, and a set of independent variables measuring the intensity of other emotions that may appear in consonance with perplexity. Our aim, at least in a first stage, is basically exploratory; that is, we do not pursue a complex mathematical model of those relationships of perplexity with other emotions, but rather a simple model describing the most significant relationships in terms that may be checked intuitively. In this sense, we found that the CRT methodology fitted this aim quite well.

Research questions and methodology

Research Questions

We particularly pursued the following research questions: Research question 1: What emotions and cognitive appraisal processes have more influence on the state of perplexity? Research questions 2:

How pleasant is being in the state of perplexity? What variables are related to the dimension of pleasantness?

Participants and instrument

Data was collected in 2014 from 100 (56 women and 44 men, aged between 22 and 23) Caucasian undergraduates working toward a BSc. in mathematics. All of the participants were in their last year of academic study, and were distributed into three training groups established by the academic institution. They were following advanced courses in several areas of geometry, algebra, probability and analysis. With regard to solving problems, the students had been introduced to the problem solving heuristics and they received training as students and in one subject related to advanced professional knowledge, practice and relationship skills relevant to teaching. They had not received any special training about backtracking heuristics.

The work dynamic was individual work and began with a paper and pencil resolution of four tasks (problems), each of one hour and a half duration. One example problem, the results of which will be analysed later in section of results, is shown below:

> Paths: How many paths consistent in a series of horizontal segments and / or vertical can be counted in the figure below (Fig. 3) (where we have indicated a possible path) so that each segment links a pair of consecutive numbers, to form, from the beginning to the end the number 1234567?

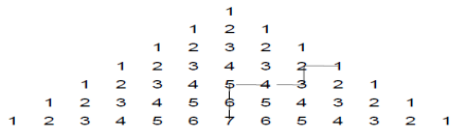

Figure 3. Possible Path

In this session, students were given the problem and asked to describe their approaches to resolving the problem using protocols including: steps in the resolution, explanations of the difficulties they might face, and strategies they would use. Afterwards, each problem was followed by a questionnaire based on the measure of fuzzy rating scales and focused on heuristics related to backwards thinking and the difficulties that are generated during the process of solving problems and emotions and cognitive processes (Gómez-Chacón, in press).

The questionnaire is based on the measure of fuzzy rating scales that used a scale of free fuzzy numbers, in which the respondent represents the same fuzzy number that most closely matches their assessment of an interval (Fig. 4). The questionnaire has two parts, one referring to the cognitive dimension and the other, the affective dimension. The cognitive dimension refers to the characterization of the personal meanings of the subjects on the cognitive dimension of heuristic backtracking, or backward reasoning, and the cognitive appraisal processes of the interaction with emotion. The studied emotions were: confusion, uncertainty, hesitation, surprise, frustration, bewilderment, boredom, and confidence. And the cognitive appraisal dimensions to differentiate emotional experiences were pleasantness, attentional activity, control (self-other responsibility/control, situational control), certainty, goal-path obstacle, anticipated effort and mental flexibility.

Data analysis

Both qualitative and quantitative methods were used to address the subject of this study. This paper presents the quantitative analysis performed on the undergraduate students' written responses to the questionnaire. The first step of the analysis was the defuzzification of data. This refers to converting the trapezoidal numbers provided at the student's responses into usual numbers that can be handled by the CRT methodology. For the purposes of our study, the average centre (also known as the centroid) defuzzification method was used. With such defuzzicated data, different regression tree analysis were performed with SPSS to uncover the prescriptive nature of the variables. Two of these regression trees, together with their associated rule models, are reported next.

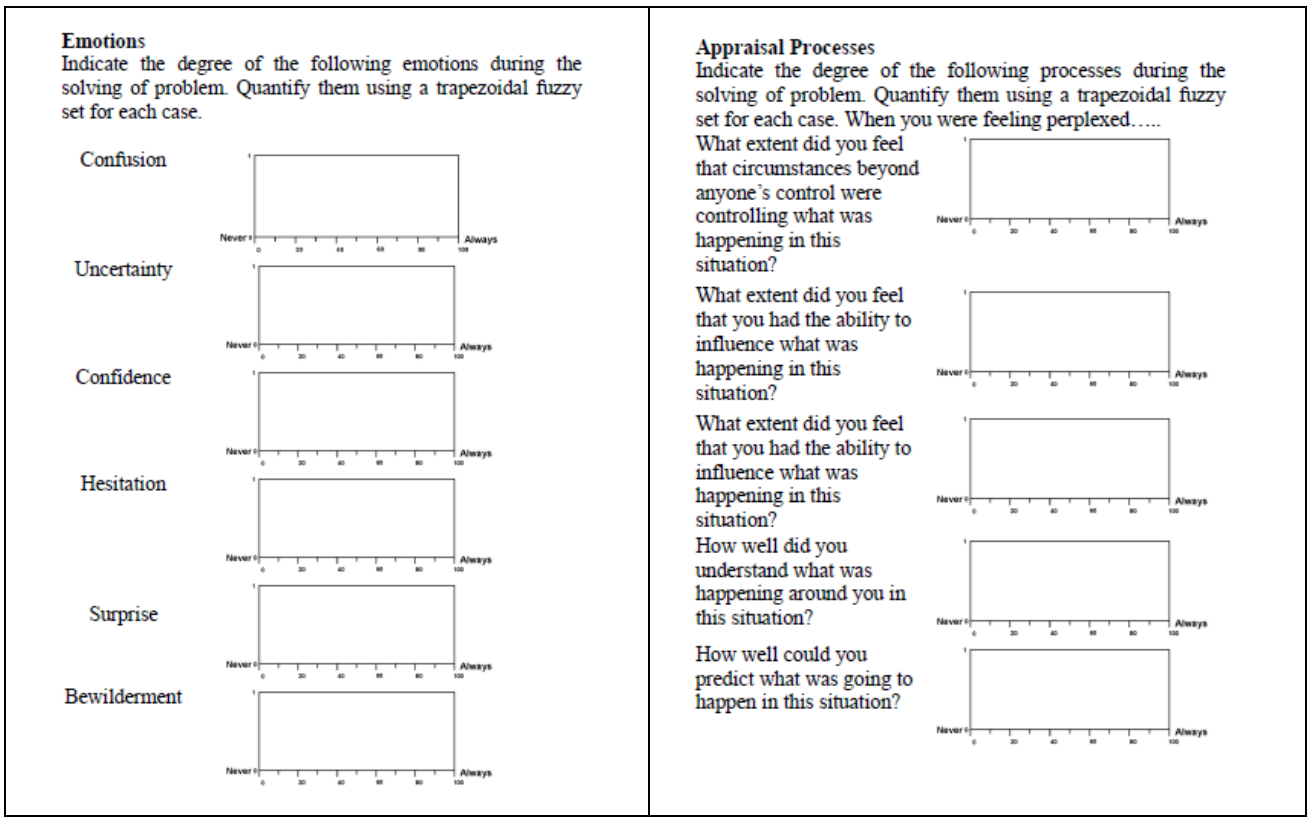

Fig. 4 Examples of some items from the questionnaire: Perplexity and backwards reasoning processes (Gómez-Chacón, in press)

Results

Research question 1. For the interpretation of the classification tree, we should go looking at the nodes and branching them until the final leaves. First, we look at the root node 0 that describes the dependent variable: Perplexity of students to solve the problem (P2). It indicates that the group mean is 49.040. Then, note that the data is split into nodes 1 and 2 depending on the variable Bewilderment (P19), indicating that this is the main predictor variable. Node 1 indicates that 22% of students who feel Bewilderment <= 21.37 has a mean of 25.04 perplexity (P2). This node 1 is again split up into nodes 3 and 4 depending on variable P8, Ability to influence (i.e. Control). We note in node 4 that the students who had Control > 67.87 experience perplexity with an average intensity of 40.28, while students at node 3 have a lower ability to influence and experience a mean perplexity intensity of 16.33. These two nodes 3 and 4 are leaves that allow us to infer rules 1 and 2 below.

Particularly, each path from the root of a decision tree to one of its leaves can be transformed into a rule simply by conjoining the conditions along the path to form the antecedent part, and taking the leaf's mean as the rule prediction or consequent. Similarly, in order to define the rest of the rules, node 2 and the following ones are studied. The profile of students who experience perplexity is defined by nodes 3, 4, 5, 9, 10, 11 and 12 through the following variables: Ability to influence (P8), Bewilderment (P19), Confusion (P13), Boredom (P21) and the ability to solve simpler problems and also goal-path obstacles (P11). The inferred rules are the following:

Rule 1 (node 3): IF ((Bewilderment (P19) <= 21.37)) AND (Ability to influence (P8) <= 67.87) THEN the prediction of perplexity (P2) is = 16.33, with a support of 14% (i.e. 14% of the participants verify the premise of this rule).

Rule 2 (node 4): IF ((Bewilderment (P19) <= 21.37)) AND (Ability to influence (P8)> 67.87) THEN the prediction of perplexity (P2) is = 40.28, with support 8%.

Rule 3: (node 5): IF ((21.37 < (Bewilderment (P19) <=64.6)) THEN the prediction of perplexity (P2) is = 48.84, with support 48%.

Rule 4 (node 9): IF (Bewilderment (P19)> 64.6) AND (Confusion (P13) <= 64.87) AND (Boredom (P21) <= 12.62) THEN the prediction of perplexity (P2) is = 67.15, with support 5%.

Rule 5 (node 10): IF (Bewilderment (P19)> 64.6) AND (Confusion (P13) <= 64.87) AND (Boredom (P21)> 12.62) THEN the prediction of perplexity (P2) is = 55.65 with support 5%.

Rule 6 (node 11): IF (Bewilderment (P19)> 64.6) AND (Confusion (P13)> 64.87) AND (the ability to solve simpler problems and goal-path obstacles (P11) <= 80.75) THEN the prediction of perplexity (P2) is = 66.99, with support 15%.

Rule 7 (node 12): IF (Bewilderment (P19)> 64.6) AND (Confusion (P13)> 64.87) AND (the ability to solve simpler problems and goal-path obstacles (P11)> 80.75) THEN the prediction of perplexity (P2) is = 78, with support 5%.

In summary, the perplexity is closely linked with the emotions of bewilderment and confusion. The bewilderment could generate a fork towards a positive path depending on the ability to influence on the problem and the ability to influence the process of resolving. The perplexity state may stem entail only high novelty, reflecting a state of uncertainty but may entail a searching of understanding when perplexity may share with appraisal dimensions linked to the ability to influence (self-control dimension) and the perception of overcome obstacles and the ability to solve simpler problems.

Research question 2. Pleasantness. Pleasantness is considered as an important dimension. It is a function of two appraisals—appraisals of what one wants in relation to what one has, and these are intrinsically pleasant or unpleasant. The mean of the group with respect to pleasure (P5) experienced during the state of perplexity is 40.92. From the classification' tree (Fig.6) can infer the following rules: **Rule 1 (node 2):** IF Confidence (P15) >55.75 THEN the prediction of pleasure (P5) = 48.07, with support 58%. **Rule 2 (node 3):** IF Confidence (P15) <= 55.75 and Understanding (P9) <= 38.37 THEN the prediction of pleasure = 16.63, with support 13%. **Rule 3 (node 4):** If Confidence (P15) <= 55.75 and Understanding (P9) > 38.37 THEN the prediction of pleasure (P5) = 37.5, with support 29. In summary, a state of perplexity could not only be a mental perturbation or anxiety, but a pleasure experience given sufficient levels of confidence and understanding.

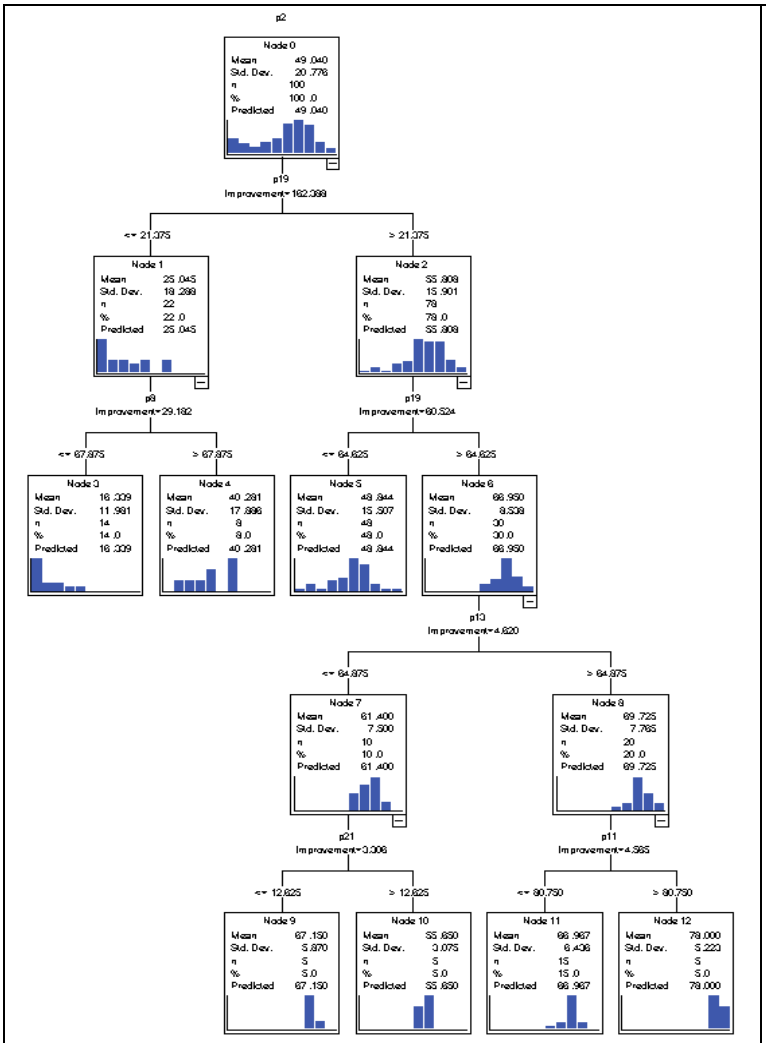

Fig. 5. Regression tree for variable 'Intensity of perplexity' (P2)

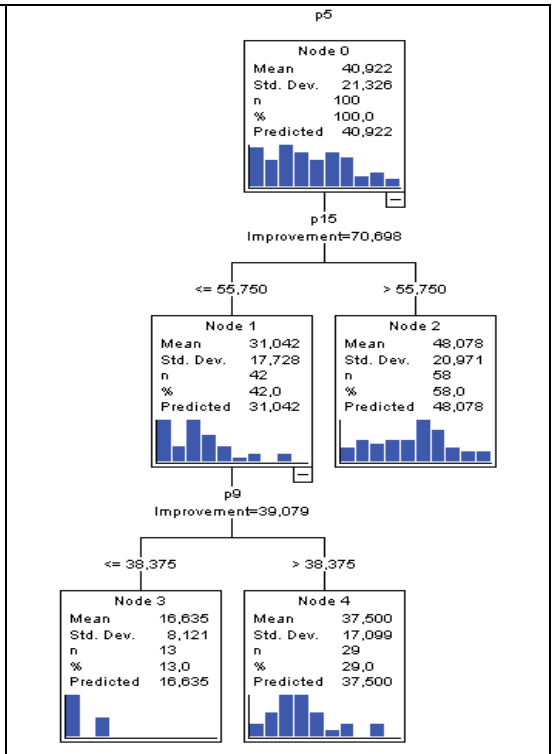

Fig. 6. Regression tree for variable Pleasure (P5).

Conclusions and discussion

The discussion and conclusions are structured around the objectives of the research, and methodological effectiveness in the use of regression trees for establishing rules.

Relative to the interplay between cognition and affect in the perplexity state the perplexity is closely linked with the emotions of bewilderment and confusion. The degree to which students associate state of perplexity with an emotion of pleasure is linked to the levels of confidence and the understanding of the problem. Likewise the perplexity state shares cognitive appraisal dimensions linked to the ability to influence (self-control dimension) and the perception of overcome obstacles and the ability to solve simpler problems. The relationship shown between cognitive appraisal dimensions and the emotions that make up the state of perplexity highlights conditions about learners who have the ability to appropriately manage their perplexity. This study shows the central role of impasse in mathematics, perplexity it is not a negative event to avoided by intellect, it is responsible for the activation of thinking (Lakatos 1976, Goldin 2000; Gómez-Chacón, 2000). Regarding the methodological adequacy of the present study, the use of a non-parametric, assumptions-free data mining model as regression trees provides a solid basis for the kind of

exploratory analysis aimed at this work. Particularly, this model allows for robust variable selection, as significant variables are identified through a greedy process in which the effectiveness of all the available variables in reducing the variability of the response variable is checked, and that obtaining the greatest reduction (or improvement) is selected. This assures that the selected variables that conform the premises of the obtained rules are relatively good explicative factors of the studied response variables.

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Impact of long-term regular outdoor learning in mathematics – The case of John

Annika Grothérus and Emilia Fägerstam

Linköping University, Sweden; annika.grotherus@liu.se; emilia.fagerstam@liu.se

This paper reports on a longitudinal study investigating the impact of long-term regular outdoor learning in mathematics in the school-grounds. An interview-based case study of John, a lower secondary school student, will be analysed. The case study will describe John's perceived experience of long-term regular outdoor learning in mathematics and its impact on affective and academic factors. The findings emphasise the positive outcomes of long-term regular outdoor learning in mathematics, indicating enhanced cooperative learning, reduced mathematics-related stress and anxiety, changed self-concept, and enhanced mathematical proficiency.

Keywords: Outdoor learning, mathematics anxiety, self-regulation, mathematical proficiency.

Introduction

The constant focus on textbooks and formal mathematical practice might invoke a view among students that mathematics is abstract, distanced and only useful in a classroom context, working only in the textbook (Boaler, 1998). If students are not given the opportunity to engage with real-life problems in mathematics they will encounter problems applying their knowledge in an outside school context (Desforges, 1995). Mathematics taught in the classroom will have limited value if it is not transferable to students' everyday life and future academic and career endeavours. Current research on outdoor learning in mathematics demonstrates positive affective outcomes and possible academic benefits (Daher & Baya'a, 2012; Moffett, 2011; Noorani et al., 2010). In this paper, by analysing John's perceived experiences of long-term (3 year period) regular outdoor learning in mathematics, we explore any possible impact on both affective and academic factors. By affective factors we mean mathematics-related stress and anxiety, and motivation. With academic factors we refer to possible academic outcomes such as application and understanding of mathematical knowledge, enhancement of mathematical proficiency, learning strategies, self-regulation and self-concept.

Theoretical background

Outdoor learning

Outdoor education can be referred to as organised learning that takes place in the outdoors and is drawn up on the philosophy and theory and practises of environmental as well as experimental education. The embodied and multisensory experiences provided by well-organised outdoor learning are believed to enhance the individual's learning and understanding within a subject, in this case, mathematics. The variation of context between the indoor classroom activities and the outdoor activities enables rich opportunities for cooperative learning in real-life situations (e.g. Jordet, 2007).

Mathematical proficiency

In the present paper we use the framework of mathematical proficiency presented by Kilpatrick, Swafford, and Findell (2001). According to Kilpatrick et al. (2001), mathematical proficiency has five strands: conceptual understanding (comprehension of mathematical concepts, operations and

relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately), strategic competence (ability to formulate, represent and solve mathematical problems), adaptive reasoning (capacity for logical thought, reflection, explanation, and justification) and finally, productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's efficacy) (Kilpatrick et al., 2001, p. 116).

Mathematics anxiety, motivation, self-regulation and self-concept

Mathematics is highly valued by society, and proficiency in mathematics has become increasingly important in order to become a fully functioning citizen, with resulting financial empowerment, making the ability to use mathematics in an out-of-school context even more important (OECD, 2004; Peterson, Woessmann, Hanushek, & Lastra-Anadón, 2011). A variety of studies have shown that many students have negative attitudes towards mathematics, which sometimes results in mathematics-related stress and anxiety (Ashcraft, 2002; Maloney & Beilock, 2012). Within a student, engaged in a mathematical activity, an on-going unconscious evaluation of the situation with respect to the student's self-concept and personal goals is taking place. Depending on the individual's evaluation of the outcome, the student may become emotionally affected being either motivated or not for further mathematical activities (Hannula, 2002; Pintrich, 2004). According to PISA (OECD, 2013), Ryan and Deci (2009) and Wigfield, Tonks, and Klauda (2009) two forms of motivation to learn mathematics within self-regulative skills exist. The two forms are intrinsic motivation and extrinsic motivation. Intrinsic motivation is the motivation to perform mathematics merely for the joy gained from doing mathematics. While extrinsic motivation is the motivation to perform mathematics due to its importance in society, usefulness and the fact that mathematical knowledge will aid future career prospects and further academic studies. An individual with mathematics-related stress and anxiety is emotionally affected in a negative sense and tends to have a low self-concept (self-related cognitions of ability that can explain as well as predict achievement related behaviour, belief in one's abilities) as well as low levels of self-efficacy (a student's belief that he/she has the capability to perform a given mathematics task at a designated level) (Bandura, 1994; Bong & Skaalvik, 2003). Furthermore, these individuals also tend to lack the ability to self-regulate, resulting in avoidance behaviour and a decline in mathematics performance. As a consequence they are assigned lower grades and tend to have limited choices of possible future academic and non-academic career paths (Wu, Barth, Amin, Malcarne, & Menon, 2012).

One important factor to promote learning is students' capability to self-regulate (Pintrich, 2004). If students' are provided with guidance on how to self-evaluate their own learning process and how to develop suitable strategies to promote their own learning during formal schooling, such guidance will enable acquired knowledge to remain updated after leaving school and to be used in an outside school context (Pintrich, 2004; Zimmerman, 2002). Self-regulation refers to the degree in which students are active participants in their own learning. It is an individual's ability to set mastery goals, mobilizing the efforts and resources the individual will need in order to achieve these goals. To reach these mastery goals, students use a wide range of self-regulatory processes and display a number of adaptive motivational factors such as self-efficacy and self-concept, where moderate levels of mathematics-anxiety may actually facilitate learning as well as performance (Pintrich, 2004).

Cooperative learning

Procedures in cooperative learning are designed to engage students actively in the learning process through inquiry and discussion with their peers in small groups, which need to be well organised with a clear structure to promote cooperative participation and learning (Davidson & Worsham, 1992, p. xii). Cooperative learning provides students with academic factors such as positive effects in mathematics performance and student achievement (Whicker, Bol, & Nunnery, 1997). Cooperative learning is a preferable method when helping individuals with mathematics-related anxiety to reduce their stress and anxiety. Furthermore, it is also an important feature of self-regulated learning because active participation is a crucial element of the self-regulation construct (Clark, 2012; Daneshamooz & Alamolhodaei, 2012).

Focus of the study

The aim of the study was to investigate John's perceived experience of long term (three years) regular outdoor learning in mathematics in terms of possible impact of integrated outdoor mathematics activities on affective and academic factors.

Methodology

This paper is a part of a larger intervention research project aiming to explore the possible impact of outdoor teaching and learning in lower secondary school (Fägerstam, 2012). Outdoor teaching and learning was implemented on a regular basis as a complement to ordinary classroom teaching during the entire lower-secondary school period of three years. The focus of this paper is to explore the possible impact of long-term regular outdoor learning in mathematics, and the case of John is presented as an example of its possible impact. The research is exploratory to its nature because there are few longitudinal studies on outdoor learning in mathematics. John is 15 years old, attending his third and final year of lower secondary school. John is a fictitious name that has been given to ensure the individual's anonymity. John's class had one of their four weekly mathematics lessons outdoors on a regular basis during the entirety of lower secondary school. The same mathematics teacher taught John during the three years of the intervention project. The school to which John and his class belong was situated in the suburbs of a medium-sized (approx. 85000 inhabitants) municipality in Sweden. The school, grade 7 to 9, was a normal sized school with approx. 450 students in six parallel classes. John was interviewed using semi-structured interview as a method. The interview was audio-recorded and transcribed using verbatim. Data was analysed thematically to identify recurrent patterns and commonalities using thematic coding (Boyatzis, 1998). Aspects of self-regulation skills were analysed based on concepts originally used in the PISA survey (OECD, 2004), namely intrinsic and extrinsic motivation, self-concept and mathematics anxiety. Representative illustrative quotes will describe the possible impact of regular outdoor learning in mathematics on self-regulation in mathematics and mathematical proficiency as well as John's perceived experience. The ethical guidelines and directives stipulated by The Swedish Research Council regarding good research have been followed (Hermerén, Gustafsson, & Pettersson, 2011).

The case of John

John is a fifteen-year-old boy who has had severe difficulties with learning and understanding mathematics since he started school. John was selected as a case due to his low levels of mathematics

self-concept, expressed mathematics-related stress and anxiety and low levels of mathematical proficiency, which were reflected in his mathematical performance.

John's overall experience: Well-planned lessons, structure, intelligibility, and time

John emphasises the importance of well-planned lessons, structure and intelligibility. It is important, he says, that one knows what is expected and that everyone knows what to do.

> It is of importance that everyone knows what he or she is supposed to do. It is important that the teacher gives a thorough briefing before the outdoor lesson. It is important that the teacher presents a well-organised picture of the task. You need to have a check before you start as well so you know what to do and that you do not just start directly and miss out on something that is of importance when solving the task that is presented to you and your group.

Through well-organised lessons and well-made tasks, it is easier to understand the mathematics and what is expected of you. John continues:

> It is crucial that you understand what you are supposed to do, what kind of theory you need in order to solve the task that is presented to you. If the lessons are not well organised, the head, the brain, you get so, you disconnect, you start to think of other things.

Time is another aspect of John's experience of the long-term regular outdoor learning. He thinks that the teacher who teaches mathematics provides time to work with mathematics outdoors. However, he questions why other teachers in other subjects do not provide time and prioritise time to have some of the weekly lessons outdoors.

> I think that you should have outdoor lessons in other subjects too. Take biology for example. In biology there are so many "outdoors and environmental issues" so it would be a great possibility to work more outdoors. But, we are never outdoors during our lessons in biology, which I think is strange.

According to John, outdoor learning in mathematics provides more time and space to understand what and why you do things.

> Indoors you seldom receive any help from the teacher. Often you just sit there for like ten minutes waiting for the teacher to have the time to help you. This results in you not raising your hand to ask for help, because it is quite meaningless. On the contrary, during the outdoor lessons in mathematics the group could either help each other or if the group needs assistance from the teacher, the teacher helps the whole group at the same time, which is really great.

Impact on affective factors

John experienced and expressed a change in self-concept.

> Outdoor lessons in mathematics make it easier for you to remember what you do and why. When you have your lesson in mathematics outdoors, the teacher explains clearly what to do. You are given a clear picture of what is expected of you and what the task at hand is about and what the aim of the lesson is. You will better understand what you do and why you do it.

Moreover, it seems like the change of learning environment reduces the mathematics related stress and anxiety.

> It is relaxing to work on a regular basis with mathematics outdoors. I get really stressed during the regular indoor lessons in mathematics and suffer from mathematics-related panic attacks. However, during the outdoor lesson in mathematics I really enjoy myself, I am more relaxed and do not suffer from mathematics-related anxiety attacks.

John experiences that he is enjoying himself and feeling more relaxed during the regular outdoor lesson in mathematics.

John also indicates that his extrinsic motivation has a tendency to hamper his achievement and performance in mathematics. He brings up the pressure to perform and the stress and anxiety the national tests in mathematics cause. However he also brings up a sense of changed perception of himself and emphasises the positive outcomes of variation of context for the learning of mathematics.

> It makes you understand everything much better. You become more engaged and motivated. Regular outdoor lessons in mathematics provide you with more input and understanding of mathematical concepts. In addition, you feel better and enjoy the mathematics lessons more.

John appears to address the idea of losing one's self-confidence, which will lead to low self-concept.

John emphasises the importance of feeling engaged and motivated. To be extrinsically motivated tends to have a negative impact on understanding and learning mathematics. It rather makes you give up because you feel like a loser who cannot manage mathematics. However, by working to engage with real-life problems in mathematics with regular outdoor lessons in mathematics solving these real-life problems together with others provides, according to John, the possibility to become aware of one's true mathematical proficiency and that it might be enhanced. As demonstrated, John experienced that he began to enjoy mathematics more. He changed from being extrinsically motivated to become more intrinsically motivated and was more ready to face new, more challenging tasks because he started to believe in yohisur own abilities.

Impact on academic factors

John also stated that he had difficulties with negative numbers when using them and understanding the concept of them. During one of the outdoor lessons in mathematics they had worked with negative numbers. Before this outdoor lesson they had, during the indoor lesson, talked about negative numbers and worked with them in the textbook.

> Well, we had one lesson when we worked with negative numbers, you know plus and minus and that kind of stuff. We did this exercise where you were supposed to run and put a piece of paper next to another on one of these big long things that looks like a row, and then there were also, you know, numbers in between and at the far end there was perhaps minus something and in the middle was zero. After a while you started to realise that it was kind of a huge thermometer. It was almost like the numbers became connected with each other. The visual picture and the practice of actually building the thermometer gave you a better understanding of the concept. It is strange but you actually need lots of self-confidence when it comes to learning and understanding mathematics.

As seen, working more visually and practically with negative numbers, strengthens John's conceptual understanding of negative numbers.

John emphasised the gain of cooperative learning. During the outdoor lessons in mathematics the students were supposed to collaborate when solving different problems they were given. During the outdoor lessons in mathematics, John was given a feeling of participation.

> The class cooperated better when working with mathematics outdoors than indoors. Indoors the class seem to be more divided into certain groups. There is that group with the smart kids, who are good at mathematics, then the rest of us who are kind of left behind. During the outdoor lessons there tends to be more cooperative work, because all of us know that you must first begin to solve the given mathematics task on your own to begin with and then help the group by discussing the problem together. It is important that everyone can join in, participate and be given the possibility to explain how to solve the task at hand. Everyone should be given the opportunity to show his or her proficiency and share one's knowledge with others. It is a way to better understand how a given task can be solved. You become more engaged and motivated if you are allowed to participate and speak your mind regarding how you believe that the task at hand can be solved.

John emphasises that cooperative learning opens up opportunities for and development of adaptive reasoning. Through the outdoor lessons in mathematics, the students are given the possibilities to reason with each other. They have the opportunity to explain and try out their logical thoughts as well as justify their thoughts and chosen solution to the task at hand by reasoning with others.

Productive disposition

John expressed experiencing that he began, thanks to the regular outdoor lessons, to recognise and realise the importance of mathematics and that it is worthwhile to make the effort to understand the concepts of mathematics. Through cooperative learning he realised that he actually possesses a certain level of proficiency in mathematics. He realised that a task can be solved in several different ways.

> You need self-confidence in mathematics and if you enjoy mathematics then you become more confident and more motivated. Indoors you just sit and understand nothing. But when you work with mathematics outdoors you understand how it all works.

Discussion

John's perceived experiences of regular outdoor learning in mathematics indicate that it is reasonable to assume that students can benefit from regular outdoor learning in mathematics. We may conclude that long-term regular outdoor learning in mathematics shows a tendency to have an impact on affective factors by altering the individual's self-concept, reducing mathematics-related stress and anxiety and resulting in a more engaged and motivated student, which is supported by the findings of previous studies (e.g. Moffett, 2011). In addition, long-term regular outdoor learning in mathematics shows a tendency to have an impact on academic factors and one major feature was the importance of variation of context. It is reasonable to assume that the possibility to work more visually and practically with real-life problems, which had first been presented theoretically during the indoor lessons, might be a key factor for enhanced conceptual understanding (Kilpatrick et al., 2001). We may conclude that engagement and motivation for learning mathematics is enhanced. Accordingly, students tend to become more engaged and motivated by regular outdoor learning in mathematics. Cooperative learning is a prominent part of outdoor learning in mathematics and is emphasised by John as a key feature for the enhancement of his adaptive reasoning (Kilpatrick et al., 2001). Previous findings have indicated that cooperative learning tends to help mathematics- anxious individuals

reduce their stress and anxiety for mathematics (Daneshamooz & Alamolhodaei, 2012). The main feature of cooperative learning is the opportunity to discuss and reason with others and justify one's mathematical thoughts on how to solve different mathematical problems. John stresses that cooperative outdoor learning in mathematics tends to make students aware of their true mathematical proficiency by being given the possibility to observe that a task at hand can be solved in more than one way and that more than one "right" solution to the problem may exist. John experienced that when mathematics became less abstract and more transferable to his everyday life, learning mathematics becomes more joyful resulting in a more positive attitude towards mathematics making him engage instead of avoid learning mathematics (e.g. Maloney & Beilock, 2012). Long-term regular outdoor learning in mathematics gave John the possibility to develop the ability to self-regulate his learning (Wu et al., 2012). Finally, it is reasonable to assume that if a student is provided with a sense of control, that student may enhance the levels of self-concept and reduce mathematics related stress and anxiety.

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Finnish educators' conceptions of the social-emotional needs of mathematically gifted high school students

Eeva Haataja[1] and Anu Laine[2]

[1]University of Helsinki, Finland; eeva.haataja@live.fi

[2] University of Helsinki, Finland; anu.laine@helsinki.fi

This article presents conceptions of social-emotional needs of mathematically gifted adolescents of certain Finnish educators. The article is based on a qualitative research study conducted with the methods of semi-structured interviews and participant observation in a Finnish high school that offers a special programme for mathematically oriented students. The study shows that the educators considered the most essential social-emotional needs of mathematically gifted students to be the need to be respected as unique personalities, to meet other gifted, to feed and guide the intrinsic motivation.

Keywords: Mathematical giftedness, gifted students, social-emotional development.

Introduction

The purpose of this study is to present and analyse the conceptions of the social-emotional needs of mathematically gifted students held by certain educators who work in a mathematically oriented high school in Finland. The research is conducted in a unique Finnish school community that is focused on educating students who exhibit particular interest in mathematics. The data of this research was collected by interviewing five educators of the school and by participant observation in the school. The research context provides valuable knowledge as it brings together the practices and theories of research on gifted education (Ambrose et al., 2010; Laine et al. 2016).

Giftedness is a very complex concept and varies both qualitatively and quantitatively within gifted persons (Passow, 2004), and multiple definitions of giftedness have been written over the decades (Ambrose et al., 2010). This chapter presents the theoretical fundamentals on mathematically gifted adolescents and summarises former research on teachers' conceptions of giftedness and gifted education.

Being gifted means possessing promising potential in a certain domain of giftedness and being able to develop this potential into actual high performance. This domain, e.g. mathematics, is the field and the context in which the gifted activity occurs (Cross & Coleman, 2014). Mathematical giftedness in particular means the ability to abstract numbers, variables and functions and the relations between them. For a mathematically gifted person, it also means courage, persistence and intrinsic motivation to go further and deeper in such mode of comprehension (Gardner 1983; Reis & McCoach, 2002; Movshovitz-Hadar & Kleiner 2009; Subotnik, Pillmeier, & Jarvin, 2009). The development of such person is not only affected by personal characteristics, but also by the structure and properties of the individual's particular domain of talent (Coleman & Cross, 2000).

Mathematically gifted students need mathematical activities, and with the support of their surroundings, they are able to take an active role in their own learning and develop into professional mathematicians (Cross & Coleman, 2014; Usiskin, 2000). They have also certain social-emotional needs… A gifted person is able to fulfil his whole potential only if his intrinsic abilities, the support of his social surroundings and the social-emotional dimensions are in balance (Subotnik et al., 2009;

Usiskin, 2000). Exceptionally gifted adolescents often experience dissimilarity and even unpopularity among their schoolmates (Mönks & van Boxtel, 1985; Rimm, 2002), especially when the giftedness takes place in the domain of mathematics (Pettersson, 2008).

Many professional educators still view giftedness as a fixed and innate characteristic of a person (Laine, Kuusisto, & Tirri, 2016). Some of teachers even consider the gifted as students who actually do not need training or instruction (Laine et al., 2016). However, recent theoretical definitions of giftedness have shifted towards contextual and malleable conceptions (e.g. Ambrose, VanTassel-Baska, Coleman, & Cross, 2010; Cross & Coleman, 2014). Various researches illustrate this gap separating the theoretical knowledge of giftedness from the conceptions of professional educators (Ambrose et al., 2010). Research further shows that there is also a gap between the teachers' conceptions of gifted education and the educational practices they conduct. Therefore there is need for in-depth research to further our understanding of teachers' conceptions of giftedness (Laine et al., 2016).

Teachers' conceptions have a significant role in supporting young gifted students in advancing their talent (Mann, 2006; McNabb, 2003; Pettersson, 2008). Teachers tend to favor quite traditional conceptions of giftedness (Moon & Brighton, 2008). Generally, according to teachers, the most determining characteristic of a gifted student in the school context is a specific difference from others, which presents itself as the gifted student's capability to perform fast, intelligent and creative learning (Kaya, 2015; Laine et al., 2016; Mattsson, 2010; Moon & Brighton, 2008). It is interesting to note that teachers also associate mainly positive social-emotional characteristics with giftedness, such as enthusiasm, sensitivity and curiosity (Kaya, 2015; Laine et al., 2016; Mattsson 2010).

There seems to be a gap between the theories and teachers' traditional conceptions of giftedness. Therefore, the purpose of this study was to examine the educators' conceptions of the social-emotional needs of the mathematically gifted adolescents in a school with successful practices. We were especially interested in the conceptions that were also enacted in their practices.

Methodology

In this qualitative research the data was collected by interviewing all the teachers who had regularly supervised the summer schools (two mathematics teachers, the biology teacher) as well as the principal and the healthcare officer. Additionally… by observing, filming and participating in the mathematics classes in the autumn of 2011 and an overnight school session in January of 2012. A qualitative semi-structured interview technique was used in order to give the interviewees the possibility to conceptualize and describe the topic in the way they prefer.

The researcher spent time with the student participants of the overnight school and participated in the social and mathematical activities of the event. Short informal discussions were conducted with the students while participating. As the data was collected in the authentic environment of the research subjects by means of participant observation, this study contains characteristics of ethnographic research (Delamont 2004; O'Reilly 2005).

The data was transcribed and analysed with inductive content analysis. Inductive content analysis means categorizing and combining units of the analysis into larger aggregates (Elo & Kyngäs, 2008). A whole statement was chosen to be the unit of the analysis. The whole statements included one thought, conception or opinion varying in length from a couple of words to several sentences. These

units were first categorized into codes. After that, as is done in inductive content analysis (Elo & Kyngäs, 2008), the codes were connected into categories and such categories into main categories, and finally the results were interpreted in the light of the theoretical background of the study.

Results

The purpose of the school is to gather together and educate adolescents interested in mathematics. The students of the school are offered a wide range of instruction in mathematics and diverse learning environments such as the Night of Mathematics and an annual summer school in Lapland. The educators interviewed in this study generally described their students as gifted. They defined mathematical giftedness as the ability to picture, learn and remember mathematical causations rapidly and with clarity. They described two types of giftedness appearing in the school: students with multiple talents and those with a single exceptional talent. The students with multiple talents were interested in societal influencing and social activities. On the other hand, the exceptionally talented tended to impress their teachers with their commitment to studying and with their high level of mathematical reasoning skills.

Principal: Roughly speaking there are those Renaissance talents who are widely talented and then those exceptionally gifted, who focus on the area of their deepest interest.

The uniqueness of the school's students was emphasized in the interviews. The interviewees were unwilling to stereotype the students and rather described their personalities, interests, social skills and profiles of giftedness as very individual.

Principal: I don't want to give any stereotyped answer here. I don't want to say that they are this kind or that kind.

According to the educators, many of the students have experiences and memories of feeling different and isolated during elementary school. Sometimes a change of school climate can be essential for a gifted adolescent.

Mathematics Teacher2: And we offer a community where you can discuss the Schrödinger equation during a break without being sneered at.

Biology Teacher: I just received a message where the parents were thankful because it has been so great [for him]. To be accepted in the group and let him be himself and encouraged and so on.

According to the interviews, the students with exceptional mathematical giftedness had more challenges in terms of social skills than those who were gifted in various fields. Moderately gifted students are usually relatively popular among their school mates and age peers, while the exceptionally gifted are more prone to being left alone (Gross 2002; Rimm 2002). Any school environment requires various social skills from students (Payton et al. 2008). According to Mathematics Teacher 1, both "social sharks" as well as those who have "obvious problems in that respect" could be found among the students of the school.

Biology Teacher: Some of them have very poor social skills. …It is often related to this narrow field of giftedness.

Every student was welcome to participate in the social activities of the school to the extent of their own preferences. According to the interviews, one important social skill for the students is tolerating of all kinds of personalities. In the interviews, the diversity among the students was seen as an important part of the school's social climate.

Mathematics Teacher1: Of course one can choose to enjoy small groups or solitude.

Mathematics Teacher1: We have a vast variety of personalities and a tight community, which means that it becomes a tolerant community.

To study and associate with other gifted students was considered one reason behind the distinctive solidarity of the school community. These views are congruent with the literature (Gross 2002; Rimm 2002). Even though a variety of social skills existed among the students of the school, the common interests made social interaction easier.

Health-care officer: To find congenial people. And I know how the teachers describe, how they [the students] make experiments in the physics lesson or somewhere, the burning enthusiasm they show.

According to the interviewees, the students of the school were able to form close friendships with each other. The class-based structure and diverse range of informal activities formed the basis for the development of friendships at the school.

Biology Teacher: And then across the groups of each year's class, because on Mondays [when extra courses in mathematics are taught] and at overnight schools, they spend time together, there are no boundaries.

Associating and studying with other gifted students are emphasized both in these results and in the literature (e.g. Subotnik et al. 2009; Rimm 2002). The positive social climate of the school was constructed upon acceptance of the dissimilarities of students, diverse social interaction, shared experiences and interest in learning mathematics. These features were also seen as suitable for enhancing giftedness.

Intrinsic motivation is one of the most essential social-emotional characteristics for the development of mathematical giftedness (e.g. McNabb, 2003; Subotnik et al., 2009). The importance of motivation was also emphasized by the mathematics teachers interviewed in this study.

Mathematics Teacher2: They are very motivated. And that is more determining than giftedness. Of course they need some kind of giftedness. But with some kind of basic giftedness you can go very far.

Even though motivation is often seen as a person's inner characteristic (Subotnik et al. 2009), the interviewees highlighted the significance of peer support in connection with maintaining motivation. The shared motivation and interest in mathematics was also apparent in the overnight school, where groups of students solved mathematical problems together while demonstrating amazing enthusiasm.

Health-care officer: It is the passion for [mathematics] that creates common good things in the class or the group or among the students.

Mathematics Teacher2: The social pressure can influence them one way or another… They support each other very much in studying.

The observations of the overnight school showed that the students were able to discuss their perfectionism, too. The conversation was humorous, and the participants were laughing at their perfectionistic characteristics.

Student1: I am not a workaholic at all!

Student2: No you're not. You only scared all the freshmen with your stories last year.

Sometimes high motivation comes with negative phenomena such as unbeneficial perfectionism. The biology teacher and the healthcare officer had seen that achieving certain objectives or failing to do so may cause stress and exhaustion.

Principal: It [exhaustion] does not occur often, but someone every year.

Health-care officer: Often great giftedness and striving for perfection and achievements are a part of the personality. There is a risk of stress and fatigue and exhaustion.

The interviewees tended to see perfectionism as a practical problem of the educational system rather than a problem in the adolescents. The perceptions of the interviewees were summed up by the healthcare officer. According to her, negative perfectionism can be prevented and treated by guiding the students, being adaptive and offering constant care to the students.

Health-care officer: Flexibility and a flexible education system are what secure the path of the adolescents somehow. And also the caring, in particular, daily care.

Studying with the other gifted students in a supporting school climate was described to help the adolescents to form a realistic self-image and a strong self-confidence as people and mathematicians.

Mathematics Teacher1: It is easy to obtain perspective, [because] some really are incredibly good. – – But nobody is the best of all.

Health-care officer: It is amazing to notice how they somehow gain self-confidence.

The gifted students were described as both ordinary and special at the same time, as they encounter the common social-emotional challenges of adolescence but also have special characteristics and needs due to their mathematically oriented and ambitious environment. The influence of the domain of mathematics on the development of mathematically gifted adolescents cannot be ignored.

Discussion

All qualitative research should be subject to realistic reflection on its general reliability (Lincoln & Guba 1985). In this particular study, the use of triangulation of data collection increases the validity of the research. Nevertheless, a longer participatory observation could have offered more profound information on the social interactions of the students. Additional reliability was achieved by presenting the interviewees with the results of this study.

This research studied educators who possess particular experience in the context of Finnish education on teaching students who are recognised as gifted. Therefore its results differ from earlier studies on conceptions of giftedness of Finnish teachers (Tirri & Kuusisto, 2013; Laine et al., 2016). The interviewees of this study described giftedness as advanced performance and a modifiable characteristic of a student as well as of the whole school community. Finnish teachers tend to relate gifted students only with positive social-emotional characteristics, such as creativity and high level

of motivation (Laine et al., 2016). The interviewees of this study were prone to discuss also social-emotional challenges related to lives of the gifted adolescents. Still, the positive attitude to giftedness, enthusiasm for teaching and pride of their students were easily heard within the interviews. Research has also noted that the amount and especially the quality of cooperation with gifted students determines the teachers' conceptions of and approaches to giftedness (Kaya, 2015).

The concept of dissimilarity is widely included in definitions of giftedness as well as in the conceptions of giftedness commonly held by teachers. If someone is thought to be gifted, she is also seen as somehow, although often positively, different from others (Shani-Zinovich & Zeidner, 2009). In this research, the school was described as a meeting place for mathematically gifted students who are, in some way, different from many other adolescents. Therefore it is significant to understand the difference as a subjective experience of a gifted adolescent. Although the society, parents and teachers usually appreciate giftedness, exceptional talents often experience isolation within their age group (Gross, 2002; Rimm, 2002).

The organization and the curriculum of national school systems should meet the needs of every student including the gifted ones (Cross & Coleman, 2014; Kaya, 2015). The school investigated in this research does not represent a common high school in Finland. Neither does it reflect general Finnish attitudes toward special education of highly performing adolescents (Laine et al., 2016; Tirri & Kuusisto, 2013). A person's individual growth and particular social environment determine the social-emotional challenges of a mathematically gifted child instead of the mathematical giftedness per se (Wilson, 2015). Finnish teachers are highly qualified and skilled at differentiating learning contents both for fast and slow learners (Laine et al., 2016). However, this research shows that even the most devoted and competent teacher cannot replace the need of meeting, studying and making friends with other congenial peers. As a conclusion, when planning education for gifted students the social-emotional aspects of gifted education and the gifted students' need for meeting congenial adolescents should be considered.

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The qualitative differences in generative characteristic of the aesthetic sensibility: Analysis of learners' problem solving

Hayato Hanazono

University of Tsukuba, Graduate School of Comprehensive Human Sciences, Japan
hanazono@human.tsukuba.ac.jp

There are some discussions about the relationship between creativity and the aesthetic sensibility. It has been a characteristic of mathematician and mathematical gifted that the aesthetic sensibility has a generative characteristic. On the other hand, some studies demonstrated that the generative characteristic also was available even by the non-mathematician or non-mathematical gifted. The purpose of this paper is to clarify the process to produce qualitative differences in the generative characteristic of the aesthetic sensibility through the analysis of learners' problem solving. For this purpose, two pairs of high school students and one pair of college students were observed during problem solving. As a result, it was clarified that it is critical whether learners have their own goal other than the given one to evoke the generative characteristic. Moreover, it was suggested that the difference of one's own goal is associated with qualitative difference in the generative characteristic.

Keywords: Aesthetic, generative, creativity, problem solving, qualitative study.

Introduction

There are two directions for the term "mathematical creativity" used widely: extraordinary creativity, known as big C, or everyday creativity, known as little c (Sriraman, Haavold & Lee, 2014, p.110). Everyday creativity also is important for mathematics educators. Silver (1997) said,

> Although creativity is often viewed as being associated with the notions of "genius" or exceptional ability, it can be productive for mathematics educators to view creativity instead as an orientation or disposition toward mathematical activity that can be fostered broadly in general school population. (Silver, 1997, p.75)

In today's national curriculum in Japan, developing *all* students' creativity is one of the objectives in high school mathematics education. So, this paper uses the term "mathematical creativity" or "creativity" as the meaning used by Silver: everyday creativity.

Since Poincaré (1908/2003) pointed the importance of the aesthetic sensibility in mathematical discovery, the interest in the relationship between creativity and the aesthetic sensibility has risen in the field of mathematics education. Some studies have claimed that the aesthetic sensibility is one of characteristics of mathematician or mathematical gifted student (Dreyfus & Eisenberg, 1986; Hardy, 1956/1992; Krutetskii, 1976; Poincaré, 1908/2003; Silver & Metzger, 1989). As the bases for such a claim, these studies had drawn attention to the process that the aesthetic sensibility worked (Dreyfus & Eisenberg, 1986; Krutetskii, 1976; Silver & Metzger, 1989). In particularly, Silver & Metzger concluded that there were several characteristics of aesthetic sensibility during mathematical problem solving only by mathematicians. On the other hand, other studies disagree with above claims in that general learner, non-mathematician or non-mathematical gifted student, can have the aesthetic sensibility and their aesthetic sensibilities have the similar characteristics with mathematician or mathematical gifted student (Papert, 1978; Sinclair, 2006a). These studies demonstrated that the

generative characteristic of the aesthetic sensibility, which works as a guide in decision making during mathematical discovery, could work in general learners' mathematical problem solving.

If the difference of such a characteristic is not due to the mathematical talent, the questions have remained unanswered what critical factor causing difference is or how such difference is caused. The purpose of this paper is to clarify the process to produce qualitative differences in the generative characteristic of the aesthetic sensibility through the analysis of general learners' problem solving.

Theoretical background

The characteristics of the aesthetic sensibility in problem solving

There is not a clear and widely accepted definition of the term "aesthetic" in mathematics education. Poincaré (1908/2003) explained using both the form of mathematical objects and the sense of the perceiver as following:

> It is the harmony of the different parts, there symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts. (Poincaré, 1908/2003, pp.30-31)

Hardy (1956/1992) and Dreyfus & Eisenberg (1986) defined it using subjective qualities like "economy", "simplicity" and "surprise". Without referring to its subjectivity, Hardy argued that mathematician would share them. In contrast to the above studies, Wells (1990) claimed that the aesthetic qualities are subjective and context-dependent.

On the other hand, some studies paid attention to the working process of the aesthetic sensibility, which is the ability to appreciate and respond to aesthetic qualities of mathematical objects, rather than strictly defining the term "aesthetic" (Papert, 1978; Silver & Metzger, 1989; Sinclair, 2006a, 2006b). Though claims about what the aesthetic is are various, the discussion about what characteristics the aesthetic sensibility have is a convergent as below. Papert (1978) was focused on the process of creation explained by Poincaré. Papert regarded the aesthetic widely and observed non-mathematicians' proving process. As a result, Papert concluded non-mathematician also could be guided by the aesthetic sensibility. Silver & Metzger (1989) classified the role of the aesthetic sensibility into two categories. First is "the guidance of decision making during problem solving" (p. 62). The viewpoint of this category can be regarded as the same as Papert's. Second is "the evaluation of the elegance of a completed solution" (p.62). In addition, Silver & Metzger observed mathematicians' problem solving and identified these roles in their problem solving. Similarly, Sinclair (2006b) classified characteristics of the aesthetic sensibility into three categories: the evaluative characteristic, the generative characteristic, the motivational characteristic. Moreover, Sinclair (2006a) observed learners' problem solving and problem posing and identified these characteristics of the aesthetic sensibility. Sinclair's generative characteristic and evaluative characteristic can be regarded as the same as Silver & Metzger's two roles.

The above results demonstrate that whether the aesthetic sensibility does work in problem solving is due not to only mathematical talent but also to something else. The existence of this "something else" is consistent with the claim of Papert. However, there has been not enough study done concerning this point. In particular, few studies have attempted to observe learners' generative characteristic of the aesthetic sensibility.

Based on the above background, this paper defines the generative characteristic of the aesthetic sensibility (GCA) as a guide in decision making during mathematical discovery, and clarifies the process to produce qualitative differences in the GCA.

Four categories of the generative characteristic of the aesthetic sensibility.

Poincaré (1908/2003) regarded the aesthetic sensibility as a thing working in the unconscious level. In contrast, some studies limited the discussion to the conscious level (e.g. Papert, 1978; Silver & Metzger, 1989; Sinclair, 2006a). This paper also limits a discussion to the conscious level.

Papert regarded reasoning without conviction or logic but with pleasure as "the problem of guidance" (p.109) by the aesthetic sensibility. Sinclair (2006b) associated such non-conviction reasoning to intuition as "capitalising on intuition" (p.94). Moreover, Sinclair (2006b) identified additional three categories of the GCA based on the mentions by mathematicians. First category is "playing with or 'getting a feel for' a situation" (p.94). This means exploration "in that the one playing is seeking to identify organizing themes and structures and to arrange the objects being played with in a meaningful, expressive way" (p.95). That is, it can be interpreted as pursuing these goals without depending on the goals of the given problem. Second category is "establishing intimacy" (p.94). This means, for example, to give a name to the considered subject. Third category is "enjoying the craft" (p.95). This is interpreted as consideration using mastered tools. Although the question remains whether it is reasonable that intuition is regarded as one of the aesthetic generating, this classification by Sinclair (2006b) can be used as viewpoints for extraction of the GCA from one's behavior in problem solving.

A study on the generative characteristic of the aesthetic sensibility in general learners' problem solving

Participants

Two pairs of high school students and a pair of college students, who had several mathematical knowledge and mathematical experience, were selected as participants, and observed during solving a problem. One pair of high school students belonged to 10th grade (Pair H1), another pair belonged to 11th grade (Pair H2). One of college students belonged to third year and another belonged to fourth year of mathematics teacher-training course (Pair C). Although they were all better than the average learners in Japan, they were not so good as mathematician or mathematical gifted student.

Pair H1 and Pair H2 belonged to the same high school in Japan. Pair H1 had learned double radical signs. However, they had learned about a particular type like $\sqrt{3+2\sqrt{2}}$ which could be transformed into other form without a double radical sign. In addition, they had not learned the relationship between the roots and the coefficient of the quadratic equation. The other hand, Pair H2 had learned same type double radical signs with Pair H1. This pair had learned the relationship between the roots and the coefficient of the quadratic equation.

Pair C did not belong to the same high school with Pair H1 and Pair H2. They had learned double radical signs and the relationship between the roots and the coefficient of the quadratic equation. Although a student belonged to third year had never "studied" mathematics in college, another student belonged to fourth year had "studied" mathematics in college for a half year. So, fourth grade

student was expected to have experienced mathematical discovery and to show the experience in the process of problem solving.

Procedure of the study

Because the GCA is "involved in the actual process of inquiry, in the discovery and the invention of solutions or ideas" (Sinclair, 2006b, p.93), participants were observed their problem solving behavior in following process.

Each pair calculated in order to clear some concrete double radical signs like $\sqrt{3+2\sqrt{2}}$ as warm-up. Then, observer showed another type which cannot be cleared double radical signs like $\sqrt{5+2\sqrt{2}}$. After that, each pair solved a problem about an abstract double radical sign (it is shown below). After they finished to solve, they were interviewed about how to solve it.

In order to analyze verbal report during solving problem as data, participants were asked to solve a problem while consulting in pairs. By this setting, it was expected to provide simultaneous and nature verbal report. All participants' verbal report was recorded on a IC recorder. (Only Pair H2 was recorded on a video camera, too.)

A problem

In this study, a following problem was chosen.

Find the conditions for clearance of a double radical sign from $\sqrt{p+2\sqrt{q}}$. However, p and q belongs in positive rational number. Moreover, q is not the square of rational number.

There are multiple conclusions in this problem as following. The participants were not informed what conditions were appropriate as the conclusion. Therefore, they were also asked value judgments for determining their finding as a conclusion. In contrast to that the multiple solution problems need participants to solve problem by more than one way — general learners usually do not so, such open-ended problems need participants' value judgments more naturally.

(Conclusion 1): If a double radical sign can be transformed into following expression:

$$\sqrt{p+2\sqrt{q}}=\sqrt{\alpha}+\sqrt{\beta},$$

then right side of above equation can be transformed into following expression:

$$\sqrt{\alpha}+\sqrt{\beta}=\sqrt{\left(\sqrt{\alpha}+\sqrt{\beta}\right)^2}=\sqrt{\alpha+\beta+2\sqrt{\alpha\beta}}.$$

From the above transformations, a necessary condition for clearance of a double radical sign from $\sqrt{p+2\sqrt{q}}$ is the existence of α and β belonging to positive rational number such that $p=\alpha+\beta$ and $q=\alpha\beta$. Conversely, if α and β belonging to positive rational number such that $p=\alpha+\beta$ and $q=\alpha\beta$ exist, then the double radical sign of $\sqrt{p+2\sqrt{q}}$ can be clear as following:

$$\sqrt{p+2\sqrt{q}}=\sqrt{\alpha+\beta+2\sqrt{\alpha\beta}}$$
$$=\sqrt{\left(\sqrt{\alpha}+\sqrt{\beta}\right)^2}=\sqrt{\alpha}+\sqrt{\beta}.$$

(Conclusion 2): Existence of α and β belonging to positive rational number such that $p=\alpha+\beta$ and $q=\alpha\beta$ is equivalence with that α and β are the roots of the quadratic equation x^2-$px+q=0$ in x. From this, the latter condition is also a conclusion of above problem.

In the following, D expresses the discriminant of the quadratic equation. If $D=p^2$-$4q \geqq 0$, then

$$x=\frac{p\pm\sqrt{p^2\text{-}4q}}{2}>0.$$

From this, if D can be expressed as the square of rational number like k^2 ($k\in\mathbb{Q}^+$), then x^2-$px+q=0$ has positive rational roots in x. Conversely, if D can be expressed as k^2 ($k\in\mathbb{Q}, 0<k<p$), then the two roots of quadratic equation x^2-$px+q=0$ are

$$x=\frac{p\pm k}{2}.$$

These sum and product are

$$\frac{p+k}{2}+\frac{p\text{-}k}{2}=p,\qquad \frac{p+k}{2}\times\frac{p\text{-}k}{2}=\frac{p^2\text{-}k^2}{4}=\frac{4q}{4}=q.$$

From these,

$$\sqrt{p+2\sqrt{q}}=\sqrt{\frac{p+k}{2}+\frac{p\text{-}k}{2}+2\sqrt{\frac{p+k}{2}\times\frac{p\text{-}k}{2}}}$$
$$=\sqrt{\left(\sqrt{\frac{p+k}{2}}+\sqrt{\frac{p\text{-}k}{2}}\right)^2}=\sqrt{\frac{p+k}{2}}+\sqrt{\frac{p\text{-}k}{2}}.$$

From above, if D can be expressed as k^2 ($k\in\mathbb{Q}, 0<k<p$), then the double radical sign of $\sqrt{p+2\sqrt{q}}$ can be clear.

(Conclusion 3): Looking back at the conclusion 2 can provide next developmental conclusion. That is, the conditions for clearance of a double radical sign from $\sqrt{p\text{-}2\sqrt{q}}$ is that D can be expressed as k^2($k\in\mathbb{Q}, 0<k<p$), too.

(Conclusion 4): Moreover, seeing $4q$ as $\left(2\sqrt{q}\right)^2$ can provide another perspective. That is, if $2\sqrt{q}$ in the $\sqrt{p\pm2\sqrt{q}}$ is replaced to $\sqrt{q}$, then a conclusion of finding the conditions for clearance of a double radical sign from $\sqrt{p\pm\sqrt{q}}$ is that p^2-q can be expressed as k^2($k\in\mathbb{Q}, 0<k<p$).

Results

Overview of three pairs' problem solving is shown in Table 1.

	Pair H1 (H1-1&H1-2)	Pair H2 (H2-1&H2-2)	Pair C (C1&C2)
Conclusion 1	04:00	H2-1 02:20/ H2-2 06:02	08:16
Conclusion 2	16:20	H2-1 05:37/ H2-2 06:59	
Conclusion 3, 4			
Finish	16:20	51:16	38:00

Table 1: Overview of three pairs' problem solving

In Table 1, each of the values show the time participants spent to get to the conclusion shown to far left, and diagonals show that participants did not arrive at the conclusion. Each participant of Pair H2 had got to the conclusion on their own before they started to consult each other. Therefore, each time of Pair H2 was shown in Table 1. Line of the "Finish" shows the time which each pair had spent solving problem. For instance, Pair H2 and Pair C kept pursuing more exact condition than their conclusion in cooperation each other. Solving process of each pair are as follows. However, the symbols used below are the same as those used by participants.

(Pair H1)

After confirming the question, each immediately got to the relationship: $p = a + b, q = ab$. H1-1 continued more investigations from the reason that "p and q cannot be found when these are the large numbers. I want another one easily puts out with only p and q.", and continued for a further inquiry. In addition, H1-2 said "it is not good to use the new a and b", and continued to explore. H1-1 associated the factorization from the above equation, and flashed that it will go well if he can factorize as following: $x^2 + px + q = (x + a)(x + b)$. H1-1 started to think about that H1-2 had been questioned: the case $x^2 + px + q = 0$ has the rational solutions. H1-1 paid attention to the discriminant of the quadratic equation, but dismissed this idea. H1-1 dazzled that if the root in the quadratic formula could be clear, it will go well. H1-2 agreed this. They established conclusion that $D = p^2\text{-}4q$ is square number, and finished the solving.

(Pair H2)

After confirming the question, H2-1 and H2-2 went ahead the discussion using a specific example. However, they did not use the peculiarities of example, generalized immediately once outlook was obtained. Up to this point, they worked on the problem at each, and got to a conclusion 2 through the conclusion 1. However, they had not been convinced that the condition they got was the one they sought for. When the observer was urged to check the progress of each other, they decided to consider in cooperation about the conditions with the p and q, which H2-1 had considered. H2-1 was looking for a simple conclusion than conclusion 2. Ultimately, they made out that D can be expressed as the square of rational number is necessary and sufficient condition for clearance of a double radical sign without deriving another "simple" conclusion.

(Pair C)

After confirming the question, C1 remembered following condition as a formula: $p = a + b, q = ab$. Then, they considered they could regard this formula as conclusion. They made sure that $a \geqq 0, b \geqq 0$, and could regard the formula as conclusion. By the "intervention" of the observer: asking them about example, they began to consider whether concrete double radical signs can be clear based on above "conclusion". C1 noticed that x, y were rational number when a double radical sign could be cleared, and he said, "I guess I should add another conditions". C2 considered that the example which could not be clear a double radical sign was really not able to clear through a specific calculation. As a result, they confirmed that it could not clear. In response to the results, they concluded as following:

Condition (ⅰ): $p = a + b, q = ab$.
Condition (ⅱ): $a \geqq 0, b \geqq 0$.
Condition (ⅲ): a, b are rational number.

The analysis

The analysis was carried out in the following procedure. First, making transcripts of participant's problem solving process. Next, judging whether some of the four GCA proposed by Sinclair (2006b) can be seen through the observation of problem solving behavior and the interpretation of the intention of the behavior. The interpretation of the intention is based on the transcripts of the problem solving process and the explanations of their process that the participant did after solution. Finally, comparing the GCA of each pair. As a result, the GCA were seen as following Table 2.

the GCA	Pair H1	Pair H2	Pair C
Removing new symbols (for Conclusion 2)	○	○	
Pursuit of exactness (for Conclusion 2)		○	○
Pursue of simplicity (for other Conclusions)		○	

Table 2: The GCA of three pairs in problem solving process

Discussion: The qualitative differences in the generative characteristic of the aesthetic sensibility

From a comparison of the three pairs, mainly three of qualitative differences in the GCA in problem solving were observed. First is the point whether they attempt to remove new symbols a and b. Both Pair H1 and Pair H2 discussed about this point, but Pair C did not. Second is the point whether they confirm the exactness of the conclusions. Pair H1 did not check it, and the remaining two pairs did. However, there was a difference in the method used by pair H2 and pair C. Pair H2 was showed exactness by considering that the found condition was necessary and sufficient condition. On the other hand, pair C confirmed the exactness by considering a concrete example. Third is the point whether they attempt to further improve the conclusion that D could be expressed as the square of rational number. This had done only pair H2. However, this pair could not obtain conclusion.

From above results, it can be presumed that the qualitative difference of the GCA in problem solving depends on participants' goal in problem solving. For instance, pair H1 wanted not only to find condition, but also find condition without new characters a and b. Pair H2 also had this goal, but pair C did not. As a result, only pair C did not pursue another conclusion. Therefore, in order to evoke the GCA it is critical whether learners have one's own goal other than the given one. In addition, even if

learners have same goal such as exactness, the difference in means of the "exactness" can cause qualitative differences of the GCA.

From above discussion, if we can make learners to have one's own goal, it is possible that we can evoke learners' GCA. However, it is not clear in this paper that what kind of goal is desirable for mathematical creation, and how can we make learners to have one's own goal. Therefore, a further study of these points should be conducted.

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Identifying subgroups of CERME affect research papers

Markku S. Hannula and Enrique Garcia Moreno-Esteva

University of Helsinki; markku.hannula@helsinki,fi

Research in mathematics related affect uses a variety of theoretical frameworks. Three different dimensions have been suggested as significant to characterize concepts in this area: (1) emotional, motivational, and cognitive aspects of affect, (2) state and trait aspects of affect, and (3) physiological, psychological, and sociological level of theorizing affect. In this study, we used the information in reference lists and graph theory to identify Graph Communities (coherent clusters) of research papers published in the affect groups of CERME conferences. The four main Graph Communities identified in the analysis were Foundation (beliefs, attitudes, emotions), Self-Efficacy, Motivation, and Teacher Development. There were six smaller Graph Communities that may suggest emerging new frameworks: Academic Emotions, Metacognition, Teacher Beliefs, Resilience, Meaning, and Identity. These results suggest that of the three possible dimensions to structure the area, the distinction between cognition (beliefs), motivation, and emotions is the most important one.

Keywords: Affect, educational theories, graphs, literature reviews.

Introduction

The affect group in CERME has spent a lot of time and energy discussing the conceptual framework and terminology, leading to more extensive theorization of the area. Three theoretical frameworks have been especially influential in CERME for structuring the area of affect. The first is McLeod's 1992 framework that identified three main topics of research in mathematics related affect: emotions, attitudes, and beliefs. Moreover, the framework suggested that emotions are the most intensive, the least stable, and the least cognitive of the three, while beliefs are at the other end of the continuum and attitudes are in the middle.

An important synthesis of discussions in the group was Op 't Eynde's graphic representation of the conceptual field at CERME 5 (Hannula, Op 't Eynde, Schlöglmann & Wedege, 2007, Figure 1). This model identifies some new constructs that had been then discussed in CERME affect group: most importantly, the model identifies motivation as a dimension separate from affect, and meta-affective constructs. Moreover, the model highlights the local (classroom) and socio-historical contexts.

The model was discussed and developed by Hannula in his CERME plenary (2011) and further elaborated in the CERME special issue of RME (2012). This cube model (Figure 2) identified three dimensions that are relevant when discussing affective constructs. The first dimension addresses cognitive, affective, and motivational types of constructs. The second dimension separates the rapidly changing state-type constructs and more stable trait-type constructs. The third dimension identifies three levels of theorizing affect: psychological, social and embodied theories.

These distinctions identify separate research areas of mathematical affect. But how separate are they? Do the studies on attitudes, emotions, and beliefs form three separate research traditions or even 18 separate research areas, as Hannula's cube model suggests? Perhaps the studies on state and trait type affects are separate? Or are the different topic areas partially overlapping with diffuse borders, as Op 't Eynde's figure depicts it?

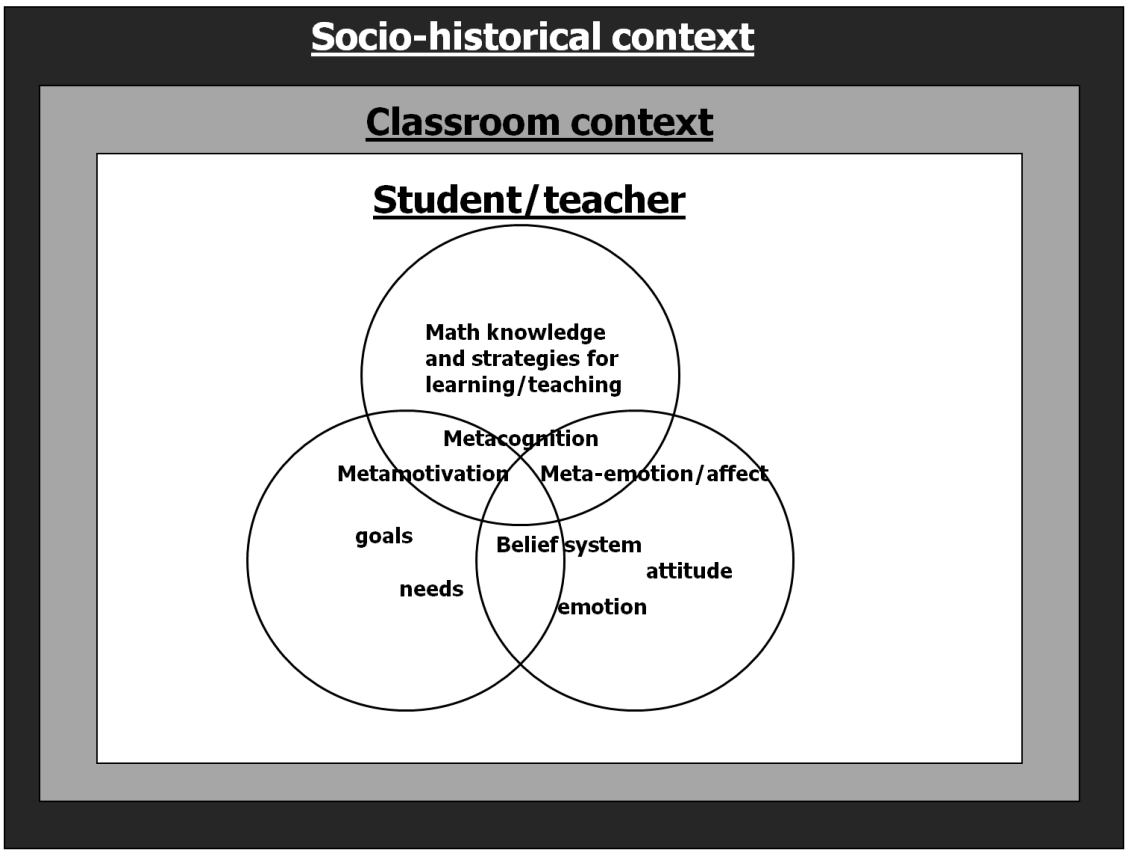

Figure 1. A graphic representation of the different dimensions of mathematics-related affect and their relationships, presented at CERME 5 (Hannula et al., 2007, p. 204)

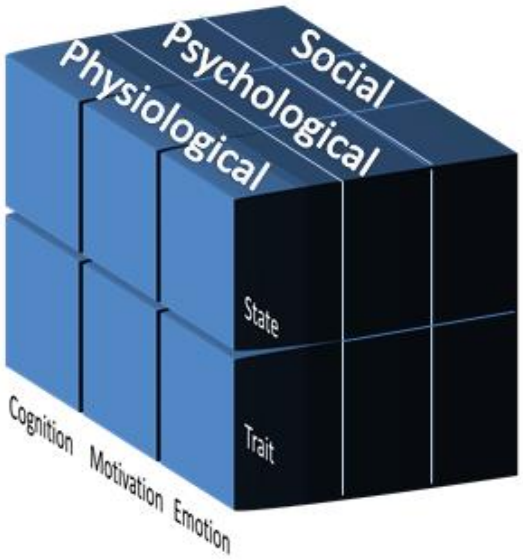

Figure 2. Hannula's (2011; 2012) cube model of the three dimensions for affective constructs

One way to analyze the question empirically is to analyze the references different research papers share. We are aware that there is a lot of cross-referencing between authors of CERME affect papers and that there are some fundamental articles that keep being cited often. However, there has not yet been any systematic analysis on the cohesion of the research papers. We are going to make a network

analysis of CERME papers in this research area. With this analysis we can identify whether there are some clusters of different research traditions within the CERME affect group. Moreover, we expect to identify important foundation works for each research tradition.

Our research questions are: (1) What possible subgroups can we identify among CERME research papers on affect when analyzing the cited authors of each paper? (2) What are the defining characteristics of each subgroup of papers?

Methods

Corpus and measures

Our corpus for analysis were the 100 research reports published in the affect groups in CERME conferences four to nine. The first two CERME conferences did not have an affect group and CERME3 one was left out because of some difficulties in the formatting that we did not have time to solve. We excluded from our analysis affect papers published in other groups, for example in groups of teacher beliefs or comparative studies.

Knowing the CERME publications in affect group, we knew that some authors had published papers that might fall into different research traditions. For example, many researchers have published papers both on teacher affect and student affect, and it could be possible that these fields would be empirically separate clusters. Therefore, we decided to search for clusters of papers published in CERME rather than identifying clusters of researchers.

When analyzing the lists of references, we had two basic options to identify links between papers. We could identify whether the same reference appeared in the reference lists of two papers, or we could link them whenever they had papers by the same author in their lists of references. Because our data corpus was modest in size, there was high probability that papers using the same theoretical framework might not share the same exact references even if they use papers by the same authors. Therefore, we decided to use the authors in the reference lists rather than the exact references as the method of connecting papers.

As a conclusion, our data consisted of research reports published in the proceedings and the authors appearing in their reference lists. For each of the papers, we created a connecting link from the paper to each of the authors mentioned in the references. For analyzing the connections between papers, we identified CERME papers and cited authors as vertices of a graph. Edges of the graph are the links that connected each CERME paper with the authors mentioned in the list of references, allowing multiple edges when a paper had several references by the same author. Hence, we produced a graph connecting all papers to their cited authors and this was then subjected to a mathematical analysis.

Analysis

The analysis included two stages: (1) To identify Graph Communities, and (2) to identify commonalities within the papers and authors of the selected Graph Communities.

We identified related papers using the FindGraphCommunities algorithm with modularity-based clustering to identify how papers and cited authors are related (Wolfram Alpha LCC, 2016). The modularity approach was originally developed by Newman and Girvan (2004) and the algorithm used in Wolfram Alpha is based on Fortunato's article (2010).

The input for the algorithm was the graph connecting research reports to authors cited in these papers. The output was subgraphs called Graph Communities, each of which consisted of some of the research reports and authors. The algorithm chooses only one Graph Community for each of the graph's vertices. In other words, although the same author may originally appear in the reference lists of papers from many graph communities, in these subgraphs each cited author belongs exclusively to one Graph Community. This accentuates the differences between Graph Communities, especially with respect to those authors who are cited in papers in several communities. In addition, we are aware that the current method does give additional weight to authors cited several times in a paper.

The next stage of analysis was to identify which Graph Communities to include in further analysis. This stage was based on a visual inspection of Graph Communities to see how well they are connected. The five biggest Graph Communities are presented as graphs (Figures 3 – 6), all affect papers published in CERME are represented as vertices with multiple edges. However, authors may have either one or multiple edges, depending on how many times they appeared in references. Another six Graph Communities are described but due to space limitation their graphs are no presented.

In the last stage of the analysis, we examined which papers and authors were represented in each of the Graph Communities.

Results

The algorithm identified 21 Graph Communities. We shall describe nine of them.

The first Graph Community 1 (29 papers; Figure 3) we call *Foundation*. It was the largest and the most cited authors in it included arguably the most influential researchers in the area of mathematical affect: McLeod (e.g. 1992), Schoenfeld (e.g. 1992) and Goldin (e.g. 2002). The most frequently cited authors in Foundation were active participants of CERME affect group: Hannula, Zan, Pehkonen, and Di Martino. Foundation is perhaps the most difficult to describe and may be best done by contrasting it with other Graph Communities. The Foundation's papers represent a large scope of research topics and theoretical frameworks, including papers that focus on beliefs, attitude, affect during problem solving, and emotions. This group also contains several papers that deal with dynamically changing affective states.

The second Graph Community (11 papers, Figure 4,) was given the label *Self-efficacy*. It had papers mostly from Cyprus and Turkey (e.g. Arslan & Bulut, 2015). The shared theoretical framework of self-efficacy was indicated by numerous references to Pajares and Bandura.

The third Graph Community (11 papers, Figure 5) we named *Motivation*. Philippou and Pantziara were influential authors in this Graph Community. Seven of the papers included at least one of them as the author (e.g. Pantziara & Philippou, 2011). Also Wæge appeared three times in this group as an author. This group shared motivation theory framework, and the most cited authors were well-known motivation theorists Midgley, Deci, Ryan, Pintrich, and Elliot.

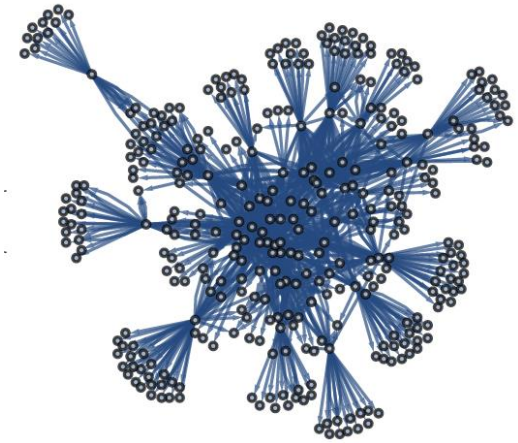

Figure 3. The Graph Community Foundation



Figure 4. The Graph Community Self-Efficacy

The Fourth Graph Community (8 papers, Figure 6) collected together papers on *Teacher Development.* Liljedahl was an important author in this group with four papers and the most cited authors include Liljedahl and Ball.

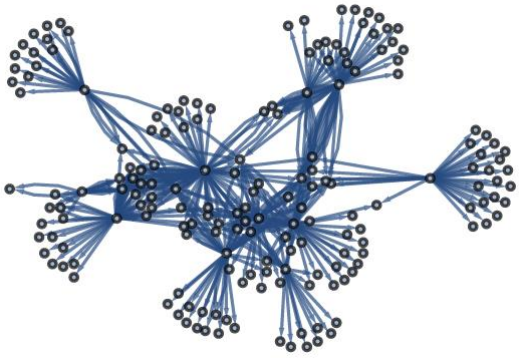

Figure 5. The Graph Community Motivation

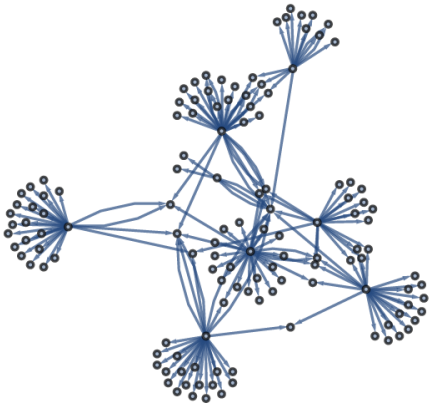

Figure 6. The Graph Community Teacher Development

The following six Graph Communities were smaller, each including 3 to 5 papers. Due to space limitations, these will be described only briefly. The four papers in *Academic Emotions* share Pekrun's (e.g. Pekrun, Goetz, Titz, & Perry, 2002) academic emotions framework and all these papers have been published in CERME8 or CERME9. All four papers in *Metacognition* were authored by Panaoura (citing e.g. Flavell, 1987). *Teacher Beliefs* had three papers from CERME8, citing, e.g. Fives & Buehl (2008). *Resilience* included 5 loosely connected papers without any frequently cited author. *Meaning* (citing e.g. Skovsmose, 2005) included four papers, and *Identity* (citing, e.g. Sfard & Prusak, 2005) four papers.

The remaining 14 Graph Communities included one or two papers each, altogether 16 papers. Nine of these papers were by authors who have ever published only once in the CERME affect group. Yet, these included also papers by frequent CERME participants (e.g. Hannula and Philippou).

Discussion

The analysis identified nine groups of CERME affect publications. Their defining features were a shared theoretical framework and often a research team. The largest group, Foundation, did not hold a theoretical framework clearly separating it from other groups. Rather, this group seemed to rely

more on the seminal works in the field of mathematics related affect and cover a variety of research topics indicating that there is much cohesion in this research field.

How much are these identified groups of papers determined by having the same authors? Most researchers with several papers in the analysis had most of their papers in a single community and only Philippou appears in three different Graph Communities. Often, having publications in different Graph Communities seems to be explained by supervisors co-authoring their students' papers that may often have quite different theoretical frameworks than their own papers.

The current method did not allow overlapping of Graph Communities, which made it difficult to identify possible authors who have a cross-cutting importance across several graph communities. However, looking at the total numbers of citations across all Graph Communities we found some such authors. The clearest examples were Ernest, who was cited 12 times in total, but not more than four times in any Graph Community, and Mason, who was cited 10 times but not more than twice in any Graph Community. Also, most authors described above as defining a Graph Community are cited in many papers of other communities. This suggests that results might identify groups more clearly, if we defined connections through specific cited research papers rather than cited authors. However, our current corpus might not be sufficiently large for that kind of analysis. Such analysis would be recommended when using a data corpus of at least a few thousand articles.

There are some methodological issues that we are aware of. We realized that summaries of the affect group from the previous CERME proceedings were cited often, inflating the number of citations by their authors. A more fundamental question is, that we have no measure for the reliability of The Graph Community analysis. With the current data corpus, our first analysis included an error that excluded 11 of the 100 papers. This was enough to produce a significantly different result: A subgroup of Foundation papers (Pehkonen and his students) was identified as a separate Graph Community and Teacher Development was not identified as a Graph Community. This suggests that the results of the analysis are quite sensitive to changes in data.

The first author of this paper has published several synthesizing articles on research in mathematics related affect. Using the graph analysis was an attempt to overcome possible personal biases in perceiving the structure of the research area. Our method of connecting CERME publications by authors appearing in their lists of references seems to have worked. It confirmed research on motivation research as a specific research domain. The analysis also identified specific research traditions on self-efficacy and academic emotions. While earlier reviews (e.g. Hannula, 2011), identified beliefs and emotions as two areas within mathematics-related affect, the current analysis identified research on beliefs in three different groups: Foundation, Self-Efficacy, and Teacher Beliefs. Similarly, the current analysis identified Academic Emotions as separate group while most emotion papers were part of Foundation. These results suggest that in the Hannula (2011; 2012) model, the distinction between cognition (beliefs), motivation, and emotions is the most important one. On the other hand, one small Graph Community, Identity, can be considered to be characterized by its theoretical background being sociological. A possible new characterizing feature for research could be focus on the dynamics of change, exemplified by the research traditions Teacher Development and Resilience.

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Tracing mathematics-related belief change in teacher education programs

Çiğdem Haser[1] and Işıl İşler[2]

[1]Middle East Technical University, Turkey; chaser@metu.edu.tr

[2]Middle East Technical University, Turkey; iisler@metu.edu.tr

Understanding mathematics related belief change in teacher education programs has been a concern due to the information it presents about the effectiveness of program experiences. The present study investigated how preservice mathematics teachers' (PMTs) mathematics related beliefs changed through a teacher education program by implementing a belief scale to a cohort of PMTs 12 times during 6 consecutive semesters. Comparison of mean scores for different time periods showed that the cumulative effect of the teacher education program is more detectable than year-long or semester-long effects. While such implementation is likely to provide a perspective for monitoring belief change and possible effects of courses, it might also address that the contents of the courses should be connected to ensure the best cumulative effect of the programs.

Keywords: Mathematics related beliefs, preservice mathematics teachers, tracing beliefs.

Beliefs and teacher education programs

Training teachers with knowledge, skills, and disposition to practice the reform-oriented teaching has been a major goal for teacher education programs as teachers are the key to the success of the reforms aiming constructivist and student-centred mathematics instruction at schools (Handal & Herrington, 2003). Hence, initiating and strengthening mathematics related beliefs parallel to the reform movements have been a concern for teacher education programs (Raymond, 1997). This study aimed to identify possible key courses and experiences in teacher education programs that influence PMTs' beliefs by monitoring the changes in beliefs for three years in a four-year middle grades mathematics education program. The other aim is to gain perspective on the information that continuous monitoring of beliefs can provide. We adopted the identification of beliefs which Muis (2004) used for students' mathematics related beliefs and considered PMTs' beliefs as *availing* if they had the potential to help PMTs to achieve the reform-oriented goals of the teacher education program and middle school mathematics curriculum, and *nonavailing* otherwise.

Preservice teachers generally start teacher education programs by nonavailing mathematics-related beliefs (Szydlik, Szydlik, & Benson, 2003) for their learning in the program. Although programs aim to initiate and strengthen availing beliefs to reach program outcomes and teach through reform when PMTs become teachers, availing beliefs are initiated and developed to a limited extent (Clift & Brady, 2005). However, nonavailing beliefs are carried to the student teaching and teaching profession and they influence teachers' practices and implementation of reform principles in classrooms (Szydlik, et al., 2003). Therefore, teacher educators face the challenge of understanding PMTs' beliefs in the beginning of programs and organize tasks and experiences to develop and strengthen the availing beliefs (Swars, Smith, Smith, & Hart, 2009). This requires monitoring PMTs' beliefs throughout programs and the possible influence of program experiences.

Several studies investigated the effects of teacher education program experiences on PMTs' mathematics related beliefs mostly by emphasizing a single course or set of courses such as the methods courses and student teaching. These studies have shown that course experiences had limited effect in terms of the extent and the duration of the developed beliefs (Szydlik, et al., 2003). Longitudinal studies which documented changes in PMT beliefs through a set of courses are rare. Swars, et al. (2009) investigated 24 preservice elementary teachers' mathematics teaching and self-efficacy beliefs, mathematics anxiety and specialized content knowledge through 3 semesters of a 4-semester teacher preparation program. They implemented the belief instruments at the beginning and end of the 2nd semester (1st methods course), at the end of the 3rd (2nd methods course) and 4th (student teaching) semesters. The analysis showed that preservice teachers' scores significantly increased after the 1st methods course, decreased non-significantly at the end of the 2nd methods course, and significantly decreased at the end of student teaching. Yet, there was a significant increase from the 1st implementation to the 4th showing that preservice teachers gained more cognitively aligned beliefs (as targeted by the program) at the end of the 4 semesters.

The present study attempted to trace PMTs' beliefs through a teacher education program beginning from the semester they were enrolled in the initial pedagogical content knowledge courses (beginning of the 3rd semester) until they graduated (end of the 8th semester). PMT beliefs were monitored by implementing a belief instrument for 12 times in 6 consecutive semesters. In this paper, we report and discuss the changes based on timing of the implementation through the following research questions: (1) How do PMTs' mathematics related beliefs change through the 2nd, 3rd, and the 4th year of a 4-year mathematics teacher education program (a) in the long term (between academic years and throughout the three years); (b) in the short term (within the academic years)? (2) What kind of information does timing of belief instrument implementation offer about mathematics related belief change for teacher educators?

Method

A longitudinal survey design was employed and data were collected from the same group of PMTs for 6 consecutive semesters to capture the possible changes in their mathematics related beliefs.

Participants and context

Turkish education system is centralized at the national level with national curricula implemented at all grade levels at all contents. All students have to take a national examination at the end of high school to attend 4-year degree programs at universities including teacher education programs. The context of the study was a four-year middle grades (grades 6-8) mathematics teacher education program (EME) at a Turkish public university. EME program had three mathematics education faculty members at the time of the study and was placed under the Department of Elementary Education with 10 faculty members. The program had mathematics courses offered by the Department of Mathematics in the first four semesters. Mathematics education courses started in the 3rd semester and were offered by the program faculty and pedagogical courses were offered by the Department of Educational Sciences. The participants of the study were a total of 33 female and 10 male PMTs who started the EME program in 2006, referred here as the "cohort".

EME program started in 1998, was renewed in 2006 and the cohort was the first to study the renewed program. The changes in the EME programs were due to a major constructivist curriculum reform in

the national mathematics curriculum in Turkey in 2005. Previous EME program offered a mandatory minor degree in science education which was removed in the renewed program. School Experience course in the 2nd semester and Textbook Analysis course in the 8th semester were removed, and 1-semester Methods of Teaching Mathematics course was renewed as a 2-semester course, which allowed the dense content be covered in more time and depth. Methods of teaching course content was combined with curriculum issues content in a new course. Two new courses on research methods and nature of mathematical knowledge, and two statistics courses from the Department of Statistics were added to the renewed program. Mathematics courses, field experience courses in the 7th and 8th semesters and most pedagogical courses were maintained. The previous program experiences were based on constructivist approaches, however, the renewed program provided more opportunities for widened and deepened experiences for PMTs.

Two studies investigated the belief change in the previous EME program. Haser and Star (2009) conducted a cross-sectionally longitudinal study through interviews with 2nd, 3rd, and 4th year PMTs. Their findings revealed that PMTs' mathematics related beliefs did not change much throughout the program. However, methods of teaching mathematics course provided PMTs a different understanding of teaching and learning mathematics, which they did not experience in their pre-college education. Haser and Doğan (2012) investigated how mathematics related beliefs differed among PMTs in different year levels. They first surveyed a total of 100 PMTs who were at the beginning of the 2nd, 3rd and the 4th year. Their analysis showed that PMTs who just started the 4th year in the program had significantly higher belief scores. Then, they focused on the effect of the general methods of teaching course in the 3rd year of the program on PMTs' beliefs.

The major changes in the EME program and the opportunity to monitor the 2006 cohort from the semester they started to take courses from the Department enabled us trace the possible influence of the renewed EME program experiences on PMTs' mathematics related beliefs.

Data collection instrument

The belief scale used in this study was developed and used in the previous study (Haser & Doğan, 2012) in order to investigate Turkish PMTs' beliefs about the nature of mathematics and teaching and learning mathematics. Mathematics-related belief scale (MBS) included 38 five-point Likert type items asking PMTs' agreement with belief statements with responses ranging from totally disagree (1) to totally agree (5). Higher scores in MBS indicated existence of more availing mathematics related beliefs. Some of the MBS items are as follows: "Problem solving should be used as a teaching method within mathematics education", "The aim of mathematics education is to obtain correct answer by using the ways previously shown in the course" and "Visual and concrete materials are used in order to set up an environment for students to investigate their ideas". The Cronbach's alpha coefficient for MBS was calculated as .85 in the earlier study.

Data collection and analysis

MBS was implemented for 12 times at the beginning and the end of each semester in the 2nd, 3rd, and 4th year in one of the courses PMTs attended. However, the number of PMTs who took the MBS in each implementation varied due to the number of PMTs present at the implementation time. PMTs completed the MBS in about 15 minutes in each implementation.

Data were analysed to investigate both long-term and short-term changes in mean MBS scores, therefore, separate analyses were conducted. Long-term changes were investigated by comparing mean MBS scores of PMTs in the beginning of 2nd, 3rd, and 4th year, and at the end of 2nd, 3rd, and 4th year by one-way repeated measures ANOVA. Mean MBS scores in the beginning of 2nd year and at the end of 4th year were compared through paired-samples t-test. Short-term changes were investigated by comparing PMTs' mean MBS scores at the beginning and the end of each year and semester by paired-samples t-test. Cronbach's alpha coefficient was calculated for each implementation and ranged between .74 and .95.

Results

The results are presented for long-term and short-term changes. First, for long-term changes, beginning of 2nd, 3rd and 4th year scores were compared to see the belief change based on the 2nd and 3rd year experiences. Then, end of 2nd, 3rd and 4th year scores were compared to see the change after 3rd and 4th year experiences. When there is a significant change in the MBS scores, it is interpreted as the effect of the EME program. MBS scores at the beginning of 2nd year and end of 4th year were compared to see the cumulative effect of the 3 years in the program. "Cumulative effect" refers to the effect of all program experiences until the mentioned implementation. Short-term changes were explored by comparing beginning of year/semester scores to end of year/semester scores. The aim was to detect possible influence of course experiences on PMT beliefs. The comparisons helped us discuss the information that the timing of the implementation might provide.

Long-term changes

In order to identify possible change in PMTs' beliefs at the beginning of the academic years through the program a one-way repeated measures ANOVA was conducted to compare scores on MBS at the Time 1 (beginning of the 2nd year), Time 2 (beginning of the 3rd year) and Time 3 (beginning of the 4th year). A total of 19 PMTs were common at all Time 1, Time 2, and Time 3 of MBS implementation. The means and standard deviations are presented in Table 1.

Time (beginning of year)	N	M	SD
Time 1 (beginning of the 2nd year)	19	3.81	.237
Time 2 (beginning of the 3rd year)	19	4.00	.246
Time 3 (beginning of the 4th year)	19	4.08	.318

Table 1: Descriptive statistics for PMTs' MBS scores at the beginning of the 2nd, 3rd and 4th years

There was a significant effect for time, [Wilk's Lambda = .550, $F(2, 36) = 10.379$, $p < .05$, multivariate partial eta squared = .366]. Pairwise post-hoc comparisons with Bonferroni adjustment ($p < .05$) showed that there was a significant mean difference between Time 1 and Time 2, and Time 1 and Time 3. The difference between Time 1 (beginning of 2nd year) and Time 2 (beginning of 3rd year) indicated that 2nd year experiences in the EME program had a significant effect on PMTs' mathematics related beliefs. Similarly, the difference between Time 1 (beginning of 2nd year) and Time 3 (beginning of 4th year) indicated that a possible cumulative of 2nd and 3rd year experiences in the EME program had a significant impact on PMTs' mathematics related beliefs.

We wanted to explore if the effect of the program differed for academic years through the program when the program experiences were rather recent for the PMTs by conducting another one-way repeated measures ANOVA to compare mean scores on MBS at the end of the academic years as Time 4 (end of the 2nd year), Time 5 (end of the 3rd year) and Time 6 (end of the 4th year). A total of 21 PMTs were common at all Time 4, Time 5, and Time 6 implementations of MBS. The means and standard deviations are presented in Table 2.

Time (end of year)	N	M	SD
Time 4 (end of the 2nd year)	21	3.86	.197
Time 5 (end of the 3rd year)	21	4.03	.251
Time 6 (end of the 4th year)	21	4.01	.239

Table 2: Descriptive statistics for PMTs' MBS scores at the end of the 2nd, 3rd, and 4th years

There was a significant effect for time [Wilk's Lambda = .543, $F(2, 40) = 5.919$, $p < .05$, multivariate partial eta squared = .228]. Pairwise post-hoc comparisons with Bonferroni adjustment ($p < .05$) showed that there was a significant difference between Time 4 and Time 5, and Time 4 and Time 6. The difference between Time 4 (end of 2nd year) and Time 5 (end of 3rd year) indicated that 3rd year experiences in the EME program had a significant impact on PMTs' mathematics related beliefs. Similarly, the difference between Time 4 (end of 2nd year) and Time 6 (end of 4th year) indicated that a possible cumulative of 3rd and 4th year experiences in the EME program had a significant effect on PMTs' mathematics related beliefs. There was no significant difference between Time 5 (end of 3rd year) and Time 6 (end of 4th year), which might indicate that the 4th year experiences did not have a significant effect on PMTs' beliefs. Indeed, mean MBS scores were slightly lower at Time 6.

The effect of the teacher education courses through the three years of the program was investigated by comparing the MBS scores between Time 1 (beginning of 2nd year) and Time 6 (end of 4th year) by a paired-samples t-test. A total of 25 PMTs were administered the MBS at Time 1 and Time 6. There was a statistically significant increase in mean MBS scores from the beginning of 2nd year ($M = 3.76$, $SD = .174$) to the end of 4th year ($M = 4.05$, $SD = .235$), $t(24) = 5.868$, $p < .001$ (two-tailed). The eta squared statistics (.59) indicated a very large effect size. This showed that a possible cumulative of 2nd, 3rd, and 4th year experiences in the EME program had a significant effect on the mathematics-related beliefs of PMTs.

Short-term changes

Three paired-samples t-tests were conducted to investigate the possible effects of year-long experiences on PMTs' mean MBS scores by comparing the beginning-of-year and end-of-year scores for each year. Table 3 presents paired-samples t-test results and the number of PMTs who were common for in both implementations of MBS for each year.

Year	N	Paired-samples t-test results
2	27	No significant difference between the beginning (M = 3.82, SD = .256) and end of 2nd year (M = 3.88, SD = .222), $t(26) = 1.035$, $p > .05$ (two-tailed).
3	25	Statistically significant increase from the beginning (M = 3.93, SD = .230) to the end of 3rd year (M = 4.07, SD = .262), $t(24) = 2.755$, $p < .05$ (two-tailed). The eta squared statistic (.24) indicated a large effect size.
4	27	No significant difference between the beginning (M = 4.07, SD = .302) and end of 4th year (M = 4.05, SD = .240), $t(26) = -.381$, $p > .05$ (two-tailed).

Table 3: Paired-samples t-test results for MBS scores and the number of PMTs for each year

Comparisons of beginning-of-year and end-of-year mean MBS scores showed that only 3rd year experiences had a significant effect on PMTs' MBS scores. The scores did not significantly change from the beginning to the end of the 2nd and 4th year of the program. However, the mean MBS scores increased in each implementation until the beginning of the 4th year.

A series of paired-samples t-tests were conducted to investigate the possible effects of semester experiences by comparing the PMTs' mean MBS scores at the beginning and end of the semester for each semester. Table 4 presents paired-samples t-test results and the number of PMTs who were common in both implementations of MBS for each semester.

Semester	N	Paired-samples t-test results
3	29	Statistically significant increase from the beginning (M = 3.83, SD = .216) to the end of the semester (M = 3.95, SD = .246), $t(28) = 3.027$, $p < .01$ (two-tailed). The eta squared statistic (.25) indicated a large effect size.
4	26	No significant difference between the beginning (M = 3.86, SD = .233) and end of the semester (M = 3.92, SD = .211), $t(25) = -1.849$, $p > .05$ (two-tailed).
5	28	No significant difference between the beginning (M = 3.96, SD = .225) and end of the semester (M = 4.00, SD = .259), $t(27) = -.973$, $p > .05$ (two-tailed).
6	32	No significant difference between the beginning (M = 4.02, SD = .230) and end of the semester (M = 4.05, SD = .263), $t(31) = -1.041$, $p > .05$ (two-tailed).
7	32	No significant difference between the beginning (M = 4.07, SD = .284) and end of the semester (M = 4.05, SD = .271), $t(31) = -.747$, $p > .05$ (two-tailed).
8	26	No significant difference between the beginning (M = 4.06, SD = .253) and end of the semester (M = 4.03, SD = .244), $t(25) = .656$, $p > .05$ (two-tailed).

Table 4: Paired-samples t-test results for MBS scores and the number of PMTs for each semester

Results showed that semester-long changes in mean MBS scores were not significant except for the 3rd semester, while mean MBS scores generally increased at the beginning each semester.

Summary of the analyses and information given by the timing of implementation

Analyses showed that there was a general trend of increase from the beginning of the 2nd year to the end of the 4th year of the EME program with slight decrease in the 4th year. This increase was significant for long-term comparisons and showed that EME program courses and experiences seemed to help PMTs develop or strengthen availing beliefs through the years.

When the analyses focused on short-term differences, the increase in mean MBS scores was not statistically significant in most of the comparisons. The comparison of mean MBS scores for the beginning and end of each academic year revealed significant increase only for the 3rd year of the program. On the other hand, semester-long comparisons of mean MBS scores addressed significant results only for the 3rd semester. These results showed that belief change might not always be significantly detectable in shorter periods. PMTs might not fully internalize course experiences only in one semester. The significance of the 3rd semester comparisons might show us that the first course on mathematics education was effective on beliefs, probably because it included methods of teaching PMTs had never experienced. The significance of the 3rd year comparisons pointed the effect of the 2-semester Methods of Mathematics Teaching courses, whose effects on preservice teachers' beliefs have often been investigated in the literature. Indeed, studies conducted in the previous program showed that courses on methods of teaching and mathematics teaching have influenced PMTs' beliefs in a more availing way (Haser & Doğan, 2012; Haser & Star, 2009). Although the decrease in MBS scores from the beginning of the 4th year to the end was not significant, it might signal for the rather undesired effect of the student teaching experiences on soon-to-be-teacher PMTs' beliefs due to the reality of classroom environment, differences in students, and lack of support from program instructors at the classrooms (Swars et al., 2009). These findings showed that detecting belief change for shorter time periods provided rather limited knowledge, but it raised issues about the effects of the program experiences for semesters or years.

The analyses reported here were conducted based on the number of common PMTs in the analysed implementations. When we compared the mean MBS scores at each point of time to the mean scores in the repeated measures ANOVA (Table 1 and Table 2) as well as the paired-samples t-tests (Table 3 and Table 4), we observed minor mean score differences between the mean scores of the PMTs who were common across the implementations and all the PMTs who were administered the survey at that implementation. These results are not given here due to space limitation. Hence, we concluded that missing cases did not impact the results of the study. Yet, it should be kept in mind that the analyses were not conducted with all PMTs for all implementations.

In summary, the results of the analyses showed that change in PMTs' mathematics-related beliefs were more detectable when the change was investigated in the long-term, throughout the program. The nature of the increase in MBS mean scores suggested a cumulative effect of the program as PMTs progressed. The short-term investigations did not give much significant results, yet they might give us clue about how courses might influence PMTs' beliefs. The significant results have addressed the possible influences of certain courses that should be investigated in detail.

Discussion

The long-term and short-term change analyses results showed that while PMTs seemed to benefit from program experiences and develop more availing beliefs through the years in the program, 3rd

year experiences seemed to contribute to the belief change the most. Course experiences were not investigated in-depth in this study, therefore how PMTs made sense of these experiences and how these experiences helped them in forming rather availing beliefs were remained unexplored in this study.

The results suggest that teacher educators should investigate change in beliefs through the teacher education programs in different ways. The first teaching related course in the program, methods of mathematics teaching courses, and student teaching courses might have relatively more weight (either positive or negative) within the cumulative effect of the teacher education programs. Considering the significant cumulative effect of the program, it is possible that this cumulative effect might get stronger when the program experiences are meaningfully related to each other to support the availing beliefs and related practices.

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Teaching self-regulation strategies with SOLVE IT to two students with learning disabilities: Effects on mathematical problem-solving performance

Keffe Paraskevi[1], Dermitzaki Irini[2], Tzivinikou Sotiria[3] and Stathopoulou Charoula[4]

University of Thessaly, Volos, Greece; [1]paraskevikeffe@yahoo.gr, [2]idermitzaki@uth.gr,

[3]sotitzivi@uth.gr, [4]hastath@uth.gr

The purpose of the present study was to investigate whether teaching self-regulation strategies via "Solve it" to students with learning disabilities could affect their problem-solving performance in mathematics. The mathematical problems involved four mathematical operations with natural and decimal numbers. Also, the present study investigated the effect of "Solve it" instruction on students' self-efficacy and value related to mathematics. It was a single-subject design with a pre-test, four repeated post-tests and a maintenance test. The results indicated that the students' problem-solving performance was improved and their self-efficacy and value attributed to mathematics were increased.

Keywords: Self-regulation strategies, mathematical problem solving, LD, self-efficacy, value.

Introduction

Learning how to solve mathematical problems plays the most important role in the promotion of mathematical thinking. Mathematical problem solving process is especially complex as it requires the use of cognitive and metacognitive strategies as well as emotional management in case of a failure (Freeman-Green, O'Brien, Wood & Hitt, 2015; Rosenzweig, Krawec & Montague, 2011). Some researchers argue that many students with LDs face difficulties in solving mathematical problems due to their deficits in metacognitive processes, such as prediction and evaluation as well as difficulties in using metacognitive strategies in order to monitor and control their learning (Babakhani, 2011; Rosenzweig et al., 2011). A recent learning approach that combines the selection and use of cognitive and metacognitive strategies, motivation for learning and successful control of emotions is called *self-regulated learning* (Wirth & Leutner, 2008).

This paper is part of a larger study which was conducted for the requirements of a Master's Degree and explores whether teaching self-regulation strategies with the program "Solve it" can influence problem-solving performance of students with LDs. This program includes the use of seven cognitive strategies and three metacognitive strategies. In this paper, it was investigated if the use of seven cognitive strategies and three metacognitive strategies in combination with self-assessment which plays the role of motivation to students can improve problem-solving performance of two students with LDs in order to reach the mastery criterion of the program. In addition, it was explored if teaching problem solving process with these strategies can affect these students' self-efficacy and value attributed to mathematics. Also, the study tried to shed further light on the metacognitive and self-regulated learning processes and their interplay with motivation in students with LDs in mathematics.

Theorɛtical framework and research questions

It is accepted that learning how to solve mathematical problems plays the most important role in the promotion of mathematical thinking. According to van Garderen & Montague (2003, p. 246)

> mathematical problems are challenging problems set in realistic contexts that require understanding, analysis, and interpretation. They are not simply computational tasks embedded in words; instead, they require appropriate selection of strategies and decisions that lead to logical solutions.

Over the last 20 years, a new approach called self-regulated learning has been developed aiming among others at improving problem solving skills. This approach has been successfully implemented in developing problem solving skills as it examines metacognitive, motivational and affective aspects of problem solving activity. A lot of researchers have tried to define the composite construct of self-regulated learning (Wirth & Leutner, 2008). Self-regulated learning is defined as

> a learner's competence to autonomously plan, execute and evaluate learning processes, which involves continuous decisions on cognitive, motivational, and behavioural aspects of the cyclic process of learning. (Wirth & Leutner, 2008, p. 103)

Research reveals that many students and especially students with LDs in both primary and secondary education face difficulties in solving mathematical word problems (Rosenzweig et al., 2011). Before proceeding to the description of these difficulties, a definition of the term "learning disabilities" should be provided. Under the Individuals with Disabilities Education Act of 2004 (IDEA), the federal law that protects students with disabilities, a specific learning disability is defined as

> a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, that may manifest itself in an imperfect ability to listen, think, speak, read, write, spell, or to do mathematical calculations, including conditions such as perceptual disabilities, brain injury, minimal brain dysfunction, dyslexia, and developmental aphasia. The term does not include learning problems that are primarily the result of visual, hearing, or motor disabilities, of mental retardation, of emotional disturbance, or of environmental, cultural, or economic disadvantage. (34 C.F.R.300.7[c][10])

These students have difficulties in using metacognitive strategies in order to monitor and control their learning (Babakhani, 2011; Rosenzweig et al., 2011).

International contemporary research has shown that teaching self-regulated learning strategies is associated with the improvement of problem solving performance of students with LDs (Babakhani, 2011; Freeman-Green et al., 2015). A self-regulation program that has been successfully implemented in interventions in order to improve problem solving performance of students with LDs is called "Solve it" (Montague, 1992). This program was introduced by Montague (1992) and it combines the use of three self-regulation strategies: self-instruction, self-questioning and self-monitoring with the following four major instructional techniques: problem solving assessment, explicit instruction of problem solving strategies, process modeling and performance feedback. The program includes the following seven cognitive strategies which correspond to seven instruction phases (Read, Paraphrase, Visualize, Hypothesize, Estimate, Compute and Check). Each of these strategies, **including** three self-regulation strategies: self-instruction, self-questioning and self-monitoring are taught. These strategies rely heavily on metacognitive processes. "Self-instruction implies telling oneself what to do before and while performing actions" (Montague, Warger & Morgan, 2000, p.111). "Self-questioning means asking oneself questions while engaged in an activity to stay on task, regulate performance and verify accuracy" (Montague et al., 2000, p.111). "Self-monitoring requires the

problem-solver to make certain that everything is done correctly throughout the problem-solving process" (Montague et al., 2000, p.111).

The "Solve it" program includes seven instruction phases and it is separated into eight lessons (Montague et al., 2000). Lesson one includes an overview of "Solve it" and a description of the cognitive strategies. In lesson two students are tested for the mastery of seven cognitive strategies. Lessons three, four and five include metacognitive strategy instruction and students solve one mathematical problem in each lesson. For example, for the cognitive strategy "Reading" there were three self-regulation strategies that had to be implemented (SAY, ASK, CHECK). The students said to themselves "Read the problem. If I don't understand, read it again.", asked themselves "Have I read and understood the problem?" and checked by saying "Check for understanding as I solve the problem". The criteria for moving to lesson six are three: remembering cognitive strategies, remembering metacognitive strategies (SAY, ASK, CHECK) and solving problems with relevant confidence. In lesson six, students solve ten mathematical problems and they can consult the diagram with the strategies which had been given to them in lesson one and think aloud. Each problem-solving process is modeled by the students or by the teacher after it has been solved. Lesson seven requires the students to solve all 10 problems before modeling the correct solutions for the problems. Lesson eight is the first Progress Check (test of ten problems). Students plot their "grade" on their performance graph and then model the solutions. From then on there will be more tests and students will plot their performance. Student progress graphs show whether students can make constant progress and move toward mastery. It is important to engage students in assessing their own progress by having them chart their performance in diagrams which motivate them to continue trying (motivation) (Montague et al., 2000). The mastery criterion of the program, which is the ultimate goal, is solving 7 out of 10 problems correct on four consecutive tests (Montague et al., 2000).

It should be noted that this program is more frequently used in secondary education (students with and without LDs) with great success (Montague, 1992; Montague, Krawec, Enders & Dietz, 2014) for solving one-, two- and three-step problems with natural and decimal numbers but as Montague (1992) states, this program can be used with younger students provided that adaptations should be made in processes and materials. In the studies where the program was implemented with younger students, they did not manage the mastery criterion as there were no adaptations. As the participants of this study were sixth grade students of an elementary school, some adaptations regarding "Solve it" were required in order to manage the mastery criterion of "Solve it". In addition, acronyms were used for the description of the strategies in order to be remembered by the students. The acronyms came from the first letter of each strategy in Greek language. Furthermore, it should be noted that there is no clear exploration of the effects of teaching self-regulation strategies via "Solve it" to LD students' self-efficacy sense and value attributed to mathematics so this is the novelty of this study.

Consequently, the purpose of this study was to investigate whether teaching self-regulation strategies with "Solve it" could affect students' with LD mathematical problem solving performance, their mathematics self-efficacy and value. Therefore, the following 4 research questions were stated as follows: 1) Will sixth grade students with LDs improve their mathematical problem solving performance in problems with four mathematical operations with natural and decimal numbers after the implementation of "Solve it"? 2) Will students' self-efficacy related to mathematics and problem solving activity change after the implementation of the intervention? 3) Will students' value attributed

to mathematics and problem solving activity change after the implementation of the intervention? 4) Will students with LDs maintain their improved performance one month after the intervention with "Solve it"?

Method

The present study was a single-subject design as two students with LDs participated in the study. In addition, an experimental design with one experimental group (two students with LDs) was implemented. A pre-test and four repeated post-tests took place. One month after the last post-test, a maintenance test was implemented. In this experimental design, the independent variable was the intervention with the program "Solve it" and the dependent variables were the following three: mathematical problem solving performance, self-efficacy in relation to mathematics and the value which was attributed to this school subject.

Participants

Two students (a male and a female) with LDs took part in the present study. The students were identified as having learning disabilities based on psychoeducational evaluations from an outside state agency. Specifically, the boy encountered specific learning disabilities of dyslexic type and speech problems and the girl learning disabilities in reading, writing and mathematics. Both students were studying in the 6th grade of an elementary school, in North-West Greece and they had difficulties in mathematical calculations and mathematical problem-solving. Moreover, they attended the subjects of Mathematics and Greek Language in a general education classroom and they additionally received resource room support on these subjects from a special education teacher. Parental consent was given for both participating students.

The students' teacher (first researcher) taught the self-regulation strategies. The teacher implemented "Solve it", designed the tests with the mathematical problems, administered and collected the questionnaires. The teacher was 25 years old female and she had met the children six months before the beginning of the intervention. She had completed her practicum with these children in the context of earning Master's Degree so she had already been acquainted with the students and that was the reason why they were selected to be the sample of the study.

Procedure

The intervention of the present study began in November 2015 and finished in December 2015.The maintenance test was implemented on 15th January 2016. The boy attended 18 sessions and the girl 23 sessions that lasted 35-40 minutes. One week before the beginning of the intervention, the pre-test was implemented. The pre-test included 10 one-, two- and three-step word problems (Montague et al., 2000). Also, the two students responded to the 2 questionnaires assessing mathematics self-efficacy and value attributed to mathematics. Afterwards, "Solve it" intervention began and included 8 lessons. The 8th lesson was the first progress check (post-test) and three additional posttests followed. In the last post-test, students responded again to the two questionnaires on mathematics self-efficacy and value. Additionally, as it was mentioned previously, an adaptation took place. Specifically, for the better interpretation of the strategies, the strategies were visualized. Specifically, each of the seven cognitive strategies was displayed with words and small pictures that showed the steps of action implied by the strategy. For example, the strategy "Read" was presented verbally, in a diagram and with this icon. 

Data collection

The mathematical problem solving performance was measured with tests which were designed by the researcher by following the suggestions offered by the creator of "Solve it". Each test included 10 mathematical one-, two- and three-step word problems which were based on the mathematical problems that students had been taught in their classroom (e.g. two-step word problem: 'Nick wants to buy three car-miniatures. Each of them costs 3.6€. He has already collected 8€. How much money does he need in order to buy them?').

Despite the small number of participants, quantitative methods for the data collection regarding self-efficacy and value were used, as the time for the completion of the intervention was limited and the school principal could not give extra teaching hours for an interview. However, some verbal questions were done for clarifications of some of the students' answers in the questionnaires. The data concerning self-efficacy regarding mathematics learning were collected with the use of a questionnaire. The questionnaire was developed by Dermitzaki and Efklides (2002) and assessed students' reported self-efficacy in mathematics with 5 items (e.g. 'I believe I will have a better mathematical problem-solving performance this year'). Answers were given on a five-point scale from 1-'Not at all true for me' to 5-'Totally true for me'. Because of the students' difficulties in reading comprehension, the questions were being read by the researcher and students were asked to circle the answer that was true for them. After the completion of the questionnaires, the students were verbally asked some questions in order to clarify some of their answers ("mini interview" for clarifications). These answers were written down by the teacher-researcher at the same time.

The data regarding value which was attributed to mathematics and mathematical problem solving were also collected with the use of a questionnaire which was made by the researcher based on Ames' scale (1983). This scale assessed students' value beliefs about mathematics as a school subject. The questionnaire included 3 items (e.g. 'Learning how to solve mathematical problems is…..') and the answers were given on a five-point scale from 1-'Not at all important' to 5-'Highly important'. Each question was asked verbally by the researcher and the students had to circle the answer that was true for them. After the completion of the questionnaire, the students were asked to clarify some of their answers ("mini interview" for clarifications) which were written down by the teacher-researcher at the same time.

Data analysis

The quantitative data that were collected from the tests were not statistically analyzed because of the small data number. However, a diagrammatical representation with Microsoft Office Excel 2010 was made. The quantitative data that were collected from the two questionnaires and mini-interviews were qualitatively analyzed. Because of the small number of questionnaires, a statistical analysis could not take place. The careful data reading and the description of the data had as a result two categories deriving from each questionnaire. Two categories were developed based on the first questionnaire. The first category included self-efficacy regarding mathematics and the second included self-efficacy regarding a problem solving activity. Similarly, two categories were derived from the second questionnaire. The first category included value attributed to mathematics and the second category included value attributed to a problem solving activity.

Results

Regarding to the first research question, the progress graph showed that both students' mathematical problem solving performance improved significantly. Specifically, the boy increased his performance from 2.6/10 on pre-test to 9.65/10 on the first post-test and the girl increased her performance from 0.5/10 on pre-test to 7.89/10 on the first post test. Additionally, both students achieved the criterion of solving at least 7 out of 10 word problems correct on four consecutive word problem tests which is the ultimate goal of "Solve it" according to Montague et al. (2000).

Concerning the second research question, the results showed that both students increased their self-efficacy regarding mathematics and mathematical problem solving activity. The boy reported that he was feeling a little efficacious in solving mathematical problems and towards mathematics before the beginning of the intervention. However, he reported that he felt very efficacious about solving mathematical problems and confident towards mathematics after the end of the intervention. The girl reported that she felt a little efficacious about mathematics and very efficacious about solving mathematical problems before the intervention. When the researcher asked her while she was completing the questionnaire "Why do you think that you will be more efficacious in solving mathematical problems?", she answered "I will read more, I will attend carefully the lessons and I will learn how to solve mathematical problems. " After the intervention, she reported that she felt very efficacious about mathematics and solving mathematical problems.

Additionally, both students attributed important value to mathematics and to the problem solving activity after the intervention with "Solve it". The boy reported that both mathematics as a school subject and problem-solving as a mathematical activity were of little importance for his life before the intervention. After the intervention, he thought that mathematics was highly important and problem solving was very important for his life. The girl attributed very important value to mathematics but she thought that solving mathematical problems was not an important activity for her life before the intervention. When the teacher asked her while she was completing the questionnaire "Why mathematics is very important for you?", she answered "Because learning the multiplication table is very important for our lives". After the intervention, she thought that both mathematics and problem solving activity were highly important for her life.

It should be underlined that both students expressed that they had developed more positive emotions such as happiness, when they solved mathematical problems after the intervention. That happened because according to them, they felt safety with the use of the strategies as the last ones had proved to be very helpful in order to solve a mathematical problem.

Finally, regarding the fourth research question, both students maintained their improved performance on the maintenance test one month after the intervention. The score for the boy was 9.6/10 and for the girl 9.05/10. It seems that the girl not only maintained her performance but also improved it more in relation to the last post-test. This finding has not been found in other studies.

Discussion

This study aimed to investigate whether teaching self-regulation strategies via "Solve it" affected students' with LDs mathematics problem-solving performance, their maths self-efficacy and reported value of maths. The results of the present study are very encouraging. In agreement with other studies (Babakhani, 2011; Montague, 1992; Montague et al., 2014) both students' mathematical problem

solving performance was considerably improved. Also, they seemed to achieve the ultimate goal of "Solve it" (7 out of 10 problems correct on four consecutive tests). This was a surprisingly good result as there was not such a result in other studies which used "Solve it" with elementary school students. As Montague (1992) states, the sixth grade students have not easily reached the mastery criterion. However, in this study students appeared to maintain this performance on the maintenance test a short while after; maybe as the result of the visualization.

Furthermore, both students reported increased self-efficacy in relation to mathematics as a school subject and in relation to problem solving activity. Additionally, both students attributed higher value to mathematics as a school subject and to problem solving activity after the intervention. As Chatzistamatiou, Dermitzaki, Efklides & Leondari (2015) state, there is a positive relationship between the use of self-regulation strategies and self-efficacy and between the use of these strategies and value attributed to mathematics by typically developing students.

Although, the effect of teaching self-regulation strategies on students' emotions regarding problem-solving activity was not examined in this study, it is important to mention that both students reported they felt happier when they solved mathematical problems after the intervention with "Solve it". More particularly, the boy said "Now I do not feel so stressed when my teacher tells me to solve a mathematical problem and I feel happy when I do it, even if I cannot find the solution". A future research could examine in more depth whether and how self-regulation strategies could influence students' emotions during problem-solving activity.

In conclusion, this study showed that "Solve it" can improve problem-solving performance not only in older but also in younger students with LDs provided that some adaptations will take place. Furthermore, "Solve it" seemed to affect positively students' self-efficacy and value attributed to mathematics. However, there are some limitations such as the limited generalizability of the results (case study), the different characteristics of the two students, the short time in which the study was carried out and the absence of a control group. Future studies could use "Solve it" in other mathematical domains such as geometry which students with LDs find quite challenging and difficult. In addition, a future study could test how teaching self-regulation strategies would influence LD students' emotions in relation to mathematics. Such data would be actually illuminative for educational research and practice.

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To err is human. The management and emotional implications of teacher error

Elizabeth Lake

UCL Institute of Education and UEA, UK

Elizabeth.lake@ucl.ac.uk

Research into errors in mathematics classrooms is often centred on student error. Whilst investigating teacher emotional expressions, using data from experienced teachers and affective pathways, I encountered examples of teacher mathematical error occurring in association with expressed emotions. Similarly, I have observed instances of new teachers making errors and hence have begun exploring the implications of teacher error management. In this paper, after describing illustrative examples, I suggest a model based on a continuum of affectively driven strategies that are likely to be familiar to secondary mathematics teachers. Each affectively driven strategy has implications for teacher student relationships and for the learning of mathematics. I discuss some of these implications in terms of longer term affective impact on classroom climate and for students. I offer the model for further discussion in relation to developing growth mindsets.

Keywords: Error handling, emotions, classroom, expertise.

Introduction

There is no doubt that we all make errors and human instinct is to avoid errors as unpleasant experiences. Yet brain analysis research suggests error, however unpleasant, is essential for effective learning; for creating neural connections. Moser et al. (2011) through studying the neural mechanisms when making numerical mistakes shows that this activity fires synapses. There seem to be two synaptic responses. Firstly, when the brain experiences conflict, without awareness, there is an increase in electrical activity. Secondly, brain signals act to draw conscious attention to the error. Conflict (such as recognising an error), also triggers emotions which means making a mistake triggers an observable emotional response. The emotions may be exaggerated as error provides not only a trigger for emotions, but also requires the person experiencing the error to engage in some form of minimising or regulation. Such a positive view of errors is supported by Boaler (2015) and others who suggest we should go further than just avoiding error; that errors should be attended to as part of addressing misconceptions and hence are necessary for learning. In mathematics teaching errors can take many forms, often numerical. Boaler suggests creating a culture where students are comfortable with handling error is beneficial for learning mathematics. Such a culture depends not only on how a teacher addresses student error, but also on how they model addressing their own errors. There seem to be international and cultural differences that merit discussion in terms of how errors are seen and addressed by teachers, from commonly using error as a teaching tool through to avoiding errors and associated discussion as damaging to self-esteem (Santagata, 2005).

Researchers investigating error focus on teacher responses to student error, and a few explore how students respond to teacher error (e.g. Borasi, 1987, Heinze, 2005; Ingram, 2014b; Santagata, 2005; Steuer et al., 2013; Tainio & Laine, 2015). There is less research on the role of error or how teacher error management occurs in the classrooms of experienced teachers. If error triggers an emotional

response, this has implications for classroom relationships and for students who may not notice error. This is important in mathematics as there are perhaps, more opportunities for error.

The definition of 'error' used here, interchangeable with 'mistake', is that error is a mismatch. A mistake demarcates the distinction between norms (such as usual behaviours) and a deviation (Borasi, 1987), (the unexpected or different) thereby defining what is false and what is correct (Heinze, 2005). Using the classic constructivist definition of emotion as a response evoked by recognition of a disparity (Mandler, 1989), it seems error cannot occur without an emotional association. This association implies that patterns of individual emotions in mistake situations is indicative of classroom culture. For this reason, Steuer et al. (2013) asked students about their teacher's handling of mistakes. In addition to identifying 'mistakes friendly' environments, they found that perceived mistakes friendly environment resulted in increased effort. Yet as recently as 2014, research suggests that UK teachers still predominantly give the message that errors are to be avoided, often through a variety of teaching strategies that rarely indicate an incorrect solution. "These strategies all give the interactional message that errors are to be avoided, or that errors are undesirable even when the teacher does not explicitly say this, or in fact explicitly states the opposite" (Ingram et al., 2014, p.40). In terms of beliefs, a teacher's stance on error may be revealed by their response to incorrect answers, but more indicative is their response to public revealing of their own errors. Such responses model expected emotional responses to error for that class. Tainio and Laine (2015) consider this in relation to emotional contagion where "Emotional contagion means that in interactions, emotions are usually shared by participants after one participant has offered public forms of emotion for others to attend to" (p.84). Further, students will mimic or synchronise (Hatfield, 1994) with their teacher's publically expressed affective pathway. According to Goldin (2000), people experience a series of emotions as they pass through the process of problem solving in mathematics. The result is an error climate. Steuer et al. (2013) see what I refer to as affective pathways as a predominantly positive (adaptive reaction) or negative (maladaptive) patterning which, if public, means displaying certain emotions enhances learning.

> Primarily, an adaptive reaction pattern is distinguished from a maladaptive pattern: An adaptive reaction pattern following errors and failure maintains learning motivation and functional affects such as joy; a maladaptive pattern decreases learning motivation and increases feelings of shame and hopelessness. (Steuer et al., 2013, p.197)

Examples of transcript and identified affective pathways (Table 1 & 2) from the classrooms of experienced teachers (> 10 years) illustrate encountering and managing of error. I examine the data from the lens of *affective pathways* (Goldin, 2000) as a potentially useful model to examine the illustrative examples. A pathway structures the interpreted emotional journey by labelling emotions from identifying a problem through to either resolution or abandonment. Examining teacher modelling of how to deal with error in conjunction with how a teacher emotionally manages error may assist in interpreting the affective impression given to students. Mandler (1989) suggests that using and modelling emotional responses to error inculcates a tolerance for error that benefits learning mathematics. How a teacher frames the handling and recovery from error can shape student experiences, and indicated preferred attitudes to error management (Santagata, 2005) especially if we consider error as a stimulus to action (Borasi, 1987).

It seems we need errors to learn, whilst teachers can support learning and affect classroom environments by modelling positive responses to error. The question addressed here is whether developing a model of error management examining how teachers model error is useful. To address this question, the data presented below is from a larger study on teacher expressions of emotion in the classroom. Both extracts are drawn from episodes of emotional expression, deemed as such through observation, measurement of galvanic skin response (GSR), (used to roughly indicate internal emotions) and confirmed in post observation discussions. They represent how experienced teachers might manage error. Other strategies, such as used by novice teachers, are drawn from the literature, collaborated by my own experience of working with teachers.

Modelling dealing with mathematical error (Adam and Bertha)

In these examples, both Adam and Bertha successfully address a small numerical error, one type of mathematical error. Both express emotions (the dominant emotion is determined by the observer, using standard emotion classifications (Scherer, 2005)), yet differ in pathway from recognising discrepancy to resolution, showing the teachers' disparate ways of modelling error management.

In the first example (Table 1), Adam accidently writes 2 for the difference between 4.5 and 3.5 when demonstrating upper and lower bounds for 5 – 4. As he reads the four possible answers aloud, '2,0,2,1', he slows, quietens his voice and movements, pauses with pen poised, whilst his head moves from side to side scanning. He then steps back and looks, appearing absorbed. Once he has identified the error, he utters the sound 'uhh', interpreted as 'never mind'. He then engages in an exchange of silly noises with a student, corrects the error, rewards a student who is quick to align, and continues in a faster pace, as at the start of the episode, quickly moving on from the error.

T	You can either do for 5..., 4.5 or 5.5, they're your two options, because that's the lower and the upper bound. So what I have written out on the board is all four different combinations of what can happen.	Confident
	Ok… [pause]	Uneasy
	I'm going to work out all of them. So I am going to get 2, 0, 2, 1. [GSR PEAK]	ERROR
S1	Blast-off…	
S2	It should be 1.5	
	Right, think about this. [pauses]	Confused
	The question was saying upper bound, the upper bound for 5 minus 4. That number is going to be as big as possible. [emphasis and shift in pace into regular time beat 1- and-2- and-3- and-4-] as | big | as | Pos-si-ble	Re-establishing certainty Interested
	Hm. [Pause with pen ready, his head moves as scans writing and then steps back] Looks like 2. [Pause] … Which 2 is it going to be? Well here... [pauses again, this is in a quieter voice]	Doubt Uncertainty Thoughtful
	Oh, that's right, I've done that wrong, that should be 1, uhh, [faster pace resumed]	Satisfaction
	Who picked up on that? [HANDS UP] Chris [S3]? Gold star[smiling]	Pleased
S2	How about me? Oh. Uh	
	Uh	Humorous
S2	Uh I said one though sir	
	Sorry, that's a 1. The biggest number is 2. How did I get 2? I took the biggest number possible here for 5, but the smallest number possible for 4. That made the difference-as-big-as-possible. 2. [firmly stated] Upper bound.	Confident

Table 1: Affective Pathway from Adam's lesson, where he makes an error

Bertha uses a well-known mathematics website to produce questions on the area of a circle (Table 2). Following the students finding the area for a given radius, Bertha enters a volunteered answer, but the website rejects this answer. As Bertha has already calculated the answer herself, agreeing with the student, the website rejection brings an unforeseen problem; she thinks she has made an error but has not. The discrepancy is between using π or 3.14, so it relates to the degree of accuracy.

S1	254.46 [student is providing answer to question on the board]	
T	.46	Neutral
S2	I've got .34 [Different answer which legitimises other students who also have different answer and they start calling out as well]	
T	Ok. Whoa, whoa, whoa, whoa, whoa.	Uneasy
	Does anybody disagree with the 254 bit?	Neutral
Many	No, yes, [some hands up]	
T	NO? Right. Can anybody think of a reason...	Neutral
	...Oh, I don't think if we can... yeah, we have. [GSR PEAK]	Uncertain
	Can anybody think of a reason why you might have different, very slightly different answers? [Terry among others raises his hand] Terry...	Confident
Terry	Is it because like one of us...um...were like... we weren't... um.... I don't know if it's right or....	
T	[Frowning]Well just, Terry, just say it, have more confidence in yourself sweetheart...	Hopeful
Terry	...some people pressed the... the pi button and some people didn't.	
T	Absolutely brilliant, well done you.	Pleased
	That's exactly right.	Satisfied
	When you press the pi button on your calculator, it uses a really accurate version of Pi. If you just put in um... 3.14, [writing something on wall behind teacher desk] then that's not so accurate...	Neutral
	...and that's the only reason.	Satisfied
	But anyway, it's coming up and telling us we're wrong.	Uneasy
	So, it says use the area of the square, we've got the area of the square as 9x9, and then multiply it by 3.14... [Does this on a calculator]	Neutral
S2	It's 254.34	
S4	Error	
T	... 3.14 .... (yeah) 254.34	Satisfied
Many	Yes!	
S4	Error	
T	...2...5...4....point... oh I see, point 3 4. Let's try it again [enters answer which is shown on the projector] Yeah!	Confused to satisfied
ALL	Yeah!	
T	OK	Satisfaction

Table 2: Affective Pathway from Bertha's lesson, where she makes an error

A proposed model of teacher error

Drawing attention to teacher error informs discussions about how such error should and could be managed, as this is not the same as when a student makes an error. Firstly, that there are emotional implications of the choices made by a teacher, including the degree of emotional labour required. Secondly, that the choices are indicative of how a teacher perceives error, both for themselves and for their students. And thirdly, that repeated over time, modelling by the teacher of error management sets the climate for students in terms of their own error management. If it is desirable to challenge behaviourist views of learning mathematics (Boaler, 2015), then discussion of error management provides an accessible route to address teacher beliefs. To support such discussions, the following continuum of error management strategies is proposed. This is derived from both the literature on error and the above data. The continuum moves from negative (associated with a mistake unfriendly environment and maladaptive error patterns) through to positive, and similarly from strategies with perceived reduced benefits, through to those which may have longer term positive impact for learning (mistakes friendly environment and adaptive patterns).

To compare and contrast with the examples above, I draw from research on and observations of trainee teachers. In this cohort, I observed embarrassment at making an error, ignoring the error, even when noticed, and errors on the whiteboard left uncorrected. I also observed surreptitious correcting when the students were engaged in another task. There is ample anecdotal and observable evidence that teachers hide numerical errors. They may later notice and leave it, or notice and amend privately. The communicated message is that errors are dispreferred (Ingram et al., 2014), and should be hidden. Similarly, public error can evoke teacher embarrassment or be correctly rapidly, where the teacher 'steps out of' rather than 'stepping into' learning through addressing the error. 'Stepping out' attaches a negative emotional association to error making. A teacher may respond to revealed error by faking, as in "I meant to do that" and publically amend. Although positive in avoiding misunderstanding resulting from the error, this choice can be negative emotionally, in that frequent repetition erodes trust in the teacher. A further response is to associate self with the error through self-depreciation. A teacher might say something like, 'What a silly error!' The effect may be that students view all error as silly, and to be laughed at. This may be positive or negative as context specific, but students may reduce contributions if they feel they may be laughed at.

There are suggestions in the literature on error that using deliberate error as a teaching strategy can be effective. This assumes a constructivist view, that the place of error is a learning opportunity, rather than a culturally located behaviourist view, that errors should be avoided (Santagata, 2005). In constructivist terms, viewing error as a learning opportunity is an ideal, yet how a mathematics teacher might cultivate such an ethos needs further research. Research from Ingram et al. (2014) found examples aimed at this ethos, but that in most cases the errors are still managed as something to be avoided. They identify a role for further strategies when positioning error in this way. They suggest either distancing self from the error since the purpose is for students not to make the same errors themselves, or apologising for an error. A more positive management might dismiss the error, but go on to associate with an emotionally positive outcome, such as thanking the student for pointing out the error. The emotional message is that criticism will not follow from error identification. However, taking the error handling strategy further, the identification may be made into an event through provoking a discussion, or may be acted out. There are many possible options, such as

expecting error due to rapid engagement in doing mathematics or that error is just to be expected. The event could show that corrections are fine, that corrections are neither good or bad, and are just a learning opportunity. The attribution of value to the error communicates a positive message.

In the examples from Adam and Bertha there is a mixture of the above that tend to the emotionally positive end of the continuum. What was not apparent either in the examples of data collection was a view that 'I never make mistakes'. Although both examples are public, Adam notices his own error as part of the next step, whilst Bertha's attention is claimed by rejection of her answer by an online website. What may be significant is the emotion work in both cases to turn the event to a positive learning experience. As for similar examples from Tainio & Laine (2015), the teachers take affective stances that display affiliation and humour (p.73). They also evoke emotionally positive responses for the students. Both show happiness in resolution of error, and give praise albeit located differently.

Bertha tells in a later interview of an occasion when a formal observation that went wrong because the questions on the website changed. A repeat experience, when again being observed, albeit for research purposes, is likely to re-evoke emotions associated with insecurity, and a need to check solutions with a form of authority, in this case the website answer. Modelling a need for accuracy, for rechecking, reinforces a product orientated 'feel' in the example, where correctness takes precedence. There is also modelling of internal thought processes. In Bertha's case public thinking out-loud for self, "[muttering quietly] ...2...5...4...point...oh. I see, point 34. Let's try it again." Possibly this indicates Bertha seeking mathematical correctness. This corresponds with what she says in interview,

> I don't see myself as a mathematician. I see myself as someone who is good at maths and you can teach me anything in maths and eventually get it, which does again sounds obnoxious but that's...you know... it might take me a lot longer with some of the things. (Bertha)

Seeking confirmation of correctness perhaps indicates a belief that 'real' mathematicians do not make errors. This example shows an intention to model what she thinks a teacher should be doing, indicating a disparity between real and aim more commensurate with trainee teachers. In contrast, Adam models shifting ownership of error by repositioning from 'we' to 'I', modelling that gives an impression that it is ok to err. He positively manages the error, moving sequentially from a point of uncertainty into exhibiting positive emotions using humour via exchanging noises and giving praise, which acts to restore lesson balance. His modelling of how to deal with error includes distraction of attention and shifting attention via assigning a social reward for correction of error. In the episode, the error becomes an object unassigned, before it is quickly shifted into a positive outcome. As an observer, it felt as if he was pinning the error somewhere distant from self. However, he included students in his happiness at resolving the error, and hence was rewarded. Either interpretation is a modelling that downplays error. As he says afterwards,

> Oh, yeah, did I put a mistake on the board to start off with? (Yeah) Yeah...I'm not fussed with that. It happens quite a lot. I always say to the students... I'll make mistakes, and they'll make mistakes...and there it goes...(Adam)

Both teachers seem to experience cognitive conflict, observable in the lesson as uncertainty, and resolve the error for themselves. They both resolve positively for their students, stepping into the error again as part of positive modelling. The data confirms that it is not the error itself but how it is

managed that has implications. Public teacher error has more impact, whereas perhaps students are expected to make errors. Underlying the difference between the examples and trainee teachers lie the issues of confidence and risk with the subsequent implications to learning climate of handling error after and during public exposure. We assume that expert teachers have confidence, but the examples show that the use of emotions is part of the restorative process, perhaps as a distraction. This seems to warrant further investigation, as does the disparity between the expert (who makes no mistakes) and Adam's declared and enacted position on error. Mandler (1989) suggests emotions activate other mental contents to deal with situations perceived as being a mismatch between what is intended and what occurs.

Conclusion and implications

In the above I have used two short episodes to illustrate teacher error management, since the research suggests that modelling of error responses plays an important role in construction of positive emotional climates in a mathematics classroom. Based on this initially small sample, the use of affective pathways seems to support exploration of the adaptive/maladaptive reaction patterns. There may be potential in exploring adaptive patterns with teachers as part of professional development. It may also be useful, in conjunction with ideas such as positive mindsets (Boaler, 2014), to consider how students might participate in error management to a greater degree, for example as springboards as suggested by Borasi (1987). The different responses of the teachers, although both successfully resolved in that student reward is given as restorative praise in both cases, have different longer term impressions. Adam says, 'gold star' for a student and quickly moves on, whilst Bertha draws attention to the correct answer, and to rewarding the student. The impact on students of repeated modelling of 'not my error, let's move on' (process interruption only) compared to 'we must get this right' (product orientation) may be significant. Shifting attention acts to distance the teacher from the ownership of error, modelling addressing error as positive. This distancing compares to a negative impact that models dealing with errors as an annoying problem, one owned by both teacher and students. There is inevitably a degree of uncertainty in relation to error management. These examples represent extremes of a management continuum from valuing error as a learning experience (modelling an expectation of error into learning), to a belief that errors are obstacles to avoid. Indeed, an emotional risk to a teacher may be in not using positive emotions to manage error.

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Students' images of mathematics: The role of parents' occupation

Ciara Lane

National Centre for STEM Education, University of Limerick, Ireland; Ciara.Lane@ul.ie

A questionnaire survey was conducted as part of a study investigating post-primary students' images of mathematics in Ireland. A definition of 'image of mathematics' was adopted from Lim (1999) and Wilson (2011). Students' images of mathematics were proposed to include attitudes, beliefs, motivation, self-concept, emotions and past experiences regarding mathematics. This paper focuses on one aspect of the study; the relationship between students' images of mathematics and parents' occupation. Some emergent findings regarding this relationship are presented and discussed.

Keywords: Image of mathematics, affect, parents' occupation.

Introduction

Mathematics education researchers have come to realise in recent decades the significance of mathematics-related affect, with consequential effects on mathematical engagement and performance (Hannula, 2016; Hannula, Ryans, Philippou & Zan, 2004; Lane, O'Donoghue & Stynes, 2014, 2016; Lim, 1999; McLeod, 1994, OECD, 2016). Affect in mathematics education has also been seen to be influenced by various factors including, but not limited to, gender, teachers, parents, peers, society and prior achievement (Frenzel, Goetz, Pekrun & Watt 2010; Hannula et al, 2004; Lane et al, 2014, 2016; Lim, 1999; Morgan, Thornton & McCrory, 2016, OECD, 2016).

There are various constructs investigated by mathematics education researchers in the field of affect. In this study, we focus on the construct 'image of mathematics'.

Although there is no universal definition of 'image of mathematics', there appears to be a general consensus that the term comprises several affective constructs such as: attitudes, beliefs, emotions, self-concept and past experiences regarding mathematics (Brown, 1995; Ernest, 2004; Lane et al., 2014; Lim, 1999). This paper derives from the author's PhD study that examined the image of mathematics held by post-primary students in Ireland (Lane et al., 2014; 2016). No previous research in Ireland had extensively examined second-level students' mathematics-related affect, although studies such as the Programme for International Student Assessment (PISA) have reported on particular aspects such as students' attitudes and confidence (Perkins, Shiel & Merriman, 2013). Lane et al. (2016) found that, similarly to the international context, statistically significant differences occurred for Irish students' images of mathematics with regards to gender, prior achievement and past experiences. Students' image formation was also reported to be influenced by their teachers, parents and peers (Lane et al., 2014). In this paper, one particular aspect of parental influence is examined, with the aim to establishing whether there exists a relationship between parents' occupation and students' images of mathematics. In particular, the author highlights parents' occupations with a strong mathematics base as a distinct occupation category, hypothesizing that these parents would hold positive views of mathematics e.g. in terms of value, and this paper aims to examine whether this positivity would appear to manifest itself in their children.

Theoretical framework

In her study on the public image of mathematics in the UK, Lim (1999) examined the influences on a person's image of mathematics. She found that images were influenced by four external factors, namely: teachers, parents, peers and society (listed in descending order of reported influence). While the relationship between students' mathematics-related affect and their teachers/the way in which they are taught, as well as the importance of peers and peer-learning in relation to affective issues, is evident in the literature (Dweck, 1986; Frenzel et al., 2010; Hill, 2008; OECD, 2016; Pantziara, 2016), the relationship between students' mathematics-related affect and parents/family is less visible.

Parents' occupation

The role of parents as an influence on students' mathematics-related affect and achievement has received some attention in the literature (Fennema & Sherman, 1976; Frenzel et al., 2010; Lane et al., 2014; Lim, 1999; OECD, 2014). Parents' influence can be explained by three underlying mechanisms according to Bosco & Bianco (2005), these being: socialization, modelling and resources. As part of these mechanisms, it is suggested that parents' values, attitudes etc. can be passed to their children, with obvious connotations with regards to students' images of mathematics. Lim (1999) and Lane et al. (2014) found parents to be the second most common influence in forming an image of mathematics. This influence occurs chiefly in the form of support and encouragement, but also indirectly from parents' own images of mathematics. In Frenzel et al. (2010), students' 'interest' in mathematics was found to be higher when his/her parents expressed higher levels of mathematics values. Similarly, ASPIRES (2013) found that a key factor affecting young people's science-career aspirations was the amount of 'Science Capital' a family has. Which includes science-related qualifications, understanding, knowledge, interest and social contacts.

With regards to the role of parents' occupations, this aspect has been found to indirectly impact on children's occupational choices, through their interests and skills (Lawson et al., 2015). The influence of parents' occupation on students' mathematical performance was examined in PISA 2012. Results indicated that across most countries, children whose parents worked as *professionals* (in health, teaching, science, business or administration) had the best results in mathematics (OECD, 2014) indicating a relationship between students' socio-economic background and their mathematics achievement. Furthermore, PISA results indicate that students from disadvantaged backgrounds tend to have a more negative mathematical self-concept than advantaged students, likely linked to lower mathematical achievement (OECD, 2016). Given the influence of parents in terms of students' mathematics-related affect (Frenzel et al., 2010; Lane et al., 2014; Lim, 1999) and also the relationship between affect and achievement in mathematics (OECD, 2016), parents' occupations may impact on student achievement not only in terms of students' socio-economic background, but also in terms of students' images of mathematics.

In this paper, we adapt the definitions of Lim (1999) and Wilson (2011) for her study, with 'image of mathematics' conceptualized as follows: a mental representation or view of mathematics, presumably constructed as a result of past experiences, mediated through school, parents, peers or society. This term is also understood broadly to include three domains:

- The affective domain dealing with attitudes, emotions, and self-concept regarding mathematics and mathematics learning experiences.
- The cognitive domain dealing with beliefs regarding mathematics and mathematics learning experiences.
- The conative domain dealing with motivation regarding mathematics and mathematics learning.

The theoretical framework for the author's study is outlined in more detail in Lane et al. (2014).

Methodology

A mixed-methodology was employed to investigate the image of mathematics held by 5th-year ordinary level mathematics students in second level education in Ireland. The main method used to examine students' images of mathematics was a questionnaire survey. The questionnaire contained both quantitative fixed-response items and qualitative open-ended questions. The quantitative aspect incorporated eight pre-established Likert scales, with a total of 84 items, to examine students' attitudes, beliefs, emotions, self-concept and motivation regarding mathematics – see Table 1. As no single scale existed to measure image of mathematics, the scales were selected that most closely resembled the elements comprising our 'image of mathematics' construct and also that fit with the other scales concisely in terms of length and layout. The five open-ended questions sought to gain further insight into students' images in terms of their influences, prior experience, use of mathematics in everyday life and their causal attributions for success/failure in mathematics. However, this paper aims to address only one aspect of the study with the following research question:

Is there a relationship between image of mathematics and parents' occupation for 5th year, ordinary level mathematics students in Ireland?

In order to address this question, we focus on the quantitative data, with students' scores on the eight Likert-type scales examined with respect to parents' occupation.

Author	**Scale**	**Image of Mathematics Element**
Aiken (1974)	Enjoyment of Mathematics	Attitude
Aiken (1974)	Value of Mathematics	Attitude
Fennema & Sherman (1976)	Attitude Toward Success in Mathematics	Attitude
Fennema & Sherman (1976)	Effectance Motivation in Mathematics	Motivation
Fennema & Sherman (1976)	Anxiety about Mathematics	Emotions
Fennema & Sherman (1976)	Mathematics as a Male Domain	Beliefs
Gourgey (1982)	Mathematical Self-Concept	Self-concept
Schoenfeld (1989)	Beliefs about Mathematics	Beliefs

Table 1: Image of Mathematics Scales

A random stratified sample of 60 schools was selected for this study, although only 23 of these agreed to participate. A total of 356 students completed the questionnaire survey. The students were aged

between 15 and 18 years and were all studying ordinary level (intermediate level) mathematics for the Leaving Certificate (end of second level state examination). The author decided to focus on ordinary level students as it was hypothesized that students in this cohort would provide a wider range of images. In addition, a majority of students (72.14% at the time of the study) took the ordinary level mathematics examination for the Leaving Certificate.

Findings

In this section, findings are presented in relation to parents' occupation and the relationship with students' images of mathematics. The quantitative data were analyzed using Statistical Package for the Social Sciences (SPSS) (version 19). In acknowledgement of the debate among researchers as to whether parametric or non-parametric methods of analysis should be applied to Likert scales (Jamieson, 2004), both methods were employed with similar findings (see Lane et al., 2014). The internal reliability of the eight scales was examined using Cronbach's alpha, with six scales found to have values above 0.8. The Value scale scored above 0.7, still a good internal consistency but the Beliefs scale was found to have a very low Cronbach's alpha of 0.21, possibly due to the short length of the scale – six items. The 84 items in total – referred to here as the combined image of mathematics scale – had a very high internal consistency of 0.94. Correlation was carried out on the scales, with each scale correlated with all other scales and also with the combined scale. The relationships between the scales was also examined using partial correlation, controlling for the effect of each individual scale on the relationship between the other seven scales. A Principal Components Factor Analysis and Multiple Regression Analysis were also employed. The analyses indicated that the Attitude towards Success in Mathematics Scale and the Mathematics as a Male Domain Scale were not found to correlate highly with the other scales used in this study and so the author has decided to not address these scales here. The median scores on each of the remaining six scales are examined according to parents' occupation. A higher score on any of the scales indicates a more positive attitude, belief, emotion etc.

As parents' occupation was an optional item on the questionnaire, this item received the lowest response rate out of the entire questionnaire with just over half of students providing an answer (n=179). Reported parents' occupations were qualitatively reviewed and grouped into categories. Occupations were grouped similarly to Lim (1999) in her study on the public image of mathematics. Due to the fact that the author wished to acknowledge occupations involving a significant knowledge of mathematics, i.e., financial banking, accountancy, mathematics teacher etc. it was decided to include a sixth category relating to this. Initially, parents' occupation was explored with categories that differentiated between one or both parents' occupations being provided. However, as some of these categories contained very few students, groups were collapsed into the 6 categories of: Professional; Managerial and Technical; Skilled (both manual and non-manual); Unskilled and Partially skilled; Mathematics; and Others (unemployed, retired and unclassifiable occupations). Median scores for these categories are also compared with those for students who did not provide their parents' occupations (Not Given).

Enjoyment of mathematics

The re-grouped categories of the parent(s)' occupation variable were first examined with regards to students' enjoyment of mathematics. The highest possible score for this scale was 55. Students with the highest median score for enjoyment of mathematics (40.5) had parents whose occupation involved

mathematics. The lowest median for enjoyment of mathematics (27.0) was found to be students whose parents' occupations were categorized as 'other'. There was little difference between the medians for the other 5 categories.

Value of mathematics

Parents' occupation was examined in relation to students' value of mathematics (highest possible score being 50) but the range of medians for the Value of Mathematics Scale was quite small with the highest median (38.0) being for students with parents grouped as Managerial and Technical as well parents grouped as Mathematics, and the lowest median (33.0) being for students whose parents' occupations were classified as Other.

Motivation in mathematics

The highest median score for motivation in mathematics (39.5 out of a possible 60) was found for the Managerial and Technical category. This was closely followed by the Mathematics grouping with a median score of 39.0. The lowest median score on the Motivation scale (30.0) was recorded for students' whose parents' occupations fell within the Other classification.

Beliefs about mathematics

For the Beliefs scale, the range of median scores was small for the parents' occupation categories. The highest median score on the Beliefs about Mathematics Scale (20.0 out of a possible 30) was found for students whose parents' occupations were classified as either Managerial and Technical or Other. The lowest median score (18.0) was recorded for students' whose parents' occupations fell within the Skilled grouping.

Mathematical self-concept

It was found that the students with the highest median score for the Mathematical Self-concept scale (43.5 out of 60) were those with parents in the Mathematics category. The lowest median score for the Self-concept scale (33.0) was recorded for students whose parents were classified as Professional. This was the first instance of the Professional grouping scoring the lowest on a scale.

Anxiety about mathematics

Finally, the parents' occupation variable was examined with regards to the Anxiety about Mathematics Scale. Students with parents who work with mathematics had a much higher median score for the Anxiety scale (43.0) compared with all other groups (meaning these students reported the lowest anxiety levels). The lowest median score for the Anxiety about Mathematics Scale (34.0) was recorded for two groups of students, those with parents in the Skilled and Other classifications.

Discussion

The most significant finding with regards to the relationship between parents' occupation and students' image of mathematics may be with regards to students whose parents are involved in mathematics-related occupations. For most of the scales, and also for the combination of the scales, the Mathematics category of parents' occupation showed the highest median scores and, in some situations, was set apart considerably from the other categories. Students with parents in the Mathematics classification of occupations were found to report the highest enjoyment of mathematics, the highest value of mathematics, a high motivation regarding mathematics, the highest

mathematical self-concept and the lowest anxiety levels with regards to mathematics. Given the suggestion in the existing research that parents can influence students' mathematics-related affect and possibly, their achievement in mathematics (Frenzel et al., 2010; Lane, 2013; Lim, 1999), it is perhaps to be expected that parents employed in mathematics-related occupations will have children with a more positive image of mathematics. In particular, it is not surprising that students whose parents work with mathematics would report a positive value of mathematics, but it would not necessarily be anticipated that these students would report higher enjoyment, higher self-concept or lower anxiety. While the PISA findings (OECD, 2014) indicate a relationship between parents' occupation and achievement in mathematics, the occupation category linked with higher achievement in that study was not specific to mathematics, although it would include mathematics related work. Their findings relating to parents' occupation and achievement are likely due to the better educational opportunities afforded to students whose parents are classified as professionals (the resources aspect of parental influence seen in Bosco & Bianco, 2005), but for the students in this study, there was no significant difference between students' reported images with regards to the type of school attended (Lane et al, 2016). Therefore, it can be hypothesized that the differences come from the parents themselves, and not from the educational opportunities afforded by parents. This ties in with the socialization and modelling aspects of parental influence (Bosco & Bianco, 2005). The significance of this finding lies in the important role parents play in positively influencing their children's mathematics-related affect. Another possibility is that parents involved in mathematics-related careers are more able to provide help with mathematics homework. This extra home support may also positively influence these students in terms of their image of mathematics. Similar findings were observed in the ASPIRES study (2013) in relation to parental influence and engagement with science and science careers. Whatever the case, parents who work in mathematics related occupations would seem to have children with a more positive image of mathematics, and while the possibility of other factors should be acknowledged, the relationship between parental influence and student affect with regards to mathematics is clearly one that requires attention. That students whose parents were employed in mathematics based occupations reported the highest self-concept and lowest anxiety regarding mathematics is particularly of note, as Pantziara (2016) highlights the predictive role of students' self-efficacy in their current and future education and course selection. Thus, the potential benefits of positive parental influence in terms of mathematics-related affect are far reaching.

Conclusion

Findings from this study suggest a relationship between parents' occupation and students' self-reported images of mathematics exist. The positive image of mathematics found to exist among students whose parents are employed in mathematics-related careers, may offer an insight into the role of parents in the formation of a student's image of mathematics, particularly in terms of the socialization and modelling aspects of influence (Bosco & Bianco, 2005). Parents with positive attitudes, beliefs etc. about mathematics, may pass this positivity to their children, thus creating a cycle of positivity and engagement with mathematics. These parents are likely to be in a better position to provide additional support with mathematics work. Given the role played by parents in terms of their children's education and future occupations (ASPIRES, 2013; Lawson et al., 2015), the influence of parents in students' image of mathematics formation (Lim, 1999; Lane et al., 2014) and the link between affect and future course selection and achievement (OECD, 2016; Pantziara, 2016), it is essential to recognize the part that parents may play in influencing students' engagement with

mathematics education and mathematics related careers. Due to the small number of students in this study with parents involved in mathematics based careers, these findings cannot be taken as conclusive and indeed further research would be necessary to clarify the relationship between parents' occupation and students' images of mathematics, and also the role of parents in influencing students' mathematics-related affect and indeed their current and future engagement in mathematics.

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Emotions and problem solving by prospective primary school teachers

Gareth Lewis
University of Leicester, UK; gl116@le.ac.uk

This paper reports on a study into the motivations and emotions of prospective primary school teachers and how they change before, during and after a problem solving task. The results highlight the need to build the emotional intelligence or cognitive-affective competences, resources and strategies to overcome negative emotion and to scaffold learning.

Keywords: Motivation, emotion, problem solving.

Background

The link between emotion and the experience of learning mathematics has long been recognised (Buxton, 1981). In particular, negative experiences, leading to negative emotions, are seen to inhibit or disable the learning of mathematics. Buxton's study provided evidence of the damaging effect of lack of confidence and competence that motivation and emotion play in the learning of mathematics. Skemp (1977) also described the part played by a range of emotions in successful and unsuccessful strategies for learning mathematics. Yet until recently, much research on emotion and mathematics has focussed on anxiety, and particularly test anxiety (Hembree, 1990). Evans (2000, p. 108) claimed "there is still little or no explicit acknowledgement of the importance of the affective – feelings of anxiety, frustration, pleasure and/or satisfaction which attend the learning of mathematics." More recently, there have been attempts to classify emotions related to learning mathematics (Pekrun, Frenzel, Goetz, & Perry, 2007). This characterisation does not account fully for the wide range of emotions reported in other research (Schorr & Goldin, 2008). Schutz and DeCuir (2010) point out the tendency of research to characterise emotion as trait, and that this in turn tends to promote more reductionist interpretations. They note the methodological problems in the attempt to study emotions, since they happen 'in vivo', and in the moment. Although not specific to mathematics, Meyer and Turner's (2010) review of emotions in classroom motivation research argues for the key role of emotions in learning, using terms such as 'essential' and 'pivotal'.

In relation to emotion and disaffection with school mathematics, there have been few studies. One such is Skinner, Furrer, Marchand and Kindermann. They view disaffection as negative engagement, or, more specifically (2008, p. 767): "the occurrence of behaviours and emotions that reflect maladaptive emotional states." Amongst the emotions accounted for are boredom, anxiety, anger and shame. The framework takes account of these emotions, but offers no theoretical account of their genesis. Lewis (2016), using a Reversal Theory structure, has attempted to widen the range of emotions studied, and has reported on the existence of a range of negative emotions that inhibit or disable learning.

Emotions, then, have received less attention than other affective constructs. In particular, emotion-as-state has been under-researched. The importance of state, as opposed to trait, or to more stable, cognitively-mediated constructs such as attitudes and beliefs, has been pointed out by Hannula. He says (2012, p. 155): "There is a clear imbalance in favour of studies that focus on traits over studies that focus on states.". He goes on to say (ibidem, p. 155): "In particular, studies that focus on the dynamics of emotional or motivational states in classroom or other learning community are still rare."

More recently, researchers have begun to pay attention to this deficit. The Cerme community, at Cerme 9, 2015, saw a number of papers address issues of emotion, and focussed on emotion-as-state, in rather innovative ways. Liljedhal (2015) collected data from 38 prospective mathematics teachers after an intensely negative experience. The results contribute to work in mathematics education that anchors emotions in a theoretical framework and links them to other constructs in the affective domain, particularly motives. Di Martino and colleagues (Antognazza, Di Martino, Pellandini, & Sbaragli, 2015) look at if, and how, young students' emotions change during problem solving, the factors behind the change, and the potential impact of certain emotional changes on mathematical activity. They investigate young students' problem-solving difficulties, and the links between affective and cognitive factors in context. They note that intrinsic causes seem to be attributed by students to positive emotions, and extrinsic causes to negative emotions.

Despite there being a number of significant differences between the studies, they share a number of features that help to address the difficulties of studying emotion 'in-situ'. They both involve polling participants before, during and after the performance of mathematical tasks, and both make creative use of open-ended responses. These studies can be taken together to form a conversation within the research community, where researchers respond to identified gaps in the field, and cooperate to move the research on at each stage. A number of points seem to me to be particularly worth exploring further. The first is the interaction between affect and cognition, which research is only now beginning to explore. Secondly, as pointed out above, we need to understand more fully how affective and emotional states help to facilitate or inhibit learning. Further evidence is needed of the dynamic progression of motivational and emotional states through the problem solving process, how these interact with cognition and cognitively-mediated constructs such as attitudes and beliefs.

The study

The aim of the study, then, is to investigate the motivations and emotions associated with the performance of a problem solving task. More specifically:

What are the motivational and emotional pre-dispositions of prospective primary teachers to performing a problem solving task? How do these change during the process of the task, and how are they interpreted after completion of the task? What is the role of self-regulatory skills in mediating negative emotions? How do motivations and emotions interact with cognition in the undertaking of the task?

Prospective primary school teachers on initial teacher training represent a category of whom many members are lacking in confidence and a facility in mathematics (See Liljedahl, 2015). There is a lot of interest in this group for this reason. Looking to take and adapt the methods and protocols from the studies outlined above, I ran a session with a group of primary PGCE students in which these ideas can be explored. The protocol involves presenting them with mathematics problems, and polling them both prior to working on them and after the task, about their affective dispositions. The task was actually a set of graduated questions involving working out terms in series in which they were given the first few terms. By using a graduated set of tasks, I was hoping that there were tasks that were simple enough for everyone to get some right, but also some that would stretch the most accomplished.

Seventeen students on the Postgraduate Certification of Education (PGCE) programme for prospective primary school teachers at a UK University volunteered to attend a lunchtime session. They were informed only that I was interested in researching affect in mathematics education. All but two of the volunteers were women, and they were split equally between those who considered being mathematics specialists, and those who did not.

At the beginning of the session, students were briefly shown the task, and then given a questionnaire. Prior to the task, they were asked to rate the difficulty of the task on a 5-point scale from easy to difficult. They were also asked how they felt about the task, what they were thinking, and why. They then undertook the written task. After completion of the task, they were asked how well they did on the task, to describe their emotions and thoughts and feelings as they undertook the task, and about their most negative emotion, and how they dealt with it. The data was content analysed according to the categories of responses, as reported below.

Results

Initial thoughts and feelings

Ten respondents assessed the task as quite easy, and no one assessed it as difficult. Consistent with the Antognazza et al. study, students who rated the task easy or quite easy felt positively about the prospect of doing the task. When asked how they felt about the prospect of the task, prior to undertaking it, a range of positive feelings were expressed:

I like a challenge; Excited; Confident; Relaxed; Curious; Anticipation

These all reinforced the apparent perceived simplicity of the task. However, there were a few other responses (and these all came from volunteers who rated the task as of medium difficulty):

> It makes me feel excited because I want to get it correct, but scared because I might get it wrong (sc)

> I think I will be able to do the first ones and then they will get harder and I probably won't be able to do them so anticipation (ja)

When asked what they were thinking, and why, most volunteers reflected confidence and excitement. The two exceptions were those again, who judged the task to be of medium difficulty.

> How can I get it right and not look silly. (sc)

> I'm thinking that although maths isn't my strongest subject, I'm not being judged and marked so I feel more relaxed (an)

In terms of the task itself, volunteer scores were evenly distributed between scores of 4, 5 and 6 out of 6, reflecting their evaluation of the task as fairly easy. After the task, volunteers were asked how well they thought they did, and most seemed to judge that they did quite well. What is clear from the narratives is that the motivation to succeed at the tasks was strong. There are multiple mentions of determination and perseverance.

> I really wanted to get the answer …… I didn’t want to be defeated (ce)

> My main emotion was one of determination (rp)

I was just determined to get the right answer (cp)

Added to this, not only is getting right answers important, but speed in doing so is also seen as a requirement. Thus we hear:

How quickly I could work it out.. (cp)

I wanted to get through it quickly (sj)

I was quite upset when I took a little bit longer to do the last one (id)

Fine until I felt rushed due to time (cw)

A number of responses suggests that getting ANY answer wrong is unacceptable, and causes negative emotion.

I didn't do well because I was unable to answer the last two questions (an)

Post task reflection

Although the questionnaire prompts participants to distinguish between emotions on the one hand, and thoughts and feelings on the other, including how they dealt with their most negative emotion, the responses seem to represent a 'package deal' in which the emotions and cognitions are conflated, thus demonstrating how intimately connected they are. In terms of emotions, a range of words and terms are used. They include: confidence; stress; panic; confused; happy; confident; uncertainty; worried; feeling worse; frustration; anger; annoyed; irritated.

This list seems to indicate quite a narrow range of primary emotions, comprising variations of anxiety and anger, on the negative side, and happiness and confidence on the positive side. Confusion and uncertainty appear to be cognitive conditions with negative valence, that result in negative emotions such as anxiety.

Again, as with Antognazza et al., making progress and getting right answers are seen as a vital condition of satisfaction.

I was just determined to get the right answer, each time I solved one I was happy (cp)

Mainly joy at being able to do the task relatively easy (gh)

The sense of satisfaction and positive emotion continues until the prospect of getting the answer right is perceived to be at risk, when the emotion turns negative. Getting answers right, and then not being able to get an answer, is expressed in emotional dualities:

Happiness, success, proud I could do it. Annoyed when I had to take a few looks at the last one (id)

The negative feelings seem to easily initiate more deep seated negative and disabling thoughts:

I felt quite happy and relaxed at the beginning when I was able to complete the sequences but later on I felt inadequate as everyone else seemed to know the answers (an)

I began to doubt myself (nd)

Frustration stimulating a negative thought process that I am not that great at maths (ao)

> I was happy and confident until I reached 'E'. At this point my uncertainty about maths re-surfaced. I have never been confident in maths and so the fact that I struggled on the last 2 questions made this emotion re-surface (nd)

Other categories of narrative expression also emerge from the data. One category relates to the mathematical or heuristic strategies employed by the volunteers either in the search for pattern or answers, or in response to negative emotional conditions. Examples include:

> My prior knowledge of sequences helped (cs)
>
> My confidence went up and down as I used trial and error, once I'd figured out the pattern that was fine (cp)
>
> Try out a variety of methods until I found the one that worked (id)
>
> My most negative emotion was before the last question when I worried 'whoa' not sure I can do this, but I dealt with it by trying to think about the problem from a different perspective and take a different approach as I had with other questions (ce)
>
> I approached it very methodically, wrote things down to help remember what I was processing mentally. As I cracked each pattern I felt more willing to try the next (rp)
>
> I was thinking about the possibilities of how to work each one out. What different methods may I need to use? (hr)

Since the tone of emotion changed from pre-task to post-task, and from mainly positive to mainly negative, it is interesting to examine how these students deal with the negative emotion. There is evidence here of significant self regulation, which is often mediated by self-talk:

> Can I do it? Can't I do it? (sj)
>
> I had to tell myself that I had tried my best (nd)
>
> Come on you can do better (cp)
>
> It's not a test so it is ok if and when I get it wrong (ja)

Self talk also plays a part in the negative case:

> I couldn't find the sequence and therefore must be rubbish at maths (sc)

The following example illustrates such a negative **pathway** or series of responses:

> I was confused as soon as I couldn't find an obvious pattern and consequently panicked and guessed. That made me feel inadequate (sc)

Note the sequence (with comments in parentheses):

1. I couldn't find an obvious pattern… [searching for pattern is a cognitive strategy – in this case unsuccessful]
2. This made me confused…[lack of success leads to lack of solution and the cognitive condition leads to high arousal]
3. Therefore I panic…[this induces negative emotion]

4. When I panic I guess…[this leads to a poor behavioural response]

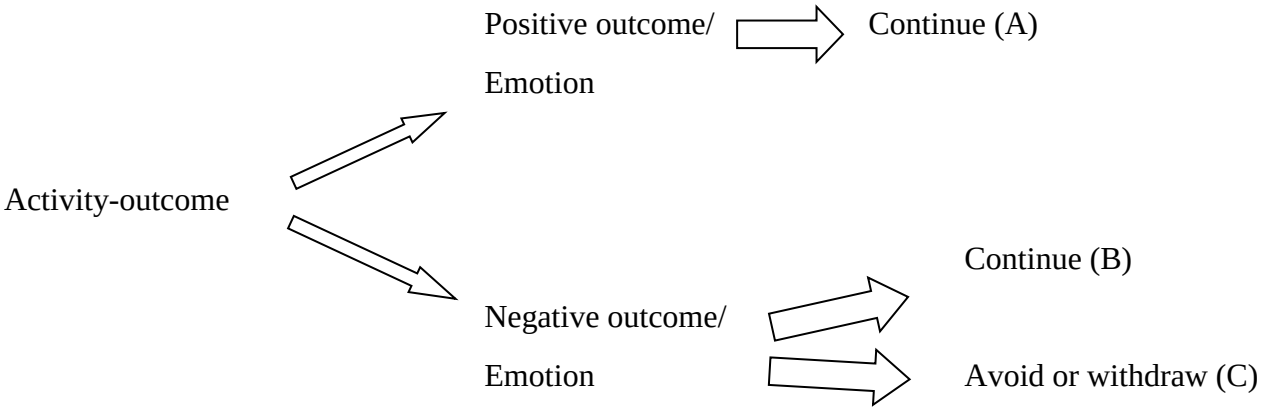
Many of the accounts talk about confusion leading to panic, and the panic relates to the very strong (but unsatisfied) need for progress, often leading to inappropriate strategies for quick solutions:

I felt panicked to try and find a solution as quickly as possible (sc)

Motivational and emotional pathways in mathematical problem solving

By assembling the evidence, it is now possible to propose a model for the possibility space of the pathways for this motivational-emotional-behavioural nexus. To do this, I will draw on the reversal theory framework, but instantiated by data from this study, and consistent with data from other, similar studies. More details about the eight motivational states, and their associated emotions can be found in Lewis (2016).

The motivational state combination determines how the experience or engagement in the task will be evaluated against the needs of the active states. If the need is satisfied, or if satisfaction is anticipated, it will lead to positive emotion. If not, it will lead to a feeling of frustration, and negative emotion determined by the specific state combinations active at the time. In this case, there is a behavioural dilemma which can be resolved in two ways. Illustrated visually, the space comprises the following pathways:



In the serious self-mastery state combination, which evidence suggests is the dominant motivational disposition of students in problem solving contexts, achieving progress, getting right answers (quickly) and the associated feeling of power or competence, is necessary for a positive affective outcome. The likely behavioural response is attraction, leading to the desire to continue (A).

If progress or positive outcome is not achieved in the serious self-mastery motivational disposition, this will result in anxiety, anger (serious), or humiliation or helplessness (self-mastery, losing), whereas in paratelic (playful) state combinations, boredom or sullenness will result if arousal is low and excitement is unavailable.

From this situation, the student has choices available. One choice is to use strategies to override or mitigate the negative emotions. One such approach is the learnt behavioural response to ‘call down’ mathematical strategies or heuristics, as evidenced above. Another available resource is the application of metacognitive skills such as determination or perseverance. This allows the student to continue, even if it is painful or uncomfortable to do so.

There is also another process that appears to take place. Negative emotions such as anxiety, anger, (possibly in conjunction with helplessness or humiliation) seem to induce a strong need for meaning, significance or explanation for the failure. Since this is unavailable in the situation, it appears to initiate a search of cognitively-mediated constructs (such as attitudes or beliefs) in order to satisfy this need. This search for meaning seems to be strongly mediated by self-talk, and may result, as the evidence shows, in evaluations of self and capability ('You can do it' or, 'I am dumb'), or evaluations or attributions related to the situation ('these questions are too hard', 'mathematics is useless'). But as Antognazza et al. point out, negative emotion is more often associated with explanations extrinsic to the problem at hand.

If such evaluations about self or the situation are positive and enabling, they provide a kind of behavioural override to the negative emotion, and encourage further qualified or reluctant attraction and engagement, as expressed, for instance, in 'come on, you can do better', as illustrated in path (B). If, on the other hand, such evaluations are negative, they result in repulsion and avoidance or withdrawal from the task, as in path (C). We can see this last option in operation in statements of the form 'I am no good at maths', 'I feel less intelligent than the others.'

Since all students will, at times, encounter negative emotions, it is important to understand in more depth what influences the choice between pathways B) and C). It seems clear that having a range of cognitive-affective resources are the key to the likelihood of students choosing pathway C). The evidence here suggests that these resources and strategies fall into three categories:

- A repertoire of mathematical or heuristic processes to enact in seeking progress in tasks and problems.
- Meta-cognitive and self-regulatory resources such as determination and perseverance in order to continue with a task when it is affectively uncomfortable or painful to do so.
- An architecture of positive or enabling cognitively-mediated structures or representations such as attitudes or beliefs, that provide a frame of confidence in which otherwise psychologically risky situations can be tackled.

Discussion

This study, then, has attempted to contribute to the understanding of affect, and particularly negative emotions and their effect on learning or not learning, mathematics. A number of interesting points arise. Firstly, further evidence to the Liljedahl study of just how prevalent negative emotion is among prospective primary teachers. Secondly, the evidence here shows just how intimately connected affect and cognition are in undertaking mathematical tasks, and the influence of affect on learning, or not learning mathematics, and indeed, the reciprocal influence of learning on affect. I have proposed a model of the mechanisms by which emotion and cognition interact when students are engaged in mathematical tasks, and in particular, ways that aspects of cognition and behaviour can be used to mitigate negative emotion, such that it doesn't disable learning.

This is a modest study which has a number of limitations, especially related to the small sample, and the fact that they were volunteers. Because of this, no attempt has been made to make quantitative generalisations from the data.

One of the key points to emerge that should inform teaching practice and pedagogy is that explicit focus and attention is needed to help students to build the emotional intelligence or cognitive-affective competences, resources and strategies to overcome negative emotion and to scaffold learning.

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On the edges of flow: Student engagement in problem solving

Peter Liljedahl, Simon Fraser University, Canada; liljedahl@sfu.ca

Engagement in mathematical problem solving is an aspect of problem solving that is often overlooked in our efforts to improve students' problem solving abilities. In this paper I look at this construct through the lens of Csíkszentmihályi's theory of flow. Studying the problem solving habits of students within a problem solving environment designed to induce flow, I look specifically at student behavior when faced with an imbalance between their problem solving skills and the challenge of the task at hand. Results indicate that most students have perseverance in the face of challenge and tolerance in the face of the mundane, and use these as buffers while autonomously correcting the imbalance.

Keywords: Flow, engagement, perseverance, challenge, tolerance.

Flow

In the early 1970's Mihály Csíkszentmihályi became interested in studying, what he referred to as, the optimal experience (1990),

> "a state in which people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it." (Csíkszentmihályi, 1990, p.4)

In his pursuit to understand the optimal experience, Csíkszentmihályi (1990) studied this phenomenon among musicians, artists, mathematicians, scientists, and athletes. Out of this research emerged a set of nine characteristics common to every such experience (Csíkszentmihályi, 1990) – the first three of which are characteristics external to the doer, existing in the environment of the activity, and crucial to occasioning of the optimal experience.

1. There are clear goals every step of the way.
2. There is immediate feedback to one's actions.
3. There is a balance between challenges and skills.

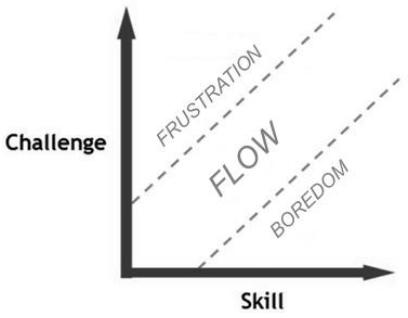

Fig. 1: Graphical representation of the balance between challenge and skill

The last of these – balance between challenge and skills – is central to Csíkszentmihályi's (1990) analysis of the optimal experience and comes into sharp focus when we consider the consequences of having an imbalance in this system. Csíkszentmihályi found that if the challenge of the activity far exceeds a person's ability they are likely to experience a feeling of frustration. Conversely, if their ability far exceeds the challenge offered by the activity they are apt to become bored. When there is a balance in this system a state of, what Csíkszentmihályi refers to as, *flow* is created (see fig. 1).

Flow is a powerful ways for us, as mathematics education researchers, to talk productively about the phenomenon of engagement in general, and the three aforementioned elements of flow gives us a way to think about the potential environments that occasion engagement in our classrooms in particular.

Williams (2001) used Csíkszentmihályi's idea of flow and applied it to a specific instance of problem solving that she refers to as discovered complexity. Discovered complexity is a state that occurs when a problem solver, or a group of problem solvers, encounter complexities that were not evident at the onset of the task, is within their zone of proximal development (Vygotsky, 1978), and occurs when the solver(s) "spontaneously formulate a question (intellectual challenge) that is resolved as they work with unfamiliar mathematical ideas" (p. 378). Such an encounter will capture, and hold, the engagement of the problem solver(s) in a way that satisfies the conditions of flow. What Williams' frame-work describes is the deep engagement that is sometimes observed in students working on a problem solving task during a single problem solving session.

Extending this work, I argued that engagement was an affective experience and used the notion of flow to look at situations of engagement extended over several days or weeks wherein students return to the same task, again and again, until a problem was solved (Liljedahl, 2006). The results of this work showed that although flow was present in each of the discrete problem solving encounters, what allowed the engagement to sustain itself across multiple encounters was a series of discovered complexities in each session linking together to form what I referred to as a *chain of discovery*.

More recently, I looked at the practices of two teachers through the lens of flow in general and their ability to set clear goals, provide instant feedback, and maintain a balance between challenge and skill in particular (Liljedahl, 2016a). From this a number of conclusions emerged. First, thinking about flow as existing in that balance between skill and challenge, as represented in figure 1, obfuscates the fact that this is not a static relationship. Flow is, in fact, a dynamic process. As students engage in an activity their skills will, invariably, improve. In order for these students to stay in flow the challenge of the task must similarly increase (see fig. 2).

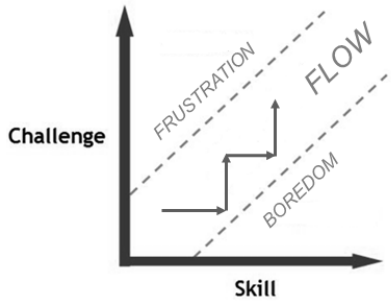

Fig. 2: Balance as a dynamic process

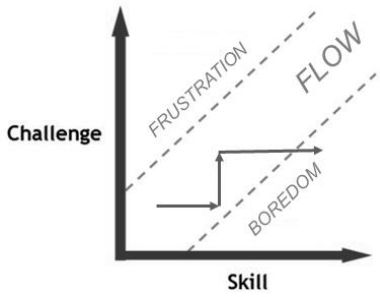

Fig. 3: Too fast an increase in skill

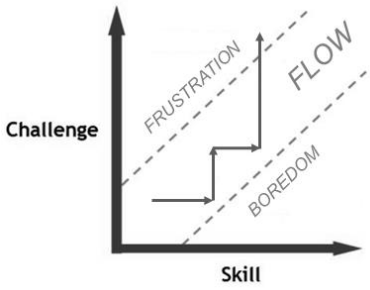

Fig. 4: Too great an increase in challenge

In a mathematics classroom, these timely increases of challenge often fall to the teacher. But this is not without obstacles. For example, if a student's skill increases either too quickly or too covertly for the teacher to notice that student may slip into a state of boredom (see fig. 3). Likewise, when the teacher does increase the challenge, if that increase is too great the student may become frustrated (see fig. 4). How teachers manage these situations of boredom and frustration is important. In Liljedahl (2016a) one of the teachers managed such situations synchronously, either giving hints or extensions to the class as a whole, usually after three groups finished or she got three of the same questions respectively. For most groups the timing of these hints and extensions was off, and not helpful in maintaining flow. The second teacher, however, managed these situations asynchronously, dealing with groups individually as they got stuck or completed a problem. Student engagement in

the second teacher's class was visibly higher as he was maintaining flow through the constant and timely maintenance of the balance between ability and complexity for each group.

What I did not learn from this aforementioned research is how students cope with imbalance when the teacher does not provide help or extensions in a timely fashion. In the research reported here I look closely at exactly this phenomenon in general, and student autonomous actions and reactions in such moments of imbalance in particular.

Methodology

To get at this behavior I chose to observe students in a problem solving settings where student work was easily visible. To this end I strategically selected two senior high school classrooms belonging to two different teachers (Cameron and Charmaine), both of whom conducted their classrooms according to a teaching framework designed to shape their classroom into a space "that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion" (Liljedahl, 2016b, p.364).

My earlier empirical work (Liljedahl, 2016b) on the design of such classrooms had emerged a collection of nine elements that offer a prescriptive framework to help teachers build such spaces. For the research presented here, five of these elements are particularly salient:

1. At the beginning of every class, students are assigned to a visibly random group (Liljedahl, 2016b, 2014) of two to four students.
2. These groups work collaboratively to solve a number of problems (usually) right from the beginning of the lesson.
3. This work is done with groups working at vertical non-permanent surfaces such as whiteboards, blackboards, or windows (Liljedahl, 2016b).
4. Students' flow is occasioned and maintained through the teacher's judicious and timely use of hints and extensions (Liljedahl, 2016a, 2016b).
5. At some point within this sequence of tasks the teacher brings the students together to debrief what they have been doing – either by going over one or more of the students' solutions or working through a new problem together with the class as a whole. This is timed so that every group is able to participate in discussion and benefit from the reification.

Taken together, both of these classrooms offered the affordances for me to easily observe students working within and an environment designed to occasion flow. The teachers were both managing engagement through the timely use of hints and extensions to maintain a balance between the challenge of an activity and the ability of each group. The student work was visible and there was enough autonomy afforded in the room that the students could take some kind of action when they found themselves in a situation where challenge and ability may be out of balance.

Data for this research were collected in Cameron's grade 12 Pre-calculus class and Charmaine's grade 11 pre-calculus class. Each class was visited five times over a seven week period in the middle of the second semester.

The data

Because the collection of video data creates such a narrow field of view, I instead used a variant of noticing (van Es, 2011) to scan the classrooms. Csíkszentmihályi (1990) characterizes flow as enjoyment, fluidity, and focus. These characteristics manifest themselves in the physicality of individuals and groups in flow and allows for the easy identification of flow and the absence of flow in a classroom. As per my research question, what I was looking for, then, were moments where an individual or a group was out of flow and where that individual or group was left to cope with this on their own. Once such a moment was identified I would focus in on that individual, or that group, taking detailed field notes and occasional photographs. When these moment seemed to wane I would conduct short, in-the-moment, interviews.

Csíkszentmihályi's theory of flow (1990) predicts that lack of flow is the result of a group of students' abilities exceeding the challenge of the task (see fig. 3) or the challenge of the task exceeding the abilities of a group (see fig. 4), resulting in the groups quitting, respectively, out of boredom or frustration. As such, flow served as the initial framework for analyzing the data. As it turns out, the theory was far from adequate for explaining all of the students' actions and reaction in the data. As such, I also used analytic deduction (Patton, 2002) to look more closely at students' actions and to group these actions into themes.

Results and analysis

From this analysis a series of six nuanced themes emerged, each marked by a different type of student action or reaction to being out of flow. In what follows I present cases exemplifying each of these themes as well as some general comments about similar cases.

When skills exceed challenge: The case of quitting

As mentioned, Csíkszentmihályi (1990) found that if a person's ability exceeded the challenge they are apt to become bored, and then quit out of this boredom. I found evidence of such behavior in Cameron's and Charmaine's classrooms.

Researcher	I notice you are not working on the assigned questions. What's up?
Mikaela	We did some of them.
Researcher	I saw that. I noticed that you did two very quickly. Took a little break from the math and then went back and did another one. I was sort of waiting to see if you would get back to it.
Allison	This stuff is easy. I'll finish it at home on my own.
Mikaela	It's actually too easy. I don't even think I will bother finishing it at home.
Allison	… Yeah. I probably won't either.

During the ten lessons I observed in Cameron's and Charmaine's classes I only managed to capture three other instances that I would say fall into the same category – quitting because the students were bored by seemingly too easy a collection of tasks.

When skills exceed challenge: The case of seeking increased challenge

Quitting out of boredom was not the only reaction to a situation where the skills of a group or of an individual exceeded the challenge of the task at hand. Some students opted, instead, to autonomously

seek increased challenge. To exemplify this I look at a case from Cameron's class captured while students were working at the whiteboards in randomly assigned groups. During this part of the lesson Cameron moved around the room helping groups that were stuck (or had made a mistake) and giving more challenging questions to groups that were done. Before a group would get his help or the next question, however, he engaged the group in conversation to assess where the group's thinking was. This took time and sometimes groups that were done were left waiting.

Researcher	So, I notice that you guys are now on question 5 and your teacher has not visited you once. How are you getting your questions?
Ameer	We just look around and see what the next question is and do that one.
Researcher	What would your teacher say about that?
Carl	Um … he'd probably want to check to see that we got the previous one before giving us the next one …
Ameer	… but we are doing that.
Researcher	Why don't you just wait for your teacher to get here and give you the next question?
Carl	We're on a roll. And sometimes we have to wait a long time.
Researcher	Do you realize that you are doing the problems out of sequence from the order your teacher is giving them?
Colton	Oh really? That's probably why some were so hard.

This was a very common reaction in both Cameron's and Charmaine's classrooms. Rather than wait for their teacher to give them the next questions groups were opting, instead, to move forward on their own by pulling the next question from groups that were ahead of them. This was facilitated by the visible nature of the work on the vertical surfaces.

When skills exceed challenge: The case of tolerance in the face of the mundane

An altogether different reaction to being tasked with doing easy and redundant questions is to just do them – without quitting and without seeking to increase the challenge. I observed such behavior in the case of Jennifer, who always worked at her desk on her own at the end of Charmaine's lessons.

Researcher	I have been watching you while I have been here. I notice that you always do a lot of questions. Can you tell me about that?
Jennifer	Yeah. I like to do a lot of questions. It's good practice. It's how I learn.
Researcher	So, are you looking for harder and harder questions to challenge yourself?
Jennifer	Not really. I just do all of them. So, if the teacher asks us to do 4a, I will also do 4bc and d and so on.
Researcher	Do you find them easy?
Jennifer	Yeah.
Researcher	How many do you do?
Jennifer	I just work the whole time at the end of class and then for maybe an hour at home.

I came to call Jennifer's behavior *tolerance for the mundane.* In my time in Cameron's and Charmaine's classes I saw two other girls who I suspect were very much like Jennifer in their approach

to learning and their tolerance for the mundane. These girls also worked alone in their desks in the last part of every lesson.

When challenge exceeds skills: The case of quitting

Csíkszentmihályi's framework (1990) predicts that sometimes students quit out of frustration. I found three cases of this in Cameron's classroom – all near the beginning of class.

Researcher	I have been watching your group for a bit and I notice that you aren't working?
Robert	We gave up. This question is stupid.
Katrina	We tried, but we weren't getting anywhere. So we gave up.
Researcher	What do you think the problem is?
Shannon	This question is too hard.
Robert	… too hard. We don't get it.
Katrina	And the teacher hasn't come over to help us.
Researcher	What kind of help are you looking for?
Shannon	You know, a hint or something.
Researcher	What would a hint do for you?
Shannon	Help us understand the question.
Katrina	… or remind us a little bit about how to do it.

For this group the question they have been asked to solve exceeded their abilities and without any help from the teacher they gave up. Interestingly, the help they were seeking was not only to reduce the complexity of the task (understand the question), but also to increase their ability (remind us of what we have done in the past). In the ten lessons I observed, I only managed to capture four instances of a group giving up out of frustration.

When challenge exceeds skills: The case of seeking help

A much more common reaction to facing too great a challenge was for students to seek help. What this looked like, however, was much more subtle than simply asking the teacher for help.

Researcher	I notice that you have been moving about the room a bit. Why?
Michael	Oh. We were just stuck so we went over there to get some ideas.
Researcher	Did it help?
Michael	Oh yeah. We got it now.

This sort of behavior was endemic in both classrooms with too many occurrences for me to track. The vertical and visible work spaces facilitated the ability for groups to check their answers and get ideas. The random groups created the porosity (Liljedahl, 2014) that made the more active interactions and movement of ideas possible.

When challenge exceeds skills: The case of perseverance in the face of challenge

But not all groups sought help when they were stuck.

Researcher	Question #5 was a tough one, huh?
Oliver	Yeah, that one took us a while.

Connor	In the end it wasn't that hard though. We were just missing something.
Researcher	Oh really. How did you figure it out?
Connor	We just kept at it and then we saw it.
Researcher	I noticed that your teacher came over to help. Did she help you?
Oliver	No, we wouldn't let her. We knew how to do it and we wanted to figure it out ourselves.

In all the lessons I observed I captured four instances where a group or an individual opted to not seek help, either from the teacher or the groups around them. I called this behavior *perseverance in the face of challenge.*

Discussion

The aforementioned six nuanced student reactions to being out of flow show that for different individuals and different groups the transitions from flow to boredom or frustration has variable immediacy. Some groups became bored or frustrated and quit. For these groups, Csíkszentmihályi's (1990) original representation of flow holds (see fig. 1).

For others, this transition was not as abrupt. Jennifer showed a great tolerance for the mundane as she spent long periods of time within a space where her ability far exceeded the challenge posed by the tasks she was working on. Likewise, Connor and Oliver demonstrating great perseverance while working on a task that presented too great a challenge for her ability. Taken together, these two cases, and the cases like them, indicate that for some students the boundary between flow and boredom and frustration is not as thin as Csíkszentmihályi's (1990) theory of flow would imply and is buffered by tolerance and perseverance (see fig. 5).

Other students used this buffer to avoid frustration or boredom as they sought to correct the imbalance between skill and challenge that they were experiencing. Carl, Ameer, and Colton used the groups around them to check their own answers and to seek out more challenging tasks when they were done. Similarly, Michael's group used the groups around them to access help when they were stuck. These groups, and the groups and individuals like them, managed to autonomously maintain the balance between challenge and ability. When their ability was too great they autonomously sought to increase the challenge (see fig. 6) and when the challenge was becoming too great they autonomously sought to increase their ability or decrease the challenge (see fig. 7). The highly visible and collaborative environments created by the use of vertical non-permanent surfaces and visibly random groups were shown by the data to be instrumental in facilitating these autonomous actions.

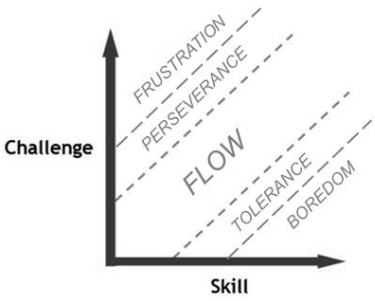

Fig. 5: Modified representation of flow

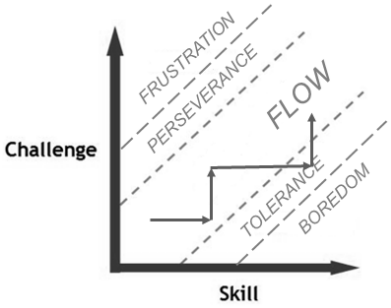

Fig. 6: Reaction to too great an ability

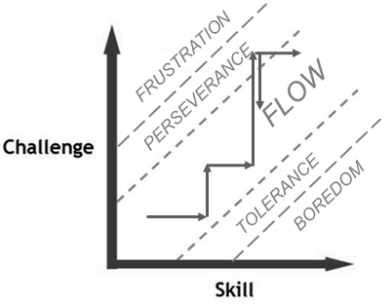

Fig. 7: Reaction to too great a challenge

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Exploring daily emotions of a mathematics teacher in a classroom: The case of Christian

Gustavo Martínez-Sierra[1], Yuridia Arellano-García[2], Antonia Hernández-Moreno[3] and Cristian Nava-Guzmán[4]

[1,2,3] Research Centre of Mathematics Education, Mathematics Faculty, Autonomous University of Guerrero, México;[1]gmartinezsierra@gmail.com, [2]yaregar@gmail.com, [3]antonia.inves@gmail.com

[4]Autonomous University of Hidalgo, México; cristiannavaguzman@hotmail.com

There is not much research on emotions of mathematics teachers other than the wide research on mathematics anxiety of mathematics teachers and pre-service teachers in elementary school. With the goal of beginning to fill this gap, this research pursues the aim of identifying the daily emotions in a classroom of a high school mathematics teacher. Data was gathered through audiotaped self-reports where the participant reported his emotional experiences during 13 mathematics classes. The data analysis show that the participant experienced diverse emotions such as satisfaction, disappointment, appreciation, happy-for, sorry-for, reproach and anger. The triggering situations for the cognitive appraisals are about the achievement of the planned activities for the lessons. The belief of the participant on the "good attitude" of students – perceived as students' "collaboration", "independence" and "participation"– supports the appraisals.

Keywords: Teachers' emotions, cognitive appraisal, self-reports of experience.

Teachers' emotions in mathematics education

In the field of mathematics education, most of the research on teachers' emotions focuses on mathematics pre-service elementary teachers. There is also some research on elementary teachers. Mathematics anxiety—"a set of negative emotions about a state of discomfort, occurring in response to situations involving mathematical tasks" (Bekdemir, 2010) — is the most widely emotional phenomenon studied in pre-service elementary school (e.g. Bursal & Paznokas, 2006; Di Martino, Coppola, Mollo, Pacelli, & Sabena, 2013; Di Martino & Sabena, 2011; Hodgen & Askew, 2007). These investigations show that "mathematics anxiety is a common phenomenon among pre-service elementary school teachers in many countries and it can seriously interfere with students becoming good mathematics teachers" (Hannula, Liljedahl, Kaasila, & Rösken, 2007, p. 153). For example, , Harper and Daane (1998) found that mathematics anxiety persists in prospective elementary school teachers, enrolled in a U.S. midsized south-eastern university, and that often, the anxiety was originated in elementary school. Causes for these students' mathematics anxiety included an emphasis on right answers and the right method, fear of making mistakes, insufficient time, word problems and problem solving.

Some other research focused in the study of specific emotions in elementary school teachers (Bibby, 2002; Di Martino et al., 2013). For example, Bibby (2002) found the presence of shame [a reaction to other people's criticisms and an emotional response to knowing and doing mathematics] related with epistemological beliefs about the nature of mathematics: absolutist/product conceptions of mathematics provide ideal opportunities for experiencing shame. Bibby found statements relating to the fear or anticipation of judgement against those standards they felt they had to measure up: "These

comments feature notions of trust, lack of trust and self-doubt, doubt, all of which indicate a fear of shame: a fear of (imagined or real) criticism, ridicule or rejection by others" (Bibby, 2002, p. 710)..

The research of teachers' emotions in mathematics education outlined here shows the strong presence of negative affect towards mathematics on elementary education teachers. There is a consensus among researchers that the main cause of all negative emotions is that most elementary and pre-service elementary teachers are not specialists in mathematics and often had negative experiences with mathematics as mathematics students in elementary or middle school (Coppola, Martino, Pacelli, & Sabena, 2012; Di Martino et al., 2013; Hodgen & Askew, 2007; Philipp, 2007). The appearance of mathematics anxiety in the first years of school is linked with the way in which mathematics is presented to pupils, with the teacher playing a central. Under these negative affective circumstances it is generally recognized that changes in mathematics education is a difficult and sometimes painful process (Hannula et al., 2007; Hodgen & Askew, 2007).

The previous review shows that most of what we know on teachers' emotions from different scholar levels is almost limited to mathematics anxiety. The intention of this research is to start filling these gaps by following the aim to identify the daily emotions experimented by a high school mathematics teacher in classroom.

Theory of cognitive structure of emotions

The theory of cognitive structure of emotions Ortony, Clore, & Collins, 1988)—known as "OCC theory" for the initials of the surnames of the authors—is an appraisal theory structured as a three-branch typology, corresponding to three kinds of stimuli: consequences of events, actions of agents, and aspects of objects. Each kind of stimulus is appraised with respect to one central criterion, called the central appraisal variable. An individual judges: (1) the desirability of an event, that is, the congruence of its consequences with the individual's goals (an event is pleasant if it helps the individual to reach his goal, and unpleasant if it prevents him from achieving his goal), (2) the approbation of an action, that is, its conformity to norms and standards, and (3) the attraction of an object, that is, the correspondence of its aspects with the individual's likes. In terms of the distinction between reactions to events, agents, and objects, we have three basic classes of emotions: "being pleased vs. displeased (reaction to events), approving vs. disapproving (reactions to agents) and liking vs. disliking (reactions to objects)" (Ortony, Clore, & Collins, 1988).

OCC theory describes a hierarchy that classifies 22 emotion types. The hierarchy contains three branches, namely emotions concerning consequences of events, actions of agents, and aspects of objects. Additionally, some branches combine to form a group of compound emotions, namely emotions concerning consequences of events caused by actions of agents. OCC theory provides specifications for each emotion type with three elements: (1) The type specification provides, in a concise sentence, the situations or events that elicit an emotion of the type in question, (2) a list of tokens is provided, showing which emotion words can be classified as belonging to the emotion type in question.

For example, 'frighten', 'scared', and 'terrified' are all types of fear (of course, 'fear' is also a type of fear): (1) TYPE SPECIFICATION: (displeased about) the prospect of an undesirable event and (2) TOKENS: apprehensive, anxious, cowering, dread, fear, fright, nervous, petrified, scared,

terrified, timid, worried, etc. In Table 1 we summarized the type specifications of all 22 emotion types.

Appraisals in terms of	Group of emotions	Types of emotions (sample name)
GOALS	Fortunes-of-others	Pleased about an event desirable for someone else (happy-for) Pleased about an event undesirable for someone else (gloating) Displeased about an event desirable for someone else (resentment, envy) Displeased about an event undesirable for someone else (sorry-for)
	Prospect-based	Pleased about the prospect of a desirable event (hope) Pleased about the confirmation of the prospect of a desirable event (satisfaction, joy) Pleased about the disconfirmation of the prospect of an undesirable event (relief) Displeased about the disconfirmation of the prospect of a desirable event (disappointment, frustration) Displeased about the prospect of an undesirable event (fear, worry) Displeased about the confirmation of the prospect of an undesirable event (fears-confirmed)
NORMS	Attribution	Approving of one's own praiseworthy action (pride) Approving of someone else's praiseworthy action (appreciation, admiration) Disapproving of one's own blameworthy action (self-reproach, shame) Disapproving of someone else's blameworthy action (reproach, rejection)
NORMA/ ATTITUDE	Well-being/ Attribution	Approving of someone else's praiseworthy action and pleased about a desirable event (gratitude=admiration + joy) Disapproving of someone else's blameworthy action and displeased about an undesirable event (anger = reproach + distress) Approving of one's own praiseworthy action and pleased about a desirable event (gratification=pride+ joy) Disapproving of one's own blameworthy action and displeased about an undesirable event (remorse = shame + distress)

Table 1: Emotion types according to the OCC theory (an extract)

Research question

Considering the above theoretical considerations, in this research we have chosen to identify the *emotional experiences* (the individuals' explicit positive or negative appraisals of the triggering situations) of a high school mathematics teacher. Thus, the research question arising from the aim of our investigation —identify the daily emotions experimented by a high school mathematics teacher in classroom— is: What are the daily individual emotional experiences of a high school mathematics teacher in classroom?

Methodology

Participant and Context

Christian, our participant and fourth author of this paper, was 35 years old by the time of the data gathering. He studied Communication and Electronic Engineering and has a master degree in mathematics education. From 2010, he ventures into mathematics teaching gaining 5 years of experience in teaching at the university and at a technical high school. High school where Cristian works is part of the national high school Mexican system, which has a dual system: it prepares students for university studies but is also engaged with those who need to enter the labour market and require a technical certificate.

Data gathering

The source of data was the daily self-informs of Christian's experiences in his high school Integral Calculus course. The diary methods "involve intensive, repeated self-reports that aim to capture events, reflections, moods, pains, or interactions near the time they occur" (Iida, Shrout, Laurenceau, & Bolger, 2012, p. 277). Christian's self-reports of experiences in class followed an event-based protocol (Iida et al., 2012). The focal experience of the participant that triggered the data collection is the emotional experience of teaching a mathematics class. After each of the 13 lessons (from October 14 to December 4, 2015) of his Integral Calculus course Christian send an audio with a smartphone to the second author of this paper via WhatsApp™ with his answers to the questions: (1) Name and date of the report, (2) What course does this report attend?, (3) What mathematics topics did you work at class today?, (4) How did you design your class?, (5) How were your students intended to learn?, (6) What emotions and feelings did you experiment today at class?, (7) Tell us about the positive experiences you lived today at mathematics class, why were they positive experiences? and (8) Tell us about the negative experiences you lived today at mathematics class, why were they negative experiences? Questions 3, 4 and 5 were designed to understand Christian's expectations and goals in each class. Questions 6, 7 and 8 were designed to know the experimented emotions of Christian in each class. The running time of the self-reports of Christian were among 1:40 y 2:54 minutes.

Data analysis

The data was completely transcribed and repeatedly read for several times. Christian did not participate in the analysis of the data but agreed that the final report adjusted in general to his experience in class. He did not propose any significant change in the interpretation made by the first three authors but made some observations on the interpretations of some fragments of his self-reports and interviews. He also suggested new elements to detail the context of the research.

Following OCC theory, we considered two aspects to identify the type of emotion: (1) ***Concise phrases that express the triggering situations*** of the emotional experiences. We highlight them with ***italic bold***, and (2) *emotional words or phrases* that express the emotional experience from the participants words or phrases that indicate the appraisal of the triggering situations. We highlight these words or phrases in *italics*. Rn (n from 1 to 13) denotes the number of the participant's report.

For the analysis we only consider excerpts that express emotional experiences; this means that it must contain at least one explicit positive or negative appraisal of the triggering situation. We interpret the

positive or negative valence according to the question reported by Christian (the valence is negative if it corresponds to questions 8 and it is positive for question 7). For example, in:

> **Christian-R2:** [The emotions and feelings I experimented in class were] Being *happy* because ***the students managed to structure answers to different doubts of their classmates*** {satisfaction-appreciation}. [The *positive experience* I lived was] ***the interest of the students in helping their classmates*** {appreciation} [I consider this experience as positive] ***because of the reflected attitude during the class, it was to help them*** {happy for-appreciation}.

We interpret "the students managed to solve doubts" as a triggering situation ('the students managed to solve doubts') of a satisfaction type of emotion (Pleased about the confirmation of the prospect of a desirable event). We interpret "the interest of the students to help their classmates" as a triggering situation ('the students help their classmates') of an appreciation type of emotion (Approving of someone else's praiseworthy action). Sometimes in the same emotional experience we identify two types of trigger situations and two types of emotions. In "being happy because the students managed to structure answers to different doubts of their classmates" we interpret as a triggering situation ('the students managed to solve doubts') of a satisfaction type of emotion and a triggering situation ('the students help their classmates') of an appreciation type of emotion.

We build a table with all emotional experiences for each report. The second and third author of this paper identified the triggering situations and the types of emotions for each self-report on separate analysis but using the same table based on the 22 types of emotions proposed the theory of cognitive structure of emotions. They worked the consensus of the triggering situations and the types of emotions with the participation of the first author of this paper. Table 2 shows an example of these analyses.

Emotional experience	Type of emotion	Triggering situation
Being *happy* because ***the students managed to structure answers to different doubts of their classmates***	Satisfaction	The students managed to solve doubts
	Appreciation	The students help their classmates
[*Positive experience*] *the interest of the students* ***to help their classmates***	Appreciation	The students help their classmates
[*Positive experience*] ***because of the reflected attitude during the class, it was to help them***	Happy for	The students have a positive attitude in class
	Appreciation	The students help each other
[*Happiness and joy]****when I realize they answered the doubts***	Satisfaction	The students solve their doubts

[Happiness and joy] ***because of the participation and enthusiasm of the students***	Satisfaction	The students participate in class
	Appreciation	The students show enthusiasm in class
[*Positive experience*]***because of the interest showed by the students helping their classmates to solve their doubts, letting them to move on*** and understand the integration techniques	Satisfaction/	Students solve doubts
	Appreciation	Students help their classmates
	Happy for	Students understood the integral techniques
[What I learn in class today] I consider that the ***help received from their classmates, in the sense that they shared different knowledge*** about the integration technique studied and this allowed their classmates to advance on the topics of the class.	Appreciation	Students help each other
	Happy for	Students make progress in the topics

Table 2. Types of emotion and triggering situations of R2 (an extract)

Results

We identified 95 emotional experiences from 7 different types corresponding to 4 groups of emotions (Table 3). We found that Christian experimented emotions of satisfaction, disappointment, appreciation, happy-for, sorry-for, reproach and anger triggered by the cognitive appraisal of 6 types of triggering situations: (1) on the achievement of the planned activity, (2) on the students' participation, (3) on the students' collaboration, (4) on the students' attitude, (5) on the students' independence, and (6) on the students' learning and understanding. During the data analysis, we identified the importance of Christian's the notion of "students' (good) attitude". We asked Christian specifically about his. We found that the supports the appraisals is the his belief about the "good attitude" of students— perceived by Christian as students' "collaboration", "independence" and "participation"— is a necessary condition to achieve participant's goals in class and for the students to learn.

Christian's most common experimented emotions were satisfaction (F=36), appreciation (F=26) and disappointment (F=16). These emotions represent 82% of his emotional experiences. More than half of the emotional experiences are *Satisfaction* (pleased about the confirmation of the prospect of a desirable event) and *disappointment* (displeased about the disconfirmation of the prospect of a desirable event). This means that they are the result of the appraisal of situations in terms of the goals Christian expressed for his lessons. On the other, *appreciation* (approving of someone else's praiseworthy action) represent more than a fourth of the emotional experiences. This highlights the

important role Christian attributes to his students' behavior ("good attitude", "participation" and "independence") to achieve their goals.

Types of emotion	*Total*	*F*	Triggering situations
Satisfaction	36	21	Students solve exercises
		6	Students participate in class
		4	Students solve doubts
		2	Students propose methods
		2	Students understand
		1	Students must be independent while solving
Appreciation	26	10	Students help their classmates
		6	Students participate in class
		5	Students are independent in solving processes
		4	Students must have good attitude
		1	Students solve doubts
Disappointment	16	8	Students do not do the planned activity.
		3	Students cannot enter the correct results in the platform
		3	Students do not participate in class.
		2	Students do not understand.

Note: ***F*** denotes the amount of times we identified the triggering situation.

Table 3. Types of emotion and triggering situation (the most frequent)

Discussion

Table 4 presents the types of triggering situations. As we said before, Christian considers the "good attitude" of a student in terms of "independence", "participation" and "collaboration". Therefore, we include all the triggering situations expressed in these terms in only one type of triggering situation named "students' attitude". In this way, we obtained that most of the triggering situations (95%) are divided in two types: (1) Students' attitude (52%) and (2) achievement of the planned activity (43%). This means that the success of an activity depends on his students' attitude above all.

Our results are consistent with those investigations that focused on teachers' emotions based on appraisal theories (e.g. Frenzel, 2014; Schutz, 2014). We believe that this consistency results from the hypothesis that emotions are the result of cognitive appraisals about what happens in class, realized in terms of goals. Our research shows that the appraisal of students' behavior, conceptualized as "good attitude", is the main triggering situation of Christian's emotional experiences. This is also highlighted in the Frenzel's (2014) "reciprocal model on causes and effects of teacher emotions". We propose the thesis that the emotional experiences of other mathematics teachers could mostly be triggered by the mathematics behavior of their students.

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The suitability of rich learning tasks from a pupil perspective

Elisabet Mellroth

Karlstad University, Sweden; elisabet.mellroth@kau.se

The construction of tasks is important to challenge pupils, but the exploration of pupils' perceptions connected to their work with tasks, is rare. This paper presents the results of a study using a tool aimed at measuring pupils' perception of joy and interest connected to 'rich learning tasks' by comparing the views of mathematically promising pupils and others. Two tasks were pre- and post-evaluated, the first by 139 and the second by 106 pupils from grade 4-9. The results indicate that the tool is suited for the exploration of pupils' views, especially as it can be deduced from the comparison that mathematically promising pupils perceived both tasks more positively than the other pupils, and that the non-identified pupils became more positive after working especially with one of the tasks.

Keywords: Mathematically gifted, mathematically promising, rich tasks, pupils' perception.

Introduction

One of the main goals in research on mathematical giftedness is to identify and foster mathematically promising pupils (Käpnick & Benölken, 2015). The construction of mathematical tasks is seen to be important for both purposes (Fuchs & Käpnick, 2009; Nolte, 2012). It is a consensus that tasks suitable to identify and foster mathematically promising pupils should, for example, be challenging, open-ended, encourage creativity and engagement, and promote enjoyment (Fuchs & Käpnick, 2009; Nolte, 2012; Sheffield, 2003). In Sweden there is no differentiation among students, every classroom is diverse and includes pupils of all abilities. Therefore, it is interesting to explore how the work with specific tasks are perceived by all pupils in the classroom. Of special interest is the perception of the mathematical promising pupils since in a diverse classroom there is a risk that they not are given opportunities to be challenged (Leikin & Stanger, 2011). A task suitable to implement in the whole class should offer a challenge to pupils at every level, which for example rich learning task are said to do (Sheffield, 2003). However, it is rare that the assessment process of the tasks appropriateness is explored, especially from the pupils' perspective. This leads to the question how tasks aimed to support mathematically promising pupils can be evaluated by the pupils. Against the background of this question this paper presents a study aiming to explore a tool in development that investigates pupils' perception of joy and interest connected to specific tasks.

This paper gives a theoretical background on tasks suitable to challenge mathematical promising pupils and other pupils. Further, the aspect of perceived joy and interest for pupils connected to work on mathematics is elaborated. The study and its results are presented and thereafter the tool used and the interpretation of the results are discussed.

Theoretical background

Pupils in a diverse classroom naturally have different levels of knowledge. Engström and Magne (2006) showed that in Swedish classrooms the mathematical knowledge of the 15 percentage lowest achieving pupils in grade nine are on the level of a grade four pupil. Also in a mathematical classroom there is a mix of pupils, some are highly motivated while others lack motivation, some are high achieving and others are low achieving (Boaler, 2006). All pupils should be given opportunities to learn and develop, and on task level there are ways to differentiate education to meet and challenge

all pupils. One way is for example through the use of rich learning tasks, which also fulfills the criteria for tasks seen to be suitable to identify and foster mathematically promising pupils (Sheffield, 2003). Because of the Swedish context with the diverse classroom the aim is to meet and develop all pupils, however, the mathematically promising pupils are of particular interest in this study. Therefore, it is important to elaborate on what is important in a task for a mathematically promising pupil as well as for pupils in general.

First, considering the mathematically promising, it is important to give them challenging tasks to help them develop according their mathematical potential (e.g. Benölken, 2015; Koshy, Ernest, & Casey, 2009; Nolte, 2012). Open-ended tasks, like rich learning tasks, are examples of tasks known to be challenging for mathematically promising pupils (Nolte, 2012; Sheffield, 2003). In addition, the joy factor is stated as important in the development process for the mathematically promising (Fuchs & Käpnick, 2009). The importance of joy in working with mathematics is further consolidated by being strived for in activities aiming to support and foster the mathematically promising, such as for example math clubs (Benölken, 2015).

Second, considering pupils in general, Taflin (2007) states that it is important that pupils perceive the problem solving process of a task as positive, challenging, and that it stimulates their creativity. Taflin actually writes that if they do not perceive this, then it is better not to implement the tasks. As to the perspective of joy, Mellroth (2014) showed that tasks aiming to evoke joy make some pupils achieve highly, even though they do not achieve highly on traditional mathematics tests. In addition, to further strengthen that pupils' enjoyment in mathematics is important, Chen and Stevenson (1995) showed that positive attitudes and interest are significantly related to mathematical achievement. And the results of Skaalvik, Federici, and Klassen (2015) show that pupils' self-efficacy in mathematics is positively and strongly related to intrinsic motivation which they directly connected to pupils' enjoyment when working with mathematics.

Based on the theory it can be assumed that pupils' perception of interest and their positive attitudes towards the task have effect on their motivation on working with the task. This is valid for both mathematically promising pupils and for others. Therefore, it is interesting and important to explore how pupils perceive working with specific tasks, especially by comparing promising and other pupils. A developed tool, easy to use, could help teachers choosing tasks that challenge and interest all pupils.

Aim

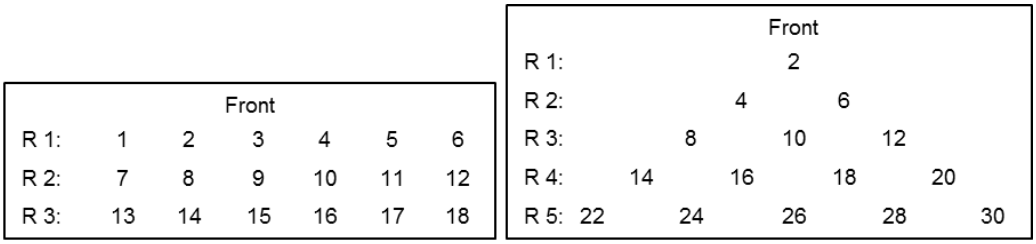
The aim of the presented study is to investigate how to identify mathematical tasks that can stimulate all pupils in a diverse classroom, including the mathematically promising. Utilizing a pupil perspective, which stresses the importance of pupil interest and joy when working with mathematics, the study provides a comparison of data from promising children and others.

The study is conducted in Sweden within the frame of a professional development program on mathematical promise for seven in-service teachers, teaching mathematics for pupils from grade 4 to 9 (Mellroth et al., 2016).

Method

Two tasks that fulfilled the criteria of rich learning tasks were implemented in seven classrooms, i.e. all pupils in "regular" classes worked with the tasks aiming to solve them. The pupils went in grade

4 (age 10) to 9 (age 15), all grades covered. The tasks considered to be suitable to challenge all pupils, specifically mathematically promising, were chosen (Sheffield, 2003), see Figure 1 (*Task 1*) and Figure 2 (*Task 2*). In the first intervention *Task 1* was implemented: 139 pupils responded on the evaluation of the task, among them 32 pupils were identified as mathematically promising[1].

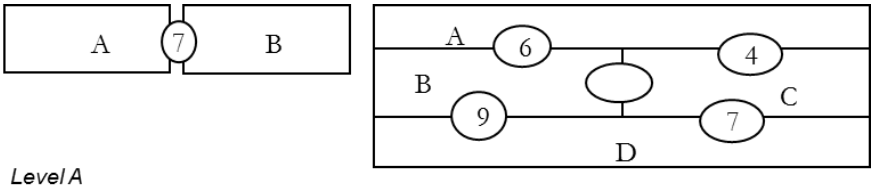


Level A
Ex. Kyla is in Seat 48. What row is she in?

Level C
Ex. I am in the last seat in a secret row. Give a rule for finding my seat number.

Figure 1: Example from *Task 1*, named Where am I? (Sheffield, 2003).

In the second intervention *Task 2* was implemented: 106 pupils responded, among them were 20 pupils identified as mathematically promising. All pupils did the interventions in the same order i.e. *Task 1* first and *Task 2* second.



Level A
Ex. How many students might there be in Field A?
How many students might there be in Field B?

Level C
Ex. What number(s) might be in the empty circle?
Is there more than one possible answer?
Why?

Figure 2: Example from *Task 2*, named Field of dreams (Sheffield, 2003); The number in a circle, denotes the total number of students in all adjoining fields.

The pupils involved in the two interventions all came from the same seven classes, 44 pupils did *Task 1* but not *Task 2* and 11 pupils did *Task 2* but not *Task 1*. Therefore 95 pupils participated in both interventions, 20 of those were identified as mathematically promising. Since the suitability of the tasks in the classroom was of interest, evaluations from all participating pupils were used in the analysis for each intervention.

Within the frame of the professional development program a tool how to measure pupils' perceptions on interest and joy connected to working with specific tasks was developed. In the development process experts on motivation and attitudes in mathematics education, and in educational psychology were consulted. The tool resulted in a pre-evaluation that utilized an emoji-note, Figure 3, and a post-evaluation, in which the emojis were changed to words, Figure 4. The reason for the change from emojis to words was to decrease the risk that pupils would chose the same emoji twice due to the short time, the time of one lesson, between the pre- and post evaluation.

[1] Selected through a synthesis of different tools, see Mellroth et al. (2016).

To collect data each teacher presented a power point slide with a picture related to each task[2] in their specific classes, without revealing the actual task. Before the task was handed out to the pupils, they were asked to mark how they felt about the task by choosing an emoji on a paper given to each one of them, see Figure 3. Thereafter the pupils were given time to work with the task.

Your teacher has presented a mathematical problem.
Which emoji best matches your feeling about this problem?



Figure 3: Evaluation note before starting to work with the task, adapted from Mellroth et al. (2016).

When the teacher ended the pupils work with the task, but before the task was discussed orally in the whole classroom, pupils were asked to evaluate the task again. This time by choosing words, see Figure 4.

What words best describes how you felt about the task while working with it?

Very interesting
Interesting
Neither or
Uninteresting
Very uninteresting

Figure 4: Evaluation note after completing working with the task (Mellroth et al., 2016).

Data from all classes were collected and summarized. For the summary process pupils identified by the teachers as mathematically promising were separated from the non-identified pupils. The evaluations, see Figure 3and Figure 4, were translated to numbers from 1 to 5, where 1 was the most positive evaluation and 5 the most negative. Thereafter, a descriptive analysis was conducted. For further details of the method see Mellroth et al. (2016).

Results

The results from each task are presented in Figure 5 and Figure 6: the graphs show the distribution of pupils' perception of the task before they started to work on it. Each bar in the graph is also split to show pupils change in perception of the task after they completed working with it. For example, in the left-hand graph in Figure 5, the bar on number 2 shows that 12 pupils, identified as mathematical promising, chose the second most positive emoji before they started to work on *Task 1*. Further, the same bar shows that of those 12 pupils, five gave the task a more negative judgement, two gave it a more positive judgement and five still gave them the second most positive judgement after they completed the work with the task.

[2] The process of choosing and analyzing the tasks are described in Mellroth et al. (2016)

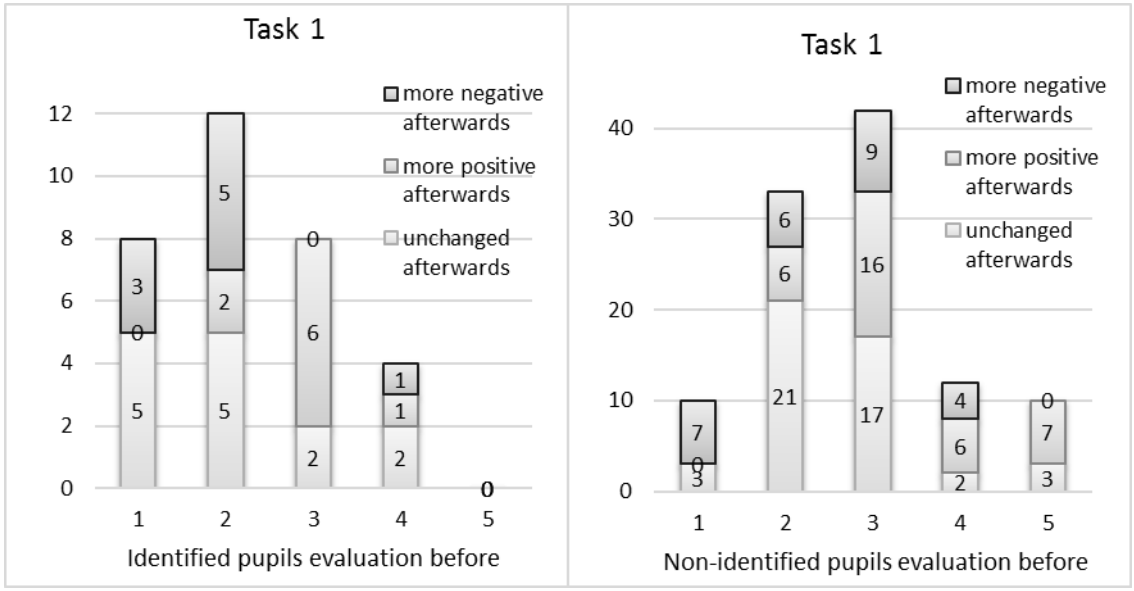


Figure 5: Pupils evaluation of *Task 1*, before starting their work on the task and after they completed their work (Figure adapted from Mellroth et al, 2016, p. 19).

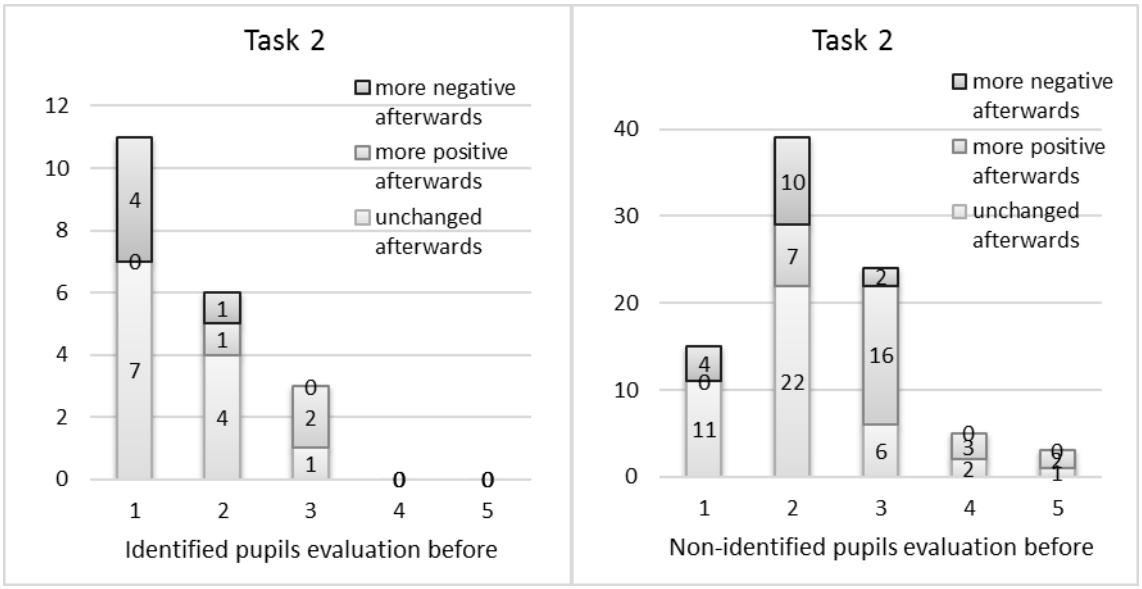


Figure 6: Pupils evaluation of *Task 2* before starting their work on the task and after they completed their work (Figure adapted from Mellroth et al, 2016, p. 19).

As both Figure 5 and Figure 6 show, through the concentration of the bars to the left, pupils identified as mathematically promising perceive both tasks more positively compared to the non-identified pupils before they started to work with the tasks. Considering *Task 1*, the two groups of pupils, identified and non-identified, did not differ much in how they changed their evaluation of the task after they completed it. 28 percent compared to 33 percent judged the task more positively after they completed it, 28 percent compared to 24 percent judged it more negatively, and 44 percent versus 43 percent judged it the same as before. As to *Task 2*, and the results of comparing identified and non-identified pupils and how they changed their evaluation of this task after they completed it show: 15 percent compared to 33 percent gave a more positive judgement afterwards, 25 percent compared to

19 percent gave the task a more negative judgement afterwards and 60 percent compared to 49 percent did not change their judgement of the task.

Interpretations and discussion

The aim of the study was to investigate how to identify mathematical tasks that can stimulate all pupils in a diverse classroom, including the mathematically promising. The two tasks used in the study were chosen because they were rich learning tasks and said to be suitable to challenge all pupils, including the mathematically promising (Sheffield, 2003). The positive evaluation given by especially the mathematically promising pupils were expected, therefore the results can be seen to verify the developed tool.

For *Task 1* the results show that the majority of the mathematically promising pupils, before starting to work on it, evaluated it as more positive: 63 percent choose the most, or the second most positive emoji, compared to 40 percent of the non-identified pupils. For *Task 2* the comparable percentages are 85 and 63 respectively. This indicates that the mathematically promising, especially, perceived the tasks interesting and joyful already before they knew the associated question. The results show that *Task 2* has this effect to a higher extend for all pupils, also the non-identified. The post evaluation of *Task 2* shows a relatively large shift to a more positive judgement of the task for the non-identified pupils, Figure 6 right graph. Altogether the results indicate that considering pupils' perception of joy and interest, *Task 2* is suitable for all pupils in the diverse classroom, including mathematically promising pupils.

The results also indicate that *Task 1* is not as suitable for all pupils. However, even if *Task 1* is not as good as *Task 2* according to the results, the mathematical promising pupils perceived it relatively positively before starting to work on it. In addition, just as many of them judged the task more negatively as those who judged it more positively afterwards. Also slightly more pupils of the non-identified judged it more positively after the completed work compared to the number that judged it more negatively. Therefore, *Task 1* might also be a suitable task in a diverse classroom even if it is not as good as *Task 2*.

According to the chosen frame for this study, tasks challenging and stimulating for mathematically promising pupils lead to that they feel joy and develop learning (Fuchs & Käpnick, 2009; Nolte, 2012; Sheffield, 2003). The identified pupils positive evaluation of the tasks, especially *Task 2*, can be a sign of that they felt the tasks challenging and stimulating. The results for the promising pupils can also be interpreted as an indication of that the developed tool fulfills its purpose to measure pupils joy and interest in a rich learning task. Furthermore, it is indicated that the tool can grade the suitability of different tasks, concerning joy and interest, in this case *Task 2* is perceived slightly more positive than *Task 1*.

It has been found that teachers rarely provide mathematically promising pupils with learning opportunities that benefit them in the diverse classroom (Leikin & Stanger, 2011), and also that positive attitudes towards working with mathematics make pupils achieve better (Chen & Stevenson, 1995). Based on this, the results show that the tasks might provide mathematical learning opportunities for all kind of pupils. Further development and verification of this tool can provide teachers with a simple way to find tasks that provide learning opportunities for all pupils in the diverse classroom, also for the mathematical promising.

Even if the simple tool has proven its use in principle, there are, of course, several limitations in this study, the investigation is simple and the tool used is not statistically verified. Nor does the investigation consider in depth how joy and interest is perceived by the pupils. In this study, pupils' motivation to work on a task is assumed to be connected to the perceived joy and interest. The teachers' evaluation of how the pupils worked with the implemented tasks is another important aspect, which this paper does not address. Within the frame of the professional development program the teachers observed and interviewed some pupils connected to their work with the investigated tasks; inclusion of this data would have strengthened the results (Mellroth et al., 2016). Also, to be able to compare different groups of pupils like for example mathematically promising and others (non-identified), teachers need knowledge on how to identify the different groups. In this study the teachers who collected the data participated in a professional development program on mathematical promise, their knowledge on how to identify those pupils can be considered as relatively deep. But it is needed to highlight this for someone who wants to repeat the study.

If further research is done to develop and validate the tool used here, it could provide in-service teachers with an easy and quick way to evaluate the suitability of tasks from a pupil's perspective. In turn this might result in mathematically promising pupils being presented with tasks that help them to develop according to their potential. In addition, complex single-case studies might explore specific aspects of tasks that are assessed highly by the pupils applying the tool presented in this study.

Acknowledgment

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Upper secondary mathematics teachers' beliefs expressed through metaphors

Marilena Pantziara[1], Marianna Karamanou[2] and Alexandra Petridou[3]

[1]Cyprus Pedagogical Institute; marilena.p@cytanet.com.cy

[2]Ministry of Education and Culture; mkaramanou@hotmail.com

[3] Centre of Educational Research and Evaluation; petridou.a@cyearn.pi.ac.cy

This study investigates grade 10-12 mathematics teachers' beliefs about their roles as mathematics teachers through metaphors. These mathematics teachers' metaphors were analysed using the categorization developed in the context of NorBa-TM project. Most of these mathematics teachers described their teaching role as didactics experts. A closer investigation of these mathematics teachers' metaphors and their teaching experience revealed some variation although not statistically significant.

Keywords: Upper secondary education, mathematics, teachers' metaphors.

Introduction

The study of mathematics teachers' beliefs and their influence on their teaching practices has gained considerable research attention. Research on teachers' thinking reveals that teachers hold coherent educational beliefs that shape their teaching practices (Handal, 2003; Thompson, 1992; Zhang & Morselli, 2016).

Metaphors offer insights into beliefs that are not obviously or consciously held (Oksanen & Hannula, 2012). A teachers' creation of a metaphor could be the result of his/her attempt to conceptualize his/her teaching. As Martinez, Sauleda and Huber (2001) state there is a need for a "shared system of interpretation and classification" of the metaphors teachers and prospective teachers hold, in order to communicate these metaphors and thus to develop them further.

Recent studies in mathematics education (Haser, Aslan & Celikdemir, 2015; Oksanen & Hannula, 2012; Oksanen, Portaankorva-Koivisto & Hannula, 2014) have used the extended model of Beijaard, Verloop and Vermunt (2000) to investigate pre-service and in-service mathematics teachers' (grades 7-9) beliefs as expressed through their metaphors. The present study aims to explore whether the extended model suggested by Löfström et al. (2010) can be used to describe and categorize Cypriot mathematics teachers' beliefs expressed through metaphors.

Theoretical framework

Mathematics teachers' beliefs

Teachers' mathematical beliefs refer to those belief systems that teachers hold regarding the teaching and learning of mathematics (Handal, 2003). These beliefs seem to be derived from personal experience, experience with schooling and instruction, and experience with formal knowledge (Zhang & Morselli, 2016).

Studies have shown that each teacher holds a specific belief system which consists of a range of beliefs about teachers, learners, teaching, learning, school settings, resources, knowledge and

curriculum. The importance of them is that they act as a filter through which teachers make their decisions and they do not rely solely on their knowledge about pedagogy and curriculum (Handal, 2003).

Beliefs and metaphors

An important trend on research regarding teachers' mathematical beliefs is the issue of teacher belief change (Zhang & Morselli, 2016). Researchers suggested a variety of methods and activities in order to investigate teachers' beliefs and also to support teachers reflect upon their experiences. One efficient way is through metaphors. Cooney et al. (1998) found that reflection played a significant role in prospective secondary teachers' growth. Researchers aiming to capture the meaning teachers ascribed to their educational experience, assessed preservice teachers' beliefs by collecting multiple data, including teachers' choice and responses to metaphors (e.g. a mathematics teacher is like an entertainer, a doctor, a gardener, a coach etc.).

Metaphors

Metaphors provide a unique way to represent the world by helping people frame the meaning of their experiences (Kasten, 1997; Zhao, Coombs & Zhou, 2010). As Martinez et al. (2001) state, metaphors are not just "figures of speech" but compose an important mechanism of the mind. The word "metaphor" is derived from the Greek word 'metaphora' (transfer) (Kasten, 1997). Metaphors refer to the understanding of one kind of object or experience in terms of a different kind of object or experience which is more familiar, concrete or visible (Lakoff & Johnson, 1980; Zhao, Coombs & Zhou, 2010). As Lakoff and Johson (1980) emphasized, a major part of the human conceptual system is structured by metaphorical relations, which are rich and complex.

In educational settings, educators are "unconsciously guided by images and metaphorical patterns of thought as recurring in the field, which can be seen as "archetypes" of professional knowledge" (Martinez et al., 2001, p. 966). In this way, metaphors reflect teachers' understanding of teaching and learning which is difficult to access in a verbatim language providing a deeper and more profound insight into teachers' beliefs and affect in relation to their teaching and the wider social context (Zhao, Coombs & Zhou, 2010). Teachers' beliefs about teaching and learning are associated with teaching roles and in this way metaphors are used to encapsulate the teaching roles (Kasten, 1997).

Categories of metaphors

Löfström et al. (2010) investigated university students' metaphors in Estonia using the Beijaard et al. (2000) model of teacher identity. This initial model identifies three distinct knowledge bases of teacher knowledge reflecting teachers' professional identity. According to the model, teachers' professional identity can be described in terms of the teacher as a *subject matter expert*, the teacher as a *pedagogical expert* and the teacher as a *didactics expert*. The results of the study indicated that the model by Beijaard and colleagues could be expanded to include three additional categories. *Self-referential metaphors*, *contextual metaphors* and *hyprids*. A description of each category of this extended model of teacher identity follows: (a) A teacher as a *subject matter expert*: The teacher has a deep and full understanding of his/her subject area and is a transmitter of knowledge to the students. (b) A teacher as a *pedagogical expert*: The teacher is someone who supports the child's development as a human being. Emphasis is on relationships, values, and the moral and emotional

aspects of children development. (c) A *teacher as a didactics expert*: The teacher has knowledge about how to teach specific subject-related content so that students can capitalise on their learning. This kind of knowledge is referred to as knowledge of didactics, and it is discipline- and subject specific in nature. (d) *Self-referential metaphors*: Self-referential metaphors do not refer to acts central to teaching, students or classroom instruction. These metaphors focus on what teaching represents for the respondents as individuals. They describe features or characteristics of the teacher's personality, with reference to the teacher's characteristics (self-referential) without reference to the role or task of the teacher. One might say that these metaphors describe who the teacher is. (e) *Contextual metaphors*: These metaphors describe features or characteristics of the teacher's work/work environment or in other ways refer to characteristics of the environment (contextual). (f) *Hybrid*s may include elements of more than one of the above categories.

The above categorization was employed in recent studies investigating teachers' beliefs through metaphors. Specifically, Oksanen and Hannula (2013) used this categorization to investigate 70 Finnish 7-9 grade mathematics teachers' beliefs regarding teaching and teachers as expressed through metaphors. The results revealed that the teacher as a didactics expert was the most frequently used metaphor (49%). The results showed no statistical significant associations between metaphors and age or gender. In the study by Oksanen et al. (2014) including 72 Finish pre-service teachers and 65 Finish in-service (grade 7-9) mathematics teachers, the most common metaphor used by pre-service teachers was self-referential (46%) while the most frequently used metaphor by in-service teachers despite their teaching experience was the category didactics expert (51%). The researchers explained that as in-service teachers gain more teaching experience, this does not affect the metaphor they use to describe their mathematics teachers' role. Finally, in the study by Haser et al. (2015) with 249 Turkish pre-service students, 29.6% used didactic expert metaphor while 26.5% used self-referential metaphors to express their beliefs.

The purpose of the current study was to explore if the categories included in the model by Löfström et al. (2010) can be used to categorize upper secondary school teachers' beliefs expressed through metaphors in Cyprus and whether the categories proposed in this model are exhaustive enough to cover all metaphors.

Context

According to the Annual Report of the Cyprus Ministry of Education and Culture (2014) the Public Secondary General Education in Cyprus is offered to students between the ages of 12 - 18, through two three-year levels - the Gymnasium (Grades 7-9) and the Lyceum (Eniaio Lykeio) (Grades 10-12). The curriculum includes common core subjects, such as Modern Greek and Mathematics and Optional Subjects. Some subjects are interdisciplinary such as Health Education and Environmental Studies. In the academic year 2000 - 2001, the institution of the Eniaio Lykeio was introduced in all public secondary schools in Cyprus. All subjects in Grade 10 are common core subjects. In Grades 11 and 12 students attend common core subjects and at the same time select optional subjects for systematic and in depth study. In Cyprus there are 38 Lycea and 7 joined Gymnasia and Lycea. Approximately 280 mathematics teachers work in Lycea. These mathematics teachers are also responsible for the preparation of students for their entrance exams in the public University of Cyprus and the public universities in Greece.

Methodology

Data collection, instruments and participants

Data for this study was gathered from mathematics teachers working in Lyceum (Eniaio Lykeio) during the school year 2015-2016. The study was conducted in the context of the international comparative study New Open Research: Beliefs about Teaching Mathematics (NorBa-TM) investigating mathematics teachers' beliefs in more than 15 countries.

A questionnaire was developed and culturally adapted in the participating countries in the context of the project NorBA-TM. The questionnaire comprised of seven parts: one of them qualitative and six quantitative (86 items). The current study used data only from two parts of the aforementioned questionnaire: Part A, that collected data on teachers' background variables (age, gender, education, teaching experience, teaching maths at Lyceum etc.) and Part H that collected data on metaphors. Specifically, Part H included two questions that invited the teachers to think and write down a metaphor characterising themselves as upper secondary level mathematics teachers and to explain their metaphor: "As a mathematics teacher I am like……" and "My brief explanation of the metaphor is as follows:….". We assume that the metaphor research is a useful social science methodology that can be used for generating authentic case study evidence in a certain field.

Data collection took place in June 2015. First, informative letters along with the questionnaire and prepaid envelopes were sent to mathematics teachers in all Lyceums inviting them to participate in the study on a voluntary basis. Teachers who wished to participate in the study completed the questionnaire and returned it to the Cyprus Pedagogical Institute without disclosing their personal data (name and school). A total of 147 out of 280 (53%) mathematics teachers completed and returned the questionnaire. Out of these 147, only 49 (33%) completed Part H by presenting a metaphor and providing an explanation.

Data analysis was performed using the categorization of Löfström et al. (2011) which was explained in detail in the manual developed for the NorBa project. The metaphor categorization was judged on a case-to-case basis using three independent raters whose coding were compared at the end. The raters compared their codes and discussed their differences. In the majority of cases, agreement between the three raters could be reached. In three cases though, consensus between raters was not reached and external researchers with experience in mathematics teachers' metaphors were involved.

Results

Categorizing teachers' metaphors

The distribution of metaphors used by the Cyprus in-service mathematics teachers is presented in Table 1.

Teacher as a subject matter expert	**Teacher as didactics expert**	**Teacher as Pedagogical expert**	**Self-referential**	**Contextual**	**Hybrid**
8 (16,3%)	12 (24,5%)	6 (12,2%)	9 (18,4%)	7 (14,3%)	7(14,3%)

Table 1: Distribution of metaphors used by Cypriot mathematics teachers

Teacher as didactics expert was the most common metaphor used (24,5%). Mathematics teachers in this category used metaphors like “a coach”, “a builder”, “an electrical wire”, “a gardener”, “a playmate, play-maker”. In their explanations, teachers emphasized their role as facilitators of students’ learning process, as mediators between students and the discovery of the new mathematical knowledge, as contributors to the construction of the new mathematical ideas. They referred to the communication of ideas and the team spirit, emphasizing a more constructivist view of learning and teaching. They made also reference to the active role of the students in the learning process.

> As a mathematics teacher I’m like a playmate in a team game that usually has the role of the play-maker. I’m trying to arrange the learning activities because we function as a team with a preset schedule of the game. I encourage initiatives but I control for the application of certain rules.

Self-referential was the second most common metaphor used (18,4%). Mathematics teachers in this category used metaphors like “a painter”, “a musician”, “a hard-working bee”, “a perfect circle”, “an angle”. In their explanations these teachers refer to their individual characteristics and their personality traits without reference to the role or task of the teacher.

> As a mathematics teacher I’m like an angle. Sometimes acute, sometimes obtuse, sometimes convex and sometimes non-curved.

Teacher as Subject matter expert was used by 16,3% of the teachers. Mathematics teachers in this category used metaphors like “a machine of knowledge”, “a guide in a journey”, “a vocation backpack”, “a well of knowledge”. In their explanations these teachers refer to their teaching role as transmitters of ready-made knowledge to the students and organizers of routines. Mathematical knowledge is conceived as predetermined knowledge that can be delivered by the teacher.

> As a mathematics teacher I’m like a well of knowledge. When I’m in class I find ways and examples to transmit the mathematical knowledge to students.

A percentage of 14,3% of the teachers provided metaphors that fell within the category Contextual. Mathematics teachers in this category used metaphors like “an actor”, “a salesman”, “the guy for every job”. In their explanations these teachers refer to the characteristics of the teachers’ work, or the characteristics of the environment the teacher works stressing that it is too demanding and multifunctional. They refer to the teacher in a social context but they do not refer to any specific aspect of the teachers’ professional knowledge or teaching.

> As a mathematics teacher I’m like a guy for every job. Mathematics teachers are like ping pong balls. They are involved in many tasks and processes in the school setting but these efforts do not lead to something recognizable or efficient.

A percentage of 14,3% of the teachers provided metaphors that fell in more than one category thus they were categorized as Hybrid. Most of the metaphors in this category include the didactics expert’s characteristics along with another category. The following metaphor was categorized as both didactics expert and contextual metaphor.

> As a mathematics teacher I'm like a director who writes a movie and participates in it. As a mathematics teacher I design the teaching of a lesson, I decide for the way to implement it in different faces and last like as an actor I perform different roles.

Finally, 12,2% of the teachers provided a metaphor that fell in the category teacher as pedagogical expert. Mathematics teachers in this category used metaphors like "a parent", "a mother", "and an eagle", "a priest". In their explanations these teachers emphasized the values, the moral and emotional aspects of students' development. They reveal a more affectional relationship and communication with the students.

> As a mathematics teacher I'm like a spiritual father (e.g. a priest). I believe that my main goal is to advice students that with hard work, healthy competition and honesty they can be better in mathematics and in society. Just by hard work.

Metaphors and background characteristics

No statistically significant differences were detected between gender, teaching experience and metaphors. Table 2 presents the distribution of metaphor used according to teaching experience.

Teaching experience	n	Subject-matter expert	Didactics expert	Pedagogical expert	Self-referential	Contextual	Hybrid
0-10	9	2 (22.2%)	2 (22.2%)	3 (33.3%)	1 (11,1%)	0 (0%)	1(11.1%)
11-20	26	2 (7.6%)	9 (34.6%)	3 (11.5%)	4 (15.3%)	5 (19.2%)	3 (14.2%)
21 and more	14	4 (28.5%)	1(7.1%)	0 (0%)	4 (28.5%)	2 (14.2%)	3 (21.4%)

Table 2: Teaching experience and categories of metaphors

As it can be observed, the most common category for teachers with the least years of experience is the teacher as pedagogical expert. The most common metaphor category for the second group of teachers is the teacher as a didactics expert. For the group of mathematics teachers with 21 years of experience and over, the most common categories are teacher as subject matter expect and self-referential metaphors. Contextual and hybrid categories are most frequently met in teachers with more years of teaching experience than those with 0-10 years of experience.

Discussion

The results revealed that the model proposed by Löfström et al. (2010) can be applied to categorize the metaphors provided by the in-service Cypriot mathematics teachers in upper secondary education (Grades 10-12) and that the categories included in the extended model were collectively exhaustive. The results showed that these teachers prioritize *didactics knowledge, self-reference and subject matter metaphors.* In particular, the findings showed that the teacher as didactics expert was the most common metaphor provided. This finding is in line with the results of other similar studies (Oksanen & Hannula, 2013; Oksanen et al., 2014) which reported that this category was the most common among mathematics teachers teaching Grades 7-9. However, in the current study the percentage of teachers who used metaphors which described them as didactics expert (24.5%) was not as high as in the other two studies (46% and 51% respectively). Self-referential metaphors were also used by participants of the current study (18.4%). This percentage is higher than the

percentages reported by the other studies in the literature, which state that this category reflects the multi-functionality of teachers' role (Oksanen & Hannula, 2013; Oksanen et al., 2014). The emergence of hybrid metaphors has been explained in other studies (Oksanen & Hannula, 2013; Oksanen et al., 2014) by the complexity of a teachers' job.

The investigation of these teachers' metaphors in relation to their teaching experience revealed no statistically significant differences similar to the results of the study by Oksanen et al. (2014). In that study the researches described that that the most common metaphor for all groups of teachers was the teacher as didactics expert. In the current study, pedagogical expert and didactics expert were the most common metaphors for the group of mathematics teachers with the least years of experience, didactics expert was the most common metaphor for the group of teachers with 11-20 years of experience and subject matter expert and self-referential were the most common metaphors for the group with 21 and more years of experience. However, these relationships were not statistically significant. Modifications in mathematics education at the university level, as well as modifications in the work context related to the Cyprus Educational Reform of 2011 could by associated with these groups of teachers' perceptions about their roles as teachers of mathematics.

The results indicated that the model suggested by Löfström et al. (2010) is a useful model that can be used to categorize teachers' metaphors. These teachers' metaphors mapped their current practice and understanding of teaching and learning and revealed what they are and how they feel about their work (Zhao et al., 2010). But how stable are these imageries provided? Will these teachers provide the same metaphor if they are asked again under different conditions, working in different school with other students or if they are under the pressure of their students' entrance exams or at the end of a stressful day? How the methodology used could be developed to include teachers' current state? Further studies investigating the stability of these metaphors are needed. Moreover, further studies could investigate the relation between mathematics teachers' beliefs and mathematics teachers' use of metaphors.

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Multiple solutions for real-world problems, and students' enjoyment and boredom

Stanislaw Schukajlow and Kay Achmetli

University of Münster, Germany; schukajlow@uni-muenster.de

Emotions are important for learning. In a previous study, we found that students who constructed more solutions for real-word problems with vague conditions reported higher enjoyment and lower boredom (Schukajlow & Rakoczy, 2016). In the present study, we had students construct multiple solutions by applying multiple mathematical procedures to real-world problems, and we investigated effects on the enjoyment and boredom. 307 students were assigned to the experimental or control group. Students in the experimental group applied two mathematical procedures, and students in the control group applied one mathematical procedure to solve real-world problems. During the lessons, they were asked to report their enjoyment and boredom. Contrary to our expectations, the results revealed no effects of the intervention on students' enjoyment or boredom.

Keywords: Emotion, affect, modelling, word problems, multiple solutions.

Introduction

Emotions are important for learning (Zan, Brown, Evans, & Hannula, 2006). Although students' academic emotions are prerequisites, mediators, and outcomes of the learning process in mathematics (Schukajlow & Rakoczy, 2016), they were neglected for decades. Thus, except for the emotion of anxiety, we do not know much about students' emotional development. Moreover, there is a lack of research on how teaching methods influence emotions. As there have been several calls for intervention studies, we decided to conduct a study that was aimed at clarifying the impact of constructing multiple solutions for real-world problems on cognitive and affective outcomes. We chose this teaching method and this kind of problem because constructing multiple solutions and solving real-world problems are emphasized in curricula in different countries. In the present paper, we taught students to construct multiple solutions by applying different mathematical procedures to solve real-world problems, and we investigated how this process affected enjoyment and boredom.

Theoretical framework and hypotheses

High-quality mathematics teaching implies that students should develop multiple solutions and compare these solutions in the classroom. Empirical evidence for the effects of constructing multiple solutions on cognitive outcomes comes from international comparative studies (Hiebert et al., 2003) and from experimental studies (Levav-Waynberg & Leikin, 2012; Schukajlow, Krug, & Rakoczy, 2015). However, the impact of constructing multiple solutions on affect is an open issue. For high-quality mathematics teaching, both cognitive and affective outcomes have to be taken into account. As we determined in the project MultiMa[1] (Multiple Solutions for Mathematics Teaching

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Oriented Toward Students' Self-regulation Learning), apart from students' achievements and strategies, there is also a need to consider their self-regulation, interest, motivation, and emotions.

Multiple solutions and real-world problems

Previous research on multiple solutions was conducted for the most part on intra-mathematical problems in different content areas such as geometry (Levav-Waynberg & Leikin, 2012) or early algebra (Star & Rittle-Johnson, 2008). Students' ability to solve real-world (or modelling) problems was not previously the focus of research on multiple solutions so far. Solving real-world problems first and foremost involves demanding transfer processes between reality and mathematics (Niss, Blum, & Galbraith, 2007). As real-world problems often include vague conditions and allow students to construct different mathematical models and apply different mathematical procedures, we distinguished between three categories of multiple solutions (Schukajlow & Krug, 2014b). The first category of multiple solutions are typical of real-world problems with vague conditions. In solving this type of problem, students make different assumptions about vague conditions and therefore arrive at different outcomes or results. Another type of multiple-solution problem occurs as a result of applying different mathematical procedures or strategies, a process that typically leads to the same mathematical outcome. The third category combines the first two categories. In the current paper, we explored the effects of applying multiple mathematical procedures while solving real-world problems for the topic of linear functions. We would like to illustrate this type of multiple-solution problem with the sample problem "BahnCard" (cf. Figure 1), which was developed in the framework of the MultiMa Project.

BahnCard

Mr. Besser lives in Hamburg. His parents live in Bremen. For the outward and return journeys with the "Deutsche Bahn" (German Rail), Mr. Besser has to pay 100 €. There are two special offers, the so-called "BahnCard 25" and the "BahnCard 50." The prices for each year and the prices for the outward and return journeys from Hamburg to Bremen for owners of the "BahnCards" are listed below.



BahnCard 25	BahnCard 50
Price per year: 59 € Price for a round-trip journey: 75 € Number of customers: 3.1 million	Price per year: 240 € Price for a round-trip journey: 50 € Number of customers: 1.6 million

Mr. Besser is going to buy a "BahnCard." When is it worth buying the "BahnCard 25" and when the "BahnCard 50"? Write down your solution.

Figure 1: Real-world problem "BahnCard" (Achmetli, Schukajlow, & Krug, 2014)

The problem solver is asked to read the problem "BahnCard," and identify the important values: price per year for each card and the amount of a round-trip journey that would be paid with each card. After mathematizing the problem, different mathematical procedures can be applied.

One mathematical procedure that can be applied is called "differences." In order to solve the "BahnCard" problem by using differences, students first have to calculate differences in the prices per year and for each round trip for owners of each card. Whereas the "BahnCard 50" is 181 € (= 240 € - 59 €) more expensive than the "BahnCard 25," each round trip with the "BahnCard 25" is 25 € (= 50 € - 25 €) more expensive than with the "BahnCard 50." The open question is how often Mr. Besser has to take a trip with the more expensive "BahnCard 50" until the cheaper prices for the

journeys pay off. This is exactly after 7.24 (= 181 € ÷ 25 €) journeys per year. This result has to be rounded up, interpreted—for example, "For up to 7 journeys per year, the 'BahnCard 25' is cheaper"— validated, and the recommendation has to be wrote down.

Another way to solve this problem is to apply a mathematical procedure "table." To apply this procedure, students must compare the costs for owners of the "BahnCard 25" and the "BahnCard 50" for different numbers of journeys per year (e.g. 1, 3, 6…). By performing this comparison systematically, they can identify that the "BahnCard 25" is cheaper for up to 7 journeys. If the owner makes 8 or more journeys, the "BahnCard 50" is preferable for him/her. Finally, students need to validate their result and write down their recommendation.

Enjoyment and boredom as achievement emotions

Emotions are typically defined as complex phenomena that include affective, cognitive, physiological, motivational, and expressive parts (Pekrun & Linnenbrink-Garcia, 2014). In the academic context, researchers are interested in achievement emotions, which occur in learning settings and are related to epistemic processes. Research on emotions in mathematics education has emerged from different philosophical traditions (Hannula, 2015) and has categorized emotions according their value (positive or negative), level of activation (activated or deactivated), or other characteristics. For example, enjoyment is one of the positive activating emotions (Pekrun, 2006). Students who enjoy problem solving are expected to report pleasant feelings. Moreover, when students enjoy mathematics, they feel activated excitement while working on a problem. The opposite behavioral and cognitive patterns are expected for the emotion of boredom. Boredom was suggested to be a negative deactivated emotion because boredom is accompanied by unpleasant feelings, and if students feel bored, they experience a state of deactivating relaxation. Following these considerations, a positive relation between enjoyment and performance and a negative relation between boredom and performance were hypothesized and confirmed in two empirical studies in the domain of mathematics (Schukajlow, 2015; Schukajlow & Krug, 2014a). Moreover, enjoyment but not boredom was found to predict students' performance in a longitudinal interventional study (Schukajlow & Rakoczy, 2016).

According to the control-value theory of achievement emotions (Pekrun, 2006), emotions are strongly determined by control and value appraisals, which arise in learning situations. In order for a positive emotion such as enjoyment to emerge, students should (1) perceive their problem solving activities as controllable and be confident that they can influence the learning situation and (2) ascribe the problem solving activities a high value. If students think that they do not have any influence over their problem solving activities, or if they view these activities as meaningless, negative emotions will emerge. For example, boredom arises if students ascribe a low value to their activities. The relation between boredom and control appraisals is complex and is proposed to be a curvilinear U-shape. This relation implies that boredom occurs when perceived control is very high (i.e. task demands are very low) or when perceived control is very low (i.e. task demands are very high). However, in the context of problem solving activities, students do not have to deal with routine tasks. Thus, a negative linear relation between control appraisals (e.g. assessed via students' performance or self-efficacy beliefs) and boredom was expected and confirmed in most empirical studies (e.g. Schukajlow, 2015).

Enjoyment, boredom, and multiple solutions for real-world problems

On the basis of theoretical considerations from control-value theory, we expected to find that constructing multiple solutions would increase students' control appraisals when solving real-world problems. Higher appraisals should increase students' enjoyment and decrease their boredom. Positive effects of constructing multiple solutions on enjoyment and negative effects on boredom were confirmed in our previous study. Students who constructed more solutions enjoyed their classes more and were less bored (Schukajlow & Rakoczy, 2016). In the current study, we sought to confirm these findings for the other type of multiple-solution problem and investigated the effects of applying multiple mathematical procedures for real-world problems on enjoyment and boredom.

Hypotheses

The hypotheses we addressed were: 1) Constructing multiple solutions by applying multiple mathematical procedures for real-world problems has a positive effect on students' enjoyment of mathematics; 2) Constructing multiple solutions by applying multiple mathematical procedures for real-world problems has a negative effect on boredom in mathematics.

Method

Sample and procedure

Three hundred seven German ninth graders from four schools with three middle-track classes each (48.26% female; mean age=14.6 years) participated in the present study. Before and after the teaching unit, students were asked about their enjoyment and boredom. The teaching unit consisted of two sessions with two 45-minute long lessons each. Each of twelve classes was divided into two parts with the same number of students in each part in the way students' mathematical achievements did not differ between the parts. Further, the number of males and females was approximately the same in each part. Eight of twenty-four groups were randomly assigned to the one-solution condition "differences" (OS1), eight groups to the one-solution condition "table" (OS2), and eight to the multiple-solutions condition "differences + table" (MS), taking into account that in each school, there had to be the same number of groups assigned to each condition, and the students in each class had to be assigned to different conditions (more details about the procedure can be found in Achmetli, Schukajlow, & Rakoczy, manuscript submitted for publication). Each group was taught separately by one of six teachers (three female, age: 27 to 60) who participated in the present study. The teachers taught the same number of groups in each condition in order to minimize the differences between conditions that might result from the influence of teacher personality on students' learning. All of the teachers received instruction manuals that included the lesson plans, problems for the students, and the solutions to these problems.

Treatment

The three treatment conditions implemented in the present study (OS1, OS2, and MS) were based on the positively evaluated student-centered learning environment for teaching modelling problems (Schukajlow, Kolter, & Blum, 2015). This student-centered learning environment was complemented by direct instruction at the beginning of the teaching unit. For the purpose of maintaining comparability between the conditions, the same order was implemented for all three treatment conditions. In the first lesson, the teacher demonstrated how real-world problems could be

solved by applying one mathematical procedure (in the OS conditions) or multiple mathematical procedures (in the MS condition). In the three lessons that followed, the students solved real-world problems by applying the demonstrated procedures according to a special procedure for group work (alone, together, and alone), presented their solutions, and discussed these solutions with the whole group in the classroom. At the end of each lesson, the teacher summarized the key points of each treatment condition. In the multiple-solutions condition, the teacher encouraged the students, further, to compare and contrast the two mathematical procedures and the mathematical results.

Students first solved four similar tasks in the one-solution conditions and in the multiple-solutions condition. The only difference between these four problems was that students in the one-solution conditions were required to apply one mathematical procedure ("table" or "differences"), whereas students in the multiple-solutions condition were required to apply both mathematical procedures ("table" and "differences"). The sample problem "BahnCard," which was given in the one-solution conditions, is presented in Figure 1. In the multiple-solutions condition, the problems were modified by adding the following sentence: "Use two different mathematical procedures to solve this problem." As the discussion of the connection between mathematical procedures required additional time in the MS condition, one additional task was offered in each OS condition. Thus, in sum, students in the MS condition solved six and students in the OS conditions solved seven problems.

Measures

Enjoyment and boredom during the teaching unit were measured after the second and fourth lessons with a 5-point scale ranging from 1 (not at all true) to 5 (completely true). Both scales included three items each (see Table 1).

Scale	Item
Enjoyment	I enjoyed task processing. I was happy during task processing. Task processing was great fun for me.
Boredom	Task processing was boring. I got so bored during task processing that I had problems staying alert. I did not want to continue my work because it was so boring.

Table 1: Items used in the study to assess enjoyment and boredom

The scales were adapted from the well-evaluated Achievement Emotions Questionnaire (Pekrun, Goetz, Frenzel, Barchfeld, & Perry, 2011). The Cronbach's alpha reliabilities were .80 and .79 for enjoyment and .81 and .83 for boredom for Sessions 1 and 2, respectively.

Treatment fidelity

To ensure the fidelity of the treatment, we videotaped the teaching unit, observed the lessons, and analyzed the students' solutions. The analysis confirmed the treatment fidelity (Achmetli et al., manuscript submitted for publication). For example, we found that students in all classes worked on the respective version of the problem (MS vs. OS) and all teachers implemented the intended methodical order in their lessons. More specifically, we found that students in the MS condition developed significantly more solutions than the students in the OS conditions (MS vs. OS1: effect size Cohen's d=4.97; MS vs. OS2: d=3.61).

Results

Preliminary results

In order to simplify the analysis of the effects, we combined the OS1 and OS2 conditions into one OS condition. Combining the two conditions did not influence the results significantly, as our statistical analysis did not show a difference at the 10% level of significance between the two OS conditions for motivational variables such as self-regulation (Achmetli et al., 2014) or interest (t(186) = 0.182; p = .856). Further, in order to ensure that the two conditions were comparable, we compared interest between the MS condition and the combined OS condition as this construct is closely connected to students' enjoyment and boredom (Schukajlow & Rakoczy, 2016). The analysis of interest at pretest revealed no differences between the MS and OS conditions (MS: M = 2.39 (SD = .90), OS: M = 2.39 (SD = .96)). This result indicates that students' emotional prerequisites were similar in the MS and OS conditions.

Applying multiple mathematical procedures and students' enjoyment or boredom

We hypothesized that constructing multiple solutions by applying multiple mathematical procedures would increase students' enjoyment and decrease their boredom. We tested both hypotheses by computing t-tests. The crucial assumption when using a t-test is that the variances are equal in the two groups. Levene's test of equality of variances was significant for students' boredom measured after the second and third lessons, indicating that the assumption of equal variances in the two groups had been violated (F(280) = 4.022, p = .046; F(279) = 4.851, p = .028). Thus, we used the adjusted degrees of freedom, t-values, and p-values for students' boredom. The descriptive statistics are presented in Table 2.

	Enjoyment first session Mean (SD)	Enjoyment second session Mean (SD)	Boredom first session Mean (SD)	Boredom second session Mean (SD)
MS	3.42 (0.92)	2.96 (0.91)	2.24 (1.05)	2.53 (1.16)
OS	3.43 (0.88)	3.00 (0.96)	2.03 (0.90)	2.30 (1.00)

Table 2: Means and standard deviations for enjoyment and boredom

Against our expectations, students' enjoyment during the first and second sessions did not differ between the MS and OS conditions (first session t(280) = 0.75, p = .940, Cohen's d = 0.02; second session: t(279) = 0.297, p = .767, d = 0.04). Thus, the enjoyment of students who solved real-world problems by applying multiple mathematical procedures was similar to the enjoyment of students who applied one mathematical procedure.

We did not find support for the second hypothesis. Our analysis did not reveal benefits of constructing multiple solutions by applying multiple mathematical procedures for students' boredom during the first or second session (first session t(165) = 1.67, p = .097, Cohen's d = 0.22; second session: t(156) = 1.62, p = .108, d = 0.22). Moreover, there was a slight (but not significant) tendency for students in the multiple-solutions group to feel greater boredom than students in the one-solution condition.

Discussion

In this paper, we aimed to analyze how constructing multiple solutions by applying multiple mathematical procedures while solving real-world problems would affect students' emotions. On the basis of theoretical considerations from the control-value theory of achievement emotions (Pekrun, 2006) and prior research that found that developing multiple solutions had positive effects on students' enjoyment and negative effects on their boredom (Schukajlow & Rakoczy, 2016), we expected positive effects of the treatment on enjoyment and negative effects on boredom during learning. However, our analyses did not confirm these hypotheses. Enjoyment and boredom in solving real-world problems did not differ between the multiple-solutions and one-solution conditions. Moreover, boredom was slightly lower in the one-solution condition compared with the multiple-solutions condition. One explanation for this finding might involve students' high control appraisals. In the previous study, the mean values for students' experience of competence, which can be taken as an indicator of students' control appraisals (sample item: "I felt confident about my knowledge of the topic today"; range from 1 to 5), were 3.85 and 3.65 in the MS and OS conditions, respectively (Schukajlow & Krug, 2014b). However, in the current study, the mean values for students' experience of competence were nearly one standard deviation higher and close to the theoretical maximum of 5 (Achmetli et al., manuscript submitted for publication). As noted in the control-value theory, if students' control appraisals are too high (or task demands are too low), they can have a negative influence on students' emotions. Thus, a future research question might involve asking whether posing more demanding real-world problems that require students to apply multiple mathematical procedures can increase students' positive emotions such as enjoyment and decrease their negative emotions such as boredom. Another research question that should be addressed in an experimental study is about the non-linear connection between control appraisals and students' emotions. This assumption of the control-value theory needs more empirical evidence from randomized studies. More specifically, the corvilinear U-shape relation between control appraisals and boredom should be addressed in future longitudinal studies. Further, it might be the case that the type of multiple-solution problem makes a difference. Whereas students enjoy making different assumptions about missing information, constructing different solutions, and comparing their results, this enjoyment might not hold when they apply different mathematical procedures. Similar effects (low level of boredom for the first type of multiple-solution problem, but no difference in boredom for the second type of multiple-solution problem) were also found for students' boredom. Another explanation for no effects of the intervention on emotions might be that students in the multiple solution condition were not offered to chouse their favorite procedure during three of four lessons. More efforts are needed to clarify the role of multiple solutions for affective measures and more generally, with respect to the effects of teaching methods on students' affect.

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Self-efficacy and mathematics performance: Reciprocal relationships

Karin Elisabeth Sørlie Street[1], Lars-Erik Malmberg[2] and Gabriel J. Stylianides[3]

[1] Oxford University, Department of Education, Oxford, United Kingdom *and* Høgskulen på Vestlandet, campus Sogndal, Norway; karin.street@stx.ox.ac.uk

[2, 3] Oxford University, Department of Education, Oxford, United Kingdom;
[2]lars-erik.malmberg@education.ox.ac.uk [3]gabriel.stylianides@education.ox.ac.uk

Several studies have looked at either the effects of students' Self-Efficacy Expectations (SEE) on their mathematics performance results, or the effect of previous mathematics performances on students' SEE. Few studies have tested the theoretically proposed reciprocal relationship between mathematics SEE and performance in mathematics. Furthermore, previous studies have not included levels of difficulty, although this is an integral part of the definition of SEE. This study applied a new measure of SEE, which included both test taking facets and levels of perceived difficulty, to investigate their relationships with students' performance on a national test in mathematics. Two models gave very good fit to the data, and supported the reciprocal effects model. These models provided estimates of the relationships between students' test performance and their level and facet-specific SEE, respectively.

Keywords: Self-efficacy, performance, reciprocal effects, levels of difficulty, test taking facets.

Introduction

Self-efficacy expectations (SEE) are important because extensive research indicates they are related to student learning and performance (Zimmerman, 2000). Studies have demonstrated the effect of mathematics SEE on performance results (e.g. Pajares, 1996) and vice versa (e.g. Pampaka, Kleanthous, Hutcheson, & Wake, 2011), but the proposed reciprocal relationship between SEE and performance (Bandura, 1997) has received little attention. Investigating this relationship empirically is important in order to understand the relationship between SEE and performance in mathematics, and the process by which SEE may influence performance in mathematics and vice versa. Few studies have investigated reciprocal effects between SEE and mathematics performance with longitudinal data, and none that we know of in relation to national tests in Norway.

Bandura (1997) argued SEE vary according to three dimensions, but we know of no studies that have included SEE *level of difficulty* to investigate reciprocal effects with mathematics. As we argued in Street, Malmberg, & Stylianides (2017), including level of SEE is important as it is an integral part of the definition of SEE (Bandura, 1997). We aimed to address this research gap by testing a reciprocal effects model including students' level and facet-specific SEE, and their scores on a Norwegian national test. To this aim, we applied a recently developed measure of mathematics SEE that includes four facets of test taking as well as three levels of perceived difficulty. We included students' SEE responses and test scores as they progressed from grade 8 to grade 9.

Theoretical background

Self-efficacy expectations (SEE) are individuals' judgments about perceived capability to perform on future tasks (Bandura, 1997), for instance students' expectations they are able to carry out the various facets involved with taking a mathematics test. Examples are problem-solving skills such as solving

a certain number of problems or solving tasks of a certain challenge, or skills in self-regulation such as concentrating for a length of time or persevering through difficult problems.

SEE affect future performance through mediating processes, influencing students' behaviors and motivations. SEE may influence individuals' tendencies to approach learning tasks, their effort and persistence while engaged in such tasks, as well as their self-regulatory processes (Zimmerman, 2000). SEE have been found to predict mathematical problem-solving after controlling for factors such as cognitive ability, mathematics Grade Point Average, anxiety, and gender (Pajares, 1996). There is also empirical support for the influence of past performances on SEE (e.g. Pampaka et al., 2011). According to Bandura (1997), SEE are formed through four sources, where mastery experiences is the strongest source. Mastery experiences stem from individuals' appraisals of previous performance situations, for instance their experiences from a previous, similar test. Importantly, previous experiences do not influence SEE directly, but are interpreted and made sense of by the individual (Usher, 2009).

A mutually reinforcing pattern of influence between self-beliefs and academic achievement is supported for several types of self-beliefs (Valentine, Dubois, & Cooper, 2004). Studies have, for instance, investigated reciprocal effects between self-concept and achievement (Marsh, Trautwein, Lüdtke, Köller, & Baumert, 2013), however only few studies have investigated reciprocal effects between SEE and mathematics performance. Williams and Williams (2010) proposed a reciprocal effects model, which fit the data well for PISA results from 26 out of 33 countries. They argued reciprocal effects might be a fundamental psychological process, however the data from Norway was excluded from the analyses due to little variation in the grade-level variable. Hannula et al. (Hannula, Bofah, Tuohilampi, & Metsämuuronen, 2014) investigated the dominant direction of effect between mathematics performance and SEE, as students progressed from grade 3 to grade 9 in Finland. Their results supported a reciprocal effects model, where the effect from performance on SEE (.30) was somewhat stronger than the effect from SEE on performance (.26). Both measures were also relatively stable over time.

Measures of SEE have typically included SEE strength and SEE specificity (also called generality). While Bandura proposed a third dimension of SEE (1997), only few studies have included level of difficulty. Students' SEE may vary between different facets of mathematics, and in regards to different perceived levels of difficulty. In relation to the task 'naming uses for common objects' Locke et al. (Locke, Frederick, Lee, & Bobko, 1984) found that SEE for tasks of medium to high difficulty were most predictive of subsequent task performance.

Theoretical model and research questions

In line with theory and empirical findings discussed, we propose a theoretical model (see Figure 1), where SEE and mathematics performance are reciprocally related. We included Norwegian students' SEE responses and subsequent test scores to investigate this model. In the model, students' SEE in grade 8 and 9 are related to their performance in mathematics in the same year. Furthermore, students' mathematics performances in grade 8 predict their SEE in grade 9. Both SEE and mathematics performance are relatively stable (mathematics performance the most stable of the two), with the grade 8 constructs predicting scores in grade 9. We further propose level and facet-specific SEE have differential relationships with mathematics performance.

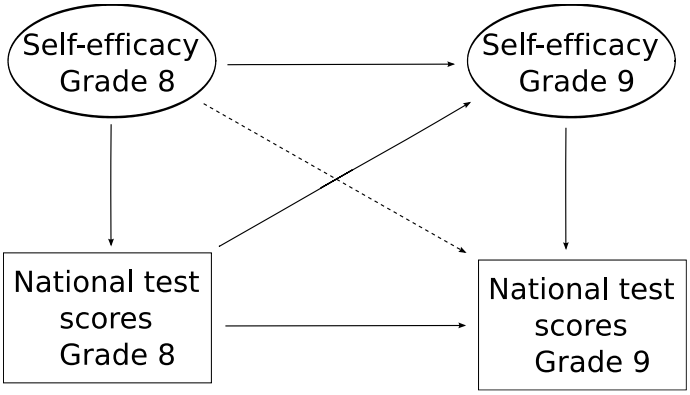


Figure 1: Theoretical model

RQ1: What is the relationship between SEE and performance on national tests in mathematics in grade 8, and SEE and performance on national tests in mathematics in grade 9?

RQ2: Do the relationships between SEE and performance on national test differ according to levels or facets of SEE?

Methodology

Participants

The participants were 95 students (44 female) in Norwegian secondary school who completed self-report questionnaires and took national tests in mathematics at the beginning of grades 8 and 9 (13 and 14 years old). The participants were part of a larger sample, selected for cross-sectional investigations (see Sørlie & Söderlund, 2015). Included in the study were schools where students had performed above and below what might be expected on national tests, considering measures of socio-economic-status. For a detailed explanation of this strategy see Langfelt (2015).

Measures

The Self-efficacy Gradations of Difficulty Questionnaire was applied to measure SEE at the two time points (see Street et al., 2017, for a detailed analysis). This is a recently developed multidimensional measure of mathematics SEE, that includes four test taking facets (facet-specific SEE) related to problem solving (complete a number of problems, solve tasks of a certain challenge) and self-regulation (concentrate, not give up), and three levels (easy, medium, and hard) of perceived difficulty (level of SEE). Each of the 14 items in the measure are related to one test taking facet and one level of difficulty within each facet (see Figure 2 for an example facet, ”concentrate”, with three levels of difficulty). For each item, students are asked to indicate their confidence (strength of SEE) on an 11-point scale from 0 “not at all certain” to 10 “completely certain”. The structural validity of this measure was tested (Street et al., 2017), and the resulting best-fit measurement model included three latent (unobserved) level constructs, with correlated uniquenesses (correlated error terms) specified for each of the four facets.

Some people can find it hard to concentrate when sitting tests. How certain are you that you will be able to concentrate during the national tests in mathematics?

During the national tests this year I will be able to concentrate:

	0 Not at all certain	1	2	3	4	5 Moderately certain	6	7	8	9	10 Completely certain
at least a little while											
at least half the time											
the whole time											

Figure 2: Example from the Self-Efficacy Gradations of Difficulty Questionnaire

The performance measure was raw scores from national tests in numeracy (Norwegian Directorate for Education, 2016). "Numeracy" is similar to what researchers generally refer to as mathematics. Norwegian students sit these tests at the start of the school year in grades 5, 8 and 9. Students in grades 8 and 9 sit the same test, which involves 58 problems, scored as either correct (1) or incorrect (0). We used unique identifiers to link the national test scores with the questionnaire responses.

Specification of empirical models

Our modeling choices were informed by the previously established factor structure of our measure, as well as our theoretical model. Street et al. (2017) found that the best-fit models were those that accounted for the multidimensional nature of SEE, through correlated latent constructs and correlated uniquenesses. In order to estimate the relationships between latent constructs in our models we used item parceling technique. Parceling involves aggregating (taking the sum or average) two or more items to manufacture an indicator of a construct (Little, Cunningham, Shahar, & Widaman, 2002, p. 152). While our measure of SEE is multidimensional, the factor structure has been tested and established in a previous study. Item parcels were formed to achieve "clean" latent constructs from each year, through aggregating items in such a way that the secondary loading was spread across parcels (Little et al., 2002). For example, the "easy" parcel contains all items related to the easy latent construct, across four facets. Similarly, the "concentrate" parcel includes all items related to the concentrate facet, across three levels of difficulty. As an example, the "concentrate" parcel in Figure 2 is created through summing the scores of the three items included.

Two reciprocal effects models were specified. Our hypothesized model (Model 1) includes national test scores and item parcels for SEE levels, representing the best-fit model from the previous study. The alternative model (Model 2) includes national test scores and item parcels for facet-specific SEE. In both models, SEE constructs in grade 8 are related to test scores in grade 9. Similarly, SEE constructs in grade 9 are related to test scores in grade 9. Correlations are specified rather than regression paths (see Street et al., 2017). SEE constructs in grade 8 are intercorrelated, as are SEE constructs in grade 9. Furthermore, corresponding SEE constructs are related across the two years (e.g. SEE easy in grade 8 predicts SEE easy in grade 9). Test scores in grade 8 predict test scores in grade 9. Finally, the reciprocal relationships tested are the path from test scores in grade 8 on SEE constructs in grade 9, and the paths from SEE constructs in grade 8 on test scores in grade 9.

Analyses

We analysed the data with structural equation modeling, using the maximum likelihood estimator in Mplus (version 7.31 for Mac: Muthén & Muthén, 2012). We used fit indices recommended by previous studies (e.g. Morin, Marsh, Nagengast, & Scalas, 2014; Schermelleh-Engel, Moosbrugger, & Müller, 2003), specifically we used the chi square (χ^2/df =<3 acceptable), the Root Mean Square Error of Approximation (RMSEA <.08 acceptable), the Standardized Root Mean Square Residual (SRMR<.10 acceptable), the Comparative Fit Index (CFI) and the Tucker-Lewis index (TLI) (CFI/TLI > .90 acceptable). To assess improvement in fit between models, we used the following cut-offs: ΔRMSEA (.015), ΔCFI (.010), ΔSRMR (.030).

Results

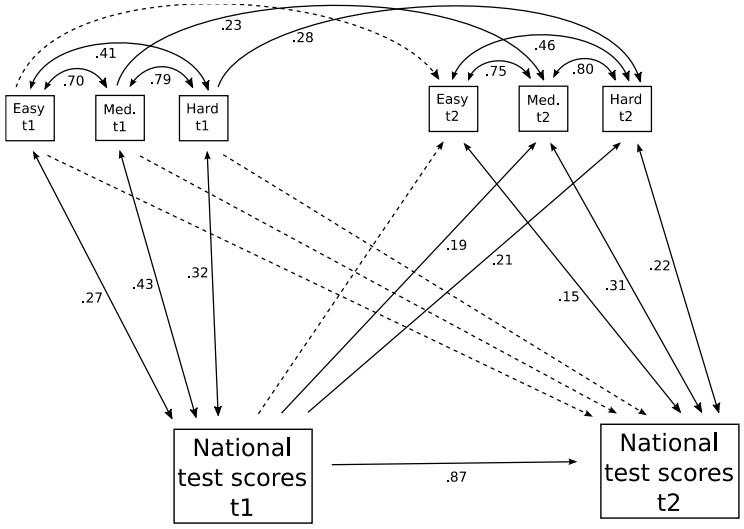
Item cross-correlations for items from the same year were consistent with previous results from the same measure. The correlation matrix indicates items are related to two types of latent constructs; levels (easy, medium, and hard) and facets (no. of problems, solve tasks, concentrate, and not give up). Autocorrelations between items in grade 8 and the same item in grade 9 demonstrate two tendencies. First, autocorrelations are significant for all items, except for two cases, both associated with the easy level (no. of problems_easy, and concentrate_easy). Second, the magnitudes of the associations are consistently weaker for items associated with the easy level, than the medium and hard levels. To illustrate this: the strongest autocorrelation of the easy level items is .21, while the weakest autocorrelations of the medium and hard level items is .26 and .29, respectively. Thus, the cross-correlations for each of the years indicate support for a multidimensional construct, while the autocorrelations indicate that students' scores on the medium and hard level items were more stable across time, than students' scores on the easy level items. The non-significant autocorrelations between the grade 8 and grade 9 easy level items might be related to a lack of variability in scores. Most students were highly confident in relation to the easy level items.

Fit indices for confirmatory factor analyses

	Model	χ^2	*p*	RMSEA	CFI	TLI	SRMR
1	Correlated levels model, reciprocal relationship with test scores	$\chi^2_{(6)}$= 4.397 χ^2/df=0.73	*p*=.623	.000	1.00	1.02	.048
2	Correlated facets model, reciprocal relationship with test scores	$\chi^2_{(12)}$= 10.890 χ^2/df=0.91	*p*=.538	.000	1.00	1.01	.061

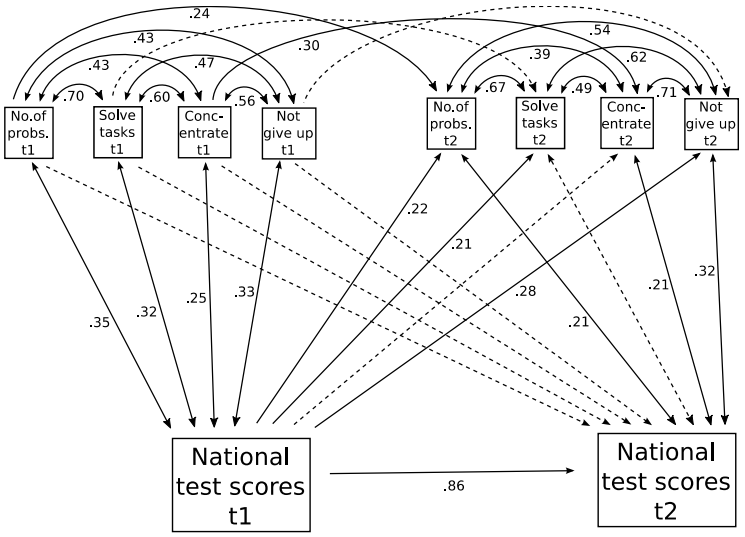
Table 1: Results from structural equation models

Results from the structural equation models are presented in Table 1. Both our proposed models resulted in excellent fit. The aim of the present study is not model comparison, but to test a reciprocal effects model, as well as to estimate the relationships between national test scores and level and facet-specific SEE, respectively. Accordingly, parameter estimates from both models are reported (see Figures 3 and 4).



Note: "Easy", "Med.", and "Hard" refer to easy, medium and hard levels of perceived difficulty. "t1" and "t2" refer to time 1 and time 2.
All estimates are standardized (STYX) estimates from Mplus (Muthén & Muthén 2012).
Paths are significant at the $p<.05$ level. Dotted paths are nonsignificant.

Figure 3: Structural equation model, Model 1 (correlated levels model)



Note: "No. of probs.", "Solve tasks", "Concentrate", and "Not give up" refer to facets of test taking. "t1" and "t2" refer to time 1 and time 2.
All estimates are standardized (STYX) estimates from Mplus (Muthén & Muthén 2012).
Paths are significant at the p<.05 level. Dotted paths are nonsignificant.

Figure 4: Structural equation model, Model 2 (correlated facets model)

As can be seen in Figure 3, results from Model 1 partially support a reciprocal effects model between levels of SEE and national test scores. Specifically, national test scores in grade 8 predict levels of SEE in grade 9 in relation to medium and hard, but not easy, tasks. The predicted relationship from SEE level constructs in grade 8 to national test scores in grade 9 is not supported. However, all three levels of SEE are associated with test scores the same year, including when previous test performance is controlled in grade 9. Two further relationships are nonsignificant: the path from SEE easy level in grade 8 to grade 9, and the path from national test scores in grade 8 to SEE easy level in grade 9.

Model 2 results (see Figure 4) are generally consistent with results for Model 1. Test scores in grade 8 predict three facets of SEE in grade 9 (not “concentrate”), while the paths from SEE facet constructs

in grade 8 to test scores in grade 9 are nonsignificant. However, facet-specific self-efficacy constructs are related to performance results in each year, including in grade 9 (not "solve tasks") when previous test scores are included in the model. Unlike the other facets, the magnitude of the relationship between "not give up" and test scores is very similar in grade 8 and 9. Similar to the easy level in Model 1, the relationships between "solve tasks" and "not give up" from grade 8 to grade 9 are not significant. Finally, we see that test scores are more stable over time than either level or facet-specific SEE.

Discussion

The current study lent empirical support to a reciprocal relationship between level and facet-specific SEE, and performance in mathematics (RQ1). It was demonstrated that students' SEE were associated with scores on a test in the immediate future, while national test scores predicted their SEE one year later. All SEE items in our study are explicitly linked with the immediately upcoming test, thus it is reasonable the SEE constructs are associated with performance results of the immediately upcoming tests, but not with the tests one year later. In two previous studies, measures of SEE and performance were collected over the course of a single day (Locke et al., 1984) or three years apart (Hannula et al., 2014). Findings from both these studies indicated the dominant effect was from performance to SEE. The type of test, and whether it is considered important by the students, might influence the relationship between performance experiences and SEE. National tests are quite "talked about" in school in Norway, and students in grades 8 and 9 are likely becoming increasingly aware of upcoming exams (at the end of year 10). This might have provided a particularly memorable context for their test experiences.

Furthermore (RQ2), differential relationships were found between national tests and easy, medium and hard levels of perceived difficulty. The associations with test scores were stronger for medium and hard levels of difficulty, which is in line with previous research (Locke et al., 1984). The relationships with test performance also varied between different facets of test taking, where "not give up" provided the strongest association.

In our sample students' SEE for medium and hard level tasks had both a stable (SEE regressed on SEE) and a dynamic (SEE regressed on test scores) component, while these estimates differed for the different SEE facets. For instance, in regard to 'concentrate', grade 9 SEE were predicted by grade 8 SEE, while in regard to 'not give up', grade 9 SEE were predicted by grade 8 test scores. This indicates students drew on their previous test experiences to a larger degree when formulating their SEE for perseverance, as compared to concentrating. This implies it is important to consider how to support student perseverance, particularly after adverse test experiences.

The current findings are limited in that we could not control for other factors that may have influenced students' SEE and test performance. Also, while the measure of SEE provided interesting information, partitioning the effects of test taking facets and levels of perceived difficulty, further replications are needed with a larger sample, and in different cultural contexts.

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Personal meaning and motivation when learning mathematics: A theoretical approach

Neruja Suriakumaran, Christoph Duchhardt and Maike Vollstedt

University of Bremen, Faculty 3, Mathematics Education, Bremen, Germany;
neruja.suriakumaran@uni-bremen.de, christoph.duchhardt@uni-bremen.de,
maike.vollstedt@math.uni-bremen.de

Personal meaning, understood as the personal relevance of an object or an action (Vollstedt, 2011), seems to be closely related to motivation. However, the structural relationships between personal meaning and motivation are unexplored yet. Two motivation theories, self-determination theory (Deci & Ryan, 2002) and expectancy-value theory (Wigfield & Eccles, 2000), are used to work out these relations. The focus of this paper lies on theoretical considerations.

Keywords: Personal meaning, motivation, self-determination theory, expectancy-value theory, theoretical considerations.

Introduction

Students are in the need for meaning when dealing with mathematics in an educational context (Vinner, 2007). As Kilpatrick, Hoyles and Skovsmose (2005) point out, the notion of meaning is blurred in the mathematics education community:

> When we consider the question of meaning with respect to mathematics education, the issue becomes even more complex, since philosophical and non-philosophical interpretations of meaning can become mixed. Thus, on the one hand, we may claim that an activity has meaning as part of the curriculum, while students might feel that the same activity is totally devoid of meaning. (Kilpatrick et al., 2005, p. 2)

This paper is interested in the perspective of the students and their individual attribution of personal relevance to deal with mathematics in an educational context. Vollstedt (2011) called this construct *personal meaning*. Theoretical considerations suggest a strong link between personal meaning and motivation drawing on the *basic needs theory* (*BNT*) and the *organismic integration theory* (*OIT*) of *self-determination theory* (*SDT*, Deci & Ryan, 2002) and *expectancy-value theory* (*EVT*, Wigfield & Eccles, 2000).

Personal meaning

With respect to the fuzziness of meaning, Howson (2005) suggests that

> one must distinguish between two different aspects of *meaning*, namely, those relating to relevance and personal significance (e.g., "What is the point of this for me?") and those referring to the objective sense intended (i.e., signification and referents). These two aspects are distinct and must be treated as such. (Howson, 2005, p. 18)

According to Howson (2005), one important interpretation of meaning is the personal one. Personally experienced meaning, again, has a wide notion of concepts: it can be understood as a personal goal, a value, an intention, a purpose, a reference, or a use that an object or an action may have for the individual (Vollstedt, 2011).

Personally experienced meaning depends on the individual and a certain context (see below). It has an endogenous character, i.e. it cannot be provided by the teacher but, on the contrary, must be constructed out of the learner's individual biography (Meyer, 2008). Regarding mathematics, the need for meaning cannot be fulfilled globally: for each mathematical learning content, personal meaning must be constantly interpreted and subjectively constructed (Fischer & Malle, 1985). Therefore, at the same time and in the same context, different students can give different meanings to the same mathematical content (Kilpatrick et al., 2005; Vollstedt, 2011).

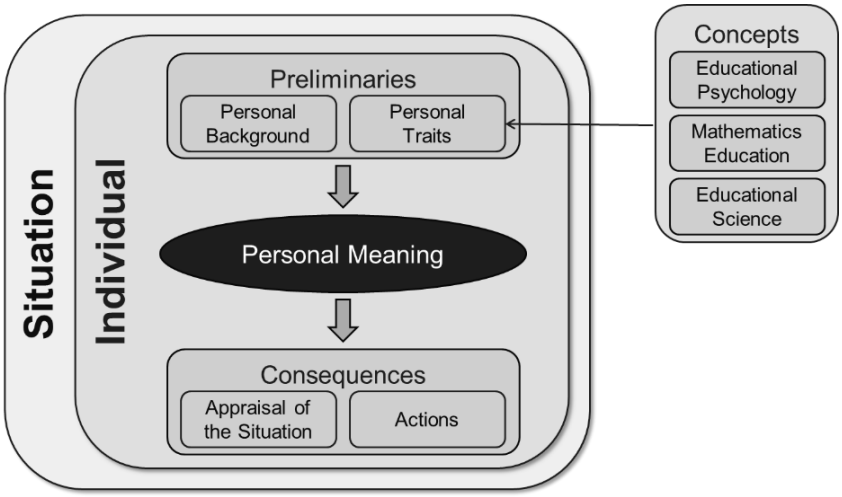

Figure 1: Theoretical framework of personal meaning (Vollstedt, 2011)

Vollstedt (2011) developed a model of personal meaning when learning mathematics and dealing with mathematical contents in a school context. In her theoretical framework (see Figure 1) she took the student's perspective, as the following two main preliminaries influence the construction of personal meaning: Firstly, the personal background of the student describes aspects which cannot be influenced by himself/herself like his/her socio-economic or migration background. Secondly, personal traits. i.e. aspects that concern the student's self, are relevant. They comprise concepts from various fields like educational psychology (self-concept, self-efficacy), education (developmental tasks), and mathematics education (beliefs). In addition to the individual preliminaries of a student, the situational context, i.e. context of the learning situation in terms of topic as well as classroom situation, is also a crucial factor for the construction of personal meaning. The theory of personal meaning developed by Vollstedt (2011) consists out of 17 different kinds of personal meaning. They were reconstructed based on interview data with students from lower secondary level from Germany and Hong Kong. These kinds vary between the duty to deal with mathematics because it is a school subject, the cognitive challenge that is contained in mathematical tasks, and the experience of relatedness among the fellow students. Note that the experience of the three basic psychological needs for *autonomy*, *competence*, and *relatedness* as described in the SDT of motivation (Deci & Ryan, 2002; cf. *Self-determination theory*) turned out to be meaningful for students. Accordingly, they were also given the status of kinds of personal meaning. The various kinds of personal meaning can be distinguished with regard to the intensity of the relatedness to mathematics and to the individual respectively, giving rise to seven superordinate types of personal meaning (see Figure 2).

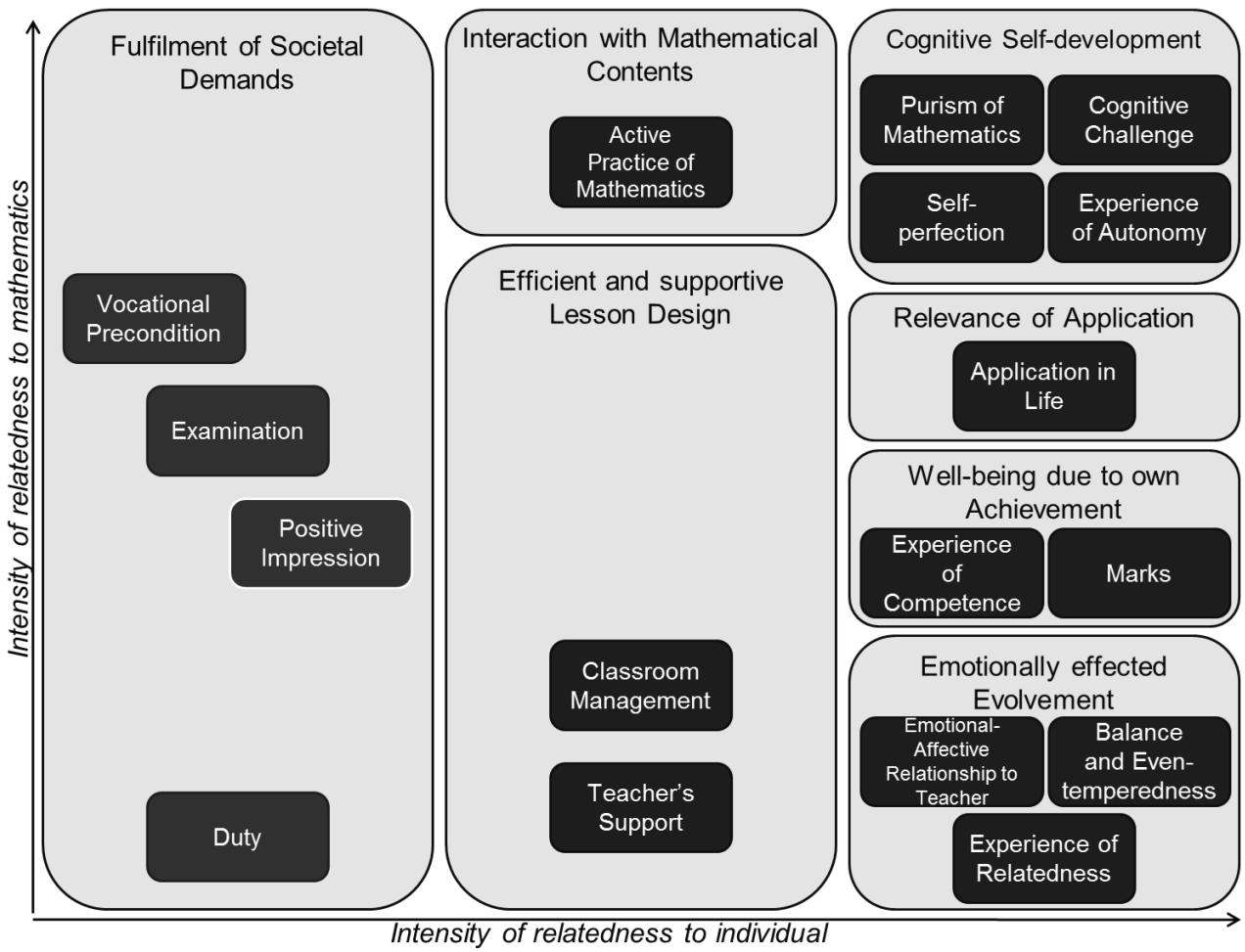

Figure 2: Model of personal meaning (Vollstedt, 2011)

A relation between the theories of personal meaning and motivation seems likely due to the obvious link via the SDT of motivation. Other links may additionally be assumed (see *Interplay between personal meaning and motivation* for further details). The structural connections between personal meaning and motivation, though, are yet unexplored.

Motivation

Self-determination theory

According to the SDT by Deci and Ryan (2002), learners have innate and constructive tendencies to develop an ever more elaborated coherent "sense of self" (Deci & Ryan, 2002, p. 5), i.e. individuals possess a tendency to promote growth or rather integration. They have the primary demand "to forge interconnections among aspects of their own psyches as well as with other individuals and groups in their social worlds". This general integrative tendency is called the organismic metatheory of SDT. Besides, SDT includes the dialectical tendency, which focusses on the interaction between the active, integrating human nature and social contexts that either nurture or impede human's effort to "integrate their experiences into a coherent sense of self" (Deci and Ryan, 2002, p. 27).

Those contextual elements can be defined by the basic psychological needs for competence, autonomy, and relatedness, which support or rather thwart motivation, performance, and well-being.

SDT embraces six sub-theories that all contain organismic and dialectic characteristics, two of which have a special significance for our study, namely the basic needs theory for competence, autonomy, and relatedness (BNT; Deci & Ryan, 2002, p. 22) and the organismic integration theory (OIT; Deci & Ryan, 2002, p. 14). These two sub-theories were formulated to clarify the interrelation between "motivation and goals to health and well-being" (Deci & Ryan, 2002, p. 10).

In BNT it is supposed that the basic needs are universal, i.e. they are valid across time, age, gender, situations, and culture. When they are satisfied, they support well-being, however when they are impeded, they might interfere with psychological health.

The OIT focusses on "internalization and integration of values and regulations (amotivated, external, introjected, identified, integrated, and intrinsic)" (Deci & Ryan, 2002, p. 14). Thereby, it defines the development and dynamics of extrinsic motivation in more detail. This process is characterized by "the degree to which individuals' [sic] experience autonomy while engaging in extrinsically motivated behaviors" (Deci & Ryan, 2002, p. 9). This taxonomy of regulation is neither a developmental continuum by itself, nor do human beings have to proceed through each level of internalization. In fact, it is possible for humans to take in a regulation at any level, when the relevant prior experience and the immediate individual climate encourage the interpersonal basic needs.

Expectancy-value theory

A second theory of motivation that stems from different theoretical roots is the EVT (Feather, 1982; Wigfield, Tonks, & Klauda, 2016). In it, motivation is described as a consequence of an interaction of *expectancy* and *value*. Wigfield and Eccles (2000) conceptualize motivation following EVT in a school context. They argue that "individuals' choice, persistence, and performance can be explained by their beliefs about how well they will do on the activity and the extent to which they value the activity" (Wigfield & Eccles, 2000, p. 68). Expectancy address the perceived likeliness of achieving a set goal or being successful on a task (self-efficacy). Value represents the extent to which a goal or an activity is desirable (Eccles et al., 1983; Wigfield & Eccles, 1992).

Wigfield and Eccles (2000) describe four subjective task values: *attainment value*, *intrinsic value*, *utility value*, and *cost*. These are also referred to as the components of achievement value. Attainment value is described as the personal importance of doing well on a task, for example on a mathematical exercise. Intrinsic value is characterized by the sense of pleasure in doing that task. Utility value defines how a task suits one's future plans or goals, such as making an effort during the mathematics lesson in order to be well prepared for an exam. Cost concerns how the decision of putting effort into an activity (e.g. doing mathematics homework) restrains opportunities for other activities (e.g. watching TV). The subjective task values serve the estimation of effort, the likelihood of task achievement, and emotional cost (Wigfield & Eccles, 2000).

Interplay between personal meaning and motivation

The conceptualizations of motivation introduced above contain various links to personal meaning. In a first approach, several networking strategies *comparing*, *combining*, *coordinating*, and *synthesizing* (Prediger & Bikner-Ahsbahs, 2014) were used to connect the different theoretical perspectives and construct an elaborated theoretical framework of the interrelation between personal meaning and motivation. This process is described below.

Common similarities and differences between parts of theoretical approaches can be identified through the networking strategy of comparing (Prediger & Bikner-Ahsbahs, 2014). Comparing the three theoretical approaches shows that in general, the theories of personal meaning, SDT, and EVT all include organismic and dialectical components, i.e. they consider the learner's biography in detail and the constant interaction with the learner's social environment (Bruner, 1991; Deci & Ryan, 2002; Vollstedt, 2011; Wigfield & Eccles, 2000). This is an essential factor and forms the basis for further elaboration of the concrete interaction between these three theories.

The networking strategy of combining makes it then possible to combine theoretical approaches even from different origin. As EVT-values and intrinsic and extrinsic constructs of SDT are examined from different theoretical perspectives they, thus, have different bases (Wigfield & Eccles, 2000; Wigfield, Tonks, & Klauda, 2016). Nevertheless, close relations could be extracted by combining the theoretical constructs of SDT and EVT: The intrinsic value of EVT is linked to the construct of intrinsic motivation as described in SDT. It refers to behaviors performed out of one's own interest, enjoyment, and the pleasure inherent in these activities (Ryan & Deci, 2002). Utility value describes more extrinsic motives to put effort in a mathematical task, such as doing a task to attain certain outcomes. Accordingly, utility value can be connected to extrinsic motivation (Wigfield & Eccles, 2000). Hence, extrinsic rewards may also help to anticipate individuals' own efforts (Spence & Helmreich, 1983). In addition, SDT points out that the regulation of motivation is important to pursue a certain goal or value. Thus, it differentiates between qualitatively different reasons for action, arguing that different types of motivation will lead to very different outcomes (Ryan & Connell, 1989).

Through the networking strategy of coordinating it is possible to clarify empirical evidence by constructing a conceptual framework grounded in different theoretical ideas. Hence, the three theories were interwoven and synthesized to link equally solid theories in such a way that a new unit of theory arises into, for example, an elaborated new theoretical approach (see Figure 3 below). To begin with, personal meaning and SDT are linked in three ways (cf. also the section *Personal meaning* above): Firstly, the three basic psychological needs for competence, autonomy, and relatedness as described in BNT are part of Vollstedt's (2011) theoretical background for the construction of personal meaning (see Figure 3 below). Secondly, their experience turned out to be meaningful for students so that there are three kinds of personal meaning closely related to the three basic psychological needs (see Figure 2 above). Thus, these two aspects directly link the theory of personal meaning to SDT. Thirdly, there is an indirect link. One of the two overall-dimensions of the model of personal meaning, namely the intensity of the relatedness to the individual, describes the degree of one's subjective involvement in the action or the content respectively.
Vollstedt's results (2011) suggest that the intensity of the relatedness to the individual is possibly interrelated with the types of regulation described in OIT. This results from the fact that the intensity of the relatedness to the individual focuses on the personal involvement of the individual with respect

to the action or object in focus. Hence, this establishes a link to self-determined behavior and internalization (Vollstedt, 2011).

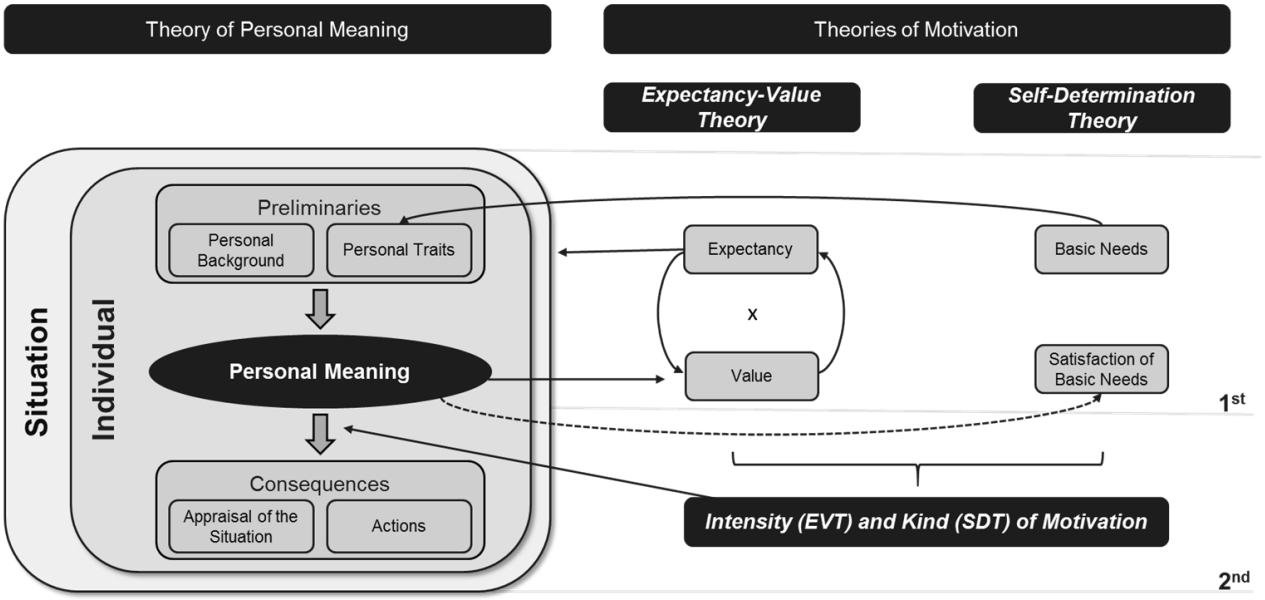

Figure 3: The role of personal meaning in the generation of motivation according to EVT

With relation to EVT, we also suppose a close link to the theory of personal meaning (see Figure 3 above). We assume that the expectancy as described in EVT is part of the individual's personal traits, i.e. they are contained in the crucial preliminaries for the construction of personal meaning (cf. the section *Personal meaning*). Besides, the subjective values of EVT are embodied in different kinds of personal meaning (cf. the section *Expectancy-value theory*). For instance, the particular nature of intrinsic value is inherent in those kinds of personal meaning which refer mostly to the relatedness to mathematics or the learning of mathematics (e.g. "Purism of mathematics", see Figure 2 above). Another assumption that can be made refers to the relation between utility value and the kinds of personal meaning which have instrumental or functional character (e.g. "Vocational Precondition"). Furthermore, attainment value may relate to those kinds of personal meaning which refer to the knowledge of mathematics being important for one's own identity (e.g. "Self-perfection").

These considerations suggest that motivation may be understood as a result from the interaction between expectancy being characterized by the preliminaries of an individual, and the values being embodied by the different kinds of personal meaning. The resulting motivation influences the consequence that results from the construction of personal meaning. Thus, depending on the kind of personal meaning and its related motivation, an action will follow that may but does not have to do with mathematics (e.g. doing homework instead of playing football – or vice versa). Hence, the inclusion of EVT may provide additional insight into the interplay between personal meaning and motivation.

When looking at the interplay of personal meaning and motivation as a process, with reference to EVT we suggest to think of personal meaning as being constructed chronologically before motivation.

In our understanding, personal meaning is the energizing factor (1st in Figure 3 above), which is significant for the students' motivation (2nd in Figure 3 above). We even assume that personal meaning is necessarily required for motivation, i.e. that the individual must think that something is

meaningful for him/her and, thus, is motivated to engage in an action that supports his/her goals and values. This suggestion gives personal meaning the status of a key factor for the theory of motivation in general (see Figure 3).

Conclusion and further perspectives

This paper provides the theoretical background to examine the relationship between personal meaning and motivation when learning mathematics. Hence, personal meaning is linked with motivation through the two motivation theories of SDT and EVT. The results of Vollstedt's (2011) study may suppose connections between personal meaning and SDT, i.e. BNT and OIT (Deci & Ryan, 2002). Three of the subjective values of EVT (attainment value, intrinsic value, and utility value) may be embodied by certain kinds of personal meaning. As cost has a negative connotation, it is not related to personal meaning denominating personal relevance of an object or action.

To conclude, we assume an interrelation between personal meaning and motivation as has been elaborated above. The model sketched above will be elaborated further in an empirical study.

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Mathematical memory revisited: Mathematical problem solving by high-achieving students

Attila Szabo

Stockholm University, Department of Mathematics and Science Education, Sweden
attila.szabo@stockholm.se

The present study deals with the role of the mathematical memory in problem solving. To examine that, two problem-solving activities of high achieving students from secondary school were observed one year apart - the proposed tasks were non-routine for the students, but could be solved with similar methods. The study shows that even if not recalling the previously solved task, the participants' individual ways of approaching both tasks were identical. Moreover, the study displays that the participants used their mathematical memory mainly at the initial phase and during a small fragment of the problem-solving process, and indicates that students who apply algebraic methods are more successful than those who use numerical approaches.

Keywords: High-achievers, mathematical memory, mathematical abilities, problem solving.

Introduction and background

Despite a growing emphasis on the identification and teaching of mathematically able pupils, much remains unknown about the abilities they display when solving mathematical problems. For reasons of social justice and equality, research has typically focused on low achieving pupils (Swanson & Jerman, 2006) while relatively few studies have observed the abilities of the gifted and high-achievers (e.g. Vilkomir & O´Donoghue, 2009) or addressed those pupils' memory functions during mathematical activities (Leikin, Paz-Baruch, & Leikin, 2013; Raghubar, Barnes, & Hecht, 2010). In particular, just a few studies (e.g. Krutetskii, 1976; Szabo & Andrews, 2017) examined the role of the *mathematical memory* in gifted and talented students' problem-solving activities.

Mathematical abilities

Our innate ability to estimate quantities, known as the approximate number system, is extremely limited (Dehaene, 1997), but an active contact with the subject may, under favourable conditions, generate mathematical abilities that are both complex and structured (Krutetskii, 1976). The nature of mathematical abilities has engaged researchers for more than 120 years; already at the end of the 19th century, Calkins (1894) presented, based on observations of Harvard students, significant information about the way mathematicians approached the subject. However, the research on mathematical abilities – mainly because of the dominance of psychometric approaches, and thereby considering abilities as innate and static – has not delivered widely accepted results during the first half of the 20th century (Vilkomir & O´Donoghue, 2009). Therefore, of importance for the present paper is the research of Krutetskii (1976), whose longitudinal observational study analysed the problem-solving activities of around 200 pupils. He concluded that able pupils' mathematical ability, while complex and dynamic, typically comprises four broad abilities. These are

- the ability to *obtain and formalise mathematical information* (e.g. formalised perception of mathematic material),

- the ability to *process mathematical information* (e.g. logical thought, flexible mental processes, clear and simple solutions, generalized mathematical relations),
- the ability to *retain mathematical information*, that is, *mathematical memory* (a generalized memory for mathematical relationships) and
- a *general synthetic component* (mathematical cast of mind) (Krutetskii, 1976, pp. 350–351).

However, while these abilities have frequently been associated with mathematical giftedness, Krutetskii (1976, pp. 67–70) argues they can also be displayed properly by high-achievers.

Mathematical memory

It is largely agreed that memory plays an essential role both in the learning of mathematics and in mathematical problem solving (e.g. Leikin et al., 2013; Raghubar et al., 2010). Thus, what seems to be crucial "is not whether memory plays a role in understanding mathematics but what it is that is remembered and how it is remembered by those who understand it" (Byers & Erlwanger, 1985, p. 261). Calkins' (1894) early study showed that the memories of mathematicians are more concrete than verbal, that mathematics students do not memorise facts more easily than other students and that, when performing mathematics, there is no difference between men and women. Some decades later, Katona (1940) found that rational methods are easier to memorise that random digits, while Bruner (1962) showed that simple interrelated representations are effective when recalling detailed knowledge. However, Krutetskii (1976) distinguished *mathematical memory* from the *mechanical recalling* of numbers or algorithms, by stressing that it is a memory consisting of generalized methods for problem solving. Hence, the mathematical memory does not retain "all of the mathematical information that enters it, but primarily that which is 'refined' of concrete data and which represents generalized and curtailed structures" (Krutetskii, 1976, p. 300). Moreover, he found that able students usually retain the contextual facts of a problem only during problem-solving and forgot it instantly afterwards, but remember several months later the methods they applied. Conversely, low-achievers often remember contextual facts but rarely the problem-solving methods (Krutetskii, 1976). Cognitive psychology studies (e.g. Sternberg & Sternberg, 2012) indicate important distinctions between different memory systems; that is, long term memory can be divided into *implicit* and *explicit* memory, based on the type of the stored information. In a mathematical context, the implicit memory contains automatized procedures and algorithms, while explicit memory retains information about experiences and facts which can be consciously recalled and explained, such as schemas for problem-solving. Thus, according to the cognitive model, we may assume that mathematical memory, as defined by Krutetskii, is explicit. Besides, it is a memory formed at later stages (e.g. Davis, Hill, & Smith, 2000) based on the ability to generalize mathematical material, because at young able pupils "the relevant and the irrelevant, the necessary and the unnecessary are retained side by side in their memories" (Krutetskii, 1976, p. 339).

The study

The present study had two aims, both based on Krutetskii's (1976) definitions of mathematical ability. The first was to identify the structure of mathematical abilities when high-achieving students solve non-routine but structurally similar problems. The second was to examine the role of mathematical memory during problem-solving activity.

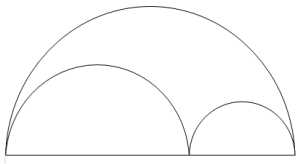
Participants

Because young children and low-achievers rarely exhibit mathematical memory (Krutetskii, 1976), participants were 16-17 years old volunteers from an advanced mathematics programme in Swedish upper secondary school who had achieved the highest grade in the Swedish national test. Prior to data collection study, to familiarize students with the study, I spent 30 hours, over a period of four months, as a participant observer in their mathematics classroom. During this period, they came to trust me as an observer of their problem-solving activities. At the end of this process, after consulting their teacher, 6 students, 3 boys and 3 girls, were invited to participate.

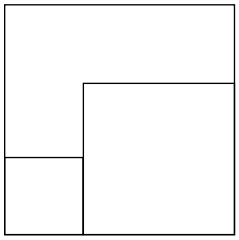
Tasks

The theoretical background indicates that an appropriate way to identify the distinct structure of the mathematical ability is to analyse the problem-solving activities of the individuals (e.g. Krutetskii, 1976). Moreover, the structure of a mathematical problem reveals the mathematical thinking which is required to solve it, because problem solving "is an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine" (Cai & Lester, 2005, p. 221). However, able students typically forget the context of a problem shortly after solving it, but, as an impact of their mathematical memory, they are several months later able to recall the methods applied to solve it. Thus, in order to complete the aims of the study, the participants solved two problems approximately one year apart. At the first observation, in order to avoid as far as possible the influence of previous experiences, the main criterion was to select a challenging non-routine task, Task 1 (T1). When selecting Task 2 (T2) – in order to emphasize the role of the mathematical memory – the main criterion was to propose a task which was non-routine, but could be solved by methods similar to those used previously.

Task 1: In a semicircle we draw two additional semicircles, according to the figure. Is the length of the large semicircle longer, shorter or equal to the sum of the lengths of the two smaller semicircles? Justify your answer.



Task 2: In a square we draw two arbitrary contiguous squares, according to the figure. Is the perimeter of the large square longer, shorter or equal to the sum of the perimeters of the two smaller squares? Answer the question without measuring the figure. Justify your answer.



Both tasks underwent substantial a-priori testing with corresponding groups of high-achievers, confirming that they were well-suited for the study and for the mathematical knowledge of the participants. This test confirmed that the students solved the proposed tasks with similar methods, that is, by applying the formulae for perimeters of circles and squares.

Observations and interviews

To avoid confounding factors during classroom interaction, which may affect pupils' thought process (Norris, 2002), every participant was observed individually and, to avoid stress, given unlimited time to solve each task. T1 was solved and approximately one year later T2 was solved. In

order to avoid participants' memories being activated mostly because of recalling the circumstances for the first observation as an unusual element in their daily activities – that is, not because of recalling the previously solved task – I continued to interact with them during their mathematics classes between the two observations. The students were invited to solve the tasks in a think-aloud manner and encouraged to describe every step in the process. To minimise participants' influence on each other, the tasks were solved during single days. The observations took place in a private room at their school and, when needed, supplementary questions were posed in order to facilitate the process. If a student neither wrote nor spoke for a while, the following questions were posed: What is bothering you? Why do you do that? What do you want to do and why? What are you thinking about? Pupils generally are not used to verbalise their problem-solving process (Ginsburg, 1981), thus, in order to avoid the risk that essential parts of their cognitive activities would not be communicated, every observation was followed by a *reflective interview*. The purpose of the interviews was to display the hidden cognitive processes at problem-solving and to evaluate the levels of competence in those processes (Ginsburg, 1981). Each observation was recorded using a technology that digitises both speech and handwritten notes; the audio recordings were transcribed verbatim. Although they were given unlimited time, no participant needed more than 14 minutes to complete a single task.

Data analysis

The piloting of the tasks on corresponding groups of high-achievers indicated that the *general synthetic component* – a typical ability of gifted students (Krutetskii, 1976, p. 351) – was unlikely to be observed during problem-solving; consequently, this ability was excluded from the analysis. The analytical framework for this study contained the following abilities from Krutetskii's framework: *obtaining* and *formalizing* mathematical information (O), *processing* mathematical information (P), *generalizing* mathematical relations and operations (G), and *mathematical memory* (M).

The digital recordings resulted in an exact linear reproduction of the students' actions, which was especially beneficial when performing *qualitative content analysis* of the material, inspired by van Leeuwen (2005). The participants´ actions were analysed by identifying, coding and categorising the basic patterns in the empirical content. This method highlighted the abilities that were directly expressed in the empirical material; each episode lasting at least one second in written solutions and verbal utterances was scrutinised for the presence of the focused abilities. Next, the data from observations were combined with data from the interviews. I exemplify this with data from Linda, who, when solving T2, didn't say or wrote anything during the initial 62 seconds, before stating:

Linda: I would like to write down, start with writing a… some nice little estimations…

After this episode she drew three squares with sides *a*, *b* and *c*, and wrote "$a + b = c$". Thus, based on the observation, the presence of O was certain, but it was not possible to decide if other abilities were also present in the actual episode. Yet, the following sentences from the reflective interview proved that she recalled another task which could be solved with similar methods:

Linda: I got blocked until I remember similar tasks, because it's a lot more difficult to solve this kind of tasks if one doesn't have a determined way to approach it… I believe I will bring up the same task as last time, with triangles and squares.

The utterances “a determined way to approach it” and “the same task as last time” indicate that Linda recalled a different task and its methods, thereby validating the presence of both O and M in the actual episode. In this way, the analysis revealed both the structure and the sequential order of the focused abilities, that is, every ability which occurred during the 12 problem-solving activities was displayed in a matrix. However, as exemplified above, some abilities (e.g. O and M) occurred closely interrelated during certain episodes and were extremely hard to differentiate.

Results

When asked, each participant confirmed that both tasks were non-routine, this being a prerequisite for the study. The analysis concluded in a matrix, with every episode of the process related to the focused abilities. As mentioned, certain abilities were closely interrelated during some episodes. As displayed (Table 1), M is present – solitary or interrelated – at 16% during the first and at 10.5% during the second observation. The most manifested ability is P, which increased from 53% to 67% a year later, while O, the second most exposed ability, decreased from 47% to 31.5%.

	O	**O & P**	**O & M**	**P**	**G**	**M**
Task 1	31%	4%	12%	49%	0%	4%
Task 2	20%	1.5%	10%	65.5%	2%	0.5%

Table 1: Average time for the focused mathematical abilities in the problem-solving process

According to the a-priori testing of the tasks, G could be revealed when numerical results – that is, solutions for particular cases – were developed into general, algebraic solutions. Thus, every student who offered purely numerical results – namely Erin, Sebastian and Larry – was encouraged to consider general solutions. Yet, when solving T1 and asked if their numerical results apply also for arbitrary semicircles, none of them could generalize (G) their findings:

Erin: I don’t know how I should prove this … if I have to do some general method.

Sebastian: I don’t know if I shall demonstrate that it should be the same thing there, for every measure. But now in my head it sounds like that it should be so.

Larry: Yes, I suppose, but I don’t know how to confirm it, it only feels that way.

Thus, the increase of G from 0% to 2% (Table 1) occurred because during the reflective interview connected to T2, when offered additional opportunities to reflect over the patterns in her numerical results, Erin performed a successful generalization of the obtained solutions, and stated:

Erin: I’ve never made a general solution like this ... But it was fun ... Especially when it concluded in something.

When concerning the efficiency of the applied methods, the analysis shows that Earl, Linda and Heather solved both tasks properly by applying general, algebraic methods. Conversely, purely numerical approaches didn’t lead to fully acceptable results. The most efficient solutions were offered by Linda, who applied the same algebraic model (and its identical steps) at both tasks.

The role of mathematical memory

The recalling of the applied methods several months after solving a problem is a typical display of mathematical memory (Krutetskii, 1976). Thus, another main criterion for the study was that both tasks could be solved with similar methods. However, only Earl and Larry associated T2 to T1:

Earl: We got a very similar task last year, when we had the circle and that semicircle.

Larry: We did a pretty similar task last time, when it was something like this, something with the radius or diameter on them.

Earl and Larry applied identical methods at the individual level when approaching both tasks. That is, Earl solved both tasks by using the same algebraic method, while Larry approached both tasks with the same numerical method. However, Earl's algebraic method gave accurate solutions while Larry couldn't solve any task properly. The other four students said that they didn't associate T2 to T1. But even though not recalling T1, they approached both tasks in identical ways at the individual level. For example, when Linda solved T2, despite stating that she didn't think at all of T1, she applied the same general method as a year before:

Linda: I will bring up the same task as last time, with triangles and squares. It is a bit the same thing ... I connect very often geometrical tasks to that. I have written that solution many times and I can see every step in the process in front of me.

As seen above, Linda refers to a generalized method which she associates to a geometrical task – about finding the side of a square drawn in a right triangle – which differs considerably from the proposed tasks. Yet, influenced by her mathematical memory (Krutetskii, 1976, p. 300) she states that "It is a bit the same thing" and applies the same method when solving both T1 and T2.

Heather as well used identical algebraic approaches for both tasks a year apart:

Heather (T1): I needed a common variable. Otherwise it will be difficult to calculate.

Heather (T2): I needed some relation among these sides in that and the large square's sides. Otherwise it will be difficult.

Also the individual approaches of Erin and Sebastian were respectively identical; Erin approached both tasks by *reasoning*, testing *numerical values* and applying *particular solutions*, while Sebastian *reasoned* carefully before requesting the use of *numerical values* at both occasions. Thus, every participant approached both tasks identically at the individual level. The analysis also shows that M is displayed mainly at the beginning of the process, for recalling mathematical relations and problem-solving methods; moreover, none of participants modified the initially selected methods.

Discussion

One of the aims of this study was to display the role of the mathematical memory (M) when high-achieving students solve non-routine tasks, which can be solved with similar methods. Despite its small proportion, M seems to play a pivotal role in problem-solving because the participants selected their methods at the early stages of the process and the methods were not changed later. Thus, by confirming earlier results (e.g. Szabo & Andrews, 2017), it seems that the choice of methods is directly influenced by M and it is critical for the success of the problem-solving.

However, unexpectedly, only two of the six participants recalled the solution process to the earlier task, contradicting Krutetskii's (1976) finding that able students recall the process but not the context of earlier problems. But even when not recalling T1, every participant approached both tasks in the same individual way. For example, Linda's method, connected to a square in a triangle and apparently very different from what is predicted, is a general approach that she uses for non-routine geometrical tasks. And even though the individual approaches of Erin, Larry and Sebastian were not successful when solving T1, they were repeated a year later. Thus, it seems that the participants rely on methods which appear to be inflexible and applied regardless of their success. The general structure of the participants' mathematical abilities indicates that O and M decrease while P increases at the second observation. Hence, it is not unreasonable to assume that the displayed stability of the individual approaches made O and M more efficient at T2, and therefore students had a larger focus on P. And even if none of the students could generalize numerical results at the first observation, Erin generalized her results during the interview after T2. Thus, when additional opportunities were offered, by evolving the patterns in her numerical results, which may be interpreted as a form of convergent thinking (Tan & Sriraman, 2017), Erin could improve the quality of her problem solving. These findings may suggest that some participants are unlikely to have experienced teaching focused on methods of generalization, because the individual structure of the mathematical ability depends on received instructions (Krutetskii, 1976). However, by confirming earlier studies (e.g. Krutetskii, 1976), the methods of this study were not able to differentiate M from O during episodes when students did not say or write anything; thus, a better investigation of the mathematical memory requires further studies. In addition, the study shows that mathematical memory has a key role during the early stages of problem-solving (e.g. Szabo & Andrews, 2017) and that individual problem-solving methods seem to be very stable and apparently independent of their efficiency when high-achievers solve non-routine tasks. Finally, the study indicates that, if given additionally opportunities to reflect over their numerical solutions, some students might be able to display their ability to generalize mathematical relations and operations.

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The mathematical values in the Turkish Middle School Mathematics Applications Course Curriculum

Gülçin Tan-Şişman[1] and Büşra Kirez[1, 2]

[1,] Hacettepe University, Faculty of Education, Ankara, Turkey

gulcintans@gmail.com; [2]ersozbusra@gmail.com

The present study aimed at investigating the mathematical values embedded in the Turkish Middle School Mathematics Applications Course Curriculum (MACC). For this purpose, MACC was carefully analyzed by considering main components of the curriculum. The findings indicated that all of the mathematical values, namely objectism, rationalism, control, progress, openness and mystery, were embedded in MACC. Although the balance among the embedded mathematical values in MACC was not equal, the findings clearly proved that the mathematical values were taken into consideration by curriculum developers, and thus it might be expected to raise awareness among mathematics teachers, students as well as textbook writers about the values in mathematics education.

Keywords: Mathematical values, mathematics applications course, curriculum analysis.

Background of the study

In almost all field of human endeavor, astounding developments and advancements are indebted to mathematics to a great extent. Among the other disciplines, "… mathematics was considered the Queen of the sciences." (Gregorian, 2009, p. ii). Mathematical literacy has always been at the heart of nations' capacity for economic growth and social welfare. In other words, there is no doubt that mathematics is not only vital to economic prosperity; but also a fundamental skill for life. Thus, it is essential for all societies to provide learning opportunities in which students will possess and use the understanding of mathematics purposively and interactively. Although mathematics is considered as one of the crucial subjects of school, it is a well-known fact that "…many students leave school with negative attitudes towards mathematics; some dislike the subject, others feel inadequate about it, still others feel it is irrelevant in their lives. This is an unacceptable outcome of school mathematics." (Education Department of Western Australia, 1998, p.9). In many classrooms, mathematics has delivered in a way that follows a textbook under the guidance of a teacher with little emphasis on affective side of mathematics. Thus, most of the students perceive mathematics as a non-creative, mechanical, value-free and teacher- or textbook writer-invented subject (Diamond, 2001). Seah, Andersson, Bishop, and Clarkson (2016) argued that such negative perceptions held by many students are not due to the nature of mathematics itself. It is most likely as a result of developing and implementing a mathematics curriculum which is full of concepts, skills, and procedures but not explicitly include the values of mathematics and the values of mathematics education. Indeed, values are powerful tools for promoting cognitive and affective development in mathematics education since they lead teachers' and students' interests, thoughts, decisions, preferences and behaviors about mathematics (Bishop, 2008; Corrigan, Gunstone, Bishop, & Clarke, 2004). The values expressed by teacher either intentionally or unintentionally in the context of the mathematics classroom may root in his/her personality, pedagogical approach and instructional materials preferred to use, etc. However, the important point is here that "what kinds

of values are embedded in the intended mathematics curriculum?" Because as a teacher, it is his or her responsibility to implement the curriculum as intended. Therefore, analysis of curriculum regarding mathematical values will shed lights on what kind of values that a teacher is expected to convey into instruction as well as what kind of an image of mathematics is going to be presented for students.

Affective issues in mathematics learning and teaching have been a prolonged and persistent interest among researchers. McLeod (1992), attributed to as the pioneer in work on the affective dimension of mathematics education (Gil, Blanco, & Guerrero, 2006), identified three main constructs of affect as beliefs, attitudes and emotions "...representing increased levels of affective involvement, decreased levels of cognitive involvement…" (p. 579). The scope of the mathematics-related affect has broadened with the addition of a fourth construct – values (sometimes including morals /ethics) – by such leading works of Bishop (2001); De Bellis and Goldin (2006); Leder and Grootenboer (2005). Considering the studies on the affective constructs such as beliefs, motivation, anxiety and attitudes, research on values in mathematics education is still insufficient (Zan, Brown, Evans, & Hannula, 2006). Leder and Grootenboer (2005) stated that the field of affective domain has been dominated by the studies on beliefs and attitudes, yet there has been a few number of study concerned with values. Although different conceptualizations of values in mathematics education have been described in the literature, the Bishop's (1988) classification of the mathematical values provides a widely-used framework while analyzing mathematical values (Hannula, 2012; Seah, 1999). Besides, there is no doubt that mathematics is the sum of the human activities, and thus the Bishop's framework (1988) not only gives room for discussing the values from the socio-cultural perspective but also corresponds to White's (1959) three components of culture – ideology, sentiment, sociology. In this respect, the present study is primarily concerned with "mathematical values" and thus the Bishop's framework was chosen to analyze the mathematical values. According to Bishop (1988), Rationalism-Objectism (the ideological component of mathematical values), Control-Progress (the sentimental component of mathematical values), and Openness/Mystery (the sociological component of mathematical values) are the mathematical values grouped as three pairs of complementary values. Table 1 summarizes each pair of mathematical values.

During the past decade, there have been the bulk of theoretical and research-based arguments on values in mathematics education (e.g. Values in mathematics and science education by Bishop, 2008; Identification of a learner's value orientations in mathematics learning by Seah, Zhang, Barkatsas, Law, & Leu, 2014), yet the number of the studies on mathematical values embedded in math curricula is still very limited. In this respect, the present study aimed to is to explore the mathematical values embedded in the Turkish Mathematics Applications Course Curriculum (MACC) by addressing the following research questions: (1) What are the mathematical values embedded in the MACC? (2) Where are the mathematical values located in the components of the MACC?

Three Pairs of Mathematical Values	
Rationalism	*Objectism*
Focuses on the development of students' mathematical reasoning through discussions, explanations, evaluating their way of solutions etc. (Bishop, 1988, p. 62).	Focuses on concretizing abstract ideas through the use of symbols and objects, promoting to use different kind of representations (Corrigan, et al., 2004)
Control	*Progress*
Focuses on control ensured by the nature of mathematics for the problems related to both natural phenomenon and social environment through application of mathematical knowledge and favors stability (Seah & Bishop, 2000).	Puts emphasis on change and progress in society by means of mathematical knowledge (Seah & Bishop, 2000).
Openness	*Mystery*
Focuses on transparency/verification aspects of mathematical ideas and conclusions through proofs (Corrigan, et al., 2004)	Focuses on mystique and unclear origins of mathematics and puts emphasis on dehumanized knowledge, intuition, wonder (Corrigan, et al., 2004)

Table 1: Three Complementary Pairs of Mathematical Values

In 2012, a comprehensive change took in the Turkish educational system. The length of compulsory education was increased from 8 to 12 years and redefined the system into three levels (12-years compulsory education covering 4-years elementary, 4-years middle and 4-years high school) and this structural reform required for the curricular revision as well. One of the results of the curricular revision was the inclusion of more than twenty elective courses offered for the Turkish middle school students who can able to choose max.6 hours in a week. Along with the fact that mathematics is one of the core subjects of the curriculum, "Mathematics Applications Course" (MAC) is one of the elective courses for middle school students since 2013-2014 academic year. According to statistics, MAC is the most popular elective course among 5th-7th graders in 2014-2015 academic year (Ministry of National Education [MONE], 2015). The main purpose of MAC is not only to improve students' mathematical knowledge and skills but also to like mathematics and develop positive attitudes towards mathematics through the learning opportunities that allow students to practice mathematics (MONE, 2013).

Keeping in mind that the concept of a value is associated with "what is desirable, preferable, worthy, important, right, or beneficial" (Bishop, Seah, & Chin, 2003, p. 723) by different scholars and they, as cultural products, depend on personal choices, preferences and decisions, the focus of the present study is on the mathematical values of the mostly chosen elective math course by the Turkish middle school students. In this respect, the present study might shed lights on what kind of values about mathematical knowledge and discipline that a teacher is expected to convey into instruction as well as what kind of an image of mathematics is going to be presented for students. Further, the results of the study might raise the awareness of curriculum developers, mathematics teachers, and textbook writers about the values in mathematics education. It is also expected that the results of this study may contribute to the math-related affect studies by giving an example from the values of mathematics in the Turkish Mathematics Applications Course Curriculum.

Method

This study was designed to explore the mathematical values in the MACC (5th-8th grade). For this purpose, data were collected through document analysis. The main data collection source of this study was MACC published by the MONE. In this respect, the focus of the present study is on the intended curriculum and limited to the MACC which is available on the MONE's official website.

Before the data analysis process, the framework for curriculum analysis was developed by considering the literature on the mathematical values and the structure of MACC as well as the research questions. In order to portray the mathematical values, namely "Rationalism-Objectism; Control-Progress and Openness-Mystery", Seah's outline of the major signals for the mathematical values was adapted (Seah, 1999, pp.110-111). Afterwards, MACC was examined carefully to find out what type of mathematical values embedded in which part of the curriculum. MACC consists of 8 main chapters as (1) Introduction part; (2) Aims of MAC; (3) The developmental characteristics of middle school students; (4) The structure of core mathematics curriculum; (5) Explanations about the implementation process of MACC; (6) Basic principles of MAC; (7) Assessment and evaluation process and (8) Learning objectives of MACC. While locating and analyzing the data, the researchers independently read all chapters line by line and coded all instances of the mathematical value statements in MACC according to the framework. To establish internal consistency, inter-coder reliability was carried out using the formula (Reliability= Number of agreements / Total number of agreements + Disagreements) proposed by Miles and Huberman (1994). A high agreement score (0.92) between the two coders (the researchers) was obtained. Then, the basic descriptive statistics including frequencies and percentages were carried out by means of IBM SPSS.23.

Results

The mathematical values in the Turkish Mathematics Applications Course Curriculum

The major focus of the content analysis here was to identify all instances of the mathematical values in the written curriculum, namely MACC. The findings indicated that totally 39 value signals of the mathematics embedded in MACC. As given in table 2, it is obvious that objectism (f = 10) was emphasized more than rationalism (f = 6); progress (f = 8) was emphasized more than control (f = 2) and openness (f = 12) was emphasized more than mystery (f = 1). According to the results, while "openness" (f = 12) was the most-emphasized value; there is only one reference to the value of "mystery" (f = 1) in MACC.

Mathematical Values	*Number of value signals (f)*	*Proportion (%)*
Rationalism	6	15.4
Objectism	10	25.6
Control	2	5.1
Progress	8	20.5
Openness	12	30.8
Mystery	1	2.6
Total	*39*	*100*

Table 2: The proportional distribution of mathematical value embedded in MACC

The place of the mathematical values in MACC

The last research question of the study aimed to portray the parts of MACC in which the mathematical values are embedded. As given in Table 3**,** the results indicated that the mathematical values mostly located in the learning objectives (f = 14). However, both in the introduction part and assessment part of MACC, only three statements referring to the mathematical values were found.

Mathematical values	*Introduction*	*Aims & Principles*	*Objectives*	*Teaching & Learning*	*Assessment*
Rationalism	1	2	1	1	1
Objectism	0	2	6	1	1
Control	0	0	1	1	0
Progress	1	3	3	1	0
Openness	1	4	3	3	1
Mystery	0	0	0	1	0
Total	*3*	*11*	*14*	*8*	*3*

Table 3: Frequencies of the mathematical values embedded in the parts of MACC

Considering the *Introduction part*, it was found that the value of rationalism was embedded in the statements explaining the importance of providing the learning opportunities to develop students' mathematical thinking as well as finding reasonable, rational and logical solutions/answers for problems. It was also found that the value of openness was conveyed in the expressions about the importance of cooperative learning. In addition to rationalism and openness, the value of progress was highlighted in the introduction part through the statements putting emphasis on making connections with mathematics and daily life. Further, the value of rationalism, objectism, openness, and progress were included in part of *the Aims and Principles* of MACC. More specifically, the value of rationalism was embedded in such statements that underline the use of mathematical reasoning; focusing on symbolization and modelling portrayed the value of objectism; putting emphasis on group working, sharing/discussing ideas addressed to the value of openness; and the statements signifying the relationships between mathematics and the other disciplines as well as daily life were found as the indicators of the value of progress. It's also noting worth that in the following statement, the attention aimed to draw on the mathematical values *"…it is important for students to develop true values about mathematics"* (p. 1). However, there was no explanation related to what are the true values about mathematics provided in this part of MACC. In order to portray the mathematical values, all learning objectives (totally 21) were analyzed in line with the framework. As given in Table 3, the mathematical values embedded in MACC were mostly located in the part of the *Learning Objectives*. The objectives putting emphasis on the use of appropriate mathematical symbols while solving problems were found as the indicators of objectism; asking students to discuss their solutions process through mathematical reasoning were the indicators of rationalism. Additionally, the learning objectives that put emphasis on the development of students' problem solving abilities through daily life math problems were considered as the indicators of progress, and the objectives aimed to promote students' procedural fluency through the tasks asking to test or evaluate the way of solution process were the signs of control. Further, the learning objectives aimed to improve students' problem solving skills through sharing/discussing ideas and pose questions/problems were found as the indicators of openness. Considering the part of *Teaching and Learning Approach*, it was found that while the value of objectism was fostered through the statements suggesting the use of different kinds of representations in problem solving process, the value of rationalism was embedded in the explanations related to the characteristics of problems that should promote students' mathematical reasoning. The indicators of control were found in the sample problem that required using of mathematical knowledge in an attempt to change the environment. The sample problem also indicated the value of progress by means of addressing the relationship between mathematics and daily life. The statements emphasizing the idea of sharing information and collaborative working were found as the indicators of the value of openness.

Besides, the statement suggesting the use of interesting mathematical games was found as the only one indicator of mystery. In the last part of the MACC, *Assessment and Evaluation Process*, three types of mathematical values were found. The value of rationalism was embedded in the statements suggesting the development of students' reasoning and logical thinking skills should be assessed by such methods as classroom observations, self-evaluation reports, etc. Besides the value of objectism was found in the statements that ask teachers to evaluate students' ability to use different symbolizations while explaining their problem solving process. Finally, the value of openness was implied in the suggestions about the assessment of collaborative work.

Conclusion and discussion

The present study was designed to examine the mathematical values embedded in MACC which is the most popular elective course among the Turkish middle school students in 2014-2015 academic year. For this purpose, the written curriculum was analyzed by the researchers in a detailed manner. In general, it can be concluded that all of the mathematical values, namely objectism, rationalism, control, progress, openness and mystery, are embedded in MACC. Although the balance among these embedded mathematical values in MACC is not equal, the findings clearly proved that the mathematical values are taken into consideration by curriculum developers. Further, the results indicated that the value of openness was the mostly-embedded mathematical value in MACC and thus, it can be concluded that the democratic side of mathematics is intended to be promoted through communicating with the mathematical ideas and questioning the mathematical facts. In other words, the intended implementation process of MAC seems more likely to focus on the appreciating the role of public sharing and discussions in facilitating understanding of mathematics. On the other hand, a considerable amount of the signals were also placed on the value of rationalism and objectism in MACC. This situation might support a widespread image of mathematics as abstract, theoretical, ultra-rational discipline (Ernest, 2004).

One of the more noteworthy findings to emerge from the present study is that both the value of progress and control were rarely mentioned in the curriculum. When considering one of the main aims of MAC that provide learning opportunities for students to experience mathematics through mathematical problems and to develop their mathematical knowledge, it is quite surprising. Besides, there was only one reference related to the value of mystery found in MACC. Such little emphasis on the value of mystery might limit the opportunities indicating the mystical, surprising, and fascinating side of mathematics. Taken together, the results of the present study indicated that the image intended to transmit through MACC might be mathematics as symbolized, open to discussion, theoretical and questionable discipline. What research has found about the mathematical values embedded in math curricula yielded the similar results with the present study (Clarkson & Bishop, 2000; Seah et al., 2016). Moreover, the following statement "*…it is important for students to develop true values about mathematics*" (MACC, p.1) might be considered as evidence that explicitly mentioned "values about mathematics" in MACC. Nevertheless, it is too vague for teachers to interpret what are the true values about mathematics, so further explanation should be provided in the written curriculum.

Considering the place of the mathematical values, the results indicated that the value signals were embedded in almost all parts of MACC. However, while the value of openness was placed in each part of the MACC, the value of mystery was only mentioned in the teaching and learning approach

part. One of the main reasons behind this situation might be due to MAC itself. In other words, MAC mainly focuses on the development of students' problem solving and mathematical thinking skills through the applications of mathematics. Thus, students are expected to learn collaboratively; to solve and pose problems from daily life and other areas of science; to share and discuss their ideas through mathematical reasoning and to test and evaluate their problem solving process rather than focusing on mystic and fascination sides of mathematics. Taken together, these results clearly indicated that the mathematical values were mostly embedded in the form of implicit statements in the different parts of MACC. Therefore, the reflection of these mathematical values from the intended to enacted curriculum might probably stay as hidden and vague. On the one hand, the image of mathematics embedded in the curriculum will shape students' future choices and career plans about mathematics; on the other hand it will be shaped by the mathematical values hold by the curriculum developers. Therefore, it is essential to raise the awareness about the values in mathematics education as well as to conduct more studies on the mathematical values in the intended, enacted and attained math curriculum.

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A case study on Finnish pupils' mathematical thinking: Problem solving and view of mathematics

Hanna Viitala

University of Agder, Norway; hanna.l.viitala@uia.no

In this article, the mathematical thinking of four Finnish pupils is reported using two temporally different data sets: problem-solving processes and view of mathematics. While the pupils seem similar on the surface level (high achievers, successful problem solvers, enjoy mathematics, motivated to learn mathematics), a closer look at their problem-solving processes and view of mathematics reveal very different strengths and weaknesses in their mathematical thinking. Most of the similarities in this study were found in individual pupils' problem-solving processes and view of mathematics.

Keywords: Problem solving, view of mathematics, affect, metacognition, meta-affect.

Introduction

Developing mathematical thinking is one of the key tasks for mathematics instruction in the Finnish curriculum (FNBE, 2014, 2004). And indeed, Finnish pupils have succeeded well in international studies that assess pupils' mathematical thinking (PISA and TIMSS; see e.g. OECD, 2014; Mullis, Martin, Foy, & Arora, 2012). However, the most recent national and international studies show that the mathematics performance of Finnish pupils is descending (e.g. Välijärvi, 2014; Rautopuro, 2013). Additionally to the alarming trend in mathematics performance, we know very little about Finnish comprehensive school pupils' mathematical thinking that go *beyond* paper tests. Thus, a quantitative research study was conducted with the aim of describing what characterises Finnish 15-year-old pupils' mathematical thinking.

On the way to describe what characterises Finnish pupils' mathematical thinking, the study reported in this article examines four high-achieving Finnish pupils' mathematical thinking through the intertwined relationships of problem-solving processes and view of mathematics. While some of the results of individual pupils' mathematical thinking have been discussed in previous publications (Viitala, 2013; 2015; 2016a), the purpose of this paper is to bring the results together, and answer *what similarities and differences related to mathematical thinking can be found between these pupils*. With this question, we can reveal some of the possible trends in skills and competences that the Finnish high-achieving pupils might have in their mathematical thinking.

Theoretical framework

Developing pupils' mathematical thinking is in the heart of mathematics education, also according to the Finnish curriculum (FNBE, 2014). While research in mathematics education does not seem to have a common understanding of the meaning of mathematical thinking, Schoenfeld (1992) recognised five aspects that are important in a study on mathematical thinking: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices. Similar findings have also been found in connection to literature on problem-solving performance (Lester 1994), and are also listed as part of final-assessment criteria in the Finnish curriculum (see FNBE 2014, pp. 433-434).

Similarly as the most recent theories on affect, mathematical thinking can be viewed through two temporally different aspects: state and trait (cf. Hannula, 2011; 2012). On one hand, mathematical thinking is always situational (state). Following Schoenfeld's (1992) categorisation, it is influenced by the pupils' knowledge base and heuristics, and guided by their metacognitive skills, affects and classroom practices. In this study, mathematical thinking is studied through problem-solving processes. In other words, 'pupils' activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking' (Viitala, 2015, p. 138).

Pupils' problem-solving behaviour is influenced by pupils' metacognition, affect and meta-affect that occur in a problem-solving situation. The successful application of problem-solving activities at the correct moment is a result of metacognitive skilfulness (e.g. van der Stel, Veenman, Deelen, & Haenen, 2010), affect influence the problem-solving situation for instance through the feeling of confidence, and meta-affect transforms individuals' emotional feelings (DeBellis & Goldin, 2006) and directs problem solving behaviour (Carlson & Bloom, 2005).

On the other hand, problem-solving situations can show patterns of thought that can be interpreted as signs of more stable ways of thinking. Some of these patterns can also be revealed through pupils' view of mathematics (see e.g. Viitala, 2016a). View of mathematics draws from psychological theories. It is a mixture of cognitive, motivational and emotional processes that include for instance beliefs, attitudes, values, feelings and motivation (Hannula, 2011; 2012). In this study, view of mathematics is studied through four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen, 1995, cf. Op't Eynde, de Corte, & Verschaffel, 2002).

Methods

Data collection

At the time of data collection, the four pupils (Alex, Daniel, Emma and Nora) were 15 years old and in their 9th and final year of compulsory school in Finland. Additionally, they were all high achievers (mathematics grades between 9 and 10 on a whole number scale of 4 to 10).

The data was collected in three cycles over the course of three months. In each cycle, one mathematical task was solved in an ordinary classroom situation as a 'main task'. The pupils solved the tasks individually but they were allowed to talk about the tasks with a friend or ask for help from the teacher. In each cycle, the pupils were video recorded while they solved the task(s) in class and their solutions on paper were collected. Below, there is an example of a main task (School Excursion, OECD, 2006, p. 87).

> A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.
>
> Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.
>
> Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

In each cycle, the pupils were interviewed individually. The interviews took place either on the same day, or on the next day after solving the task in the classroom. The interviews contained two parts. The first part concentrated on affective traits and treated the following themes: pupil's background, mathematical thinking, and pupil's view of mathematics (following the categorization of Pehkonen, 1995; see example questions in Table 1, Viitala, 2016a, p. 1295). This part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009).

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking	What does mathematical thinking mean? / How do you recognise it?
Mathematics	What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

Table 1: Interview themes and example questions.

The second part of the interview was about problem solving. The classroom data was used as stimuli when the pupil's problem-solving process was discussed. The pupils were asked to explain their thinking and actions during the problem-solving situation and additional questions were asked (e.g. what are you thinking now? Why are you doing so? What did you feel when you read the task? Did you think about your own thinking when solving the task?).

Finally, in each interview, the pupils were asked to assess their confidence before, during and after solving the problem, as well as their confidence in school mathematics using a 10 cm line segment (scale from 'I couldn't do it at all' to 'I could do it perfectly'). All interviews were video recorded.

Analysis

Following the state and trait aspects of the study, the analysis was divided into two sections: problem solving and view of mathematics. The problem-solving processes were analysed first by going through the problem-solving phases introduced by Carlson and Bloom (2005): orienting, planning, executing and checking (cf. Polya, 1957). Then the results on problem-solving behaviour were complemented with metacognitive activities (orientation, planning, evaluating and elaboration van der Stel et al., 2010), affect (state and trait, as well as cognition, emotion, motivation; Hannula, 2011; 2012) and meta-affect (DeBellis & Goldin, 2006) emerging in problem-solving processes. Finally, the pupils' confidence to solve the problems was analysed.

The first analysis of the pupils' view of mathematics followed the themes of data collection (Pehkonen, 1995). After condensing the results, a pupil profile was created to be used as background information about the pupil. Pupil profile is a short description of the pupil that is based on the pupil's mathematics grade, motivation to learn mathematics, and the core of his view of himself as a learner of mathematics (ability, success, difficulty of mathematics, and enjoyment of mathematics, following Rösken, Hannula, & Pehkonen, 2011).

In the end, the results of problem solving and view of mathematics were compared to see if there are similarities in pupil's problem-solving skills (state) and competences found through pupil's view of mathematics (trait). More details of the methods used in the study are reported for instance in Viitala (in press).

Results

On a surface level, Alex, Daniel, Emma and Nora seem quite similar: they are all high achievers in mathematics, they enjoy mathematics, and they are motivated to learn mathematics (see excerpts in Table 2). They are also successful problem solvers, that is, they could solve all the problems given to them in the study and justify their answers and solutions. However, a deeper look at their problem solving and view of mathematics introduce four pupils with a very different skills and competences. In the following, the key results of each pupil will be introduced individually.

Alex is very fluent and thorough mathematics learner and problem solver. He can move naturally between different phases of problem solving. He is aware of his own thinking and fluent in explaining and justifying his cognitive and metacognitive actions in problem solving. Similarly, when explaining his learning of mathematics, he says he is actively seeking for connections between new knowledge and prior knowledge, and he is able to spontaneously give examples of this behaviour. He says he trusts his own thinking more than his calculations, and shows to be able to direct his behaviour according to his affects in problem solving. He is confident in school mathematics but in the interviews, he constantly compares his abilities to mathematics as a science and recognises that there is much more than school mathematics (more results in Viitala, 2013; 2016b).

Whereas Alex seems to be very fluent in every aspect of mathematical thinking studied in this research project, from a similar starting point, Daniel shows somewhat different strengths in mathematics. Unlike any of the three other pupils, he is extremely confident in mathematics. He says that mathematics is easy for him, and he shows to be very aware of his success in mathematics. His confidence seems to guide also his problem-solving processes. He is able to move fluently back and forth between problem-solving phases and is skilful in performing metacognitive acts. However, even though (or because of) learning mathematics and solving problems are easy for him, he cannot explain the processes he goes through in or for learning, and he has problems in explaining his problem-solving actions after the problem-solving situation. An illustrative example of this situation is Daniel's explanation about how he learns mathematics: pieces just click together or things become familiar (more results in Viitala, in press).

Similarly as in Daniel's case, also Emma's learning of mathematics and problem solving are strongly influenced by her confidence in mathematics, or more precisely, her lack of confidence. Because of the uncertainty in mathematics, for Emma, learning takes time and effort. She says she learns every topic as a separate entity, and she is able explain the steps that are needed for her to learn a new thing. Similarly, she uses a considerable amount of time for orienting and planning in problem solving. After understanding the problem and the given data, she is able to follow her plan through and check her solution. It seems that Emma‘s uncertainty in mathematics makes her work harder, and through hard work, she succeeds in mathematics. Moreover, she says that succeeding in mathematics and understanding it, makes it worthwhile studying. On the other hand, affect can also be an obstacle in her problem solving, since she does not seem to have efficient tools to overcome the feeling of getting stuck (more results in Viitala, 2015; 2016a).

Also for Nora, learning mathematics takes time and effort but after learning something, applying is easy. She says that she is quite confident in mathematics and likes learning mathematics very much.

She is capable in explaining her thinking and problem solving, and connecting mathematics to her own life. She also has a diverse view of mathematics as a science. In problem solving, she is flexible in directing her actions based on the affective states occurring in problem-solving situations. She is also fluent in moving between orienting, planning and executing in problem solving. However, given the choices she had made while planning, she is happy with the first answer she gets, and does not check her results (more results in Viitala, 2015).

	Ability and success	**Difficulty of mathematics**	**Enjoyment of mathematics**	**Motivation to learn mathematics**
Alex	Confident in math; deserves the high grade: knows school math quite thoroughly	Learning 'a separate thing' is easy, connecting it to 'other things' might take time	Learning math is fun and interesting; routine learning is boring	Good grade and future studies, also understanding the issue at hand
Daniel	Very confident in math; can do math well; deserves the high grade (active learner, succeeds in tests)	Learning math is easy and it does not take much time or effort	Math is enjoyable, even fun	Math is needed through life; the most important school subject
Emma	Not confident in math; could not get a better grade in math	Learning math takes time and effort	Learning math is irritating and tiring; succeeding and understanding is fun	Wants to succeed in mathematics and be proud of herself; future studies
Nora	Quite confident in math; not perfect in math but deserves the high grade in school math (active learner, succeeds in tests)	Math can be easy or difficult, more on the easy side; learning takes time and effort, applying after that does not	Learning math is interesting, likes math very much	Good grade; wants to learn math

Table 2: Examples of pupils' own statements about their view of mathematics (cf. pupil profile).

Some reflections of the results

In addition to forming descriptions of pupils' mathematical thinking, and showing pupils' strengths, the study also revealed issues that pupils could work with in order to develop their mathematical thinking. For instance, even though Alex was fluent in problem solving and school mathematics, he did not relate the problems to real life and his view of mathematics outside school was quite limited (see Viitala, 2013, 2016b). Recognising mathematics more in his own life could enrich Alex's view of mathematics, and through that, also his understanding of school mathematics might develop. Daniel, on the other hand, had problems explaining his thinking after the problem-solving situation and had similar problems with explaining his mathematics learning (see Viitala, in press). Problem solving and learning mathematics might be easy for Daniel in compulsory school, but what happens if (when) the situation changes? Becoming aware of his own learning and problem-solving processes could help him cope in new situations and develop his metacognitive skills not only in mathematics but also in other school subjects.

Emma's weak point was her uncertainty which she had turned into success in problem solving and learning of mathematics. She had overcome some of the uncertainty with the support of her family (see Viitala, 2016a). However, because she was not confident in mathematics, she learnt every topic in mathematics as its own entity, and did not connect it to prior knowledge. This might also hinder her learning. Hence, supporting Emma emotionally could open doors to more thorough learning and understanding of mathematics. Finally, Nora's results were not always correct, and both her activities and explanations showed that she does not evaluate her problem-solving process or check her results (see Viitala, 2015). Supporting her to look back, and perhaps exposing her more to, for instance, open problems, might help her to become more reflective user and learner of mathematics.

Summary and discussion

The purpose of the paper was to answer the question what similarities and differences related to mathematical thinking can be found between the four Finnish high-achieving pupils. Mathematical thinking was studied through two temporally different data sets: problem-solving processes (state) and view of mathematics (trait). The results showed that the similarities between the pupils were found to be mainly on a surface level: all the pupils liked mathematics, were motivated to learn it, enjoyed doing mathematics and were successful problem solvers. However, after a deeper look into their problem-solving processes and view of mathematics, the study revealed a great deal of differences between the pupils, and showed different competences: Alex is a very conscious thinker and learner of mathematics, and excellent in justifying his thinking and actions in mathematics. Daniel is extremely confident and metacognitive skills are prominent in his problem solving. Emma is an unsure but very thorough problem solver and learner of mathematics. Nora is fluent in expressing her thoughts and connecting mathematics to real life.

In addition to the strengths found in these four pupils, the framework also revealed some of their weaknesses. The strengths, together with the weaknesses can be used to support individual pupils' development in mathematics. For instance, Alex seemed to see mathematics only as a tool to solve something and his view of mathematics outside school was quite limited (see Viitala, 2013, 2016b). This knowledge can be used to develop pupil's mathematical thinking. Four years after the data collection of this research project, I met Alex again. At this point, Alex was as a university student. He explained that only after realising the tool value that mathematics had for him, and learning that mathematics is not just calculations but also ways of thinking, he began to see mathematics everywhere in his real life, and he began to use his mathematical thinking more creatively (see Viitala, 2016b).

All in all, the results showed that even though the pupils seem similar on the surface level, on a closer look, they have very different skills and competences in mathematics. This is an indication that the framework allows different pupils to show different strengths, and also different weaknesses in problem solving and learning of mathematics. Hence, the framework could assist also teachers to pay attention to the aspects that pupils might need help with in developing their mathematical thinking, which in turn can help the pupils to recognise the knowledge, skills and affects that might need further developing (cf. FNBE, 2014, p. 377; Viitala, in press; see also Viitala, 2015). An example of how teachers can use this framework to support their teaching is presented in Viitala (in press).

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Exploring affective reactions of children to Scratch programming and mathematics: The case of Kim

Johanna Carvajal

UCL Institute of Education, University College London, United Kingdom; j.benitez.14@ucl.ac.uk

Keywords: Affect, Scratch programming, mathematics.

Background

Around four decades ago the use of programming as a medium for mathematical learning became the focus of research for some mathematics educators (Papert, 1980; Noss and Hoyles, 1992). The new computing curriculum in the UK with a strong focus on programming inspired the ScratchMaths project which rescues this idea and aims to exploit the potential of programming in Scratch to support mathematical thinking (Benton et al., 2016). Using this project's designed materials the purpose of my research has been to investigate the kind of affective reactions that emerge from the experience of Year 5 children working with the activities during their computing lessons. Given the connection of the materials with mathematics, and the relationship between mathematical and computational skills, those children that underperform in both subjects have been of particular interest for my study. Three Year 5 classes (sixty 9-10 years old in total) from a primary school in London and taught by the same teacher were observed for sixteen weeks during computing lessons. 11 children were purposively chosen to participate in the research.

Building mainly on the literature on affect and mathematics education (McLeod, 1992; Hannula, 2012) and based on results from the pilot study, three categories concerning efficacy beliefs and emotions were explored: perceived efficacy, general emotion and emotion enjoyment. The three main research questions were:

- What kind of experiences do participants have when working with the activities?
- What changes can be observed (positive/negative evolution of the learner's perspectives)?
- On what might a positive or negative evolution of the learner's perspective depend on?

Methodology

The research used a case study approach and combined different methods for data collection: questionnaires (a short questionnaire administered lesson by lesson during 12 weeks and the attitudes towards Scratch questionnaire), field notes, students' work and critical incident (CI) interviews. The short questionnaire allowed the researcher to capture systematic information along the three categories explored regarding different types of tasks. Indicators above 5 were considered positive reactions, below 5 were considered negative reactions, and equal to 5 were considered neutral. Key moments were identified and were then triangulated with data from other sources.

Results

Kim's responses to the short questionnaire (Figure 1, left) suggest that for her, positive affective reactions tended to happen when the activities were of an exploratory nature and negative reactions

when the activities had a more explicit mathematical focus (Figure 1, right). Nonetheless, she had positive or neutral experiences around 80% of the times that the questionnaire was applied.



Figure 1: On the left, graph of Kim's response to the short questionnaire. On the right, examples of a task with mathematical focus (lesson 5) and a more playful and exploratory task (lesson 6)

Data from the affect towards Scratch questionnaire administered in January and June of 2016 indicates that she had had a negative evolution of perspectives as a learner in this context. Data triangulation from all sources suggests that Kim had noticed a connection between the activities and mathematics and that this connection might have played a crucial role in her learning experience. Children who struggle with mathematics may be in disadvantage when tackling some of the ScratchMaths activities on their own in classroom conditions. In the case of Kim, the involvement of mathematical skills in the activities seemed to trigger negative affective reactions and the belief that she just could not do it. However, when extra material and individual support was given during the lesson she was able to solve the task, help other students in the class, and moreover, she regarded that particular lesson as her best moment working with the activities.

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Documenting metacognitive activity in qualitative interviews with high school graduates

Sebastian Mungenast

Wuerzburg University, Germany; sebastian.mungenast@uni-wuerzburg.de

Keywords: Metacognition, calculus, university.

Theoretical background

The poster gives an overview of the research and theoretical background of an ongoing dissertation project the major focus of which is on metacognition in mathematics education. A central objective is the documentation of metacognition during mathematical activity in prospective mathematics university students and using the results of this study to expand upon existing category systems of metacognition – especially for the field of Calculus.

Metacognition is generally understood as knowledge about and cognitive processes dealing with (one's own) knowledge and cognitive processes (Flavell, 1976) as well as regulating those.

Metacognition is usually divided into a declarative and a procedural component and their respective sub-categories. Declarative metaknowledge signifies available, explicable knowledge a person possesses about the workings of their own cognitive functions and knowledge. Further specifying the concept, declarative metaknowledge can be subdivided into the three categories *person knowledge*, *task knowledge* and *strategy knowledge*. Procedural metacognition on the other hand is focused on planning, cognitively monitoring, reflecting and evaluating cognitive activity as well as regulating the latter. (Schneider, 2010)

It is expected here that in order to be metacognitively active during mathematical learning or problem-solving processes two or more of these (and possibly further) subcategories "interact".

The usefulness of metacognition (for learning and doing mathematics) has been documented in the past.

On the mathematical side of the project, the field of Calculus/Analysis was chosen due to its potential for metacognitive activity. For example, developing a stable idea of the concept of limit demands for a change in perspective when dealing with its dynamic and static aspects and integrating them into a coherent idea. It seems likely that the ability to metacognitively reflect and regulate one's own learning processes is beneficial to enable this change and for mathematics learners and practitioners in general.

Research questions

1) Which kinds of metacognitive activity can be observed in high school graduates/ prospective mathematics students when dealing with mathematics?
2) Can these activities be fully described using existing category systems or can they be used to expand those?

3) How and where can metacognitive activity be beneficial with regard to the field of Calculus/Analysis? What will a category system look like that classifies the term metacognition for that field and expands existing – and possibly more general – models?

Empirical study

Prospective mathematics students' pre-existing use of metacognition was documented via a qualitative interview study at Wuerzburg University. As a sample group, eleven prospective students from mathematics study courses were selected on a voluntary basis. Five participants were invited for a single interview (one participant, one interviewer) in order to avoid inhibitors between two or more participants (such as shyness, different levels of extroversion or different levels of (perceived) "competence"). Six participants were interviewed in pairs (two participants, one interviewer) to reduce inhibitors between interviewer and participant (such as shyness, artificiality of the situation or different levels of mathematical experience) and to give the students the possibility to interact with each other – such as mutually explaining ideas and one's understanding of a mathematical concept, motivating each other to go on, detecting each other's errors and correcting them, discussing strategies, etc.. A partially-structured interview design was chosen, focusing on metacognitive activities during high school and on developing problem-solving strategies for a Calculus problem at hand. The interviewer had a manual at his disposal to guide the participants through various metacognitive topics, but in order to reduce influencing effects direct questions were avoided and the conversation was mostly left to the participants. The resulting interview transcripts are currently being evaluated by means of Qualitative Content Analysis methods (Mayring, 2010).

Perspectives

It is the aim of the project to specify the term metacognition and its sub-categories with regard to the field of Calculus/Analysis, building upon existing category-systems and expanding them. It is hoped that such an expanded category-system can be used as a base for introducing metacognitive activity in Calculus classes (at school and university level) in a structured way and to precisely evaluate students' metacognitive "abilities". Documenting prospective mathematics students' pre-existing metacognition should help to both further specify the afore-mentioned categories and possibly add more sub-categories to the system, as well as to gain information about which metacognitive skills students may already possess and actively apply in their learning practice and which kinds of skills need to be strengthened and/ or introduced into a "metacognitively-supported" curriculum.

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TWG09: Mathematics and language

Introduction to the papers of TWG09: Mathematics and language

Núria Planas[1], Jenni Ingram[2], Frode Rønning[3], and Marcus Schütte[4]

[1] Autonomous University of Barcelona, Spain, Nuria.Planas@uab.cat

[2] University of Oxford, UK

[3] Norwegian University of Science and Technology, Norway

[4] Technical University of Dresden, Germany

Introduction

Within the context of mathematics education research there is strong agreement on the importance of language for learning and thinking, and on the centrality of being able to communicate mathematically for learning and teaching school mathematics. These standpoints are particularly idiosyncratic of TWG09 and the group of papers presented therein on the occasion of CERME10. It is assumed that developing more knowledge about language and language processes can aid the field in terms of a better understanding of what is involved in mathematics learning, teaching and thinking. In the centre of important debates around which theories and conceptualizations of language to take, there is a growing awareness that dialogue between theories will help to refine our approaches to the various phenomena embedded in mathematics education and language research. Within the context of TWG09, this awareness has been present in many ways over the course of past editions (e.g. Planas, Chronaki, Rønning & Schütte, 2015; Rønning & Planas, 2013). Also in the TWG09 sessions at CERME10 participants did not restrict themselves to 'defending' their positions. They were interested in exploring common ground and opportunities to take the field forward.

The T of TWG09 stands for a number of topics, themes and theories. As a group, we cover frameworks drawing on linguistics, cultural and social semiotics, sociolinguistics, positioning theory, functional grammar, theory of didactical situations, social interactionism, and content analysis, to list only a few. The main idea we want to share in this short introduction is precisely the possibilities of dialogue between theories opened up to the group and to the domain by the existence of such theoretical diversification –i.e., the fact that theoretical perspectives mostly construct their identities by differing from others. Biehler, Scholz, Strässer, and Winkelmann (1994) recommend talking about diversification instead of diversity. Dialogue is seen to be one of the positive and productive outcomes of diversification, which can keep the domain moving in several ways.

Diversification and dialogue in TWG09

In this section, we take the collection of TWG09 papers at CERME10 to illustrate the line of argument of a landscape of diversification and dialogue. By commenting on the joint discussions within different groups of papers, we will claim that both diversification and dialogue were present in our working sessions. Throughout these sessions, the emerging common themes showed that dialogue, even if it sometimes remains elusive, is worth pursuing. Dialogue between perspectives was actually made possible because people from different perspectives worked together.

In the first session, we had a discussion of four classroom-based papers by: Brandt and Keuch; Häsel-Weide; Ingram, Andrews and Pitt; and Tatsis and Maj-Tatsis. All these papers have in common that they represent studies of social interaction, each on the basis of different theories and methods. The number of differences visible in the use of terms competing with each other –e.g., deviations, mistakes and opportunities– turned into a collaborative search for common themes. One theme emerged regarding the relationship between long-term mathematics learning and short-term language accuracy in mathematics teaching and learning. Patterns of corrective responses and markers of authority, as reported in some of the papers, were viewed as indicators of discourses of language accuracy at the intersection with processes of meaning construction and negotiation, as reported in all four papers. A related issue in the discussion was the extent to which the suppression –if possible– of certain discourses of language accuracy was necessary for the development of mathematical activity in classrooms. Either explicitly or implicitly, the different analyses presented in the papers reveal this tension between mathematics learning and language accuracy.

The second session brought many of the methodological issues explored by the group to the fore. Four papers by Farrugia, Schubert-Meyer, Ní Ríordáin, Flanagan and Brilly, and Wessel each highlighted the back and forth flow between conceptual development in mathematics and language learning within classrooms with varying degree of linguistic diversity. The papers were each offering a different perspective on the relationship between word and use, including within which language the word is used, and learning mathematical ideas including subtraction, fractions, relative frequency and undergraduate mathematics. Again each of these papers drew from different theories and methods, and researched different settings, but each raised methodological questions at the core of language research within mathematics education. The discussion focused on how integrative frameworks can be developed that draw upon the different approaches that are grounded in the study of language in mathematics.

The third session included five papers that focused on higher grades in mathematical education. The paper presented by Wille dealt with the topic of the shift from difference quotient to the derivative, moving from algebraic to analytic concept formation explored through imaginary dialogues. With this method, different perspectives (horizontal and vertical) could be identified with preservice teachers, which helped focus on the diversity of conceptions later in class. Related to this topic, Zweidar also worked with the topic of functions and its implicit meanings in the classroom. Her research focused on mathematics lessons through a lens that shows the invisible demands of mathematical discourse. Ulises pointed his research also in the direction of mathematical discourse. He examined the signs of vector quantities and their corresponding gestures in regard to novice teachers and he raised awareness of the semiotic dimension. Schlager examined the connection between language proficiency and achievement in mathematics with 10th grade students, who work on tasks with different linguistic characteristics. The results demand further research but suggest that extremely difficult linguistic structures should be avoided to reduce the achievement gap. Finally, Arce, Ortega and Planas researched students' mathematical notebooks. They analysed comments into different groups of knowledge to later conceptualize them as a learning resource. Especially interesting in this session was the focus on higher grade mathematics education. These papers all showed that mathematics and language is not a topic that is solely important in early education. Ideas of interactionistic learning theories of mathematics (Krummheuer 2015; Schütte 2014) are not just about building a foundation for later learning but also can be used in higher classes with exceptionally

more complex topics. As all papers stated, research in the specific fields has to be extended to draw broader conclusions but the results look promising.

All four papers presented in the fifth session deal, in various ways, with learning by participation in practices. Another theme, common to most of them, is that they are concerned with explanation and logical reasoning. Logic is central to mathematics but in the paper by Ludes and Schütte the authors take this out of the context of mathematics when they discuss a project which aims to include computer science in primary education. An important aim is to look for possibilities to integrate computer science and mathematics and in the paper, competencies in mathematics from the German core curriculum are listed alongside relevant competences from computer science. Carotenuto, Coppola and Tortora also report from a project which is about logic. In the project the students are working with logical riddles, which are not about mathematics but where logical reasoning is needed to solve the riddles. Erath is interested in how students learn to participate in mathematical practices, and in particular how they participate in explaining practices in whole class discussions. The paper is based in interactional discourse analysis and builds on data from grade five classes. The paper by Fetzer and Tiedemann is of a more theoretical nature. Their interest lies in reconstructing mathematical learning processes with a special focus on the interplay between language and objects. They discuss and compare three theoretical frameworks: by Aukerman on language and context, by Bauersfeld on domain-specific learning and by Latour on objects as actors.

In the sixth session connections between the modality of the language used and the learning of specific mathematical concepts became the focus. The relationships between informal everyday language and formal mathematical language, between informal gestures and sign language, between visual, dynamic and verbal modes are explored, considering not only how the mathematics is learned, but also how the mode influences how the mathematics is conceptualised. Here the links between diversification and dialogue are readily apparent. Each paper draws upon different frameworks, with Ferrari drawing upon Systemic Functional Linguistics, Khalloufi-Mouza drawing upon the Theory of Semiotic Mediation, Krause drawing upon the Theory of Embodied Cognition, Mizzie drawing upon Cummin's model of language use, and Rønning and Strømskag drawing upon the Theory of Didactical Situations, and indication of the diversification within the field. Yet the dialogue within the group focused on the commonalities between each of the papers, that is the relationship between the mode of language and the conceptualisation of the mathematics.

Four posters were also part of our group. Using a meta-analysis, and a qualitative analysis of its results, Dyrvold investigates demanding textual features of mathematics tasks, and the relevance of these features to the mathematical content. Rauf and Schmidt-Thieme sketch the required linguistic competencies of mathematics teachers, and outline a "language curriculum" recently introduced for future mathematics teachers at the University of Hildesheim. Similarly, Krosanke presents a study investigating the effect of integrating inclusive language teaching into the education of mathematics teachers in Hamburg, using analysis of interviews and video-vignettes. Kenton, meanwhile, examines the role of metaphor and language in the development of individuals' understanding of risk, confirming that this understanding is enhanced when probability is expressed in natural language.

Old debates, contemporary challenges

Debates regarding dialogue between perspectives are not new and are not unique to our research domain (Bikner-Ahsbahs & Prediger, 2014). In particular, in mathematics and language research, the risks of moving towards a fragmented domain cannot be underestimated. The last decades of increasing research on mathematics and language have provided a serious and valuable diversification of theories and lines of interest, inside (Morgan, 2013) and outside ERME (Pimm, 2014). We are progressively including work of a review nature in the agenda in order to recognize what different theoretical perspectives have in common. As a group, we are mature enough to know that the multiplicity and richness of theoretical positions go with articulation and dialogue.

Throughout the reading of the following collection of papers, we invite you to look for common grounds emerging from contemporary ERME research on mathematics and language. Although it may be easier to grasp differences rather than commonalities between papers, careful attention to questions, approaches and methods will offer evidence of similar problems and theoretical challenges. Hopefully some of these challenges have been discussed in this introduction.

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Rethinking students' comments in the mathematics notebook

Matías Arce[1], Tomás Ortega[1] and Núria Planas[2]

[1]University of Valladolid, Spain; arcesan@am.uva.es, ortega@am.uva.es

[2]Autonomous University of Barcelona, Spain; nuria.planas@uab.cat

We illustrate and expand some findings from research on students' mathematics notebooks in four classrooms (Arce, 2016). Through methods of interpretive content analysis, we discuss what we call comments and group them into a typology of three regarding whether they are primarily instances of: 1) conceptual knowledge, 2) procedural knowledge, or 3) practical knowledge in which it is difficult to recognize the exploration of either conceptual or procedural knowledge of mathematics. We indicate that comments are relevant in the investigation of: 1) How different they are according to the representation of mathematical knowledge, and 2) How influential they are according to the development of student mathematical activity. We anticipate that the analysis of comments may be essential in the conceptualization of notebooks as tools and resources.

Keywords: Student mathematics notebook, written communication, student comments, texts.

Introduction

The mathematics notebook of the student (MN) is a common tool in many classrooms and has been an object of study in former CERME papers (see, e.g., Segerby, 2015). This is the material instrument in which lesson notes are taken, and mathematical work is developed and revised. In our study, we deal with contents that belong to the private domain of students (Fried & Amit, 2003). In their notebooks students produce a language for communication with themselves about the contents to be developed and learned, and they are the ones ultimately responsible for the small units of information that conform the written outcome. Thus, notebooks somehow act as windows into student work and understanding in the mathematics classroom. Despite this significance of notebooks, research on them is still rare (together with Segerby, 2015, some exceptions are Fried & Amit, 2003; Yau & Mok, 2016), in part due to an expanded "illusion of transparency" (Villarreal & Borba, 2010). Notebooks are often seen as learning resources *per se* (i.e. MN represents information used in the student learning experience). Our study contributes to research on mathematical communication (Morgan, Craig, Schütte, & Wagner, 2014) by focusing on the under-researched idea of MN as a communicational tool in the relationship between learner and learning content.

We follow up on a larger study with eleventh-grade students in four Spanish mathematics classrooms. The lessons in these classrooms consisted of the initial exposition of contents and tasks by the teacher, followed by student work on the resolution of tasks. A type of notes emerged as distinguishable during the processes of MNs content analysis. Regarding the teaching dynamics, it was expected that the students wrote down blackboard contents –from the stages of exposition and resolution of tasks– and oral contents dictated by the teacher. The students, however, wrote annotations that do not fit into the former types of content. These annotations constitute what we name *comments* (see Arce, 2016; Arce, Conejo & Ortega, 2015). Regarding comments, in this paper we address two research questions:

- To what extent are comments different depending on the mathematical knowledge they represent?

- To what extent do comments seem to have influence on the development of students' mathematical activity?

Framework for analysis of comments as texts

Arce (2016) applied a complex system of ideas to analyse what students seem to be saying and communicating in their mathematics notebooks. In this section, we address the theoretical grounding for some of those ideas in order to cover the notions more directly linked to the present research questions. Before making sense of the students' written comments from the perspective of their implications for learning, we need to organize the information represented in these comments.

Arce (2016) makes use of the first two types of writing out of the three discussed by Britton, Burgess, Martin, McLeod and Rosen (1975): transactional, expressive and poetic. The transactional type is particular of public domains (Fried & Amit, 2003) in that it communicates meanings intended to inform and persuade an audience. In contrast, the expressive type is particular of private domains in that it communicates personal reconstructions of public meanings. Transactional and expressive writing relate to each other in the articulation of theoretical, algorithmic, logical, methodological and conventional contents (Shield & Galbraith, 1998). All these contents are modulated by the language used, either particularized or generalized, and procedural or descriptive. What we have in comments are all sorts of combinations of contents and forms in the writing.

Types of writing, contents and forms intersect with types of knowledge, namely, conceptual and procedural knowledge of mathematics (Hiebert & Lefevre, 1986). This is a distinction that emerged as fruitful in the analysis of how, and how much mathematical concepts and procedures are present and related in MNs. Due to the complex overlap involved, we consider these two types of knowledge as critical in the analysis of comments. By the conceptual type, we refer to mathematics knowledge that is rich in semantic relationships between concepts and propositions, while by the procedural type we refer to mathematics knowledge that is rich in syntactic relationship, often expressed by means of rules and algorithms. Again, what we have in MNs and particularly in comments are all sorts of combinations of types of writing, contents, forms and types of knowledge.

In his analysis of comments in MNs as representational texts, Arce (2016) adds the differentiation of issues not strictly related to the communication of mathematical knowledge. We strategically put these multidimensional issues in one group because they all fit into a type of comments where the focus on either conceptual or procedural mathematical contents is difficult to recognize. All these issues are grouped under the name of practical knowledge to provide visibility to and promote discussion of the use of comments that may be facilitating their role in student learning even though they do not primarily address conceptual and procedural contents particular to mathematics.

Context, background and methods

For the examination of comments under the conceptual-procedural-practical framework in line with the first research question, we scanned data from the MNs of 41 volunteering students that had been collected in four high school classrooms chosen by availability. We selected contents of the teaching units on functions, limits of functions, and derivatives. With inspiration on deductive methods of manifest content analysis (Krippendorff, 2004), the first two authors examined comments according to types of mathematical knowledge (i.e. conceptual and procedural), contents involved in the writing (i.e. theoretical, algorithmic, logical, methodological and conventional), forms involved (i.e.

particularized, generalized, procedural, descriptive), and other issues regarding organizational, pedagogical and personal aspects. These were the codes useful in the initial stage of the current research in order to get a first general picture of comments as tools.

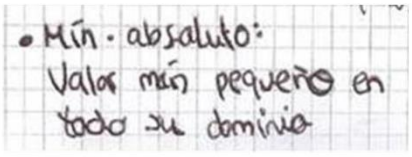
In line with the second research question and together with the findings about differences across comments, the collaboration with the third author helped to rethink the previous analysis with data from eight interviews with pairs of participant students. The selection and pairing of students followed criteria, decided in the context of the larger study, about comparison and contrast among MNs in terms of written contents. By applying inductive methods of interpretive content analysis (Ginger, 2006) to MNs and interview data, some possibilities of comments as sources of mathematical activity for the production of mathematical texts in notebooks were preliminary inferred. When there is mention of or allusion to the comments' impact on either the development of MN contents or the performance of mathematical activity, the third author mentioned the possibility of comments playing a role in the development of student mathematical activity. In the next section, we present some instances of comments to reproduce partially the analyses.

Are student comments more than texts?

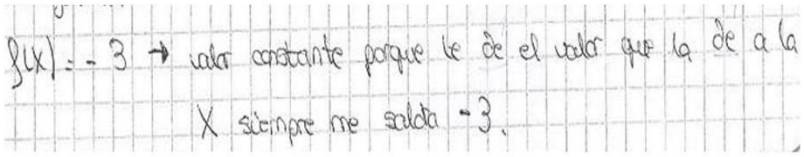
The groups of comments below are not exclusive of each other. The groups may occur in various combinations although we present them separately by choosing comments that more clearly illustrate the dominance of some particular features over others. With their characterization, we want to make the argument that they must be taken into account in student written communication. Moreover, these groups of comments are sufficiently important to be included in the investigation of relationships between the knowledge communicated by students in their writing and the mathematics learning opportunities created and eventually explored by them.

Comments related to conceptual knowledge of mathematics

There are comments in which we find the name of the concept, a more or less formal narrative for its definition or an explanation about the concept definition. Figure 1 shows two instances of this: on the left, a generalized comment on the definition of the absolute minimum of a function; on the right, a particularized comment on the definition of constant function applied to f(x)=-3.



"Absolute min.: Smaller value in all its domain"



"Constant value because whatever the value I assign to X, I will always obtain -3"

Figure 1: Examples of conceptual comments on definitions

There are other comments from this group with an emphasis on relationships between concepts, properties and rules like those in Figure 2: on the left, a comment on a relationship between different kinds of asymptotes; on the right, a comment on the relationship between the derivative of the identity function and the power rule. In this last case, and different from the comment about asymptotes, as the rules had been independently presented by the teacher, we see expressive writing.

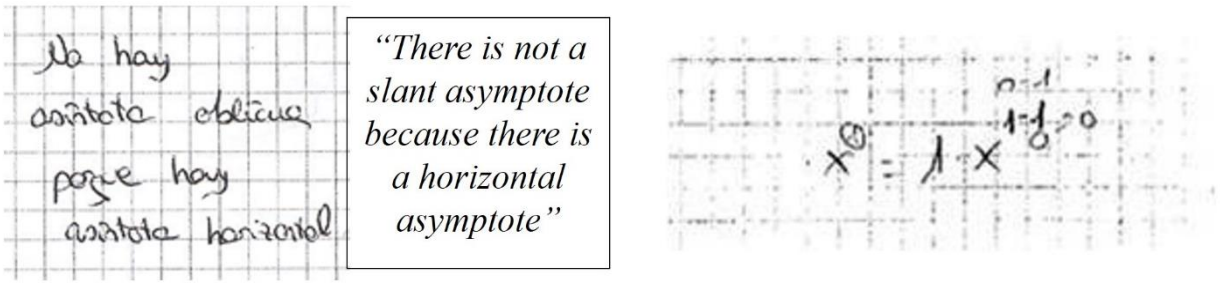


Figure 2: Examples of conceptual comments on relationships

Other conceptual comments are centered on the requirement and justification of conditions for a specific process to be applied, and in this way they may express attention to logical aspects of mathematics. Still some other comments anticipate the need to introduce a new concept or technique, and therefore communicate broader networks of concepts and relationships.

In all these comments of a conceptual type, the overall focus becomes student work on concepts and eventually on relationships between concepts involved in the construction of mathematical knowledge. Different uses of these comments by the students emerged in the interviews. Some students said that comments are useful to "clarify", "evoke" or "remember" concepts, as well as to "support their study at home." Similarly to the reflections made by Morgan (2005), we see in some of these texts (e.g. Figure 2, on the right) the reconstruction and use of mathematical concepts in ways that allow students to learn mathematics. This is confirmed in the interviews in which students refer to this type of comments as an aid for developing their learning of concepts and relationships.

Comments related to procedural knowledge of mathematics

In this group, we place the comments in which we see procedural knowledge of mathematics, that is, comments about recalling or clarifying the application of procedures such as algorithms, rules and techniques, as well as the conventions around them. There is an emphasis on algorithmic aspects when mentioning steps or actions that constitute a procedure such as: the calculation of the domain of a function, the resolution of limits with indeterminate forms and the representation of elementary functions. Figure 3 shows on the left, two comments of a student recapitulating how to solve two indeterminate forms, specifically "$\to 0/\to 0$" and "$\to \infty/\to \infty$", and on the right, the comment of another student who indicates the steps to be followed to examine any function.

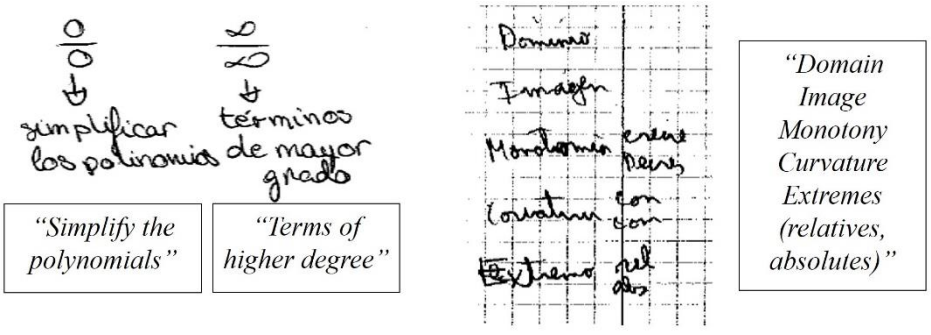


Figure 3: Examples of procedural comments on methods

There are comments with verbal and symbolic marks that recall or clarify mathematical properties applied in the development of the steps of a procedure. Figure 4 shows two instances of this kind: on the left, a comment recalling the calculation of the cube root of any real number; on the right, some

marks by means of arrows clarifying how to operate in a quotient of fractions. In all these comments of a procedural type, the overall focus becomes student work on methods and eventually on the symbolic and formal representation of properties and procedures. As said by the students in the interviews, they provide different uses to their procedural comments, namely: highlighting and clarifying procedural aspects found more difficult (Figure 4), and acting as an aid in order to "mechanize" a mathematical procedure (Figure 3, on the right).

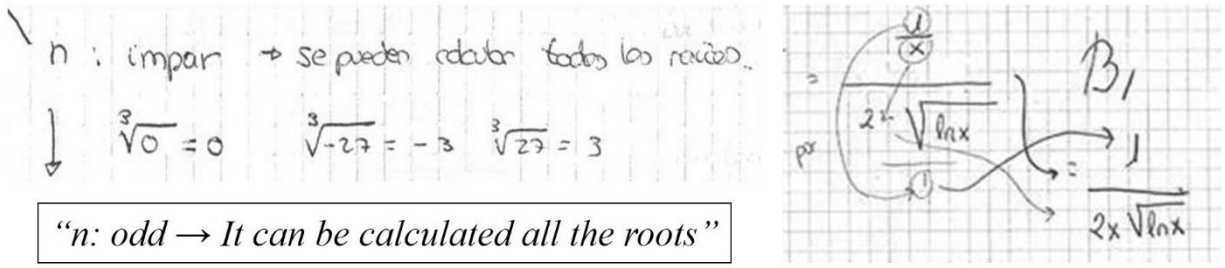


Figure 4: Examples of procedural comments on properties

Together, conceptual and procedural comments point to an important presence of expressive writing with modifications and connections to the mathematics communicated by the teacher in the lesson. We interpret the engagement with expressive writing as evidence of some mathematical work. Drawing on this, these comments may be acting as resources in that they develop written representations of student understanding. One could possibly expect to see in representations of understanding at different moments some traces of different learning stages.

Comments related to practical knowledge

The third group of comments does not directly refer to contents about types of mathematical knowledge communicated in the notebooks; instead, it refers to contents that privilege pedagogical and organizational knowledge, among others. This is a "big" group of comments in this report whose detailed deconstruction in codes can be found in Arce (2016).

Some comments record texts that teachers said during the lessons concerning the organization of curricular issues and other forms of pedagogical support. On the left of Figure 5 there is an instance of an indication about the school time for a curricular content to be considered. Other comments highlight processes that are "tricky" as said by a teacher in a lesson; here we find rules of action as ways to manage difficulty. On the right of Figure 5 there is an instance of a rule for the generation of a table of values in order to represent a function with three positive and three negative values.

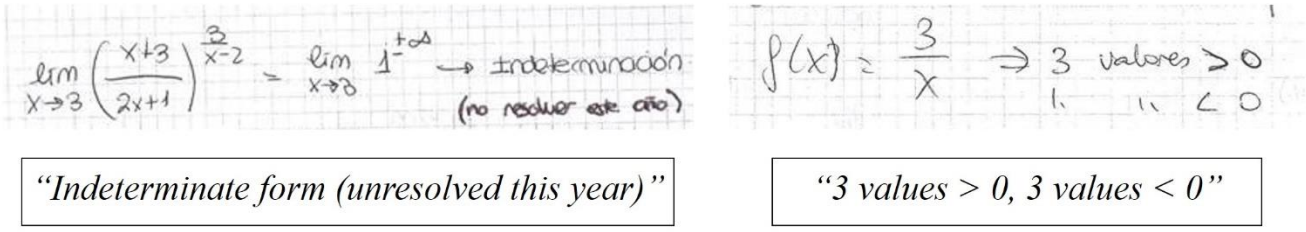


Figure 5: Examples of practical comments on organization

There is also practical knowledge in comments about the forms and contents of evaluation in the subject. This is clear in the two instances of Figure 6 with references to an exam and to the expected contents to appear in it. Comments such as those in Figures 5 and 6 are difficult to be recognized as

representations of mathematical understanding. They also contain/are mathematical texts. Some sort of practical knowledge with purposes of optimization of opportunities as learner is communicated.

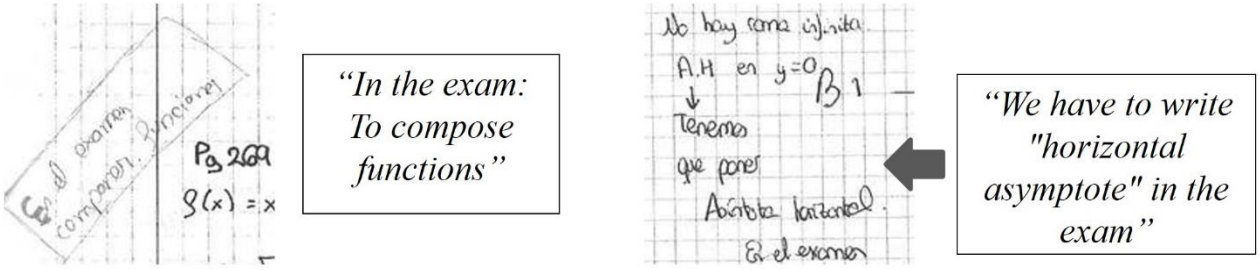


Figure 6: Examples of practical comments on evaluation

Similarly to the instances in Figures 5 and 6, the texts in Figure 7 do not provide evidence of the creation of opportunities for mathematics learning, but rather seem to indicate the creation of opportunities for the student writer as learner. In Figure 7 we find question marks on the left, and sentences to indicate doubts and uncertainty on the right. Other comments documented in Arce (2016) use impersonal forms of language to recommend "reviewing", "studying", or "asking".

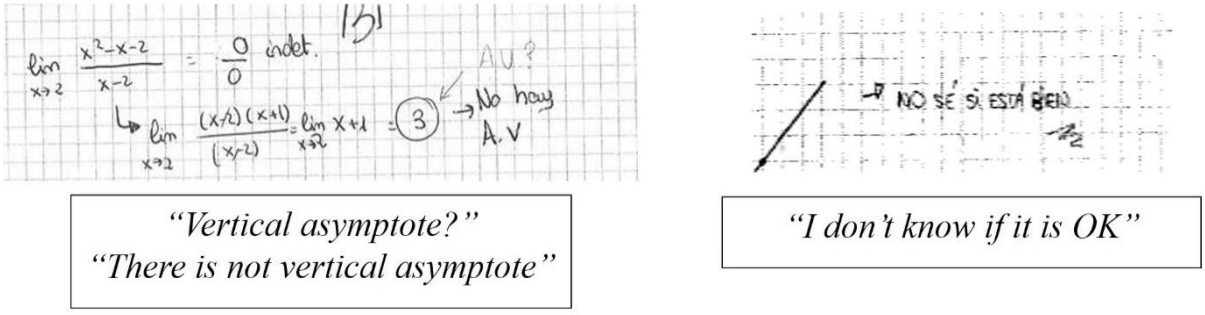


Figure 7: Examples of practical comments on cognition

There is some evidence of personal meta-cognition and control over the learning process in all these comments. We found in the interviews that some students add comments like those of Figure 7 in the resolution of tasks to focus and increase their attention when the teacher makes corrections in the classroom or to ask directly the teacher about a doubt. However, except for Figure 7 (on the left), there are no concrete initiatives in the MNs aimed at exploring or clarifying what has been indicated as mathematically difficult to understand.

We claim that all these practical comments are also valuable; they communicate part of the knowledge that the student needs to develop, together with particular mathematical knowledge, in the mathematics classroom. These comments provide an opportunity for students to make their own judgments on, for example, what needs to be known (i.e. what is to appear in the exam, what is planned for another school year) and who they are as knowers of mathematics (i.e. what is not mathematically clear to them, which learning requires revision).

Rethinking comments as more than texts

We are in the position to adventure some initial thoughts and questions about the conceptualization of student comments as resources. Conceptual and procedural comments like those exemplified in this report (Figures 1 to 4) facilitate the exploration of mathematical knowledge, even though there may be some instances of a routine writing activity enabling the development of reproductive or

imitative patterns (also studied by Yau & Mok, 2016). However, our analysis of comments includes one more practical function about prescriptions of what is to be learned (Figures 5 and 6) and about judgments on what is not known (Figure 7). It is not easy to elucidate whether the learning process would be similarly facilitated if particular types of comments were not present. We anticipate that all types of comments –conceptual, procedural and practical – are necessary in order to create learning opportunities oriented to both the creation and communication of mathematics learning.

We agree with Fried & Amit (2003) that there is a need to further investigate the role and use of MNs, and not only of comments, as a learning resource in the mathematics classroom. A significant question for which this area of investigation might provide understanding is why some notebooks are better facilitators of opportunities for mathematics learning. This is an issue related to the broader question of the role of mathematical writing in mathematics learning. As noted already with the analysis of comments, the written elaboration of conceptual and procedural knowledge of mathematics seems to indicate work relevant for the development of student mathematics learning. However, these comments alone may not be enough in the construction of mathematics learning. Other types of comments, like those that constitute what we have called practical knowledge, may be required in the process that goes from student written communication to mathematical understanding and from here to student written communication again. Especially important may be the ways in which all these types of comments appear combined and related.

Acknowledgement

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The duck is the biggest – kindergartners talking about measurement and magnitudes

Birgit Brandt and Sarah Keuch

Technical University of Chemnitz, Germany

birgit.brandt@zlb.tu-chemnitz.de; sarah.keuch@zlb.tu-chemnitz.de

This paper deals with aspects of language learning in settings planned for mathematical learning by kindergarten teachers. Using qualitative and linguistic analysis tools, we reconstruct patterns of language use and the language sensitive organization of kindergarten teachers. We mainly focus on the children's language use, particularly on semantic deviations in utterances in relation to the mathematical negotiation process.

Keywords: Language usage, measurement, early childhood education.

Introduction

The importance of language for cognitive (subject-specific) learning processes is undeniable and well established with regard to research in early mathematics education. Scientific language proficiency is seen as an important factor for successful education and schooling. There are still unsatisfied needs for Germany to appropriately support children with disadvantageous starting conditions (for example migration, socio-economic background, developmental speech disorder), in order to give them an equal chance to participate in education processes (Gogolin & Lange, 2010).

Early education in kindergarten, which puts emphasis on supporting language education, could provide a remedy. Prediger (2015) suggests that academic language education processes start as early as possible, to design them age-appropriately and to orientate it to specific contents. However, Germany is particularly lacking language education approaches that integrate subject-related learning processes and not only selective training single academic language terms. Rudd, Satterwhite and Lambert (2010) describe how mathematical learning and language learning can be combined in (natural) kindergarten situations. They introduce the concept of Math-Mediated Language (MML). This means that mathematical learning is embedded in dialogues, which include mathematical as well as linguistic knowledge (Rudd et al., 2010). They give examples for different mathematical topics, e.g. how to foster complex counting strategies by modeling them in concrete situations or by requesting them from children using corresponding questions. Even though the concept of MML emphasizes mathematical learning in kindergarten, it points to the need that kindergarten teachers consider both the mathematical context and linguistic effort involved in the dialogues – and address this connection in their planning as well as in spontaneous situations. Thus, MML deals with the integration of language education and subject learning in everyday activities for kindergartners.

MML requires a certain amount of language awareness. For pre-service early childhood educators, Moseley (2005) found out that their perceptions of MML is restricted to technical terms and basic mathematical terminology. In our qualitative-empirical project, we are interested in kindergarten teachers' language awareness in everyday situations. We put our focus on the support of language

learning in settings planned for mathematical learning. This idea corresponds to the underlying idea of supporting language development within the subject (Leisen, 2013; Prediger, 2013; Prediger & Wessel, 2013) as it is discussed in the schooling context. Often, these concepts trace back to the Immersion Model for bilingual education for children with migration background in school contexts (e.g. Cohen & Swain, 1976).

Kindergartners are not only 'subject learners' but, independently from their language background, always 'language learners'. Hence, they sometimes have difficulties expressing complex facts and their language productions often show deviations from the standard language (Volmert, 2005). In this paper, we want to deal with deviations from standard expressions that can have an impact on mathematical learning processes. Since we are dealing with spoken language, which often includes aspects of dialectal variation and language change phenomena, it is not always trivial or even possible to decide whether one utterance is correct or not. In German for example, there are nouns with locally varying genders (cf. der Joghurt: male or das Joghurt: neuter, both possible in standard German; and in eastern parts of Austria die Joghurt: female).[1]

In principle, mistakes can be divided into lexical (neologisms and wrong pronunciation), syntactical (wrong conjugation or flexion, word order) and semantic (inappropriate choice or combination of words) ones. In this paper, we concentrate on semantic deviations, which we list as a separate category since the meaning of utterances does not always depend on the choice of single words or grammatical constructions alone. Meanings rather tend to exceed verbal boundaries, which also has to be taken into consideration when looking at inappropriate utterances (Brandt & Keuch, in Press).

In particular, our aim is to reconstruct the empirical language in use, to detect aspects of language support, and to show the connection to specific meanings and concepts that are negotiated in certain situations. In our prior analyses, we found different kinds of language support and correction strategies (Brandt & Keuch, in Press). Similar to Moseley's results (2005), when using language in everyday situations, kindergarten teachers put special emphasis on technical terms and only a limited focus on complex language structures. Thus, in this paper we will concentrate on semantic aspects of the empirical language and the corresponding questions:

- What kind of semantic deviations can we identify in the field of measurement?
- Which impacts for negotiation processes about measurement can we deduce from these deviations?

Research design

The data basis for our analysis consists of mathematical situations designed by kindergarten teachers and taken from the project erStMaL (early Steps in Mathematical Learning) (Acar Bayraktar, Hümmer, Huth, & Münz, 2011). Methodologically, our project is based on grounded theory (Glaser & Strauss, 1967). We figure out the negotiation of meaning in the interaction processes through the interaction analysis (Krummheuer, 2007), which is a sequential analysis and is organized as an extensive turn-by-turn interpretation. Further, we determine linguistic features

[1] Duden, 2013; 26. Aufl., Dudenverlag, Berlin.

that originate from a linguistic valence analysis (Herbst & Götz-Votteler, 2008) by looking at the relation between verbs and their objects (Brandt & Keuch, in Press)[2]. Our aim is to create a category system of difficulties and deviations, and their corresponding reactions and support from the kindergarten teachers. According to qualitative content analysis (Mayring, 2000), these categories are generated inductively. Based on these analysis methods, we will present case studies that point out the empirical language use in this partial corpus in the following paragraph.

In this paper, we refer to five situations, which kindergarten teachers designed and realized to support mathematical learning. Besides the general topic *measurement* and the involved children, there were no content-related or structural prompts for the realization of mathematical situations.

Situation	*Teacher*	*Children*	*Magnitude*
A	Doris (MA[3])	Nikola (f): 4;2 / BL[4]; Orania (f): 3:10 / L1 (Greek); Regina (f): 4;4 / L1; Uwe (m): 3;11 / L1	length
B	Sabine (MA)	Mona (f): 5;5 / L1; Omara (m): 4;11 /L2 (Tamil); Sadira (f): 5;11 / L2 (Urdu); Theresa (f): ? / L2; Oslana (f): 5;3 / L2 (Croatian)	length and volume
C	Berna (L2 /unknown)	Bella (f): 6;0 / L1; Can (m): 6;0 / BL (Turkish); Denis (m): 6;0 / L1; Friedel (m): 6;0 / L1	length
D	Johanna	Ona (f): 5;6 / L2 (Turkish); Tamila (f): 4;10 / L2 (Pasto / Afghan)	length
E	Linda	Irvin (m): 5;0 / L1; Torben (m): 5;5 / L1	weight

Table 1: Basic information on the focused situations

Difficulties and deviations in language usage

Example from situation D: The kindergarten teacher and the two girls are building towers with colored rods and building blocks of different sizes. One tower of the teacher's construction falls down, which she comments on: "huu jetzt is es gefallen \" (huu now it has fallen). Ona takes up this structure: "deiner war nicht gut meiner hat nich gefalln." (yours was not good mine has not pleased). Her utterance is grammatically correct. However, using the auxiliary "hat" (to have) instead of "ist" (to be) like the kindergarten teacher for the perfect tense, Ona expresses the meaning of 'pleasing' instead of 'falling'. Certainly, this was not Ona's intention. Thus, semantic deviation can only be determined by focusing one's attention to the context.

[2] For more details see our analyses in the next paragraph.

[3] MA: trained in mathematics.

[4] L1 means, the child learned and uses German as a first language; L2 means, the child learned another language than German as a first language, now learns, and speaks German as a second language; bilingual (BL) means, the child learned German and another language as first languages and now uses both languages at home.

According to Bishop (1988), "measuring (...) is concerned with comparing, ordering, and with quantifying qualities" (p. 34). Comparing, ordering, and quantifying qualities ask for a differentiated language usage, including certain technical terms and grammatical structures. In the next sections, we illustrate semantic deviations in this context. That means we look for language productions that are syntactically correct but their initial meaning does not fit with the context of actions.

Verbal constructions with *to measure*: Measuring (yourself) with something or someone

In Brandt and Keuch (in press) we explain how linguistic valence (Herbst & Götz-Votteler, 2008) can be used to explain the emergence of a cognitive concept of measuring and the acquisition of case endings in relation to the verb *to measure*. With the verb *to measure*, you normally use a subject (someone who measures), something that is measured (the accusative object) and a tool you use for measuring (the dative object). There are, however, situations in which children as well as kindergarten teachers use this expression in a slightly different way.

In situation C, measuring the children's body lengths occupies most of the situation. The children lie down on the floor and have the position of their head and their feet marked with chalk on the floor. Subsequently, the distance between those two chalk lines is measured with different devices:

Berna	you can actually measure it with all those things here
Can	wait . I measure it with the chalk \ here it starts (draws a line from one limiting line to the other)
Berna	so Can / now wait \
Can	sooo \ (.) up to my line \
Berna	up to your line

While Can's utterance is syntactically correct, his actions do not fit with its meaning. If he was measuring a certain length with the piece of chalk he carries in his hand, he would aim to find out how often that piece of chalk fits into that length. The group had used a building block before in a similar way. What he does, instead, is to draw a line from one point to another. Since he incorrectly uses the verb *measure* in this context, probably synonymously to *draw* or even *connect*, we consider his utterance as a semantic deviation.

In German, as well as in other languages, certain words used as a collocation in combination with certain prepositions or complements can have a different meaning than the original word, often metaphorical or figuratively. For the verb *measure*, if used with a reflexive pronoun, it gets the meaning of competing with someone (in any possible way, not limited to magnitudes). In situation B, the kindergarten teacher Sabine asks Oslana to stand back-to-back with Sadira and compare their sizes. She accompanies her request with the words "Willst du dich jetzt mit der (.) Sadira messen?" [Do you want to measure yourself / compete with Sadira?]. Sabine does not seem to notice the ambiguity in her utterance on the one hand and the children do not seem to notice the figurative meaning on the other hand. In the course of the situation, Sabine leaves out the reflexive pronoun. She now asks Omara "Whom do you want to measure with?" While the meaning is probably relatively clear due to the unambiguous situation, the dative object is no longer a measuring tool but a person, which could lead to confusion. One could also argue whether the sentence is really any

longer syntactically correct. Mona (the only child whose mother tongue is German in this situation) finally takes up Sabine's sentence structure and says, "I want to measure with you". In contrast to the usual valence, the dative object ("with you") does not represent the measuring device but it rather works as an adverbial phrase, expressing the kind or manner how the activity of measuring shall be done.

The use of personal pronouns with comparisons

In almost all situations, the groups address (direct) comparisons of sizes. When it comes to someone's own body length, competitive situations emerge quite often. For the children it is important to know "Who is taller than the other?" or "Who is the tallest?". This aspect of rivalry is especially obvious in Situation E, when Irvin and Torben compare different things with a beam balance. The kindergarten teacher has prepared different building blocks and plastic figures, which possess certain weight proportions. The main idea of Linda's arrangement seems to be producing balance with these special objects. Both children use one scale together and each child fills the balance pan on their side. In their first attempt, in Irvin's balance pan there is one green stone and two blue ones in Torben's balance pan. The scale is in balance. The kindergarten teacher asks the children to compare the stones:

Irvin	ahh / that that is small and I am big /
Linda	right \ this is a bit smaller / and this is a bit bigger \

Irvin's sentence structure is perfectly correct from a syntactic point of view and in principle as a statement as well, since Irvin really is big in contrast to the building block on the scale. However, he probably wants to express that the green stone on his side is "big" in contrast to the blue stone on Torben's side. In this sense, Linda paraphrases his statement. She indirectly corrects his verbal expression (Brandt & Keuch, in Press), by formulating the relational connection "smaller – bigger" on the one hand, and the personalization "I'm big" connected with the pointing gesture to the actual object of comparison. While Torben uses correct possessive pronouns with corresponding comparisons ("then mine are / heavier\"), Irvin consequently uses the personal pronoun and therefore figuratively makes himself the object of comparison. Finally, Torben picks it up. With the following utterance, Irvin and Torben alike refer to the fact that the content of 'their' balance pan is heavier. Nevertheless, through the context of actions, both children are able to understand each other:

Irvin	then I'm heavier \
Torben	now I'm stronger hihihaha \
Irvin	yooo I'm the strongest \
Torben	no / I'm stronger \
Irvin	there I'm heavier \

Using the words *strong* and the related forms of comparison *stronger* and *the strongest*, the children focus on the idea of competition. However, at least Torben would be able to express himself correctly in such situations. Irvin as well uses the correct possessive pronouns at the end of the situation to explain, why "his" balance pan with the smaller (and therefore lighter) piece of cardboard is up: "Because this is very big / and mine is very small \" – interestingly this is a

situation in which he would not be the ‘strongest’. This competition, generated through language, gains momentum and prevents the original request to balance out the different objects through skillful placing.

Scale values and their verbalization

In most situations, the kindergarten teachers measure the children‘s body length and name and record them in different ways (some write them down, others document them with woollen strings, (Brandt & Keuch, in Press). When you capture body length with standardized measuring tools, you read the numbers on the measuring tools as a scale value. With measuring tools, the scale value indicates the corresponding measuring value based on a certain scale unit; for ordinary leveling boards or carpenter’s rules, that is centimeter.

When using measuring sticks and carpenter’s rules, the kindergartners on the one hand are confronted with measuring units (meter and centimeter), whose meaning they rarely comprehend and only hesitantly take over into their active vocabulary (Brandt & Keuch, in Press). On the other hand, they also have to deal with numbers that exceed their actively mastered range of numbers. The kindergarten teachers seem to be willing to make the numbers consciously perceivable as scale values with different circumscriptions and complements. In the following example, the focus on the meaning of the scale becomes obvious, when Doris refers to the animal symbols on the leveling board:

Doris	okay / look here \ one meter are you \ (.) hee \ one one meter one \ up to there \ [unintelligible] at the monkeys right \
Nikola	up to here \
Doris	exactly at the monkey \ and Uwe / (.) at what have you / [unintelligible] [at the sea lion\]

The kindergarten teacher therefore uses the animal symbols here as scale values; the connection with the local preposition “up to there” points to the distance from the floor to the symbol as a representation of the body length. The children take up the animal symbols on the leveling board for their comparisons of size:

Regina	the biggest ehm \
Uwe	is the duck

The generated verbal co-construction is a grammatically correct utterance: The duck is the biggest one in relation to a (not further specified here) selection of reference objects. This statement, however, is neither correct for the mentioned animal symbols (sea lion, monkey, duck) nor their real counterpart. Still, Uwe does not formulate a ‘wrong’ statement. A few minutes before, Nikola determined that the duck stands for the scale value 116 (Regina’s body length). Therefore, Uwe related with “the duck” to the corresponding scale value without using the corresponding local preposition. Regina is indeed the tallest child, as Doris confirms shortly after “Regina has [unintelligible] is the tallest”. The statement “The biggest is the duck” stands for the comparison of body length and gives an answer – at first with reference to the measured values – to the question: Who is the tallest? Concerning the linguistic means, Uwe treats the scale value ‘duck’ syntactically

like a representation of the measured length: “The biggest is 116 centimeter.” Interestingly, we also find comparable deviations in the language usage of our kindergarten teachers:

Sabine	now I measure you \ that means the hand is now on this / (.) und you are one meter and ten centimeters \ look \ and you are exactly (.) as big as this red number is \

Here as well Sabine is eager to make the numbers comprehensible for the children. On the carpenter’s rule used in this situation, the scale values are marked in red every ten centimeters, while all other numbers are black. The red number thus references the measured body length. Similar to Uwe, Sabine syntactically uses the red number as a representation for the measured size value 110 centimeters.

Conclusion

In this article, we looked at semantic deviations concerning verbal constructions with *to measure*, the use of personal pronouns in comparisons and the verbalization of scale values. Each of the analyzed sentences were syntactically correct, the semantic deviations, however, emerge from prepositions, pronouns; and additions and omissions of phrases. In everyday situations and action settings, these sentence constructions rarely lead to misunderstandings. In the analyzed situations too, the action flow is preserved. However, it remains unclear which conceptual understanding of measuring, comparing or scale values the children develop, which goes beyond the actual action context. The vague and imprecise use of *to measure* immediately concerns the meaning of measure as an activity, as well as the associated behavior patterns in relation to measuring devices. “Measuring” becomes the hypernym for the whole situation and is not delimited from other activities. By means of personalization, the comparison in the balance beam situation becomes a competition, and the semantic deviation becomes a play on words with its own dynamics. For linguistically less competent children, the pun might not be accessible and therefore they do not get a chance to improve their linguistic competences. The negotiation process related to the mathematical content stays at the surface, since it is overlapped by the play on words.

Ambiguity and change of meaning by using different prepositions as well as adding or omitting certain objects can lead to confusion in more in-depth negotiation processes. With regard to the development of less context-dependent language registers, one has to look critically at the observed reactions by the kindergarten teacher. Although all kindergarten teachers show pedagogical as well as didactic competences, in relation to our investigated difficulties and deviations we only observed minor language awareness. On the one hand, we just find a few reactions to semantic deviations in the children’s language productions. On the other hand, even our kindergarten teachers produce such deviations. Especially for learners of German as a second language, figurative language constitutes a specific problem. In this area, we still perceive a major challenge in order to establish educational equality via early education.

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Educating to rationality in a narrative context: An experimentation

Gemma Carotenuto[1], Cristina Coppola[1] and Roberto Tortora[2]

[1]Università degli Studi di Salerno, Salerno, Italy; gcarotenuto@unisa.it ; ccoppola@unisa.it
[2]Università Federico II, Naples, Italy; rtortora@unina.it

The focus of the paper is the analysis of written argumentation in solving logical-linguistic riddles by 6th and 7th grade students. This is part of a larger path dealing with the introduction of some logical contents, in which all the activities are immersed in a narrative framework. In analyzing students' productions, we pay great attention to the interplay between logical-scientific thinking and narrative thinking, with the awareness that a rigorous mathematical argumentation can be obtained only at the end of a path starting from different, often not rigorous, forms of reasoning.

Keywords: Language, written argumentations, logical riddles, narrative and scientific thought.

Introduction

In this paper we focus on the analysis of written argumentations produced by 6th and 7th grade students to solve logical-linguistic riddles. This kind of activities is part of a path carried out within a project in a secondary school near Salerno (Italy), during the year 2014-15. The aim of the whole project was to reconcile students with low level of mathematical skills with the subject. In accordance with the teachers of the school, the focus was on linguistic competences in a scientific environment, with particular attention to the development of the argumentative competence.

The starting point was the didactic path described in Tortora (2001), consisting of 15 structured worksheets. Its aim was to bring the contents of classical propositional logic to the students, through a fantastic and attractive way. The innovation with respect to the initial idea, favoured by the introduction of the logical-linguistic riddles (Smullyan, 1978), is the great attention devoted by us to students' reasoning. To give importance to students' answers, we have let them naturally emerge from a learning set in which discussion had a central role. In analysing students' productions, we can observe how their spontaneous reasoning is a first step toward the development of their argumentation skills. We are aware that a rigorous mathematical argumentation can be obtained only at the end of a long path that starts from different forms of reasoning, often not scientifically rigorous. This does not mean that different forms of rationality should be dismissed in favour of the specific mathematical rationality. We know that for each of us all the forms of rationality coexist more or less in our life, but what is important is the possibility given to all students as early as possible to acquire the special kind of mathematical rationality.

This is the specific purpose of this work, where we analyse some of students' productions in solving the riddles and we trace the development of their reasoning. Therefore, our main research question is to what extent, and by means of what specific didactic mediations, the use of logical riddles with their linguistic challenges, can favour the development of argumentative competences and of scientific language and thought.

Theoretical background

Language and in particular linguistic competencies are considered very relevant issues in mathematics learning. For example, Sfard (2000), to quote just a single seminal work, interprets thinking as a form of communication and considers languages not only as vehicles of pre-existing

meanings, but as builders of the meanings themselves. These competencies are the basis of many cross abilities, argumentation, communication, problem solving and so on, recommended as essential in all the official documents (for example, MIUR, 2012, Italian Ministry of Education).

In our works (e.g., Coppola, Mollo & Pacelli, 2010) we have often used logic in educational contexts, just because, in addition to being an important learning goal in itself, it has a special role in relation to language. In fact, logic appears as a privileged field for analysing the relation between language and interpretation, for identifying, studying and using linguistic manipulation rules and especially for the dual role of object and tool of investigation that language plays within logic (Ferrari & Gerla, 2015). The attention to the distinction between language and metalanguage is evident in our study, where the language is in the logical riddles and the metalanguage occurs in the discussions and the written argumentations used to solve them. However, no educational use of mathematical logic can be exhausted in its strictly disciplinary or formal aspects. These aspects may at most be considered a point of arrival, bearing in mind that in any case the way leading to the formalization is long and arduous. Along this road, the language takes on different forms and levels and the argumentations meet various needs. The importance of the contexts in which communication occurs and of the different forms of language has been widely recognized by the research that has put into the foreground the pragmatic aspects of language (Ferrari, 2004).

In general, the topics of pragmatic are deeply connected to the critical points of the research on learning and teaching mathematics (Ferrari, 2004). In our study, we use these tools to interpret some of the students' behaviors, elsewhere classified as 'irrational'. On the contrary, according to (Zan, 2007), we believe that the behaviors of the subjects 'getting wrong' may appear consistent when considered in relation to contexts and purposes other than those strictly adhering to rigorous logical reasoning. For this reason, we prefer to speak of two forms of rationality, rather than counter the rationality of mathematics with other behaviors that obey to different pulses. For example, according to one of the central issues of the pragmatic (Grice, 1975), in a particular context it is possible to make interpretive inferences based on the belief that who speaks or writes respects the Principle of Cooperation, according to which the communication is a collaborative process among those who are involved. These inferences, called conversational implicatures, differ by the logical implication, which relies only on the semantic content. Moreover, in making inferences in a certain context, it is frequent and legitimate to resort to one's own *encyclopedic knowledge,* that is the general knowledge of a person about the world (Zan, 2007).

The aspects of language brought to the fore by the studies on pragmatics are intertwined with the Bruner's distinction between two kinds of language or thought, the narrative and the scientific ones (Bruner, 1986). The scientific thought categorizes reality, recognizes the order of things, and produces demonstrative argumentations. It comes up in linguistic forms which are typically impersonal and timeless. Narrative thought, instead, interprets human facts: actions, intentions, desires, beliefs and feelings. It comes up in linguistic forms in which actions are performed by individuals and are accomplished in time. The acquisition of the first kind of thought and language, necessary for the understanding of science and mathematics in particular, is slow and it requires a careful didactic mediation, whereas the narrative way is more spontaneous and within everyone's means. For this reason, in many researches there are several suggestions for using narrative forms or even invented stories as a way to present mathematical contents (see, e.g. Zazkis & Liljedahl,

2009). Hence our decision to use a fantastic setting to introduce abstract concepts and present logical tasks. Our choice also depends on other reasons. It prepares the students themselves for using narrative modes. This establishes a working setting in which teachers and researchers can use and interpret students' answers in order to guide them through the gradual acquisition of forms of scientific language. Moreover, a third reason, regarding the logical contents we introduce, led us to design a fantasy narration. In fact, we agree with what Eco (2009) says about the relationship between narration and the notions of true and false:

> "[…] every statement in a novel draws and constitutes a possible world whence all our judgments of truth or falsity will refer not to the real world but to the possible world of that fiction [...] The epistemological function of such fictional statements is that they can be used as a litmus paper for the irrefutability of any other statement. They are the only criterion that we have to define what the truth is" (Eco, 2009, translated by the authors).

It is only in the context of an invented story that true and false are incontrovertible: for example, Rome could stop being the capital city of Italy, but Juliet will never stop loving Romeo.

Methodology

The study involved eighty 6th and 7th grade students of the same school with medium-low level of mathematical skills for about two months. The activities were carried out outside school time, in the presence of a mathematics teacher and a researcher (one of the authors of this paper). In a Vygotskian perspective, according to which the reasoning ability increases in the interaction among peers under the guidance of an expert (Vygotsky, 1934), the children participated in the activities working in small (2, 3 or 4 people) cooperative groups. Moreover, in accordance with the notion of *didactical cycle* (Bartolini, Bussi, & Mariotti, 2009), the activities were carried out with the alternation of different phases: exploration of the artefact, problem solving and collective discussion guided by the researcher. In our case, the artefact is the text of the riddle, so a linguistic artefact.

The students alternated their work with structured worksheets and with logical riddles. All the texts are adapted from the tales of "the knights and knaves island" (Smullyan, 1978): an imaginary island populated by two kinds of inhabitants, the knights who always tell the truth, the knaves who always lie. The activities on the structured worksheets, already used in the original path (Tortora, 2001), were proposed in the first part of every lesson. Their aim was to introduce in each lesson some of the basic elements of logic, e. g. the notion of proposition, the truth values, the distinction between simple and compound sentences, the logical connectives. The worksheet activities also gave to the researcher the opportunity to involve students in reflections about the differences between mathematical logic and common sense, as well as about the relativity of the notions of true and false and their dependence on the available information, the context and in some cases the judgment of the evaluator. The second part of every lesson was devoted to the solution of logical-linguistic riddles, as an application of the notions and the abilities acquired. From a formal point of view, the resolution of this kind of riddles requires that the students succeed in determining the only model[1] coherent with the dialogues in the text of the riddle.

[1] We use here the term 'model' to mean a correspondence that assigns to each character in the story the category he belongs to (knaves or knights).

All the collected data, that is, the students' written argumentations and the audio-recordings of their interaction within the group, have been analyzed. Here we refer only to the analysis of some written protocols, produced as answers to a single riddle. The task we examine is the solution of the first riddle, proposed at the end of the first lesson, after having introduced the notions of logical proposition and true values. We find this task the most interesting in order to reflect on how the students switch from one kind of thought and language to another, since in this first phase they were totally free from the influence of any didactic contract in solving linguistic riddles. Protocols have been examined on the basis of the awareness that different contexts and aims activate different forms of rationality and different linguistic styles (narrative vs. scientific). This aspect is crucial since we required students to logically solve linguistic riddles situated in a narrative environment. The analysis was carried out recognizing students' behaviors just on the basis of these categories.

The task that we examine is the solution to the riddle described in Figure 1.

Team:

Riddle 1

Oreste is in the knights and knaves island and he meets two persons, Alberto and Bernardo.

Alberto claims: "One of us is a knave, at least"
What can we say about Alberto and Bernardo? Can we establish which kind of inhabitant is Alberto? Can we know what kind of inhabitant is Bernardo? Discuss about this with your team mates. Then, write your reasoning.

Figure 1: Riddle 1 - The right answer is: Alberto is a knight and Bernardo is a knave

Analysis of protocols

We report four protocols[2], which seem to be meaningful and representative. We have selected them among the others, to show a spectrum of resolutions starting from a completely narrative approach until a prevalently scientific one. In them we have found also many interesting examples of conversational implicatures.

Protocol G1 (Fig. 2) gives us an example of completely narrative resolution of the riddle, without any explicit argumentation. In the first part, the students attach to Alberto the identity of a knight: probably, since Alberto speaks with Oreste, they affirm that "he seems sincerer". From that they deduce that Alberto is a knight and Bernardo is a knave.

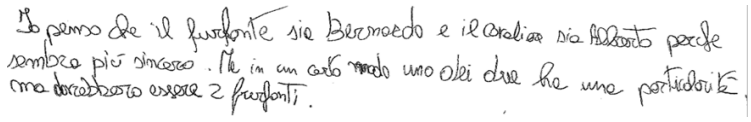

Figure 2: Protocol G1 - Answer: Alberto is a knight and Bernardo is a knave

G1: [First part] I think that Bernardo is the knave and Alberto is the knight because he seems sincerer. [Second part] In a certain way one of the two has got a particularity, but they should be 2 knaves.

[2] The labels G1, G2,…indicate the protocols of the different groups (Group 1, Group 2,…). In Figures 2 to 5 we show the original protocols (including some erasures), and then just below we report our English translations.

Many groups attribute conversational purposes to knaves and knights. In particular, here students seem to believe that the knaves prefer not to intervene in the dialogues, because they do not want to risk, exposing themselves to reveal their nature, whereas the knights speak freely, because they have nothing to hide.

In Protocol G2 (Fig. 3), the possibility that Alberto be a knave is excluded on the basis of encyclopedic knowledge: in fact, the group imagines that in that case Alberto would have said something different. Thus we have an example of narrative thinking, with an argumentation.

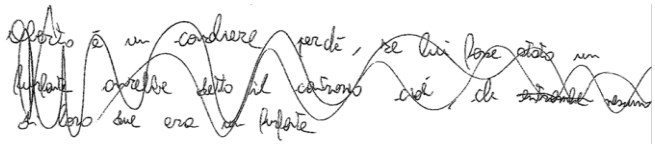

Figure 3: Protocol G2 - Answer: Alberto is a knight

G2: Alberto is a knight because, if he were a knave he would have said the opposite i.e., that ~~both~~neither of them was a knave.

Very often the students' priority in solving their first linguistic riddle seems to be to preserve the coherence of the story, based on their daily life experience. In this protocol, for example, we note a *change of script*, which is one of the most frequent phenomena we found in the first approaches to the riddles. By this we mean an argumentation used to exclude cases that appear to the students inconsistent with the narrative. When the students judge a case as inadmissible, they try to make examples of what the characters would have said in a case coherent with the narration ("*this case is not possible, because otherwise the character would have said so…*"). This change of script is in accordance with the cooperative purpose often attributed to the knaves.

Nevertheless, already within the activity of resolution of the first riddle, it is possible to notice the emergence of a different form of rationality. In many protocols, there is a transition from a first response, corresponding to an involvement of only the narrative thinking, to subsequent responses, in which the students bring into play simple forms of logical-scientific thinking. This evolution was supported by collective discussions, which took place during the activity. For example, in Protocol G4 (Fig. 4), we can read three successive different kinds of resolutions: exactly what we intend for a complete spectrum of different approaches to the riddle resolution.

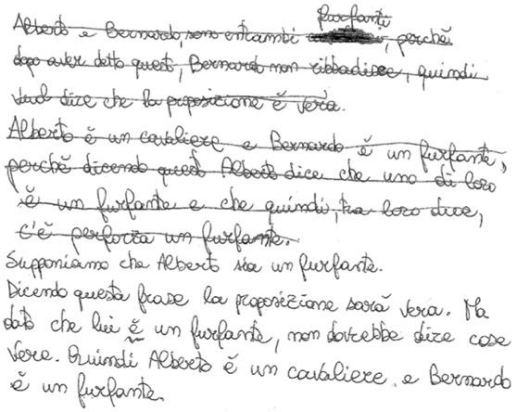

Figure 4: Protocol G4–First version answer: Alberto and Bernardo are both knaves. Second and third versions answers: Alberto is a knight and Bernardo is a knave

G4: [First version, erased in the protocol] Alberto and Bernardo are both ~~knights~~ knaves, because, after that saying, Bernardo does not rebut, therefore this means that the proposition is true.

[Second version, erased too] Alberto is a knight and Bernardo is a knave, because, saying what he says Alberto affirms that one of them is a knave and so, among them, there is necessarily a knave.

[Third version] Let us suppose that Alberto is a knave. By saying this, his sentence will be true. But, since he is a knave, he should not tell true things. Therefore Alberto is a knight; and Bernardo is a knave.

In the first version the answer is wrong and the argumentation seems ascribable to a totally narrative approach, with reference to personal encyclopedic knowledge. By saying "*Bernardo does not rebut*", he wants to express that "*Bernardo does not justify himself*": according to their life experiences the students interpret this attitude as an admission of guilt. In the second version the answer is correct. Nevertheless, in their attempt to argue, the students only explain the meaning of the sentence pronounced by Alberto, with special attention to the crucial expression "at least", which was examined during a short collective discussion. Finally, the third version, which maintains the correct answer, contains a "scientific" argumentation. It comes after a longer collective discussion, in which the researcher, comparing the productions of the different groups, pursued two principal objectives: to support students in re-situating the activity in the mathematical context, introduced in the first part of the lesson; and to build a shared more rigorous language. It can be said that the discussion favored the appearance of words like "*sentence*", "*true*", "*true things*" and, at the same time expressions like "*let us assume*", "but", "since" and "therefore", in this way supporting a complete "reductio ad absurdum" form of reasoning. A similar evolution can be found also in other groups, as we can see in Protocol G7 (Fig. 5).

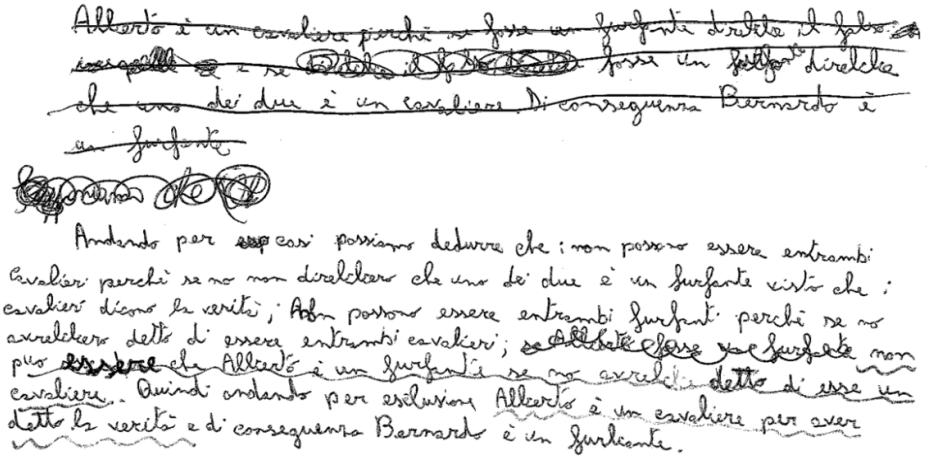

Figure 5: Protocol G7 - The answer in both versions is: Alberto is a knight and Bernardo is a knave

G7: [First version, erased] Alberto is a knight because if he were a knave he would tell the false and if he were a knave he would say that one of them is a knight. Consequently Bernardo is a knave.

[Second version] Going by cases, we can deduce that: they cannot be both knights, otherwise they would not say that one of them is a knave, since the knights say the truth; they cannot be both knaves otherwise they would have said

> to be both knights; ~~if Alberto was a knave~~ it cannot be that Alberto is a knave otherwise he would have said to be a knight. Therefore by exclusion Alberto is a knight having told the truth and consequently Bernardo is a knave.

In both versions the answer is correct. In the first argumentation, we can notice a clear predominance of the narrative thinking over the logical one, leading to a change of script, coherent with the conversational purposes attributed to the characters. On the contrary, in the second one, which comes after the long collective discussion, we notice the use of a scientific language, while the content is at the same time narrative and logical. In fact, the possibility that Alberto and Bernardo are knights is ruled out by means of a logically correct argumentation. The other two non-admissible cases, instead, are excluded on the basis of narrative argumentations, through changes of script. For example, the change of script "*they cannot be both knaves otherwise they would have said to be both knights*", is based on the conversational purpose that the knaves team up to hide.

Discussion and conclusions

It is well known that the logical formalism, although necessary, to a certain extent, for a full acquisition of mathematical notions, may constitute a difficult obstacle for students, due to its distance from common sense and to its exasperated exactness. The awareness of this risk was the starting point of our research and experience. For this reason, a first decision was to introduce the didactic activities by means of riddles, which are a sort of game. Secondly, these puzzles were immersed in a fictional context, using explicitly a narrative mode. But in the experimentation analysed in this study, the role of narrative has been twofold. On the one hand, as we have said, following a well-established research trend, we have benefited from the context of an invented story and its appeal to introduce some not easy logical-mathematical concepts; on the other hand, we have paid special attention to the narrative mode adopted by students in their oral and written productions. It seems to us that our experimentation has brought some interesting results. First, the path supported the students toward a strengthening of the metalinguistic control over the texts, spurring a reflection on the relationship between language and metalanguage. In our context, the object language corresponds to the sentences pronounced by the knights and the knaves, whereas the metalanguage is the one used in group discussions and in the production of written argumentations. Thus the students became aware of the dual role of language, as a communication tool and an object of manipulation. In addition, our way of introducing the activities allowed the students to grasp a first sense of the logical formalism, although deliberately not rigorous. Addressing the proposed activities, the students were gradually able to experience directly how a rational management of statements, at a first glance uninformative, could be very efficient. The narrative dimension has played a key role in providing a criterion of truth, which was naturally accepted by the students and which allowed to (partially) approach what logicians call a 'model'.

During the steps towards the resolution of the riddles, alternated with the collective discussions about the students' argumentations, it seemed to emerge a gradual evolution from a purely narrative approach toward an approach where some form of scientific thinking appears. Even in the solution of the first riddle, where the two kinds of rationality are intertwined, it emerges a shift towards a more conscious management of the two forms of thinking, spurred by the resolution of the logical tasks. This kind of evolution is supported by the emergence of a more and more rigorous language.

Our further step will include a deeper analysis of the oral discussions among the students, in order to try to better understand if and how the peer discussion, the comparison of different views and the necessity to write down the shared conclusions favour the transition toward a more and more sophisticated use of a scientific language. If this will be the case, there will be room for designing and experimenting further didactic proposals also in order to observe whether, in a longer period of time, there are positive repercussions on mathematical competences.

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Implicit and explicit processes of establishing explaining practices – Ambivalent learning opportunities in classroom discourse

Kirstin Erath

TU Dortmund University, Germany; kirstin.erath@math.uni-dortmund.de

Participation in mathematical practices is widely accepted as important for students' meaningful learning of mathematics. But how do students learn to adequately participate in these practices? This paper addresses the question for the specific case of oral explaining practices in whole class discussions. The study is theoretically based on merging an interactionist and an epistemological perspective to describe explaining practices as interactive processes in a classroom microculture while simultaneously keeping in mind the development of the broached mathematical content. The identified implicit and explicit processes of establishing explaining practices are exemplified and discussed with respect to ambivalences in the differing learning opportunities they offer.

Keywords: Explaining practices, student participation, discourse, interaction, implicitness.

Introduction

This study is based on the assumption that students' learning is inseparably linked to participation in classroom interaction, which is mainly based on verbal communication. Therefore, learning of mathematics is conceptualized as "a process of enculturation into mathematical practices, including discursive practices (e.g., ways of explaining, proving, or defining mathematical concepts)" (Barwell, 2014, p. 332). In this study, the discursive practice of explaining in whole class discussions is further investigated. Erath, Prediger, Heller, and Quasthoff (submitted) show that explaining is the most frequent discursive practice in German grade 5 mathematics classrooms. Furthermore, explaining has an important role for the meaningful learning of mathematics since it serves to communicate about more isolated pieces of knowledge, including talking about meanings and connections (Prediger & Erath, 2014). But how do students learn to participate in explaining practices and the corresponding epistemic processes? This question is investigated in this paper on the level of the interactive processes of establishing explaining practices in four microcultures.

Theoretical background: Explaining as practices of navigating through different epistemic fields

Following Interactional Discourse Analysis, explaining is understood as multi-turn units which are interactively co-constructed, contextualized and serve to convey or construct knowledge (Erath et al., submitted). This definition is extended and intertwined with an interactionist and an epistemological perspective from mathematics education: From an *interactionist perspective*, explaining can be conceptualized as a mathematical practice (Cobb, Stephan, McClain, & Gravemeijer, 2001) that is interactively established in a classroom microculture and that allows talking about collective mathematical development. Here, this concept is used descriptively to talk about identified ways of collective explaining processes in whole class discussions that are interactively established by students and the teacher. But not every explanation constitutes an own practice. To clarify this point, the notion of mathematical practices is enriched by the following definition from general educational science: "Practices are […] understood as rule-governed, typecasted, and routinely recurring

activities" (Kolbe, Reh, Fritzsche, Idel, & Rabenstein, 2008, p. 131; translated from German by the author). Therefore, mathematical explaining practices are conceptualized as *recurrent* ways of explaining that are treated as matching the classroom microculture from the participants' perspective.

The *epistemic matrix* was derived from research in mathematics education from an *epistemological perspective* (Prediger et al. 2014) and is used to further specify the notion of 'recurrent ways of explaining'. Different possible objects of explanations in mathematics are systematized in the lines of this matrix (not readable in Figure 1 that focusses on the depicted pathways), called *logical levels* (from top to bottom): Concepts, propositions, representations, and models (conceptual logical levels) and conventional rules, procedures, and concrete solutions (procedural logical levels). Each of these mathematical aspects can be explained by different means that are distinguished in the columns of the matrix (Prediger & Erath, 2014), called epistemic modes (from left to right): Labeling & naming, explicit formulation, exemplification, meaning & connection, and purpose & evaluation.

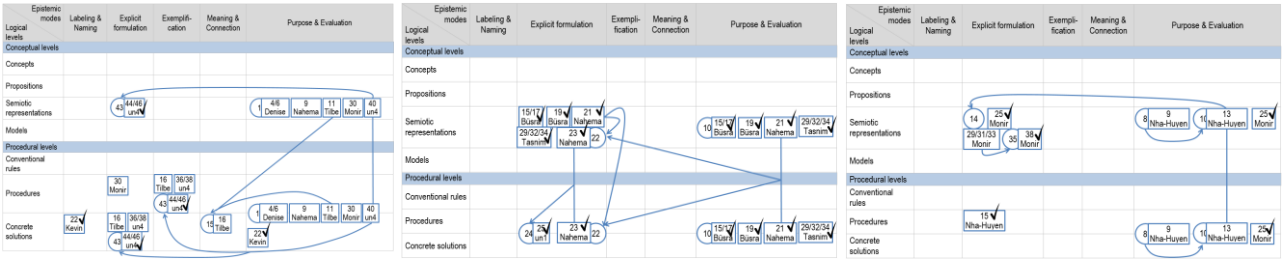

Figure 1: Three explaining pathways contributing to the practice of explaining 'good' representations

On the one hand, the epistemic matrix is used to characterize the utterances of students and teachers in explanations by analyzing which cells (called *epistemic fields*) of the matrix they address. On the other hand, the matrix is used to depict the *explaining pathways* that are interactively established by mapping these characterizations of utterances in the matrix (see Figure 1): Students' utterances are depicted as rectangles (including turn number and name), the teacher's utterances as circles with turn numbers. These pathways give access to the underlying mathematical structure of explaining sequences, which are especially characterized by the teacher's navigations through different epistemic fields (indicated by arrows in the pathways). Therefore, the matrix is used as tool of analysis and at the same time the associated language of 'explaining pathways' serves to specify the definition of explaining practices: Altogether, explaining in mathematics classrooms is conceptualized as practices of navigating through different epistemic fields, which are identified by building categories of pathways with similar structures (Erath, 2017). Hence, different practices are constituted by different defining patterns of their pathways describing the 'recurrent ways of explaining'. For example, all three pathways in Figure 1 have entries in the lines of "representations", "procedures" and "concrete solutions" in common and after working in the column of "purpose & evaluation" (more on the right) the teacher navigates towards the column of "explicit formulation" (more on the left). This recurring pattern defines the practice of explaining 'good' representations in one classroom: "navigating from the evaluation of a student's concrete solution and suggestions for improvement to deriving more general hints for drawing and characteristics of good representations".

But how do students learn to adequately participate in these interactively established explaining practices? Studies on the establishment of related norms in mathematics education (e.g. Yackel & Cobb, 1996) and discourse analysis (Heller, 2015) identify implicit and explicit processes as oppor-

tunities for students learning how to explain in their classroom. This work is extended to the processes of establishing explaining practices in this paper.

Methodology of the study

Larger data corpus. In the larger project INTERPASS, video data was gathered in 10 x 12 mathematics and language lessons (each 45-60 min.) in five different grade 5 classes (age 10-11 years) in an urban area of Germany. Eight lessons were observed in the beginning of the school year directly after the transition from primary school since it could be expected that processes of establishing practices are more explicit in this time of getting to know each other. Further four lessons each were gathered in the middle of the school year in order to get a long-term impression.

Sampling for the case study of this paper. The presented study builds on data of four mathematics classrooms chosen due to the following different characteristics in order to observe a broad range of interactions (Erath, 2017): Two higher tracked secondary schools (German: "Gymnasium") and two normal secondary schools (German: "Gesamtschule") and within each of these subgroups one classroom with students from a privileged and one with students from an underprivileged quarter.

Data analysis. This paper is based on analyses done for the PhD thesis "Mathematical discursive practices of explaining in different classroom microcultures" (Erath, 2017) enrooted in the larger project INTERPASS. In this context, all explaining sequences in whole class discussions were transcribed and analyzed by means of the epistemic matrix resulting in explaining pathways for each sequence. In a second step, explaining practices were identified in each classroom by developing categories of pathways with similar structures. In this way, three to five different explaining practices were explored in each of the four classrooms. In order to answer the question "How do students learn to adequately participate in the explaining practices of their classroom?" the interactive processes of establishing practices were further investigated. More precisely, it was explored if these processes were explicit or implicit and which turn teachers use in order to express their expectations and if this makes any difference for students learning opportunities.

All presented transcripts were translated from German and simplified (capital letters indicate stressed words, round brackets indicate phrases difficult to understand in the video data).

Empirical results: Processes of establishing explaining practices

The investigation of processes of establishing explaining practices in four German grade 5 classrooms (Erath, 2017) shows that there are some explicit but primarily implicit processes that contribute to the establishment of explaining practices. Furthermore, it comes to the fore that teachers (implicitly or explicitly) explicate their expectations in the turn of demanding an explanation as well as in the turn of responding to a student's utterance.

Explicit processes

Out of 16 identified explaining practices, only the practice of "explaining a concrete solution by means of a conventional rule" in Mr. Maler's classroom is recurrently established in an *explicit* way. The following transcript from the sequence "rounding on tens" (see Prediger & Erath, 2014 for a longer extract of the sequence) exemplifies how the teacher explicates his expectations for a 'good' explanation in his response to Kosta's explanation why 63 can be rounded on 60:

8	Kostas:	°hhh [*articulated clearing his throat*] Well, if you are rounding DOWN the sixty-three on TENS; then it comes, it gets, there must be ALWAYS a zero at the end, it MUST be,
9	Teacher:	[hm_hm]
10	Kostas:	[when you are rounding.]
11	Teacher:	On TENS yes.
12	Kostas:	And then there, if you take AWAY the three and shift the ZERO to it. So, you could DO that, but actually it's WRONG. You just have to round down and nea.. nearest number with a ZERO you have to write there.
…		
20	Teacher:	[…] and you already implied WHY; but does any of you know a RULE, HOW one has to proceed here, and when one here, when the ten stays the SAME? In this case, and the place BEHIND, which is rounded, goes to ZERO? Ha; [*4.5 sec. break*] Katja.
21	Katja:	With zero one two three FOUR you are rounding down and with five six seven eight NINE you are rounding (up). [*3.5 sec. break*]
22	Teacher:	Did EVERYBODY understand that?
23	Class:	YES [*affirms in choir*]

Kostas explains his solution by referring to the meaning related model of distance and closeness on the number line. This is implicitly rejected as a matching explanation by the teacher in #20, immediately followed by questioning the class about a rule that could be applied to explain the solution. In this way, Mr. Maler explicates his idea of a 'good' explanation of a concrete solution, which is underlined by his reaction to Katja's formulation of the corresponding rule. This sequence is an example of explicating expectations in responding to a student's explanation by navigating to the epistemic field (explicit formulation of a conventional rule) that would match for an explanation from the teacher's perspective and directly demanding an answer in this epistemic field. Another conceivable possibility would be that the teacher explicitly talks about which mathematical aspect of an explanation he values or which part did not match from his perspective.

These kinds of explicit processes can also be observed in Mr. Maler's demands for explaining a solution. The following extract from "rounding on thousands" in the context of a homework on rounding the length of rivers illustrates the case of explicating expectations in the turn of asking for an explanation by pointing to the expected epistemic field (explicit formulation of a conventional rule):

6	Tabea:	SIX thousand
7	Teacher:	GOOD; but now my QUESTION is, HOW did you arrive at this six thousand? Since we also want the RULE
8	Tabea:	Because from e:r,
9	Teacher:	to be CLEAR
10	Tabea:	Well up to five, well up to four, you have to round DOWN, and from five six seven eight nine you have to round UP.
11	Teacher:	EXACTLY. […]

After naming the right number (#6), Tabea is asked by the teacher in #7 to explain her solution. In this turn of demanding an explanation he directly states that she should refer to the conventional rule (shortly interrupted by Tabea): "HOW did you arrive at this six thousand? Since we also want

the RULE […] to be CLEAR" and in this way explicitly points to the expected epistemic field. Tabea follows this navigation (#10), which is explicitly evaluated positive by Mr. Maler in #11.

In both ways, the teacher explicates his expectations and reveals the recurring, typical structure of explaining a concrete solution by means of stating the related conventional rule. That is, this structure is made accessible to all learners and not only to those who can interpret the implicit processes of establishment (see below). Hence, the teacher's explication of expectations in the turns of demanding for and responding to an explanation are major learning opportunities for explaining a concrete solution adequately in this classroom (this must not hold for other classrooms since every microculture establishes different practices). Especially the way of explicating in the demand for an explanation seems to be important: This allows all children to contribute in the subsequent explanation even though they might not yet recognize the recurrent pattern of the underlying practice.

Implicit processes

Explicit establishments (see above) have only been found in rare cases in the data corpus. Instead, processes take course implicitly. Three different ways of *implicit* processes contributing to establishing explaining practices were identified and are exemplified in the following: (1) marking match or mismatch in responding to a student's explanation without giving reasons for the evaluation, (2) picking up only particular aspects of a student's explanation without explication, and (3) navigating recurrently to specific epistemic fields without revealing the underlying (intended) pathway.

An example for evaluating a student's explanation without further comments is the sequence "distinguish lists" from Mrs. Bosch's classroom. During revision at the beginning of the lesson, students are asked to distinguish the concepts of tally sheets and frequency tables.

12	Teacher:	[…] Now, WHAT was tally sheet, WHAT was frequency table, this PART, Barbara,
13	Barbara:	Tally sheet is where you did strikes; and frequency table is er [*4.0 sec. break*] er-
14	Teacher:	Can you HELP Maria?
15	Maria:	YES, when you did it all count up and then wrote it DOWN with numbers
16	Teacher:	EXACTLY. Well CAUGHT. […]

Mrs. Bosch marks Maria's explanation explicitly as matching (#16) but does not reveal the underlying pathway in her response: The analysis of several sequences on explaining concepts in this classroom unfolds that in this microculture a concept is adequately explained by means of addressing an epistemic field on the level of procedures, which means formulating an instruction for generating a representation of the concept. Therefore, the sequence is an example of an implicit process that contributes to the establishment of an explaining practice by marking an explanation as matching or mismatching without commenting on the reasons for the evaluation.

The second kind of implicit processes (picking up only particular aspects of a student's explanation in a response without explication) is concretized by the sequence "function of diagrams" from Mr. Schroedinger's classroom in the context of talking about different ways of presenting data.

1	Teacher:	[…] WHY they're doing quite frequently in printed media but also um on TV in the news, um why they're not giving a LIST like that […]

2	Nikolas:	um because maybe because this CATCHES one's eye much faster and um well that you can SEE this faster; so that something is BIGGER; because this is also bigger from its SIZE. So it's MORE because it's BIGGER from its size.
3	Teacher:	[*nods*] [Markus]
4	Marcus:	[Because you] can CATCH it very fast. For example um now up RIGHT I think there are such PERCENTAGES; because (that they) CATCH that well it's actually even BETTER than this; (also how many) PEOPLE;
5	Teacher:	hm_[hm]
6	Marcus:	[How] many SIBLINGS they have, because then in parts they would maybe have to always go THROUGH our classroom that small.
…		
9	Teacher:	THIS exactly meets the point, these two utterances. THEREFORE you normally do it in the form of such diagrams, because of the clarity actually […]

In his evaluation in #9, the teacher explicitly marks that the students' answers match but in his subsequent summary, only specific aspects are picked up: The teacher takes on the aspects related to functionality but he does not refer to the further issues of meaning (#2, "it's MORE because it's BIGGER") or examples (#4/6, number of siblings) addressed by the students. This selection is in line with the practice of explaining that is established during several sequences: In Mr. Schroedinger's classroom, concepts are adequately explained by referring to purposes and functionality.

The third way of implicit processes contributing to establishing a practice is the teacher's repeated steering to specific epistemic fields without revealing the underlying (intended) pathway. This directly refers to the definition of explaining practices that forms the basis of this study. Figure 1 shows three pathways of sequences that show the same regularities. These kind of pathways are identified five times in Mr. Schroedinger's classroom in the context of explaining how 'good' representations are designed. But, as with the two ways illustrated before, the underlying pathways are not revealed. The following extract from the sequence "lists of pets" exemplifies the navigation from evaluating a concrete solution to formulating hints for generating 'good' representations.

1	Teacher:	What would you SAY which ADVANTAGES, DISADVANTAGES [*break 1.3 sec*] have these particular ways of writing it down; […]
34	Büsra:	Well, in my POINT of view number two is BEST [*break 1.7 sec.*]
35	Teacher:	WHY; [*break 1.2 sec.*]
36	Büsra:	Yes because it doesn't that much TIME and em like Monir-Zohir already SAID em it only takes you like two MINUTES or so-
37	Teacher:	hm_HM, [*break 1.3 sec.*] But with number two TOTALLY obvious- something is MISSING in order to make it as clearly arranged as POSSIBLE […] WHAT is missing TOTALLY obvious with number two so you can SAY yes THIS makes somehow sense- this there you need LITTLE time- this is SOMEWHAT clearly arranged
…		
40	Uwe:	the NUMBERS; [*break 1.7 sec.*]
41	Teacher:	SAY again- WHY does it make sense to write numbers behind it?
42	Uwe:	So that you don't always have to count THROUGH;

43 Teacher: [*writes on the transparency*] EXACTLY; […]

Mr. Schroedinger initiates (#1) the evaluation of the representations and after some students stated pros and cons for the different representation, he navigates to formulating suggestions for improving the lists (#37), which helps clarifying how good lists should be designed. By repeatedly starting from a student's concrete solution and navigating from its evaluation and suggestions for improvement to deducing more general hints for drawing and characteristics of good representations this explaining practice is established across several sequences.

Although the three presented ways of establishing explaining practices must be characterized as implicit processes, they serve as opportunities to learn how to participate adequately in whole class explanations, at least for some learners. But the examples also show how challenging it is for other students to interpret these implicit processes (see e.g. Gellert, 2009, for further discussion of divergent learning opportunities related to implicitness). In the cases of marking matches and mismatches or picking up particular aspects without further comments, it might be challenging for students to follow since they must relate the teacher's evaluation to a classmate's utterance that is not present any more. The third way (recurrent teacher's navigations) allows all students to contribute in the explanations as long as the teacher explicitly demands for the shifts of epistemic fields. However, it probably takes students several sequences of one practice to recognize that there is a pattern and that knowing "how explaining works" is important for adequately taking part in whole class discussions.

Conclusion and discussion

The distinct dominance of implicit processes found in the qualitative analysis of the video data corpus is in line with research that identifies criteria of "successful participation" and "expected student contributions" (Gellert, 2009, p. 131, translated from German by the author) as often staying implicit. The presented study deepens these findings for explaining practices. Furthermore, the epistemic matrix offers a possibility to talk about mathematical aspects of 'good' explanations and to make the hidden regularities visible and discussable by means of the pathways. The dominance of implicit processes also suits the observation that explaining (as well as oral communication in general) is not treated as an explicit *learning goal* by the teacher (Erath, 2017). Instead, explaining serves as *learning medium* that is used without talking about adequate participation beyond general social behavior, i.e. the mathematical aspects of 'good' explanations. Moreover, it becomes apparent that learning how to adequately participate in whole class explanations is a learning process (across several sequences and lessons) and hence especially not a feature that students bring to the classroom but a competence that can be acquired in the interaction of collective explanations guided by the teacher.

The explicit and implicit processes of establishing explaining practices relate to different learning opportunities for students as explicated above: More explicit processes are eligible since they reveal the underlying patterns of the pathways and provide more students access to this mathematical aspects of participation, not only those who are able to also interpret the implicit processes. However, this does not imply a call for direct instruction: Talking about language on meta-level while simultaneously talking about mathematical content may ask too much especially from weaker students. Instead, it is about making the criteria for matching and mismatching utterances in relation to a practice accessible in responses to explanations or in the turn of demanding for explanations. Thereto, teachers need to be sensitized for the role of oral explaining as a learning goal in order to be able

to deliberately initiate particular practices. But as a first step, research in mathematics education should further specify which explaining practices are reasonable (also related to general discourse acquisition) or rather necessary in which grade in order to help teachers to enable even more students to actively participate in oral explanations and the corresponding epistemic processes.

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Young children's appropriation of mathematical discourse: Learning subtraction in a plurilingual classroom

Marie Therese Farrugia

University of Malta, Malta; marie.t.farrugia@um.edu.mt

I describe a teaching experience I carried out in Malta with a class of 5-year-old children of different language groups. The language of instruction was English and the topic subtraction. I explicitly taught mathematical expressions and sentence frames and planned class and paired activities wherein the children themselves would use the language to express the concepts at hand. The theoretical framework underlying my interpretation of the children's efforts is learning-as-participation. More specifically, I used Krummheuer's empirical model designed for interpreting classroom interaction in terms of producers and recipients. My teaching experience illustrates that with careful attention to both mathematics and language objectives, young learners in plurilingual Maltese classrooms can appropriate and use mathematics discourse within structured activities. However, more research is needed with regard to how students might use this language to author novel contributions.

Keywords: Learning-as-participation, mathematical discourse, subtraction, elementary education.

Introduction

Atlhough the academic language for mathematics in Malta (an ex-British colony) is English and written texts are in English, interaction in classrooms is usually conducted through both Maltese and English. However, the number of non-Maltese students in classrooms in increasing over time, prompting teachers to use more English during lessons; although English may not be the non-Maltese children's home language, it is more likely that they would familiar to some extent with English rather than Maltese. Anecdotal evidence suggests that teachers generally view this situation to be problematic, as they need to cater linguistically for Maltese and non-Maltese children. I carried out a study in the form of a teaching experience with a mixed language group of children with the aim of encouraging students to learn and use topic related language. My reasons for carrying out the study were (1) to apply to the Maltese context, the recommendation from the field of mathematics education for teaching mathematics language explicitly; (2) to focus on specific aspects of the mathematics register. I hoped that the reflections would aid me in my discussions with local colleagues, trainee-teachers and policy-makers on medium-of-instruction issues. My research question was: *How can children of different language groups be supported to learn English mathematical language*? By 'learning' I mean that the children would be able to express mathematical ideas through appropriate language during and after a series of lessons wherein specific language structures were emphasized.

Learning mathematical discourse

Several researchers, among them Gibbons (2015) and Murray (2004), make a strong case for teaching students explicitly how to talk about mathematics. Learning the language of mathematics allows individuals to express the ideas and concepts that form the discipline of mathematics, and by learning the language pupils begin to be enculturated into it (Lee, 2006). Thus, children learn the *discourse* of the discipline. This includes more than just learning subject specific vocabulary. Rather, it involves

learning - and using - the 'ways of saying' particular to the subject. For example, Sammons (2011) states that students need to learn how to formulate questions, make inferences and predictions; Murray (2004) and Gibbons (2015) recommend reflective journal writing as part of learning mathematics while Gerofsky (2004) considers word problems to be a genre forming part of the discourse. Whereas learning mathematical discourse is important for all learners, second-language learners have the dual task of learning the second language and content simultaneously (Bresser, Melanese & Sphar, 2009). Consequently, the teacher of second-language learners faces the challenge of not only making the mathematics lessons comprehensible for students, but also of ensuring that the students have the language needed to express their grasp of mathematics concepts. Coggins, Dravin, Coates and Carroll (2007) and Melanese, Chung and Forbes (2011) offer suggestions for classroom activities to support these students in learning 'academic' language, while Gibbons (2015) suggests that when planning lessons, both subject and language objectives should be listed.

In her consideration of mathematical language as a 'register', Morgan (1998) highlights grammatical features such as the imperative (command) and nominalization (nouns derived from verbs like *rotation* and *construction*), the passive voice (e.g., a line is drawn), being concise and the use of symbols. Whereas Morgan refered to written mathematics, these features also contribute to spoken mathematical discourse. Prediger and Wessel (2011) link the verbal register with what they call non-verbal registers including concrete, graphical and symbolic-numerical representations. They note that cognitive development of mathematical concepts is deeply connected to the ability to relate concepts in different representational modes. In this study, particular features of the register were addressed and developed by using them alongside non-verbal represenations.

Theoretical framework

Lave and Wenger (1991) proposed that learning can be considered as participation in a community of practice. As students progress in their learning of an apprenticeship, they move from what Lave and Wenger call 'peripheral' to 'full' participation (p. 37), which involves learning the tools of the activity and gaining autonomy. Dealing with a similar theme of apprenticeship, Rogoff (1995) writes about two concepts: *guided participation* and *participatory appropriation*. The former refers to the mutual involvement of individuals, including communication, in a collective valued activity. 'Participatory appropriation' refers to the process by which individuals transform their understanding of, and responsibility for, activities through their own participation. In the context of the mathematics classroom, 'learning' may be taken to be the participation in the practice of the discourse of mathematics. Krummheuer (2011) has queried how the sensitizing notion (Blumer, 1954) of learning-as-participation can be described or represented on an empirical level and he proposes a framework for this purpose. He widens the dyad 'speaker/listener' to *multiple* roles so as to account for the multiple individuals normally participating in a group conversation such as that of a classroom context (ibid, pp. 84-85). For participants who are listening (recipients), he proposes the following roles; the first two refer to direct participation, while the latter two to indirect participation:

- Conversation partner (addressed by the speaker);
- Co-hearer (unaddressed by the speaker);
- Over-hearer (tolerated by the speaker);
- Eavesdropper (excluded by the speaker).

For participants who are speaking (producers), Krummheuer suggests these roles:

- Author (responsible for the content and the formulation of an utterance);
- Relayer (not responsible for either content or formulation; echoes the author);
- Ghostee (takes over the identical formulation and uses it to try to express an original idea);
- Spokesman (expresses the same idea with his or her own formulation).

As Krummheuer points out, this model has the advantage of accounting for the process of moving from 'legitimate peripheral participation' (as evidenced by the roles of eavesdropper, over-hearer, co-hearer and relayer) to 'full participation' (role of author), through the intermediate stages of spokesman and ghostee. Using this model, it is possible to interpret classroom interaction and participation, and hence learning, in a more specific way. Whereas Krummheuer (2011) used the model to interpret a small-group discussion, I apply it to a whole-class setting.

Research context and method

An ethnographic approach was considered suitable since I wished to describe the practices of an educational community. Since my focus was the details of a teaching/learning context, I considered the case study method to be appropriate (Yin, 2014). I approached a school where the Head of School was an acquaintance of mine, and she put me in touch with a Grade 1 teacher, whom I call Ms Jenny. Most of her children were 5 years old at the time of the study. The class comprised 22 children and exemplified a 'superdiverse' context (Barwell, 2016). There were nine children of whom both parents were Maltese; seven children had one parent Maltese and the other non-Maltese, namely Australian, Irish, Bulgarian, Serbian, South African, two Libyans. Five children's parents came from varying countries: Italy, Greece, Hungary, Ecuador, South Africa. One boy had a Finnish father and a Kenyan mother. All the non-Maltese children understood English with different levels of confidence; I am not in a position to know their exact language experiences through which they learnt English, but from the conversations I had with some of the children it transpired that those who had one Maltese parent used English with this parent. Some of the non-Maltese children understood some Maltese and could say a few words. The Maltese children spoke Maltese fluently and understood English, but their speaking proficiency varied. Ms Jenny's class situation prompted her to use English as the medium of instruction. Some of the Maltese children used Maltese when communicating on a social level. I did not hear any of the other languages during my time in the classroom, although I cannot exclude that the two Libyan children might have used Arabic to communicate outside the classroom. Ms Jenny and I agreed on the topic to be taught: Subtraction. 9 one-hour lessons were given and these were video-recorded. The focus of this paper is on the 5 lessons on subtraction as separation or 'take away'. Parental and the child's own consent was sought for children to show up on the cameras. If either withheld consent, the child was placed out of camera view. I interviewed some children individually before the lessons, asking them about the languages they spoke; I also confirmed that Subtraction was going to be new to them as a school 'topic'. Three days after the lessons ended, I spoke to the children again, asking what they recalled. Children were chosen on the basis of consent obtained from both themselves and their parents; six were interviewed prior to the lessons, seven after. The interviews were audio-recorded.

I wished to present the children with mathematics/language and to see if, and how, they would appropriate the targeted discourse. The mathematical objectives of the lessons were: subtraction as

separation using pictures, blocks and fingers; symbolization (5 – 2 = 3) and translating story problems into subtraction operations.The related language objectives included the following structures: making statements using specialist vocabulary with regard to items ("Five [blocks] take away one is four"), using the imperative ("Take away two!"), asking a mathematical question ("How many left?"), 'reading' consicely symbolization (5 - 1 = 4 read as "five minus two equals three"), and formulating story problems orally (" Ms Jenny has 5 cookies. She eats 2 cookies. How many are left?"). Resources included a story book, pictures, blocks and fingers. As suggested by Bresser et al (2009), I modelled sentence frames during whole-class discussion, then set paired tasks during which the children were encouraged to use similar language. Some pairs (based on consent) were recorded using an audio-recorder. Individual worksheets were also set, of which photos were taken after the children completed them. In order to analyse the data, I studied the lesson, pair-work and interview recordings in detail, together with the completed worksheets. I focused especially on children's contributions, now interpreting my original, general aim of getting children to use mathematical language in terms of Krummheuer's framework.

Teaching and learning subtraction as separation

Classroom interaction is a complex acitivity, with participants' roles interweaving. However, for the sake of presentation, I here tackle the roles separately. Names are pseudonyms. In the transcripts, the language is presented as stated by the children, and so in some cases may differ from standard English.

Teacher as author, children as recipients and relayers

In order to introduce the children to the new expressions, I first authored them myself within a whole-class discussion. In these situations, the children took the roles of either conversation partners or co-hearers, since I could not interact with all children simultaneously. In a typical 'whole-class' style of interaction, I sometimes drew on particular children ("This one's for Andrea") while at other times I selected children with raised hands, or allowed a chorus answer.

The following is an illustration of how I introduced an expression and encouraged the children to relay it back. The conversation follows Ms Jenny's reading of the story 'Monster Musical Chairs' (Murphy & Nash, 2000) during which I had used statements such as "Three monsters take away one monster leaves two monsters", while showing up large number cards.

MTF:	(*Refering to the monster pictures attached to the whiteboard*). I'm going to say something important: "Six take away one leaves five". (*MTF simultaneously removes one card*). Now I need Dragan to say "five take away one leaves four".
Dragan: (Serbian/ Maltese)	(*As MTF removes another card*). Five take away one leaves four.
(…)	(*A short while later with reference to three attached monster cards*).
David (Maltese)	Take away …
MTF:	*(Indicates the three cards attached to the whiteboard*). First say how many there are. (*Slowly*) Three – take –away – one – leaves – two.
David:	(*Saying it with me)*. Three take away one leaves two.
Children:	(*Some children in the class say it with myself and David*).
MTF:	I want to hear you say it.
Children (chorus):	Three take away one leaves two.

For some children, relaying was not a trivial matter. For example, in the second lesson I introduced the question "How many left?" or "How many are left?" During this lesson, Lili (Hungarian) relayed this as "How many is?" but the following day I overheard her ask the question correctly to her task-partner. Initial difficulty may be due to the fact that English might not be a child's first language. Age may also have an impact on how quickly a child might pick up a new expression; these young children were still developing general language communication skills. Once a key phrase was practised a number of times, I encouraged children to offer their own examples, thus giving them the opportunity to act as ghostee or spokesman.

Children as ghostees and spokesmen

Taking the role of ghostee (identical formulation, original idea) first occured during class discussions. For example, in the second lesson I showed up picture cards, starting with six, decreasing to zero, each time asking a child to express what we had observed during the monster story. By this stage in the lesson, we were using both expressions *take away leaves* and *take away is*.

MTF:	What shall I do with the picture?
Kylie:(Australian/Maltese)	Take away!
MTF:	So what shall I say?
Kylie:	Three take away one is two!

Although Kylie's idea was not 'original' as such, I still consider that Kylie had progressed a step ahead of simply repeating after me, or with me. By Lesson 3, the children had picked up a lot of confidence, sometimes using the formulation to 'jump the gun'. For example, in one activity I was asking children to show up a certain number of fingers, then take away (put down) a number of them. I had previously set two examples, guiding them with questions.

MTF:	OK, another example. Seven fingers…
Child 1 (unseen):	Take away four!
Sofia (Bulgarian/Maltese):	It's three!
MTF:	Listen carefully!
Child 2 (unseen):	Five! Five!
Child 1 (unseen)	No, four!
Children (chorus):	Seven take away four is/leaves three.

Another context in which the children took the ghostee role was during a structured paired activity. For example, in the excerpt below Sofia and Lennie were using monster pictures.

Sofia (Bulgarian / Maltese)	(*Puts out six pictures, removes one*). Six take away one is five.
Lennie (South African/Maltese)	(*Removes a picture*). Take away /
Sofia:	(*Interrupts*). FIVE take away.
Lennie:	Five take away one is four.
Sofia:	(*Removes a picture*). Four take away one is three.
Lennie:	(*Removes a picture*). Three take away one is two.

It took some time for some children to get accustomed to stating the original number; as Lennie did, they might say "Take away two is three". I drew their attention to stating the first number; this was important as a preparation for the standard symbolization $5 - 2 = 3$ that was introduced in Lesson 4.

It is not possible, nor necessary, to insist on identical formulations when working with mathematical discourse, since it would render the language-use artificial. Indeed, taking the role of spokesman (same idea, varying formulation) played a crucial part, since it allowed the children to express themselves freely and to draw on English *as they knew it* to express the mathematics at hand. This helped create an inclusive context, build up their confidence in English and mathematics, and allow me to gauge their understanding. For example, during a paired activity with blocks, the children were required to give an instruction to their partner (use of imperative, e.g. "Show six blocks"), then ask "How many (are) left?" Some children asked the question differently, for example: "How much is there?" (Sofia, Bulgarian), "How much is there now?" (Andrea, Ecuadorian), "How many blocks there left?" (Shania, Maltese) and "How many is the answer?" (Ritienne, Maltese). During the paired activities, Ms Jenny and I had monitored the children's work, using the new expressions as we interacted with them and, through questioning, encouraged them to use the expressions themselves. Plenary sessions in which we reviewed a lesson also allowed the children to express themselves as they wished, while using the new expressions.

MTF:	What was that special word we were using today?
Ian (Maltese):	We were taking away.
MTF:	Can somebody remember what we were doing when we were playing teacher?
Sven (Finnish/Kenyan)	We was … we was …we was asking to show … to show …to show ten blocks.
MTF:	Good! We were asking our friend to show blocks. And then, Sven, what did we ask them to do?
Sven:	To … to … to … to take away.
MTF:	And then what did we ask our friend? … Luca?
Luca (Italian):	How many there left?
MTF:	Very good! How many are left?

A number of the children had another opportunity to take the role of ghostee or spokesman during informal individual interviews I carried out with them. In this context, I asked open questions like "Tell me what you remember", prompting them to use the language – or similar - that we had focused on in the lessons. Following are two examples. In relation to word problems, Dragan used the story problem formulation to offer an example about himself, while Mohammed articulated a story sum with varying formulation, drawing on his knowledge of English as best he could.

Dragan	Dragan has five cookies and he ate three, and it's two.
MTF:	Do you remember that question we were asking?
Dragan:	Yes. 'How much are left?'

Mohammed:	Ms Farrugia is, have a … like … eleven biscuits and he eats … em … six biscuits.

Children as authors

The role of 'author' is one that implies original input by the speaker and hence the role implies a certain autonomy. In the lessons on subtraction as 'take away' I did not recognize instances when children acted as authors in the sense of them coming up with novel input that could shape the

discussion, or influence other chidlren's learning of subtraction. I believe that the reason for this was the structured nature of the activities. Due to my intention to stress and develop specific mathematical language, the whole class conversations were shaped by myself. Furthemore, the paired activities had particular instructions to follow. Although children offered their own subtraction examples on the worksheets, including drawings of their choice, I would still say that they were following quite closely the structures I had taught them. Of course, the children did digress in their talk during the pair work and as they worked out the written examples on the sheet. However, this alternative talk tended to be social talk, such as "Look, my monster is green" or "Hey! Don't take our blocks" and so on, as one might expect from children this age.

Conclusion

From a researcher perspective, my study served the purpose of supporting international research that highlights the benefits of giving explicit attention to academic/English language with non-English speakers. It provides an example of focusing on specific features of the register. My study also illustrates an attempt at addressing mathematical language in a plurilingual classroom, and an application of Krummheuer's (2011) framework. From a teacher perspective, I concluded that the strategies I had planned had been effective in reaching my aim which was to enable the children to use features of mathematics discourse, namely specialist vocabulary, the imperative, asking questions, interpreting symbolization and the story problem genre. Thus I went some way in guiding a group of children with very different language backgrounds and differing proficiency levels of English to appropriate the 'academic' mathematical discourse and hence to increased participation in the discipline (Lave & Wenger, 1991). However, according to Krummheuer (2011), *full* participation in the practice is achieved through authorship. Due to the structured nature of the class activities, I cannot say that the children fulfilled the role of authors; it is likely that open-ended style activities are required to allow for authorship opportunities.

In conclusion, I note that on one hand the explicit attention to language can help to set up a reciprocity of conversational English and mathematical discourse, with potential benefit for both aspects. This would seem to be an important teaching strategy for mixed-language groups. On the other hand, the attention to language in itself can be restricting unless further opportunities are provided for more open-ended tasks. In the latter tasks, one might hope that the language expressions learnt during structured activities might then be utilized as students offer novel ideas. This would result in students' authouring by using the mathematics register - surely the 'fullest' verbal participation that one can expect from mathematics learners. Of course, appropriating mathematical discourse is not something that can be achieved over a few lessons, especially in the case of very young children learning mathematics in a second or foreign language. The next stage in my line of research is to explore how newly learnt language structures can be encouraged in a plurilingual classroom to author original contributions to the development of the mathematics at hand.

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Language in argumentation and solution of problems with graphs

Pier Luigi Ferrari

Università degli Studi del Piemonte Orientale, Dipartimento di Scienze e Innovazione Tecnologica, Alessandria, Italy, pierluigi.ferrari@uniupo.it

The paper reports an investigation about undergraduates' argumentations to justify answers to elementary calculus problems involving the recognition of relationships among graphs, verbal texts and formulas. The examination of the texts produced over more than ten years highlights serious language difficulties and suggests that we cannot exclude that language is a key factor for the quality of arguments. The main goal of this study is to gain a better understanding of how language difficulties (depending on both competence and attitudes) affect argumentations.

Keywords: Language, problem solving, register, argumentation.

Introduction

The paper focuses on how undergraduates justify their answers to elementary calculus problems involving relationships among graphs, verbal texts and formulas, in the frame of an introductory mathematics course delivered in Italian to biology freshman students. The course is short (48 hours of lectures and 24 hours of optional tutoring sessions) and taught by two instructors, one of whom is the author. It is usually attended by more than 400 students, coming from various regions of Italy and Eastern Europe. The students' language competence is much varied. Due to the goals, the attendance, and the lack of time, I did not develop a standard course moving from the basic definitions of Calculus to get to theorems and applications, but focused instead on a few concepts such as graph symmetries and slope. Most of the tests administered as working material or examination papers require the linking of different representations of functions (symbolic expressions, graphs, verbal texts, tables of numbers). Through these activities participants are systematically asked to explain and justify their answers in writing, even informally. This requirement is aimed at discouraging guessing or rote learning and help the students to better understand the concepts involved. Morgan (1998) provides a description and discussion of the 'writing-to-learn' paradigm, and highlights the benefits of the use of writing as a means for learning. Very appropriately, she suggests (with the expression 'learning-to-write') that nobody needs to assume that students of any age have achieved the level of linguistic competence required in order to produce texts adequate to their goals, and challenges the assumption, more or less implicit in a number of studies, that language naturally develops and there is no need for deliberate language teaching (Morgan, 1998, pp. 37-49).

The scrutiny of the papers written by students over the years highlights serious difficulties with language and suggests that we cannot exclude at all that language competence is a key factor for the quality of arguments. If we admit that there is a link between language and thought, there is no reason at all to rule out the hypothesis that the quality of the texts a subject can produce or interpret could deeply affect the quality of her/his thinking, and thus of the arguments s/he can produce.

An investigation of this topic requires dealing with the texts involved as objects, not only as means to develop a discourse within a given context. The focus should be on the organization of the texts, not just on the corresponding communication process.

The main goal of my research is to get a better understanding of how language difficulties (depending on both competence and attitudes) affect the production of arguments by undergraduates. In this specific study I focus on problems involving graphs.

Theoretical framework

Research on argumentation has produced a large number of papers from a wide range of perspectives. Some researchers, such as Crawshay-Williams (1957) and van Eemeren et al. (1996), in different ways, have underlined the role of context in argumentation and the links between argumentation and language. Crawshay-Williams (1957, p.3), for example, claims that his work on argumentation "enquires how we use language as an instrument of reason" and argues that "[i]t is only possible to determine whether an empirical statement is true or false if the context of the statement is known." On the other hand, van Eemeren et al. (1996) relate the theory of argumentation to the pragmatic theory of speech acts (Austin, 1962), which takes into account not just the *propositional content* of a statement (i.e. the part of its meaning, based on vocabulary and grammar, that allows the receiver to identify the referents and possibly to establish whether the statement is true or false), but also the *speech act* (i.e., the fact of expressing a proposition in a specific context, which conveys also speaker's (or writer's) beliefs, attitudes and commitments, possibly influencing the hearer's (or reader's) ones. Toulmin's framework (2003), on the contrary, although it is widely adopted in research on argumentation in the context of mathematics education, seemingly pays very little attention to language and context.

As far as language is concerned, I adopt Halliday's (2004) account of the relationship between scientific language and science and his Systemic Functional Linguistics (SFL) (Halliday, 1985, 2004; Leckie-Tarry, 1995; O'Halloran, 2005). Halliday, whose research is in the field of pragmatics started by Austin (1962), argues that there is no learning of science without some learning of its language (2004, p. 160). The adoption of the SFL framework is justified by the opportunity of focusing on the functions of language in mathematics education, where the needs for effective representations of concepts and their relationships and algorithms is unavoidably at odds with those of effective communication. Multisemioticy is an important feature too, as the interplay among verbal, figural and symbolic representations is stronger in mathematics than in other fields.

In order to analyze the protocols, I am using the idea of *register* as a linguistic variety related to use (Halliday & Hassan, 1990). An enlightening discussion on registers in an SFL framework has been provided by Leckie-Tarry (1995). Morgan (1998) and Ferrari (2004) have applied this idea to mathematical language. Any individual has at her/his disposal a range of registers that s/he uses according to circumstances. The most relaxed registers, used in spoken (but sometimes also written) everyday communication are classified as *colloquial*, while those adopted in written (but sometimes also spoken) communication among educated people, for example in institutional, educational, literary, research contexts are referred to as *literate*.

Colloquial registers, in short, are characterized by their strong dependence on the *context of situation* (i.e., according to Leckie-Tarry, 1995, the space and time in which the exchange takes place, the participants…), which allows participants to negotiate meanings and makes it unnecessary to produce accurate and unambiguous statements from the beginning. Literate registers are less related to the context of situation. In colloquial registers the meaning of words is mainly taken from previous

experience, and most often much precision is not required to achieve the goals of the exchange, whereas in literate ones words have precise meanings, based on definitions (the so called *lexicalization*). In colloquial registers syntax is customarily relaxed, while in literate ones it is tighter. In colloquial registers there is an extensive use of *iconicity*, i.e. the analogy between the form or organization of a representation and its meaning. Iconicity is opposed to *arbitrariness* and can involve factors such as order (the order of facts matches the sequence of the representations). In literate registers representations are less iconic and more conventional. As a consequence, the interpretation and production of texts and representations in colloquial registers are quite unstable, since they depend on factors difficult to control (how the subject is accustomed to use words, how s/he interprets images, the mental models s/he uses in place of the definitions, the personal experiences s/he tries to recall, …). In literate registers, the lesser dependence on the specific situation, the reference to defined meanings (thus more objective and verifiable) and the role of syntax (objective and verifiable too) make the interpretation and production of texts more stable.

Even a quick analysis of what is described above should make it clear that most of the registers used in mathematical settings share the features of a literate register in an extreme way: in mathematical registers the interpretation of a text depends little on the context of situation in which it is produced. I am not referring to the processes of learning or communicating mathematics, but on the organization of mathematical texts, as they can be found in any mathematics textbook from primary to graduate schools. The dependence of mathematical language on the *context of culture* (any kind of systems of knowledge related to the participants and the topics of the exchange), on the contrary, is very strong (think of definitions, conventions, theorems…), as well as lexicalization and conventionality (there are not many other semantic domains where definitions play as important a role as in mathematics). The same holds for syntax: in a mathematical text, either symbolic or not, a minor variation (e.g., the displacement of a parenthesis or of a comma) can change its meaning. The interpretation of texts in mathematical registers is stable: in some cases it can be performed automatically. The use of colloquial registers is essential for learning as well: nobody could ever learn anything if s/he should use literate registers only. So, in learning mathematics the trouble is not the use of colloquial registers, but the failure to adopt literate ones when necessary.

Methodology

A large number of argumentative texts produced by freshman students to justify their answers to problems involving the interpretation of graphs, both in examinations and in tutoring sessions (including online ones) have been scrutinized. In this paper I take into consideration only texts produced for one specific examination. To understand the argumentations it is necessary to regard them related to the problem-solving context they are produced within, considering the solutions produced as well. This study is not aimed at testing a particular model but rather at understanding the difficulties of a relevant number of students with different backgrounds, cultures, attitudes, and levels of competence, also in order to improve our teaching and tutoring strategies. For these reasons I have used a large number of protocols taken from a real examination, as most often the weakest students are not willing to take part in other activities, such as special tutoring sessions.

Some of the participants have been interviewed after the test. For each participant I tried to classify answers and errors, if any, such as: use of pseudo-rules or of mathematically inappropriate models, wrong reading of the data, miscalculations, and language errors. I have also classified the kind of text

produced (basically, the register adopted, by means of the indicators suggested by Leckie-Tarry, 1995) to see if and how linguistic competence might have affected the answers. In some of the excerpts both the original Italian text and an English translation are given. The kind of analysis I want to carry out does not allow me to refer to an English translation only, which, even if it may convey with fair approximation the ideational component of the text, unavoidably it cannot but fail in conveying other aspects of the text, such as register or improper uses.

It is never easy to understand whether an error depends on the language (e.g., a proper idea wrongly expressed), on contents (e.g., a wrong idea truly expressed) or on both. For example, the (wrong) claim that function *g* below is decreasing in [0, 10] might depend on a poor understanding of the definition, or on the improper use of 'decreasing', or even on a wrong interpretation of the graph. This in turn might be affected by the everyday use of the same words. The analysis of cohesive devices (i.e., the linguistic resources used to link the parts of the text), as carried out by Alarcon and Morales (2001) is a classical way to deal with argumentation in a SFL setting. In the analysis of the protocols, I have applied two criteria: the appropriate use of cohesive devices (contrasted to improper use or no use at all) and the vocabulary (lexical vs colloquial use of words).

The problem

Here I focus on problems involving graphs, such as problems requiring to associate a formula to a graph, or a graph to a formula, or to link the graph of a function to the graph of its derivative. All the protocols (about 200) used in this study come from the following problem.

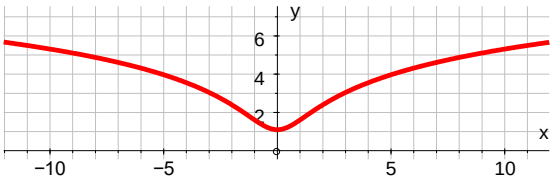
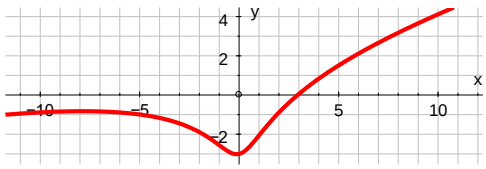
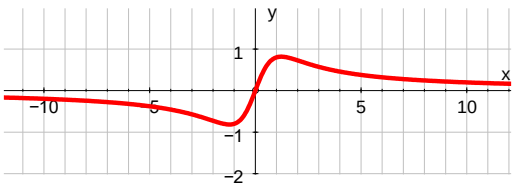
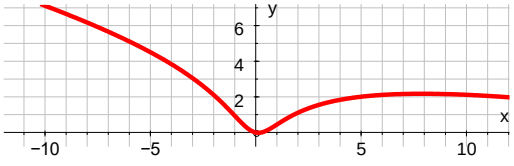
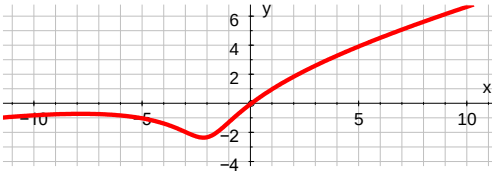
<table>
<tr>
<td>Consider the graphs A, B, C, D drawn below and choose three of them which, in the interval displayed, do not correspond to the derivative of the function g drawn on the right.
Justify your answer.</td>
<td>g
</td>
</tr>
<tr>
<td>A)
</td>
<td>B)
</td>
</tr>
<tr>
<td>C)
</td>
<td>D)
</td>
</tr>
</table>

Table 1. The problem

The problem is in negative form, as participants are required to identify three graphs that do not correspond to the derivative of the function *g*. It is manifest that in a problem of this sort it is not

possible, from the scrutiny of the graphs only, to decide that a graph does correspond to the derivative of a given function. On the other hand, it is possible, in many cases, to decide that a graph does *not* correspond to the derivative of a function. Considerations of this kind hold for a great deal of mathematical problems involving graphs. The negative wording of the problem has proved a source of trouble although it explicitly refers to the need for excluding three graphs.

I have regarded as acceptable all answers excluding the three appropriate graphs with appropriate justifications, i.e. argumentations where the properties of *g* and of the graphs the answer is based on are explicitly mentioned. For example, a text like "*g* is increasing in (0, +∞), so its derivative must be positive in the same interval, so I exclude graph A, which is partly negative in the same interval", has been considered a sufficient justification in order to rule out graph A, although the student has made no explicit reference to any theorem or rule. It is uncommon among freshman students to find explicit reference to some general property to justify an inferential step.

Outcomes

Although this is not a quantitative study, I often give some quantitative indication about the size of the groups adopting some behaviors. It might be interesting, from a teaching perspective, to know if a behavior is adopted by a small group of students or it is more general.

Correct answers equipped with acceptable arguments usually range from 20 % to 40 %, according to the sample and the task. In this experiment correct answers have been a bit less than 25 %.

Student A02 marks the graphs A, B, D and gives the following argument:

> "Non corrispondono, perché g è in positivo mentre A, B, C sono sia in positivo che in negativo."
>
> ["Do not match, because *g* is in positive while A, B, C are both in positive and in negative."]

The argument, which seems aimed at ruling out graphs A, B, C, is inconsistent with the marks on the diagram, which rule out graphs A, B, D. The argument adopted seems to fit graph D better than graph C, and one can imagine that the subject wrote 'C' in the argument by mistake. Errors of this kind are quite frequent. Second, the subject only deals with what s/he is looking at and makes no reference to mathematical properties connected to the problem, nor any attempt to link to each other the data s/he has mentioned. In other words, the argument is completely bounded within the context of situation, with no attempt to put it in a framework of knowledge, i.e., in a context of culture. Third, the text is inaccurate: the main verb has no subject, the expressions "in positivo", "in negativo", which are circumstantials of (spatial) location, are used in place of the more correct attributes ('positivo', 'negativo') and the expression "sia in positivo che in negativo" is vague.

Student A03 marks graphs B, C, D and produces the following argument:

> "in $x>0$ la fne della derivata dev'essere positiva quindi non è sicuramente la B, in $x<0$ la fne della derivata dev'essere negativa (decrescente) quindi non può essere la D e la C non è sempre crescente nell'intervallo (0,+∞). Penso sia il grafico a."

'fne' is an informal abbreviation for 'funzione' (function). In order to help reading, I translate it as the whole word.

["in $x>0$ the {function} of the derivative must be positive so surely it is not B, in $x<0$ the {function} of the derivative must be negative (decreasing) so it cannot be D and C is not always increasing in the interval (0, +∞). I think it is graph a."]

In this text some connection is hinted at but not developed, the student states (in her/his way) that the derivative must be positive for $x>0$ and negative for $x<0$ but s/he does not explain why. S/he seemingly identifies "increasing" with "positive" and (explicitly) "decreasing" with "negative", as s/he rules out graph B which is positive for $x>0$ but not increasing in most part of that interval. Moreover, s/he inconsistently does not rule out graph A, which is the only one with negative values for some $x>0$. The identification of "increasing" with "positive" and "decreasing" with "negative" may depend on poor understanding of the subject matter, but most likely it has linguistic roots, as this student seemingly makes no distinction between the words and most likely s/he refers to everyday-life uses, according to which "positive trend" might mean "increasing trend".

Student A36 marks graphs A, B, C (with some erasures) and writes:

"La funzione tra [0, +∞[$f(x)>0$ quindi la funzione è crescente quindi la B non è crescente. *[erased words]* funzione $g(x)$ è tutta positiva da da]-∞,0] è decrescente mentre da [0,+∞[la funzione è crescente l'unico che cresce sempre di più è la D."

["The function between [0,+ ∞[$f(x)>0$ so the function is increasing so B is not increasing. *[erased words]* function $g(x)$ is all positive from from]-∞,0] is decreasing while from [0,+ ∞"[the function is increasing the only one that always increases is D."]

The linguistic quality of this text is very poor. There is a bad coordination between the verbal and symbolic parts, the given function is referred to as 'f ' instead of using its proper name 'g', the conjunction "quindi" ["so"] is used twice inappropriately, in the second occurrence to introduce some data taken from a graph. There are a number of erasures and repetitions, and some of the last clauses are linked neither by discourse markers nor by punctuation.

Student A39 marks graphs A, C, D and writes:

"Escludo la C perché nell'intervallo (10; 0), la funzione decresce perciò la sua derivata dovrà essere negativa. Escludo la D perché la funzione è pari mentre il grafico D è dispari. Escludo la A perché la funzione g è crescente nell'intervallo [0;10) e quindi il grafico A dovrebbe essere positivo mentre è negativo per $x\in[0;3]$."

["I rule out C because in the interval (10; 0) the function decreases so its derivative will be negative. I rule out D because the function is even whereas graph D is odd. I rule out A because function g is increasing in the interval [0; 10) and so graph A should be positive, whereas it is negative for $x\in[0;3]$."]

In this case the choice of graphs is the correct one. Most likely in the expression (10;0) the subject has forgotten to write the sign '-' before '10' (although other participants wrote reversed intervals too). The motivation to rule out D is inappropriate, for it would have been necessary to recall that the derivative of an even function, if any, is an odd function and that graph D does not correspond to an odd function but it is neither odd nor even). On the contrary, the subject proceeds by analogy (f even $\Rightarrow f'$ even), missing the classification of graph A: s/he claims it is odd. Maybe s/he means that it is not even, but is misguided by the meaning of odd/even in the frame of integers.

Student A17 marks graphs A, C, D and writes:

> "Non corrispondono i grafici A-C-D. Possiamo escludere il grafico C perché per esempio nell'intervallo (-10;0), la nostra funzione *g* risulta decrescente mentre in quel tratto il grafico C risulta positivo (dovrebbe invece essere negativo). Possiamo escludere la A perché per esempio nell'intervallo (0;3), la funzione *g* risulta crescente mentre il grafico A in quell'intervallo è negativa anzi ché positiva. Escludiamo anche il grafico D perché nella funzione *g* la concavità è verso il basso tra (1;5) quindi nello stesso intervallo il grafico dovrebbe essere decrescente mentre la D è crescente."
>
> ["Graphs A-C-D do not correspond. We can rule out graph C because, for example in the interval (-10; 0), our function *g* results decreasing while in that stretch graph C results positive (it should be negative instead). We can rule out A because, for example in the interval (0;3), function *g* results increasing while graph A in that interval is negative instead of being positive. We rule out graph D too because in function *g* the concavity is downwards between (1; 5) so in the same interval the graph should be decreasing, while D is increasing."]

This excerpt underlines the difference between those who can use language in a mathematical setting and those who cannot. The text of A17 is not perfect, but language for her/him is a tool good enough to understand the problem, find a solution and justify it. The text is explicitly organized with conjunctions and discourse markers ("while", "so", "for example", "instead") and each statement is equipped with its own domain of validity ("…in the interval (-10; 0) …"). The general properties the argument is based on are not explicitly mentioned, but the subject adds some remarks that highlight the connections between the parts of her/his argumentation and make it unambiguous ("… it should be negative …", "… while D is increasing."). Although the subject does not write down some general rule or property, s/he underlines the critical points of her/his argumentation and uses language (including grammar) to organize and clarify her/his answer.

In the optional interviews performed in the week following the experiment, subjects A02, A03, A36 and A39 could not reconstruct their thinking and explain their answers. This is a general behavior: a great number of students cannot reconstruct the meaning of the text they have produced, even if they have it before them and are given time to read it with no pressure.

Discussion

The protocols examined have been chosen as representatives of diffused patterns of argumentation. In particular, the lack or improper use of connectives and discourse markers (i.e., of cohesive devices) is a serious problem: the links between the clauses are not made explicit or are expressed in a vague and improper way; even if the subjects, while writing down, may have some nice idea in mind, the lack of an explicit and effective objectification through language, prevents them from reconstructing and developing it afterwards. Behaviors of this kind are common.

Some students (such as A02) seem not to be able to recall the necessary pieces of knowledge and work on the data of the problem by creating pseudo-rules (e.g., *g* increasing/positive/even $\Rightarrow$ *g'* increasing/positive/even). Models of this kind are very robust. It is possible that these models are consequence of the practice of not interpreting the text of a word problem in order to reconstruct the problem situation, but to search for keywords that might suggest the proper.

The difficulties mentioned above all increase the instability of the processes of interpretation and production of texts, which might explain some apparently inconsistent behaviors; an example is protocol A39: the student answers correctly and correctly rules out graphs A and C reasoning on the basis of known properties of functions; to rule out D as well, s/he properly focuses on the evenness of *g*, but, probably in the attempt to apply the pseudo-rule "*g* even $\Rightarrow$ *g'* even" claims that D is odd; a number of students (more than 30 % of the sample in this experiment) correctly rule out A and C but use wrong or inconsistent arguments to rule out D too; the fact that in order to rule out D some 'rule' different from "*g* increasing $\Rightarrow$ *g'* positive" is required is enough to trouble the subjects and induce them to provide wrong answers.

Although much research is needed to determine the exact role of language in argumentation processes, it seems to me that the outcomes of this study suggest that it cannot be disregarded at all, in spite of the fact that a number of current studies on argumentation do not take the role of language into account. On the other hand, SFL seems a promising framework to better understand students' linguistic behaviors in a mathematical setting, disregarding neither the factors related to interpersonal communication nor those related to the specific features of mathematical language.

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Talking with objects

Marei Fetzer[1] and Kerstin Tiedemann[2]

[1]Goethe-University Frankfurt, Germany; fetzer@math.uni-frankfurt.de

[2]Bielefeld University, Germany; kerstin.tiedemann@uni-bielefeld.de

Both language and objects seem to play an important role in mathematics learning. In our research, we focus on their interplay: How do language and objects support students' development of mathematical ideas? In order to develop a framework of 'talking with objects', we draw on three approaches. First, we adopt the idea from Bauersfeld that learning is a domain specific process. It is always bound to a very specific situation and context. Second, Aukerman's approach of re-contextualization supports our insight in the link between language and context. Third, Latour's Actor-Network-Theory helps us to better understand, how concrete objects take part in the process of constructing social reality in mathematics lessons.

Keywords: Language, objects, recontextualization, domains of subjective experience (DSE).

Introduction

Children are supposed to learn what a 'number' is or what we mean by 'addition'. The challenge of mathematics learning is to construct abstract mental objects that can neither be touched nor seen. Even if we cannot see the mathematical objects themselves, there is a lot acting and handling of and with concrete objects to be observed in everyday mathematics classes: Children write, read, and work with different concrete objects like bead frames, hundred boards or Dienes blocks. And they speak: They ask questions, explain their ideas and discuss about different mathematical interpretations. Obviously, both objects and language play an important role, when students are learning mathematics. Children and teachers use physical representations of mathematical objects in order to clarify what they are talking about and what they are referring to. These physical objects help to coordinate children's mathematical communication and their learning processes (Sfard, 2008, p. 147).

As mathematics educators, we see already quite clearly that language is an important aspect of mathematics learning. But how do objects come into play? And how do language and objects interact as means of representation? These questions lead our main research interest. In our research project, we intend to reconstruct mathematical learning processes with a special focus on the interplay of language and objects (Fetzer & Tiedemann, 2015). To begin with, we concentrate on primary school children who learn arithmetic in different German primary schools. We collect data in several schools so that we cover different social and cultural backgrounds and get an impression of our research topic that is as broad as actually possible.

Theoretically, our study is based on two main assumptions. First, we assume, together with many other researchers, that *mathematics learning is a social process* (Bauersfeld, 1988; Jungwirth & Krummheuer, 2006; Miller, 1986). Children do not construct abstract mathematical objects without any suggestions from their environment, but rather in permanent exchange with it. In processes of social interaction and collective argumentation, mathematical objects are constructed, negotiated, and clarified. In this sense, children create abstract mathematical objects on the basis of social

processes. But who are the players in these social processes? Usually, mathematics educators think of students and teachers as actors. However, concrete objects influence the ongoing interaction, too. And, in our opinion, it will mean missing important opportunities to support mathematics learning if we neglect them. Especially in primary classes, objects play an important role in the process of abstraction.

Second, we conceptualize mathematical abstraction as a process of *becoming aware of similarities in different experiences* (Skemp, 1986). According to that assumption, children have to grasp the similarities in different representations, which they encounter in the context of arithmetic. To make this clear, we can consider children playing with little game figures. Four are sitting in a train and two more are getting on. What do these little figures have to do with the drawing of a number line, with six fingers of our hands, with the arithmetical task "4+2" written on a sheet of paper or with specific arrangements of didactical material in mathematics classrooms? (Compare Fig. 1)

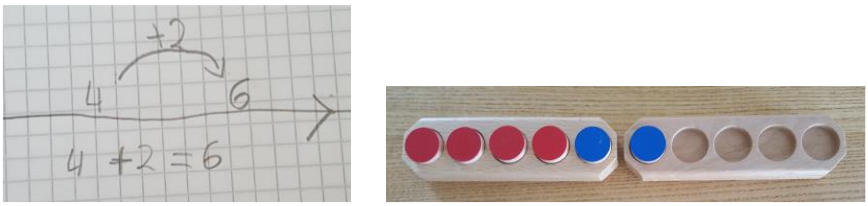

Figure 1: Different representations of 4+2

In order to express what is similar in all those representations and to come to a social agreement on those similarities, children and teachers need language. It is a tool, which allows individuals to share their interpretations of reality with each other. They can express what they 'see' in a certain representation and can, in this way, develop a shared interpretation. Within that interplay of language and objects, children construct their concepts of addition or number. It is for that reason that children have to develop appropriate language skills in mathematics classes, i.e. that their language has to become suitable for describing similarities in different representations.

We present the theoretical framework that we have developed so far for our research project. It consists of three parts. First, we refer to Bauersfeld's (1988) framework of domains of subjective experience (DSE). He points out that learning is a domain-specific process, i.e. that children's mathematical constructions are always bound to the situation in which they were developed. Second, we focus on the aspect of language and our fundamental assumption that every linguistic utterance, how concrete or abstract its content may be, always refers to a context (Aukerman, 2007). Third, there is the question of objects and the role that they may play in the process of mathematics learning. In this regard, we refer to Latour's Actor Network theory (ANT) (2005) which offers a new perspective on objects and their contribution to mathematical communication.

On domain-specific learning: Bauersfeld (1988)

Bauersfeld's (1988) approach of Domains of Subjective Experience (DSE) elaborates how individuals organize their construction of mathematical knowledge. He assumes that children do not organize their remembrance of experiences in a hierarchical way, but rather accumulatively in separate domains. Each experience is stored with reference to the very specific and complex situation in which it was made and, accordingly, in its own domain. These different domains of

knowledge are called “domains of subjective experience (DSE)”. They include their own meaning, language, actions and objects. To illustrate this approach, Bauersfeld (1983, p. 3) reports from Ginsburg’s (1977) work about eight-year-old Alexandria. She is not able to solve the task “8:4=” which is written on a piece of paper, She only suggests 0 or 1 as possible solutions. But, surprisingly, she can solve another task without any apparent effort: “Imagine you have 5 dollars and there are four children. How many dollars will each child get?” In fact, this second task is more difficult from a mathematical point of view. So we might ask why Alexandria did not transfer the initial task “8:2=” to the money-world herself. Why did she not solve it with reference to the domain that is obviously much more familiar to her? Bauersfeld’s answer points to an important characteristic of DSEs: They are not linked automatically. Thus, from Alexandria’s point of view, two different DSEs are affected which are unconnected up to now. In the paper-world, you have to cope with mathematical signs that are written on a piece of paper. In the money-world, you have to cope with banknotes and coins and think about buying attractive goods. Language, actions, objects, but also interests, motivations and feelings are fundamentally different in both DSEs. For that reason, Bauersfeld (1983, p. 6) doubts fundamentally, whether Alexandria regards the number word ‘eight’ which appears in both domains as the same at all.

According to Bauersfeld, mathematics learning can be understood as a process of constructing, deepening and connecting DSEs. However, how can those separate domains be linked? How does mathematics learning proceed? Bauersfeld (1983, p. 31) describes that individuals cross the borders of a DSE by trying, creating and negotiating. In order to link two different DSEs, they have to build a third DSE that exclusively aims at comparing the two already existing ones. Solely in such comparative DSEs, it becomes reasonable to develop a comparative language. In fact, it is this comparative language that allows students to talk about similarities, which they ‘see’ in different representations. This means that all parts of a DSE, including language and objects, can help children to link DSEs and to get access to abstract mathematical objects. In the following paragraph, we focus on the language at first.

On language and context: Aukerman (2007)

Aukerman (2007) points out that it is quite misleading to talk about a ‘decontextualized’ language because no “text, and no spoken word, ever exists without a context” (p. 630). This approach puts the main emphasis on the content level of a linguistic utterance: Every utterance refers to a context, no matter whether this context is concrete or abstract, close or far, accessible to observation or only hypothetical. It is important to notice that Aukerman does not make any statement about the setting in which language is used or how language is used in it, but rather about the point of reference. Utterances in mathematics classes may be produced in many different ways, e.g. with gestures or not, with a parallel action or not, with pointing at something in the closer environment or not, etc., but they are all produced with the intention of talking *about* something. Subsequently, we always talk and listen to others with regard to a specific context. We think about a specific context and produce an utterance. We hear an utterance and interpret it against a background that we deem appropriate. Thus, no matter whether we are the ones who speak or the ones who listen, we relate every utterance to a context that we regard as adequate at that very moment. Aukerman (2007) refers to the process of connecting utterances with contexts as *recontextualization*. In the process of recontextualizing, speaker and listener have to agree to a certain extent on the context of their

conversation: What are we talking about? Thus, when students are expected to talk about mathematical objects, they have to re-contextualize their language and match it with rather abstract contexts. Seen from that perspective, the question is no longer, whether a student is able to decontextualize his or her language, but the question is whether students and teachers succeed in finding a shared context: Do their *re*contextualizations fit together sufficiently?

On objects as actors: Latour (2005)

When students and teachers are negotiating a shared context for their constructions of DSEs, they can get help from concrete objects, which have a lot to offer. Objects as actors? This conceptualization appears to be unfamiliar at first sight. Nevertheless, we think that it can be very useful to adopt Latour's (2005) sociological proposal for accepting objects as actors in the course of action. According to him, they participate in the emergence of social reality.

Latour (2005) goes beyond the traditional understanding of the social, widens the perspective and redefines the notion of 'the social'. He takes a closer look on who and what assembles under "the umbrella of society" (p. 2). As a consequence, he defines sociology as „the tracing of associations" and thus "reassembles" the social (p. 5). In his view, the social refers to any kind of networking: humans with humans, but also humans with any kind of things. Heterogeneous elements that are not necessarily social themselves associate in different ways. According to Latour, all these different associations create social reality. Thus, in his *Actor Network Theory* (ANT), he extends the list of potential actors in the course of action fundamentally and accepts all sorts of actors: "Any thing that does modify a state of affairs by making a difference is an actor" (p. 71). Consequently, objects *participate* in the emergence of social reality, too. In this sense, Latour asks for a broader understanding of agency. "Objects too have agency" (p. 63). They are associable with one another, but only momentarily. To say it with Latour's words, they "assemble" (p. 12) as actor entities in one moment and combine in new associations in the next one. Following Latour, there are no longer stable and pre-defined associations and actor entities.

Again, following Latour (2005), objects participate in the emergence of classroom reality. In fact, this is true for *all sorts* of objects: Paper and pencil, as well as manipulatives or even the bottle of water on the table. Should we as researchers in mathematics education not focus on a certain kind of object, on didactical material? From a theoretical as well as from a methodological point of view, we clearly deny that restriction. Just imagine that the bottle was open, and would drop. Not only the table, but also the paper would get wet, the pencil might fall on the floor. This would surely influence any process of social interaction. "Any thing" (p. 71), a human or non-human actor, might become associated with other actors in the course of action, but only momentarily. The association might be dissolved the next minute. However, in that very moment these actors, no matter who and what they are, contribute to the ongoing process of social interaction.

Looking through Latour's sociological glasses, we can see clearly that concrete objects do play a role in the emergence of social reality. This appears to be especially true for manipulatives and other didactical material. They participate in the negotiation of a shared context and, in this way, offer help in the social process of constructing and connecting DSEs. However, how *do* they contribute? Earlier research revealed different modes of participation that objects might take or have in ongoing classroom interactions (Fetzer, 2013). Our current research on the interplay of language and objects

goes one-step further. Now, we try to get hold of objects' contributions on the content level. In our opinion, their most important contribution is to offer various contexts for re-contextualisation from which students and teachers may choose. A short example that we could observe in a second grade class might illustrate this variety of possible offers. The students and the teacher talk about the question what *the* diagonal might be on their hundred board. On that special hundred board (compare figure 2), the numbers from 1 to 100 are covered with red and blue pieces of paper.

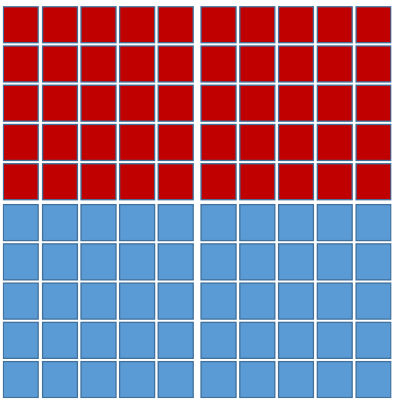

Figure 2: Hundred board covered with red and blue pieces of paper

Here are some of the offers that the students accept and express in linguistic utterances - always in association with the hundred board in front of the classroom:

1) "The diagonal runs from 10 to 91."
2) "The diagonal runs from one corner of the hundred board to the opposite one."
3) "The diagonal runs from one corner of a square to the opposite corner."

We see that the hundred board suggests a wide variety of contexts, which might be suitable for re-contextualization. Most important to us is the fact that those offers range from "concrete" to "abstract". Thus, on the one hand, objects support the opportunity to construct new DSEs because they make a very concrete offer. They are concrete in nature so that students can associate with them and refer to their rather concrete offer: The 10 is at the top right, the 91 is at the bottom left. On the other hand, they make offers that appear to be suitable for comparing. The hundred board has properties that other geometrical forms have as well. It is a square and in every square, "the diagonal runs from one corner […] to the opposite corner." However, in order to grasp that similarity in different representations, you have to construct a DSE of a specific square at first and then a comparing DSE, which aims at comparing different geometrical forms. Those offers of objects may support the construction and connection of DSEs and may support the language development which goes along with it, too.

Integrating the concepts: Talking with objects

According to Bauersfeld (1988, p. 178), the subjective realization of a mathematical object remains always bound to the context of experience, i.e. to the objects and language used in the situation of construction. This approach gives us a clue that we will understand mathematics learning better, if we concentrate not only on language or on objects but on the interplay of both. How do language

and objects interrelate in the process of mathematics learning? How can we talk with (the help of) objects? Aukerman's (2007) approach of re-contextualization and Latour's (2005) Actor Network Theory seem to be useful background theories to tackle these questions.

According to Aukerman, every spoken word in mathematics classrooms refers to a context and is re-contextualized by the recipient. In order to achieve a shared understanding, the interlocutors have to agree on the 'right' context. What can serve as a solid ground, which a linguistic utterance can be related to? At this point, objects come into play. According to Latour, humans and non-humans associate with one another and create social reality. In terms of mathematical learning processes, children and objects interact in the social process of learning. Objects make offers that students and teachers can accept and refer to in order to coordinate their mathematical communication. The "steely quality" (Latour, 2005, p. 67) is a solid ground that allows individuals to experience reality. Objects are not a mere tool in students' hands that can easily be manipulated. Objects are participants in their own social right and contribute to the ongoing classroom interaction: Objects make offers and students 'listen' to those offers. Students talk *to* objects, and become associated with one another. At this point students 'talk' together *with* objects in a combined action.

But how does that work? Objects are concrete in nature. Nevertheless, in their concreteness they prove to be not a limitation, but a chance for development of (mental) mathematical ideas. Indeed, objects offer a variety of possible contexts for re-contextualization ranging from concrete to abstract. Sometimes, objects may provide the context for very specific experiences. In these cases, objects can help to construct a new DSE or to deepen already existing DSEs. At this stage, students try to find words with which they can express the particularity of this specific context and to negotiate it with others. Language is probably the most important tool for such a negotiation: What do I 'see' in that object? What do you 'see'? In a second step, students have to become aware of similarities in different experiences. They are in permanent exchange with their social environment. They listen to as well as talk to and with participating actors. In doing so, they construct new DSEs that aim at comparing already existing DSEs. Again, objects profoundly contribute. They offer a context for comparisons: What do I 'see' as the same in different objects (or in different actions with objects)? What do you 'see' as the same? Where are differences? Do we agree? In this sense, objects help not only to coordinate mathematical communication, but also to develop language more and more. Students are challenged to match their language with a concrete experience at first and with a comparison afterwards.

On closer inspection, we see that objects are actors that students can talk *to* and talk *with*. In fact, objects contribute to the process of negotiating mathematical meaning. In most interactional situations, it is not only the child who is responsible for a linguistic utterance. Words are not the only means to negotiate mathematical meaning. Instead, students and objects often associate and convey a mathematical idea together. In these cases, the object actor takes over part of the act of re-contextualization (Fetzer & Tiedemann, 2016). Students talk together *with* objects. The boundaries between language and objects almost seem to merge.

Discussion

The theoretical framework that we have sketched in this paper raises awareness of some aspects that are not new in mathematics classes, but that are new in our thinking about content-related language

use. When students want to express their mathematical interpretation of reality, they are not restricted to the words they have at their disposal. Thus, they can accept one of many offers that the objects in their close environment make. In this process of assembling, the objects achieve two things. They offer their help, but at the same time, they challenge the children to move further in their mathematical development and in their improvement of content-related language use.

For that reason, the framework does not only make us sensitive to the importance of objects in the process of language development, but it points in a direction that might be productive for our further research. We have not only to analyze objects that we use in mathematics classes, but we have to analyze children's associations with them, too. Which offers do the children accept? Moreover, how do these offers support their language development in mathematics classes? These are the questions, which will lead our further steps in that project about the interplay of language and objects.

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Occasions for productive interactions in inclusive mathematics classrooms that arise following mistakes

Uta Häsel-Weide

University of Paderborn, Institute of Mathematics, Germany;
uta.haesel.weide@math.uni-paderborn.de

When learning mathematics, mistakes can serve as productive occasions if a mistake is followed by a process of negotiation and insight. This paper addresses the question of the extent to which the informal occasion of a "mistake" leads to productive interactions in inclusive mathematics classrooms. Within the context of the project LUIS-M video-based qualitative analysis of cooperative learning situations in inclusive classrooms are made with focus on formal and informal occasions for productive interaction. In the paper the negotiation processes which follow mistakes are exemplified and discussed with respect to opportunity for learning processes.

Keywords: Cooperative learning, discourse, interaction, inclusion, mistakes.

Introduction

Learning mathematics requires interaction; cooperative learning leads to interaction. This is the hope which is associated with cooperative learning in teaching mathematics. Therein, learning is understood as a co-construction process in the context of social processes and/or interactions. As such, not only social learning objectives but also cognitive learning objectives are of importance in cooperative learning. From a mathematical-didactical point of view the epistemological learning processes based on the exchange of different interpretations are of interest (Steinbring, 2005).

Cooperative forms of learning also gain in importance in the context of fostering teaching, both for children with and without (mathematical) learning weaknesses. Meta analyses in the elementary school area prove that cooperative forms of learning and peer-supported learning show stronger effects with respect to subject matter performance than traditional forms of teaching (Rohrbeck, Ginsburg-Block, Fantuzzo & Miller, 2003). However, the effects are dependent on both the composition of the pairs as well as the degree of structure of the cooperative learning environment.

For the inclusive teaching of mathematics, the expectation being associated with cooperative learning environments is that children support and correct each other. Further it can be assumed that children with a lower learning performance develop their own interpretations due to the more elaborated interpretation of other children.

This report focuses on the question of in which way children's mistakes in cooperative learning settings of inclusive mathematics teaching can serve as an occasion for productive interactions. For this, transcribed interaction flows are analysed in the sense of a qualitative research paradigm, and theoretical insights are derived in an abductive process. However, before the research design and the results are presented, the essential theoretical foundations and empirical insights will be outlined.

Theoretical and empirical starting points

Cooperation and interaction in mathematics education

When students learn mathematics, they can benefit from sharing their ideas with their peers, especially when their peers have a different point of view. Several studies mention that differences between students' explanations and procedures can increase mathematical understanding and create new insights (Pijls, Dekker, & van Hout-Wolters, 2007). However, not every interaction leads to new insight, the interaction must take place in a productive discourse. Dekker and Elshout-Mohr (1998) work out that verbalising, explaining, defending, asking and arguing are key activities for productive learning processes. Cooperative learning situations seem to be suitable to make such key activities come up. It is expected that children associate ideas, discuss solution possibilities and discover mistakes when they work in small groups.

Especially, it seems to be productive to develop ideas collaboratively and to find a common way for problem solving. In these phases of cooperative learning a high frequency of communication can be observed and the interacting partners change within the group (Brandt & Tatsis, 2009). It can be assumed that the children show a high rate of key activities like verbalising, asking or arguing by sharing ideas. Therefore, the cooperative learning situation should be based on tasks and problems which allow a range of solution procedures, so that children *need* to verbalise and argue with their partner. This could be achieved when the problem is complex and the students do not have a routine which allows them to solve the problem easily (Dekker & Elshout-Mohr, 1998). Open problems, which allow solutions on different levels, can lead to many key activities as well.

In an epistemological view Nührenbörger and Schwarzkopf (2015) mention that it is helpful to use tasks which create a "productive irritation" concerning the experiences of the children. While working on the tasks the children discover something surprising which then becomes an amazing phenomenon. As a consequence, the children may provide phenomena that they do not expect, so that "they have to reflect on the given structures and see the need to re-interpret the experienced mathematics behind the problem" Nührenbörger et al. (2015). Activities like »comparing« and »sorting« can help initiate such processes e.g. when children reason about the criteria for sorting (Häsel-Weide, 2015). Only if children feel the need or the pleasure to share ideas, mathematical communication takes place.

In addition to the cooperation processes initiated through the cooperative learning technique, the tasks and the work assignment, mistakes can lead to interactions and discourse between children (Götze, 2014; Häsel-Weide, 2016). Götze analysed interaction processes of children in so-called "math conferences", where the children share and discuss their solutions. She points out, that a suggestion of a wrong solution can induce an exchange of arguments and a verbalisation of prior statements - as such key activities take place, that are initiated by the incorrect solution. In my own study regarding the replacement of persistent counting strategies, the children verbalised different interpretations of a structure initiated by mistakes. This interaction had a productive effect and enabled new insights for some children (Häsel-Weide, 2016). The expectation, furthermore, is that non-comprehension or non-knowledge leads to questions and then, as a result, to explanations, and therefore moments of »gaps« and/or »helping« become productive.

Cooperative mathematics learning in inclusive beginners' classes

Inclusive teaching always is in between the poles of common ground and individualisation. It is necessary to take the individuality of each child into account, his/her competences and difficulties, and to foster it individually. At the same time, inclusive teaching has the objective of joint learning, i.e., an objective of participating in content, of the joint learning of mathematics that goes beyond [just] being a part of a class.

In order for children to be able to work together on content, the learning environment must allow working on different levels. At the same time, a joint focusing on a core idea that can be worked on by all children and that is at the centre of the exchange is necessary. Concepts of cooperative learning furthermore recommend a significant methodological structuring as well as the creation of a positive dependency between the cooperating children, e.g. through role distribution, limiting of the material, or time allowances (Johnson, Johnson, & Holubec, 1994). As such, a field of tension exists between a limitation and a common focus for a successful outcome of cooperative processes and the enabling of working on an individual level. In addition, the mathematical competences and verbalising and presenting interpretations, as well as interpersonal competences such as a constructive dealing with conflicts that are necessary for the interaction and cooperation must first be learned.

Objective and design of the study

In project “Learning environments for inclusive mathematics teaching –LUIS-M”, learning environments were developed for the entry phase of elementary school and tested in six classes of the first and second school year (children age 6 to 8). The focus in each class is on a cooperative learning environment which is processed in partner work and accompanied by other offers for individual and cooperative learning. Thematically, the learning environments pick up the basic mathematical topics, which means in the first and second school year the presentation of numbers and operations. Children with different competences in mathematics are supposed to interact and cooperate with each other. Two objectives are being pursued by this: in the spirit of mathematical didactics as a design science (Wittmann, 1995), learning environments are developed for and with teachers and scientific insights are gained via the analysis of the learning processes of the children. We are interested to explore if and how children work together, which different levels of understanding can be reconstructed, and which occasions during the cooperative learning lead to a productive interaction. In this paper the following questions are discussed:

How can aspects and moments causing a productive discourse be characterised? In what way do mistakes induce productive interactions?

The teachers were asked to choose pairs with different competences in order to work together. Two pairs were video recorded in each of the six participating classes. Chosen for the video were pairs of children where one child shows low mathematical competences. The corresponding transcripts were interpreted by a group of researchers. The analysis was compared in an interactive way with empirical findings of other studies and theoretical approaches. As a result, insights about the communication of children in heterogeneous groups could be constructed. This procedure allows the development of new theoretical elements analysing individual cases.

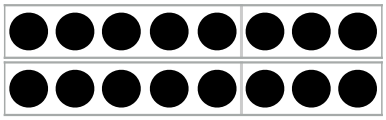
Analysis of episodes

In the following two examples are presented and analysed where it was possible to reconstruct mistakes. Based on the interpretation of the examples, the theoretical conclusions are being worked out and subsequently presented, going past the examples. In this, analyses of additional scenes are included in the development of a theory.

Case 1: Jana's incorrect interpretation of the dot-strip image

The episode is from the second school year and was recorded during the introduction to the multiplication tables. The children work in a learning environment that aims at comprehension of multiplication as repetition of equal groups, and the connection between addition and multiplication. The children lay out dot images with strips of two, four, and eight dots and interpret them additively and multiplicatively. For each dot image only one type of strip may be used. The children work focused on each other, i.e., one child lays out a dot-strip image and the other child states the matching addition and/or multiplication task. These are jotted down and then sorted in a second step. In this process the children can already recognise first relationships between the tasks.

In this scene, Kadir – in accordance with the assignment – lays out two strips of eight in the following dot-strip image. Jana is now required to state the task.



1	Kadir:	(*lays out 2 strips of eight*)
2	Jana:	(*counts the dots individually*). Sixteen.
….		
		Okay, sixteen times two.
3	Kadir:	(*lays his head on table)*
4	Jana:	(*laughs*) Sixty times two?
5	Kadir:	Oh God! You can't read that. Five (*points to the first five dots of the strip of eight*) plus three (*points to the last three dots of the strip of eight*) two times the eight, oh God.
6	Jana	(notes down $2 \cdot 8$)

In her first statement, Jana mentioned the total number of dots, “sixteen”, and the multiplication, “times two.” Since she determined “sixteen” dots by counting, she appears to link the number of dots with the operation in her phrasing. In this, she is phrasing a typical sequence of »number operation number«. Kadir – probably as a reaction to Jana's statement – lays his head on the table. Jana seems to interpret this as a sign that he considers her statement to be incorrect. She then corrects herself and states “sixty times two”. She does not change the structure of her statement, but rather the number.

Jana is able to determine the number of dots correctly and seems to know the sentence structure of a multiplicative phrase. Besides, she shows a behaviour that is typical for children with mathematical learning difficulties: She determines the quantity by counting in ones, but she seems to have difficulties to find a fitting multiplicative expression and to distinguish between the number words sixteen (“sechszehn”) and sixty (“sechzig”) (Anghileri, 1989; Gaidoschik, 2015).

Kadir's statement (5) makes it clear that he considers Jana's interpretation to be incorrect and furthermore shows that he is applying a different way of reading (“You can't read that”) the strips than Jana. As such, a disagreement exists between Jana’s interpretation and his own. Kadir now interprets the dot-strip image himself and in the process phrases a matching multiplicative

interpretation. At first glance, this approach seems to be of little help. Kadir does not provide Jana with any indications how she can correct her interpretation, but rather solves the task himself. However, taking a closer look at his remark, it becomes apparent that he formulates two aspects that can be a learning opportunity for Jana in the spirit of co-construction.

First, he explicates a structured way of grasping the number of dots on a strip. Jana has determined the total number by counting them in ones. Kadir points out to Jana the power of five in the strips, because he separates the dots in the five dots on the left hand side of the mark and the three dots on the right. So he explicates the part/whole structure 8=5+3. He demonstrates and verbalises an option of the structured grasping of the quantity. Through his gesture he additionally makes it clear that only the number of dots on one strip needs to be grasped since these are strips of equal magnitude. In the second part of his statement, Kadir formulates the multiplicative structure and phrases “two times *the* eight”. With the nominalisation, he emphasises the two elements “number of strips” (multiplier) and “number in each strip” (multiplicand).

Therefore, at second glance, it becomes apparent that Kadir is not only providing the correct result, but that there are opportunities of insight for Jana located in his verbalisation and his gesture. The key activities of verbalisation, explaining, and pointing out, described as productive by Plijs et al. (2007), can be reconstructed in the case of Kadir. He shows an understanding of multiplication and an elaborated strategy in the grasping of the quantity. His statement is caused by Jana’s solution, which he recognises as being incorrect, and which he introduces in the interaction. To what extent Kadir's interpretations are picked up by Jana cannot be identified in this scene since Jana notes down the solution without it becoming apparent to what extent she is exclusively translating the verbalisation into a term or to what extent she can comprehend it with respect to the dot-strips.

Case 2: Marie suspects a mistake in Milene's solution

In the first school year, the children were given tasks to create “simple” subtraction problems with the subtrahends 1, 5 and 10. In this, the children freely select the minuend, by laying down corresponding dots in the field of twenty, draw an action card, »minus 1«, »minus 5«, or »minus 10«, solve the problem by removing the number of dots or mentally, determine the result and note down the problem on a card. After a while, the cards are sorted in accordance with self-selected criteria, which could be e.g. the minuend or the subtrahend or the size of the numbers. The relationships between the problems become apparent by sorting them, e.g. according to the minuend, the subtrahend, the difference or the size of numbers. So, on the one hand, the focus is on the basic topic of »simple subtraction problems«, on the other hand the structure between problems can be discovered and described. Both the activity of sorting, as well as the potential discoveries are suited to stimulate productive interactions. The step to find the problems can be worked on either individually by each child or the children work with distributed roles. In this case, one child specifies the minuend with dots while the other does the subtraction by taking away one, five or ten dots and notes down the term. Working while focused on each other allows for more cooperation between the children and for a direct reaction in case of erroneous solutions or questions, but also leads to a limitation of the individual level of processing.

Marie and Milene are working on the assignment with distributed roles based on the division of labour, i.e. one pupil lays out the minuend on the field of dots and draws the card with the subtrahend

(c.f. Figure 1); the other pupil finds the difference and notes down the task. Maria has already placed a strip of 10 and 3 chips on the field of twenty in the following way:

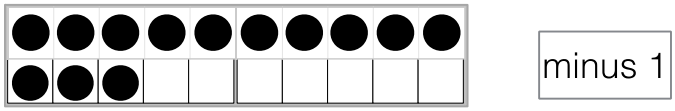


Figure 1: Depiction of the assignment situation

1	Milene:	(*lifts away the third dot from the bottom row of the field of twenty*)
2	Marie:	Now you have to guess how much that is
3	Milene:	One, two. Twelve! (*looks at Marie, puts the chip back onto its original spot on the field of twenty)* So, and now I write down the assignment (*takes a piece of paper and a pen*)
4	Marie:	(*4sec*) (*shakes her head*) You got this wrong, you misunderstood.
5	Milene:	(*notes down the task 13-1=12 on the piece of paper*) Thirteen minus one equals twelve. (*places the piece of paper on the stack with the other tasks*)
6	Marie:	(*5sec*) (*points with the finger to the dots*) That equals thirteen (*grabs Milene by the arm and points to the 3 dots in the bottom row of the field of twenty*). That equals thirteen.
7	Milene:	(*lifts away the third dot from the bottom row slowly once more, directly looks at Marie, and nods*)
8	Marie	Oh (*turns away from the field of twenty*).
9	Milene:	Understood?

Milene, in accordance with the action card, »minus 1«, takes one dot away from the field of twenty, holds it in her hand and states the correct difference (1 & 3). She then places the dot back onto the field of twenty so that thirteen dots lie on the field once again. Marie voices the suspicion that Milene has made a mistake (4) and phrases “That equals thirteen” (6) which probably traces back to the thirteen dots lying on the field. In this situation, there is disagreement between the children.

Both pupils defend their interpretation in the further process: Marie grabs Milene's arm, maybe in order to be heard, repeats her interpretation, and points to the dots in the field of twenty. By taking away one dot, Milene seems to illustrate the operation »minus 1«, looks at Marie and nods and thus demonstrates that the difference of the task is twelve. The sequential carrying out of the actions corresponds to the process of subtraction so that Marie potentially recognises that the 13 dots represent the minuend, one dot was taken away correctly, and the difference of 12 dots was determined. However, due to the replacement of the dot (3) it was not the difference that was visible, but rather the minuend. Marie seems to be able – through the gesture of Milene (7) – to comprehend that the difference was determined correctly, nevertheless. Her statement “Oh” and her turning away from the task could be a sign of agreement and an acceptance of the result.

The suspected mistake with respect to the solution of the subtraction task leads to the connection between quantity, change, and result being demonstrated on the material. The trigger of this elucidation was the dissent between the interpretation of the 13 dots that Milene considered to be the minuend whereas Marie saw the difference in the field of dots. Here, the supposed mistake leads to negotiation and defence of different interpretations. Both children show key activities, pointing out

and verbalising their interpretations of the material. In this way, they appear to defend their respective point of view and to negotiate the correct interpretation.

Interpretation and conclusion

Both scenes exemplarily show that mistakes and suspected mistakes can cause verbalisations, demonstrations, and defensive actions. If (supposed) mistakes are recognised in the interaction, they appear to not only be corrected but key activities also take place in the negotiations (Table 1). The children utilise, for example, the material to illustrate something, or (re)phrase it so that other aspects become clear. In this way, (supposed) mistakes function as a trigger for a productive interaction process. Therefore, it does not seem to matter whether a mistake was actually made or if this was only suspected. The children not agreeing in their interpretation – meaning that there is dissent – is of central importance. In this, two cases could be distinguished. (1) One solution is incorrect and this is recognised by the partner child who in the context of the cooperation has therefore solved the task himself (correctly). (2) A correct solution is interpreted as incorrect by the partner child. Here too, the partner child must solve the task herself. Prerequisite is, in this case, that both partners solve the task and arrive at different results.

Trigger	Prerequisite	Effect	
		Occasion for negotiation	Negotiation
(supposed) incorrect statements or approaches are being verbalised	recognising the (supposed) mistake in one's own / parallel solution of the task	verbalising the irritation / non-agreement	explaining, defending, demonstrating the point(s) by one (both) child(ren)

Table 1: Productive interaction triggered by a mistake

The activities in the negotiation are gestures and verbalisations which are utilised as justifications and defences. Properly formulated chains of reasoning are not observed in the inclusive beginners' classes. This could be because the children are not yet equipped with respective competencies or also because this does not correspond to the interaction of children among each other and requires a focusing and moderation by a teacher (Gellert & Steinbring, 2012).

Regarding inclusive mathematics teaching, it can also be assumed that the cooperation of children is productively becoming an informal learning occasion through mutual helping and correcting. The analyses show that children in the first and second school year not only take corrective action but also display key activities in their correction which can lead to learning processes for themselves and also potentially for the partner children. This, however, requires that mistakes are recognised during the learning process. Yet, this prerequisite does in no way always exist, not even when pairs of children with different competences work together. On the one hand, the children are in part simultaneously busy with their own tasks, while on the other hand even in case of working focused on one another not all mistakes are being recognised and explored. As such, it is not sufficient to trust in informal occasions. Productive interaction should rather be stimulated through suitable activities and through phenomena yet to be discovered.

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Revisiting the roles of interactional patterns in mathematics classroom interaction

Jenni Ingram, Nick Andrews and Andrea Pitt

University of Oxford, UK. Jenni.Ingram@education.ox.ac.uk

The ways in which teachers and students interact about mathematics in lessons can be more powerful in influencing learning than the materials and resources that teachers use. Interactional patterns structure all interactions and there are many such patterns that occur frequently in mathematics lessons. This paper focuses on one such pattern, the funneling pattern, which is widely discussed in the literature. Three distinct examples described in the literature as a funneling pattern are examined in order to examine the different roles sequences of closed questions can have and the opportunities these patterns can provide or constrains to students in the learning of mathematics.

Keywords: Interactional patterns, funneling, questioning.

Introduction

The aim of this paper is to contribute to the discussions around the roles different interactional patterns have in the teaching and learning of mathematics. The simplest, and most prevalent, pattern that is discussed widely in the mathematics education literature is the IRE pattern of teacher initiation, student response, and teacher evaluation (Mehan, 1979). Many authors describe this pattern as doing little to encourage students to reason, give explanations or articulate their thinking (e.g. Cazden, 1988; Nystrand, 1997). Yet the discussion has now moved on, with authors pointing out that it is not the IRE pattern itself that is the issue, but rather how it is used. This IRE pattern continues to dominate classroom interaction because it enables students to know when to speak, how to speak and about what to speak (Ingram, 2014; Wood, 1998). This pattern can be used by mathematics teachers to convey and establish different norms. It creates opportunities for students to communicate in classroom interaction, but it is largely teachers who can both constrain or enhance their students' opportunities to communicate mathematics or to communicate mathematically.

This paper focuses specifically on another interactional pattern called the 'funneling' pattern (Bauersfeld, 1980; Wood, 1998) that comprises of a series of IRE sequences. Four extracts that have many of the features of the 'funneling' pattern are discussed with a view to illustrating here that again it is not the pattern itself that focuses student thinking on "trying to figure out the response the teacher wants instead of thinking mathematically himself" (Wood, 1998, p. 172), but rather it is the way that it is used by the teacher that can affect student thinking.

The funneling pattern

The funneling pattern was initially described by Bauersfeld (1980, 1988) and consists of a series of teacher questions and student responses that has particular features. The sequence follows an incorrect answer from the student, or some other form of difficulty with the mathematics. The teacher uses "more precise, that is, narrower, questions" (Bauersfeld, 1988, p. 36) to lead the student to the correct answer. This narrowing effect of questions towards a particular correct answer (hence the term funneling) contrasts with sequences of questions that leads students step-by-step through a process (e.g. Herbel-Eisenmann, 2000), however both invite students to do little more than complete the

teacher's sentences (e.g. Franke et al., 2009). These different examples have led to further terms, such as leading questions (Franke et al., 2009), guiding questions (Moyer & Milewicz, 2002) and scaffolding (Wood, Bruner, & Ross, 1976), becoming associated with funneling. The distinction between these terms and the precise relationship with funneling is often not made but we believe it is an important one as we outline below. As a result, funneling has become used more broadly in the literature to describe any sequences of IRE patterns that lead students through a series of specific narrow questions, often only requiring short factual response from the students.

Particular concerns have been raised about the implications of such funneling interactions. For example, Brousseau (1984) refers to the Topaze effect in which the sequence of 'funneling' questions disguise the mathematical knowledge that is being targeted by the interaction as a whole. Indeed such interactions in which students do not need think about mathematical relationships, patterns or structures in order to answer the teachers' questions (Wood, 1998) the most frequently cited instantiations of 'funneling' patterns. Wood (1998) argues that in funneling patterns of interaction students are only responding to the surface linguistic patterns in order to respond appropriately to the teacher's initiations. However, Temple and Doerr (2012) have shown how this aspect of the funneling pattern can be used by mathematics teachers to activate prior knowledge and offer them opportunities to talk about newly learned concepts. This indicates the possibility that the funneling pattern can have a variety of roles within the classroom, some of which support students' learning and communication of mathematics. Wood also connects funneling to "certain beliefs about the nature of mathematics and the relationship between teacher and students" (p. 175) but we would suggest that it is not the pattern itself that indicates these beliefs, but how it is used by teachers.

Data and methods

The data used in this paper to illustrate the different functions of a funneling pattern of interaction comes from two sources. The first is transcripts from two videos of mathematics lessons collected as part of a larger project looking at the role and use of language in mathematics teaching and learning. Both lessons were from the same school, a small inner-city comprehensive secondary school with high levels of students in receipt of free school means and over 50% of the students with English as an additional language. The lessons are taught by two different teachers and the students are aged 11-12 years old. The second source is transcripts from a published article (Drageset, 2015) focusing on categorizing language use in mathematics classrooms that also uses conversation analysis as its methodology. A conversation analytic approach is taken in the analysis of the transcripts, which is an approach that focuses on the identification of patterns of interaction. Conversation analysis (CA) looks specifically at what participants are doing in their turns at talk through a careful analysis of how the turn is designed, both in terms of its content but also in terms of how it is spoken, i.e. quickly, hesitantly, emphasizing particular words. A key feature of any analysis based on CA is the reflexivity of turns at talk. Each turn is designed in response to the turns that it follows and affects the turns that follow. This makes it a particularly useful approach for examining the relationship between teacher questions and student responses.

The roles of funneling

In this paper we will outline three distinct patterns of interactions described in the literature as funneling. The first is used by the teacher to make assumed knowledge publicly available. The second

offers students the opportunity to use recently introduced vocabulary. The third involves two extracts that are used in combination to draw attention to structures within a sequence of mathematical interactions.

Making assumed knowledge publicly available

The funneling pattern of interaction does not occur very often in the lessons collected as part of the larger project, which contrasts with other studies looking at mathematics classrooms (e.g. Temple & Doerr, 2012; Franke et al. 2009). Yet using a conversation analytic approach in the analysis of this pattern reveals that each instance of the pattern is doing different things. For example, in the first lesson the students have discussed the meaning of some key words on the whiteboard associated with probability. The extract in Figure 1 follows this discussion and then is followed by an activity where students are tossing a coin twenty times and then combining the results. No connection is made between this interaction and the tasks that came before it or after it.

```
T:   …lovely.  so what would be the probability of rolling an e:ven (.)
     number.  how many e:ven numbers are there?  s1 (0.4) how many even
     numbers are there (0.3) o:n a: dice.
     (1.1)
S1:  u:m ↑three
     (0.6)
T:   what are the total (0.4) numbers on a dice?
     (0.5)
S2:  six?
     (0.8)
T:   okay so the probability (0.6) of rolling an e:ven number would be
     what,
S3:  three sixths.
     (0.9)
T:   pardon
S3:  three … or [a half]
S4:             [half. ] a ha[lf]
T:                          [thr]ee sixths.  (1.2) which you're quite
     right i:s a half. (1.9)  okay are we happy with that?  are we ha:ppy
     with how to find the probability of an event.
S:   ye::[a[h ]        ]
S:       [yeah]
T:         [(beautiful)] okay. …
```

Figure 1: Calculating the probability of getting an even number

See Jefferson (2004) for details of the transcript conventions used here

The teacher asks a series of questions requiring short factual answers, which are given by the students. These questions lead the students through a step-by-step process for calculating a probability. The fact that these responses are given hesitantly, as indicated by the pauses, ums and phrasing the response as a question, is ignored by the teacher. The sequence of questions focuses on the identification of the numerator and the denominator when identifying the probability and this is emphasized through the teacher's choice of accepting the answer three sixths rather than the half, which is acknowledged but not treated as the answer to the probability of rolling an even number.

The interaction ends with the teacher checking that the students are happy with this process and treating them as such by moving on to the next task. Yet there is little in the interaction to indicate that the students as a whole could calculate the probability themselves. This is a feature of the funneling pattern that Wood (1988) draws attention to: that it can give the impression of learning even though it is the teacher that has done the cognitive work. However, what the teacher has done through this interaction is explicitly to make the process public and has involved the students in this process (as opposed to just telling them how to calculate the probability). The ability to calculate the probability of an event is taken as assumed knowledge in the following task where the students have to calculate the relative frequency of getting a head when tossing a coin. So, whilst there is no evidence that the students are doing more than responding to the immediate initiations, the funneling pattern of questions and responses does make public knowledge that is needed later in the lesson. So the teacher's questions are doing something other than just assessing whether students have the required knowledge. This is demonstrated further in other examples where incorrect responses are ignored such as the second student's suggestion of larger in the second extract (Figure 2), which comes from a lesson focused on solving linear equations.

Opportunities to use terminology

The majority of the lesson on linear equations is spent with students working independently through a set of differentiated exercises. At the start of the lesson a student asks a question about the difference between an expression and an equation and the extract follows this question. Again, the teacher leads students through a series of closed questions requiring short factual responses from them. This example shows the teacher using questions that offer students opportunities to talk about newly learned concepts and new terminology in a similar way to the example offered by Temple and Doerr (2012). The questions serve to support the students in recalling processes and words introduced in previous lessons such as simplifying and collecting like terms. Each use of a technical word is connected to the specific example, $3x + x$ becoming $4x$, and $3x + x = 4x$ being an expression is contrasted with $4x = 12$ being an equation. Throughout the interaction student responses that do not fit with the use of the language the teacher is focusing on are ignored or built on by the teacher who turns them into a form that does fit. This sequence of questions again is doing other than assessing students' knowledge. The questions are providing students with the opportunity to use mathematical terminology and hear it used in a mathematical way by the teacher. This sequence could be considered a form of scaffolding (Wood, Bruner & Ross, 1976) if the support the teacher is giving, through his questioning and phrasing of his responses, is withdrawn over time until the students are using the language in their own descriptions of their work on mathematical tasks.

T: …wha- I mean what process, what mathematical process essentially has, has S1 demonstrated here by doing three ex and ex

S1: addition.

T: addition. and in which case what's happened to that expression.

(1.0)
S1: i[t's been answered]
T: [it's become,]
(0.5)
S2: ↑larger
(1.5)
S3: um
S5: °it wha[t°]
S4: [simpli]fied.
T: ↑simplified. absolutely S4. you've got three ex plus ex, (0.8) we've si:mplified that. by collecting our,
(0.9)
S: like [terms].
S: [terms]
T: like terms good. and we've managed to say three ex plus x has become fou:r ex. so that's simplified. but it's ↑still an expression. (1.2) whats- how can I turn this fou:r ex here, how can I turn that four ex from an expression (.) into an equation.
S: make x three.
(0.7)
T: right, so if I make ex three, four times three would be twelve. that's still an expression.
(2.0)
S6: give er: would it be give what ex i[s]
S7: [↑oh] take away the e[x and ((inaudible))]
T: [so (.) go on then S8] hang on
S8: would it u:m you would do ex equals, °you would- can't remember°
T: so (0.6) if I did four ex equals: (2.8) twelve?
S: yeah
(0.9)
T: that's no:w (.) an equation.

Figure 2: The difference between an expression and an equation

Drawing attention to regularities

The last two transcripts are taken from Drageset (2015) and have been coded as "closed progress details" which is one of the "main elements of funneling" (Drageset, 2014). In each extract the teacher takes a step-by-step approach in posing questions and students are only required to give short factual responses to the question asked immediately before:

Teacher: How much is one of… one-fifth then of … of twenty-five?

Student: Five.

Teacher: It is five, yes. How much is two-fifths?

Student: …ten.

Teacher:	Then it becomes ten. How much is three-fifths
Student:	Fifteen.
Teacher:	How much is four fifths?
Students:	Twenty.
Teacher:	And how much is five fifths:
Students:	Twenty-five.
Student:	One whole.
Teacher	One whole, yes. Yes, good. Great.

Extract 1: Extract 1 from Drageset (2015, p.260)

Teacher:	Yes. So if I have thirty chips here and then divide them into six equal piles, then how many are there in each pile then?
Student1:	There are five (hold up five fingers).
Teacher:	Five. But how much is two-sixths of thirty, then?
Student2:	Ten.
Teacher:	Ten. How much is three-sixths?
Student2:	Fifteen.
Teacher:	And four sixths?
Student2:	Twenty-five.
Student1:	Twenty, twenty
Teacher:	and f…six sixths?
Student 2:	Thirty.
Teacher:	Yes. And… six sixths, how much do I have then?
Student2:	One whole.
Teacher:	One whole. And then, this time the entire quantity was?
Student2:	Thirty.
Teacher:	Thirty yes.

Extract 2: Extract 5 from Drageset (2015, p. 265)

Extracts 1 and 2 are not just narrowing sequences of questions, but are also specific, structured, and lead to a mathematical regularity within the sequence of questions itself. It is also the repetition of the pattern of interaction itself that offers students an opportunity to see the relationship between the fractions and the quantities. This is pointed to by the teacher in their penultimate turn with the phrase "and then, this time". So, whilst the teacher does not explicitly talk about the meaning of 'one whole' the sequence of questions identifying each of the fractional parts goes in order, and stops when one whole is reached. The teacher does not ask what seven sixths is, and also does not stop at four sixths

for example. In both extracts the total number, twenty-five and then thirty, is said alongside one whole. The sequence of closed questions is leading students through a process in a similar way to the example offered by Herbel-Eisenmann (2000, p. 182). However, it is also the repetition of the sequence that makes this process more explicit and affords students' attention to be drawn to it.

Conclusion

In this paper we have explored three different interactional patterns referred to in the literature as a type of funneling pattern: one example of a narrowing pattern, one example of step-by-step pattern, and one example of connected step-by-step patterns. Each pattern includes a sequence of closed questions requiring short factual responses from the students. Each sequence is leading the students to a particular answer. However, we question whether the mathematical knowledge is always being disguised (c.f. Brousseau, 1984). Each pattern is doing something different to the other patterns and, in the final example the repetition of the pattern itself can be used to support the students' thinking. We have shown the possibilities for how teachers can use these sequences of questions to make assumed knowledge publicly available for subsequent work, offer opportunities to use technical vocabulary, and perceive regularity in mathematical processes. Each of these functions is an important part of the teaching and learning of mathematics. Whilst in each of the examples offered it is the teacher who is controlling the content, in is possible to imagine situations where the teacher is using a similar sequence of closed questions about a student's idea. The funneling pattern can and does have a role in the teaching and learning of mathematics but it is how it is used, rather than the structure of the pattern itself, that can offer or constrain opportunities for students to engage in mathematical thinking and communicating.

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Constructing mathematical meaning of the cosine function using covariation between variables in a modeling situation in Cabri

Faten Khalloufi-Mouha
Faculté des Sciences de Bizerte, Université de Carthage, Tunisia; fkhalloufi@yahoo.fr

Many studies argue that dynamic features of the technological artifact Cabri may function as instruments of semiotic mediation. Our study is situated in this perspective and aims to analyze the impact of the artifact Cabri on the construction of the mathematical meaning. Using the theoretical approach of the semiotic mediation, a teaching experiment was designed with the aim of introducing the notion of trigonometric function as a covariation. Assuming that the use of a semiotic system in social interaction as the natural language contribute to the emergence of internal process (such as concept formation), this paper analyses the verbal signs elaborated and used by the students and the teacher in order to describe the evolution from students' personal meanings to mathematically shared meanings of the trigonometric function cosine.

Keywords: Verbal signs, semiotic mediation, covariation, collective discussion, Cabri.

Introduction and research problem

The notion of trigonometric function is related to three mathematical domains: trigonometry, algebra and function. Understanding the idea of a trigonometric function requires students to realise the relationship between the length of an arc and the measure of the angle that is subtended by the arc, the relationship between the real line and the unit circle, the covariation between a point moving on the unit circle and its projection on one of the axes, and the idea of the graph of a function (Khalloufi-Mouha, 2009, Khalloufi-Mouha & Smida, 2012).

From a historical perspective, the idea of the functional relationship between the real line and the unit circle is related to the notion of motion as a variation in the space and time. (Falcade, Laborde & Mariotti, 2007) So, this idea could be understood from a dynamic perspective, referring to the wrapping of a line around the unit circle and to the variation of a moving point. Many studies (e.g., Laborde, 1999; Laborde & Mariotti, 2002) claim that the key to understand the dynamic aspect of function is the notion of trajectory. According to Laborde (1999) the graph of a function could be considered as the trajectory of a moving point, representing the dependent variable according to the variation of a variable point on the axis of abscissas, so, the covariation of these two points becomes a relationship between two variations depending on time. Adopting a teaching approach based on the idea of covariation in the semantic field of space and time, Laborde et al. (2002) and Falcade (2006) assume that the teaching and learning of the notion of function should start in a dynamic environment. They claim that the use of Cabri allows access to the ideas of variation, functional relationship and covariation through the movement of a point on the screen. According to the Vygotskian (1978) perspective of semiotic mediation, Laborde et al. (2002) and Bartolini Bussi and Mariotti (2008) argue that some of Cabri's features can be viewed as potential tools of semiotic mediation carrying mathematical meanings. Cabri's tools can be thought as external signs referring to a specific mathematical meaning, and may become tools of semiotic mediation Laborde et al. (2002).

Following these studies, we propose a teaching experiment integrating the technological artifact Cabri, aiming to introduce the trigonometric function cosine as covariation.

Theoretical framework

According to Vygotsky (1978), human intelligence is defined by the ability to use various types of tools with signs playing the role of mediators. Vygotsky distinguishes between tools (technical tools) and signs (psychological tools). Tools are externally oriented and aimed at controlling the process of nature. Signs are internally oriented and aimed at mastering the individual's behavior and cognitive processes. Vygotsky claims that, through the process of internalization, a tool may be transformed in a sign, and then may function as a tool of semiotic mediation. Vygotskian theory supposes that the development of behavior and cognitive processes is the product of activities practiced in the social institutions of the culture in which the individual grows up. For Vygotsky, language is the most important semiotic mediator allowing the passage from the interpsychological level to the intrapsychological level. The theory of semiotic mediation (TSM), developed by Bartolini Bussi and Mariotti (see, for instance, Bartolini Bussi & Mariotti, 2008), is situated in the Vygotskian perspective. Two key notions support this theory, the *zone of proximal development* and the *internalization.* Vygotsky introduces the concept of the zone of proximal development as *"the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers"* (Vygotsky, 1978, p.86). This concept models the learning process through social interaction. In fact, the collaboration between one individual, whose cognitive attitude has the potential to facilitate change, and another individual (or group) who intentionally cooperate to accomplish a task or to pursue a common aim may generate a possible development (Bartolini Bussi & Mariotti, 2008). The transposition of this concept in the school context is possible thanks to the intrinsic asymmetry of the relationship between students and teacher in terms of knowledge. Therefore, the teacher's actions should be within this zone for achieving a didactic objective through the use of the signs and tools as semiotic mediators. In this zone of proximal development, cognitive development is modelled by the process of internalisation.

According to the TSM, the social use of an artifact in the accomplishment of a task leads to the emergence of signs expressing students' personal meanings, i.e. "*the conscious sense created by each individual to express the direction of his or her action"* (Bartolini Bussi, 1998, p.69). The evolution from personal meanings towards mathematical meaning, i.e, what describes the properties of the concepts under scrutiny inside a system of theoretical knowledge (Bartolini Bussi, 1998, p. 81), is an educational aim that can be realized promoting the evolution of signs, expressing the relationship between artifact and tasks, into signs expressing the relationship between artifact and knowledge (Bartolini & Mariotti, 2008). The teacher can guide this evolution through collective discussions.

Rationale for the teaching experiment

Our teaching experiment aims at constructing the meaning of the cosine function and of its graph using the notion of covariation (Khalloufi-Mouha & Smida, 2012.) The experiment has four parts. The first two parts use the situation *A rope on a wheel* (Genevès, Laborde & Soury-Lavegne, 2005), which is a modelling situation in Cabri: a thread that can be wound around a wheel through the Dragging tool. The experiment involves description and prediction tasks intended to provide students with opportunities to realise that in the unit circle the length of an arc is equal to the measure (in radian) of the angle that it subtends. In addition, wrapping the thread on to the wheel aims to make students construct, for a given real *x*, the point M on the unit circle with the arc $\widehat{IM} = x$. Thus, they

could realise the functional relationship between the real line and the unit circle. Mathematically, this relation is interpreted by the existence of a surjective group homomorphism between the set of real numbers and the unit circle. Throughout these activities the role of the teacher is crucial. He has to establish strategies fostering the development of meanings of functional relationships and variation. The third and fourth parts of the experiment aim to analyse how the Dragging tool, the Measurement tool, and the Trace tool function as semiotic mediators for the ideas of variation, the functional relationship and covariation. The cosine function is introduced as a relationship between two variations depending on time, and its graph as the trajectory of a moving point.

The teaching experiment involves different typology of activities aimed to develop different components of the complex semiotic process: working in pairs and collective discussions generally initiated and guided by the teacher. When working in pairs, students use the artifact to accomplish the given tasks. This type of activities promotes social exchange through the use of verbal signs (both oral and written), gestures, drawing... Consequently, this provides students with opportunities to construct personal meanings linked to the mathematical target concepts. Students' personal meanings may evolve to the shared mathematical meanings through collective discussions. These discussions involve all students and allow the confrontation between their personal meanings and the mathematical meanings. These collective discussions may reach the status of mathematical discussions, in the sense of Bartolini Bussi (1996), and can also involve phases of institutionalization (Brousseau, 1998) and the introduction of new formal notations. The analysis of the verbal signs used by the students and the manner in which the teacher exploits them can highlight the evolution process from the students' meanings towards mathematical meanings. According to the TSM, the status of the signs belonging to the different categories vary in the evolution process, they can be used as an index of the move from personal sense to mathematical meaning. (Bartolini Bussi & Mariotti, 2008).

Methodology

The sample of this study consisted of 16 students from a class of 2nd year (16-17 years) in a high school in Bizerte, Tunisia. The students were familiar with analytic trigonometry and they regularly used computer software in class. The teaching session was carried out in the computer laboratory. Students were grouped in pairs and asked to produce a shared written answer on a worksheet. The discussions within pairs and the collective discussions have been audio-recorded.

The analysis is based on several kinds of data that were collected: protocols produced from the student working in pairs (audio-recording, texts, and drawings), audio-recordings of collective discussions and the teacher's and observer's notes.

The impact of the use of Cabri on the construction of mathematical knowledge was analysed by identifying the development of the students' and the teacher's verbal signs related to the target mathematical concepts. We distinguish the two types of signs identified by Falcade (2006): simple signs and complex signs. Simple signs are "*easily recognizable by representations of almost atomic type (words or specific formulations*)" (p.202). For example, the word "circle" is a simple sign. Amongst the simple signs Falcade (2006) distinguishes three types, i.e. artifact-signs (arising directly from the use of the artifact, their meanings are personal and commonly implicit. The artifact-signs are strictly related to the use of the artifact to accomplish the task), mathematical-signs (referring to the culture of mathematics and constitute the goal of the semiotic mediation process.), and pivot-signs which play a pivotal role between the semantic and mathematical fields. "The characteristic of these

signs is their shared polysemy, meaning that, in a classroom community, they may refer both to the activity with the artifact; in particular they may refer to specific instrumented actions, but also to natural language, and to the mathematical domain." (Bartolini Bussi & Mariotti, 2008).

The complex signs refer to relationships between families of relatively 'simpler' signs. For example, the sentence "the set of points that are equidistant from a given point" is a complex sign that refers to the mathematical-sign circle. Complex signs are subdivided into four categories: characterisations, definitions, interpretations and instantiations. Characterizations tend to highlight some characteristics that could be interpreted in mathematical terms. Nevertheless, characterizations are not real definitions because in the mind of the speaker the statement is not precise. A definition for an object provides a "boundary in words" which was until then unknown or little known. They are not definitions in the mathematical sense, but can be considered as being part of the process towards a mathematical definition. The interpretations concern explicit links between two families of signs which belong to two different semantic fields. Instantiations are signs which concern the establishment of an interpretative link between artifact-signs and mathematical-signs. Falcade (2006) elaborates that "Instantiations are of the same nature as the interpretations. However, the latter concern universal, while instantiations refer directly to the specific activity in Cabri" (p.210).

In our analysis of the development of students' and teacher's verbal signs, all the relevant mathematical notions and all the different signs used were identified. The classification of Falcade allows us to analyse the development of students' personal meanings. Every evolution from an artifact-sign to a pivot-sign will be interpreted as a step towards the construction of personal meanings. The use of complex signs of the type characterization or instantiation will be interpreted as an attachment to the artifact. The use of the complex sign interpretation will be interpreted as a step in the process of semiotic mediation. Finally, the complex sign definition will be interpreted as a step in the process of internalisation of a mathematical definition of the object.

Findings and discussions

The first two parts of the teaching experiment focus on the situation "a rope on the wheel".

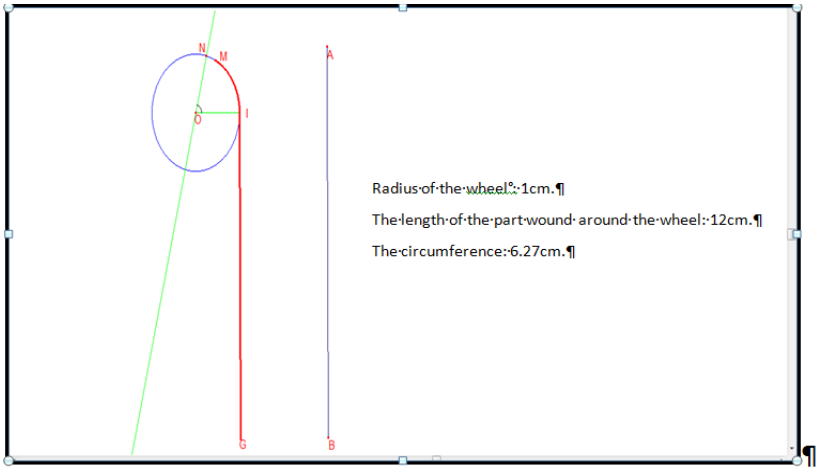

Figure 1: The situation *A rope on the wheel*

From the first task, when working in pairs, students used a large number of artifact-signs related to the ideas of motion and numerical domain, such as "to unwind", "to turn", "to move", "to increase",

and "to decrease". For some students, we identified initial pivot-signs related to variation such as "to change" and "to vary", and related to the meaning of functional relationship such as "depends on" and "being a function of". The pair discussions indicate that the use of the artifact facilitated students' use of complex signs of the instantiation type: "If the length of the arc changes, the point N will change*",* and of the interpretation type: "Then the length of the arc varies as a function of the angle". These complex signs show the construction of personal meanings of the relationship between the length of an arc, the measure of the angle and the radius of the wheel. The collective discussions initiated by the teacher led to the use of these signs in the process of constructing mathematical meanings. The role of the teacher was crucial in the collective discussion. Taking into account individual contributions, the teacher engaged students through juxtaposing their personal meanings, and encouraged them to disregard the artifact and then to focus on the mathematical concepts.

The second part of the experiment aimed at exploring the metaphor of the winding of the real line around the unit circle. During the working in pairs, we observed artifact-signs such as "put the point", "move the point", and pivot-signs as "length of the arc". This characterizes a first step related to the recognition of the numerical and geometric variation. Students used also mathematical-signs "For every x, we can construct a point" and interpretations of the functional relationship between the real line and the unit circle. This can be interpreted as a second step: the identification of the covariational relation at a perceptual level. During the collective discussion, the teacher used the rope as a semiotic mediator to introduce the idea of the functional relationship between the set of real numbers and the points of the unit circle. The collective discussion highlights the use of mathematical-signs such as "for every real x in ..", "symmetry", "absolute value" and "a function of". In addition, we also observe an interpretation of the idea of the functional relationship: "For every positive real x we can construct a point on the unit circle". It seems that students found themselves in a familiar mathematical environment and had no difficulties in generalising the use of the artifact. They were engaged spontaneously in the intended process of mathematical meaning construction.

A definition of the cosine function

The third part of the teaching experiment focused on the ways the Cabri-tools Dragging, Measurement and Trace function as semiotic mediators for the development of the definition of the cosine function as a covariation, (i.e. the relationship between two variations depending on time), and the construction of its graph as a trajectory of a moving point.

Students were asked to construct a point *N* on the x-axis *(1,0)* with abscissa x. Using the Measurement tool, they should then construct the point *M* of the unit circle such that $\widehat{IM} = x$. Finally they should construct the point *K* on the x-axis*(1,0)* with the same abscissa as *M*, and the point *H* on the y-axis *(0,1)* with the same ordinate as *M*. Using Dragging tool, students were asked (a) to describe the relationship between the points *N, M, H* and *K* when *N* is moving, (b) to determine the values taken by the abscissa of *M* when the point *M* is moving on the unit circle and (c) to identify the relationship between *x* and the abscissa of *M*.

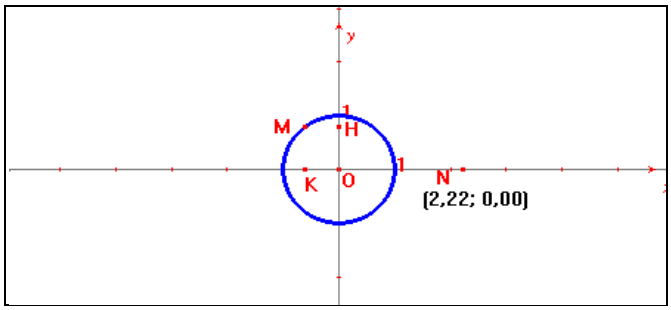

Figure 2: The figure related to the third part of the teaching experiment.

The evolution of the verbal signs elaborated and used by the students and the teacher attest the existence of four steps towards the definition of the trigonometric function cosine.

1st Step: Recognition of numerical and geometric variation

The use of the artifact to accomplish the task promoted the emergence of verbal signs related to the idea of motion. The first step is characterised by an important use of atifact-signs stressing the attachment to the activity with the artifact. In fact, students used verbs of action as "to move", "to vary", "to turn", "to change".... and used expressions as "we draw the point M on the unit circle and we move it" or "we can choose different values for x..."

2nd Step: Identification and recognition of covariation at a contextual level

This step is characterized by the emergence of complex signs "interpretation" linking two variations. In these signs the reference to space and time was eliminated and we noticed the use of expressions related to the indirect variation and to the simultaneity of variations such as "When X varies then Y varies" or "If X varies then Y varies"

3rd Step: Interpretation of covariation as a functional relationship.

This step is characterized by the use of pivot-signs "depend on", "a function of" and "relation between" to replace the expressions "When X varies then Y varies". Students identify the functional relationship as a relation between two variations. This was interpreted as an evolution of students' personal meanings related to the notion of function.

In the collective discussion the teacher used the artifact to promote the emergence of the students' simple and complex signs related to the ideas of functional relationship and covariation "*T : Well. Now what is going to change if you move N?"* The artifact is used as a semiotic mediator supporting the transition of signs expressing the artifact-tasks relationship into signs expressing the artifact-knowledge relationship. As a result of the guidance of the teacher, students expressed mathematical-signs such as "the function which associates M to N", and others related to the idea of covariation.

4th Step: The mathematical definition of the notion of cosine function

When exploring the relationship between *x* and the abscissa of *M through the use of Dragging*, we observed students' interpretation of the movement of point *K* as a variation of the abscissa of *M* and the description of the functional relationship between *M* and its abscissa.

The use of mathematical-signs related to the meaning of the cosine function illustrates the development of personal meanings related to the mathematical meaning of the cosine function as covariation. The collective discussions, guided by the teacher, allowed for the construction of the accepted mathematical meaning, i.e. the mathematical definition of the cosine function. (Khalloufi-Mouha, 2012) The main goal of the teacher's actions in the collective discussion is managing the

students' discourse, in order to support them to move from the artifact and place them in a mathematical environment which allows the recognition of the cosine function. The discussion shows a large commitment of students to the process of meaning construction. By using many mathematical-signs, they succeeded in linking their mathematical and physical knowledge. In this case, we can say that the artifact was a powerful resource for the construction of the mathematical meaning based on the activity with the artifact.

Conclusion

The epistemological importance of introducing the notion of trigonometric function as covariation to make the link between trigonometry and trigonometric functions led us to design and experiment a teaching sequence integrating the artifact Cabri. The analysis of the process of constructing the mathematical meaning of the cosine function, through the analysis of the verbal signs, allowed us to identify that this process begin by a first step related to the recognition of the numerical and geometric variation. This step is characterized by an important use of the artifact-signs and an attachment to the task with the artifact. The second step is the identification of the covariational relation at a perceptual level. The third step marks a mathematical interpretation of the activity with the artifact following the collective discussions orchestred by the teacher. The process is carried out following the interventions of the teacher, by the mathematical definition of the notion of cosine function.

The teacher's role is very important in the process of constructing mathematical meaning among students using a technological artifact. The teacher has to use the semiotic potentialities of Cabri and orchestrate the discussions in order to guide this process towards the construction of mathematical meaning. For this reason, further investigations into the role played by the teacher are required for a better description of the process of construction of mathematical meaning among students using a technological artifact.

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DeafMath: Exploring the influence of sign language on mathematical conceptualization

Christina M. Krause

University of Duisburg-Essen, Germany; christina.krause@uni-due.de

Sign languages are performed in a modality other than spoken languages, using the entire body in a spatial-visual-somatic way. With reference to spoken language, performance of the language in terms of articulation, but also perception and interpretation, changes in the medium of sign language as a visual means of expression. Considering mathematical discourse and social interaction as an important factor in the learning of mathematics, this paper discusses theoretical approaches of a research program, currently underway, that aims at getting a better understanding of how the use of sign language may influence the learning of mathematics. From this a more profound basis shall be derived for developing didactical strategies responding to the special needs of deaf learners and understanding the role of bodily language in mathematical conceptualization.

Keywords: Sign language, deaf learners, gestures, mathematical discourse and social interaction.

Introduction

Research in the area of Deaf studies and Deaf education points at the special challenges deaf students face when learning mathematics. Their lack of basic mathematical skills—deaf children lack several years on average behind hearing peers (Nunes, 2004; Traxler, 2000)—is considered to be mainly caused by social and linguistic aspects.

Deaf children do not 'pick up' informal knowledge (Ginsberg, Klein, & Starkey, 1998) about mathematical concepts in early childhood as easily as hearing children do, due to growing up in an environment that is primarily aligned to auditive social experience (Nunes & Moreno, 1998). For example, everyday phrases of "mathematical conversation" (Gregory, 1998) just as 'It is five to twelve' or 'Turn right in three quarters of a mile' can provide a first contact to numbers that is not accessed 'en passant' by deaf children. Not necessarily growing up in a deaf community, they may also lack everyday interaction with peers that may initiate first instances of problem solving in playful situations, e.g. dividing a quantity in equal parts. Furthermore, deaf learners struggle with reading, understanding and processing written word problems (Hyde, Zevenbergen, & Power, 2003). Their challenges are partly explained by a decreased short term memory in serial recall of linguistic material, by a problematic comprehension of certain language structures like conditionals, comparatives, inferentials and lengthy passages (Rudner, 1978), and by the semantic understanding of the written language as a second language (Barham & Bishop, 1991; Traxler, 2000).

Hence, and probably as no surprise, language is considered a main factor influencing the learning of mathematics for deaf learners. However, language has mostly been considered a problematic condition that impedes deaf students' learning rather than investigated as an integral part of the learning process itself. As a spatial-visual-somatic language, the sign language used by the Deaf provides access to mathematical ideas different than that of spoken language. But what exactly does this mean for the learning of mathematics? And what can we learn from looking at how learning under these special conditions takes place?

The approach presented takes into account the specificity of sign language to encounter the peculiar characteristics of mathematical discourse and social learning processes in the deaf classroom. Furthermore, I support the claim that the modality of language not only affects how mathematics is learned, but that it also influences how mathematical ideas become conceptualized by impacting the structure and process of thinking (Healy, 2015). This contribution therefore outlines theoretical approaches and possible implications of a new research program that aims at developing a better understanding of how mathematics is learned using the medium of sign language.

Sign language(s) and gestures

Sign languages are visual languages that are formed by several components such as the configuration, movement and orientation of the hands and their location in space, body posture, facial expression and the viseme (or 'mouthing': the movement of the mouth). These aspects shape what is considered the utterance in sign language and are, just as spoken language, more or less conventionalized. These conventions distinguish the manual expression from the gestures defined in the style of McNeill. While he defines co-*speech* gestures as "idiosyncratic spontaneous movements of the hands and arms accompanying speech" (McNeill, 1992, p. 37), I adapt this definition for an understanding of co-sign gestures as being 'idiosyncratic spontaneous movements of the hands and arms' *accompanying the signed discourse*. Signers use non-conventionalized gestures in addition to the signs and both types of gestural expression can hardly be distinguished (see also Healy, Ramos, Fernandes & Botelho Peixoto, 2016). Being performed in the same visual-gestural modality, signs and gestures are deeply intertwined in their use and in their interpretation, probably even more intertwined than in the case of spoken language.[1]

Cognitive aspects of the influence of sign language on the learning of mathematics

Embodied cognition

Following the theory of *embodied cognition,* our (mathematical) thinking is deeply influenced by how we experience the world as physical beings (Lakoff & Núñez, 2000). How we act in and perceive the world structures our thinking and shapes to large extent our conceptual understanding:

> Human concepts and human language are not random or arbitrary; they are highly structured and limited, because of the limits and structure of the brain, the body, and the world. (Lakoff & Núñez, 2000, p. 1)

A slightly more cautious claim is stated by Wilson and Foglia in the *embodiment thesis:*

> Many features of cognition are embodied in that they are deeply dependent upon characteristics of the physical body of an agent, such that the agent's beyond-the-brain body plays a significant

[1] This also becomes a methodological issue. It is almost impossible to translate from sign language to written language, even if using lexemes for the notation. Gestures contribute naturally to the interpretation of the utterance such that the analytical distinction between which aspects are signed and which are gestured cannot be made as clear as analytical distinctions between the spoken and the gestured. Neither can be considered separately.

> causal role, or a physically constitutive role, in that agent's cognitive processing. (Wilson & Foglia, 2011, paragraph 3)

More precisely, Wilson and Foglia distinguish three roles the body can play in cognition: It can constrain cognition, distribute cognitive processing and regulate cognitive activity (Wilson & Foglia, 2016, paragraph 3). In sum, "such determinate forms of the Embodiment Thesis can ascribe the body either a significant causal role, or a physically constitutive role, in cognition" (Wilson & Foglia, 2016, paragraph 3).

However, the "body as constraint" is not to be understood with a merely negative connotation as one may get at first sight, taking into account two further implications provided by Wilson and Foglia (2016):

- Some forms of cognition will be easier, and will come more naturally, because of an agent's bodily characteristics; likewise, some kinds of cognition will be difficult or even impossible because of the body that a cognitive agent has.
- Cognitive variation is sometimes explained by an appeal to bodily variation. (paragraph 3)

This view on embodied cognition is coherent with the approach taken by Healy and colleagues who understand bodily organs as tools in the sense of Vygotsky, influencing structure and process of thinking (Healy, 2015). As instrumental tools, the sensory organs can be substituted among each other, which "is expected to cause a profound restructuration of the intellect" (p. 299).

Such a substitution comes into play for deaf learners, where the lack of auditive perception becomes substituted by other sensory experiences. In the hearing classroom, information and ideas are shared to a large extent verbally while deaf students acquire information and interact by means of visual modes of expression, just as sign language. Following the theoretical approaches laid out, such a variation concerning the process of learning mathematics should alter cognitive structures and thinking processes, perhaps also leading to differences in conceptualization of mathematical ideas.

Features of sign language

Research in the field of Deaf Studies in fact indicates that deaf people 'think differently' (Grote, 2010, 2013). Grote emphasizes that the modality of language—whether it is communicated in vocal language or in sign language—influences processes of conceptualization. She identifies two features of language modalities with such influences: *Articulation* and *iconicity*.

While information is strung together sequentially and linearly in vocal language, sign language offers the possibility to represent different aspects of the utterance simultaneously. This can compensate for the greater time required by spatial articulation in sign language over that of verbal articulation (Bellugi & Fischer, 1972; Grote, 2013). However, sign language can represent only those concepts simultaneously that stand in a *syntagmatic relationship*, that is, concepts that consist of several aspects connected through linguistic contiguity. Signs that bundle these aspects by using a particular handshape to express additional information are sometimes called *polycomponential signs* (Grote, 2013), *classifier predicates*, or *depicting verbs* (Liddell, 2003). In contrast to this stand the representation of concepts from the same paradigm, e.g. concepts that are connected in hierarchy (see Fig. 1).

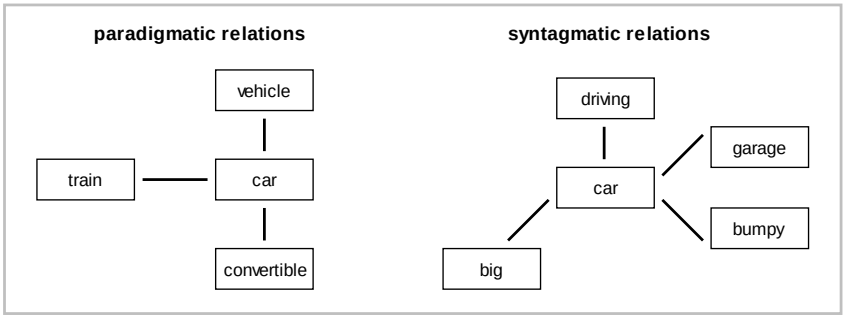

Fig. 1: Example for paradigmatic and syntagmatic relationships (following Grote 2013, p. 313)

These *paradigmatic* (or 'associative' (Saussure, 1983)) relationships need to be articulated linearly, just as in verbal language (Grote, 2010, 2013). Grote (2010) claims that this may lead to a preference for communicating those ideas that stand in a syntagmatic relationship and gives empirical evidence that this preference may engender the establishment of a stronger link between these relations over paradigmatic ones.

Furthermore, gestures often show a certain resemblance with what they signify; they evoke an iconic relation to its referential object. This relationship, however, needs to be established since it is not self-evident. Related to the process of conceptualisation, Grote claims that

> assuming that epistemic processes are processes inherently mediated by signs, the similarity that forms the relationship between icon and referential object is constituted actively. This means that in the process of iconisation, there is a focus on specific features of the semantic concept which probably become stronger linked and get an exposed position in the semantic net. (Grote, 2010, p. 312, translated by the author)

When conducting verification tests, she found remarkably shorter reaction times for those pictures that showed the feature that was iconically reflected in the sign. This pointed to a stronger semantic link between this feature and the signed concept and provided evidence that "those features that are reflected in the iconic moment of sign language get a specific relevance for the whole semantic concept" (Grote, 2010, p. 316, translated by the author).

So what might this mean for the learning of mathematics?

Learning mathematics is not perceived as a purely cognitive phenomenon but can be understood as a social process in which individuals co-construct mathematical meaning and knowledge within the social interaction that is constituted by the use of signs. These signs can be of written, spoken, or gestural form or anything else that can be considered a semiotic sign, performed in any modality. In this sense—and taking into account the embodied approach outlined earlier—learning is understood "as a multimodal process" (Arzarello, 2006, p. 1), influenced by production and perception of signs within social interaction. The use of sign language plays part in both, production and perception.

Based on this, possible issues that can arise are the following:

- A preference of communicating syntagmatic relationships may lead to place special emphasis on these when carrying out social epistemic processes in social interaction and therefore, may lead to make syntagmatic relations conceived as being more important for the related mathematical concept.

- Knowledge about which relations are ‘linked’ linearly and which simultaneously can influence teaching methods. While in the learning of deaf students there needs to be emphasis on developing paradigmatic inner-mathematical relations, the use of co-speech gestures may support strengthening syntagmatic links also in the regular classroom. Theoretical foundations for such an approach are provided by the results on gestural specification of the verbal utterance in processes of constructing mathematical knowledge in social interaction, as described in Krause (2016).
- Providing ‘mathematical signs’ as nonverbal terms to students, it needs to be noted that the iconicity of the sign may lead to an exposed position of the aspects that become visually reflected in it. Oftentimes, official and conventionalized ‘mathematical signs’ do not exist or are not known so that a ‘suitable’ mathematical sign may develop hand in hand with the knowledge during the learning process in the mathematical classroom (see also Fernandes & Healy, 2014; Krause (2018)). To support the conceptualization of mathematical ideas, it is therefore important to take a closer look at which aspects of a mathematical idea are reflected iconically in a mathematical sign, and how meaning develops in the respective signs in a process of iconization while the ideas become encountered. Within this process, the iconicity of the gesture may inform about the signer’s current conceptionalization of the mathematical idea. This may be used for the purpose of assessment and fits the development of the ‘associated gestures’ found in hearing learners’ social processes of constructing mathematical processes (Krause, 2016).
- Many mathematical concepts are shaped metaphorically so that the mathematical concepts are understood through something familiar or more illustrative (Lakoff & Núñez, 2000). These metaphors cannot be represented iconically in a direct way, the developing sign/gesture rather refers to an ‘underlying’ meaning (see again Fernandes & Healy, 2014). Gestures developed by deaf students while constructing mathematical knowledge in social interaction may therefore indicate possible approaches to these ideas and concepts. Knowledge about these approaches can also help in cases of learning mathematics in a second language since linguistic approaches to metaphors may not be accessible.

The research program “DeafMath"

These considerations motivate my research program in which I investigate the influence of sign language on the conceptualization of mathematical ideas, focusing on two main aims:

- Contributing to the development and further elaboration of a theory on the role of the body in the conceptualization of mathematical ideas,
- Providing theoretical foundations for developing didactical methods and strategies that involve the body in processes of teaching and learning.

Furthermore, another goal lies in the development and evaluation of methodological approaches that take into account the specificity of the research setting when working with deaf children. The crucially different characteristics of sign language as a visual-gestural language, as well as the students’ difficulties with written language, demand an adaption of methods for collecting, preparing, and analysing data. This becomes especially important with respect to qualitative studies that follow interpretative and reconstructive methods since the holistic representation in sign

language cannot be captured merely in written form that can only reflect linear and segmented language. A (more) suitable methodological approach might place a greater emphasis on the coordination of written transcripts, pictures, and videos for means of analysis, but also for the documentation of the results.

Potential long-term goals with respect to implications on teaching methods and strategies concern the following aspects:

- The identification of challenges that are specific to deaf students and countering them in their core: Is one challenge grounded in their understanding of (some) mathematical concepts as deviating due to the deviating modality of their language?
- Understanding the inclusion of deaf learners and their way of communicating as actual surplus in the inclusive classroom. Results gained from these studies can point out how an actual inclusion of hearing-impaired students can enrich the entire classroom.
- Using representational gestures in a goal-directed way as didactic means. In Krause (2016) I describe how the use of representational gestures can influence the collective formation of mathematical concepts in a beneficial way by its various representational functions. Results derived from the here described study may give insights in how these representational gestures may look like.

This program therefore considers ‘barriers and chances’: While the different kind of communication may lead to specific challenges when learning mathematics, taking into account these differences entailed by the spatial-visual-somatic and embodied medium of sign language might help to “become better able to respond to their particular needs, but also build more robust understandings of the relationships between experience and cognition more generally” (Healy, 2015, p. 289).

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Fostering of interdisciplinary learning through basic education in computer science in mathematics in primary education

Peter Ludes and Marcus Schütte

Technical University Dresden, Germany; peter.ludes@tu-dresden.de

Digitalization is a topic that is ubiquitous in everyday life. The technological revolution progresses with high pace. Technology/web/mobile companies all over the world are flourishing but primary schools (in Germany) do not follow the economy's development and has not made computer science part of its curriculum yet. This is either due to technical equipment limitations or to the inability or insecurity of teachers to include technology and the underlying principles in their classroom. Furthermore, it is not officially integrated into the core curriculum which prohibits the implementation of a universal standard for computer science competences that have to be taught in primary schools. The paper will present the underlying project and the development of specific learning environments that are designed to nurture cooperation and interaction between participants to have a closer look at the in-between, where learning occurs. This will help to solidify the claim that computer science in primary school has a right to exist.

Keywords: Computer science, interaction, mathematical competences, digitalization.

Introduction

Computer Science, digitalization or technology are only three out of many other terms concerning the digital age that ubiquitous in today's society. Nearly every professional branch has to deal with technology and digitalization at some point. Employees have to learn how to deal with these digital structures mainly on their own, because they never experienced a fundamental education in computer science. Many schools in Germany offer Computer Science only as a subject in secondary school although coding is seen as the language of the 21st century. Whoever is able to "speak" a programming language will be understood all over the world. Many IT unicorn businesses have their roots in a teenager's room or the parents' garage. Often, these businesses came to be because exceptional skill and auto didactical training come together. The masses only seem to use the technology that emerges from these businesses without understanding the basic underlying structures that enable the creation of a product. No matter which idea came to be, it has almost certainly been realized through programming. The choice of programming language is rather secondary as they all follow the same basic rules. Children, from first grade onwards, know about digital technology and computers in different shapes and formats, but when they enter school, these digital tools often cease to exist. Therefore, a responsible use of these technologies and learning in computer science is not fostered in primary schools. Tablets, laptops, smartphones and gaming consoles are at least present once in every household. According to the KIM study, most families own more than one of these devices and use them more than once a week (e.g. the percentage of smartphone/laptop ownership is 97%). In fact, 41% of children from 6-10 own a smartphone themselves. Children learn from an early age how to use e.g. tablets. The functions of these devices are often self-explanatory. The underlying structures, e.g. how is software developed, what rules does a computer follow or how to troubleshoot problems, are not that clear to most children although these structures are relatively similar across all digital devices, this also is true for most teachers, who did not grow up with technology and are now supposed use it in the classroom and teach its responsible use. Compared to mathematics this would

be similar to learning specific calculations with specific numbers or materials but not understanding how to transfer and apply e.g. long division to other tasks. The paper does not aim to defy media competences and how they can be accomplished, it is rather a claim to go further and to embrace computer science itself into primary education and see where and when interaction and negotiation of meaning between children occurs and how it fosters the learning of computer science and mathematics competences. This would empower children to accomplish much more with their knowledge about media and their use (knowledge, they undoubtedly gained while growing up in a digital world).

Status Quo: Media competences, a watered-down term

Computer Science is a topic that is present in almost every secondary school, either as a mandatory or a facultative subject. Either way, the subject is taught by teachers, who have a higher education degree in computer science. To become a primary school teacher in Germany, as in many other countries, the selection of specific subjects is not necessary, as the primary teacher is trained to teach all subjects that are part of the primary school curriculum (although many universities offer the opportunity to choose a core subject (e.g. mathematics) that will give the student the ability to enjoy an even more focused education in this specific area. Computer Science as a subject is not present in either primary school or primary school teacher education. Future teachers rather can archieve something that is often called media certificate or media training. Mostly, this is integrated into the courses that are already taken at university. E.g. one course would include a topic like “use of digital media in geometry in primary school”. This is neither standardised, nor does it aim at specific competences. It is rather a subject specific realization of the use of technology for a certain topic. Today, most schools decide themselves how and when to use digital media and often one feels that it is mostly aimed at helping teachers to keep up with what children already know. If they decide to do so it is often done under the term of media competency. Krauthausen (2012) mentions that the term “media competences” is colorful and can be interpreted very differently according to specific interests. For some schools media competency means that the teacher uses e.g. an iPad, for others that the children learn how to use word processing software. Other projects try to foster the use of specific types of media such as podcasts to facilitate learning process for mathematics (cf. Schreiber, 2012). This is far away from being similar to what computer science would request children to learn. Computer science is considerably more than just knowing about the functions of a computer. It is learning about logic, about algorithms, about programming & robotics and (late-breaking) cryptography. Questions like: *What is an algorithm? What is logic? What technology can I use to facilitate my work?* can be answered by young children to improve their learning. But not only will the acquisition of such skills foster the expert knowledge, it will also nurture competences that can be interdisciplinary used. This is important, as we have to ask whether our traditional cultural competences are still sufficient in a more and more digitalized world or whether we need to teach computer science as well. Some projects try to accomplish this, e.g. Herper and Hinz (2009) through computer science education in primary school and Weigand (2009) with his project: “Algorithms in primary school” (title translated by the author). Weigand especially shows that to work on basic principles of computer science (in this case algorithms) a computer itself does not necessarily have to be available, as he uses pen and paper to work on algorithmic processes. Primary schools do not offer the subject of computer science and it will not be easy to implement it into the curriculum as an independent subject. To achieve this goal, a back door has to be found to sneak computer science into

primary school. Then, once its benefits have become obvious the step from being part of another subject to being an independent subject is only a small one. The answer to the question what subject should be used to include computer science in its contents could not be answered more easily. Mathematics seems to be the ideal candidate, as competences in both subjects are similar right up to identical. We will now focus on these similarities in more detail.

Competences in Mathematics and Computer Science

The German core curriculum provides two types of competences for the subject of mathematics: the general mathematical competences and the content-related mathematical competences (KMK Bildungsstandards, 2005). General mathematical competences include arguing, problem solving, communicating, modelling and the presentation of mathematics. Although these competences will equally be taken into account, the main focus here will be placed onto the content-related mathematical competences, which are numbers and operations, space and shape, pattern and structures, sizes and measurement and data, frequency and probability. These content-related mathematical competences provide the framework for the contents of the learning environments. To justify these contents, a side by side comparisons of some chosen content-related mathematical competences and where to find them in computer science will be done. The mathematical competences will be presented as written down in the German core curriculum, then computer science content that also matches these competences will be provided.

Mathematics	**Computer Science**
Understand the relationship between and the representation of numbers, Understand and master calculations	Algorithms use calculations, Loops have to be counted, Sorting algorithms, Types of variables integer, float, double

Table 1: Competences in Mathematics and CS: Numbers and Operations

Mathematics	**Computer Science**
Spatial orientation	Program robots, Define an area of movements,Plan with obstacles and predict motion sequences

Table 2: Competences in Mathematics and CS: Space and Shape

Mathematics	**Computer Science**
Recognize and characterize regularities	Structure and plan algorithms, Plan processes and translate them into a programming language, logic, Sorting algorithms

Table 3: Competences in Mathematics and CS: Pattern and structures

Mathematics	Computer Science
Have the ability to imagine sizes, Have the ability to use sizes in specific situations	Determine the step range of a robot, Determine run time

Table 4: Competences in Mathematics and CS: Size and Measurement

Mathematics	Computer Science
Understand the relationship between and the representation of numbers, Understand and master calculations	Algorithms use calculations, Loops have to be counted, Sorting algorithms, Types of variables integer, float, double

Table 5: Competences in Mathematics and CS: Data, Frequency and Probability

(KMK Bildungsstandards, 2005)

This selection of competences from the core curriculum shows that to each content related mathematical competence a related topic in computer science can easily be found. This is highly interesting, as it suggests that mathematical competences are similar to those that are required to perform tasks in computer science. Thus, mathematics can fulfil the requirements to integrate elements of computer science into its curriculum. To prove this claim, it would hardly be possible to modify the existing mathematical lessons to include this content. Rather, specific learning environments have been developed to show that learning computer science topics nurtures the competences that are necessary for both mathematics and computer science.

The pilot project

Partner school

The search for partner schools was far from easy. Many schools were not interested. One school on the other hand was immediately willing to participate and upon further information managed to interest 19 children from grade four to take part in the project. As we did not expect such an overwhelmingly large number of participants from one single school, we cancelled all further efforts to acquire more schools and decided to work exclusively with the just one partner school. The school was very open to our project and supported us from day one. The first feedback suggests that all children are highly enthusiastic and motivated. A claim that we can only support after our first three weeks. In agreement with the school, we chose two time slots of 90 minutes in the afternoon twice a week (Tuesday and Thursday) over the course of six weeks. During the time slots the children worked on the learning environments in pairs (some tasks required two groups to come together to discuss) supervised by a student from the seminar or Peter Ludes.

Learning environments

To examine whether certain competences are used during working with tasks, specific learning environments have been developed that have a computer science topic as a core topic. The learning

environments have been developed during one of Peter Ludes' empirical seminars for future elementary school teachers. The students developed the first draft of the learning environments on their own and after a review process finalized them collectively during the seminar. The main topics are: logic (general and propositional), algorithms, cryptography and programming/robotics. The main challenge for the students was to develop learning environments with core topics that are not (yet) part of their actual studies, as computer science is not a topic that is taught in elementary school. Therefore, an extensive introduction into the field of computer science has been necessary.

A second demand that had to be kept in mind during the development has been a focus on interaction. The tasks should be designed in a way that -at least partially- fosters interactional processes between the participants as we focus on learning through collective argumentation and participation (Krummheuer, 2011). The first learning environment focused on the topic of logic. The primary task always aims to get a first impression of what the participants already know (or think to know) about the concept. This very open question (e.g. *What does logic mean?* or *When does a person have to think logically?)* provides a wide variety of possible answers without any pressure for right or wrong. The supervising students are always advised to let the children speak freely as much as possible, unless a lively discussion does not occur, general questions or guidance is to be avoided. Although the content is important, the actual focus for this pilot project is not to survey content learning, but rather the learning that occurs between individuals whilst discussing and arguing about the specific topic. Learning in primary school is initially dialogical learning. That is, learning is seen as a dialogical process in contrast to learning as a monological process, which would be rooted in the individual (cf. Miller, 1986). The learning of mathematics can be seen as an increasing autonomous participation in collective argumentations that are produced and nurtured collectively by the group itself (Krummheuer, 1992; Voigt, 1995). This idea of learning can be transferred to the learning of basic competences in computer science. To build upon this concept, all ways of communication and interaction between the children has to be supported as much as possible. To ensure an efficient way to videotape the children working on the learning environments, we designed the environments to be worked on in pairs or groups of three, sometimes with a closing task, that included a larger group discussion in groups of up to six children. Working in smaller groups for us provides the advantage, that single children are not able to extract themselves from group tasks or discussions but rather encourage them to participate. Every learning environment is designed to cover three timeslots of 90 minutes. Here, two sessions of 90 minutes are planned for the actual content-related activities and one session of 90 minutes reserved for documentation and evaluation of learning processes though tools like learning diaries, wikis and storyboards. Learning diaries can e.g. be used to recapitulate the learning process, correct misconceptions and enables the child to visualize what learning progress it has made. We chose the storyboard as an adequate tool for the programming/robotics learning environment. This environment will be realized with LEGO Mindstorms EV3 Educations sets. The main goal after developing and building the robot itself will be to program its specific actions. These actions should be planned beforehand because the children can choose from a variety of actions and sensors with endless combinations. For this task, a storyboard is the ideal candidate as it enables the children to structure their thoughts and plan the movements and sequences that the robot has to fulfil. It also makes trouble-shooting rather easy as the children can always compare their plan to the robot's actual movements and actions.

Analysis

The analysis of the results of the work on the learning environments is based on methods of interpretive classroom research, such as interaction analysis. As we place much importance onto interaction, cooperation and discussions this method seems most appropriate to us. It helps to find more suitable tasks that help children to collaborate in a productive way. The negotiation of meaning is a key element that has to be focused (cf. Krummheuer, 1992). How does the negotiation of meaning in collective processes of argumentation occur and how can it be supported thought the task itself? According to our perception, learning does not occur inside the individual but during the interaction between individuals, whilst discussing, talking, arguing and also justifying the own answers and the answers of others (cf. Krummheuer, 1992). The underlying concept is founded on the ideas of the symbolic interactionism (cf. Blumer, 1969) that will allow us to examine learning as the increasing autonomous participation in collective argumentation in computer science discourse. Possibly this could lead to the definition of participation profiles (cf. Brandt, 2004) for different students, specifically tailored to the computer science classroom. This could not only benefit primary education, but also computer science classrooms in secondary schools. If a specialized and standardized computer science education in primary school was mandatory, secondary schools would have a contact point and a profound basis on which they could build and focus their curriculum.

First impressions

The first impressions of the pilot project are consistently positive. One major question during the development process was whether children could be unchallenged or overwhelmed with the tasks as they work with topics that are not being taught in primary school and are therefore unfamiliar. The individual knowledge of the children concerning this specific topic could therefore be developed very differently. The children engaged immediately with the tasks and lively discussions occurred. In the very beginning, a certain insecurity was noticeable. This was expected as the children naturally try to give correct answers. This was not possible as most questions are designed in a way that multiple answers and also very individual answers are possible. After the first sessions, the children got more and more used to the types of tasks and felt more comfortable when joining the discussions.

Interestingly, mathematical discussions occurred very often, whether they were planned or not. One example where a rather simple question led to a lively discussion occurred during the learning environment: logic. The children had to decide whether a statement is right or wrong. One of the statements claimed: If something is round it is not pointy. Right or wrong? The children then engage into a discussion, where they have to decide what qualifies as being round:

S1:	If something is round it is not pointy\ [reads]
	That does not have to be true\
S2:	(It is not\) [laughs].
S1:	That does not have to be true maybe it is so to say a pointy circle\
I:	How does a pointy circle work?\
S1:	So . wait (4) [draws into his folder]
S2:	[looks into S1’s folder]

S1: Like this and so on and on \ [points at the drawing in his folder]

S2 Yes, but if something is round [draws a circle into the air with his pen] so this here [points at S1's drawing] that is not round\ that is pointy\

S1: Yes, it is **roundly** but fi it is round then you are right\

First, S1 states that he is not convinced that this statement is true and he tries to find a shape that is pointy and round. He therefore draws a shape that has corners but is not just one straight line, similar to a part of a polygon. S2 then looks at the drawing and defines the shape as pointy because it has corners. S1 then again discusses the word round and seeks for a better word to describe his drawing and proposes the word roundly for something that is not a straight line but follows the form of a circle part although it has corners. S1 and S2 clearly have a dissent in what qualifies as being round and then shift this disagreement into a consensus through the introduction of a new vocabulary. It is rather interesting that the S1 and S2 discuss to find a consensus and the questions remains, what would happen if they could not find one. Here, the material itself proposes a research question: How do consensus and dissent influence cooperative learning. Is learning through collective argumentations also possible on the basis of a dissent or does a consensus have to be reached in order to complete/move on with the task itself? The structures in cooperative learning opportunities could be fundamentally different if the necessity of reaching an agreement would not be mandatory. This question will be focused during further analysis as the concept of computer science learning environments provides a perfect frame: It has clear and visible connections to mathematics but is fundamentally new and different, so that children can learn a new topic in which they do not have prior knowledge to build on.

Prospects

The learning environments with the topics cryptography, algorithms, programming/robotics are also completed by now. The videos and writings of the children are being extensively analysed using methods of interpretive classroom analysis. The analysis will enable us to look at the negotiation processes in more detail and determine, where the learning of individual computer science concepts and understandings occurs and how meaning is negotiated during the task itself also in regard to cooperative learning and the underlying structures. Through this, it will be possible to rework the learning environments to tailor them even more specifically to their purpose. The reworked learning environments will then be used in a more extensive main study to examine our claims and to strengthen the position of computer science as a key part of a profound and forward-thinking education that will not only benefit children's abilities in this specific subject but also strengthen their mathematical competences

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On the nature of spatial metaphors: Dimensions of spatial metaphors and their use among fifth graders

Angel Mizzi

University of Duisburg-Essen, Faculty of Mathematics, Essen, Germany; angel.mizzi@uni-due.de

This paper reports on a study about the use of spatial metaphors among students solving spatial tasks. The aim is to describe the nature of spatial metaphors and analyze the use of spatial metaphors under consideration of other factors, such as students' language proficiency and spatial ability. Sixteen fifth grade students, chosen according to a theoretical sampling, were required to describe to other students how to build a pre-designed spatial object made up of building cubes. The data was analyzed and different dimensions of spatial metaphors were identified and described. Whereas the overall frequency of use of metaphors in spatial discourse does not differ substantially among different groups of students, findings show that the use of spatial metaphors might differ if one analyzes the functions and the conceptions of the underlying metaphors.

Keywords: Language, metaphors, spatial ability, geometry.

Introduction

Metaphors and their role in shaping the teaching of mathematics has been an area of research in the domain of language and mathematics education for a long time (e.g. Lakoff & Núñez, 2000). Further research is required to show the individual use of metaphors among students learning mathematics. The aim of this paper is to show the nature and diversity of spatial metaphors by describing and illustrating the different dimensions involved when students use spatial metaphors to verbalize their spatial thinking. The use of the spatial metaphors among different students chosen according to background factors (language proficiency and spatial ability) will also be an object of discussion in this paper. Hence, the research questions of this paper are as follow:

1. What are the different dimensions of metaphors used by students in their spatial discourse?
2. Does the use of spatial metaphors differ significantly among students with different background factors (language proficiency and spatial ability)?

Theoretical considerations

Language in mathematics classroom

Several researchers of language in mathematics education have pointed out the different planes of language which co-occur in the learning and teaching of mathematics. A bi-planar model about the use of language has been developed by Cummins (2000), who differs between Basic Interpersonal Communication Skills (BICS) and Cognitive Academic Language Proficiency (CALP). BICS describes the use of language in everyday-life context, for instance, in order to communicate with friends or family. In contrast, CALP refers to the language used for academic purposes, which includes a more complex syntax and specific vocabulary, used to describe abstract ideas, such as mathematical concepts. There are no clear-cut boundaries between BICS and CALP, since both can

influence each other. Hence, both planes should be visualized as a continuum of language in the mathematics classroom, rather than as separate entities.

Metaphors

Metaphors play an important role in constructing mathematical discourse and are present in the language of mathematics. Lakoff and Núñez (2000) use the term conceptual metaphors, which are metaphors used to understand abstract ideas by referring to concrete objects or experiences. In their notion of metaphors, properties are transferred from the source object to a target object. If one considers the metaphor "container", which is an object of everyday use, it can also be used to describe the notion of class in mathematical language. Therefore, certain characteristics are transferred from the source ("container") to the target domain ("class") (Lakoff & Núñez, 2000).

Spatial ability and spatial language

Along with language proficiency, spatial ability is another factor which plays an important role in mathematics performance (Büchter, 2011). From a psychological perspective, spatial ability or knowledge is considered to include all abilities needed to navigate through space, visualize objects from different angles, and recognize space and spatial characteristics and other abilities needed to solve spatial tasks (Gardner, 2006). The solving of spatial tasks requires several cognitive processes, such as perception of figure and ground, which is the ability to identify a figure in space, spatial relations, which is the ability of identifying the spatial relation between two spatial objects, and position in space, which deals with the identification of the object's position in space under consideration of one's own body (Maier, 1999).

From a mathematics education perspective, spatial ability is incorporated in geometry teaching and is considered an important domain for successful acquisition of geometrical understanding. This is the case in Pinkernell's (2003) spatial ability model, which consists of three main categories of abilities which play a major role in solving spatial tasks: *spatial-visual operations*, which concerns mental and real actions performed on spatial objects, geometrical thinking, which refers to the abilities of recognizing and describing spatial objects by referring to their geometrical properties, and *visual abilities*, which includes the abilities of constructing different forms of representations of spatial objects and being able to interpret them in space.

When analyzing the notion of spatial language and the different ways of representing spatial objects, Mizzi (2016) states that spatial metaphors play an important role in shaping students' language when describing the construction of spatial objects. Spatial language can be described as the language required for talking about spatial objects and their underlying spatial characteristics. An analysis of spatial language can reveal more about the students' spatial thinking and their spatial concept images (Landau & Jackendoff, 1993).

Conceptions of mathematical notions

Sfard (1991) states that mathematical concepts have a dual nature – the structural and the operational conceptions. In the former, a mathematical concept is treated as an abstract object and is mostly likely to be conceived as a static entity. In the operational conception, the individual "speaks about *processes, algorithms, and actions* rather than about the objects" (Sfard, 1991, p. 4). Hence, mathematical objects can be perceived as objects with static properties, which should be denoted as

static conception, or as a sequence of actions on the mathematical object, denoted as dynamic conception. An integration of both conceptions is considered to be important for concept formation in learning and teaching of mathematics.

Methodology

Task design

In order to analyze spatial metaphors, a method is needed to investigate the interplay between language and spatial abilities, and to allow the verbalization of students' spatial thinking. The reconstruction method, a data collection method in which two or more learners seated in a back-to-back position communicate with each other to solve a task using learning manipulatives, was chosen. In this data collection method, one learner (the describer) is given a spatial object, designed by the researcher and he/she must describe to the second learner (the builder) how to build the same object, as in the following student instructions given by the researcher:

> In this experiment you [the describer] will be given an object made up of these building blocks, which can be put together. You must give him/her [the builder] instructions on how to build this object, so that he/she [the builder] can reconstruct the same object. The colour of the building blocks is not important and whilst you [the describer] are describing you can also touch and move the object as you like, but the object structure needs to remain unchanged. At the end, the objects' structure must be identical.

The above spatial task and instruction required an appropriate spatial object which should be described by the describer. In order to possibly obtain as many metaphors as possible, two spatial objects were used in the study.



Figure 1: Spatial object I



Figure 2: Spatial object II

The criteria for spatial object design included *three-dimensionality* (a requirement to describe along three dimensions), *breakdown* (different possibilities of breaking down the object) and specific *spatial relations* between parts of the object. Both objects consisted of building cubes (see Figure 1 and Figure 2), which were provided for the builder to build the described object.

Data collection and analysis

Thirty-two students attending the fifth grade were chosen to participate in this study. In order to consider different factors which might play a role in solving the task, a theoretical sampling was used for choosing the describing students based on two dichotomies (since the describers are considered to be the ones mostly contributing to the spatial discourse in the underlying spatial task): high vs. low language proficiency (LP+/LP-) and high vs. low spatial knowledge (SK+/SK-). The students' language proficiency and their spatial knowledge were assessed using C-Tests and Pencil-and-Paper tests (Büchter, 2011), respectively. These dichotomies were established after considering which factors could possibly influence the solving of spatial tasks, which created four sample groups, each consisting of four describers, as illustrated in Table 1.

	LP+	**LP-**
SK+	Group 1 (four students)	Group 3 (four students)
SK-	Group 2 (four students)	Group 4 (four students)

Table 1: Theoretical sampling under consideration of two student's background factors

Sixteen other students were chosen to act as builders for the sixteen describers chosen according to the theoretical sampling. The describers were given the first spatial object (Figure 1) and the instructions were given by the researcher. The task was repeated by using the second spatial object (Figure 2). The students were video recorded and their discourses were transcribed for the data analysis. Based on an interpretative qualitative approach, the collected data was analyzed to establish categories for metaphors based on the theoretical considerations about language and spatial abilities. The frequency of the metaphors identified in the spatial description of both spatial objects was analyzed in terms of the students' language proficiency and spatial knowledge.

Results and discussion

About the nature of spatial metaphors

Spatial metaphors can be described as metaphors in terms of Lakoff and Núñez (2000), whereby target objects are spatial objects. The spatial metaphors used among students to describe the spatial objects in the reconstruction method can be characterized by three dimensions: the linguistic, the spatial and the conceptional. In the linguistic dimension, spatial metaphors are characterized by the use of everyday (E), letter-based (L) or mathematical (M) language. Everyday language denotes the use of language from everyday situations (based on BICS). Letter-based language is the use of symbols or letters from written language in spatial discourse. Mathematical language denotes the use of mathematical terms or concepts in spatial discourse, which are more likely to be acquired in mathematics classroom (based on CALP). Again, these three categories should not be considered as entirely separate, but rather as a movement along a continuum with two poles from "concrete" to "abstract" (E – L – M respectively) and vice versa (M – L – E).

The spatial dimension of spatial metaphors involves the function which the metaphors serves from a spatial content perspective. The functions are: structure (ST), spatial position (SP) and spatial relations (SR). In the third dimension, conceptional, the spatial metaphors can either be of a static (S) or of a dynamic (D) nature regarding the conception of spatial objects. In Figure 3 spatial metaphors and their dimensions are represented in a three-dimensional coordinate system.

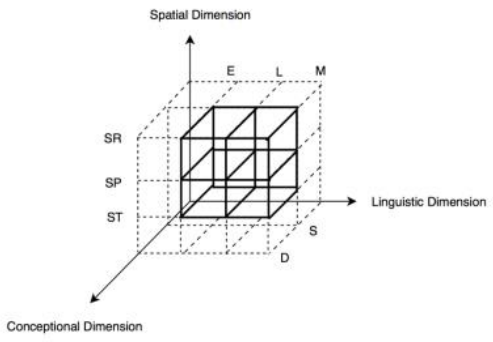

Figure 3: Representation of spatial metaphors and their three dimensions

In the following, I will give some examples of spatial metaphors which can be represented as different points in the coordinate system visualized in Figure 3. Consider the following transcript of a student during his/her description of spatial object I:

> Student A: "Do an L only with one, two, three, four, (…), six pieces. (…) And then do again one, two, (…), five, do five again, so that it looks like an L".

The spatial metaphor 'L' used by student A can be categorized as letter-based on the linguistic dimension of spatial metaphors, because the properties of a capital letter in the Latin alphabet (source domain) are transferred to the spatial object (target domain) (however, Student A does not mention "capital" in the reconstruction method). If one thinks of the distinction between everyday language and mathematical language as a continuum in terms of abstraction and of specific knowledge acquisition, the use of symbols from written language could conceivably be located in between, because symbols represent abstract thinking in terms of lines and distances between them, which are, nevertheless, used in everyday life. From a spatial meaning perspective, Student A uses the spatial metaphor to describe the structure of internal parts of the spatial object and can therefore be assigned to the rather static conception of the object in terms of the conceptional dimension. Therefore, Student A's spatial metaphor 'L' can be assigned to the vertex (L, ST, S) in Figure 3. A more explicit trigger of use of letter-based spatial metaphor ('H') is given by the structure of spatial object II, which has proven students' strong emphasis on spatial metaphors for describing spatial objects rather than on other spatial-geometrical characteristics, such as dimensions etc.

The following transcript excerpt shows another example of the use of spatial metaphors when describing spatial object I:

> Student B: "And now do three steps at the other staircase which you have done. Place it in a way in front of you as if you would walk up (…) and then (…) and now take the other stairs which you have done now, and set it in the most front (…) in a way as if you would walk up at the front and then go down again the other staircase".

The first spatial metaphor used by Student B is 'staircase' and can be assigned to the everyday language to describe the structure of the spatial object and to the static conceptional dimension (E, ST, S). The next spatial metaphor used by Student B is 'walking up or down on the stairs', which is used to rather describe the spatial position of the object, so in which position the 'staircase' is in space under consideration of the subject's own body. In contrast to spatial metaphor 'staircase', the student includes his body in the description to perform an imaginary action on the spatial object. Since the spatial metaphor of 'walking up and down on the object' consists of actions and processes, this metaphor can be described as having a dynamic conception, hence it can be assigned to the vertex (E, SP, D) in Figure 3.

Another example of a spatial metaphor coded from the mathematical language in the linguistic dimension can be observed in the following transcript excerpt:

> Student C: "At the right. And then make this one to the foursome like a triangle to it (…)".

In the above utterances, Student C uses the spatial metaphor 'triangle', which conceivably originates from mathematical language, to describe spatial object I. Student C states that an internal part of the spatial object should be linked to another (the foursome) like a triangle, which indicates

that the spatial metaphor ‘triangle’ primarily describes the spatial relation between both parts. The student seems to refer to a type of triangle (right-angled triangle), using its properties to describe the spatial relation between the two internal parts of spatial object I. Since the spatial relation of the two objects is rather fixed in this context, this particular spatial metaphor ‘triangle’ can be assigned to the vertex (M, SR, S). More examples of spatial metaphors can be found in Mizzi (2016).

Use of spatial metaphors in spatial discourse

Spatial metaphors are characteristic elements of spatial language and they are used very frequently among students in description of spatial objects, as can be observed in Figure 4.

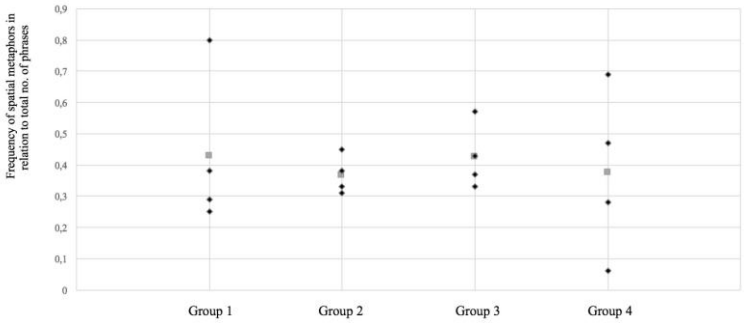

Figure 4: Frequency of use of spatial metaphors in students’ spatial discourse among the four groups

At a first glance on Figure 4, one can see that the use of spatial metaphors in the students’ discourse has the largest variation among the four students in Group 4. However, on average (represented by the squared light grey dots in Figure 4), spatial metaphors tend to be used on average in approximately 36 % to 40 % of the total phrases of the spatial discourse by the students regardless of their assigned group. So the general use of spatial metaphors among the students do not differ significantly among the different four groups.

Use of language and spatial metaphors in students’ spatial discourse

The type of language used in spatial metaphors by students is worth looking at. The underlying hypothesis is that the use of the linguistic dimension of spatial metaphors differs among students with different language proficiency. Students with high language proficiency may use more spatial metaphors, since they may have more vocabulary (in mathematical language) at their disposal.

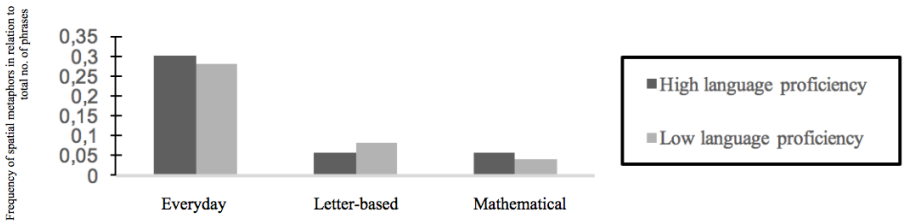

Figure 5: Use of language and spatial metaphors by students’ language proficiency

Figure 5 presents the different uses of language in spatial metaphors among students with low and high language proficiency. It shows that students with high language proficiency use slightly more spatial metaphors from everyday and mathematical language, and slightly less letter-based spatial metaphors than students with low language proficiency.

Functions of spatial metaphors in students' spatial discourse

The next step is to analyze the functions of the used spatial metaphors among the different students. Figure 6 shows the number of spatial metaphors used in relation to the average students' total number of phrases by their spatial functions: ST, SP, and SR.

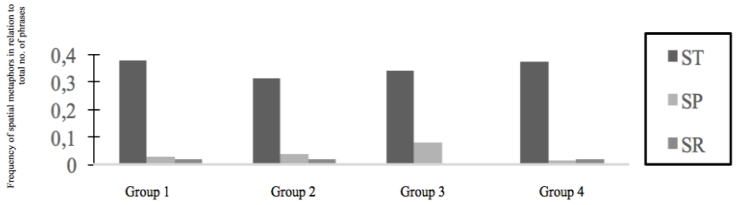

Figure 6: Frequency of the used spatial metaphors classified according to their function

According to Figure 6, most of the spatial metaphors were commonly used to describe the structure of the spatial objects. On average, every metaphor with ST-function occurs in at least every third phrase of the student's discourse. Students in Group 4 relatively used relatively less spatial metaphors to describe the spatial position of spatial objects. Moreover, one can notice the relatively higher use of spatial metaphors with SP function in comparison to SR, especially among students with high spatial knowledge (students in Groups 1 and 3). In contrast, spatial metaphors with SR function were very rare and if used they tend only to describe the spatial relations between two object parts after breaking down the spatial object in the description.

Conceptions of spatial metaphors in students' spatial discourse

Consider the conceptions of the spatial metaphors used by the students, which were based on the theoretical framework of Sfard (1999). It is worth mentioning that although most utterances were action-based and therefore of a dynamic nature, spatial metaphors were analyzed as distinctive elements conveying an idea which is either of static or dynamic nature. The former tends to be more property defining or transferring, whereas the latter entails a movement in the spatial metaphor itself. Considering the conceptional nature of spatial metaphors used by the participants, one can conclude that most of the spatial metaphors used are predominantly static in nature, regardless of the two dichotomies in the theoretical sampling. The high occurrences of static and the low occurrences of dynamic spatial metaphors seem to be linked to the function of the underlying metaphors. Since most of the students used spatial metaphors to describe the structure of the spatial object (see Figure 6), the corresponding nature was static. Whereas, if more spatial metaphors were used to describe the spatial position of the spatial object, then there is a higher tendency of the spatial metaphor being dynamic. However, this does not imply that all the spatial metaphors of S and SP function are static or dynamic respectively, as the following transcript excerpt of Student D describing object I shows.

> Student D: "At first it not much far, and then forth and forth. And at the other edge as well, and then they meet each other at the top. It is almost four cubes to the top and four steps".

Student D uses the spatial metaphor of 'meeting' to describe the structure of the spatial object, which she has broken down in two parts (two 'edges'). The metaphor 'meeting' is used to convey the convergence to one point of the structure of the object created by the two parts or 'staircases'. Hence, this metaphor is of a dynamic nature and an example for the vertex (E, ST, D) in Figure 3.

Conclusion

This paper has offered an insight into the nature of spatial metaphors which fifth grade students use when describing particular spatial objects. The different dimensions of spatial metaphors reflect the integrated language and spatial content learning which is important in mathematics education research. The analysis of groups of students according to theoretical sampling shows that spatial metaphors are common features of spatial language. Regarding the function of spatial metaphors used, spatial metaphors were prevalently used to describe the structure of the spatial object. However, on average, students with high spatial knowledge tended to use more spatial metaphors to describe the spatial position of an object or its parts. In terms of conceptual dimension, most spatial metaphors used by the participants were of static nature, which is consistent with the finding that most spatial metaphors were used to describe the structure of the object. These findings about spatial metaphors reveal some characteristics about fifth grade students' spatial thinking, i.e. the predominant role of associations from everyday-life and the preference of static conception of spatial concepts to master spatial tasks which require verbalization. These results provide an initial step toward understanding the under-researched relationship between spatial thinking and language.

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Ensuring methodological rigor for investigating the discourse development of bilingual mathematics learners

Máire Ní Ríordáin[1], Eílis Flanagan[2] and Claire Brilly[3]

[1]University College Cork, School of Education, Ireland; maire.niriordain@ucc.ie;

[2]National University of Ireland, School of Education, Ireland; eilis.flanagan@nuigalway.ie;

[3]National University of Ireland, School of Education, Ireland; claire.brilly@nuigalway.ie

Given the nature of investigating bilingual mathematics learners and learning environments, a key concern is how we can ensure that the rigor of our research is matched by the rigor of methodological frameworks and approaches employed. Our goal is to develop a theoretical framework and associated methodology and methods, in practice, in order to ascertain their suitability for investigating bilingual mathematics learners in an educational context. Moschkovich (2016) identified four key recommendations for conducting research on language: utilising interdisciplinary approaches, defining central constructs, building on existing methodologies, and recognizing central distinctions. Utilising Moschkovich's framework, this paper provides an appraisal of the methodology and methods to be employed in a research project examining bilingual mathematics learners.

Keywords: Methodology, methods, bilingualism, discourse, framework.

Introduction

Investigating mathematics and languages is a complex process. Therefore, the authors argue that there is a need to develop appropriate research methods in order to investigate language use and its impact on mathematics learning. In particular, we believe that the role of language(s) should be examined within mathematical activity and in situ (Barwell, 2016). This paper draws from the researchers' current study, which explores the potential for developing a coherent and integrated interpretive theoretical framework to examine whether differences in languages, and their use, by bilingual mathematical learners have a differential impact upon cognitive mathematical processing, while recognizing the social aspects of learning. The project, entitled 'M²EID: Mathematical Meta-level developments in English and Irish language Discourses', is a mixed-methods study, comprising video-recorded observations, questionnaires and cognitive interviews. The research project is being undertaken with first year, undergraduate students, who choose to study Mathematics through a bilingual approach (English and Irish) during their first year of undergraduate education at the National University of Ireland, Galway (NUI Galway). This option runs parallel to its English-medium counterpart, which typically receives a large intake (at least 150) of students. Four weekly lectures are provided in the Irish language with all terminology given bilingually. In addition, lecturers may opt to describe more complex concepts (such as limit of a function) bilingually. The lectures are supplemented by the provision of a weekly workshop in English in addition to an Irish-medium workshop.

Given the nature of investigating bilingual mathematics learners and learning environments, a key concern of this paper is to describe and discuss how we can ensure that the rigor of our research is

matched by the rigor of methodological frameworks and approaches employed. It is imperative to review epistemology and associated underlying assumptions in order to make meaningful the methodology and methods of the research being undertaken in a bilingual mathematics education context. Grix's (2004) definitions of 'method' and 'methodology' are valuable for interpreting these constructs. A 'method' refers to the procedures or processes by which data is gathered; whereas, a 'methodology' refers to both the theory applied to inform the research and the data analysis strategies employed as appropriate to the data collected (via the specific methods). While Grix's definitions regulate our M^2EID research study, this paper focuses on possible methodological constructs that can frame such practice-based and context-driven bilingual classroom research. Consequently, the purpose of our paper is to describe and discuss the M^2EID research methodology and methods utilising Moschkovich's (2016, p.1) recommended constructs for conducting research on language use and learning in mathematics. These are: (1) using interdisciplinary approaches, (2) defining central constructs, (3) building on existing methodologies, and (4) recognising central distinctions while avoiding dichotomies. The paper is structured in accordance with these four recommendations and outlines their application to the main research study (M^2EID).

Using interdisciplinary approaches

Research on language and mathematics needs to consider interdisciplinary approaches in the development of methodology and methods and should be grounded in classroom discourse as well as language and bilingualism (Moschkovich, 2016). Therefore, this necessitates the development of integrative frameworks for examining, in situ, both the cognitive and social constructs of mathematics learning through and with languages.

In terms of the mathematics as a composite register comprising content, languages (e.g. English and Irish) and shifts between everyday and subject-specific registers, the authors emphasise the social and interpersonal aspects of language use and bilingualism in mathematics. Such aspects include the use of modes and gestures for communicating understanding and in particular, engagement in the situated and sociocultural practices of mathematical Discourses (Gee, 1996; Moschkovich, 2002). Further, the M^2EID study is aligned with the perspective that learning mathematics is essentially a discursive activity in which learners form and actively participate in a community of practice (Lave & Wenger, 1991; Lemke, 1990). Therefore, learners develop unique sets of mathematical practices and modes of communicating with each other using all of the social, cultural and cognitive resources available to them. Consequently, a democratic process of learning emerges through a continuous cycle of negotiations in relation to views, beliefs, knowledge and meaning making (Moschkovich, 2002). So, by adopting this comprehensive sociocultural perspective of learning and language use in mathematics, this study requires an interdisciplinary approach to research within this educational field. Based on the sociocultural nature of mathematical concepts and how we understand and communicate this nature, it is vital to consider how various disciplines contribute to mathematics education. In order to address the aims of this study we will draw on the principles of Discursive Psychology, Cognitive Psychology, Semiotics, Pedagogy and Anthropology to progress a unified approach to researching learning and language use within mathematics. Due to the multi-ontological nature of this grounding framework for the M^2EID research project, it is essential to develop a dynamic and multifaceted methodological approach to the research, data collection and analysis strategies, which this paper focuses upon. Drawing on the body of relevant literature in this

regard, the authors designed a methodology for investigating bilingual mathematics learners that is underpinned by Sfard's (2008) commognitive framework for examining learning. This framework, described later in this paper, is founded on the premise that thinking is a form of (interpersonal) communication, and that learning mathematics entails extending one's discourse.

Defining central constructs

Moschkovich (2016) emphasises that research studies need to be clear and explicit in relation to the key constructs utilised. Considering the centrality of discourse to the commognitive approach, it is important therefore, that our perspective of discourse is outlined first. Discourses encompass more than verbal and written language and the use of technical language; discourses also involve communities, points of view, beliefs, values, and pieces of work (Gee, 1996). Accordingly, we perceive mathematics as a discourse and a complex form of communication (Sfard, 2012). Gee's concept of Discourse will inform the examination of conceptual mathematical development of bilingual learners, linking both the cognitive and social aspects of language use.

Equally difficult and demanding is the task of defining bilingualism and in particular defining whether a person is bilingual or not. To illustrate these concepts further we employ Grosjean's (1999) model of a continuum of modes with monolingual and bilingual occupying opposite endpoints; this continuum reinforces an understanding of bilinguals using their languages independently and jointly depending on the context/purpose in which the language(s) is being employed. Appropriately then, we support a non-deficit view of bilingual learners, combining everyday and mathematical registers and view language(s) as a resource and a support for learning. Our research is particularly concerned with the role of bilingual students' languages in mathematics teaching and learning. We consider mathematical language as a distinct 'register' within a natural language and each language will have its own distinct mathematics register, encompassing ways in which mathematical meaning is expressed in that language. Specifically, we are concerned with conceptual mathematical activity. This encompasses a knowledge of what it means to understand a concept and an appreciation of how such an understanding can be constructed by a student, thus providing a model of cognition for the concept (Asiala et al., 1996). Given that language influences thought and thinking and that each language will have its unique manner of constructing the concept, it is critical to develop an insight into the role and effect of bilingualism/languages on conceptual mathematical learning. In addition, language(s) facilitate the development of a student's mathematics register and participation in discourse. Consequently, it is an essential instrument of thought and it is vital for understanding and combining experiences and for organising concepts (Vygotsky, 1962). We propose that there are differences 'between linguistically distinct versions of "the same discourse"' (Kim, Ferrini-Mundy & Sfard, 2012, p. 2) which correspondingly impact on mathematical learning. Therefore, it is the use of language as an instrument of thinking that is of importance, as well as its effect on cognitive processing.

When examining bilingual mathematics learners, it is important to address the social use of language within the learning context, not just its role in cognition. As previously noted, Moschkovich (2012) emphasises the importance of learning being illustrated within the sociocultural practices of a certain setting. These practices involve a process of describing learners and communities and considering culture as a set of practices, which actively involve participants

(Gutiérrez & Rogoff, 2003). Hence, bilingualism is described in terms of learners' participation in and use of language(s) for different purposes and particularly in the context of mathematical discourse. Similarly, Moschkovich (2012) emphasises the importance of discerning between the *conditions of* learning and the *processes for* learning, and the importance of describing the curriculum, courses/programmes and teaching and learning approaches utilised that yield successful outcomes for different groups of learners.

Due to the multifaceted process of investigating bilingual learners' use of language in mathematics education, it is vital that an extensive research methodology is developed to facilitate examination of central constructs such as discourse, bilingualism, and language use.

Building on existing methodologies

Research examining the development of mathematical learning and its relation to language draws on multiple theoretical frameworks to support investigations and accordingly methodological approaches (Moschkovich, 2016). Adopting Sfard's (2012) commognitive approach, data collection and analysis must adhere to its five methodological principles. These principles have been expounded upon to reflect our investigative framework and are 1) Operationality, 2) Completeness, 3) Contextuality, 4) Alternating Perspectives and 5) Directness. First, Operationality refers to the provision of a balanced account of the process through the sharing of practical, unambiguous stories that emerge from the study. Second, Completeness of the research emphasizes that the unit of analysis must comprise the entire discourse related to the topic. The researchers extended this principle for M²EID to include the documentation of such discourses (plausible developmental trajectories) in both the English and Irish languages. Third is Contextuality, which encompasses the premise that all interaction can be characterized as a learning event. We extend this, in the given context, to the need to examine when and how bilingual students/researchers use their language(s) in interactions. The fourth principle is that of Alternating Perspectives and explains the interchangeability of the researcher's insider/outsider methods of using words. This is intensified within a bilingual context because consideration must be given to both languages, their use within the given context as well as the possibility of significant differences between researcher and participant discourses. Fifth, the principle of Directness affirms that all descriptions of the study should commence with the specific raw data from the participants rather than the researcher's interpretation of that data. The application of these distinctive methodological standards will provide unique insights into the processes of bilingual mathematics learning and potentially contribute to the development of an empirical research base to ensure rigor in examining whether differences in languages, and their use, by bilingual mathematical learners have a differential impact upon cognitive mathematical processing.

Further to adopting Sfard's approach, it is vital to consider that epistemological assumptions inform methodology, which subsequently engender the methods employed to collect data. Therefore, aligned with the interdisciplinary foundations of the M²EID research project, the following are the proposed methods to be utilised in the study in order to ensure that a robust methodological framework and approaches support our inquiry.

1. *Discourse models:* This study will map the plausible developmental trajectories in both the English and Irish languages with respect to students' learning in various mathematical topics

–e.g. functions– as consistent with the NUI Galway undergraduate module. The purpose of discourse models is to examine how language nuances and use affect learning (Kim et al., 2012).

2. *Videographic evidence*: This study will identify and explore when and how bilingual learners at NUI Galway employ each language (English and Irish) when engaged in mathematical learning. Specifically, the research will examine the cognitive functions of code switching and language use within a natural educational context, while also providing for the social aspects of learning. Videography is an effective method of examining teaching and learning experiences in naturalistic contexts and the affordances of modern technologies provide opportunity to document, share and analyse cases of particular practice (Derry et al., 2010). All lectures and tutorials relating to the bilingual mathematics module in NUI Galway will be recorded and analysed as appropriate.
3. *Questionnaire*: The purpose of the first part of the questionnaire is to gather participants' background data. The second part of the questionnaire will engage participants in discourses related to particular mathematical topics (linked to the developed discourse models) with the option of utilising English or Irish or both languages. The Cognitive Aspects of Survey Methodology (CASM) model will guide participants in an activity series involving thinking-aloud their thought processes as they recall prior knowledge and experiences of mathematical discourses while answering the questions (Desimone & Carlson Le Floch, 2004). The focus will rest on conceptual mathematical activity based upon a variety of constructs, both familiar (such as functions and their analysis) and new (such as logical form, equivalence relations and classes, and related number theoretic constructs). A primary mathematical objective of the first year module in NUI Galway is to facilitate and develop advanced mathematical thinking.
4. *Video-recorded Cognitive Interviews:* Cognitive interview methods will be employed to explore respondents' explanations of the answers in order to acquire comprehensive knowledge about how well respondents comprehend, appreciate or even misinterpret the specific mathematics concepts central to the study (Desimone & Carlson Le Floch, 2004). Participants will engage in paired discussion of mathematical tasks (the same as in the questionnaire) and justify their answers where appropriate.

It is proposed that the combination of the above methods facilitates a progressive and incorporative investigation into the cognitive aspects of bilingual mathematics learning and to evaluate the impact of languages on mathematics learning in practice.

Recognizing central distinctions while avoiding dichotomies

With Sfard's (2008) commognitive framework undergirding the approach, the following are key aspects of the methodological framework under investigation (Ní Ríordáin & McCluskey, 2015):

- **Discourse changes**: If assuming the premise that mathematical learning involves initiation into the discourses of mathematics, then learning mathematics involves substantive discursive changes for learner. Sfard (2012, p. 3) distinguishes between two types of mathematical learning (change in discourse) as follows: *object-level learning* (expansion of

what is known already and is mainly accumulative) and *meta-level learning* (change of meta-discursive rules and is a more radical and complex change). Within the proposed framework, development refers to a change in discourses. Accordingly, we refer to the development of students' mathematical discourses as opposed to the development of the students themselves.

- **Sociocultural perspectives**: Discourse is more than just language. We utilize Gee's (1996, p. 131) work which refers to Discourse as incorporating both talk and non-talk modes of participation such as gestures and artifacts, as well as participation in a social group. The employment of this definition synchronises with the concepts of discourses inherent within the sociocultural and Community of Practice perspectives.
- **Community of practice**: Within the framework, thinking can be defined as the activity of communicating with oneself. Accordingly, mathematical thinking can be viewed as a discourse, which in turn is a form of communication and involves being part of a mathematical community. Taking this view, the language or languages in which mathematics is being learned becomes an important issue for consideration.
- **Conceptual learning**: Given that language influences thought and thinking (Vygotsky, 1962) and that each language will have its own way of constructing the concept, insight into the role and effect of bilingualism/languages on conceptual mathematical learning is critical. We consider languages and registers as vital resources and skills for learning and language use in mathematics. Grosjean's (1999) concept of a continuum of modes will be employed to trace bilinguals' use of languages in situ.
- **Linguistic relativity hypothesis**: It is the use of language as an instrument of thinking that is of importance, as well as its effect on cognitive processing. The linguistic relativity hypothesis proposes that the vocabulary and phraseology of a particular language influences the perceptions and thinking of speakers of that language (Whorf, 1956). Accordingly, each language (e.g. English or Irish) has a different cognitive system that influences concept formation and development. The study adopts the premise that a language influences, rather than determines, our mathematical thinking, and is cognisant of the impact of linguistic distinctions in a particular discourse on mathematics learning (Kim et al., 2012).
- **Meta-discourses**: The proposed framework is primarily concerned with meta-level developments in mathematical discourses. Since our focus is on bilingual mathematics learners, it is important that an analysis of the language(s) in which the discourse is taking place is conducted. In particular, the successive meta-discourses relating to mathematical topics of interest will be documented and compared between languages.
- **In situ research**: Since the development of discourses is essentially a product of collective human actions, the specific contexts must be acknowledged. Hence, learning and language use in mathematics will be analyzed within the social, cultural and cognitive practices of the particular learning context (Moschkovich, 2012).

Conclusion

Utilising Moschkovich's framework, this paper has provided an appraisal of the methodology and methods to be employed in the M²EID project, which is concerned with examining bilingual mathematics learners in situ. We assume that methodology is inclusive of both theory and methods. Accordingly, it is of importance to outline the underlying theoretical assumptions relating to the M²EID project, as well as how we plan on documenting, describing and explaining these phenomena. Hence, a core consideration for our project is what data to collect and how to collect such data. Therefore, a key aim of the M²EID research project is to evaluate the proposed methodology and methods in practice in order to ascertain their suitability for investigating bilingual mathematical learners in an educational context. In particular, the project will evaluate whether differences in languages, and their use, by bilingual mathematical learners have a differential impact upon cognitive mathematical processing, when engaged in conceptual mathematical activity.

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Entering the mathematical register through evolution of the material milieu for classification of polygons

Frode Rønning and Heidi Strømskag

Norwegian University of Science and Technology, Department of Mathematical Sciences, Trondheim, Norway

frode.ronning@math.ntnu.no heidi.stromskag@math.ntnu.no

This paper is based on classroom observations from Norway with 7-8 years old children working on geometrical shapes. The intention is that the children shall classify different polygons according to their number of edges. The observations are part of a teaching sequence that is designed using principles from Brousseau's Theory of Didactical Situations (TDS). From the teaching sequence we identify certain challenges in the children's development of scientific terms and the observations allow us to conclude that these challenges to some extent are connected to specific semantic features of the Norwegian language. The use of TDS is instrumental in revealing the challenges that occurred and explaining what was changed to overcome them.

Keywords: Register, language of nearness and distance, polygons, adidactical situation, milieu.

Introduction

This paper reports on a teaching sequence within the project Language Use and Development in the Mathematics Classroom (LaUDiM)—an intervention study carried out in collaboration between researchers at the Norwegian University of Science and Technology and two local primary schools in the period 2014-2018. The main objective of the project is to study pupils' development and use of mathematical language in order to gain knowledge that will help the teachers to develop their teaching—aimed at pupils' increased proficiency in expressing mathematical ideas, mathematical reasoning, arguing and justification. Teaching sequences are designed in collaboration between researchers and teachers, where the design is guided by principles from the theory of didactical situations (Brousseau, 1997).

In this paper we study a teaching sequence at one of the project schools including pupils from Grade 2 (7-8 years old) and their teacher. The main aim of the teaching sequence—consisting of three sessions—is that the pupils shall develop their language use about polygons, with the specific aim that they shall be able to classify polygons based on seeing visual images and that they can discern different parts of a polygon (vertices and edges). The classroom observations (recorded on video) give insights into pupils' evoked concept images (Tall & Vinner, 1981) of vertex and edge, and how discrepancy between these and the scientific definitions of them constrains the teacher's goal of the first session. Further, we show how this is resolved by the teacher in the subsequent sessions.

Theoretical framework

According to Halliday a register is "a configuration of meanings that are typically associated with a particular configuration of field, mode and tenor" (1985, pp. 38-39). Halliday compares *register* to *dialect* and states that "dialects are saying the same thing in different ways, whereas registers are saying different things" (Halliday, p. 41). So changing between registers can mean that the same

word gets a different meaning. The mathematical register is characterised by the property that words have very precise meanings and sometimes the same word may be used in the mathematical register and in the register of everyday language but with different meaning. This feature is language specific in the sense that for a given word it can be present in one language but when this word is translated to another language it may lead to one word in the everyday register and another word in the mathematical register.

Koch and Oesterreicher (1985) make a distinction between language being *conceptually oral* or *conceptually written.* They refer to the first category as *a language of nearness*, where the interlocutors are in direct contact and can comment on each other's utterances and directly refer to the given situation, for instance by using gestures. The second category they refer to as *a language of distance*, where the sender and the receiver are not necessarily in contact and the language therefore has to be more precise. The mathematical language is in its nature conceptually written because of the way it strives for precision and unambiguity. However, much of the communication in the mathematics classroom has many of the features characterising a conceptually oral language, a language of nearness. In particular, when working with young children the communication is characterised by dialogue, face-to-face interaction and a desire to avoid complexity, features characterising a language of nearness. However, one of the aims of schooling is to develop the mathematical language into a language characterised by greater precision, compactness, density of information, features characterising a language of distance Koch et al., 1985, p. 23.

The theory of didactical situations in mathematics, TDS (Brousseau, 1997) is a scientific approach to the problems related to teaching and learning of mathematics, where the particularity of the knowledge taught plays a significant role. Its methodology—for a targeted piece of mathematical knowledge—is based on creating a *situation* with a problem to be solved, where the knowledge aimed at is the optimal solution to the given problem. In the following, based on Brousseau (1997), we explain some concepts of TDS that are relevant for our analysis.

An *adidactical situation* is a situation in which the student takes a mathematical problem as his own and tries to solve it without the teacher's guidance and without didactical reasoning (i.e., not trying to interpret the teacher's intention with it). The *milieu* models the elements of the material and intellectual reality on which the students act when solving a problem—these elements are conditions for the students' actions and reasoning. The milieu may comprise: the problem to be solved; material or symbolic tools provided (artefacts, informative texts, data, etc.); students' prior knowledge; other students; and, arrangement of the classroom and rules for operating in the situation (determinative of who is supposed to interact with whom). The milieu of an adidactical situation is called an *adidactical milieu*. An appropriate adidactical milieu provides feedback to the students, whether their responses are adequate with respect to the knowledge at stake.

After *devolution*, a phase where the teacher has (temporarily) transferred responsibility for solving the problem to the students, four situations (or phases) follow: Situations of action, formulation, and validation are (intentionally) adidactical situations, whereas the situation of institutionalisation is a didactical phase. The situation of *action* is where the students engage with the given problem on the basis of its inner logic, without the teacher's intervention. The students construct a representation of the situation that serves as a "model" that guides them in their decisions. This model is an example of relationships between certain objects or rules that they have perceived as relevant in the situation.

The situation of *formulation* is where the students' formulations are useful in order to act indirectly on the (material) milieu—that is, to formulate a strategy enabling somebody else to operate on the milieu. In this situation the teacher's role is to make different formulations "visible" in the classroom. The situation of *validation* is where the students attempt to explain some phenomenon or verify a conjecture. In this situation the teacher's role is to act as a chair of a scientific debate and (ideally) intervene only to structure the debate and try to make the students use more precise mathematical notions. The situation of *institutionalisation* is where the teacher connects the knowledge built by the students—through adidactical interaction with the milieu—to the scholarly and decontextualised forms of knowledge aimed at by the institution.

Methodical approach

Each teaching sequence in the project starts with a planning session where teachers and researchers work together to plan the activities for two classroom sessions, and in particular set the learning goals for the classroom sessions. Activities and actions are planned according to the phases of TDS, devolution, action, formulation, validation, and institutionalisation (Brousseau, 1997). Some days later, the first classroom session takes place, immediately followed by a reflection session, where experiences from the first classroom session are discussed and adjustments are made for the second session, taking place yet a couple of days later. Researchers are also present in the classroom. Observations from all sessions (planning, classroom implementation and reflection) are recorded on video, and additional audio recording is used to secure the quality of the sound. In the classroom sessions selected pupils working in groups (2-3) are video recorded, as is the teacher in whole-class sessions. Tasks given to the pupils in the observed sessions and written material produced by the pupils are also data sources. After completing a cycle of planning, reflection and classroom sessions, teachers and researchers meet to watch parts of the video recordings from the classroom. This represents the first step in analysing data, where interesting sequences from the classroom are identified. In the planning session and the video session, teachers from both schools are present.

This paper is based on a teaching sequence on geometrical shapes, consisting of three classroom sessions[1]. Data from the teaching sequence form the basis for answering the following research question: *What conditions enable or hinder pupils' opportunities to categorise polygons according to their number of edges?*

The utterances reproduced are excerpts from a transcript of the video recorded whole-class discussion in the second session. The camera faces the teacher at the board and it captures the dialogue between the teacher and the 14 pupils who are sitting in a semi-circle close to the board. Parts of the dialogue between teacher and pupils have been transcribed and translated from Norwegian into English. In cases where it is important for the analysis to emphasise the meaning of a particular word in Norwegian, the Norwegian word is included in square brackets in the transcript.

Our analysis is based on ethnomethodological conversation analysis, focusing on the thematic development of an interaction rather than on its structural development (Holstein & Gubrium, 2005). This gives the possibility to analyse the relationship between language and the figures with

[1] The analysed teaching sequence consists of three sessions (instead of two which is common in the project). The third session involves pupils' interaction with a milieu designed so as to give feedback in the devolved adidactical situation.

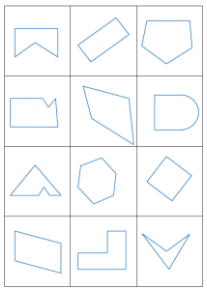
their components while teachers and students negotiate mathematical meaning (Fetzer & Tiedemann, 2015).

The teaching sequence analysed here is chosen because: (1) it illustrates how the phenomenon of words having different meanings in the mathematical and everyday registers constrains pupils' conceptual development; and (2) it illustrates how an evolution of the milieu gives a rationale for *using* the target knowledge.

Analysis of the teaching sequence

First session—classification

In the first session of the teaching sequence, pupils work in pairs on sheets of paper showing 12 shapes, as presented in Figure 1 (one pupil has blue, the other has red figures). The task they get is that each pupil shall (individually) sort the figures into groups (cutting the individual figures from the sheet) and give a name to each group (ACTION). Then they are supposed to compare (in the pairs) how they have sorted the figures and agree on a way to sort them and also agree on a name for each group (FORMULATION). The final result from each group is a sheet of paper on which the pupils have glued on figures from the same group and with the text "These are <__> because <____>" and the pupils have filled in the blanks (VALIDATION). After the session the teacher collects the worksheets and she uses them as background for a whole-class discussion in the second session (INSTITUTIONALISATION).



In Norwegian, polygons are named literally after the number of edges, using the standard Norwegian number words, so that a triangle is called a "three edge" (trekant), a quadrilateral is called a "four edge" (firkant), and similarly for the others. An accepted name for the generic concept *polygon* is 'mangekant' which literally means "many edge". Learning names of polygons, and understanding the reason for the names, is therefore not considered to be a challenge for Norwegian students.

Figure 1: Shapes to be classified

This is in contrast to the situation in English where it is not obvious from the everyday language that for instance a pentagon is a shape with five edges. The teacher has seen from the collected worksheets that all groups have given names to the shapes based on the number of edges and they have written for instance "these are 'five edges' because they have five edges". However, from the discussion in pairs she has observed that even if all the pupils talk about *edges* (kanter), the way they point at the figures indicates that some counts the edges but others count the vertices. The Norwegian language has no precise scientific word for *vertex*, the word which is used is 'hjørne', which (also) means *corner*.

Second session—the meanings of edge and corner

In the institutionalisation phase the teacher asks pupils to come to the board and explain their reasoning. She has observed Oliver and Amelia counting the vertices and Thomas, Daniel and Sophie counting the edges. Among the figures is a quadrilateral with three acute angles and one reflex angle (Figure 2), where we have inserted the letter A for reference in the dialogue. Although Oliver and Amelia have grouped this among the quadrilaterals,

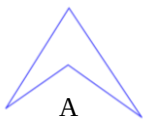


Figure 2: Non-convex quadrilateral

Oliver expresses some doubt when he is called to the board to explain how he and Amelia have thought.

Oliver:	If we had pulled this out a little (pointing to vertex A with the reflex angle) it would have been a “four edge” [firkant].
Teacher:	OK, but still you have grouped this among the “four edges”.
Oliver:	One, two three, four (pointing to the vertices).
Teacher:	So what is an edge?
Oliver:	That is the pointed parts [spissene].

When Thomas is called to the board he uses a rectangle as his example and clearly points to the edges, counting “one-two-three-four”.

Teacher:	What is the difference between what Oliver did and what Thomas did?
Megan:	Thomas counted the lines [strekene] and Oliver counted the pointed parts [spissene].
Teacher:	So actually we did not quite agree on what an edge really is.

The dialogue above reveals that there are different opinions among the pupils as to what the word ‘kant’ means. All the pupils claim that they are counting the edges but when asked to explain what they have counted, Oliver points to the vertices and Thomas points to the edges. Megan makes the observation that the two boys have actually counted different parts of the polygon.

The teacher has in many sessions talked about “what mathematicians do” and that they for instance decide and agree on what names to give to mathematical objects. At this stage the teacher says that now we have to agree on something—as the mathematicians do—so that we have a common understanding of what an edge is. The teacher has also observed that some pupils use the word ‘hjørne’ and she draws their attention to this. One pupil, Jessica, says that they had talked about ‘hjørne’ but they did not know what it was, so they had written ‘kanter’. The teacher asks the pupils to explain what a ‘hjørne’ is and encourages William to come to the board.

William:	That inside is a corner and those outside are edges.
Teacher:	Can you show us?
William:	This is a corner (points to vertex A with the reflex angle in Figure 2) and that is an edge (points to one of the acute angles).
Teacher:	But what about this (points to the rectangle)?
Oliver:	Edge, edge, edge, edge (points to each of the four vertices).

Then the pupils continue to discuss the inner and the outer angle at a vertex, and that one is a ‘hjørne’ and the other is a ‘kant’. Thomas says that “the corners are inside and the edges are outside” and Chloe agrees that the corner is inside but she refers to the outside as the “pointed parts” (spissene). William is making a distinction between the vertex at the reflex angle of the non-convex quadrilateral (Figure 2), which he refers to as a ‘hjørne’, and the vertices at the acute angles, which he denotes by the word ‘kant’. Oliver, using the rectangle as his reference context, refers to all the vertices by the word ‘kant’.



To get the pupils to agree on *one* name for the same object the teacher brings in a reference context from their everyday life, a mini-pitch. The picture in Figure 3 is shown on the whiteboard. Using this picture as the reference context, the teacher asks questions like "If I say that you should place yourself on the edge of the mini-pitch, where would you be standing?" or "…place yourself at the corner, where would you go?"

Figure 3: The mini-pitch

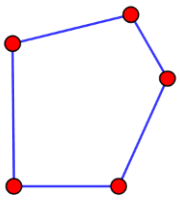
There is still some confusion among the pupils, so the teacher says that she will tell them "what the mathematicians have decided". She holds up a rectangular sheet of paper (A4) and says:

> Teacher: Corner (vertex), that is where two sides meet. When we talk about edge, we can also call this the side-edge [sidekant], and where two edges meet, that is a corner (vertex). There is the corner (points to a vertex of the sheet).

The last part of the session was completed at the mini-pitch in the schoolyard, where they played the game of the teacher telling where to go—by using the concepts of edge and corner—and the pupils went to a place which (supposedly) fulfilled the teacher's command.

The teacher's reference to "work like mathematicians" and that this entails giving precise definitions indicates that she intends to introduce her pupils to the mathematical register. However, the communication is hindered by the fact that some of the words have different meanings in the mathematical register and the everyday register. In particular this is the case for the word 'hjørne' which can mean vertex (mathematical register) as well as corner (everyday register). In everyday language the corner is a spacious area, somewhere you can stand, but in mathematics it is a point, the intersection between two lines. The word 'kant' has the same connotation as English edge, and in everyday language this is used as something that is sharp. This may explain why 'kant' is used to denote both the side, or edge, and the vertex when it is approached from the outside.

Third session—a milieu that affords feedback



As a follow up, a game with 12 tiles was developed. On one side of the tile was depicted a polygon where the edges had one colour and the vertices were marked with another colour. On the reverse side was written "<*name of shape*> with <*colour*> edges" or "<*name of shape*> with <*colour*> vertices". An example is shown in Figure 4. On the back of this was written "Pentagon (femkant) with blue edges".

Figure 4: Pentagon with blue edges

This game was played in pairs of pupils both having the full set of tiles. One pupil reads the text and the other one is supposed to pick the correct shape. After picking he/she can turn the tile and read the text to see if the correct shape has been picked.

Discussion

The target knowledge of the teaching sequence was that the pupils should develop the scientific language for naming 2D shapes and become aware that these names are based on the number of edges in the shape. To know the difference between edges and corners (vertices) will then also be

part of the target knowledge. A condition that hinders pupils' opportunity to categorise polygons according to their number of edges, is the ambiguous use of the concept of edge. Many of the pupils thought that edge ('kant') referred to corner/vertex, and since the number of edges equals the number of vertices it gave meaning to classify polygons according to the number of corners.

The material milieu in Session 1 did not have an adidactical potential for categorisation according to the number of edges, since it was possible to solve the task apparently correct, without the pupils having a common understanding of what is an edge and what is a corner/vertex. There was no feedback from the milieu that could have told them whether they used the desired concept to classify: In action they counted either edges or corners (which gave the same answer); in formulation they compared their categorisations (and if they had a figure that was categorised differently, they used either edges or corners as a basis for categorising jointly and agreeing); in validation, if they reasoned on the basis of different attributes (edges or corners), they concluded that it did not matter which attribute to use.

During Session 1, the teacher realised that the pupils had other connotations of edge and corner/ vertex than the scientific ones. In institutionalisation (Session 2), the teacher let the different connotations be displayed, and—with reference to mathematicians—she introduced the scholarly meaning of the concepts, in the mathematical register. Further, she connected them to the pupils' everyday register, through the mini-pitch context.

Based on results from Sessions 1 and 2, the teacher designed a new material milieu (tiles) that has an adidactical potential (see Figure 4). The game will produce a win if the pupil uses the scholarly meaning of edge and corner, and a loss if not. Hence, the pupil will *need* the target knowledge to act on the milieu—a principle at the core of TDS' instructional design. The evolution of the milieu described here is a condition that *enables* the pupils' opportunity to categorise polygons according to their number of edges.

The teacher's desire to introduce precise mathematical terms also points to introducing a language of distance. However, the situation is such that the pupils are able to express themselves clearly using gestures together with oral language, thereby using a language of nearness. However, in the game with the tiles it is necessary to use a language of distance in order to pick the correct tile. Hence, the intended language development is stimulated by the activity's adidactical potential.

It has been observed earlier that Norwegian children focus on the vertices when naming polygons (Rønning, 2004) but the observations made in this paper show that they may use different words depending on whether they approach the vertex from the inside or from the outside. We have also seen that they may use different words within the same shape, as with the non-convex quadrilateral in Figure 2. This shape is also interesting in the sense that it is not really accepted by Oliver as a quadrilateral but it would have been if "we had pulled this [the vertex with the reflex angle] out a little". We interpret this as Oliver's inclination to distinguish between convex and non-convex polygons. In future learning of geometry, the concept of a convex polygon will be introduced and this example indicates that early exposition to non-convex shapes can be important for making pupils familiar with these shapes.

Distinguishing between edges and vertices can also be seen to be important for future learning. For polygons, the number of edges is equal to the number of vertices so to name a polygon one may just

as well count the number of vertices instead of the number of edges. However, for polyhedra the number of vertices, edges and faces are not the same, and the naming is based on the number of faces.

The results presented here are relevant for mathematics teachers and teacher educators: They present challenges and affordances related to teaching of properties of polygons—with emphasis on language and characteristics of the milieu with which the pupils interact when solving a problem.

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The discourse of a teacher as a means to foster the objectification of a vector quantity: The case of a novice teacher

Ulises Salinas-Hernández[1], Luis Moreno-Armella[2] and Isaias Miranda[3]

[1]Cinvestav, Mexico City, Mexico; asalinas@cinvestav.mx

[2]Cinvestav, Mexico City, Mexico; lmorenoarmella@gmail.com

[3]CICATA; Mexico City, Mexico; imirandav@ipn.mx

This article presents the analysis of the discourse of a novice teacher when he tries to clarify what the sign of a vector quantity is. The elements considered for the analysis are language (speech and gesturing) and reference system concept as mediators within the process of meaning-making. Our analysis shows the novice teacher has difficulties promoting the understanding of the (negative) sign of a vector quantity and its relationship with the convention used to solve problems of motion of objects. The results shown here are part of a wider ongoing research concerning discourse analysis and teaching practice in grade 11 of two teachers with different profiles, expert and novice, from the theoretical approach of semiotics–the theory of objectification.

Keywords: Novice teacher, classroom communication, objectification theory, sign of a vector quantity, semiotic approach.

Introduction

From the discussion on the state of research in mathematics education which arose during the 10th *International Congress on Mathematical Education* (ICME-10), and as different works indicate (e.g., da Ponte & Chapman, 2006; Adler, 2000), there is growing attention to the teaching practices compared to what occurred in the past, when such practice was not a primary concern. Sfard (2005) states that it was particularly during the first decade of the 21st century when there was a decisive change towards the study of teachers' practices. From the works presented in the work group of mathematics and language in the past CERME9, and related with the interests of this research, it is of great importance to point out the interest on communication and interaction in the mathematics teaching and learning processes. In these processes, the use of gestures is included both in a communicative role and as a resource during teaching practice. Thus, we must highlight the works by Nachlieli and Tabach (2015), who show the elements of teaching practice that promote learning, and the work by Farsani (2015) on the role that deictic gestures play as a communicative tool. There are the works regarding the way in which meanings are produced in the classroom and the role of gestures as mediators of such process (e.g. Miranda, Radford & Guzmán, 2013). In this way we seek to contribute to the discussion on the relevance of interaction and communication on teaching and learning processes from an analysis perspective of semiotic orientation.

Research problem

Da Ponte and Chapman (2006) agree that, in the 90s, Vygotsky's work came to prominence and evolved in a number of research lines with respect to teaching practices. Considering Vygotsky's concept of semiotic mediation –defined as usage of means [artifacts and signs] by which the individual receives the action of social, cultural and historical factors, and acts upon them (Vygotsky, 2009)–, Mariotti (2009) conducted a research with the aim of observing the teacher's role and the use

the teacher gives to artifacts in order to develop mathematical signs in the students during the teaching-learning processes. Mariotti considers that the teacher, playing the role of cultural mediator, is responsible for introducing specific terms and using his or her judgment to recognize what may be referred to as mathematical concepts. Morgan (2006) considers that: "An important starting point for a social semiotic perspective is the recognition that meaning making occurs in social context and language use is functional within those context." (p. 220). In the same work, the author emphasizes the multimodal characteristic of communication in which, besides language, gestures and the use of other resources are found. In this sense, Arzarello, Paola, Robutti and Sabena (2009) highlight the dynamic process that takes place during the multimodal semiotic activity of the subjects.

Then, the research firstly goes back to the interest on teacher's practice (a novice teacher data are reported here), and additionally it considers the use of a sociocultural approach of semiotic orientation to observe teachers' practices. Our objective is to analyze the teacher's discourse at the moment when he talks about the sign of a vector quantity in a physics class, in which the use of language and gestures is essential during the process of meaning making and awareness. Therefore, we pay close attention to the semiotic means of objectification [language, gestures and signs] that the teacher uses and encourages in the interaction with the students.

Conceptual framework

The research is supported by the theory of objectification (Radford, 2014; 2008) that includes Vygotsky's notion of semiotic mediation as well as the importance of the use of artifacts and gestures in the processes of knowledge production. Radford (2014) considers that the main objective of the theory of objectification (TO) is that of mathematics education as: "[A]s a political, social, historic and cultural effort with the aim of creating ethical and reflective individuals who take a critical position in mathematics practices historically and culturally constituted." (Radford, 2014, p. 135–136, free translation). Thus, this formation of the individual involves an analysis between being and knowing in which both of them are closely interrelated. The principle of *labor* or activity represents the fundamental principle of the TO (Radford, 2014). It is through *labor* that the individuals are developed and continuously transformed and that we find the Other and the world in its conceptual and material dimensions. Through labor we find the systems of ideas of culture (systems of scientific, legal, and artistic ideas, etc.) and cultural forms of being as well. Radford and Roth (2011) introduce the concept of joint action, which implies more than a spatial notion where the interaction takes place. It represents the place in which the students and the teacher think and act together in pursuit of a common goal. It is important to emphasize that from TO approach, what mediates is the activity. Where both students and teacher are immersed. However, artifacts and signs continue play a relevant role. They are also part of the activity and are defined as semiotic means of objectification; which are: "These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making process to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (Radford, 2003, p. 41).

This approach also revisits the notion of consciousness as something concrete; it is a subjective reflection of the world. Then, any consideration regarding learning must also comprehend the field of consciousness in which the students' thought and emotional orientations are included. Consciousness can be captured through its manifestations: discourse, gestures and all the other sensual actions. In order to recognize the forms of expression, action, and reflection, that are the

mathematical objects, the student goes through a social and physical process of awareness, which is mediated, in turn, by the activity; and where the artifacts y signs both physical and psychological belong to this activity (Radford, 2014; 2008). Therefore, gestures and artifacts act as important elements of the activity and are essential to the reflection processes. In this way, in TO, knowing and individuals are produced in the classroom through labor or activity. One way of identifying how meanings of mathematical objects are produced is through language and gestures.

Within the aim of this work, we include the teacher's practice to seek to characterize how the novice teacher promotes the objectification of the sign of vector quantities. In other research works, the role of gestures and the character of artifacts and signs as mediators has been developed; Roth (2000) particularly points out the importance the use of gestures has in the relationship with speech and in the road towards the scientific language. Roth indicates that, in the absence of an appropriate scientific discourse, gestures help to explain and describe the phenomena among the students. Additionally, he stresses that, during the emergence of the [scientific] discourse, both the iconic and the deictic gestures precede the spoken words associated with them. For their part, Moreno-Armella and Sriraman (2010), consider that the access to [mathematical] objects is not direct, but through mediation. The way in which we interact with our environment and the rest of the people—for instance, through language—is part of our symbolic nature. They state that: "Only humans possess, (…) what can be termed *explicit cognition* that allows us to go from learning to knowledge. Explicit cognition is symbolic cognition. The symbol refers to something that, although arbitrary, *is shared and agreed by a community*." (Moreno-Armella & Sriraman, 2010, p. 216).

Method

This is a qualitative research performed through a case study. The pilot study was carried out in a high-school (grade 11) from Mexico City. The participants were two teachers (expert and novice) who teach physics and who have over 20 years and 2 years of experience, respectively. This article reports the data collected from the novice teacher. The instrument to collect the data was non-participant observation of the *Physics I* classes. In the classes, the teacher addressed mechanics topics, specifically, Newtonian dynamics. The teacher considered the concepts of force, displacement, and interpretation of Cartesian graphs. The classes lasted two hours (twice per week) and one hour (once per week). We observed 12 sessions and obtained 20 hours of recording. We used two cameras controlled by the researcher. One camera remained fixed and was directed to the board while the other was moved to focus on the interactions during the students' participations. In addition, we used a voice recorder placed on the teacher to obtain audio recordings of the classes. After the data were collected, we watched the videos from the classes to identify moments when key concepts had been addressed. Once the moments (class segments-excerpts) were identified, we transcribed what occurred in those segments. Our analysis is based on those transcriptions.

Analysis and discussion of results

Below we present excerpts that show the discourse of a novice teacher who tries to clarify the purpose of using the sign in a vector quantity on a free fall problem. In its entirety, the teacher's discourse lasts around 10 minutes. To carry out the analysis, we identified three main excerpts that deal with the teacher's discourse regarding the concept of a vector quantity. The excerpt starts after a student [who does not take part in the dialog] goes to the board to write a response and uses the value of

acceleration of gravity ("*g*") with a positive sign (see Figure 1-Photo 2). It is two students (S1 and S2) have a question about the sign that the teacher's explanation starts. The excerpts correspond to a class in Spanish, in such a way that a translation in English is presented, trying to maintain dialogues fidelity.

Excerpt 1- Is gravity negative?

S1: Teacher, is gravity negative?

Teacher: It is negative.

S2: Is it?

Teacher: Gravity will always be negative, right? But in this case (…) I'd told you that acceleration was a vector, right? Then, for example, if you want to speak in, let's say, a vector manner, you must express gravity with its negative. Because it will always point down [*makes a gesture; see Figure 1-Photo 1*], right? But in this case, if you place it like this, in a scalar manner (…) we're only looking at the magnitude of the gravity. Which would be 9.8. I mean, gravity will always go down [*A student says: "but not now"*] on the axis and down. Let's leave it at that for now [*with the positive sign*].

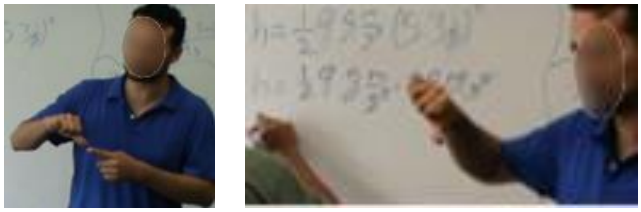

Figure 1: Photos of gestures used by the teacher to represent the sign of gravity in two moments (Photo 1-left; Photo 2-right).

The intention of the teacher is that the students understand the sign of *g*; that is to say, the students have to be aware of the meaning of the sign of gravity. The teacher seeks to encourage this awareness through a speech in which he includes gestures. However, from this point it is evident there is no articulation between the teacher's verbal arguments and his gestures. The teacher stressed that *g* "will always be negative." However, the argument the teacher uses gesturally links *g* with the type of motion (free fall) and not with the mathematical relationships of the function of motion (position with respect to time). When he says "Because it will always point down", the teacher does not explain that "down" —or "up", given the case—depends on a frame of reference involving a starting point (origin) from which measurements and directions (orientations) are taken to obtain numerical values. That is, the set of conventions used is arbitrary.

Excerpt 2- The system of reference

S3: And if I did it using the minus nine point eight? [*referring to* $g = -9.8 m/s^2$].

Teacher: If you did it with the minus, that means that, what does it mean? That when you were talking about this problem… [*He is interrupted by another student*].

S1: But you said that it was if it was falling, then…

Teacher: I'm telling you "*g*" will always be negative, right? [*See Figure 2-Photo 1*] Now, you will take a point of reference (…) [*the teacher draws a system of coordinated axes; see Figure 2-Photo 2*]. If you take a point of reference here. Here, it would be *y* [*vertical*], *x* [*horizontal*], right? Then, if you take the point of reference there, what is the value of this point? [*pointing at the origin of the Cartesian system he drew*] It is the origin, what is its value? [*S1 answers: "zero, comma zero"*] Right now, we are only acting on *y*, then the value will always be zero at *x*. Then, if this [*the stone*] is falling towards here [*simulates the fall of the object with respect to the diagram; see Figure 2-Photo 3*], that is why we have a negative value in *y*. Because *y* that goes down is negative. (…) Because the point of reference, we are up here [*points at the origin of the Cartesian system*] and we are measuring how the little ball falls down, but from my point of reference [*makes a gesture using both hands; see Figure 2-Photo 4*]. Which would be from the bridge. I won't be measuring this in the water, right? Then, that's why it is negative in this case [*the distance (height)*] and that's why I'm telling you that this [*acceleration of gravity*] is negative.

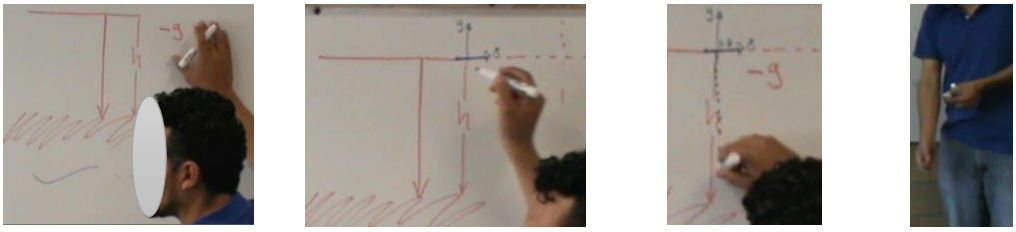

Figure 2: From left to right, photos of gestures used by the teacher to represent: the phenomenon (Photo 1), the system of reference (Photo 2), the motion of the object with respect to the system of reference (Photo 3), and the measurement of the distance (Photo 4).

S1 goes back to the notion that the sign and the values obtained depend on the direction of the motion observed. Later, the teacher incorporates a conceptual resource he considers necessary to understand the sign of *g*, that is, the concept of system of reference. It is observed that the teacher determines it [system of reference] from the system of coordinated axes (Cartesian graph) and its usual directions (positive: up and to the right; and negative: down and to the left). At this point of his speech, the teacher mainly uses the mathematical concept of system of reference. His use of the language makes him focus on conveying the mathematical meaning of the problem while he uses gestures only when addressing the physics phenomenon. What is observed is that it considers mathematical thinking and physics separately. For instance, the teacher seems to use gestures to exemplify frames of reference oriented negatively down only. The meaning of his gestures only depends on the particular motion of the object (Photos 3 and 4). Thus, the teacher is explicit when he says: "That is why we have a negative value in *y*. Because *y* that goes down is negative." Therefore, with respect to the language used by the teacher in this excerpt, it is unclear how students can be aware of the sign of *g* from the use of reference systems when the teacher includes the system of reference in his speech.

Excerpt 3- Two signs for the same problem

S3: But I still don't understand the thing about gravity.

Teacher: I'm telling you that, in this case, the acceleration is a vector. And if the acceleration is a vector, the acceleration of the gravity will also be a vector, ok? Then, this here [*points out at the sign in Figure 2- Photo 1*], the negative of the gravity is indicating where gravity is always directing to. Then, it would be something like this [*draws an arrow pointing down on the board*]. It will always be directed downwards. Now, this will always be [*writes: "g = –9.8m/s²*], this will never change. Now, if you do not want to express this to me [*referring to the acceleration of gravity*], then give me the scalar, I mean, give me the magnitude of your gravity. Then, if you give me the magnitude, it would only be this here [*see Figure 3-Photo 1*], yes? I mean, without the negative, 9.8. If you tell me where it is headed to, you're giving me the direction [*see Figure 3-Photo 2*], which is downwards, really. And in that same way, to get the distance covered. If you tell me, are you going to say it in distance? Or are you going to say it in displacement? Displacement is supposed to be a vector, too. (…) Then, when you get the magnitude, it will always be a magnitude like this [*covers the negative sign of the acceleration of gravity again*], positive.

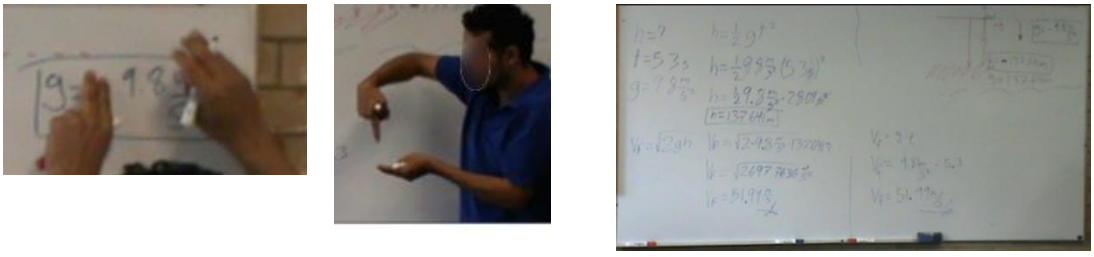

Figure 3: Photos of gestures used by the teacher to represent: the magnitude of a vector (Photo1-left) and the direction of a vector (Photo2-center); additionally, a photo of the board (Photo 3-right)

What S3 says at the beginning of the excerpt indicates that, so far, the "relativity of the sign" has not been understood and that it depends on the frame of reference used to analyze the physics phenomenon. The teacher goes on with the discourse, explaining that knowing the sign of a quantity means knowing the direction of the motion, and says: "the negative of the gravity is telling [us] where gravity is always directed to." However, gravity does not go "upwards" or "downwards", but to the core of Earth, which to our perception is "falling down". The difficulties arise when trying to explain why. Then, the teacher focuses his attention on the magnitude of a vector (see Figure 3-Photo 1). Again, using a gesture, he hides the negative sign of *g* to refer to a probable positive value, yet he relates such value to a scalar quantity and not to the direction of the vector in a system of reference. The teacher implies that one can make reference to the two signs in one quantity in the same problem, which results in an ambiguity to the student.

In teacher discourse, it is important to realize the use he makes of the board, it is noteworthy saying the teacher only writes numbers, symbols (Cartesian graph, vectors) and formulas, but fails to write a single word; and that creates a gap between his discourse (spoken language and gestures) and symbolic language. The students are used to writing down what information is on the board, without adding elements from the spoken language. Therefore, when they go back to check their notes, they can hardly remember the exact words the teacher used, instead, they only see abstract symbols. Thus, reconstructing both the discourse and the discussion that unfolded can be difficult for them. Thus, it would be convenient to carry out research aimed at an analysis of the use of resources by the teacher.

Conclusions

In this work, we observe the roles language and gestures play in a novice teacher's discourse and the difficulties he faced when trying to stabilize awareness on the meaning of the sign of *g*. Then, the way in which the meanings of the mathematical concepts are displayed and understood involves the mobilization of gestures and signs. This is because gestures, artifacts, signs, and the process of meaning making in the classroom, have a semiotic nature. We observed the complexity and the importance of articulating language and other semiotic resources as gestures and concepts—system of reference—in the processes of meaning making. Particularly, we observed there was no satisfactory coordination between the teacher's gestures and language. While the teacher consistently used gestures to point out the negative sign of the gravity when the object "falls down", he was vague when trying to explain why the sign was negative from its vector character. Then, the teacher focused his language only on the mathematical characteristic of the problem, but he focused his gestures on the physics description of the problem. We observed, however, an attempt to coordinate language and gestures in the excerpt in which he includes the use of the concept of system of reference. It follows that determining the system of reference to solve a given problem in advance is essential. Therefore, the system of reference used as a semiotic resource may allow this articulation between the mathematical meaning and the physics motion of objects to be understood. And this motivates us to conduct further research on this line.

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Effects of linguistic variations of word problems on the achievement in high stakes tests

Sabine Schlager, Jana Kaulvers and Andreas Büchter

University of Duisburg-Essen, Germany

sabine.schlager@uni-due.de; jana.kaulvers@uni-due.de; andreas.buechter@uni-due.de

In Germany, as in other countries, language proficiency impacts the achievement in mathematics. Linguistic features may constitute possible obstacles for students solving word problems. This study explores the interplay of language proficiency and achievement in mathematics with tasks in which linguistic characteristics were varied experimentally. 640 10th grade students solved the tasks. An analysis of the shift in difficulty of linguistically varied tasks indicates an irregular scheme: Only tasks with extreme variation of linguistic aspects, for example many nominalizations, show a significant shift. Data collected from student interviews using the think-aloud method show that linguistic aspects can be obstacles and therefore have an impact on the solution process, even if these linguistic aspects may not have an impact on the results in the quantitative part of the study.

Keywords: Language proficiency, difficulty level, item analysis, language obstacles, mathematics achievement.

Language and mathematics

Impact of language proficiency on mathematics achievement

Research on the relationship between language and mathematics has a long tradition in Germany and internationally (Barwell et al., 2016; Maier & Schweiger, 1999). In Germany, research on differences in achievement in mathematics due to background factors is relatively new when compared to research in Anglophone countries. Internationally, the strong influence of language proficiency on mathematics achievement, especially for word-problems and tests that follow the literacy approach, has been an issue for decades (Abedi, 2006). International surveys, such as PISA, show that there is a strong connection between mathematics achievement and family background in Germany, especially regarding the socio-economic status and the first language (Gebhardt, Rauch, Mang, Sälzer, & Stanat, 2013). In a recent German study, Prediger and colleagues have identified that "language proficiency is the background factor with the strongest connection to mathematics achievement, among all social and linguistic background factors" (Prediger, Wilhelm, Büchter, Gürsoy, & Benholz, 2015, p. 77; see also Prediger, Wilhelm, Büchter, Gürsoy, & Benholz, 2013).

Linguistic aspects causing difficulty in mathematics

For English, Abedi and Lord (2001) showed that word problems with low linguistic complexity can reduce the achievement gap evoked by differences in students' language proficiency. However, Martiniello (2008) reports that research about effects of linguistic features is inconsistent. For example, items with many polysemous words, pronouns or prepositions are difficult only in some grades and the number of subordinate clauses and passive-voice sentences had no significant effects. In her own study, she highlights complex sentence structures with embedded adverbial and relative clauses, long noun phrases and limited syntactic transparency on the syntactic level and

unknown or polysemous words on a vocabulary level as linguistic features creating difficulties (Martiniello, 2008).

For the German language context, there are only a few studies about linguistic obstacles. For example, noun phrases and the number of academic language words seem to be difficult. The identified obstacles of word problems in the study by Prediger et al. (2015) included prepositions, complex syntax and nominalizations. The obstacles did not only occur in the context of reading, but more often they were related to conceptual understanding. In this study the specific linguistic features which made the text complicated could not be isolated as they interacted with each other as well as with the mathematical content. With the experimental approach in our study, we strived to address this issue. For the specific context of the German language, we wanted to specify to what extent linguistic aspects could explain the achievement gap between students of higher and lower language proficiency by investigating the questions:

Q1. What impact do different linguistic aspects have on the achievement in literacy-based tests?

Q2. Which linguistic aspects are difficult for (lower language proficient) students?

Research design and methods for the mixed-methods study

Context of the study

The mathematical tasks of the study were oriented towards the literacy-based high stakes exam "Zentrale Prüfungen am Ende der Klasse 10 (ZP10)" [central examinations at the end of 10th grade] of North-Rhine-Westphalia (NRW), the most populous federal state of Germany. The ZP10-exam is designed by QUA-LiS NRW, the "Qualitäts- und UnterstützungsAgentur – Landesinstitut für Schule NRW" [Quality and support agency – Institute for school NRW] that supports the Ministry of Education and lifelong learning in NRW. The exam does not only assess mathematical competencies acquired in 10th grade, but also the competencies acquired during grade 5 to 10. For that reason, our tasks are required to illustrate different mathematical levels.

Design for the quantitative part of the study

The sample consists of 640 students from ten different comprehensive schools in the metropolitan area of Rhine-Ruhr. As independent variables, we collected information about students' language proficiency, cognitive capabilities and family background (cf. Table 1). Another important variable was the linguistic variation in the tasks. The dependent variable was mathematics achievement.

Language proficiency was measured using a standardized C-Test. Cognitive capabilities were measured by the extended standardized German adaption CFT-20R of culture fair intelligence tests. Information about family background (socio-economic status, immigrant status, languages spoken at home) was collected by questionnaire.

The mathematics test was designed referring to the ZP10-tests. Members of QUA-LiS NRW assured that our test would be acceptable for the high-stakes exam. The 90 minutes long test contained the typical topics of ZP10 – functions, descriptive statistics and percentage calculation – which appear in the exam every year. These tasks were also chosen since the study of Prediger et al. (2015) revealed many language obstacles in tasks about these topics. These observations are supported by results of Martiniello (2008) who identified data analysis, statistics and probability as topics that were difficult for English learners with Spanish as native language in the US.

The test was presented in three versions (A, B and C) with six identical anchor tasks (21 items) and six tasks with linguistic variation (13 items). The tasks with variation always existed in three different versions; the context and mathematical content remained unchanged at all times. In the initial version, we avoided difficult linguistic structures in the tasks, especially those structures that were varied in the other versions. Besides this initial version, the tasks were varied using two of the four linguistic features context vocabulary, verbs followed by prepositions or separable verbs, nominalization and density of the text, and reference structure, which were chosen out of possible linguistic obstacles in word problems reported in Prediger et al.'s study (2015).

On the word level, we chose the criterion "context vocabulary", which often causes problems but – in contrast to technical terms – can be changed without having influence on the mathematical content. Concerning the syntactical level, interviews in the study of Prediger et al. (2015) confirmed the linguistic supposition that it is difficult for students to connect information given in sentences with separated separable verbs or between a verb and its preposition. Since Uesseler, Runge and Redder (2013) also state that separable verbs influence the understanding negatively, we selected these two features as a variation-category. As mathematical texts are often very dense and their length and density play an important role in student's understanding, the feature "density of the text" was chosen. This goes hand in hand with the feature "nominalization", because a dense text implies (in German) the use of many nominalizations which have a negative effect on the understanding (Uesseler et al., 2013). Therefore, these two features form one variation-category on the text-level. As mathematical texts try to avoid repetitions, the reference structure is often unclear. This led to consideration of "reference structure" as another category on the text-level. An example of a varied task will be presented later.

The groups of students sitting for the tests A, B or C were systematically formed based on the results from the C-Test and the CFT-20R. The variations were dispersed equally to the three versions of the test. There were no statistically significant differences between the groups concerning language proficiency, cognitive capabilities and mathematics achievement.

For data analysis, different statistical analysis procedures were applied. We split the group in half depending on students' results on the C-Test into groups of students with lower (C0) and higher (C1) language proficiency. The mathematical items were scaled using a Rasch model. An analysis of variance (univariate ANOVA) and a regression analysis were used to identify the background factors with the highest impact on mathematics achievement and to determine the explained variance. The shift in difficulty from the initial version of a word problem to its linguistic variation was determined by an analysis of the change of the WLE for these items on the Rasch scale.

Design for the qualitative part of the study

The purpose of this part of the study was to gain a deeper understanding of the quantitative results through a qualitative approach. For this reason, four tasks were presented to N=32 students with different levels of language proficiency from different comprehensive schools. The students were required to solve the designed tasks independently using the think-aloud method. Once the students solved the task, there was a discussion about the task between interviewer and student. All processes and discussions were videotaped, transcribed, and analyzed interpretatively with respect to whether the linguistic variations created difficulties for the students.

Examples of linguistic variation of tasks

An example of the linguistic variation of a particular task is represented in the translated initial version (1) and its variation "nominalization/dense structure" (2) of the task "Bathtub".

> (1) A bathtub has one cold water tap and one hot water tap. The bathtub can be filled with 135 liters of water. If both water taps are open, it takes 9 minutes until the bathtub is filled completely. If only the cold water tap is open, it takes 7.5 minutes more than with both water taps open. How much water runs out of the opened cold water tap per minute? Note your calculations.
>
> (2) A bathtub **with one cold and one hot** water tap can be filled with 135 liters of water. **Opening** both water taps, the **filling** of the bathtub takes 9 minutes; **exclusively opening the** cold water tap, it takes 7.5 minutes more than **by opening** both water taps. Which **amount of water** runs out of the opened cold water tap per minute? Note your calculations.

The bolded words or phrases show the variations in the task for this article. The translation of the tasks can only give an idea of the linguistic aspects that have been changed during variation, because most characteristics are inherent to the German language and sentence structure. For example, if you try to nominalize "the cold water tap is open [der Kaltwasserhahn ist geöffnet]" into "**opening the** cold water tap [Öffnung des Kaltwasserhahns]", in German, it implies the genitive case of "the cold water tap" tagged by an additive "s" at the end of the word ["Kaltwasserhahn**s**"].

The variation of the items was not possible in every case, because of the structure of the German language. Sometimes it was impossible to use alternative formulations using the defined categories and employing them at positions in the text that are significant for the mathematical solution. The variations were limited due to the common use of language, the context and the linguistic realization of mathematical concepts. In addition, we had to accept the fact that variation of isolated linguistic aspects is almost impossible. In the German version of "If only the cold water tap is open [Wenn nur der Kaltwasserhahn geöffnet ist]", we find the participle II of the verb "open [geöffnet]" (in the English sentence it has the function as an adjective) because of the use of passive voice "is open [ist geöffnet]". This has to be nominalized. In the nominalized version, it is linguistically not possible to continue to use the familiar adverb "only [nur]" but necessary to use the less frequently used adjective "exclusively [ausschließlich]", which imposes another possible lexical obstacle.

Selected results

Impact of background factors

To check if linguistic features can explain the achievement gap due to different levels of language proficiency, we first analyzed the impact of background factors in our study. The results provide additional support for evidence of the impact of language proficiency on mathematics achievement. Students with higher intelligence or higher language proficiency, without immigrant status, and students who only speak German at home had statistically significant better results in the mathematics test (see Table 1, the number of the test persons varies because not all students gave the questioned information).

Regression analysis shows that language proficiency and cognitive capabilities as isolated background factors have the highest impact on mathematics achievement. They both explain about

15 % of the variance. The other background factors explain much less: Univariate variance analysis showed only low explanation potential of immigrant status (12 %), languages spoken at home (8 %) and socio-economic status (1 %). As language proficiency has such a high impact in our study, we investigated if it is possible to explain the achievement gap by the introduced linguistic obstacles.

Background factor	**Specification of groups**	**Distribution of groups**	**Mean score (WLE), m(SD)**	**Significant differences**
Students	10 schools	n=640	-1.37 (1,16)	-
Version of mathematics test	version A version B version C	219 (34.2 %) 214 (33.4 %) 207 (32.3 %)	-1.28 (1,16) -1.49 (1,14) -1.34 (1,16)	-
Socioeconomic status (SES) (n=626)	low SES medium SES high SES	187 (29.9%) 174 (27.8%) 265 (42.3%)	-1.52 (1,12) -1.34 (1,17) -1.24 (1,16)	high&low: 0.046
Cognitive capabilities (CFT-20R) (n=577)	lower scores higher scores	289 (50.1%) 288 (49.9%)	-1.68 (1,13) -1.00 (1,09)	< 0.001
Immigrant status (n=568)	1st generation 2nd generation 3rd generation no	42 (7.4%) 287(50.5%) 56 (9.9%) 183 (32.2%)	-1.64 (1,12) -1.72 (1,07) -1.17 (1,09) -0.83 (1,09)	2nd& 3rd: 0.007 no & 1st: <0.001 no & 2nd: <0.001
Languages spoken at home (n=616)	1: German + x 2: no German 3: only German	269 (43.7%) 106 (17.2%) 241 (39.1%)	-1.58 (1,17) -1.75 (0,95) -0.94 (1,08)	2 & 3: < 0.001 3 & 1: < 0.001
Language proficiency (C-test, n=578)	low proficient high proficient *all C-tests*	289 (50.0 %) 289 (50.0 %) *578 (100 %)*	-1.76 (1,08) -0.94 (1,11) *-1.35 (1,17)*	< 0.001

Table 1: Distribution and differences among groups

Shifts in difficulty due to linguistic variation

All results for the different variations have been scaled in a Rasch model. Shifts in difficulty due to the linguistic variation were identified especially in Item 6 “Bathtub”. In the initial version, 69 % of the C1-students and 48 % of the C0-students solved the item correctly. This shows that the initial version was easy for C1-students and common for C0-students. In the nominalized version with dense structure, half of the C1-students solved the item correctly compared to only 37 % of the C0-students. The varied version was common for C1-students and difficult for C0-students.

Figure 1 depicts the shift in difficulty for the variations in comparison to the initial version. In comparison to the initial version, the denser text containing nominalizations became 0.6 WLE more difficult on the Rasch scale, comparable to three out of 34 correctly solved items in our study. Other items of this variation had no significant shift in difficulty (see Figure 1, left side). Students who only did tasks formulated like Item 2b would solve one item less correctly (0.2 WLE = 1 item); students who only had tasks formulated like Item 2a or 8a would solve the test basically the same. One explanation for the increasing difficulty of “Bathtub” is that this item is linguistically more difficult than others with the same linguistic variation, as the Flesch-Reading-Ease index shows.

The variation “reference structure” equally evoked a shift of 0.6 WLE for Item 6 “Bathtub”. Other items with variation showed inconsistent shifts. Some items became more difficult, other items less

difficult. A possible explanation is the higher relevance of mathematical content in contrast to language in particular items. For example, language has less effect in Item 8a, "What is the median?". In this case it is more important that the concept of median has been taught in mathematics classroom. Similar irregular shifts are visible for the other variations.

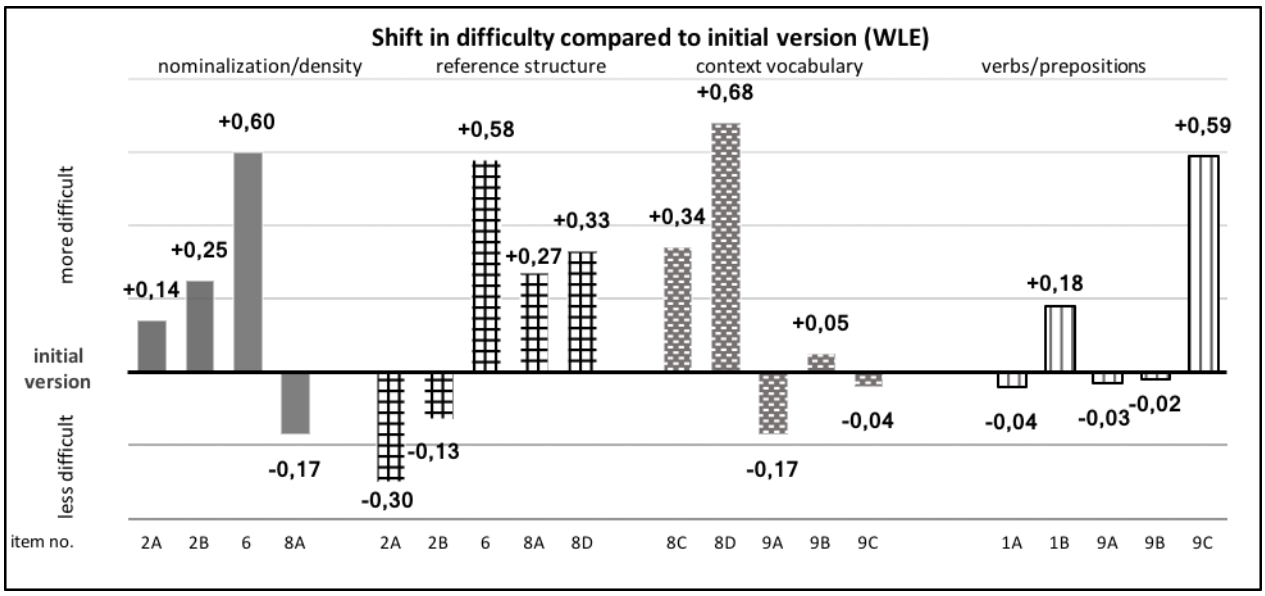

Figure 1: Shift in difficulty compared to initial version (in WLE)

In summary, linguistic variations make word problems more difficult, when applied often and when the general difficulty for understanding the word problem (cf. Flesch-index) increases greatly because of the application of these variations. In these cases, we talk about "extreme variations". Slight variations with only few modifications, which do not increase the difficulty for understanding, do not have statistically significant shifts. As the effect fluctuates substantially between different items, we can presume that the topic of a task is very important.

Linguistic variations as source for mathematical difficulty

The fact that the difficulty does not increase significantly for all items (with linguistic variation) does not imply that linguistic variation does not evoke obstacles for students, as our qualitative analysis highlights. Furthermore, we have observed that linguistic aspects do not create difficulty on their own, but in combination with other characteristics of the task as illustrated below.

In addition to the quantitative data, 16 tenth grade students solved the item "Bathtub" in the version "nominalization/dense structure" during the interviews. In particular, students with lower language proficiency had difficulties in understanding the word "exclusively [ausschließlich]", which had to be used due to the nominalization. While reading the text aloud, several students were puzzled by this word and explained they would never use it. This problem has already caused wrong solutions in the tests. One student wrote: "I do not understand the question. Does 'exclusively opening the cold water tap' mean that only the cold water tap is used?" He knew the correct meaning of "exclusively" but he was not sure about it. Probably, this was the reason he did not solve this item. The nominalization "Opening [Öffnung]" of "to open [öffnen]" and the following genitive case of "the cold water tap [der Kaltwasserhahn]" also caused understanding problems, as the genitive case of "cold water tap [Kaltwasserhahn]" finishes with an 's' "[Kaltwasserhahns]" which in German is sometimes also a cue for plural of a word. In this case, the plural would be [Kaltwasserhähne].

Student: I was confused by the word “Kaltwasserhahns”. Does it mean two water taps or only one?

Apart from the linguistic variation, we identified other difficulties in our interviews. In particular, students with lower language proficiency had problems in understanding the relation “more than” and made wrong calculations. One student thinking aloud said (after three minutes of task solving):

Student: Ah, now I made a wrong calculation […] I thought that … the filling takes 7.5 minutes, but it takes 7.5 minutes MORE. I have to do a new calculation.

In the final discussion, many students could identify the relevant information “more than” and “exclusively opening the cold water tap” on their own or after a question from the interviewer. This shows that, even if there are difficulties in understanding the text, most of the students could solve the task if they had enough time and could verbalize their thoughts aloud. In contrast, in the tests, students often stopped solving a task because of uncertainty concerning their understanding.

Conclusion and consequences

Our outcome concerning Q1 is that linguistic variations had significant impact when the task varied in an extreme way. The fact that in German isolated variations have no overall significant impact evokes the hypothesis that they create difficulty in combination with each other or with other characteristics of the tasks. This hypothesis is supported by the qualitative results showing that not only the linguistic feature itself, such as nominalizations, but also the linguistic structures evoked by this feature, such as genitive cases, create difficulty. However, this has to be analyzed further.

In the context of Q2, in German, nominalization/dense structure, academic words such as “exclusively [ausschließlich]” or genitive cases are obstacles for several students, especially those of lower language proficiency. This was, for example, prominent in the results of lower language proficient students for the item “Bathtub”. Students struggled in understanding these structures, which sometimes lead to not solving the task. When students had a lot of time and could speak out their thoughts, they reflected on these difficulties.

Consequences of this study for test construction are that high stakes tests should try to avoid extremely difficult linguistic structures, e.g. due to the frequent use of nominalizations, to reduce the achievement gap. Our study also showed that linguistic difficulties cannot always be avoided because there are constraints in the language. A consequence for mathematics classrooms that has emerged from students’ ability to overcome linguistic difficulties by talking about them in our interviews, would be to explicitly address language issues and not to avoid linguistic difficulties.

The fact that our results pertaining to the effects of linguistic variations cannot explain the achievement gap, entails further and deeper research on the interplay between language proficiency and mathematics achievement in order to determine the nature of this correlation. A hypothesis that came up in our study and will be investigated is that the selection of strategies by students and their processes of solving tasks differ according to their language proficiency.

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Multilingual learners' opportunities for productive engagement in a bilingual German-Turkish teaching intervention on fractions

Alexander Schüler-Meyer

TU Dortmund University, Germany; Alexander.Schueler-Meyer@math.tu-dortmund.de

A teaching intervention should provide learners with opportunities for productive engagement with mathematics. Teaching learning situations are organized in patterned ways along culturally shared expectations of how these situations unfold. However, in multilingual classrooms, expectations of language learning can compete with mathematical expectations. Drawing on positioning theory, I will reconstruct these expectations – storylines – in a bilingual Turkish-German teaching intervention, specifically for the case of two learners, Akasya and Ilknur. I will show how the teacher establishes two expectations, one of "deciding who has downloaded more gigabyte" and one of "giving a Turkish explanation". After some effort on forming an explanation, Ilknur's and Akasya's shift their activities towards fulfilling the second expectation, which results in them being less agentic for the mathematics at hand. Instead, Ilknur and Akasya focus on the form of their explanation.

Keywords: Mathematics education, multilingualism, positioning theory, agency.

Introduction

As learners construct mathematical knowledge by participating in social practices, they need to engage in interpretations and reflections on the meaning of mathematics, rather than being mere "receivers of predetermined knowledge" (Boaler & Greeno, 2000, p. 179). Productive engagement is an engagement in the classroom where learners are making each other understandable and where the students are responsible for the mathematics at hand, in ways that allow the learners to actively develop their interpretations of knowledge involved in a teaching-learning environment (Turner et al., 2013).

Multilingual mathematics learners are especially at risk to not engage productively. For example, it has been documented that many multilinguals do not actively engage in classroom discussions because of the demand to speak in the language of instruction (Planas & Setati, 2009) or that a focus on correctness in the language of instruction excludes multilingual learners and interrupts the engagement with mathematics (Planas & Civil, 2015, p. 47). Such mechanisms of inclusion and exclusion from productive engagement can be understood with positioning theory, where individuals in a conversation dynamically take up specific roles – positionings – and act according to the duties and rights that come with these positionings. To make sense of their positionings and their rights and duties, learners relate to storylines, where storylines stem from culturally shared patterned ways in which conversations are organized (Wagner & Herbel-Eisenmann, 2009). For example, in a storyline where the teacher asks questions and evaluates students' answers, the teacher is positioned to evaluate these answers. The learners are positioned with the duty to give answers, but also with the right to not contribute for a certain time. When the learners perceive this storyline as "I may be negatively evaluated", they may refuse to contribute, and might be less productively engaged with mathematics.

This study addresses two questions: How does the use of two languages in a bilingual teaching intervention influence the learners' positionings, and how does this affect their productive engagement with mathematics? Accordingly, the paper presents the here employed theoretical

framework of positioning theory; analyzes data from a bilingual Turkish-German teaching intervention on fractions, specifically the case of Ilknur and Akasya; and shows that by switching the understanding of the storyline of the intervention "giving a Turkish explanation" towards "giving an answer with the form of an explanation", the learners are less agentic for the mathematics at hand.

Positioning theory for investigating productive engagement in multilingual settings

In a teaching-learning situation, teacher and students fluidly take up certain roles, similar to the roles actors take up in a stage play, where these fluid roles are called positionings. "Positioning […] is the discursive process whereby selves are located in conversations as observably and subjectively coherent participants in jointly produced storylines" (Davies & Harré, 1990, p. 48) The individual in a conversation will take up, resist or reframe positionings as it tries to be subjectively coherent in his or her actions. At the same time, the actions of the individuals become intelligible to the others as they assume that the individual's actions are coherent with his or her position (Langenhove & Harré, 1994, p. 362). The theoretical construct of positioning resonates with classical interactionist perspectives on classroom interaction in its assumption that learning, identities and even competence are a result of social interactions (e.g. Cobb & Bauersfeld, 1995). Positioning theory has been used in mathematics education to investigate classrooms in regard to issues of opportunities to participate.

Storylines are jointly produced, they "stem from culturally shared repertoires" (nurse/patient; coach/athlete) (Wagner & Herbel-Eisenmann, 2009) and the culturally shared patterned ways in which conversations are organized. Different storylines provide learners with different means for productive engagement (Herbel-Eisenmann & Wagner, 2015). The use of more than one language reinforces these issues, because it brings more diverse storylines to the classroom – for example of the political value of languages or storylines of language learning – that might affect mathematics learning. Studies on multilingual mathematics learning from the perspective of positioning theory suggest which storylines might lead to productive engagement, and which not: in his research with Latino learners in the US, Domínguez (2011, p. 325) shows that students' home language "figures as a language to discuss, argue, take risks, and learn with others, whereas English tends to be reserved for enacting more traditional schoolwork". Storylines that allow learners to position themselves as mathematically competent while relating to everyday experiences tend to lead multilinguals to productive engagement (Moschkovich, 2002). Norén (2015) gives an example where multilingual students are productively engaged because the teacher establishes a storyline that allows for talking in non-formal ways and making use of out-of-school-experiences. On the other hand, multilingual students may position themselves and others based on storylines from previous classrooms where they have been positioned as less proficient language learners (cf. Civil & Planas, 2004).

As storylines are a product of the individuals' interpretations in a conversation, more than one storyline can coexist in a given situation. These different storylines mirror themselves in the different positionings that are on the one hand assigned, and on the other taken up by the individuals: while in a teaching learning situation the storyline of teacher/learner is jointly produced, the individuals in the conversation might have different notions of what this storyline is and how they are positioned in it. In this paper, I will analyze the storylines that are jointly produced and the notions the students have of this storyline. I investigate the question: *How does a jointly produced storyline in a multilingual teaching intervention influence the students' productive engagement, and hence, their agency?*

A bilingual teaching intervention on fractions

The here presented case study is part of the research project MuM-Multi which investigates how to foster multilingual learners in the mathematics classroom. A multilingual Turkish-German teaching intervention on fractions was conceived. The teaching intervention builds on a previous project (Prediger & Wessel, 2013) in that it combines a conceptual learning trajectory with a language learning trajectory. Furthermore, it builds on the relating registers approach (Prediger, Clarkson & Bose, 2016). In this approach, the mathematical-technical-register, the academic-school-register and the everyday register are continually interlinked. For example, the multilingual students are asked to reflect on the Turkish way to express fractions as '5 de 3' ('5 therein 3') (Turkish mathematical-technical register) for which they are asked to employ the graphical representation of a fraction bar and the activated everyday contexts (sharing a bar of Baklava-cake). The relating registers approach serves as a heuristic tool for designing activities and learning opportunities in which the multilingual learners can access their multilingual resources for developing conceptual understanding of fractions.

Each session of the teaching intervention starts with an everyday, out of school contexts that connect in culturally sensitive ways to the multilingual experiences of the learners. For example, the first task starts with a short story about a traditional Turkish figure, the "Nasrettin Hoca" and children sharing a Baklava with him in an unfair way. The following tasks in a teaching intervention are then anchored in the culturally sensitive out-of-school context, so that the students can relate to their everyday experiences while working on the mathematical tasks. During all sessions, the fraction bar is implemented as a central graphical representation. All worksheets are given in both Turkish and German, assigned by the teacher.

41 multilingual students in grade 7 with low mathematics achievement and heterogeneous German and Turkish language proficiency participated in eleven small groups in the bilingual teaching intervention lasting five sessions of 90 minutes each. The intervention was conducted by four specifically trained bilingual Turkish-German teachers who were sensitized for specific affordances of teaching in multilingual settings, for example strategies of revoicing. However, the teachers were not aware of the specifics of positioning students in conversations.

The eleven groups of the bilingual intervention were videotaped over the course of the intervention, and specific groups and sessions were selected for transcription based on considerations of key points in the learning trajectory that occur after the students got familiar with speaking Turkish (Sessions 2 - 4). The video material was transcribed and, where necessary, translated into German. The data is analyzed qualitatively in a turn-by turn analysis of the learners reflective positionings and the teachers interactive positioning, that is, the positioning of learners themselves within a conversation (reflective) and the positioning of the learners by the teacher in the conversation (interactive) that affects how the learners position themselves (reflective).

I illustrate the case of Akasya and Ilknur, as examples of students who successfully engage with mathematics in Turkish language in the last intervention session. I employ a positioning analysis that focuses on the students' reflective positioning in reaction to the teachers' interactive positioning in the unfolding mathematical conversation. By investigating the positionings the storylines of learners and teacher can be reconstructed. Productive engagement is analyzed with the construct of agency, according to which students are coded as productively engaged in moments when they are

- influencing the direction of the discourse
- asking for clarifications or clarifying
- taking charge of ideas
- establishing competence

This analysis allows investigating in which storyline the students are highly productively engaged, and when less. The analyzed transcript belongs to the first task of the second teaching intervention (Figure 1), in which ordering fractions for different referent wholes is approached by the everyday context of downloading movies.

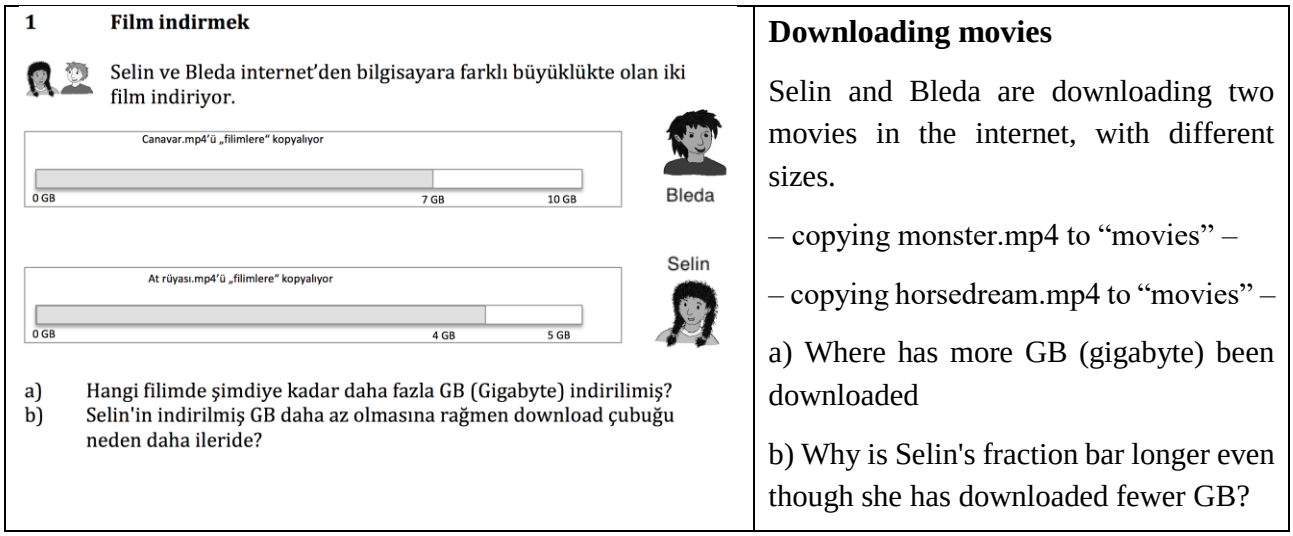

Figure 1: First task of Session 2 of teaching intervention

Analysis: Storyline of Turkish speaking and less productive engagement

Two girls, Akasya and Ilknur, work on the first task in Session 2 of the intervention. Briefly, I will refer to another group of two students who work separately at another table on the same task. I will show here how the teacher changes and refines the storyline that guides his actions over the course of the task. As storylines are jointly produced, the students have to actualize their storylines in line with how they perceive this changed storyline. While Ilknur adapts her notion of the storyline to the newly established storyline of the teacher, Akasya seems to subsume this change under her notion of the storyline as "the teacher guides the discourse and will give the resolution of the task in the end".

Episode 1: The teacher changes the storyline

The teacher introduces the subtask b) to the students, and with it, also changes the previous storyline of "deciding who has downloaded more". The students have signaled, by hand signal, that they have a solution to subtask a). The teacher now establishes a storyline which is not centered on finding a solution anymore, but one where explanations in Turkish are valued.

In the following transcript, the teacher comes to Ilknur and Akasya, after he talked to the other two students at another desk. His utterance marks the first change of the storyline:

75	Teacher	[coming back to Akasya] Ben size ehm yanlış, hayır hayır yanlış var mı diye hiç korkmana gerek yok.	I have you, ehm, something wrong, no, no, you don't have to be afraid if it is wrong. You just have to write down

	Sen sadece düşünceni yazacaksın. Tamam? Yani siz Bleda kazandı diyorsanız Bleda kazandı yaz ve neden kazandığını düşündüyünü yaz. Tamam?	your idea. Okay? So, when you say that Bleda has won, then write down that Bleda has won and justify why he has won. Okay?

The teacher metadiscursively addresses the rules of the conversation. In this way, he positions Akasya, and implicitly Ilknur, as competent regardless of the correctness of their solution, as long as she writes down her ideas, and justifies "who has won". The students are explicitly positioned as explainer of their mathematical thinking in line with a storyline of "formulating a written explanation of your decision". This is in contrast to the previous storyline of the teacher, where the students were asked to cooperatively decide whether Bleda or Selin has downloaded more.

In the following utterance, sometime after the previous transcript but still in the beginning of the work on subtask b), the teacher gives a reason why the students need to explain their thinking:

83	Teacher	Aber das ist gut, weil dann lernt ihr auch. Dann könnt ihr euch auch überlegen, wie ihr das auf Türkisch schreibt. Das ist eine gute Sache.	But that is good, because then you also learn. Then you also could think about how to write that in Turkish. That is a good thing.

The teacher asks the students to think about how to "write [their explanation] in Turkish". Again, he makes the rules of the mathematical discourse explicit by valuing the Turkish language and adding "That [Turkish] is a good thing", which he repeats in the next turns. Valuing the Turkish language is a recurring theme in this storyline (turns 69, 83, 85) and is tightly interlinked with working on subtask b). In summary, by explicitly positioning the students in these ways, the teacher establishes a storyline where students are required to explain their thinking in (written) Turkish, perhaps best described as "Solving subtask b means giving a Turkish explanation of your solution". This is a refinement of the new storyline.

Episode 2: Ilknur takes over the teacher's storyline

Ilknur seems to adopt the storyline established by the teacher. She makes several attempts to explain why, from her standpoint, Bleda has downloaded more. Her choice of language, her pauses and by "ehm", suggest that she specifically tries to explain her thinking in Turkish:

86	Ilknur	Çünkü o .. on gigabyte [...]	Because he ... ten gigabyte
88	Ilknur	ehm yaptı	um, he did

Ilknur attempts an explanation in Turkish, positioning herself in a way that acknowledges the position offered by the teacher (see turn 75). At first, she seems to be uncomfortable with explicitly using only Turkish – previously she only spoke Turkish in a mixed German-Turkish mode with focus on German – which is indicated by the hesitant way in which she forms her explanation. All her following task-related utterances after this turn are in Turkish (105, 108, 110), which suggests that she positions herself as Turkish speaker and that the storyline "giving a Turkish explanation" guides her actions.

Episode 3: Students agency and potential for productive engagement

In engaging in Turkish, the students Akasya and Ilknur do not appear as agents for the content of their explanation. Ilknur explicitly positions herself as less competent with the Turkish language: When discussing which language their worksheets should have, Ilknur explicitly wants German worksheets, and she positions herself as being more proficient in French than in Turkish (94). This is reinforced by Ilknur and Akasya's expressing the amount of effort it takes them to form a Turkish explanation

(see turns 86-88 and linguistic markers “Üfff“ (Boah), “Heh?“ in turns 106, 110). Above that, Ilknur and Akasya’s struggles also make them less competent in the eyes of the teacher. On the one hand, the teacher addresses those students who have already generated a Turkish explanation and positions them as competent by praising their solutions. Ilknur and Akasya, on the other hand, are addressed with “He şimdi kızlar (So, now, gals)” (turn 107) as a result of them not yet having generated a written explanation, this way positioning the students as being behind in the group.

In the following episode, Ilknur and Akasya seem to explicitly give up on their position as Turkish mathematics learners, and take up a position where they are responsible for the form of their solution. It seems that they give up their position because of mathematical difficulties, not language difficulties.

113	Akasya	Aber zehn ist doch mehr.	But ten is still more.
114	Ilknur	Ist so. Ich weiß überhaupt nicht, baah	Its like that. I don’t know at all, wah
115	Akasya	Ich schreib einfach was.	I will just write something.

When Ilknur and Akasya do not make progress in generating a Turkish explanation, they explicitly position themselves as learners who struggle with their task. This goes hand in hand with Ilknur switching back to German (turn 114). Akasya, in line with her storyline where the teacher guides the discourse, assumes that the teacher is likely to correct them in the end, so that only the form of their explanation matters, but not the content. It might be that in accordance with this storyline, Akasya and Ilknur will take up positionings where they exercise less agency for the mathematics at hand.

In summary, over the course of working on subtask b), the students have less capacities to engage with the mathematical content of the task within the storyline “giving a Turkish explanation”. They position themselves as insecure and less Turkish proficient, while they receive no help from the teacher or their peers, but are instead positioned as being behind the group. As Ilknur and Akasya have not yet written down their explanations, and as the teacher was not present, their efforts are not valued. In this sense, turns 114 and 115 can be read as a way to reclaim their agency: Within the teacher’s storyline of generating a Turkish explanation, the students change their activities towards writing “something” that conforms with the form of an explanation, but give up improving the content of the explanation. This is consistent with Akasya’s take on the storyline - in which the teacher is the guide of the conversation and responsible for the content of the explanations - seems to be taken up. The potential for productive engagement, that is, the potential for becoming responsible for the mathematical content, diminished in subtask b) as Ilknur and Akasya did not engage with finding a mathematical explanation in the task, but instead reclaim their agency with producing a Turkish sentence that complies with the form of an explanation. The storyline of the teacher allows for this, as the teacher gave a loophole with saying that “you don’t have to be afraid if it is wrong” (turn 75).

Discussion

In order to be coherent with the teacher’s expectations of giving an explanation in Turkish, the students in the episodes try to form a Turkish explanation. This leads the students to focus solely on forming a correct Turkish sentence with adequate formal words, while the mathematics behind the explanation is not in focus anymore. As the students are not successful in forming a Turkish explanation, the students might take up a position of being less agentic for the content, resulting in less productive engagement with mathematics. It seems that the teacher, by asking for a Turkish explanation and by valuing the Turkish language, and the students, by taking over these positionings,

jointly produce two foci: one on the mathematical content in Turkish and one on the form of a Turkish explanation. In shifting to the second focus, the students lose their focus on the content of the explanation – which interestingly goes hand in hand with a code-switch back to German. It may be that the two foci are a product of valuing Turkish without giving the students the support they seem to need for talking about the content of the task.

Positioning theory and storyline analysis here proved to be a useful tool to reconstruct the participation of multilinguals and the opportunities for productive engagement and mathematics learning, especially in regard to the multilingual teaching-learning situation. However, it is an open question whether the students multilinguality is a resource for the students in regard of what the storylines tell us in this research. Data suggest that difficulties with mathematics led Akasya and Ilknur to move the focus of their activity away from the content of the explanation. Previously, however, Ilknur struggles with the Turkish language. It is an open question whether Turkish might have added to Ilknur's mathematical difficulties in form of increasing cognitive-load, or if Ilknur is in a natural translanguaging mode where her choice of language is not relevant and where her difficulties would have remained if she had used German instead of Turkish.

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Authority structures in preservice teachers' talk

Konstantinos Tatsis and Bozena Maj-Tatsis

University of Ioannina, Greece; ktatsis@uoi.gr

University of Rzeszow, Poland; bmaj@ur.edu.pl

In the paper we draw on the authority structures that were observed in preservice teachers' talk while working with collaborating pairs of primary school pupils. We have found some interesting structures that emerged, especially when the preservice teachers had to consider the goals of their working sessions with the pupils, as conceived by them. Particularly, the preservice teachers switched to a more authoritative stance, usually when there was time pressure.

Keywords: Authority, preservice teachers, pair work.

Introduction

In any learning environment when there are more than one participants present, various interactions take place. Among these, the verbal ones have been the focus of numerous studies in the last decades. Most – if not all – of these studies originate from linguistics; in one the first papers that explicitly refer to the relationship between language and mathematics we read that "the processes of learning and communication are closely inter-related and both present the investigator with problems of bewildering complexity." (Austin & Howson, 1979, p. 162). Today, almost 40 years since that statement was made, researchers in mathematics education are still struggling to interpret the complexities of learning and communication. However, the main interests of research remain on how students learn (at the same time, what constitutes learning may be by itself the topic of investigation) and what (can) teachers do to enhance their students' mathematical thinking. In our view, mathematical thinking relates to effectively participating in the mathematical classroom community (Yackel & Cobb, 1996). This in turn requires a particular distribution of authority in the classroom: the teacher is expected to "step back" from the official role and give room to the students to make conjectures, pose problems and generally think like mathematicians (Mason, Burton & Stacey, 2010).

Based on the above assumptions, we designed our study with preservice mathematics teachers. We engaged them in a research-type activity: they were asked to find a mathematical problem and use it as a tool to enhance pupils' creative thinking. Our main interest was in the ways these preservice teachers will manage the interactions and particularly their authority in order to achieve that aim. Our theoretical framework is presented in the next section, followed by our methodology.

Theoretical framework

Language use in the mathematics classroom – or, more generally in mathematics learning contexts – can convey various aspects of the interactions or the mental processes that occur. There are at least two ways that we can view and analyse language use: at an object level, we may focus on the mathematical register and its use among the students and the teacher. From a teacher education point of view, this can be also related to mathematical knowledge (Rowland & Ruthven, 2011). Then, if we move to a meta-level, we realise that "sense-making in school mathematics is not solely a matter of private interpretation within some absolute, secure reality of 'real' objects ('people' and the like); it is also one of linguistic enculturation, of initiation to discursive practice". (Walkerdine, 1988, p.

128, as cited in Rowland, 2000, p. 194). Mathematics learning is thus seen as a social process, regulated by rules and norms (Yackel & Cobb, 1996), in which the participants are continuously negotiating meaning by exercising their agency (Cobb et al., 2009). This requires a distribution of authority by the teacher. This is because verbal interactions between teachers and students are usually characterised by the teacher's authoritative stance: the teacher represents the mathematicians' community, thus s/he is *an* authority because of his/her content knowledge; at the same time, s/he is *in* authority, because of position (Wagner & Herbel-Eisenmann, 2014). Thus, depending on the didactical approach, the students may only have to follow the teacher's guidance, or they may be given space to actively participate in the learning process. For the purpose of the present paper we adopt the following definition for authority:

> Authority concerns the degree to which students are given opportunities to be involved in decision making about the interpretation of tasks, the reasonableness of solution methods, and the legitimacy of solutions. Authority is therefore about "who's in charge" in terms of making mathematical contributions. (Cobb et al., 2009, p. 44)

The analysis of authority can be done by the use of positioning theory (van Langenhove & Harré, 1999) as implemented by Wagner & Herbel-Eisenmann (2014). According to this, there are some pervasive "lexical bundles" (i.e. speech patterns) in classroom talk, which are related to participants' positioning and authority. The main categories of these are: personal authority, discourse as authority, discursive inevitability, and personal latitude. Personal authority, which is the most common in the classrooms, relates to teacher's own authority; in other words, the students are expected to follow (the authority of) their teacher. A characteristic case of personal authority is when the teacher says, e.g., "I want you to solve this equation for me". Discourse as authority relates to the rules which must be followed in the interaction, "which come from outside personal relationships, [and] may be attributed to the discipline of mathematics (or perhaps school mathematics)." (Wagner & Herbel-Eisenmann, 2014, p. 873) An example of discourse as authority is the utterance "We need to follow the multiplication rules". Discursive inevitability relates to events or actions that are verbally presented as inevitable; as Wagner & Herbel-Eisenmann (2014) note: "there is no explicit reference to obligation, but rather a sense of predetermination" (p. 873) An example may be the utterance: "We are going to calculate the average of the given values". Finally, personal latitude relates to the students making their own decisions during the interactions, thus exercising their own authority; this is related to *conceptual agency* (Cobb et al., 2009) and mainly expressed by students' questions.

The above are usually accompanied by other verbal strategies; speakers sometimes feel the need to convey other messages related to power or solidarity. Politeness strategies (Brown & Levinson, 1987) have been the focus of a number of studies, either solely (Tatsis & Rowland, 2006) or juxtaposed with authority structures' analysis (Tatsis & Wagner, 2016); these studies stress the fact that students and teachers continuously interpret each other's actions, in order to perform their own actions.

Summing up, although various verbal patterns coexist in the mathematics classroom, there are situations in which particular patterns are prevalent – these patterns characterise the interactions and influence its outcome to a significant degree. Thus, in our study we were expecting to locate and identify such patterns in preservice teachers' discussions with pupils. Our methods of analysis are described in the next section.

Context of the study and methodology

Our research was based on the work of six preservice teachers, at the third year of their studies in mathematics. They already had three weeks of practice in a primary school, during which they had taught for 18 hours per week. They had attended – among other courses in pedagogy, psychology, psychology of mathematical thinking, and didactics of mathematics – a course led by the second author of the paper in problem solving, during which they solved problems and analysed pupils' solutions and discussions; they had also watched and analysed a video with a study that involved a pair of young children (see Maj-Tatsis & Tatsis, 2015).

For the purpose of the present study, the preservice teachers were asked to find an interesting mathematical problem (the only clue given was it should offer the possibility for mathematical explorations) and give it to a pair of pupils from the class that they had already taught. The ages of the pupils varied from 10 to 12 years. There were no restrictions on how to choose the pair; this resulted in pupils from varying backgrounds and of various mathematical attainment. Other instructions provided to the preservice teachers were related to the distribution of their authority: they were asked to let the pupils talk and, generally to try not to impose their own way of thinking. The sessions lasted from 20 to 45 minutes and were videotaped. Then, the preservice teachers were asked to transcribe their sessions and analyse them, according to whether they had achieved their aim (as perceived by them) and the possible reasons or events that were responsible for that – including their own decisions and actions.

For the purpose of the present paper, we mainly focused on preservice teachers' transcribed discussions with the pupils; this data was complemented by their interviews given to the second author of the paper, after the completion of the sessions. Our analysis was a meta-level analysis (Rowland, 2000), since we focused on utterances related to the participants' interactions and we searched for manifestations of authority structures. Following Wagner & Herbel-Eisenmann's (2014) framework we located utterances or exchanges that accounted for personal authority, discourse as authority, discursive inevitability, and personal latitude. Particularly, personal authority "was identified by the presence of first- and second person pronouns together" (Wagner & Herbel-Eisenmann, 2014, p. 873). Discourse as authority was identified by the presence of modal verbs such as "need to" and "have to", which explicitly express a strong obligation; it was also identified by the use of "they", which refers "to a non-specified entity or group who have potentially made decisions about the mathematics students encounter" (Wagner & Herbel-Eisenmann, 2014, p. 873). Discursive inevitability was identified by utterances like 'you are going to' and 'it is going to'. Finally, personal latitude was mainly identified by the presence of questions by the pupils. These initial categories were then utilised in order to establish the various *authority structures*; these were not predetermined, but established during the course of the analysis.

Additionally, we looked into other verbal phenomena, such as the use of personal pronouns, such as "we" and "you" (Rowland, 2000). During our analysis, there were instances when we had to "make a step back" into an object level analysis, by looking into the mathematical concepts and processes that were discussed and established during the interactions; this was deemed necessary in order to fully comprehend the participants' actions and to study their effect on the authority distribution.

Results

As we mentioned in the previous section, our analysis focused on the verbal interactions of the preservice teachers with the pupils. We will firstly present the analysis of the interactions between Tomek and two grade 6 boys (12 years old). We present this case because it demonstrates in a clear way the authority structure which was the most prevailing among the cases we analysed.

The case of Tomek: Problem posing that leads to generalisation

The situation given to the pupils by Tomek included the following text:

John is creating "chairs" by using chips:

The text was followed by these drawings (Figure 1):

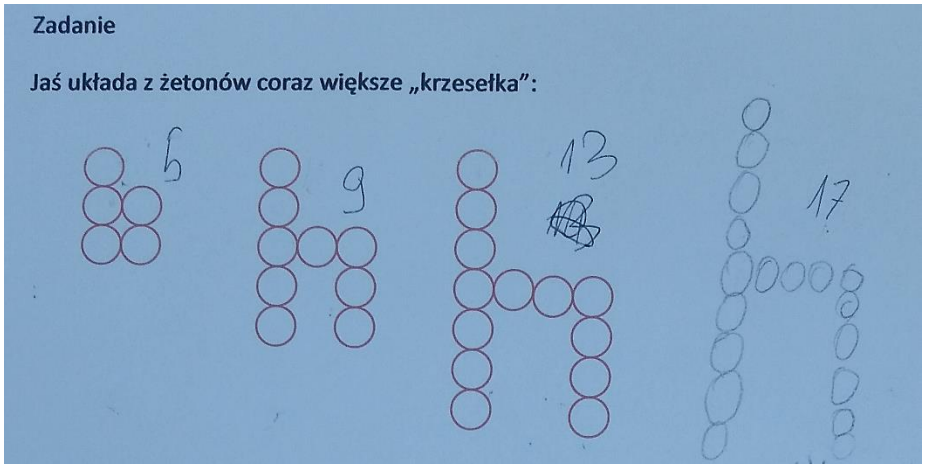

Figure 1. The "chairs" made by chips.

Below we read the initial discussion[1] between Tomek and Pupil 1:

1	Tomek:	OK, the task is simple: firstly, read it. Have you ever seen something like this?
2	Pupil 1:	Hmm, no, but I know how to do it.
3	Tomek:	OK, so now think up of some questions. What questions would you ask? [small break] Usually, there are ready questions, and here…

Then, the pupils started working on making questions for the given situation. In turns 4-35, Tomek talked only four times, and among these only the next was related to the problem posing process:

10	Tomek:	Just write your ideas.

In the next turn there is the following exchange:

35	Pupil 2:	Do we have to answer every question? [which they have posed]
36	Tomek:	Hmm, later we will choose two questions which we will answer.
37	Pupil 1:	For example, what would be the sum of the second and the third figures? What would be the sum of the chips of the second and the third figures?
38	Pupil 2:	[He writes Pupil 1's question to their worksheet]

[1] All the discussions were translated from Polish by the second author of the paper.

39 Tomek: You can always draw more figures.
40 Pupil 1 or 2: Aha.
41 Tomek: You are not limited to three [figures] only.

Then, until turn 59 the two pupils are working without any intervention by Tomek; they are making figures and discussing on the pattern. Then they draw the fourth chair shown in Figure 1.

60 Tomek: Are you sure about that chair?
61 Pupil 1: Why not?
62 Tomek: Calculate once again.

The preservice teacher started his research by asking pupils to put questions to the presented situation. By such way of guiding the process he gave them the possibility to engage in problem posing by formulating their own questions. Although his request is an expression of discursive inevitability, it resulted in the pupils being interested in the task and eventually the authority was passed to the pupils (personal latitude). Up to turn 59 Tomek did give space to the pupils, by letting them discuss. By his question in turn 60 he exercised personal authority, he used direct formulation "are you sure?", and then asked them to correct it (62). We may claim that it was a good moment for an intervention, since the pupils had made a mistake and this would result in an improper rule and generalisation. However, that could be done in a less explicit way, e.g. "how did you know how many chips should be in the fourth chair?". After the boys corrected the mistake, Tomek gave them again space to work, in other words, he let them exercise their personal latitude.

In turn 70 Tomek suggested: "So maybe let's do the second question [which was formulated by the pupils and written at their worksheet: "How many chips will the next chair have?"] because I think it is the most interesting". In this utterance we can find two different authority structures: personal latitude (inclusive imperative) and personal authority ("I think"). By that intervention Tomek directed the work of pupils into generalisation. But still the question was formulated by the pupils so it can be considered as an appraisal of their work.

The next intervention of Tomek took place when the pupils put a false hypothesis:

96 Pupil 1: So we can conclude from it [the number of chips in chairs 1, 2, 3 and 4 were calculated by the pupils as follows: 5, 9, 14, 17 – note that 14 is incorrect] that every figure increases by 4. So, the next figure will have 21. And so on.
97 Tomek: So, for example, how many will be in the tenth chair?
98 Pupil 1: 21 times 2 equals 42. Because it is times two. Because it is the fifth chair [number 21].
99 Tomek: Are you sure?
100 Pupil 1: Yes, for sure. Yes, right? [calculating] 21 times 2 is 42, the tenth chair is that. So, it's like that.
101 Tomek: [movement by his head that it is wrong]
102 Pupil 1: No?! But how, if it is…
103 Tomek: Calculate precisely.

In the above transcript we see that Pupil 1 has formulated the assumption that the number of chips increases proportionally to chair's number, thus the number of chips of the tenth chair is double the number of chips of the fifth chair. This incorrect assumption was formulated despite Pupil 1's correct

observation that "every figure increases by 4 [chips]". Tomek decided to resolve that situation by offering a counter-example, which falls into the discursive inevitability category:

111 Tomek: So, in that case, chairs 2 and 4 should… so, the fourth should be twice bigger than the second.

112 Pupil 1: Oh! That's right. Something went wrong (…)

It is obvious that Tomek's intervention helped the students realise their false assumption. Generally, the previous excerpts demonstrate the prevailing authority structure in Tomek's verbal actions: he mainly let the pupils exercise their personal latitude and he exercised his personal authority and discursive inevitability structures only at crucial moments (according to him, but also according to our interpretation). In other words, if he did not intervene at these moments, the pupils could have spent much time in explorations that would not lead to the desired generalisation. However, this authority structure changed significantly in the 16th minute of the session, when the person who was handling the video camera (who was Tomek's colleague) informed him that it is the 16th minute (turn 171). Tomek switched his behaviour to a more authoritative structure, by asking closed questions, prompting the pupils and generally drastically increasing his interventions. This is eloquently demonstrated in Tomek's own account of his actions:

> Generally, making such a research was very tiring. Despite my earlier preparations I did not expect that it will go in such a way. I expected that they [pupils] will calculate chair x just by adding. And they started multiplying from the beginning. I made many mistakes, it could be conducted in a better way. Thankfully, the children liked it. Close to the end, more or less at the 14th minute, I started prompting them too much because I was afraid that I will not manage in time. Because of that [prompts] I limited their mental actions.

Tomek's case demonstrated how we performed our analysis over the six chosen cases. The following authority structures were identified:

a) Structure 1: Personal latitude accompanied with mainly personal authority and discursive inevitability; followed by a blunt switch to personal authority, eventually accompanied with discursive inevitability.
b) Structure 2: Personal latitude accompanied with mainly personal authority and less frequently with discursive inevitability.
c) Structure 3: Personal authority.

The above structures are presented according to their frequency; the first one was found in four cases, while the other two appeared in one case each.

Conclusions

Our study was inspired by studies on the authority structures that can be identified in the mathematics classroom (Wagner & Herbel-Eisenmann, 2014); we were interested to see whether similar structures would appear in preservice teachers' talk. Our results have shown that the preservice teachers of our study initially had been less authoritative, thus allowing the pupils to work collaboratively and exercise their personal latitude. However, during the interaction, and due to some events, they switched into more authoritative structures. One of the basic reasons for that switch was the time pressure together with the obligation to fulfil the request of their educators. The interesting thing was though that all of them were aware of this switch, as it was expressed during their interviews.

Following Cobb et al. (2009), we agree that the distribution of authority is closely linked to the ways that students exercise their agency in the mathematics classroom, including, as our study has shown, collaborative work. Thus, the teacher should be aware of that fact, in other words, s/he needs to know when it is time to exercise authority, and when s/he can deviate from the lesson plan; this is related to the notion of *contingency* as presented in Rowland, Huckstep and Thwaites's (2005) "knowledge quartet" framework. Particularly, the teachers' responses to students' ideas as well as their ability to deviate from their agenda affect the knowledge construction in a significant way. These considerations should be discussed and analysed in pre- and in-service teachers' training courses, preferably by involving teachers in self-monitoring with regards to the distribution of authority during their interactions with the students.

In our study, the preservice teachers' awareness of the nature and the effect of their own verbal actions was a clear result of their training courses. However, this did not stop them from exercising their personal authority at particular moments. Thus, there is still lots of work that needs to be done in the field of authority distribution by mathematics teachers in order to establish an active learning community in their classrooms.

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How do students develop lexical means for understanding the concept of relative frequency? Empirical insights on the basis of trace analyses

Lena Wessel

TU Dortmund University, Germany; lena.wessel@math.tu-dortmund.de

In order to enhance students' proficiency in the academic language, building up more formal language on the basis of individual and everyday language is claimed as a fruitful approach. However, there is little empirical research on how students adopt and develop lexical means of the academic language. This paper addresses this field of research for the case of concept-specific lexical means for relative frequencies by presenting the applied design principles for learning opportunities as well as empirical insights into initiated concept- and language development processes.

Keywords: Academic language, relative frequency, scaffolding, design research, trace analysis.

Introduction

Due to the cognitive and epistemic function of language, the academic language can be challenging for many students (Schleppegrell, 2004). These challenges are also relevant for mathematical learning, as shown in various empirical studies. Thus, studies in mathematics education concentrate on designing and researching learning environments that integrate mathematics and language learning (Prediger & Wessel, 2013; Prediger & Pöhler, 2015).

For the field of understanding relative frequencies, the presented study relies on analyses of design experiments in small group settings focusing on one lesson within the larger intervention study of the DFG project MUM-MESUT (Grant PR 662/14-1 to S. Prediger). Detailed analyses of students' conceptual development and language development against the background of intertwined conceptual and lexical-discursive learning opportunities serve to structure the relevant lexical means and give insights into how students become proficient in the academic language for relative frequency.

Theoretical background: Design principles for learning opportunities

Academic language proficiency has repeatedly been shown to influence achievement in mathematics and this general finding also applies for the mathematical topic of understanding fractions (Wessel, 2015). As a consequence, some current design research studies focus on developing and investigating content- and language integrated instructional approaches for fostering students with low language proficiency (Prediger & Wessel, 2013). The following paragraphs deal with the major design principles that were implemented in the presented design research study.

Design principle "Macro-scaffolding". The general structure of intended lexical learning trajectories is well described in the principles of *macro-scaffolding*, namely from students' everyday resources to academic and formal technical registers (Gibbons, 2002). However, its topic-specific realization is still an urgent need of research, as well as explorations of students' individual learning pathways (Prediger & Wessel, 2013). Previous research shows the relevance of phrases and syntactical constructions needed to express the meanings of a mathematical concept in view, which is also relevant for understanding the concept of fraction (ibid). Wessel (2015, p. 327) shows the importance of understanding how macro-scaffolding and interactional moves on the micro level relate to each other for moving students beyond their zone of proximal development. Here, the principle of *macro-*

scaffolding by coordinating conceptual learning opportunities with well-structured language learning opportunities on the lexical level is applied.

Design principle "Pushing students' output by realizing discursive practices". Given the sociocultural perspective on the learning of mathematics as participating in mathematical practices, mathematical activity is to a great extent mediated by language and interaction. In the context of mathematics learning of English language learners (ELLs) and with a perspective on extending academic language proficiency, Moschkovich (2013) stresses the relation between the lexical and discursive level of language: "The question is not whether students who are ELLs should learn vocabulary, but rather how instruction can best support students to learn vocabulary as they actively engage in mathematical reasoning about important mathematical topics" (Moschkovich, 2013, p. 46). This theoretical assumption leads to an extension of the design principle so that we use the principle of *macro-scaffolding* by coordinating conceptual learning opportunities with well-structured language learning opportunities on the lexical level *in addition to rich demands and language initiation on the discursive level.*

Design principle "Relating registers". Pushing the students' output and applying scaffolding strategies can be supported by the *design principle of relating registers*, according to which the graphical, the symbolic and the different verbal registers (everyday, academic, and technical register) are related systematically to achieve conceptual understanding (Prediger, Clarkson & Bose, 2016). For the lessons of the presented intervention, activities of relating registers have been realized with the fraction bar and bar board as a prominent graphical representation. In order to activate students' individual and everyday language resources, typical contexts of downloads, fair share and soccer competitions have been implemented (for detail see Prediger & Wessel, 2013).

In their combination, the formulated design principles allow to integrate theoretical aspects on developing learning opportunities on the conceptual, lexical and discursive level. However, while the integrated analysis of initiated learning processes on conceptual and discursive levels are well-established in mathematics education research, only rarely empirical studies reconstruct lexical learning processes (exceptions e.g. Prediger & Pöhler, 2015, for the field of percentages). That is why Schleppegrell (2010, p. 107) demands more respective research which goes beyond analyzing short interactional sequences: "More research is needed that takes a developmental approach (…). We need rich studies of how language and ways of talking about mathematics evolve over a unit of study, focusing on more than brief interactional episodes and fragments of dialogue". The presented study aims at minimizing this research gap for the field of relative frequency.

Research questions

On the basis of the theoretical background and the research gaps listed above, the developmental work and analyses of the learning processes are guided by the following two questions:

1) *On the level of design outcome:* How can conceptual and lexical-discursive learning opportunities for understanding relative frequency be intertwined and designed in a sequence of rich mathematical activities?
2) *On the level of initiated learning processes:* Which lexical means do students activate and how are those lexical means intertwined with individual conceptual development when working on the learning opportunities towards relative frequency?

Methodological framework and research context

The research was conducted in the methodological framework of topic-specific didactical design research (Prediger & Zwetzschler, 2013) in which the analysis of teaching-learning processes takes place in carefully designed teaching experiments. The design outcome, namely the consolidated intertwinement of conceptual and lexical-discursive learning opportunities, is next described.

Design outcome: Learning opportunities towards relative frequency (research question 1)

In order to combine conceptual, lexical and discursive learning opportunities according to the design principles described above, the larger intervention with five lessons for fostering conceptual understanding of students with diverse language proficiency in the language of instruction aiming at enhancing understanding of fractions was designed. For answering research question 1 the designed learning opportunities towards relative frequency as a design outcome are presented in the following section.

The intended *conceptual learning opportunities* were adapted from Prediger (2013). It starts with students' individual approaches and everyday experiences to compare three groups with different relative frequencies in the context of a soccer competition (see Table 1, Task 5). It then proceeds to constructing meaning of the given relative frequencies by introducing the bar board (Task 6). At this point, students' informal strategies for comparison are elaborated by focusing the need for normed referent wholes (here fraction bars of normed length) and the necessity of including every group's number of shots (not only number of strikes) to refine the concept of relative frequency which finally aims at the flexible use of relating number of shots and number of strikes.

The intended *lexical-discursive learning opportunities* focus on the vocabulary required for the conceptual learning process of thinking in relative frequency which is mainly the prepositional "of- or thereof construction" ("to score … of … shots", "… shots, thereof …") (see Table 1), which can be conceptualized from the so-called 'basic meaning-related vocabulary' (Wessel, 2015). Students are asked to give reasons in the setting of discussing ways of fair or unfair strategies to rank the three groups. It starts from students' individual resources as well as with offering the relevant "of-construction" already in Task 5.

Conceptual learning opportunities	**Tasks and mediator bar board**	**Lexical-discursive learning opportunities**
Initiation of *individual* approaches for comparing relative frequency conceptualized as strike rates in soccer competition	**5. Who scored best?** In class 7c three groups took part in a soccer competition. *The group of boys scored 4 of 5 shots.* *The group of girls scored 8 of 10 shots.* *The group of teachers shot 20 times and didn't score 4 times.* a) Who won the competition? Write your answer on a card. b) Put your cards in the middle of the table. Do you agree? Give reasons for your answer.	Initiation of *discussing* individual approaches and *giving reasons* Introduction of *lexical* means "to score / not score … of … shots"
Investigation of individual hypotheses in the bar board: Comparing with fraction bars of normed length	**6. Who scored best?** Use the bar board in order to commonly find out whether one group scored better. The boys have already been marked. Add the results of the girls and the teachers as well as the speech bubbles.	*Reflecting* and *discussing* fitting of fraction bars and groups ("This bar fits to the boys because …")

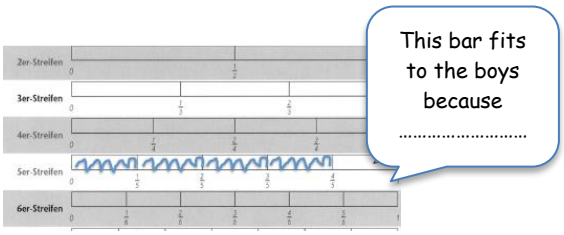
Necessity of including number of shots (not only number of strikes) to refine concept of relative frequency		Activating lexical means for marking strike rates in the bar board focusing number of shots as a referent whole, number of strikes and strike rate
Systematize and deepen understanding by giving reasons for all groups scoring equally well	**7. And the winner is...** In the bar board you have found out how well the different teams scored. Which group won the competition? Give reasons for your answer.	Written reasoning on equivalence of relative frequency in the three groups Applying introduced lexical means

Table 1: Conceptual and lexical-discursive learning opportunities (not necessarily strictly sequenced)

The tasks in Table 1 illustrate how conceptual and lexical-discursive aspects are intertwined. On the discursive level, students are encouraged to verbalize and discuss their own ideas and structures. The vocabulary for these discussions is bound to the bar board as well as the context of the scoring situation, which always allows students to relate the vocabulary to its meaning. In Task 7 the students are free to note their reasons either with reference to the bar board, to the context or to the formal level of expanding and reducing fractions.

Methods for data gathering and selection: Design experiments

Design experiments were conducted and video-taped within the larger research project MuM-MESUT with N = 343 mathematically low-achieving mono- and multilingual students in grade 7. For the detailed analyses in this paper, a group of three students was selected according to their German language proficiency (measured with a German C-test) and language background (mono- or multi-lingual, operationalized by "speaks at least one other language than German with a parent or grandparent"), with the aim to have a linguistically heterogeneous sub-sample for conducting case analyses (in total the below presented method of analysis was applied in detail to n=16 students). Due to the larger study, we can also draw on fraction test scores of the students (Wessel, 2015).

Methods of data analysis for reconstructing conceptual and lexical development

In order to qualitatively reconstruct the students' lexical pathways and how their lexical means relate to the initiated discourse and individual concept development (research question 2), the following three steps were applied:

Step 1. Conceptual analysis. For reconstructing the students' conceptual development, strategies for comparing the given three groups of girls, boys and teachers and steps on the pathway to understand the concept of relative frequency have been identiefied by analysing transcripts and video data.

Step 2. Trace analysis. Concept-specific lexical means (words and phrases) which the students activated were inventoried and coded whether the students used them in oral or written language and whether they self-initiated the use or whether they adopted them from the material, the teacher or another student (for detail of the method "trace analysis" see Prediger & Pöhler, 2015). In this paper the focus is on oral language.

Step 3. Relating conceptual development and language. On the basis of step 1 and 2, the results of conceptual and language analysis were related and contrasted to reconstruct prototypical learning pathways and critical steps on the pathway under the perspective of different language backgrounds.

Empirical insights into the initiated conceptual and language learning processes

On the level of initiated learning processes, research question 2 asks for lexical means that students activate and how these lexical means are intertwined with individual conceptual development. By contrasting the inventory of lexical means of two students the first part of the research question is addressed in the next paragraph.

Concept-specific language production: Qualitative overview and comparsion

Makbule and Kiran (working in a group of three together with Vehbiya) are multilingual learners in year 7 of a German secondary school. In a German C-test Makbule's score is at percentile rank 37 and Kiran's at 84. In the fraction test Makbule's score is at percentile rank 7 and Kiran's at 15 (percentile ranks for both tests for full sample of N=1124 seventh graders). While Kiran is the more language proficient student according to the C-test results, Kiran and Makbule started at comparable low levels of fraction proficiency.

In Table 2 the actual orally activated concept-specific lexical means in the analyzed transcript (23.46 minutes of video data) of Kiran and Makbule are contrasted. While Makbule activates 26 different concept-specific lexical means in the course of the process and 76 in total, Kiran activates 15 different concept-specific lexical means and 21 in total. As Makbule generally talks the most in this lesson, relating these numbers to each student's individual rate of participation will be a further step in the data analysis.

It becomes apparent that Kiran uses all lexical means correctly, which fits to his high percentile rank in the German C-test and which is not always the case for Makbule. Also, while Makbule uses many of the lexical means various times (which leads to the high number of lexical means in total), the list of Kiran can give a hint at the possibly *sufficient* language for working on the given tasks and developing the concept of relative frequency.

Makbule		Kiran	
Concept-specific lexical means in oral language production in chronological order of first use in process, (frequency in brackets, semantically not correct lexical means in italics)			
#14 best (1)	#143 stripe (9)	#26 had … shots (2)	#182 divided in the middle (1)
#15 won (5)	*#151 similar won* (1)	#26 did not score (1)	
#25 because (9)	#151 this big (1)	#28 score (2)	
#25 had … shots (4)	#151 to divide in the middle (1)	#34 won (2)	
#25 did not score … times (3)	*#151 divide small* (1)	#60 fraction (2)	
#25 scored (2)	*#153 shoot similar* (1)	#67 took … shots and scored … of them (1)	
#58 scored ... times (3)	#180 similarly big (2)	#104 score … times (1)	
#76 … of … (1)	#180 normal big (4)	#124 tie (1)	
#95 bar (13)	#180 separated in the middle (5)	#128 the same (1)	
#101 ... times shots (1)	#186 separated (1)	#175 bar (3)	
#101 shoot … times (2)	#188 line (1)	#175 fits to (1)	
#103 took … shots and scored … of them (1)	#188 the same (1)	#175 because (1)	
#125 tie (1)		#175 as good as (1)	
#141 similar (2)		#177 as long as (1)	

Table 2: Variety of concept-specific lexical means in comparison

However, the transcript analysis of Makbule's conceptual development indicates that the additional lexical means like "to divide in the middle", "normal big" and "separated in the middle" in Makbule's inventory are of great importance for her learning process towards understanding the idea of expanding fractions as refining the structure in the fraction bar. As a first conclusion, the comparison of the results from Makbule and Kiran shows that the question of the required language seems to vary between the students and demands further analyses of different cases.

Makbule's process of adopting concept-specific lexical means when relating registers

As a conceptually relevant step in the learning process, it is important to move from the strategy of comparing the three groups' results (girls, boys and teachers) on the basis of the absolute number of strikes to experiencing the necessity of and applying the concept of relative frequency as a fair strategy for comparison (Prediger, 2013). How this pathway can be related to the activated and required lexical means becomes apparent in the following two excerpts taken from the corresponding learning process initiated by Task 6. The transcripts were translated from German and shortened to relevant utterances of Makbule which are needed for tracing those concept-related language means in focus (// indicates interruption).

When answering Task 5 (see Table 1), Makbule focuses on the three groups' absolute numbers of strikes and claims that the teachers won the competition. When working on Task 6, the following process was initiated by reflecting on why the chosen fraction bar of fifths fits to the boys group:

Excerpt I: Kiran stresses the idea of relative thinking

56	Teacher:	So why does this fraction bar fit to the boys, the one that is marked?
58	Makbule:	Because they, because they took yes, ehm, five shots and have only scored four times.
59	Teacher:	And why does the fraction bar fit?
61	Makbule:	Because it's four fifths.
66	Makbule:	Because they//
67	Kiran:	//took five shots and scored four of them.

On the lexical level, Makbule uses the coordination "and" to relate the number of shots and the number of strikes to each other (#58). As the teacher again asks why the fraction bar fits, Makbule focuses on the representation of rates as fractions, namely "four fifths" (#61). When starting an additional explanation ("Because they", #66) she is interrupted by Kiran, who finishes the sentence with "took five shots and scored four of them" (#67). This utterance in #67 is assumed to be a relevant trigger for the following discourse in the group, which becomes clear in the next excerpt.

Excerpt II: Makbule adopts Kiran's "of them" construction

70	Makbule:	*reads her written answer:* because they took five shots and scored only four of them.
73	Teacher:	So now the girls and the teachers. Where do we mark them?
74	Vehbiya:	The girls in the bar of tens.
75	Kiran:	Eight tens.
76	Makbule:	so 8 of, 8 of 10.
99	Makbule:	Because they took ten shots and have scored eight times.

101 Makbule: So they have yes, ehm, they had 20 times to shoot, so could shoot 20 times. And they only, so they had, so they didn't score four of them.

In #70 Makbule reads out her written answer in the speech bubble next to the bar of fifths. She adopts Kiran's mathematically more adequate construction for the relation of shots and strikes by using the prepositional sentence structure ("take … shots and score ... of them"). Further in the process, Makbule also adopts the "of-construction", which had been introduced by the material in Task 5, in order to reason the fitting of the bar of tens to the results of the girls (#76). Considering Kiran's utterance of the fraction in #75, it can be assumed that Makbule purposefully links the fraction with its meaning-related conceptualization "8 of 10". In #99 and #101, when reasoning the fitting of the bar of tens and bar of twentieths, she once again uses the coordination "and" as well as the prepositional "of-structure". However, in her written products she constantly applies the mathematically preferred "of-structure". Thus, it can be assumed that thinking relatively as well as having lexical constructions for expressing relative frequency meaningfully anchored in her mental lexicon was successfully achieved for Makbule. It is assumed that Kiran's introduction of the sentence structure "take … shots and score ... of them" was supportive for Makbule's conceptual and lexical learning pathway.

Conclusion

To summarize, the empirical insights show how rich and demanding discourse practices can be initiated in small group settings by means of the design principles of macro-scaffolding and relating registers. The dual focus of the applied macro-scaffolding on the conceptual learning opportunities intertwined with language learning opportunities on the lexical level has to be emphasized as this builds the basis for the analysis of the initiated lexical learning processes. On the developmental level the presented design outcome thus helps to answer the question of how instruction can support students to learn vocabulary as they engage in mathematical reasoning (Moschkovich, 2013), here with a focus on relative frequencies. Moreover, the case of Makbule implies that offering and relating various mathematically intended lexical constructions could be supportive for becoming more proficient in the formal language of schooling. This can be implemented more prominently in the material by activities of reflecting and discussing concept-specific lexical means, which again would be an intertwined conceptual and lexical-discursive learning opportunity.

So far, analyses of learning processes on the lexical level are quite rare in mathematics education research. Applying the method of trace analysis (Prediger & Pöhler, 2015) reveals details of students' language production and development on the lexical level. For Makbule and Kiran differences with respect to which and how concept-specific lexical means have been activated and adopted became apparent. However, further insights into the processes of the other groups are necessary to ensure the first empirical results and formulated hypotheses.

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Conceptions of the transition from the difference quotient to the derivative in imaginary dialogues written by preservice teachers

Annika M. Wille

Alpen-Adria-Universität Klagenfurt, Austria; annika.wille@aau.at

The transition from the difference quotient to the derivative is a step from the algebraic to the analytic concept formation. The aim of this article is to analyze the conceptions of this transition that can be traced in preservice teachers' imaginary dialogues, a form of mathematical writing.

Keywords: Elementary analysis, derivative, imaginary dialogues, preservice teacher education.

Introduction

An often formulated critique of the teaching of analysis in school is that syntactical calculus is used too early and there is a need for more content-related understanding of central concepts (cf. e.g. Hahn & Prediger, 2008). Therefore, there is a need to sensitize preservice teachers to this issue and strengthen their own understanding of analytical concepts.

The course "Didaktik der Analysis" at the Alpen-Adria-Universität Klagenfurt in 2015/2016 addressed this purpose. The preservice teachers often reflected on central concepts of elementary analysis, their relations and their meaning for classroom teaching. The reflections were written down as *imaginary dialogues*, a form of mathematical writing where a single student composes a written dialogue between two protagonists who discuss a mathematical task or question (Wille, 2008). This article focuses on one reflection task that addresses the transition from the difference quotient to the derivative. The preservice teachers' imaginary dialogues will be analyzed in order to determine their conceptions of this transition.

Theoretical framework and research questions

Within Anna Sfard's *theory of commognition* (Sfard, 2008) *thinking* is seen as self-communication, an individualized version of (interpersonal) communication. The term *commognition* is used for both, the processes of thinking and communication, which are considered as two "manifestations of the same phenomenon" (p. 296). In commognitive research the *discourses* are precisely the main unit of analysis (cf. p. 276). Discourses are different types of commognition that "draw some individuals together while excluding some others" (p. 91). Sfard names as a precondition for learning the "recursivity of linguistic commognition" (Sfard, 2008, p. 116). An example of recursivity is *communicating-on-communicatin*g, such as "reports on what somebody else has said, remarks on her own thoughts, or reflects on other interlocutors and their communicative actions" (p. 103). The recursivity allows for a look from the outside in order to reflect, abstract and reason.

Conceptions of central concepts of elementary analysis have been described in various studies (e.g. Thompson & Thompson, 1994; Hahn & Prediger, 2008; Roh, 2008; Greefrath, Oldenburg, Siller, Ulm & Weigand, 2016). Danckwerts and Vogel (2006) list interpretations of the difference quotient and the derivative in overview tables (p. 57 and p. 85). Figure 1 shows an adapted and translated version of these tables.

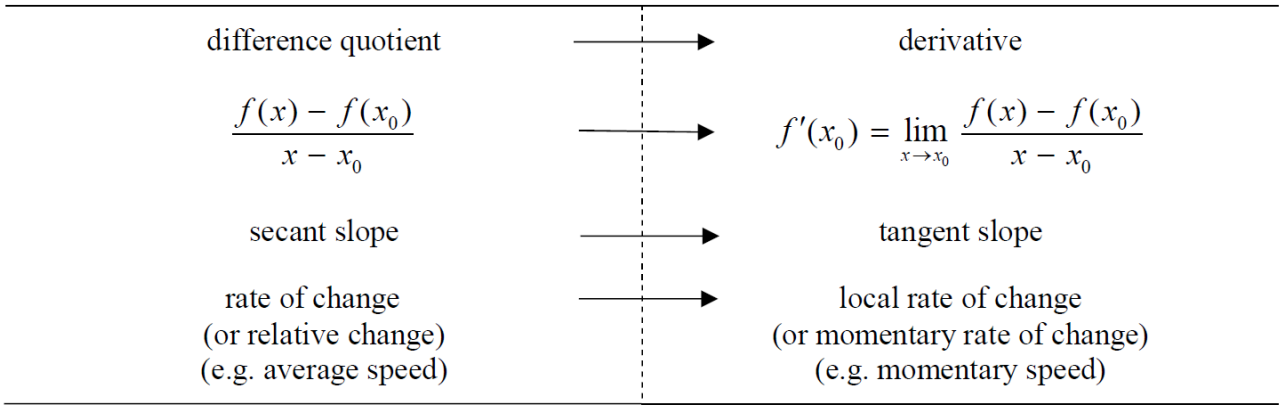

Figure 1: the transition from the difference quotient to the derivative

Here, the vertical dashed line denotes the transition from the algebraic to the analytic concept formation, which is, according to Danckwerts and Vogel, particularly difficult to realize in the mathematics classroom (p. 85).

In the author's view, this is why it is particularly important for the preservice teachers to develop a profound understanding of this transition from various perspectives. Therefore, the focus of this paper is not on a single analytical concept, but on the whole figure, in particular on the arrows going from left to right in Figure 1. Thus, the focus is on the *conceptions of the transition from the difference quotient to the derivative* that can be traced in the preservice-teachers' imaginary dialogues. The main question is: *What are the aspects of the preservice teachers' conceptions of the transition from the algebraic to the analytic concept formation?*

In the light of the framework of commognition, in order to initiate preservice teachers' reflection processes and investigate their conceptions of elementary analytical concepts and relations, a form of communication seems appropriate that allows for a "look from the outside". Imaginary dialogues, where one person writes what two protagonists are discussing, is such a form of communicating-on-communicating. Furthermore, imaginary dialogues are *written* works of a single student (or preservice teacher), but they consist of written *oral* dialogues. Therefore, imaginary dialogues display characteristics of both, written and spoken language (cf. Wille, 2017). In particular, the *processuality of spoken language* comes into play and students tend to write *how* they understand something instead of only what. Therefore, imaginary dialogues are one appropriate method to approach research questions that concern a transition process. Another reason is practicability: by using imaginary dialogues it is possible to receive reflections of each student of a course several times within the semester, which – although they are written – contain attributes of oral language. Moreover, the lecturer can react to the reflections within the course.

Method

Altogether 40 preservice teachers participated in two parallel courses "Didaktik der Analysis" at the Alpen-Adria-Universität Klagenfurt in Austria in the winter term 2015/2016. The content of the course was oriented on the book "Analysis verständlich unterrichten" (translation: teaching analysis comprehensibly) by Danckwerts and Vogel (2006). Five times within the course lectures, exercises, and reflections alternated. As reflection tasks the preservice teachers were given written initial dialogues that each of them had to continue in the form of an imaginary dialogue. In the following,

the second reflection task, the transition from the difference quotient to the derivative is in the focus. The initial dialogue given to the preservice teachers was the following:

Two preservice teachers S1and S2 are talking with each other. Continue the dialogue.

S1:	Can you please explain to me one more time what was important during the last hour?
S2:	Do you have a specific question? We talked about three approaches to the concept of derivation: one geometrical, one application-oriented and at the end, in addition, local linearization.
S1:	In the application-oriented approach there was the local change rate?
S2:	Yes, exactly.
S1:	That seemed to be particularly important for analysis teaching at school. But why?
S2:	You go from the actual amount via the absolute change to the relative change and – schwupp – you have the local rate of change!
S1:	Schwupp???
S2:	Excuse me, the "schwupp" really is not that simple and, actually, it is the crux of the matter. I'll explain it to you one more time precisely and after that why it is important for the teaching of analysis in school.
S1:	Thanks, because I cannot make any use of all these terms at the moment.

Figure 2: Initial dialogue task (the original German text was translated by the author)[1]

Of all participants, 30 preservice teachers wrote an imaginary dialogue by continuing the initial dialogue above. These written texts serve as the *data material* for the analysis. In the following the research questions that will be addressed are how the preservice teachers describe the transition from the algebraic to the analytic concept formation, and in particular, what the aspects of their conceptions of the transition are. The percentages named below are meant only as informative "mini-statistics", because of the small number of participants. Within the *data analysis* categories of aspects were built inductively. In a second step these aspects were categorized two perspectives. For a discussion regarding the depth of the preservice teachers' reflections, see Wille (2016).

Findings

Within the imaginary dialogues of the preservice teachers several aspects of the previous named transition can be detected. In what followed they will be explained and exemplified (all examples were originally written in German and translated by the author).

A1: The calculus aspect – describing the calculation of concepts with formulas

Eleven preservice teachers (31.4%) described with formulas how to calculate concepts from the left or right side of the transition. For example, a preservice teacher, Lydia, describes in her imaginary dialogue how the difference quotient and the derivative can be calculated with the help of formulas:

> "The next step is to build the difference quotient. As the numerator you calculate: $f(x) - f(x_0)$ and as the denominator $x - x_0$. The result is the relative change in the time span from x_0 to x. The relative change is also denoted by rate of change.

[1] "schwupp" in the sense of "voilà"

> And now the derivative comes into play. You build $f'(x_0)$. Now, $f'(x_0)$ is the limit (x approaches x_0) of the rate of change. The result is the momentary or local rate of change at the time of x_0."

In a similar way Paul writes about the calculation of the rate of change:

> "Okay. Now, we can determine the rate of change by a formula. The formula you can find in your documents, of course."

Or Maria writes about the momentary rate of change:

> "Now, we reached the last step. The momentary rate of change at the time of x_0. That is the first derivative, thus, $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$."

This aspect – i.e. describing the calculation of concepts with formulas – will be denoted as *calculus aspect* (A1), similarly to the calculus aspect of variables (cf. Malle 1993).

A2: The concept identification aspect – different concepts are identified

In the writings of Lydia and Maria another aspect becomes apparent: They are identifying concepts by their language use. Lydia identifies the concepts "difference quotient", "relative change", and "rate of change". Afterwards she identifies the concepts "derivative" and "momentary or local rate of change". Similarly, Maria identifies the concepts "momentary rate of change" and "first derivative". Altogether 16 preservice teachers (45.7%) identify concepts likewise.

Identifying different concepts is denoted as the *concept identification aspect* (A2).

A3: The limit aspect – the right side of the transition is described as a limit

17 preservice teachers (48.6%) describe concepts of the right-hand side as a limit, sometimes with mentioning a process of approaching (see aspect A4), sometimes as a "finished product"[2]. The first case (including the description of a process of approaching) appears in Elena's imaginary dialogue:

> "S1: I see. Now, I understand it and now it is clear for me why one writes $\lim_{\Delta x \to 0} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$. While letting Δx tend towards 0, or x towards x_0, respectively, while choosing the interval steadily smaller (note: here she crosses out "choosing it that small to be almost 0") one obtains the local rate of change, thus, the momentary speed at a certain point of time, if one brings the limit into play at this point.[3]"

Likewise, the process of building a limit is expressed by Jan: "Thus, the secant becomes, so to speak, a tangent by building a limit." In contrast, as a "finished product" the limit is used in the following writing: "And in step 3 the tangent slope will be calculated as a limit."

Describing the right side of the transition as a limit is denoted as the *limit aspect* (A3).

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[2] For distinguishing mathematics as a product with mathematics as a process (Danckwerts & Vogel, 2006) or concerning the duality of processes and objects (Sfard, 1991).

[3] The German original text is: "wenn man an dieser Stelle den Grenzwert ins Rennen führt".

A4: The approach aspect – the transition is described as a process of approaching

More than the half of the preservice teachers, 20 (57.1%) described the transition as a process of approaching. This can be seen for example in Elena's imaginary dialogue above ("While letting Δx tend towards 0, or x towards x_0, respectively, while choosing the interval steadily smaller", see aspect A3) or in Patrick's imaginary dialogue below ("we let the secant become shorter and shorter until it vanishes completely for $x = x_0$", see aspect A6).

This aspect – i.e. describing the transition as a process of approaching – will be denoted as the *approach aspect* (A4). It is *not* a subaspect of A3, because there are students who describe a process of approaching without describing the right side of the transition as a limit and the other way round.

A5: The distinction aspect – distinctions between the left-hand side and the right-hand side of the transition are described

17 preservice teachers (48.6%) described various distinctions between the left-hand side and the right-hand side of the transition. For example, the distinction of having first $x \neq x_0$ and then $x = x_0$ is described as follows:

S2: Think about it: You let x come closer and closer to x_0. What will happen then?

S1: They are equal eventually.

In another two examples the distinction between having a secant at first and then a tangent is described and additionally, having first two points and then only one on the function graph:

"From the secant it arises a tangent that does not intersect the function any longer, but touches it now in one point."

"(...) you don't want to have the slope of the secant between two points, but the slope of the tangent in only one point."

Finally, the distinction of having at first two points in time (in order to measure speed) and then one, a preservice teacher describes with these words:

"(...) we see that the secant expresses time intervals and we calculated within those the average rate of change[4]. The point P that we approach, is therefore the momentary rate of change at a certain point of time."

This aspect – i.e. describing distinctions between the left-hand side and the right-hand side of the transition – will be denoted as the *distinction aspect* (A5).

A6: The problem aspect – distinctions are considered problematic

Out of the 17 preservice teachers who described distinctions between the left-hand side and the right-hand side of the transition, 10 preservice teachers (28.6%) considered these distinctions as problematic (in their language use). For example, Patrick uses phrases like "we claim that it would be a smart result" or the adjective "mysterious":

[4] Original German text: "sehen wir, dass die Sekanten die Zeitintervalle ausdrücken und wir in diesen die mittlere Änderungsrate berechnen."

"S2: (...) Now, we have 2 problems, for one thing we begin our observation with $x \neq x_0$, but afterwards we set $x = x_0$ and claim that it would be a smart result. Secondly, we let the secant become shorter and shorter until it vanishes completely for $x = x_0$, but we claim that it became that way an infinitely long tangent (...) that is a somehow very mysterious and at least not illustrative (...)"[5]

Another preservice teacher let one protagonist face the impossibility of covering a distance at a point of time.

S2: How many seconds or minutes, respectively, had passed at the point of time?

S1: Well, none, that is why one names it POINT of time.

S2: And how can you cover a distance in 0 seconds?

S1: That was mean! Okay, nearly no time had passed, or rather so little time that it makes no difference.

This subaspect of A5 – i.e. considering the described distinctions problematic – will be denoted as the *problem aspect* (A6).

One could go more into depth at this point to differentiate the conceptions of the preservice teachers. For example, it is interesting to see which words are used to describe the process of approaching. But this would exceed the size of this article and therefore is left for a further and more detailed analysis. Similarly, misunderstandings that occurred in the imaginary dialogues cannot be addressed here, because of the length of the article. Instead, the detected aspects shall be compared and evaluated. Table 1 gives an overview of the aspects discussed above:

A1	*calculus aspect*	the calculation of concepts is described with formulas
A2	*concept identification aspect*	different concepts are identified
A3	*limit aspect*	the right side of the transition is described as a limit
A4	*approach aspect*	the transition is described as a process of approaching
A5	*distinction aspect*	distinctions between the left-hand side and the right-hand side of the transition are described
A6	*problem aspect* (a subaspect of A5)	distinctions are considered problematic

Table 1: aspects of the preservice teachers' conceptions

Discussion – vertical and horizontal perspectives

The aspects of the preservice teachers' conceptions detected above indicate two perspectives, *a vertical and horizontal perspective* regarding Figure 1. The aspects A1 to A2 are oriented vertically,

[5] This argument already goes back to Newton's academic teacher Isaac Barrow (1630-1677) (Danckwerts & Vogel, 2006, p. 48).

meaning that concepts or formulas are vertically linked (A1) or different concepts are identified either on the left-hand side or on the right-hand side (A2). Additionally, aspect A3, i.e. describing a limit, is oriented vertically, if the limit is seen without regarding the process of approaching. In contrast, the aspects A4 and A5 have a horizontal perspective, as does A6 as a subaspect of A5. The left-hand side of Figure 1 is linked with the right-hand side, when the transition is described as a process of approaching (A4), when describing distinctions between the left-hand side and the right-hand side of the transition (A5), or when considering those distinctions problematic (A6).

In the author's opinion, those texts that display the aspects A4 to A6 show a *content-related understanding* by the preservice teachers. In particular, when omitting the aspects A4 to A6, the understanding of the transition seems reduced to a temporal "and then" in the sense of: "first you calculate this and then you calculate that". Regarding the reflection depth of the imaginary dialogues (Wille, 2016), exactly those preservice teachers whose imaginary dialogues did not indicate horizontal-oriented aspects (31.4% of the participants) displayed the shallowest reflection depth.

In the course "Didaktik der Analysis" with the help of the imaginary dialogues, it was possible to react to imaginary dialogues, aspect diversity and misunderstandings during the semester. Additionally, some participants themselves regarded the time exposure negative, but mostly they perceived the writing of the imaginary dialogue positive as one preservice teacher wrote in a comment: "It was positive that first, while writing the reflection, it became really clear whether a topic was understood or not."

In summary, the method of imaginary dialogues turned out to be helpful to detect different aspects of the conceptions regarding the transition from the difference quotient to the derivative. In the author's opinion, a profound understanding of preservice teachers should display various perspectives, horizontal and vertical, in order to be able to address the diversity of the students' conceptions later in class.

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Functions: Implicit

Vivica Zweidar

University of Bremen, Germany; vivica@math.uni-bremen.de

What is the role of the implicit in teaching and learning functions? Functions are a central topic in mathematics classrooms in Germany. The mathematical concept of a function is typically represented by graphs, tables of data, algebraic expressions or by verbal descriptions. Formal definitions of functions only play a minor role in school mathematics. This is one reason why the mathematical content of what a function is, remains implicit when teachers and students talk about functions. Another reason for being implicit about functions in classroom conversation is inherent in the structure of language: in many utterances the implied meaning differs from the literal meaning. I use Swan's (1982) model of translation skills and Grice's (1975) concept of implicatures in order to analyse videotaped classroom conversation in a unit on functions. In this paper, I sketch the concept of implicitness in mathematics lessons.

Keywords: Functions, classroom conversation, discourse analysis, implicit meaning, implicature.

Introduction

In my research on the implicit in classroom conversation about functions I seek to describe the discourse of learners and teachers in order to reveal the hidden, the unsaid, the implied meaning that has to be decoded to create and follow lessons sensibly. I want to characterize the interplay between the implicit in the mathematical object and the implicit in language when talking about functions. My research questions are: How are functions brought up in classroom conversation? And: What is the role of the implicit in classroom conversation in a unit on functions?

It is common sense that language is important for all school subjects. Mathematics is no exception (see for example Meyer & Prediger, 2012) – it is needed to get access to mathematical matters and to express mathematical concepts. Participating in classroom conversation and getting access to the mathematical content of mathematics lessons can become difficult for the participants on different levels. Apart from obstacles that are immanent in mathematical concepts there may be vocabulary or grammar difficulties, difficulties in the semantics as well as difficulties in a pragmatic sense when learning new contents. In my work, I focus on a pragmalinguistic perspective on classroom conversation about functions. This perspective takes into consideration that not only words or sentences are meaningful when talking but also the context of an utterance. Looking at language that way promises to reveal an unsaid meaning that can be found between the lines. The idea of making the unsaid or the unsayable come to the surface by analyzing the classroom discourse seems conveniently suitable to approach the processes in mathematics classrooms when abstract objects like functions are tried to be made accessible.

Didactical analysis

In order to link my language analysis to the mathematical content, I analyze the lessons and especially the classroom conversation on a didactical level at first. I seek to figure out what is presumably intended to be taught and learned about functions when working on different tasks and when talking about them.

The mathematical object *function* is very abstract. Looking at the historical development, the understanding of what a function is demonstrates how much functions are attached to their representations (Malle, 1996). Till today in German classrooms functions are a central topic in the curriculum where traditionally the representations like algebraic expressions, graphs, tables of data and (realistic) situations of functional dependencies play an important role. Swan (1982) developed a model to describe the different activities, the so-called translation skills that have to be performed when transforming one representation into another. In my work, I categorize the classroom activities into sequences according to those translation skills. I focus especially on the skills that afford translating a representation of a function from or into situations that are generally closely linked to verbal descriptions. The activities when a representation is transformed from a situation can be summarized as modeling skills and activities when a representation is transformed into a situation can be summarized as interpretation skills.

Furthermore, different aspects of a function can become central in different tasks: you can classify the thinking about function into the following categories (Vollrath, 1989): functional dependency as a point wise relation, functional dependency as a dynamic process and functions viewed as objects or as a whole. The point wise relation takes into consideration that one independent element of a set gets mapped to one dependent element of a set, the dynamic process stands for the change that is produced in the dependent values when changing the independent values and functions as objects look at the given or produced correlation as a whole. Each activity or rather each translation skill can be performed on all of these three levels and form the second dimension for the didactical analysis of the lessons in my investigation.

In *The example*, I demonstrate how I work with this analysis in combination with the linguistic analysis in order to get a description of the implicit in the classroom conversation on functions.

Linguistic analysis

For the linguistic analysis, I use Grice's *theory of conversation* (Grice, 1975) as the background theory. Central in his theory are so-called implicatures – very generally speaking an implicature is existent when something that is literally said by an utterance differs from what is meant by that utterance. An implicature analysis, as for example suggested by Hagemann (2014), reveals the implicit and what is likely to be meant and understood by the participants. Pimm (1994) claims that Grice's implicatures are also relevant in classroom discourse and Rowland (2002) presents analysis of students' utterances in mathematics discourse using Grice's ideas as a starting point for his theoretical framework.

Grice's concept of implicatures is based on the cooperation principle:

> Make your conversational contribution such as is required, at the stage at which it occurs, by the accepted purpose or direction of the talk exchange in which you are engaged (Grice, 1975, p. 45).

Under this assumption nothing that is uttered by a participant of a conversation is assumed to be meaningless and the speaker of the utterance supposes that the conversation partners can decode the utterance meaning. The cooperative principle is formulated very general and leads to more specific aspects about how understanding is produced in conversation. These aspects are the so-called

conversational maxims. Grice formulates twelve maxims[1] that are categorized into the maxims of quantity, quality, relation and manner. Whenever one of the maxims is violated this is a hint at the actual meaning of an utterance. While talking the flouting of maxims is an intuitive process that leads to understand an utterance in a specific way. An implicature analysis explains how a particular intuition is evoked. The following example illustrates the formation of an implicature by infringing the maxim of relation *be relevant*: The setting of the following conversation between two students is a lesson at school. The students are supposed to write down a solution to a textbook task.

(1) A: I do not have a pencil.
B: My pencil case is on the table over there.

In example (1) A implicates that she or he needs a pencil and B implicates that A can find one in the pencil case on the table. Both utterances only indirectly hint at the presumably intended meaning. The literal meaning is simply that one student does not have a pencil and that there is a pencil case on the table. The literal meaning is not very likely to be the intended meaning as this meaning would not be relevant in the given context.

To confirm the presence of a conversational implicature in Grice's sense implicatures need to have two characteristics – non-detachability and cancelability. These attributes can be tested as follows (Korta, 1997): the implicature is non-detachable if you find synonym ways of saying the utterance without changing the implicature. In example (1) you could change A's utterance into "I do not hold a pencil." The implicature then remains that A needs a pencil. And the implicature is cancelable if you can neglect the implicature without producing a conflict with the literal meaning. For A in example (1) it is possible to say: "I do not have a pencil, but I do not mean that I need one. In fact, I do not want to write something down."

Both utterances in (1) could have been expressed explicitly, for instance by the following exchange:

(2) A: Please lend me a pencil.
B: You can use one of my pencils – take one out of my pencil case on the table.

In this case, the literal meaning and the intended meaning coincide and no implicature is used in this excerpt.

Hagemann (2014) formulates an implicature analysis that refers to Grice's theory of conversation. It follows three steps, which I also use for my analysis: Firstly, contextual elements such as disambiguation and referent assignment have to be determined. Secondly, suspected implicatures have to be tested on cancelability and detachability. Thirdly, a sequence analysis confirms what implicatures are most likely. In example (1) the implicature could be as already suggested above "I need a pencil", but it could also be "I cannot (and I do not want to) write down my solution to the textbook task". The answer of B allows both possible implicatures, but depending on further reactions that may follow this little conversation it can be that one or the other is more plausible. In other words: in the third step you look at what happened before and after the utterance to find out what is most likely intended and understood.

[1] Grice's (1975) maxims of conversation: make your contribution as informative as is required, do not make your contribution more informative than is required, try to make your contribution one that is true, do not say what you believe to be false, do not say that for which your lack adequate evidence, be relevant, be perspicuous, avoid obscurity of expression, avoid ambiguity, be brief, be orderly.

The example

In the following paragraph, I want to demonstrate how I analyze sequences of lessons in order to give a detailed description of what remains implicit in classroom conversation on functions.

The setting of the discussion

Participants in the presented classroom conversation are the members of a course in secondary school with 29 students and their mathematics teacher. The students are in the 11th grade on a high level, preparing for the last two school years in order to get the Abitur[2]. The classroom is equipped with a black board that is used frequently during lessons by the teacher as well as by the students. Tasks are generally handed in in written form on worksheets. The teaching methods and the social forms vary depending on the task. The researcher participates during the lessons only as an observer and does not interfere. The lessons are designed by the teacher without any collaboration with the researcher in order to observe lessons as close as possible to usual lessons.

The task – Walking functions

The task *walking functions* and the excerpts of the conversation on that task are taken from the first lesson in a unit on functions in the course introduced above.

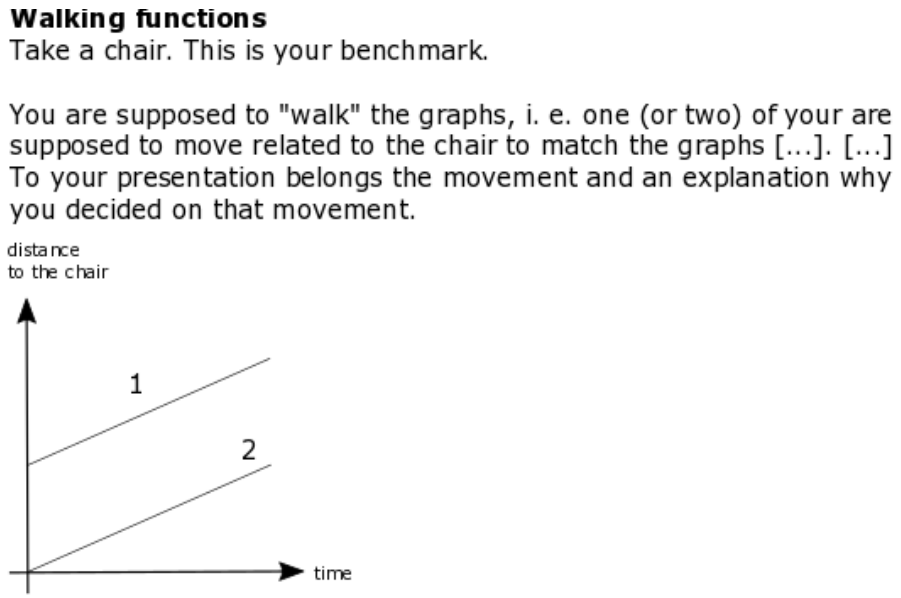

Figure 1: The task – *walking functions*

The task *walking functions* requires transforming a graphic representation of a function into a suitable realistic situation. That means given graphs have to be interpreted. In the setting when a group presents its movements also the reverse is required for the observers: movements have to be referred to graphs. In this task interpretation skills are in the focus while modeling skills only come

[2] The Abitur is the highest graduation in the German school system that forms the general qualification for university entrance.

up to a lesser extend. The transformation of a graph into a movement demands to have a view on the function as a whole and not only on a certain point of the underlying function. In the example above (see Figure 1), the meaning of straight lines, the meaning of the intersections between the graphs with the ordinate as well as meaning of the parallelism of the graphs must be considered. Also the change of distance to the chair in relation to the change of time has to be taken into account.

When having two representations of a function, it is not immediately obvious that those two representations stand for the same function. In this example it can only be seen implicitly by making clear what certain characteristics of a graph mean in the context of the movement in relation to the chair. The second part of the task asks to give an explanation why the group decided on their specific movement. It is asked to make the relation between two representations explicit.

In my analysis of the classroom situation I seek to reconstruct what is made explicit and what remains implicit when an accepted interpretation of the translation from graph into a situation is debated.

The conversation about the task

During the lesson the students worked in groups of three on the task *walking functions* for about 15 minutes. After working on the task in the groups, three different presentations of movements and explanations were performed. All three groups had different ideas:

In the first group, two students started their movement a few meters away from the chair and walked constantly to the chair. One of them stopped at the chair, the other one stopped at the same time but at some distance to the chair. Their explanation was the following: “Well, we did it that way- well, Paula was the one and Paula started to walk a little earlier because the graph does not start at the zero point. And then Luna came along and then they walked in parallel because they run parallel.”[3]

In the second group, two girls both started in line with the chair and walked vertically to that imaginary line away from the chair keeping the same distance between them. They commented on their movement. A: “Well, not the way that we walk apart, but parallel. But it goes (up) then.” B: “[…] Because we thought that the x-axis is sort of the position of the chair.”[4]

In the third group, two students walked one after another constantly away from the chair. They started and stopped their movement at the same time and explained: “Well, firstly they did not start at the same point because ehm well you have you see on the y-axis that someone starts further

[3] Translated from the German transcript by me. Original: “Also, wir haben die so gemacht weil- also Paula war die eins und Paula ist ein bisschen früher losgelaufen, weil der Graph ja nicht im Nullpunkt startet. Und dann ist Luna mitgekommen und dann sind die parallel gelaufen, weil die parallel verlaufen.”

[4] Translated from the German transcript by me. Original: A: “Also so halt nicht, dass wir auseinanderlaufen, sondern parallel. Aber es geht dann (hoch).“ B: Weil wir gedacht haben, dass ja die x-Achse sozusagen die Positon vom Stuhl ist. Dann geht das ja davon weg.”

ahead, well that he is further away from the chair. […] And they walked at the same speed because the graphs run quasi parallel against each other."[5]

In the sequence outlined above the students discuss different aspects of the functions in the task. At that point it is not clear which movements and explanations are accepted as a correct translation from the graphic representation into the movement. Subsequently to the presentations, a classroom conversation starts and it is clarified what is considered to be a correct solution to the task. At the end of the discussion, the third group is considered to have presented the best matching movement.

In the following paragraph, I try to reconstruct the process to come to this conclusion and thereby I focus on what remains implicit. The expressions in curly brackets are my interpretation of the intended meaning or rather the implicatures of the utterances.

Finding a suitable movement for the graphs in Figure 1 is not distinct and the different presentations show that the students pick out some characteristics of the graphs and refer them to the movement. All three groups relate the parallelism of the graphs. The first states: "[…] they [the girls] walked in parallel because they [the graphs] run parallel." Presumably they implicate "they [the two girls] walked in parallel because {that is the meaning when} they [the graphs] run parallel." The second group also points out that walking in parallel is the correct transformation: "Well, not the way that we [the two presenting girls] walk apart {by mistake}, but parallel." The third group is the first that relates walking at the same speed to the parallel running graphs: "And they [the two presenting students] walked at the same speed because {that is what is meant when} the graphs run quasi parallel against each other." In all three explanations they mention different characteristics of the functions apart from the parallelism. That is a hint that they see the function as a whole even though not entirely as they miss mentioning some of the functions' relevant characteristics that could be observed in the graphic and situational representation. By their performances and their utterances, the groups made clear what they assume to be the correct relation between the representations. They claim this connection without making explicit why that relation is supposed to be correct.

After these three presentations the teacher asks several questions: "What is the origin of the coordinate system? What is on the x-axis? What is the y-axis? And what does parallel mean?"[6] The teacher seems to pick out the characteristic of the functions she considers to be relevant in this task. When looking at the conversation after these questions you can identify the implicatures: "What is {the meaning of} the origin of the coordinate system {in relation to the situation}? What is {the meaning of} the x-axis {in relation to the situation}? What is {the meaning of} the y-axis {in relation to the situation}? And what does parallel mean {in this context}?" She implicitly focuses the conversation on the relations between the representations. The relations between the representations are discussed for about four minutes. Then the teacher also brings a new possible

[5] Translated from the German transcript by me. Original: "Also erstmal, ähm, sind die Beiden nicht vom gleichen Punkt gestartet, weil ähm, ja weil hat man halt auf der y-Achse sieht, dass jemand weiter vorne startet, also dass er weiter vom Stuhl entfernt ist. Und dann halt so bald neben dem Stuhl. und die sind beide gleich schnell gelaufen, weil die ehm Graphen halt quasi parallel aneinander laufen."

[6] Translated from the German transcript by me. Original: "Was ist der Ursprung des Koordinatensystems? Was ist auf der x-Achse? Was ist auf der y-Achse? Und was heißt das mit dem Parallel?"

relation not mentioned before into the discussion: “The graphs go up. That means you walk away from the chair. But it [the movement] is getting faster because {that is meant when} it [the graph] goes up. What do you think of this argument?”[7] This utterance brings the students to see the rising of the graph as the speed of the movement and finally to declare that the movements of group three are best matching to the graphs. At no point in the discussion the parallel graphs are related to movements with the same speed again as suggested by group three at the beginning. Although the parallelism is brought into discussion several times the relation between the parallel graphs and the movements with the same speed remains implicit till the end of the conversation sequence about the task *walking functions*.

Conclusion and discussion

Such a dense description of the activities in the classroom with the focus on the implicit has the aim to make the implicit accessible. It can be seen as an objectification of the assumptions and normative demands that resonate subliminally in the lessons. In the presented classroom conversation, the convention that parallel running lines in a path-time diagram stand for movement at the same speed is initiated. This relation between the two representations is declared correct by employing many implicit relations and implicatures. This shows that decoding the implicit in the classroom conversation is relevant. An analysis as demonstrated above can help to understand the demands on teachers and students in the classroom to create and follow lessons sensibly.

This analysis of the episode in this paper is just the beginning of my research, my starting point of looking at mathematics lessons with a specific lens – with a lens, which seeks to make invisible demands in mathematics discourse visible. Comparing the analyses of different situations with a similar mathematical content can reveal how the implicit is used in mathematics classroom systematically and give insight into the role of the implicit in teaching and learning of functions. With the help of these analyses I want to develop a local theory on implicitness in classroom talk on functions that links Grice’s vast theory of implicatures to the mathematical topic of functions in mathematics classrooms. Whether the evolving theory holds will not be part of this research, but should be probed by using it for broader classroom observations for example.

Didactical implicatures for mathematics lessons deriving from insights in the meaning of the implicit cannot be given at this point of my research. Only as a final result of further analyses ideas for teaching and learning may be devised. For now, I can conclude that implicatures play a role in classroom conversation on functions and that I am curious about finding out more about the relevance of the implicit in mathematical classroom conversation.

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[7] Translated from the German transcript by me. Original: “Die Graphen gehen ja hoch. Das heißt, man geht vom Stuhl weg. Aber es wird schneller, weil es ja hoch geht. Was sagt ihr zu diesem Argument?”

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Which textual features are difficult when reading and solving mathematics tasks?

Anneli Dyrvold

Umeå University, Sweden; anneli.dyrvold@umu.se

Keywords: Mathematics tasks, reading, solving, textual features.

Despite the digital revolution much of the mathematics practiced in schools is still tightly bound to two-dimensional texts. This emphasis on text is neither surprising, nor inadequate, since mathematics has developed through a long history with the use of written text, consisting of natural language, mathematical notation and images. Natural language is our native language consisting of letters and words (see e.g., www.oed.com). Different features of the mathematics text are also important in written tests, since reading the text is part of the assessment. If the text is hard to read, that difficulty can be relevant as part of assessing the communicative competence in mathematics. Crucial is, however, whether potentially difficult textual features are part of what the assessment aims at. This issue is investigated in the current study, using a synthesis of statistical results and qualitative analyses of task text.

A critical question is where to draw the line between necessary and unnecessary reading demand and how to judge which textual features are irrelevant and therefore should be avoided in mathematics assessments. In the current study this aspect of reading demand is addressed through a small meta-analysis of four studies where different textual aspects in task text are analyzed in relation to task difficulty and task reading demand. The theoretical starting point for the current research is an understanding of language as an essential part of mathematics. It has been argued theoretically that the understanding of a mathematical object develops as the student develops her or his discourse on that object (see e.g., Sfard, 2008). An understanding of mathematics discourse as part of what mathematics is, is in line with the theoretical interpretation of the statistical measure for *demand on reading ability* (DRA) used in the studies included in the meta-analysis conducted in the current study. DRA is a measure of the unnecessary reading demand in a mathematics task, and within this interpretation lays also an assumption of a kind of reading demand that is relevant in mathematics tasks (see also Dyrvold, Bergqvist, & Österholm, 2015). The purpose of the study is to contribute to the knowledge about which textual features in tasks are demanding and whether that difficulty is a mathematics relevant difficulty. The research questions are: i) what conclusions can be drawn regarding reading demand in mathematics tasks in relation to textual features?, and ii) how can the conclusions based on statistical analyses be interpreted in relation to a qualitative analysis of mathematics task text with a high reading demand?

The study consists of a meta-analysis and a qualitative analysis of tasks that stand out in the quantitative analysis. Only four studies are included in the meta-analysis but even such a small meta-analysis do contribute to the development of knowledge since the analysis enables conclusions to be drawn that would not be possible to draw with-out such an analysis. The qualitative analysis has a systemic functional perspective (Halliday & Matthiessen, 2014) and includes also images and mathematical notation.

The meta-analysis focuses on textual features in relation to two quantitative measures; task *demand on reading ability* (DRA) and task *difficulty*. Results in relation to those variables (difficulty and DRA) are relevant to interpret together since they represent different aspects of how a task can be demanding. The measure DRA is obtained through a principal component analysis (PCA) on students' results on PISA reading and mathematics tasks. The result of the PCA is several components that explain different parts of the results on the tasks. The components are statistically disjoint, and therefore the DRA represents demand on a reading ability that is not part of a mathematical ability (see also Dyrvold et al., 2015). Through the analysis, every PISA mathematics task obtains a loading value on that component, a value interpreted as the tasks DRA.

The results reveal several features of the natural language that distinguishes tasks with a high DRA, but also that the images are more tightly integrated with the sentences in tasks that have a low DRA but are difficult to solve. For tasks with high DRA, the sentences are knitted together through the Themes (the topic of the sentence) and Rhemes (what is presented in relation to the theme) something that is not as pronounced in task with low DRA (Theme and Rheme are explained by e.g., Halliday and Matthiessen, 2014). One example of that can be found in the following sentences. The Themes are underlined. "The sculpture is a half circle with the radius 2m. The half circle is inscribed in a square." Those sentences represent a linear progression since the Rheme of the first sentence becomes the Theme of the next sentence.

The results from the meta-analysis reveal other features than the natural language (words and letters) that are related to difficulty but not to DRA. Tasks with high DRA and tasks with low DRA are alike when it comes to presence of natural language, images, and symbols but for tasks with a low DRA there are more references within natural language and between natural language and images or symbols. In summary, the textual analyses reveal features of the text in tasks with high DRA that enlighten what the high reading demand may stem from, since the textual analyses indicate that the progression between Themes and Rhemes can be a distinguishing feature for tasks with high DRA, whereas references to images may not play such a role.

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Language and risk literacy

Dianne J. Kenton

Brock University, Faculty of Graduate Studies, Canada; dianne.kenton@brocku.ca

Keywords: Language, risk literacy, developmental research project.

Recent studies on adult literacy and numeracy raised several key issues some of which are (a) over the last 20 years there has not been significant improvement in adult literacy and numeracy in Canada; (b) regions within Canada will not have enough post-secondary graduates with sufficient literacy skills to fill the jobs created by the Canadian economy; (c) declining numeracy may be due to gender discrimination and skill mismatch in the workplace which in turn, contributes to the shrinking pool of skilled labour; and (d) currently, there are no existing numeracy programs for non-STEM students at the postsecondary level to address the issue of innumeracy or risk illiteracy (OECD, 2013). Like statistical literacy, risk literacy requires familiarity and comprehension of the statistical and risk lexicon that is often confused and/or misused in the public sphere. Bolton (2010) remarks that published statistics are inappropriately used when they are presented without context, use confusing terminology, and misuse or use ambiguous terms. He adds further that, the concept of *uncertainty* is often "lost, forgotten or ignored by authors" creating misleading and/or inaccurate estimates that do not reflect the entire scenario.

Theoretical framework

Mathematical instruction and comprehension of abstract concepts such as risk, are executed through language and use conceptual metaphor. Language clarifies meaning, making meaning more precise, and executes an operative (process) role in thinking. According to Jurdak, Vithal, de Freitas, Gates, and Kollosche (2016), any level of mathematics instruction is mediated through language. Nunez (2007) explains that, mathematics is predominantly metaphorical and to make abstract mathematical concepts concrete requires the use of conceptual metaphors which are language and cognitive devices. He defines a conceptual metaphor, as a "cognitive mechanism that allows us to reason about one kind of thing as if it were another", more specifically it is a "grounded, inference-preserving, cross-domain mapping, neural mechanism that allows us to use the inferential structure of one conceptual domain (e.g. geometry) to reason about another (e.g. arithmetic)" (pp .4-6). The instructional resource examines language and conceptual metaphors used in risk instruction focused on developing skills in decoding and understanding publicized risk in legal, social, financial or medical contexts.

Methods and findings

The methodology of this study was guided by McTighe and Thomas (2003), *Understanding By Design* (UbD) framework for curriculum planning and design that is to (a) identify the desired skills or outcomes required (b) determine the assessment evidence and (c) plan the experience or instruction. The content of the instructional resource comes from the results of literature review and web research, information interviews, document analysis, government publications, curriculum documents, expert critique and informal information gathering from practitioners in the field. Cumulative and summative assessment rubrics in the resource were developed based on Facione

(2011) critical thinking assessment. A small group of independent reviewers examined a lesson on risk and were asked to provide independent written feedback to the lesson questionnaire. The resource was further revised with respect to its structure, content and recommended instructional methodology and reviewer feedback was included in the pilot lesson.

Reviewers commented that the instructional resource was relevant, well-constructed, and raised awareness of risk miscommunication. Respondents agreed the lesson had a logical flow, demonstrated progression and that the background information was indeed necessary to provide context, to follow presented arguments, and to comprehend and make sense of data. All respondents agreed with the type and appropriateness of the lessons assessment also remarking that the self-assessment was very important to verify their own understanding. The findings from this development study confirmed that basic risk comprehension can be achieved by rephrasing risk and expressing probability as a natural frequency within a specified context.

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Professional competences of future mathematics teachers concerning the role of language in mathematics teaching and learning

Nadine Krosanke

University of Hamburg, Germany; nadine.krosanke@uni-hamburg.de

Keywords: Language, mathematics teacher education, vignettes.

The state funded project 'Professional teaching practice to promote subject-related learning under changing social conditions' (ProfaLe) aims at improving teacher education at the University of Hamburg at various levels: One objective is to sensitize future mathematics teachers to language difficulties in mathematical learning processes and tasks as well as to promote professional competence in helping students to tackle possible learning barriers based on language difficulties. For this purpose, courses, which integrate aspects of language learning into mathematics education, are going to be developed and their effects will be evaluated in an ongoing PhD study using video-vignettes and interviews.

International large-scale studies like PISA and TIMSS and other studies have repeatedly shown that a connection between the first language, language proficiency and the mathematical performance of students exists. Especially skills referring to a language register described by Gogolin (2009) as academic language (so-called 'Bildungssprache') have proven to be vital for educational success. In mathematics classes different language registers are needed and used: everyday language, 'Bildungssprache' and mathematical language. Halliday defines register as "a variety according to use; a register is what you speak at the time, depending on what you are and the nature of the activity in which the language is functioning" (Halliday, 1978, p. 31). Consequently there is no translation from one register into another without shifts in meaning and function, but it is often not possible to make a clear separation between the registers. The concept of 'Bildungssprache' refers to the ability to employ language skills in order to access knowledge and participation in education discourse (Gogolin, 2009). Language in mathematics teaching and learning has not only a communicative, but also a cognitive function (Maier & Schweiger, 1999) and is for instance very important for developing mathematical concepts. Although teachers should promote language skills according to the German national standards, language is seldom addressed explicitly in ordinary classroom activities (Schütte & Kaiser, 2011) and future mathematics teachers at the University of Hamburg are currently not obliged to attend courses focussing on the subject-specific role of language in teaching and learning. For this reason, elements of inclusive language teaching are going to be developed, tested and implemented into two consecutive courses at the master studies in mathematics teacher education.

The first course refers completely to the role of language in teaching and learning mathematics and offers opportunities to learn about linguistically diverse students from the perspective of intercultural education, applied linguistics and language teaching. The second course accompanies the school internship and, therefore, is more general about mathematics education and pedagogical aspects such as classroom management, nevertheless continuously considering language and its learning. The future mathematics teachers will be enabled to recognise different registers and related to that potential language barriers in written mathematics texts as well as in text-production tasks. For that

aim future teachers analyse mathematical tasks and text-vignettes as well as video-vignettes displaying mathematical solving processes of students. The future teachers will be familiarised with the approach of *Scaffolding* (Gibbons, 2002), which has proven to be effective for language-based learning difficulties and plan lessons or parts of it combining language and mathematics learning. In the second course the future teachers are developing even more awareness of the role of language by reflecting their own or joint classroom observations i.a. in the accompanying course at the university.

The ongoing PhD project aims at examining the changes concerning noticing and beliefs of the future teachers. The main research question is: to what extent can relations be reconstructed concerning the awareness of the role of language in mathematics teaching before and after an intervention? Before and after every of the two courses semi-structured interviews will examine which beliefs about the relationship between language and mathematics teaching the future teacher hold and how these beliefs may change. Based on the situated approach for measuring competencies by Blömeke, Gustafsson and Shavelson (2015) and the concept of noticing (Sherin, Jacobs & Philipp, 2011) the project presented here aims at coming closer to the measurement of the performance of the future teachers by evaluating situation-specific skills with a video-vignette. Interviews (pre-and-post) based on a video-vignette will provide data which linguistic aspects of teaching and learning mathematics future teacher notice, how they would decide to react in a specific situation and how this may change due the two courses. All data will be analysed by qualitative text analysis (Kuckartz, 2014).

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Getting language awareness: A curriculum for language and language teaching for pre-service studies for teachers of mathematics

Carina Rauf and Barbara Schmidt-Thieme

University of Hildesheim, Germany; rauf@imai.uni-hildesheim.de; bst@imai.uni-hildesheim.de

Keywords: Language, teacher education, teacher professional knowledge.

Teaching and learning mathematics is based on the use of language: for introducing and defining new mathematical objects, discussing different ways of calculating, documenting the results of a proof or explaining how to handle teaching materials, different representations are used, but almost always accompanied by language. Language as a medium and an aim of mathematics lessons at school is a well-known object of didactic research. In the last years one can observe a focus on differentiations and transitions between different languages (or registers of one language) used in mathematics (Duval, 2006) or questions of teaching in multilingual classrooms (Ahrenholz, 2010; Prediger, Clarkson & Bose, 2016). On the other side, the professional knowledge of teachers, and questions on how this can be developed in pre-service and in-service lessons gained considerable interest. Beside others "Explaining" was recognized as an important factor of effective instruction (Kunter et al., 2013; Vogt, 2009). Bringing this together, the question is, which language-related competencies a teacher has to have as part of his professional knowledge and when, where and how he can achieve this. Therefore, several universities in Germany designed and developed exemplary trails in preservice teacher-studies in the last years. In most of the cases these are extra, but obligatory courses on "German as second language". Another approach is followed for example by a project called "Umbrüche gestalten – Sprachen bilden Niedersachsen" (languages educate Lower Saxony) (http://www.sprachen-bilden-niedersachsen.de/index.php/projekt.html) of all universities with teacher education in Lower Saxony.

A "Language Curriculum" (SuM_MaSt)

Following the approach of the project "Umbrüche gestalten" we developed a pre-service "language curriculum" for teachers of mathematics (based on experiences of the implementation of language studies in teacher studies mentioned above) following four assumptions for learning opportunities (used to guide the curriculum): Learning tasks should:

(a) be spread from the beginning to the end of the academic studies, like a vertical spiral curriculum,

(b) be integrated in mathematical lessons, for the technical language of mathematics is best learned by doing mathematics; certainly, there are also a couple of explicit language-related courses,

(c) evoke an active and reflective handling with language in learning situations and be applied and tested in authentic situations,

(d) include individual feedback and allow some comparative measurements.

And in addition: It should be transferable to other designs of academic studies for teachers at other universities and to other subjects.

Among the language and language learning competencies are the ability to use the language of mathematics and some of the other languages used in schools (e.g. colloquial language or "Bildungssprache"). This needs to be done adequately to learners and learning situations. After some pre-studies the design based research project SuM-MaSt started fully 2016 at the University of Hildesheim. Around 20 learning task were implemented:

- tasks integrated in mathematical lectures, for example using different registers and representations while explaining main topics, verbalize formulas and write down explanations for pupils, deal with historical sources in the original language (feedback/evaluation (qual.): correctness of the mathematical content and adequacy of language)
- explicit courses about "Language and Mathematics" contenting representations of mathematical objects and concept building, communication and argumentation, language sensitive teaching material; (feedback/evaluation: analysis of the results during the lessons)
- theoretical inputs in lectures, for example "Explaining" in the lecture "arithmetic" (feedback/evaluation (qual.): describing the development of linguistic competencies)
- practical exercises in schools.

Beside further designing, implementing and evaluating of language-focused tasks in the mathematic curriculum in the next terms we will take first steps to a transfer of these tasks to other subjects, beginning with the natural sciences.

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TWG10: Diversity and mathematics education: Social, cultural and political challenges

Introduction to the papers of TWG10: Diversity and mathematics education: Social, cultural and political challenges

Lisa Björklund Boistrup[1], Nina Bohlmann[2], Javier Díez-Palomar[3], David Kollosche[4], Tamsin Meaney[5]

[1]Stockholm University, [2]Freie Universität, Berlin, [3]University of Barcelona, [4]Goethe University Frankfurt, [5]Western Norway University of Applied Sciences

Scope and focus

Thematic working group 10 is interested in discussing diversity and mathematics education within the realms of the societal, the cultural, and the political. In the work of the group, mathematics education is assumed to refer to more than the encounter between an individual and a mathematical object and is considered to occur in wider contexts than just classroom settings. The group is specifically interested in discussing research that addresses how diversity affects students´ possibilities to learn in mathematics education. Diversity also occurs in relationship to who is doing the research and who is being researched, posing methodological issues of an ethical nature. Hence, multiple diversities intersect, and in so doing pose challenges to intended and actual learning and teaching practices in their multiple forms.

Organisation of TWG 10's work

In the seminars during CERME10, papers were presented in a similar way to what had occurred previously, in that the authors did not present their own paper. Instead each paper was presented by another author giving a description of the main ideas from the perspectives adopted in the paper. The author(s) then had a few minutes to add to or comment on the presentation, with the possibility of pointing out or emphasising important aspects. In the end of each session, there was time for discussing the presented papers. These discussions firstly occurred in small groups and then were shared in the whole group. A poster session with 5 posters was held, in addition to the general CERME poster session. Here each author had 3 minutes to describe the content of their poster. The poster authors then positioned themselves next to their respective posters to engage in discussions with group members.

The papers discussed

Below are the papers presented during our sessions with the respective presenter(s).

Papers and authors	**Presenter**
How Sámi teachers development of a teaching unit influences their self-determination by Anne Birgitte Fyhn, Tamsin Meaney, Kristine Nystad, Ylva Jannok Nutti	Anette Bagger
Importance and possibility of integrating gender competence as a key qualification in mathematics teacher education by Anina Mischau, Katja Eilerts	Helena Roos
Cultural diversity as a resource or an obstacle for teaching practices	Tamsin Meaney

in multicultural milieu: Experience of a training course for Italian teachers about Chinese Shuxue by Benedetto Di Paola, Giovanni Giuseppe Nicosia	

Table 1: Session 1.

Integrating critical theory and practice in mathematics education by David Swanson, Laura Black	David Kollosche
Incepted neoliberal dreams in school mathematics and the Chilean experience by Melissa Andrade-Melina	Anne Birgitte Fyhn
Towards cultural responsiveness in mathematics education by Swapna Mukhopadhyay, Brian Greer	Hilary Povey
Content-related and social participation in inclusive mathematics education by Judith Jung, Marcus Schütte	Colin Jackson
Social class and "ability" grouping in mathematics in English secondary schools: A review by Colin Jackson	Sabrina Bobsin Salazar

Table 2: Session 2

Subjective theories of teachers in dealing with heterogeneity by Elisa Bitterlich, Judith Jung, Marcus Schütte	Benedetto di Paola
"No, it just didn't work": A teacher's reflections on all-attainment teaching by Colin Jackson, Hilary Povey	Javier Díez-Palomar
The socio-politics of teacher explanation in mathematics education by David Kollosche	Elisa Bitterlich
Diversity in an inclusive mathematics classroom by Helena Roos	John Keogh
“It is only a test”: Social aspects of displaying knowledge in mathematics for Second Language Learners (SLL) by Anette Bagger	Melissa Andrade-Melina

Table 3: Session 3

Polish parents and mathematics education in Swedish preschools by Dorota Lembrér	Judith Jung
(Wanting to do) Ethical research in a shifting context by Andrea Eikset, Trude Fosse, Troels Lange, Johan Lie, Magni E. H. Lossius, Tamsin Meaney, Elena Severin	David Swanson
Mathematics at the enterprise: Industry, university and school working together to facilitate learning by Hans Kristian Nilsen,	Brian Greer

Anne Vegusdal	
The context of workplaces as part of mathematics education in vocational studies: Institutional norms and (lack of) authenticity by Lisa Björklund Boistrup, John Keogh	Alex Montecino

Table 4: Session 4

School mathematics education through the eyes of students in Ghana: Extrinsic and intrinsic valuing by Wee Tiong Seah, Ernest Kofi Davis, Monica E. Carr	Anina Mischau
Teaching practices in a mathematics classroom and their connection to race and racism in the United States by Sabrina Bobsin Salazar	Dorota Lembrér
Exploring Roma learning Mathematics: A sociomathematical view by Javier Diez-Palomar	Wee Tiong Seah
The mathematics teacher's quasi-Darwinism. Problematizing the mathematics education research by Alex Montecino	Hans Kristian Nilsen

Table 5: Session 5

Themes discussed in the TWG

As at previous CERMEs, TWG10 discussed political aspects of mathematics education intensively. One way of addressing political aspects of diversity in mathematics education is by assuming research as always, in one way or another, constituting political acts. This assumption rejects a naive idea of research as politically neutral, providing objective data that is used to rationally guide policy. An example is Fyhn, Meaney, Nystad and Nutti (this volume) who address cultural responsive teaching of mathematics in relation to Indigenous (Sámi) teachers' self-determination. The acknowledgement of the political nature of research constitutes a recognition of how the issues we write and talk about as researchers are inextricably political and framed by world-views. Political aspects also concern how the broader political context of mathematics education, as it is performed in a variety of contexts, affects the teaching and learning of mathematics. Two papers looked at the intersection of school and workplace and the impact of mathematics education (Nilsen & Vegusdal, Boistrup & Keogh). Kollosche (this volume) discusses the role of teacher explanation for student passivity, also in relation to the discipline of mathematics. Another example of addressing political aspects derives from Sweden, where Bagger (this volume) addresses the effects from national testing on students "in special need". Political aspects present in TWG10 also concern how diversity among learners may have consequences in terms of unequal access to the learning of mathematics. This research may include critical investigations of the impact of socio-economic or cultural backgrounds, as well as other background factors, as grounds for unequal mathematics education, because of a sorting of students so they receive different learning opportunities (e.g. Salazar, this volume). A tension here is previously presented research to the group which confronts "the official discourse, which posits inclusion and equity as fundamental goals of mathematics education" (Straehler-Pohl & Pais, 2013, p. 1792, see also Valero, 2013). There are more recent papers

addressing ways of overcoming such tensions in research through collaboration between researchers adopting complementary perspectives, with teachers and students (Black & Swanson, this volume), hearing the voice of parents (Lembrér, this volume) or bringing individuals' voices to the discussion, highlighting potential opportunities to overcome such tensions (Díez-Palomar, this volume). Still, the work in the TWG, with its developments and tensions, can be viewed as constituting part of a resistance to a rampant global homogenization that is central to the neoliberal agenda, which stands in ideological opposition to the group's commitment to valorizing diversity (Mukhopadhyay & Greer, this volume).

The ethics of doing research, in relation to diversity of various forms, has been addressed in the TWG (e.g. Eikset et al., this volume). The TWG is united in a strive for social justice, inclusivity and variety. Consequences of an engagement in ethical considerations is reflexivity in research, where also the researcher's acts are critically observed (Montecino, this volume). Ethics also includes the impact of our actions within the context in which we are conducting our research and many addressed that we must pay attention to the consequences of our research on the end-users' (students, teachers, families, etc.) opportunities to improve their chances to learn mathematics and/or to legitimize their own social and cultural knowledge about mathematics (Mukhopadhyay & Greer, this volume). Diversity as a concept, and the connotations hereof, were problematized in the TWG (e.g. Roos, this volume). One aspect here is that diversity as a concept may assume a norm that there is something normal from which, for example, diverse students deviate, while "diversity" instead should be viewed as the norm itself. A connected matter here are words that mean something similar to diversity, but perhaps with other connotations; difference, heterogeneity, multiplicity, variety, and connected words: democracy, inclusion/exclusion, segregation/integration, empowerment. With an interest in the broader context of settings for mathematics education, the political times of today with neoliberal agendas affecting the framing of mathematics education, has been part of the work of the group during the last CERMEs (e.g. Andrade-Melina, this volume). Aspects here are political decisions governing towards mathematics education to be effective and market based, which are forces where diversity among students may be disturbing, rather than part of the responsibility of the system. We expect such issues to be elaborated more during future meetings, while also addressing the roles of mathematics education in times where populistic policy making is becoming more common.

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Incepted neoliberal dreams in school mathematics and the 'Chilean experience'

Melissa Andrade-Melina

Aalborg University, Denmark; melissa.andrade.mat@gmail.com

This work aims at portraying a rhizome of circulating naturalized truths about who citizens should be and how they should act within neoliberal governmentality. It does this by a historization of an incepted belief entangled in diverse social spheres. It unfolds how the ideas of human capital and welfare become a top right in mathematics education. The 'Chilean experience' is used as an example to construct a rhizomatic historization of events, strategies and technics of government that enabled the inception of neoliberal dreams into school mathematics.

Keywords: Governmentality, neoliberalism, historization, rhizome, school mathematics.

Introduction

> *An idea. Resilient... Highly contagious. Once an idea has taken hold of the brain it's almost impossible to eradicate. An idea that is fully formed—fully understood—that sticks; right in there, somewhere (Dominic Cobb, Inception).*

It is intriguing how highly perceived the Chilean models are—economy, education or health systems—to other countries. According to Taylor (2003), Chilean systems have been taken as models 'worthy of emulation'. Is Chile doing something marvellous? The country has been seen as an example of organization and 'proper' policies for economic progress and welfare (Silva, 1993). Its policies are considered as trendsetters among privatized pension systems (see Mesa-Lago, 2012), among health care reforms (see Bruce, 2000), and it was one of the first countries implementing neoliberalism as a framework in education (Aravena & Quiroga, 2016). The results in PISA, particularly in mathematics literacy, have progressively increased over the years—2000 (384); 2006 (411); 2009 (421); 2012 (423). And so, Chile has risen to be seen as one of the most developed countries in Latin America (Gregorutti, Espinoza, González, & Loyola, 2016). Chile is considered, by the World Bank's annual reports on development, the proven example of the benefits embedded in 'conforming' to a neoliberal approach to social policy.

> [Chile] is often viewed as a trendsetter in introducing fundamental and far-reaching neoliberal reforms […] the Chilean example as been heralded as proof of the success to be gained from an uncompromising commitment to neoliberal policy prescription (Taylor, 2003, pp. 21-22)

But… it is not all sunshine and roses! By building on Foucault's work, this paper aims at portraying how neoliberal discourses about mathematics education have been (re)produced and how they have circulated amongst diverse spheres of human interaction, (re)shaping citizen ways of being and acting in the world. It does this by taking "a critical attitude towards those things that are given to our present experience as if they were timeless, natural, unquestionable" (Rose, 1999, p. 20). This paper deploys a historization of the present of entangled historical events, strategies and techniques that made possible to incept the neoliberalism into school mathematics in Chile. This narration is not a critique about the implementation of educational policies in Chile; rather it is the tracing of naturalized truths in mathematics education as an assemblage of diverse governmentality techniques (Foucault, 1991).

These naturalized truths are traced in five moments. First, regarding the introduction of neoliberalism as a set of political movements. Second, regarding neoliberalism as a system of reason for economic improvement. Third, regarding the specific type of citizen that the new economy requires, a consumer of goods. Fourth, regarding the productive subject of schooling for the market, a competitive subject. And finally, regarding how school mathematics becomes the vehicle to shape the desired subject for economic growth.

The plot of the movie "Inception" inspires the style of writing of this paper. In this movie a series of dreams are unfolded. Each dream should be understood as a new and deeper dream occurring inside the previous one. The dreams do not follow a chronological arrangement. It is not a lineal story; it is a rhizomatic construction. A rhizomatic network allows a non-hierarchical multiplicity of entryways, of dimensions, lines; it has no beginning or end, but always a middle: "The rhizome is altogether different, a map and not a tracing […] The map is open and connectable in all of its dimensions; it is detachable, reversible, susceptible to constant modification" (Deleuze & Guattari, 1987, p. 12). Hence, all dreams are connected not as a sequence of events, but as continuities and discontinuities. All narrations are entangled, even in different times, even in different spaces, and even in different voices. The paper is written in this form in alignment with Foucault's rejection of causality.

> We consider the understanding of the way one event succeeds another as a specifically historical issue, and yet we do not consider as an historical issue one which in fact equally so: understanding how two events can be contemporaneous […] History is quite frequently considered as the privileged site of causality […] But we have to rid ourselves of the prejudice that history without causality would no longer be history. (Foucault, 1999, p. 92)

First dream: The Cold War and the neoliberal revolution

It is the late 60s, in a country apparently far from the War, but close enough to be in the spotlight. There is the danger of it becoming the first socialist nation in South America, and this is threatening for the US. Silent voices were saying: "under no circumstances should Allende be elected!" But, he was… Salvador Allende became the first democratically elected socialist president in the Western hemisphere. What a revolutionary! Fighting for the people! Chile has begun to increase its role in the provision of social services.

> By subsidising the reproduction of the labour force through allocating resources to the development of state systems of health, education, housing, staple-food subsidies and social insurance, universalistic social policies tended to reinforce the purchasing power of wages thereby expanding domestic markets for industrial goods. (Taylor, 2003, p. 23)

Something is starting to go extremely wrong in Chile. Suddenly there commenced a crisis that led to most of the population clambering for improvement. The, so-called, socialist experiment "united capitalists, landowners, the middle classes, and their political party allies against labor, peasants, and leftist part" (Silva, 1993, p. 535). Apparently the US government, also pressuring the World Bank and the Inter-American Development Bank to do the same, minimized the aid they provided to Chile… And so, Chilean foreign reserves plunged from $400 to $13 million in one year (Moreno, 2008). Discontent people wanting the president out are growing in number. It is socialism versus capitalism… "$7 million channelled to anti-Allende groups", according to a report of the US senate (Moreno, 2008, p. 93). And he was overthrown on September 11th 1973, by the military force

commanded by General Pinochet. Now, neoliberal ideas are being forced into Chilean minds that are afraid all the time, afraid for their lives, afraid to raise their voices. Meanwhile, those in favour of the new regime are enjoying the pleasures of the new order (Salazar, 2003).

Second dream: The Chilean experiment, Friedman and the School of Chicago

The year is 1950; the place, Chicago. Milton Friedman is developing a new approach to economy theory. This new theory is in opposition to socially conscious economies, which have been prominent in Western governments after 1929. Friedman believes that "economic benefit could best be optimized if the individual has the autonomy to pursue his or her own self-interest" (Moreno, 2008, p. 92). This new theory was the hope for a group of technocrats that moved to Chicago, the "Chicago Boys" (García & Wells, 1983). In the 70s, Pinochet decides to leave the economical management of Chile on the hands and knowledge of the Chicago Boys. This is going to be the first time that a group of Friedman has "an opportunity to influence governmental policy and put their theories into practices […] They already have a complete programme aiming to re-structure the economy and to reverse Allende's social reforms" (Moreno, 2008, p. 94). The military regime and the Chicago boys established neoliberal economic and social policies here (Salazar, 2003). "[T]he market supplanted state intervention in the economy, except in labor relation" (Silva, 1993, p. 527).

Within the first six years of dictatorship, the 'shock therapy' was the only approach to curb social policy and state expenditure (Huber, 1996). Chilean reform "has been led by both the advocates of monetarism, located principally in US institutions and universities, and by the Chilean reformers themselves" (Taylor, 2003, p. 22). Neoliberal ideas were taken as a sort of 'second independence' and, also, an entrance to the first world of developed countries (Salazar, Mancilla, & Durán, 2014).

Third dream: Consumerism as the ever-growing economy

Here, in this place of earth, everything could be marketed, everything could be sold, and most people would feel the urge to buy it. Health and education are, by constitution, social rights to every citizen. But here, those basic social rights fade into consumer goods. Public and private enterprises competing with each other, providing services for customers willing to pay for them, after all it is their choice (Taylor, 2003). Parents have the opportunity to choose freely the type of school—municipal, subsidized private or fee-paying private schools—and the type of education they want for their children (Mizala & Romaguera, 2000). Free choice… if they can afford it!

Public against private institutions… In a place where private institutions have the right to charge in excess to ensure better and better quality. Private schools enjoy, without any guilt, "having greater resources, enabling a stronger quality of education to be taught, and thereby reinforcing the desire of parents with available income to send their children to such schools" (Taylor, 2003, p. 34). After all, the more you pay the better you get; the less you pay the worst you obtained. In a time and place where education policies are transformed into economic policies of education (Castiglioni, 2001).

Fourth dream: Competitiveness in schools, education and freedom of choice

After the introduction of 'welfare' as a method to increase efficiency, "the element of competition and the response of enterprises to public desires as indicated by market forces were suggested to create an optimal allocation of resources throughout welfare provision" (Taylor, 2003, p. 26). The reform of the 80s, under the military regime, changed Chilean education system. Decentralization

was key to encourage private providers to enter the market (Mizala & Romaguera, 2000). And there was more, so much more than that. This reform involved a reformulation of the interplay between state and schools, a voucher system that indirectly funded schools by assigning the resources to students (Parry, 1997). This measurement left schools receiving financial aid depending "on the number of students that they could attract […]. If schools were unable to compete in this new marketplace environment, they would be allowed to fail and face dissolution" (Taylor, 2003, p. 33). A highly competitive system generated by an educational market and by the policies aiming at improving the quality of education (Mizala & Romaguera, 2000) was shaped. And so, state accountability systems were able to reward and/or punish schools by allocating resources regarding the performance of each school (Elacqua, Martínez, Sontos, & Urbina, 2012). A system in which, schools, teachers, students are constantly competing and being assessed.

Fifth dream: The sky is the limit! Mathematics to the people

Welfare and mathematics, always hand by hand. Here, mathematics has been granted with a great importance and status. In the 60s, logic was taken as the foundation of every science, reasoning accurately and rigorously was the core of any argumentation and of critical thinking (Diaz & Giudici, 1970). Mathematics was the one that helped to develop reasoning and logical thinking and reading proficiency was thought as a tool to better understand mathematical instructions (Ministerio de Educación & CPEIP, 1967). In the 80s, the military regime reformed the curriculum and school textbooks to reflect the regime's doctrine: "education was recast to promote studies functional to the new productive structures of Chilean society, whereas traditional arts and humanities studies were discouraged" (Taylor, 2003, p. 32). It was indispensable to embody in individuals certain knowledge skills—mathematical knowledge—, and attributes to facilitate the creation of personal, social and economic well-being (OECD, 2001). Economic growth was about human capital.

National assessment started to be taken as the key to achieve economic progress, on the one side, to test current policy changes, on the other, as a mean to set standards. And so, competitiveness and accountability, within school mathematics testing, led to higher performances, higher incomes, higher social mobility and welfare (OECD, 2014). Nowadays, by knowing students' numeracy proficiency in PISA it is possible to predict, amongst many others, their likelihood of being employed or to calculate how different their hourly earnings would be (OECD, 2015). And so, the promised state of welfare is side by side with mathematics proficiency. Mathematics is now the key for a brighter future, all students have to do is to be good at math and the sky will be their only limit! ***[End of dream 5]***

The standardized test SIMCE has been a key element to promote competitiveness and pressure to the system. Since its results are publicly published, it becomes an objective indicator to assess school performances (Mizala & Romaguera, 2000, p. 393). It also enables parents, as consumers, to demand better services for their children (Meckes & Carrasco, 2010), for students to be successful and entrepreneurs. ***[End of dream 4]***

School mathematics is now an investment! Reforms have shaped education into a capitalist marketplace, by promoting entrepreneurial profit-minded investment and by remodeling education to consolidate the productive structures of economy (Taylor, 2003). ***[End of dream 3]***

And the so-called 'economic miracle', product of the economic growth in the late 70s, helped raising the prestige of neoliberalism "under the banner of 'the Chilean model'" (Taylor, 2003, p. 25). By

now, Chile has become famous for its neoliberal restructuring followed under General Pinochet (Silva, 1993; Aravena & Quiroga, 2016). ***[End of dream 2]***

This is it! Chile is no longer an underdeveloped country (Salazar et al., 2014). Chile is now part of the first world, the "tiger" of Latin America (Teichman, 2016). ***[End of dream 1]***

Incepted neoliberalism

You create the world of the dream. You bring the subject into that dream and they fill it with their subconscious (Dominic Cobb, Inception)

From a Foucaultian perspective, conduct is governed through diverse techniques, strategies, and devices (Foucault, 1991), within a space of government that "is always shaped and intersected by other discourses" (Rose, 1999, p. 22). In doing so, each individual conducts him/herself by (re)shaping his/her own modes of being and acting in a space of 'regulated freedom' and under a promised state of welfare. In this sense, "people are governed by and through their own interests" (Cotoi, 2011, p. 113). This is precisely the idea behind the 'inception' of a neoliberal mentality. A set of naturalized truths circulating amongst diverse times and places, knitting a web to govern the self and to regulate habits and desires of cultural and historical subjects through school mathematics. These discourses help governing productive citizens, in the sense that intend to insert subjects in regulatory practices that (re)shape their conduct "without interdicting their formal freedom to conduct their lives as they see fit" (Rose, 1999, p. 23). Reforms, according to Dussel (2003, p. 94), "have to be understood as part of government technologies that intend to shape the way people are to act, think, and feel about the world, that combine the old and the new in unique ways".

One possible narrative to understand the success of neoliberalism in Chile could be grasped through the articulation of certain discourses about consumerism and competiveness. SIMCE in mathematics, for example, became the first step of knowledge consumerism and of a marketable education/society. SIMCE's results are publically published in national newspapers and widely discussed through other means of public communication, so parents and society could judge schools by their performance in standardized tests. 'Judge' in the sense of deciding which school is the best option for their children's future. This marketing of schools and teachers leads to the most utopian non-sense practices. For example, within the belief that welfare is only achieved by a high quality education, parents, in order for their children to be enrolled in those schools with "higher quality"—with good scores in national tests—are willing to stay all night in line, outside a school, to submit the admission application. Figure 1 shows a Chilean newspaper, *Las Últimas Noticias,* reporting the news: "Parents slept on the street under -3,6º Celsius because of enrolment. They are trying to enroll their children in pre-school for next year in Santa María School in Osorno". One of the parents, who waited in line for 12 hours said: "It is demeaning but what else can we do. This school is good and affordable. I have three kids and they all need to study".



EL DÍA

Viernes 2 de septiembre de 2016 / Las Últimas Noticias

"Es denigrante pero qué le vamos a hacer. Es bueno y está al alcance del bolsillo", dice Camilo Martínez.

Postulan a sus hijos al prekínder del próximo año en el Colegio Santa Marta de Osorno

Apoderados durmieron en la calle con 3,6° grados bajo cero por una matrícula

VALENTINA ESPEJO D.

No importó que la temperatura en Osorno, a las 7:07 de la mañana de este jueves, haya llegado a los -3,6 grados Celsius.

A esa hora, la vereda de la Avenida Manuel Antonio Matta, en la zona central de esa ciudad de la Décima Región, estaba copada por una larga fila de personas.

Algunos arropados con frazadas y otros tomando café o té en un termo. Pero todos esperaban lo mismo: que el Colegio Santa Marta abriera sus puertas para dar inicio al proceso de admisión 2017 de los prekínder.

Incluso algunos padres llegaron al colegio la noche del miércoles y se amanecieron frente al recinto.

Al parecer, sabían que no sería sencillo: había 188 interesados en una matrícula para 2017 y sólo 70 cupos.

"Es denigrante pero qué le vamos a hacer. Es bueno y está al alcance del bolsillo. Yo tengo tres hijos y todos tienen que estudiar", cuenta Camilo Martínez, uno de los apoderados que esperó en la fila durante 12 horas. El pago mensual en el Colegio Santa Marta es de $36.300.

Carolina Negrón, que también hizo la fila, dice que el colegio es barato en comparación a otros de la zona. "Este jueves hacía mucho frío. Había papás que se amanecieron con braseros y frazadas. Justo tocó una de las noches más heladas de las últimas semanas. Pero la recompensa es una educación de calidad", asegura.

Los apoderados aguardaron toda la noche.

A las 8:30 de la mañana se abrieron las puertas del recinto. Ahí, a los 188 interesados sólo se les pasó un número de espera para inscribirse en la postulación. Fue el primero de varios pasos a seguir para lograr el anhelado cupo. Además de la gran demanda, los postulantes tienen que lidiar con los criterios de prioridad que tiene el colegio para asignar las matrículas nuevas: en primer lugar, los hermanos de estudiantes actuales; luego, los hijos del personal del establecimiento e hijos de ex alumnos.

Desde el recinto informaron que, como nunca, vino mucha gente a inscribirse. "Si se espera que la gente esté un rato antes de la apertura, pero no que se amanezcan", dijo Claudia Antias, asistente de educación del colegio.

El próximo 15 de septiembre, el colegio dará a conocer la nómina de los afortunados alumnos seleccionados.

Figure 1: News about parents staying all night outside a school

These discourses are not isolated from the ones of school mathematics. The importance that mathematic literacy has within OECD's indicators help move research to be about how to improve students' performances in mathematics. For example, in order to be successful in SIMCE, students should not be allowed to miss classes. If students are "absent 9 days during the school year (the sample average of absences) reduced performance by at least 23% of the standard deviation of the score on the SIMCE mathematics test" (Paredes & Ugarte, 2011, p. 199). Welfare can also be measured in relation to students' performances in national tests, by correlating SIMCE scores in mathematics to predict a student's future income (Bharadwaj. Giorgi, Hansen, & Neilson, 2012).

The Chilean Ministry of Education released the "Learning standards" in school mathematics to help teachers evaluate "what students should know and are able to do for displaying, in national tests, appropriate levels of achievement" (MINEDUC, 2013, p. 4, my translation). These learning standards categorized students in three levels of achievement that, at the same time, predict their future outcome in SIMCE in mathematics. So, if students do not want to be label at the lowest level, they have to engage in regulatory school practices, they have to compete with their classmates and with themselves. In this fashion, SIMCE in mathematics also operates as a technique to generate 'self-entrepreneurs', "individuals that self-regulate, self-direct and are continuously in a process of redefining their competences" (Cotoi, 2011, p. 116).

As portrayed within the dreams, Chilean neoliberalism have been (re)producing discourses that circulate within diverse time and places in order to obtain economic growth, progress and welfare through school mathematics. School mathematics, since it was thought to shape productive citizens, was taken as the key for Chile to become a developed country. Mathematics needed to be a good that people wanted and were willing to consume. With marketable school mathematics, whomever wanting to achieve welfare would have to pay for higher quality. And, therefore, Chilean economy should increase. This would not have occurred without the dictatorship and Friedman thoughts: Economy would be best optimized if people have the freedom to pursue their own self-interest.

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"It is only a test" – social aspects of displaying knowledge in mathematics for second language learners

Anette Bagger

The Department of Science and Mathematics Education, Umeå, Sweden; anette.bagger@umu.se

This article discusses social dimensions connected to assessment in mathematics for second language learners in Sweden. The data consist of two semi-structured interviews with students in the ninth grade of compulsory school. Foucault's thinking on discourse and positioning was advocated as a frame for analysis. The units for analysis were students' statements about caring and the other in connection to the display of knowledge in mathematics. Results show that caring of and for others are important resources in managing assessment and believing in the future.

Keywords: Second language learner, assessment in mathematics, opportunity to display knowledge.

Introduction of the problem area

Measures of achievement are often situated as measures of quality in education (Lundahl & Tveit, 2014) that promote striving towards high quality, yet also threaten equity at times (Llewellyn & Mendick, 2011). An example of this are recent educational reforms including earlier and extended testing in Sweden (Regeringen, 2006). These reforms have led to an enhanced focus on the measures of knowledge, while at the same time school agencies generate reports on inequalities in the measured knowledge and grades in mathematics between schools and groups of students. Differences are connected to gender, class and ethnicity (e.g., Skolverket, 2015). Achievement in national tests is central in grading as the results of the national tests in mathematics often are used to indicate students' grades (Skolverket, 2016). Not having an approved grade in mathematics at grade nine in Sweden means that a student does not have access to public programs at the upper secondary school in the following year. The opportunity to display knowledge then becomes a critical point of departure for the individual's possibilities for positive development and positioning in mathematics. This turns the assessment grade in mathematics into a gatekeeper to get access to higher education, and is a gatekeeper that keeps out second language learners (SLL) more often.

This portrayal of circumstances alludes to societal, historical, social and political discourses influencing an individual's mathematical development and life-choices by making some positions available and other positions not available. Researchers are currently engaged in issues of for whom education functions and thereby which students can have access to success in education and life (Au, 2008; Peters & Oliver, 2009). The focus on disadvantaged groups of students then affords new ways of understanding and approaching mathematics education (Gutiérrez, 2013). Alternative ways of relating to and understanding assessment are needed and in this, listening to the students as opposed to only labelling them, are of core importance (Hodgen & Marks, 2009). Research that is paying attention to students' stories also contributes to knowledge about the lived and social dimensions of assessment that needs to be paid attention to according to Black and Wiliam (2010).

In this article, I strive to recognize and study the production of (in)equity in connection to grading and national assessment in mathematics. I highlight aspects that might be foreseen in the general debate and work in the area of mathematics education. Through this, I contribute to the identification of possible resources that could promote paths for increased equity. In regard to the concept of equity,

I draw on Boaler's (2006) concept of relational equity. In a school where participants (do not) learn to respect differences and each other in the mathematics classroom, relational equity is (not) lived and taught. This understanding can be applied on both the micro-, meso- and macro-levels of education and opens up studying equity beyond the gap-gazing of diverse achievement levels in mathematics (e.g., Gutiérrez, 2008; Rodrigues, 2001) something that actually might counteract equity and hold disadvantaged groups of students behind (Gutiérrez & Dixon-Román, 2011). Following from this point of departure, resources for producing equity are to be found in lived, diverse, relational and social aspects in the process of assessment and displaying knowledge. I contribute to the identification of some of these resources through the study of social and relational aspects connected to assessment and the opportunity to display knowledge in mathematics for SLL. In this paper, I specifically contribute to the act of listening to and exploring experiences of students within subordinated groups by highlighting experiences of SLL in connection to displaying knowledge in mathematics. The purpose of this paper is to contribute to knowledge about social dimensions of displaying knowledge for SLL in mathematics and how this can be related to future prospects connected to the subject. The investigation examines three research questions: Q1) How do statements about the other and caring appear in talk of displaying of knowledge? Q2) How do statements about the other and caring appear in talk of future prospects connected to the subject? Q3) What discourses are activated and what positions are available in the students' talk? Here, caring refers to both care for oneself and for/from others.

Theoretical framework

A statement works as a mediator of knowledge and truth that exists in a field of power-relations and is embedded in discursive formations with other statements (McIlvenny, Klausen & Lindegaard, 2016). Discourses are understood as governing and positioning individuals through power and knowledge (Foucault, 1994), and positioning is understood as "the discursive process whereby selves are located in conversations as observably and subjectively coherent participants in jointly produced story lines" (Davies & Harré, 2001, p. 264). Therefore, analysing students' statements will reveal discourses that are activated in connection to the display of knowledge and assessment in mathematics. In order to capture social aspects of displaying knowledge in mathematics, the role of *others* and *caring for/by others* are used in this project (Black, Solomon, & Radovic, 2015). Black et al. (2015) have shown that these phenomena may be powerful cultural resources in shaping a positive identity in mathematics. Black et al. (2015) have drawn on Bakhtin. Instead, I use the concepts as signal-words, which reveal the representation of a social and lived aspect of displaying knowledge in mathematics.

Method and selection

This paper presents the analysis of two interviews with SLLs. Their names are fictional. Amina and Ahmed are 15 years old and have both struggled with their learning in mathematics since third grade. They did not pass several of the goals in mathematics in their third and sixth years and have had special support during this time. Amina achieved the lowest passing grade (E) in her ninth grade and was given support in the form of special instruction. In the ninth grade, Ahmed did not receive any special support or adaptions in mathematics and got a D, which is the average grade. The data were collected right before the final choice of program to the upper secondary school was made and the final grade in compulsory school was given. Amina and Ahmed had just finished their last national

tests in mathematics. Interviews conducted on this occasion were assumed to contribute to a concentration of their experiences of displaying knowledge and assessment. This selection of students is meant to bring specific questions concerning assessment in mathematics for SLLs into the forefront.

The interviews were semi-structured. This approach was used to promote the student's possibilities to talk freely and to display as much of their understanding and experience as possible (Kvale & Brinkman, 2014). The students could talk about anything they chose in the areas of support, assessment, national tests, grades, mathematics and the future. For the most part, I asked open-ended questions to follow each theme. Questions were, for example: why do you take these tests; what did you think/feel; did you talk about it to anyone and how; was it possible to get help; if so, how? These open-ended questions were followed up with more specific questions connected to what the participant had expressed, in order to get as good an understanding of and rich information about their experiences as possible. The positions and activated discourses were constructed through analysing statements about caring and the other in the context of displaying knowledge in mathematics. Key markers in students' talk were statements involving others as for example peers, friends, family and school-staff and statements regarding caring about or being cared for by others. These statements also had to relate to assessment and/or displaying knowledge in mathematics. An interpretative reading of statements was done back and forth in order to identify the discourses and positions involved. For this purpose, an adaption of Foucault's (2011) description on how to find discursive formations was used: 1) First, statements regarding caring and others were identified. 2) Secondly, the form of these statements was described. 3) Thereafter, the relations between these statements were described and the correlations and contradictions between these statements were explored. 4) Then, the statements were grouped and the correlations and contradictions between these groups were explored. 5) In the final step of the analysis, the discourses were construed.

Ahmed and Amina: Interview data

The interview data connected to the first two research questions are presented here. This presentation derives from the first four steps of the analysis. Overall, these statements concerned family or peers, and notably teachers were not mentioned at all in connection to statements concerning caring and the other.

Statements about the other and caring in talk of displaying of knowledge

The students talked about care of and from others primarily in relation to peers and as peers as a point of reference for the achievement or possibilities to succeed. Ahmed mentions that the girls talked about when the tests were, and, in a way, that could make each other nervous: "The girls are like: it is mathematics (national test) tomorrow, tomorrow! They mentioned it several times" (transcript 005). But he was not nervous himself but rather preferred to take it all in time and put the test into a larger context of living: "If I make it, I make it... There is no point in worrying, life will continue anyway" (transcript 005). Even if he was at ease, he expressed concern about a friend who did not manage Swedish well enough in relation to the support given and the construction of the tests. The friend was very good in mathematics in his homeland but after coming to Sweden he almost did not pass: "He is not so good at Swedish so he thought it was a Swedish word he asked for help with. The teacher could not do anything but read it aloud again. In his homeland, he had like a high grade and here he barely passed" (transcript 005). Ahmed talks about care of himself in relation to effort and

outcome on the tests: he is at peace himself with not being able to solve all the tasks, since they are constructed for all levels of difficulty. This circumstance also makes it hard for him to know if he passed the test, "I might think it is hard but I am on an E, the ones being on an A may not think it is so difficult. So, I would say it was ok, even if it was hard for me" (transcript 005). In this way, Ahmed refers to peers as a point of reference. Ahmed says that he made a deliberate choice not to study before the test: "I have myself to blame if it went bad, I accept my choices" (transcript 005). Amina talks about care of herself in relation to her knowledge in the subject, her effort and grades. She thinks mathematics is hard but does not think the grades reflect her experience of the subject as interesting and of herself as someone who is interested, learns and works hard: "I think math is easy, or easy, it is hard but I think it's fun. What comes out shows in the grade… I put a lot of effort in math but I do not get good results. It does not show in the grade. It makes me feel disappointed, but at the same time it challenges me" (transcript 002).

Statements about the other and caring in talk of future prospects

The students' statements regarding the future prospects are often connected to the family's care about them. Both students expressed that parents and relatives had high expectations and beliefs in them and their engagement in mathematics. Amina connects the big expectations she has for herself to her parents' expectations: "I think my expectations come from mum and dad, they expect big things from me" (transcript, 002). The family stressed that they should do what they could to enter upper secondary school. For example, if Ahmed did not get the lowest approved grade, the family would encourage him and not let him give up: "They would be grumpy with me and they would think that I should go back and keep on fighting and not stop" (transcript 005). In particular, Ahmed's brothers had given him advice on how he should choose a program at the gymnasium in relation to mathematics and also had given him a good trust in upper secondary school, the mathematics involved in the program he chose and the teachers: "I have lots of expectations since I have a family from whom I have taken like a lot of advice. All have said very good things about the school and the one (brother) who studied construction has said a lot of good things about a teacher working there" (transcript 005). He also compared and talked about his siblings and how they succeeded and what they had done in their time at the upper secondary school: "You know, my brother, he says that there are three days of practice a week and that you get to learn a lot out in the field. Three weeks before finishing school he was offered an employment… He is 19 and he has a job" (transcript 005). Motivation was in this way connected to talk about parents' and relatives' anticipation of and belief in them and their engagement in mathematics. Although Ahmed could feel that they nagged at him, he understood and appreciated the advice to put effort into the learning in mathematics: "I understand their arguments and so and I really appreciate that they help me there and I understand the point. It seems to be important to get a grade in math" (transcript 005).

Statements about the other and caring in talk of future prospects

Care about themselves in connection to the future were expressed in relation to belief, struggle, worries and seriousness. Ahmed worried about the test a great deal afterwards, if some of the harder tasks would deprive him of his grade, his time in the upper secondary school and stop his journey in life and companionship with friends: "This is life, this is it. I would be very disappointed if I did not pass. Then I will miss a whole year… I do not want to wait a whole life for life to continue" (transcript 005). The students meant that future choices may be limited depending on their knowledge, which

made them both choose a program at the upper secondary school with a low level of mathematics. Statements about expressing care for others concerned peers and primarily gatekeeping functions in the assessment in mathematics, but also their own learning in positive anticipation regarding their ability to develop. Amina talks about her peers as participants in discussions about the grades, something that has been intensified over time as it is connected to mathematics as the gatekeeper to the upper secondary school: "We did not talk as much about it (earlier) but more later. That it is the grades that decide if we get into upper secondary school" (transcript 002). Here, Ahmed points towards the gatekeeping function in the national assessment of mathematics, which is worrying: "If you fail in math, then you are done… That is why I have been lying awake at night and thought about the test" (transcript 005). Both students anticipate that math will be hard in the upper secondary school but they are confident that they will learn. Amina says she is eager and ready to take on the challenge: "I am going to study at the upper secondary school and I have to be prepared that mathematics is the hard thing. I am very excited" (transcript 002). Both Amina and Ahmed are very confident that they can learn the mathematics they need when they finally begin the upper secondary school. Ahmed for example states "but I think that when I finally go to the construction program I will learn it, how to count with area and stuff" (transcript 005).

Analysis

The analysis answers the third research question and explores the discourses that are activated and the positions available. A *discourse on managing assessment* (connected to statements about peers), a *discourse on progress* (connected to statements about family), a *discourse on future challenges in mathematics* (connected to care about oneself) and a *discourse on fairness* (connected to care about oneself), were construed from the analyses of statements connected with *caring* and *the other* in connection to displaying knowledge and mathematics. The activated discourses led to some available positions for the students.

The discourse on managing assessment concerns support, comparison and monitoring of support, grades and tests. Ahmed talked more about his peers and talked overall more than Amina. Amina referred to peers as a help in focusing on the grades and Ahmed expressed care for others. He then positioned both himself and his peers as *disadvantaged test-takers* due to language and the settings and construction of the national test. He also positioned girls as more nervous in their monitoring of test occasions.

The discourse on progress circles about responsibility, advice and expectations stemming from the families. Expectations were then blended with demands on focus and progress. *Hard working* and *you can if you want to* were positions connected to the family discourse. These positions had connotations of personal responsibility, achievement and future prospects.

The *discourse on future challenges in mathematics* held statements in which caring about oneself was connected to a position in which limitations in knowledge blended with striving to learn and the outcome that learning was to be conquered.

The *discourse on fairness* connects the student's individual responsibility, effort and knowledge to the achievement and assessment in mathematics. If the effort is made, the knowledge should be retrieved and following from that, the grades should be accordingly high or low. In *the discourse on future challenges in mathematics* both students positioned themselves as capable of trying and

working on improving, although within certain limits. A position of *struggling while learning* was identified, this position is possible to connect to the *hard-working* and *you-can-if-you-want-to* positions in various combinations. In *the discourse on fairness,* a position of *choosing your achievement* was shown in Ahmed's talk as he chose not to study and accepted the consequences. Amina also spoke out from a discourse of fairness as she questions the grade, and that the hard work should have been seen in the grade. At the same time, she capitulates and says that it is hard for her to learn and remember for example the methods to use – so the grade may be fair after all. This could be described as a position of *being unable* to succeed.

Discussion

The semi-structured interviews contributed to a trustful and open climate for conversation. This made it possible for the two students to display important social dimensions of displaying knowledge and learning mathematics. One example of this was when Ahmed talked about how his brothers and family supported him: "I took their advice… It seems to be important to get a grade in math" (transcript 005). This happened on an occasion when he was actually skipping school for a day and hanging out with his brothers. This occasion proved to be an important moment in his positioning as a mathematics learner. The main concepts in the analysis were statements about *the other* and *caring* (also see Black et al., 2015. In many ways, it is possible to assume that there are many differences between Roz, the adult mathematician in Black et. al.'s (2015 study and the two students in this study when it comes to opportunities to learn and display knowledge. What they have in common is that the female mathematician and immigrant students are groups both governed by different types of gatekeeping functions in their access to the subject. Interestingly, the same socio-cultural resources as found in Black et. al.'s (2015 study, seem to work well in illuminating prerequisites for a positive development of identity in mathematics among students that have struggled with their learning. The statements about assessment in mathematics and the future were in many ways a narrative drawing on the community of the family and the peers. This could be a sign that relations outside school, in families or between peers, are important resources in the building of a positive identity in and a relation to the subject mathematics.

Both students in this study were willing to learn and develop their skills in mathematics, although they knew that they were in some way limited because of their lack of knowledge at some times. Although struggling with mathematics, they still had a positive way of approaching the subject, which is not always the case for students in need of support in mathematics after nine years in school. The families' expectations and talk with their children about the future and the role of mathematics in it may have contributed to a discourse about struggling when learning. This discourse could contribute to a positive identity in mathematics rather than devaluing the struggling student as a learner in mathematics. This research contributes to knowledge about the social dimensions of testing for SLL. Social and lived dimensions of assessment may get lost in translation if the measures on achievement are interpreted without taking the social, cultural, political and relational contexts into account. Conclusions from assessment that mainly focuses scores and levels of achievement might reveal differences and tendencies of segregation or lack of knowledge but without affording means to counteract these inequalities. Therefore, more research in the socio-political area of mathematics education is needed. This paper joins the socio-political research in mathematics education as I contribute to knowledge about social dimensions of displaying knowledge in mathematics for

disadvantaged groups of students. The aim was to identify valuable resources in providing access to and success in mathematics for all students. I also emphasise an alternative way of understanding assessment beyond measures of knowledge and quality, namely as a means of promoting social and relational aspects of becoming more mathematically able. Since, as the students' statements revealed, it is (not) only a test of knowledge but also an occasion of caring about oneself, caring for others, and being cared about by others.

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Subjective theories of teachers in dealing with heterogeneity

Elisa Bitterlich, Judith Jung and Marcus Schütte

Technical University Dresden, Germany; elisa.bitterlich@tu-dresden.de

This article presents initial results of a research project which investigates subjective theories and typical action strategies of teachers and student teachers in dealing with heterogeneity in school with a focus on the subject mathematics. These results are ultimately intended to contribute to the development of future teaching approaches. To reconstruct subjective theories of teachers and student teachers, group discussions were carried out. The initial results show the aspects of heterogeneity the participants deem important and the possible actions they discern for coping with pupil diversity.

Keywords: Diversity, heterogeneous grouping, teacher beliefs, group discussion.

Fundamental theoretical considerations

The fact that children and young people often differ in terms of their needs and preconditions for learning, and that this heterogeneity of learners sometimes presents teachers with significant challenges, are not new phenomena (Trautmann & Wischer, 2011). The heterogeneity of learners relates to different dimensions, such as cognitive performance, age, gender, linguistic-cultural background, social class and many more (Hinz, 1993). In accordance with current political discourses and social developments, these different dimensions are given unequal attention in pedagogical discussions. The impetus for a renewed focus on heterogeneity in Germany was provided by the results of international comparison studies (in particular PISA, 2000), which highlighted especially the sizeable differentiation in pupil achievement, the alarmingly high number of very-low-achieving pupils, and a close relationship between social background and academic success (Trautmann & Wischer, 2011). The UN-Convention on the Rights of Persons with Disabilities, which came into force in Germany in 2009, made the inclusive schooling of children with and without disabilities the subject of renewed debate. In addition, the phenomenon of increased linguistic-cultural differences among learners has come into focus in the last two years by the increased number of refugees entering Germany. For most student and practising teachers the heterogeneity of learners represents an important problem area in planning and teaching lessons which seems to be complex and fundamental. Askew (2015) expounds that the teacher's ways of thinking and talking about heterogeneity impact how they react to the differences that learners bring to the mathematics classroom. The following questions arise: "Why have so many, essentially well-founded pedagogical ideas not been realised? What prevents teachers from seeing heterogeneity as enriching, and dealing with it productively?" (Trautmann & Wischer, p. 9, translated by the authors). In order to answer these questions and create concepts for future seminars, not only a scientific reflection of this topic is supposed to be considered. Especially the perspective of those facing heterogeneity daily in their pedagogical work is to be included by analysing their subjective theories about heterogeneity, too.

In the framework of the Germany-wide "Qualitätsoffensive" for the improvement of teacher

training, the project "Synergistic teacher education in an excellent framework"[1] at the TU Dresden includes the sub-study "Heterogenität in der Lehrerbildung von Anfang an" (Heterogeneity in teacher training from the start). Based on qualitative questionnaires, group discussions, and participatory observations of everyday teaching in schools, subjective theories[2] and predominating patterns of action among teachers and student teachers will be surveyed in different kinds of school. On the basis of the survey results, the project intends to develop concepts for teaching events to make student teachers sensitive to the different facets of heterogeneity.

The concept of 'Heterogenität' (heterogeneity) is defined in various ways in the relevant German-language scientific literature, and indeed is often used without specific definition. A number of terms are used synonymously, ranging from 'Vielfalt' (plurality) to 'Unterschiedlichkeit' (difference), 'Unbestimmbarkeit' (indeterminableness) and 'Beliebigkeit' (arbitrariness), or English words like 'diversity'[3]. In many scientific articles, the focus is placed on only one aspect of heterogeneity (such as language, culture, gender, or disability), and the relevant definitions insinuate a polarisation between 'normal people' and 'the others'. However, authors like Prengel (2006) and Krüger-Potratz (2011) articulate a different understanding of the concept of heterogeneity. We share this understanding of the concept of heterogeneity, which finds placing the focus on a few 'dominant' characteristics to be a reductive approach (Krüger-Potratz, 2011).

> Diversity education is based on the 'indeterminability of people'; it is therefore unable to diagnose 'what somebody is' or 'what shall become of somebody'. It [diversity education] opposes all reification in forms of definitions of what a girl is, or a boy, a behavioural deviant, a Turkish woman… If people must be characterised, then this must be based on their dynamic development and the context of their environment. (Prengel, 2006, p. 191, translated by the authors)

To analyse teachers' subjective theories, and to develop concepts for teaching events that are based on the teachers' views, against our understanding of the concept it nevertheless appears sensible to establish a theoretical categorisation of some of the individual facets of heterogeneity. Some of the existing studies on 'beliefs', as well as on teachers' implicit or subjective theories about heterogeneous contexts in school, show a focus on selected individual aspects of heterogeneity in this way. On the aspect 'heterogeneity', in an interview study on belief systems of primary-school

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[1] This project is part of the "Qualitätsoffensive Lehrerbildung", a joint initiative of the Federal Government and the Länder which aims to improve the quality of teacher training. The programme is funded by the Federal Ministry of Education and Research.

[2] In the literature, concepts such as subjective theories, implicit theories, or naive theories, as well as teachers' ideas and attitudes, or "beliefs", are sometimes used synonymously in accentuating different aspects, and cannot be clearly differentiated (Törner, 2002). In this paper, we will discuss teachers' subjective theories based on Heymann's (1982, p. 146, translated by the authors) understanding of "the totality of knowledge elements and orientations affecting teachers' actions in the conducting of lessons".

[3] While in the English speaking area, the Term „diversity" is preferred and well-known, in Germany many publications tend to use only "heterogeneity". But in some more actual publications we can see that "diversity" is more and more a common term for describing the plurality of pupils.

teachers working with children with special educational needs concerning cognitive development, Korff (2014) identifies a central challenge for didactic activity in mathematics teaching in the establishing of links between different approaches and varying levels of representation. On the aspects of gender and ethnicity, the ProLEG study[4] addresses the question of how ethnic-cultural and gender-related perceptions influence teachers' educational activity (Winheller, Müller, Hüpping, Rendtorff & Büker, 2012). The results show that the respondents overwhelmingly see questions of children's gender as unimportant and considered themselves to be sufficiently competent in this area. In relation to ideas about ethnicity, the respondents attached equally low importance to 'Intercultural Education'.

> Topics such as 'Individual Support', 'Social Learning', and 'Inclusivity' are given primary importance, while culturally sensitive approaches have the lowest priority, followed by 'gender-aware education'. (Winheller et al., 2012, p. 10, translated by the authors)

However, Zobrist's (2012) investigation attempts to approach heterogeneity from a broad perspective, without restriction to particular aspects of the concept. Using semi-structured interviews and 'simulated recall' in addition to teaching observation, the author attempts to produce a comprehensive view of the ways teachers deal with heterogeneity in mathematics teaching in secondary school. The results show that teachers tend to define heterogeneity especially in terms of different preconditions for learning and different kinds of social behaviour. Furthermore, special educational needs assessment is seen as highly relevant in dealing with diversity, but personal competences in this area are considered inadequate. Schönknecht and de Boer (2008) point out that in describing heterogeneity student teachers often seem influenced by an idea of polarisations and dichotomisations, as well as a limited perspective focusing on supposed 'problem children'. Also in relation to perspectives on the dimensions of heterogeneity, little differentiation is evident, with the use of a number of generalising (stereo)types (e.g. 'normal' and migrant children). They summarise:

> Thus, addressing the construction of normality contributes to dealing with heterogeneity and difference, and can be a significant building block towards realising equality of opportunity in school. (Schönknecht & de Boer, 2008, p. 258, translated by the authors)

Regarding to Garmon (2004), both dispositional factors (like openness to diversity as well as receptiveness to others' arguments and ideas, self-awareness and self-reflectiveness, and commitment to social justice) and experiential factors (like intercultural experiences, support group experiences with individuals who encourage another person's growth and educational experiences) influence teachers' attitudes towards diversity.

Method

In order to gain a first impression of student teachers' ideas about the concept of 'heterogeneity' in the school context and to discover how they encounter diversity among children, a qualitative questionnaire with three questions requiring written answers was distributed to around 80 student

[4] "Professionalisierung von Lehrkräften für einen reflektierten Umgang mit Ethnizität und Geschlecht in der Grundschule" (Professionalisation of Teachers for a Considered Relationship with Ethnicity and Gender in Primary School)

teachers training to teach mathematics. The students answered the following questions in their own words in running text or bullet points: What does the word ‘heterogeneity’ make you think about in the school context? To what extent have you addressed this topic (in your studies)? To what extent do you feel prepared to deal with heterogeneity in the school context?

In addition, we held group discussions with student teachers[5] on the topic of ‘heterogeneity’. The group discussions were video recorded and transcribed and evaluated using the documentary method (Bohnsack, 2010). Group discussions can help to identify and analyse the implicit or tacit knowledge of the participants while they talk about a specific topic (e.g. heterogeneity). Between three and seven participants talked about a given topic for around 60 minutes. They have some special experiences in common or commonalities in their history of socialisation, thus sharing a “conjunctive space of experience” (Mannheim, 1982). The momentum of the discussion process, uninterrupted by the researcher, is important to discover these conjunctive spaces of experience, which become visible through “focusing metaphors” in which the group adjusts itself to those specific topics that are most relevant in their common experience (Bohnsack, 2010).

> Concerning group discussions, the immanent meaning comprises that stock of knowledge which is made explicit by the participants themselves. This has to be distinguished from knowledge of experience, which is so much taken for granted by the participants that it must not and often cannot be made explicit by themselves. The participants understand each other because they hold common knowledge without any need to explicate it for each other. (Bohnsack, 2010, p.103)

Results

For reasons of space, this article will highlight a few clear results of the qualitative questionnaires as well as extracts from the overall transcript of the group discussions with teacher students.

Qualitative questionnaire

The students gave highly diverse answers to the question “What does the word ‘heterogeneity’ make you think about in the school context?” While one respondent (female, 23 years, fourth semester) answers only with a few key words (“diversity, differentiation, boys and girls, high-achieving, low-achieving”), other students give more complex answers, making clear their awareness of the unbounded, indefinable nature of the concept:

> Every class is different (age, background, etc.). Every child therefore has different preconditions for learning, which one should include in the teaching. Differentiation is important (natural differentiation, internal differentiation, external differentiation). The application of learning environments to enable different approaches (with different difficulties/materials, etc.). (female, 21 years, sixth semester).

This student’s response also suggests how she would deal with diversity among children and where she thinks particular emphasis should be placed. On the question of how well prepared the students

[5] Student teachers in this way includes teachers who have finished their university degree and are now in training for one or two years before receiving their final teaching license and students who want to become a teacher and who still have lessons in university.

feel to deal with heterogeneity in the school context, the majority of responses are sobering. Most of the students complain about a lack of practical experience, stating that although the university education in many respects provides a lot of theory, there are few opportunities to reflect upon the ideas and to try them out in practice. Furthermore, it is criticised that not enough attention is paid to the topic of heterogeneity (overall) in the study course; it is often covered quickly as a "marginal topic", but not "dealt with in depth" (female, 22 years, fourth semester). On the basis of their experiences in the course of their studies, several of the students differentiate between the different teaching subjects; for example, one female student (21 years, sixth semester) remarks: "I feel better prepared in maths than in German. For example, by the '(Maths) Learning Under Conditions of Heterogeneity' course".

Extracts from the group discussions

The participants in the group discussions presented below were student teachers in training for primary-school and high school with the subject mathematics. The students are in the middle to last phase of their studies or have finished their university degree and are in training before receiving the final teaching license, meaning that all have already completed placements in schools. During the 60-minute discussion on the topic "What experience do you already have of diversity respectively heterogeneity among children in school and in teaching?" it becomes clear that those facets of heterogeneity that are dominant in the social discourse, such as native language, disability, social status, achievement and migration background also dominate the students' discussions.

After a group of five female teacher students for primary school have talked about topics such as German as a foreign language and the meaning of academic language, one of the participants turns to the topic of inclusivity and the schooling of children with special educational needs in regular teaching. The following extract is a part of the discussion that develops on this point. The following transcript extract (the original version is in German) shows how the students encounter diversity among children, the challenges and opportunities they see in such diversity, and what experiences they have already gained in dealing with it.

Tina: I also think it's very important how children gain another view of what is actually normal. A person sitting in a wheel chair is just as normal and can also move around. And that simply this acceptance and tolerance can develop amongst each other. That you simply know how to deal with the person and that it becomes natural from an early age on.

Bianca: I think this is also easier for children. I also always like that about children that they very openly go to other children who are a little bit different. I also think that this should be encouraged but it is also a fact that there are also mentally disabled children. I don't know if they are also affected by inclusion?

Sarah: Yes.

Bianca: Well, I think that is difficult. Well, I was at a school for children with special needs and sat in on classes and I thought it was really bad.

Sarah: Well, otherwise, in front of the same class plus children with special needs that is not possible, I think. Then also structurally things would have to be changed.

Diana: Well, for all of us it is a challenge to stand in front of a class after finishing our studies. Even if they are top-performing and are all a relatively homogenous group.

Tina: But you will never have this homogeneous group (smiles and shakes her head).

Diana: (nods) Yes. You also don't have that in society. The whole society is extremely heterogeneous.

Extracts from a discussion between four female student teachers for high school shows a similar view on heterogeneity, however with a greater focus to specific problems of the subject mathematics.

Wiebke: The heterogeneity of the teachers also effects the lessons and what the children learn in the end. Therefore, also the teacher's competence of explaining.

Nathalie: (Laughs) Yes, especially in mathematics.

Tamara: And I also think what attitudes the teacher has towards heterogeneity. Meaning, is my attitude that I take everyone along or do I only take the top 50 percent along? Or drastically said, what is my opinion about somebody from a migrant background? That also plays a huge role.

Vera: Yes, that's true.

Wiebke: I mean, at the university this is addressed but how I should really deal with it … It is nice to say that you need difficult tasks for those who are good and easy ones for those who are not so good at it. Yes, but in the end, all of them take the same test and are marked according to the same grading system.

A third extract is from a group discussion of four female primary-school student teachers which have already been in training at school for a few months. This brief passage of the discussion shows that student teachers who are already teaching in school seem to be more aware that there are differences between the theory they have learned in university courses and the dealing with heterogeneity in real life. Additionally, they critique some aspects of the education in university.

Linda: The only option is individualized teaching, if you want to give every single child the chance to take part in the lesson and to have fun.

Beate: And you have to accomplish this without straining oneself.

Isabel: Exactly! And I would like to know, how that can work (laughs). How can I differentiate without constantly feeling stressed at home?

Linda: Well, when I was at university, I often thought, „Bla, bla, heterogeneity, differentiating. How can this work?" They [the university teacher educators] always treated this like a big cloud but they never told us specifically. And then, in school, you think, "Well, how does this work?" And only through experienced teachers you understand "Oh, this is how you can approach this!" And it doesn't always have to be three different worksheets. A more open form can work as well.

But I think that in university it was something which existed somewhere up high in the universe but which cannot be implemented.

Denise: In my point of view, such opened instruction was seen [in university] as a kind of sanctuary and I always thought, „No. I can't do it. I somehow am not able to do it at all!" Because it is a Utopia to do this. But in the end, it is somehow possible and [at university] one should have used that as a starting point. Even though we heard keywords like "weekly schedule work", we never spoke about this in depth. It was more like scratching the surface.

Isabel: Then we received lots of academic texts about this which we were supposed to read. Afterwards I knew as much as beforehand because the time to really understand the content was too short. This was easier with conversations. But I did not have a concrete plan either.

Beate: But now, in the courses for trainee teachers, we recognized that we all open our lessons. We do not carry out frontal teaching as we always imagined. That's why it is helpful to have somebody with experience, who has stood in front of children for many years and who was able to teach this to us in a normal way.

It becomes clear during the group discussions that the students are aware of the problem of how to judge children fairly despite potentially enormous diversity. They principally discuss the questions of how fairness can and should look in the school context, how it can be realised, and which obstacles and problems can exist in its realisation. They argue that differences play a secondary role for children and that "it is easier for children" to accept and tolerate each other because they are "a lot more open in that". The participating students seem to agree that it is important to develop a broader perspective of what is 'normal' as early as possible. At the same time, they consider it to be difficult to hold collective lessons for 'normal' children and children with the need for special education, especially when there is only one teacher in class. Furthermore, they are aware that there is a dilemma between the need to differentiate in school and the society being extremely heterogeneous. While in a school context, every child is supposed to receive the best possible support suitable for its own needs, in society, diversity is nearly never considered or discussed.

Perspective

The results collected so far already provide initial insights into student teachers' subjective theories and guiding ideas on heterogeneity in the school context. Further group discussions will be carried out in the near future, and questionnaires distributed. The framework will be expanded to include practising mathematics teachers in different kinds of school in Saxony. Thus, data on teachers' and student teachers' subjective theories on diversity in pupil populations will be available for comparison. Consequently, individual teachers who participated in the group discussion will be selected to receive classroom visits. Through participatory observation we hope to be able to undertake a comparison between the collective opinion emerging in the group discussion and the models of activity that are actually applied by teachers for dealing with diversity. All these data will ultimately serve the development of concepts for events for student teachers with the aim of making them more sensitive to heterogeneity and more prepared to deal with it.

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Teaching practices in a mathematics classroom and their connection to race and racism in the United States

Sabrina Bobsin Salazar

University of Michigan, United States of America; ssabrina@umich.edu

In this paper I introduce critical realism to investigate the relationship between a mathematics classroom and the broader social context in which it is inserted. To ground the new approach, I focus on race and racism, and use critical race theory to guide the use of critical realism in this investigation. This is a report in an ongoing study about the work of teaching and social change with respect to race and racism in the United States. Data comes from a laboratory mathematics classroom held every summer by a large research university in the United States. Although more analysis is still necessary, initial results reveal how broader social context can influence and be influenced by broader structures of race and racism. Moreover, the framework shows potential to illuminate the relationship between classroom interactions and social systems of inequality.

Keywords: Racism, teaching practices, critical race theory, critical realism.

The problem

This study starts from a perspective that education and schooling play a role in our society that can serve both to sustain and reproduce dominant social structures or to challenge them and promote change (Freire, 2015). It is generally accepted that our social world is unequal and unjust, and under such a view, education is frequently seen as a source of and/or a solution to social inequalities. It is therefore important to educational research, mathematics education research in particular, to better understand the relationships between educational and schooling practices and the broader social context they are situated in.

In this study, I want to zoom in on classroom practices and investigate how mathematics classroom practices can reproduce or challenge macro social systems of inequalities, which I understand as social structures that privilege certain social groups over others. Racism in the United States is for example generally characterized by a set of social privileges rather than by individual acts of hate or prejudice (Bonilla-Silva, 2006). Even though individuals being overtly racist still exist, I am referring to a different problem, in which Whites enjoy better social opportunities only because they are White, such as African Americans being incarcerated in disproportionately higher rates than Whites (Barish, DuVernay, & Averick, 2016), and Latinxs having disproportionately fewer opportunities to obtain higher educational levels in comparison to Whites (Yosso, 2006).

The main purpose of this paper is to report the application of critical realism in an ongoing study to offer new possibilities to understand the relationship between classroom interactions and the broader social context they are inserted in. I am particularly investigating race and racism in this first exploratory study, because it brings interesting dynamics between local social situations and broader social institutions. Such dynamics come from the fact that racism is deeply ingrained in American society (Ladson-Billings, 1999). I hope that a critical realist lens can help to illuminate how racism occurs within classroom interactions, specifically I want to investigate how racism can be challenged or disrupted in mathematics classrooms through teaching practices. In this study, I am particularly trying to answer the research question: How can the critical realist concept of norm circles help us

better understand how teaching practices in a mathematics classroom (can) challenge and/or disrupt structural racism? In this paper, I will present the framework, describing a few core concepts of critical realism and critical theory and discussing how these two theories can be combined to better explain the connection between classroom teaching practices and structural racism, and then I will illustrate the use of the framework with initial findings.

Conceptual framework

Mathematics education research has focused its interest on the social and political dimensions of mathematics and education for quite a while now. To better understand such dimensions, some researchers are now foregrounding power relationships in research and using a variety of critical perspectives and methods in what has been called a sociopolitical turn (Gutiérrez, 2013). In her argument, Gutiérrez (2013) describe some of the theories used by critical scholars to address power relationships in society. Critical realism is not directly indicated in her article, but it is a possible sociopolitical theory. Critical realism is a philosophy of science originated out of the need for better theories to understand the social world; in particular, critical realism explores power relationships in the social world. Specifically the focus of this study, critical realism proposes a new approach that can illuminate the relationship between social structures and micro social interactions by elaborating an analytical mechanism that focuses on the interaction between individual agency and social structure (Elder-Vass, 2010). Such a mechanism has its foundation in the critical realist concept of emergence and is operationalized in terms of norm circles. These theoretical constructs however, are not tied to any specific method to conduct research though under a Critical Realist approach. What I expect in this study is that Critical Race Theory can provide theory and methods to investigate the relationship between mathematics instructional practices and racism, and that Critical Realism can complement CRT supporting the explanation of mechanisms of (re)production and disruption of racism.

Critical realism

The basic premise of critical realism is that the world is made by real things that have real causal powers (Bhaskar, 2008). Phenomena are interpreted as outcomes of causal powers of such real things. At a first glance these ideas look very similar to positivist ideas, but they are not. In positivism phenomena can be completely determined by scientific laws, whereas in critical realism, phenomena are only influenced by scientific laws. The main idea is that these laws impose constrains and prevent possibilities otherwise available, describing a tendency rather than a certain outcome. The example cited by Bhaskar (2008) is that the path of his pen does not violate any law of physics, nevertheless it is also not determined by such laws (p. 95). There is a limitation of what a pen can do that is described by the laws of physics, yet such laws do not determine what is being traced by the pen. What is important in these basic ideas is that the world, which includes the social world, is made by real things; and that scientific laws, even social laws, describe tendencies rather than determination.

One concept that is central for critical realism and that will be very relevant for this study is the concept of emergence. Here, I am particularly adopting the compositional version of emergence as described by Elder-Vass (2010). In this version, the real things in the world can be combined in a way that, because of their structure and not only its individual properties put together, a new thing emerges in the world. Elder-Vass (2010) also refers to this new thing as an ‘entity’ or whole, and it

possesses "properties or capabilities that are not possessed by its parts" (p. 4). The idea is that the whole is not just the sum of its parts, but it is something else, with a new causal power that is, of course, derived from the individual properties of its parts, but not only this, the way the parts interact and relate with each other is also responsible for the emergence of the new thing.

The concept of emergence is what forms the layered or laminated view of the world under the critical realist perspective. A particular whole is said to be in a higher level or layer than its parts. The same whole, however, can be a part of another emergent structure; in this case the whole is in a lower level than the new emergent structure. In our social world, an individual can be interpreted as the lowest level, and the whole society as the highest level, with many intermediate levels in between, such as social institutions. The immediate higher level to an individual is, in Elder-Vass' (2010) definition, a norm circle. The norm circle is defined by the group of individuals who hold a normative belief of endorsing a social practice. By endorsing, he means that each individual in the norm circle acts to reinforce the norm and discourage behavior that does not conform to the norm. Elder-Vass (2010) argues that the shared endorsement of a norm

> when combined with these sorts of parts, provide a generative mechanism that gives the norm circle an emergent property or causal power: the tendency to increase conformity by its members to the norm. The property is the institution and the causal power is the capability that the group has to affect the behaviour of individuals. That causal power is implemented through the members of the group, although it is a power of the group, and when its members act in support of the norm, it is the group (as well as the member concerned) that acts. (p. 124)

With this argument, Elder-Vass (2010) is explaining why the norm circle is actually an emergent structure rather than only a group of people. He is explicitly pointing out what is the new causal power by showing the tendency it describes: to increase conformity to the norm.

Particularly relevant for this study is how agency is viewed in this critical realist account of the social world in which the social wholes with causal powers are norm circles. It is important to consider that causal power in critical realism describe tendencies, so the fact that a norm circle enforces compliance with a particular norm indicates that someone in this norm circle will have the tendency to act in conformity to such norm, but this is not determined. With respect to the focus of this study, race and racism, and the guiding provided by critical race theory, a central aspect of this investigation will be, in a context of racialized circumstances, the situation of an individual being in different and conflicting norm circles, i.e. individuals that are part of norm circles that enforce opposite norms, and how the study of this conflicting positions can reveal routes for social change.

Critical race theory

Critical race theory is a theoretical framework for research that foregrounds race, racism, and racialized experiences. Critical race scholars ground their argument in the idea that racism is a social construction and is different than individual prejudice. Gloria Ladson-Billings (1999) reports that race is a complex social construct that goes beyond the color of skin, citizenship, and individual acts of prejudice, forming what can be called a system of racial inequality usually hidden under a colorblind discourse (Bonilla-Silva, 2006). To overt the practices of colorblind racism, critical race scholars anchor their work on five tenets: permanence of racism, Whiteness as property, interest convergence, critique of liberalism, and counter-storytelling (DeCuir & Dixson, 2004). I will briefly

describe three of these tenets because they will be more salient to the preliminary results to be presented later on this paper.

Whiteness as property refers to the idea that that Whiteness can be viewed as a set of social (privileged) possessions, that can operate similarly to property in a capitalist society. In the context of education, Whites have some sort of control of what is valuable knowledge and who gets access to it, which can be interpreted as a kind of intellectual property. Critique of liberalism is a direct critique to liberal economic-based ways of understanding and living in the world, grounded in free-market ideologies, under which people believe that best outcome for all is achieved when there is no external regulation of the market. Meritocratic and individualist discourses are frequently associated with liberalist discourses (Solomona, Portelli, Daniel, & Campbell, 2005). Counter-storytelling is the main methodological strategy used by critical race scholars to challenge inequality and White privilege. One important aspect of counter-storytelling is the double-consciousness or angled vision attributed to individuals living in the margins of society (Anzaldúa, 1999) that is going to be one of connections between critical race theory and critical realism.

Before I discuss the issue of marginality in society in light of critical realism and critical race theory, I will briefly elaborate in another connection between the two theories. Such connection will not be the focus of this study, yet it is necessary to understand why the two theories are compatible and suited to be thought together. Ladson-Billings (1999) says that racism describes a norm in current American society and "because it is so enmeshed in the fabric of our social order, it appears both normal and natural to people in this culture" (p. 12). This idea is the gist of *permanence of racism.* So, in a critical realist account of it, there must be a norm circle enforcing and endorsing racist practices. To present one example of a norm in this circle, I will need to unpack the idea of discourse within the critical realist framework.

Dave Elder-Vass (2012) elaborates on the concept of discourse as discussed by Michel Foucault (1969). He emphasizes the idea that Foucault is concerned with the content of what we express using language norms. Moreover, he refers to Foucault asserting that there are normative practices that dictate what we can say and what we should not say, and, in some way, they also dictate how we should act. Elder-Vass (2012) constructs a realist ontology explaining how discourse in this sense can have causal powers. He argues that discourse is shaped by the normative rules enforced by members of a discursive circle through the discourse such members produce. Discourse, therefore, is the means to what the causal power of a discursive circle exerts causal effect.

With this idea of discourse, I will point the emergence of a racist norm circle to the slavery system in the colonial period. As an example of a racist norm originated in the colonial United States, I point to the normative discourse that says "African American are less than Whites". Specifically in educational contexts, such a discourse is reported in the autobiography of the former slave Frederick Douglass (1892) as a way to justify slavery. Once the norm circle reinforcing this discourse had emerged, it started to operate downwards, constraining the individuals in the circle to act accordingly. The way the discourse is reinforced has changed throughout time. For example, in the beginning of 20th century, IQ tests (Karier, 1986) helped to disseminate the idea that Blacks were less intelligent than Whites; and recent research reports such as Robert Berry (2008) reveals that African American boys are more likely to be placed in lower track courses in mathematics, in comparison to their White peers, as they advance their studies.

Critical race scholars ground their argument in this kind of normative racism in opposition to individual and overt prejudice. Theoretically, they understand this kind of racism as a social construction that brings real consequences to people of color (Chapman, 2013). This view gains a total new meaning under a critical realist perspective: Social constructions are real and have real causal powers.

Now, focusing on norm circles and searching for routes for social change, I will discuss the double-consciousness or angled-vision usually explored through counter-storytelling by critical race theory (Anzaldúa, 1999). This is usually a characteristic attributed to individuals that live in the margins of society. This place is viewed as a space of conflict of identity, a space of belonging and not belonging at the same time (Anzaldúa, 1999). In a critical realist perspective, I interpret the angle vision as a product of participating in conflicting norm circles. It is at this conflicting space that critical realism leaves room for human agency to act in a way that might not conform to a social norm. Elder-Vass (2012) discusses that when an individual participates in two or more conflicting norm circles, the outcome in terms of individual behavior can be very poorly predicted in the sense that the individual can decide for either norm, or can even create an innovative action to escape the ambiguous situation:

> In contexts of complex normative intersectionality, skilled social performances depend upon the possession by the individual of a sophisticated practical consciousness of the diversity, applicability and extent of the normative circles in which they are embedded, and indeed of others to which they are exposed, even though they may not be parts of them. Whether or not they are able to articulate this consciousness discursively, members of such societies depend upon it whenever they act. (Elder-Vass, 2010, p. 133)

My idea in this study is to explore the norm circles with respect to race and racism that exist in a mathematics classroom environment. Particularly I want to investigate what are the norm circles the teacher participates in, what are norm circles created in the context of the classroom interaction, and how particular teaching moves are interpreted in light of agency within complex normative intersectionality.

Methods

I am conducting my research as a secondary case study on a summer program held by a large research university in the United States, in which an experienced teacher publicly teaches lessons to a group of elementary students. This summer program serves different purposes: one is to be a site of learning for practicing teachers and teacher educators, because of the nature of the public teaching and the professional development sessions that follows it; another is to be a site of research for both student learning and teaching practices.

The student body of the laboratory is composed by a sampling of students from one school district in the Midwest United States. It is made to represent the demographic distribution of this district. Its composition counts with students of different ethnicities, but mostly are African American; all (or most) students come from low-income households; the students have different levels of English proficiency; and their mathematical proficiency is homogenously low.

Because the EML is a site of research for student learning and teaching practices, different types of data are collected by the research team organizing the EML. The data set includes video records of instruction, video records of pre-brief and de-brief meetings with learning teachers, copies of students

notebooks, pictures of classroom records such as charts, lesson plans, etc. Since I am analyzing data already collected, I did not engage in a relationship with the participants. I am observing and respecting their voices the best way I can by triangulating different sources of data. The data corpus is composed by video records of classroom interaction (approximately 2.5 hours per day across 10 days), detailed lesson plans for each class, copies of student work (notebooks, homework, and assessments), and photos of every collective record produced in classroom (such as charts and white board records). The high quality documentation of the laboratory classes allows detailed observation of classroom interaction that are usually not possible in regular school settings. Also, the composition of the student body, the qualification and experience of the teacher, and the laboratory setting provide a fruitful environment for the existence of multiple norm circles operating in this same space, which is important for this research.

I am following an analytic induction approach (Erickson, 1986) to code and interpret the data. The videos are being watched with a focus on interactions that involve students of color. Written records similar to fieldnotes are produced and used to identify episodes. Each episode is then re-watched; better-detailed descriptions are produced and interpreted according to the conceptual framework. In the next section I will illustrate how the episodes are interpreted in light of the conceptual framework. I will focus particularly on episodes that feature three Black girls.

Illustrating the framework

The initial analysis was focused on the interactions involving three Black girls, and findings show they can engage in mathematics discussions, and are interested in learning mathematics. These findings firstly challenge the idea that girls, specifically Black girls, are not suited for mathematics. In one example, one Black girl is behaving in a way that could have been interpreted as if she was not listening to what the teacher was discussing about the conditions of a problem they were about to work on. She was not looking to the teacher at the board and she was not quiet in her seat, but she promptly raised her hand when the teacher asked for a “wrong answer”. She correctly shared a solution to the problem that violated one of the conditions, making it a wrong answer. This example reveals a Black girl accessing mathematical content, usually a White intellectual property (Moses, & Cobb, 2001). This girl, however, was only accessing such content because the teacher ignored her apparently disruptive behavior and believed in her raised hand. The calling of the student to share her answer was a choice of the teacher; she had other students with raised hands to call at that moment, still she decided to call on that Black girl. She was at that particular moment deciding her action based in different norm circles, one that told her the student was not paying attention to the lesson, and another that told her the student knew something and wanted to share with the class.

Initial findings also reveal that the teacher consistently engages in what Boaler (2008) calls assigning competence. This means that the teacher calls a student to publicly share an answer to a problem to raise their mathematical status among their peers. To do that, the teacher specifically points to an aspect of the proposed solution and indicates why that aspect is mathematically relevant. In one example, the teacher called on a Black girl to share how she had recorded an answer to a mathematics problem in her notebook. The teacher focused in how she wrote the answer using complete sentences, and that writing complete sentences foster writing clearly and is an important mathematical practice. This girl was not positioned as a competent mathematics learner or doer, other students in class frequently did not listen to what she was publicly sharing. In this example, the teacher had to intervene

so all students were actively listening while she was showing her notebook. The students were also asked to comment what was good in her answer. With this move, the teacher raised her status as a mathematics learner and doer among her peers. The teacher was actively pushing back some kind of liberalist practice in classroom. In a liberalist practice, the teacher might pursue a meritocratic ideal and let the classroom "regulate itself", without intervening in the social relationships being (re)produced there. By raising the mathematical status of some students, in particular two Black girls, the teacher might be changing the power relationships being established among students, which can promote some sort of local socio-mathematical justice. By consistently engaging in assigning competence, this teacher is actively challenging liberalist norm circles.

Concluding comments

The implementation of this new framework still brings methodological challenges. As it stands, the framework shows potential to illuminate the relationship between mathematics classroom practices and the broader context they are inserted in. The theoretical articulation of critical race theory and critical realism can help to refine both theories, providing methodological tools to apply critical realism in empirical research, and providing new analytical tools to understand how racism can be challenged. Particularly, the first example showed how the concept of norm circles was helpful to interpret a teacher action that counter racist discourses, and the second example show how a teaching practice can be used to counter liberalist practices that promote racism. I expect that such initial results can be elaborated with the further development of this study, providing more refined analysis. Finally, I expect that this study, when completed, can provide a more detailed description of classroom interactions in light of the social context they are inserted in.

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The context of workplaces as part of mathematics education in vocational studies: Institutional norms and (lack of) authenticity

Lisa Björklund Boistrup[1] and John Keogh[2]

[1]Stockholm University, Department of Mathematics and Science Education, Stockholm, Sweden; lisa.bjorklund@mnd.su.se

[2]Institute of Technology Tallaght, Dublin, Ireland; john.keogh@ittdublin.ie

This paper presents an analysis of institutional norms and the authenticity of out-of-school contexts that are reflected in a film about mathematics teaching in a comprehensive in-service vocational studies programme. The analysis was performed through the lenses of a theoretical framework of learning in different contexts and an empirically derived framework of mathematics as part of workplace complexity. The comprehensive in-service programme, which is available on internet, aims to improve mathematics teaching and learning, nationwide in Sweden. Through our analysis, we highlight a dissonance between the pedagogical approach displayed in the film, and how its authenticity may be compromised from the perspectives of our analytical frameworks.

Keywords: Workplace mathematics, vocational studies, institutional norms, authenticity.

Introduction

A key assumption in this paper is that school, as an institution, has much to learn from workplaces. We adopt the view where mathematical notions are not only applied *at* workplaces but also are developed *in* workplaces (Wedege, 2013). One argument presented is that the context of any workplace stands in stark contrast to a formal mathematics classroom, where calling on authentic everyday experience may inhibit school mathematical sense-making (Gellert & Jablonka, 2009). Simultaneously, the performance of different jobs may be underpinned by the similar mathematics concepts as in the school context, but often interwoven in complex activities. The difference is not only in the type of work done, but in the sophistication of their contexts, such that unproblematic transfer of knowledge and skills may be highly unlikely. From an educational perspective it is rather about recontextualisation (Bernstein, 2000, see also FitzSimons & Boistrup, 2017). Through recontextualisation, a practice (for example mathematics) is transformed, rather than transferred, to an education context. In support of professional development for mathematics teachers in vocational programs in upper secondary school, Skolverket (Sweden's Education Authority), produced, among articles and the like, a number of short films, each demonstrating ways of teaching and, in the case of this paper, taking relations between contexts of school mathematics and workplaces into account. Our contribution is concerned with how the institutional norms present in one such film may restrict authentic meeting points between workplace mathematics and school mathematics, and the possible implications for such a film. Our research question was: "What institutional norms may be construed through analysis of such a film, and what are the implications for vocational students of mathematics as part of their future work?" Through a description of our analysis, we highlight the dissonance between the pedagogical approach displayed in the film and how its authenticity may be compromised from the perspectives of our analytical frameworks.

Literature on mathematics: In work and in school

That the mathematics knowledge, skills and competence that underpin work may be denied, or dismissed as common sense may be a testament to its invisibility rather than its absence from the workplace (Boistrup & Gustafsson, 2014; Keogh, Maguire, & O'Donoghue, 2014; Williams & Wake, 2007). Mathematics may be something that workers use to solve problems or manage mathematics-containing situations, and may be observable as numerate behaviour (e.g. Gal, van Groensteijn, Manly, Schmitt, & Tout, 2003). In contrast, school mathematics is topic-specific, determined by curriculum, shaped by prescribed text, built on layers of escalating complicatedness, taught to the rhythm of the semester, and formally assessed as a *passepartout* to the next level of complicatedness. Whether its transferability is enabled by the context in which it is learned or contained by it, is disputed (Evans, 2000; FitzSimons & Wedege, 2007). The introduction of 'real-world' mathematics to the classroom offers the prospect of easing the transformation from formal mathematics learning to its application in work situations . Nevertheless, it is often constrained and reduced to mathematics word-problems surrounded by 'real world' conceits, that may be "inauthentic... fragmented, static bits of tasks.. that are neither contextualized nor intellectually challenging" (Wiggins, 1989, p. 711).

In Keogh, Maguire, and O'Donoghue (2016), a Workplace Context Complexity Protocol (WCCP) is presented. It identifies workplace competence, in relation to mathematics, as being shaped by strands such as familiarity and stressors. The ability to capture and calibrate the features of many workplaces, informed by the students, may enable teachers to build a platform to establish authentic, credible and familiar workplace contexts in which to set and explain mathematics as part of workplace complexity. In Boistrup (2016), a case of a nursing aide is presented drawing on different sociological frameworks, indicating connected notions as in Keogh et al. (2016). In the analysis of the study presented in this paper, we drew on findings from these two studies.

Analytical framework

Selander (2008) presents a design theoretical perspective of learning when he explains how learning may take place in relation to its communicative constituents and associated processes, and where the design of a setting affects possible learning. This perspective is underpinned by Institutional Theory (e.g. Douglas, 1986), which emphasizes how society is surrounded by institutions which condition the actions that are possible to take. The perspective is similarly influenced by multimodal social semiotics as described by Van Leeuwen (2005) and others, where emphasis is put on the ways in which communicative resources are used, such as speech, artefacts, gestures displays and the like, and the roles they might occupy in the setting. From this perspective, learning is considered to take place in all kinds of settings, for example, the situation as reflected by the learning required to buy a bus ticket in an unfamiliar city. It could also encapsulate a workplace, where, typically, the main institutional norm is to "get the job done", part of which includes some learning.

Selander (2008) also specifies this theory in terms of a formal setting of learning, such as a mathematics lesson. In this paper we draw on the following selection of facets of this theory, highlighted in italics:

- We analysed the *institutional framing* of the lesson of the film, and the purpose of the chosen *curriculum*. We also paid attention to the *learning resources* that were present.
- We analysed the *social interaction* between teacher and students in terms of displayed *interest* in the communication, informal *assessment acts* and *teacher interventions*.
- We analysed how, and through which communicational resources (e.g. speech and hand gestures), teacher and students *presented* the *processes* undertaken during a problem solving activity. In addition, we analysed how, and with what focus, students were *meta-reflecting*, i.e. reflecting on their own actions.

In the analysis, we adopted the WCCP protocol by Keogh et al. (2016), using attributes from the strands comprising the model, e. g., *accountability*, *clarity*, *familiarity*, *stressors* and *volatility*, to capture the matrix of factors that are thought to define, enable and constrain performance in work.

Methodology: Case study and analysis

This is a case study (Yin, 1984) wherein the primary data source is a film that is part of a 'module' on mathematics in vocational studies, upper secondary level, within a larger nationwide in-service program. The vocation of the students participating in the film is not made clear. However, as the context is introduced by the teacher as being concerned with a patient and her medication, we found it reasonable to infer that the students in the film are prospective nursing aides. We do not seek to analyse what the teacher is saying and doing *per se*, but rather to examine the messages being projected by the Swedish Agency for Education to teachers, albeit through the medium of an abridged video provided for teachers in the agency's official website. Secondary data, analysed for interpreting the film's message from its institutional context, stem from a discussion template to guide teachers' in discussing the film, an interview with a home-caring nurse, potential parts of the module where the vocation of home-caring nursing aides are described, and the national syllabus for mathematics in upper secondary school, vocational studies (Skolverket, 2013).

The design theoretical perspective of learning was used in the analysis of the different processes in the film, which helped both in describing the lesson, and in the construal of institutional norms as reflected through the filmed lesson. The theory offers a framework with which to analyse learning with a particular focus on instances of institutional norms that are 'there' from the 'beginning', shaping, and perhaps containing the possible setting and learning activity. The perspective does not focus, specifically, on the content being taught and learnt. In our analysis we coordinated the design theoretical perspective with the framework by Keogh et al. (2016) in order to address how the teaching objective being illustrated, e.g. relations between workplace mathematics and general mathematics, is presented.

We transcribed the film multi-modally (Van Leeuwen, 2005), documenting what was being said and done, by whom, and identifying the artefacts that were being used and how. In the first step of analysis we both analysed the data through the two different frameworks i.e. Selander's perspective and Keogh's WCCP (Selander, 2008; and Keogh et al., 2016) separately. In this analysis, we

compared the transcript of the film with the concepts of the frameworks and with the secondary set of data. In a second step of analysis, we focused specifically on institutional norms while building on the findings from the previous analysis. The kind of institutional norms that we construed concerned relations between school mathematics and workplace mathematics. In this paper, we account for our analysis of the teacher's introduction and our analysis of the first student's contribution and whole class discussion.

Analysis and findings: A mathematics lesson claiming to draw on workplace 'reality'

This section describes our first step (description and analysis) and second step (construal of institutional norms) of analysis while indicating elements in the analytical frameworks using italics.

Description and analysis of film title and the teacher's introduction

The purpose of *curriculum*, in this case the topic claimed by the title of the film, is to go from the particular workplace mathematics to general mathematics. One central *learning resource* in the lesson is a screen at the front of the classroom where the teacher's computer is mirrored. In the beginning there is a picture of, and a short text about, a patient, called "an Irma" by the teacher. The problem is presented in writing as follows:

> Irma who is 90 years old has high blood pressure. In order to lower the blood pressure, Irma takes medicine with the active substance bendroflumetiazide. She has been prescribed 15 mg, which she should take each morning.
>
> Each morning, she should take 9 pills in total. Help Irma to fill the 'dosett' [a dosage unit]: HOW MANY PILLS OF EACH KIND SHOULD SHE TAKE?

The teacher checks with the students whether they are familiar with the active substance (*social interaction*). The students do not respond to this, and there is no sign of communicated *interest* in the topic from the students. There is more evidence of engagement when the teacher asks them to say the name of the dosage unit. There is not much *assessment* communicated to the students in this first part.

When we analysed the data with the WCCP framework (Keogh et al., 2016) one main point to consider was *familiarity*. The familiarity strand of the framework refers to how *specific* an activity is and to what extent the activity is a *regular* or irregular *occurrence*. In our interpretation, the context in which the problem is set is totally *un*familiar with regard to the students' future vocation, especially in connection to potential *accountability*. Firstly, it is not a nursing aide's *responsibility* to distribute pills into dosage units. Secondly, it is not feasible to restrict the patient to an exact number of different types of pills. This kind of *decision making* is not authentic. Thirdly, the way the teacher in the film begins his introductions, by referring to the picture of the patient as "an Irma", is not interpreted as reflecting any *familiarity*. This way of objectifying patients is not authentic from the perspective of a nursing and caring workplace (see also Boistrup, 2016).

Institutional norms construed from the title of the film and the teacher's introduction

The *institutional norms* that frame the setting of the lesson are construed from the title of the film and from how the teacher introduces the work. One institutional norm is construed as (1) "It is

important to introduce general mathematical methods based on particular workplace problems". The basis for this construal is the title of the film, which is presented at the agency's website where teachers are invited to find and use the film. A second institutional norm is construed as (2) "It is important to find mathematics in the context of the students' future vocations and build on that". This norm is construed from the teacher's way of starting the lesson with the patient, Irma, where he describes the problem with her pills. This norm is part of the institutional framing also, since it is stated in the national syllabus that mathematics in vocational upper secondary school should be strongly influenced by the future vocations of the students (Skolverket, 2013). A third institutional norm is construed as (3) "It is not really important to secure the authenticity of contextualized tasks". The basis for this is described above in relation to Keogh et al. (2016) and Boistrup (2016), where it is clear that this is an example of un-familiarity with a workplace context, rather than a lesson where the mathematics teacher is making an effort to acknowledge the context of future workplaces of the students.

Description and analysis of first student and whole class discussion

After the introduction the teacher starts a new sequence (*teacher intervention*), where he invites the students to discuss, in pairs (*social interaction*), possible solutions to the problem. At this point, the teacher is still standing at the front of the classroom, looking at the students. The students are looking to their front. The teacher repeats the problem and tells the students to discuss it for 30 seconds. The *resources* the students are offered to use are speech, writing, and calculators etc.

The teacher *intervenes* and initiates a whole class session. The teacher asks a student to *present* how she solved the problem. The student *represents* her previous solving of the problem with *resources* as speech and gaze when *presenting* her solution to the teacher and the class. Her first *meta-reflection* occurs when she says that her answer may not be correct. She then *presents* that she doubled the pill with 1.25 mg substance, to make 2.50 mg comprising 2 pills. She then added one of the 2.50 mg pills, to make 5 mg comprising 3 pills. She then took this times 3 which makes 15 mg and 9 pills, as required in the problem. In the film there is no further *discussion*. Instead the teacher *intervenes* telling the class, while pointing at the screen, "The method that Jasmine mentioned here is building on those numbers being easy". He elaborates for 10 seconds on this and continues, "but there is a general method. And this we call a system of equations, i.e. to set up a system of equations." This can be viewed as feedback, which constitutes an implicit *assessment*.

When analysing with the framework by Keogh et al. (2016) one point was again *familiarity*. The student in focus solves the problem in a way that could be the case for a nursing aide (Boistrup, 2016). (Here, briefly, we assume that the problem is authentic.) *Accountability* attributes are present in her reasoning, for example she takes the *initiative* and she uses the *concreteness* that suits this particular situation. The teacher stresses *theoretical concepts* and *abstract thinking* while stressing that the student's method only works with "easy numbers".

Institutional norms construed from first student and whole class discussion

The instance of the first norm regarding the high relevance of introducing general mathematical methods, is construed as present in the end of the sequence described. Implicitly, we regard the third norm, i.e. the lack of relevance to secure authenticity, to be apparent in the sequence. The teacher indicates that the student's solution is not good enough since the method would not work well with

other numbers. However, it was taken for granted that a problem with a certain number of pills was relevant in the presumed context. We also construed a fourth institutional norm: (4) "All relevant power and knowledge resides with the teacher". The basis for this construal derives from the way that the teacher assessed the student's answer, seemingly dismissing her practical and correct solution to the problem as posed. In addition, the teacher did not allow much opportunity for the students to *meta-reflect* and discuss during the whole class session.

Conclusions and discussion

The findings derive from our analysis of a filmed lesson, which is part of a library of resource material for in-service training for mathematics teachers. Authenticity is key from the vocational mathematics learner's point of view. In summary, we conclude that the workplace context presented in the instructional video is transparently contrived, dehumanised and shaped by an unrealistic aim, e.g. that a patient should be required to take exactly 9 pills. Furthermore, it is not the nursing aid's job to decide medication. That this description of the setting is patently false to the students, risks discrediting the underlying content and, ultimately, may undermine the intended learning outcome.

In the film, a teacher introduces how vocational students can use a system of equations when solving problems in their future work as nursing aides. From our analysis and findings, we conclude that the film depicts a mathematics lesson where little time is spent on student interaction and student meta-reflection. The teacher is the most active person who intervenes most often. The overarching content promotes the benefits of solving workplace problems with formal school mathematics. Our analysis and findings reveal that the claimed workplace context, in fact, is not authentic to any significant extent. Nothing of the workplace complexity, shown in previous research (e.g. Keogh et al., 2016), is present and the teacher does not use the students' knowledge about the work as nursing aide as a help to achieve authenticity.

We view the construed institutional norms as a series of essential messages (from the National agency to mathematics teachers), that can be interpreted from the film as:

1. It is important to introduce general mathematical methods based on particular workplace problems
2. It is important to find mathematics in the context of the students' future vocations and build on that
3. It is not really important to secure the authenticity of contextualized tasks
4. All relevant power and knowledge resides with the teacher.

To a large extent the construed institutional norms coincide with a typical school mathematical discourse as described in Gellert and Jablonka (2009). Looking at the first norm, we argue that it is important not to exclude vocational students from general mathematical methods, but whether it is always important to build on workplace problems or not may be disputed. If it is essential to acknowledge the complexities of mathematics in work, as described in the first sections of the paper, it may be more suitable to introduce systems of equations intra-mathematically rather than drawing on inauthentic problems. The inherent risk of inauthentic contexts is it may serve to undermine rather than support the learning outcome. Regarding the second norm, there are a number of studies (e.g. FitzSimons & Wedege, 2007) revealing that in order to secure the authenticity of workplace mathematics, a way forward is enabled by cooperation between the stakeholders,

including mathematics teachers, vocational students, and workplaces. Such projects may create opportunities to find relevant pathways from workplace problems and ways to solve them with general mathematical methods.

The primary data source of our analysis was a filmed example for a module on mathematics in vocational studies within a larger nationwide in-service program. The film is used in this study as a starting point for a discussion. Our analysis acknowledges that there is not much in the module, as a whole, that reflects a balanced account of the work done by nursing aides. Hence, we find it hard to see that a teacher-discussion based on this film alone, without, for example, alternative films for comparison and contrast, would challenge a school mathematical discourse, where out-of-school contexts are not meant to be taken as 'real' (cf. Gellert & Jablonka, 2009). Rather, we see a risk that a film like this perpetuates existing institutional norms. We concur that much more work is needed in the future, both in terms of professional development and research, to enable and facilitate mutual learning opportunities between school practice and workplace practices. In such work, our use of the design theoretical perspective of learning (Selander, 2008) may be of use in the analytical description of mathematics lessons with an interest in institutional norms, teacher interventions, communication and roles of modes. In addition, the framework is useful for comparisons within and between workplaces since the perspective also includes informal and semiformal settings, such as workplaces. Additionally the WCCP framework (Keogh et al., 2016) may be useful for identifying authentic mathematics with reference to the complexities of workplaces with an interest in the factors that define, enable and constrain performance in work.

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Exploring Roma learning mathematics: A socio-mathematical view

Javier Díez-Palomar

University of Barcelona, Faculty of Education, Spain; jdiezpalomar@ub.edu

In this paper, I discuss six case studies of Roma people who have overcome inequalities to learn mathematics. I explore which elements may explain their success, as well as which ones appear to be barriers that makes learning more difficult for students belonging to an ethnic minority. I draw on their testimonies using a qualitative methodological approach. The analysis of the data reveals that social representations about Roma have a major impact on these six Roma people attitudes and beliefs towards mathematics and schooling, which also affects their strategies to learn mathematics. The six narratives that I discuss in this paper suggest that success happens when Roma children are not segregated from the mainstream, but receive the same mathematics curricula as their peers.

Keywords: Roma, mathematics learning, successful learning trajectories. SLT.

Roma within the literature on mathematics education

In the field of mathematics education there are few studies about how Roma children perform in mathematics (Chronaki 2005, 2008; Stathopoulou & Kalabasis, 2007). Stathopoulou and Kalabasis (2007) analyse the relation between Romanó (Roma language) and the learning of mathematics in Greece. According to them, language is a form of cultural identity for Greek Roma children, and they use it to resist the homogeneous discourse of the school. Stathopoulou and Kalabasis define the academic Roma culture as an "oral" culture, which they use to justify reputing the claim about a "lack of written Roma language". Starting from this statement, they argue that Roma children are proficient at oral calculation methods connected to their cultural roots, because Roma culture is mainly based on oral tradition.

Chronaki (2008) suggests introducing hybrid practices[1] as a way to break with the hegemonic discourse of the school (Matusov, 2009). Drawing on Bakhtin's (2010) concept of polyphony, Chronaki argue that we need to develop dialogic practices to include the voices of all students in the classroom, including students from minority groups, not only the ones who share the monologist hegemonic discourse.

There is a lack of scientific literature on the type of actions performed by Roma children that appear to be successful in terms of achieving academic success and developing, what I call, successful learning trajectories (SLTs). Drawing on previous research (Flecha, 2014), I define SLTs as the set of practices and interactions conducted by an individual to pass his/her exams, successfully obtaining his/her school certificate(s). I use the "grading of a test, exam or any other assessment procedure" as indicator of success, for the lack of a better indicator of "learning." According to the European

[1] *Hybrid practices* has been used in linguistic and cultural studies to characterize situations within the school (or the classroom) in which participants draw on different social and cultural backgrounds (Gutierrez, Baquedano-López & Tejada, 1999). *It* includes formal practices (school like practices) and practices related to what Luis Moll calls *funds of knowledge* (Gonzalez, Moll and Amanti, 2006). Chronaki (2008) uses the term in that sense, in the context of Roma culture.

authorities in education, the minimum level of education expected for everyone is compulsory education (goals of the *Horizon 2020* Program), generally up to age 16[2]. For this reason, "early leaving" and "dropping out" of compulsory education is considered "failing" in this approach.

Methodology

This study was part of a larger research project investigating the response of individuals from ethnic minorities to overcome the social inequalities they face in the formal educative system when learning mathematics. This research interest emerged in the frame of analysing how children at risk of facing these inequalities can find their way to gain successful scores at school. The data was collected in Barcelona and its metropolitan area. When conducting this study, I realized that Roma people developing SLT (in mathematics) made use of strong support from their relatives. In order to better understand their process of learning, I decided to conduct a series of interviews with six Roma individuals, previously identified as "successful cases" in the sense that all of them had obtained good grades in mathematics during their academic trajectories (until the last course taken) (see table 1).

Pseudonym	Year of birth	Academic trajectory	Current situation
Federico	1984	University degree	PhD candidate
Joana	1990	University degree	PhD candidate
Joaquim	1988	Elementary degree	Access > 25 years old
José	1979	VET degree	Access > 25 years old
Antonio	1978	Secondary degree	Access > 25 years old
Aroa	1996	Upper secondary degree	Working

Table 1: Description of the participants in the study

To collect the data, I used personal interviews, an instrument with questions oriented to identify the elements that explain participants' success in learning mathematics, according to their personal (subjective) point of view. Drawing on Chase (2005),

> Contemporary narrative inquiry can be characterized as an amalgam of interdisciplinary analytic lenses, diverse disciplinary approaches, and both traditional and innovative methods –all revolving around an interest in biographical particulars as narrated by the one who lives them. (p. 651)

This method of inquiry is rooted in previous work, of researchers such as Thomas and Znaniecki (1918/1927), Garfinkel (1967) and Mills (1959). It involves collecting the testimonies of participating people using a number of instruments of data collection, including life stories, self-reports, oral biographical memoires, testimonies, in-depth interviews, recorded narratives and life review (Mertens, 2009). This method of inquiry starts from an epistemological approach of individuals as primary sense-making agents (Giddens, 1991; Riley & Hawe, 2005). Personal narratives allow us to see first hand the interpretations made by the protagonists of their own life experiences. As Bruner (1990) claimed, we understand our world through the lenses of personal narratives. In addition, we also make meaning (and even build meaning) through those narratives. This type of method (inquiry)

[2] Almost in one of three European countries "compulsory education" goes up to 16 years old. For more information, see: http://eacea.ec.europa.eu/education/eurydice/documents/facts_and_figures/compulsory_education_EN.pdf

led me to include the voices of the participants within the wider study. All information was transcribed for further analysis.

I selected six participants for the purposes of this paper. All met the requirement of having developed a SLT. As we can see in table 1, two of them are now enrolled in PhD programs. Three more, Joaquim, José and Antonio, are preparing to access university degrees, through the exams set up by the Government for people over the age of 25. Finally, Aroa is a girl (the youngest one in the group) who after finalizing her studies in the high school, decided to start working.

In order to analyse the data, I used narrative and discourse analysis (Mertens, 2009), drawing on a communicative methodological approach (Aubert 2015, Sánchez, Yuste, de Botton, & Kostic, 2013). This approach focuses on the analysis and interpretation of the discourse from the dialogue with the participants, using validity claims (Habermas, 1984). The interpretation is organized in two different dimensions of analysis: transformative and exclusionary. Transformative dimension includes all aspects that will enable the subject of the study to answer positively to the research question. Exclusionary dimension has the opposite meaning: it includes all aspects that will avoid (or make more difficult) answering the research question. For the study reported in this paper, the transformative dimension includes all aspects leading the participant to achieve a SLT, whereas the exclusionary dimension refers to all aspects making difficult (or even avoiding) the subject to achieve a SLT.

		Attitudes	Beliefs	Contents	Strategies	Social Representations
Individual	T*	1	2	3	4	5
	E**	6	7	8	9	10
Teacher	T*	11	12	13	14	15
	E**	16	17	18	19	20
Family / Community	T*	21	22	23	24	25
	E**	26	27	28	29	30

* Transformative dimension
** Exclusionary dimension

Table 2: Coding categories for data analysis

Interviewers and interviewees established a dialogue to explore the data collected, which allowed us to identify meaningful topics regarding interviewees' learning trajectories. In a subsequent interview, I asked them to further clarify those topics. Drawing on this dialogue I elaborated a key to codify all data collected (see table 2).

The categories (codes) emerging from the discussions include: attitudes, beliefs, content, strategies and social representations. The first two concepts were defined in McLeod's (1992) terms. Content is referring to the mathematics itself (the curriculum). Strategies are defined in Maehr's (1983) terms

(see also Ames & Archer, 1988). Social representations correspond to the Moscovici's (1981) definition.

Discussion

Next I provide first an inductive analysis of the data collected through the interviews. Then I try to create a tentative model to make sense all the relevant variables identified by the participants in the study, explaining their 'learning process'.

First step: Inductive analysis

Federico provides a good example illustrating the type of answers obtained from all six Roma interviewed. He claims:

> In my case, the key aspects were, one I was passionate [about mathematics] and I really like science, so my motivation was somehow 'natural', or 'intrinsic'. In addition, my teachers of mathematics and biology used to give me extra homework. I remember that in 4th grade - ESO [secondary education – middle school] I used to solve problems of 1st grade - *Bachillerato* [secondary – high school], and this was very motivating for me. It was the opposite in Catalan or English, teachers used to reduce my tasks so I could pass them [with no work]. Not the case for Spanish: I also used to get difficult homework to do in the classroom or at home. (Federico)

In his words, we can see that aspects particularly positive in his academic career include effort, a challenging curriculum, a personal motivation (passion) for some of the academic topics (such as mathematics or biology), whereas no enthusiasm at all is devoted toward other ones (Catalan or English). From a very tentative entry point, we can infer here that 'motivation' is, somehow, connected to 'grading' (as learning 'indicator'), given that Federico (for example) got excellent grading in mathematics and biology and, in fact, he started 'biology' for his minor at the university. But he also declares that his family (specially his mom), took a relevant part in his learning process.

> In an informal way, my mom taught me mathematics in the kitchen, or when shopping. Topics like volume, arithmetic, counting, prices, change, […] My dad, as he was in the construction sector, taught me how to calculate budgets, how to calculate the price for square meters, how much does it cost the staff, rates. (Federico)

He and Aroa, Antonio, José, Joaquim and Joana as well, explain that their good grades in mathematics were consequence, mainly, of their family involvement. Their parents were the ones teaching them to develop mental calculation skills, estimation skills, etc., which were highly valued in the school framework. Stathopoulou and Kalabasis (2007) also provide clear examples in the case of Greek Roma, confirming this finding. However, this is not a main strategy for learning mathematics for any of our six cases. What they claim is that in order to obtain good grades in mathematics, they basically had to study hard, to put a lot of effort, using academic strategies to better learn: outlines, summaries, use of key words, or other mnemonic strategies, practice, problem solving, homework every day, etc.

However, drawing on the data that I collected, it seems that certain social contexts could become sometimes a barrier limiting the opportunities of some Roma individuals to successfully perform in their grades. Joana, for instance, felt isolated from the 'Roma world', whereas Joaquim resists and rejects school because "[school] it is not Roma". Teachers are crucial. According to Federico, some teachers really do not help Roma students because they feel that Roma are not interested in education.

Joana holds a similar view. She explains how for many of her Roma friends teaching was just a matter of "being happy attending the school", rather than "being a place for expand their learning". According to Joana, that was the consequence of some teachers' prejudices against Roma children:

> Not always, but in some cases, yes. It was not the case with me; my teachers always encouraged me to continue studying. But most of the times they did not identify me as Roma, and I have come to hear some pejorative comments towards my people from my teachers. In the classroom, I never had so many Roma peers. But I know other Roma girls who explain to me that, in their high schools, teachers, instead of teaching them the lesson, they lead them to see the *telenovela* [soap operas on TV]. I guess that it is the easiest solution and they think that doing so, the girls would be happy of attending the school every day. But I think that it is the opposite in fact, because the girls and their families know that they are not learning anything, so attending the school is useless. If things are like that, then they can stay at home and see the *telenovela* over there. (Joana)

Looking at the testimonies of the six individuals interviewed, what they highlight as the most exclusionary factor is segregation, the separation of Roma students from the mainstream. This is due either to stigmatization from some teachers, or because genuine wishes of some teachers to better help them, hence they use "separation into homogeneous small groups" as a way to 'concentrate' additional support. However, according to Federico and Joana, those efforts are useless, since taking Roma children away from the mainstream does not help them to better learn mathematics, but the opposite; this practice leads the Roma students to be labelled by their peers and teachers, creating a stigmatized social representation about Roma. Therefore, this process somehow "announces" the academic performance of Roma children even before they conduct the tests. It is like a "self-fulfilling prophecy". This perverse effect has been proved many times in the field of mathematics education research with children from vulnerable groups (Secada, Fennema, & Byrd-Adajian, 1995; de Abreu, Bishop, & Presmeg, 2001). When the school creates ability groups to segregate vulnerable groups children from the mainstream, creating low level groups, or designing segregated paths tracking, the consequence is that these children fail.

Second step: Creating a model of analysis

Drawing on the analysis of these testimonies, after coding them with the qualitative software package *Atlas.ti*, I produced a tentative model to describe SLT. I called this scheme *Learning Core Matrix* (LCM), as in figure 1, drawing on the variables identified by McLeod (1992), Maehr (1983) and Moscovici (1981), as reported earlier in this paper. The sum of all these components forms what I call LCM. I use this scheme to understand how every SLT works. On the subjective level of analysis, personal attitudes are shaped by social representation of being Roma. Social representations include values, ideas, metaphors, and beliefs (in the sense of Moscovici's notion) related to learn mathematics being Roma. Strategies refer to the practices that individuals (Roma students, teachers, etc.) perform to teach, learn, resist or avoid mathematics. Attitudes include positive or negative evaluation of people, objects, events, activities, ideas, etc. in the frame of teaching and learning mathematics.

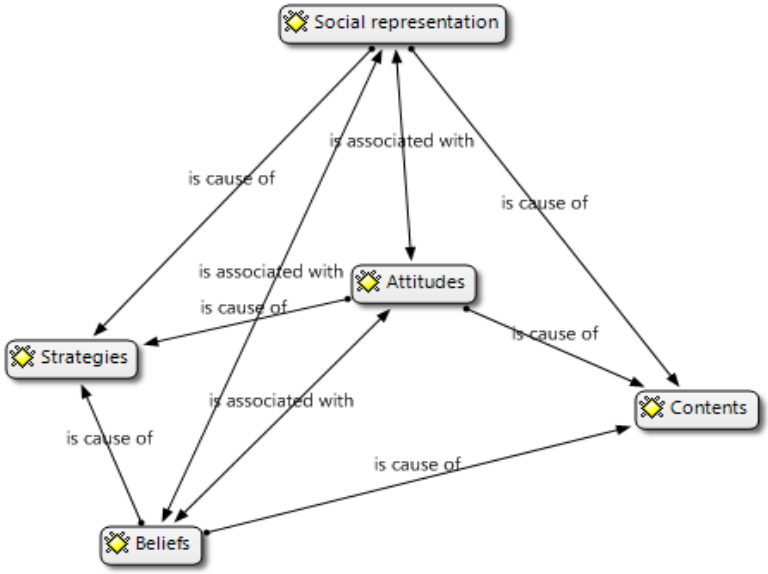

Figure 1: Components of the Learning Core Matrix (LCM) associated to a SLT

Beliefs include personal mental states regarding people, objects, events, activities, ideas, etc. Finally, contents refer to the mathematics curricula. According to the data collected, it seems that a 'positive' social representation of Roma may explain why some teachers do not segregate Roma addressing them to low-achievement groups with poor content (in mathematics), whereas other teachers holding 'negative' social representations about Roma use segregation strategies with them, lowering the curriculum, for instance. It is also the case that the same teacher may also project good expectation in one particular Roma student, whereas segregating other ones. Having a positive or negative social representation depends on the teacher attitude towards Roma identity, which is closely interlinked to teachers' personal beliefs. From the student point of view, the model works accordingly: confidence in the school (positive belief) is attached to a positive attitude in the classroom, as well as to the use of a variety of strategies to learn mathematics (such as prepare exams some days ahead, do homework, look for extra work, etc.). This is associated to a teacher's positive social representation of Roma as successful learner (in mathematics), as well. The individuals who show positive components in this LCM use to hold SLTs more likely than the ones who, at some point of their lives, had a (mainly) negative LCM.

Conclusion

The six narratives I have discussed here suggest that success happens (or is more likely to happen) when they have had positive LCMs. When all of the five components of the LCM (or most of them) are positive, then it is more likely that an individual would develop a SLT. On the contrary, when the negative component is prevailing, then is hard to see SLT as a result. Joaquim, for instance, at some point of his life dropped out the school because he was feeling resistance against the school institution. According to him the main reason to explain such attitude was his negative social representation of "school" as something alien to his identity. This feeling could be, somehow, the result of being segregated by certain teachers holding negative expectations towards Roma students.

The role of the family plays a crucial role to overcome the difficulties and barriers that some Roma students face along their school trajectories. A significant amount of these barriers is connected to prejudice and negative social representations about Roma. Family may be a resource. However,

sometimes this is not true because the members of the family did not have any opportunities or possibilities to study themselves. For this reason, they cannot become 'resources' to help their children to solve their mathematics assignments. But, according to Hoover-Dempsey et al. (2005), the families can look for further resources to reinforce their children's learning. Recent studies suggest that family engagement in the school has major impact on learning than just appointing family members in the school to 'report' on children's behaviour (Díez-Palomar, Santos, & Alvarez 2013).

LCM may have the potential to explain both Roma's SLTs or the failure of many Roma children in the school, as narrated by Joana, Federico, Joaquim and the rest of their peers. In their narratives, they explain how many of their Roma peers used to be re-allocated to low-level classrooms, how teachers use to decide to lower the grades for them, cutting down on the curriculum, or asking them to do *naive* work (like painting) rather than problem solving or other high-mathematics-oriented tasks. However, examples like the narratives by Federico or Joana may help us to understand how holding a positive identity connected to showing positive attitudes in the school, using different strategies to overcome the difficulties related to mathematics itself (epistemological, ontological, etc.), combined with high quality curricula and classroom organization, may end in the development of SLTs.

The model that I presented here is not generalizable. We need further quantitative studies to either accept or reject this approach. This would be the next step in the near future.

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(Wanting to do) Ethical research in a shifting context

Andrea Eikset, Trude Fosse, Troels Lange[1], Johan Lie, Magni Hope Lossius, Tamsin Meaney[2] and Elena Severina

Western Norway University of Applied Sciences, Norway; [1]trl@hvl.no; [2]tme@hvl.no

This paper describes the features of ethical research and how we attempted to undertake this kind of research. Our contention is that the aim of ethical research should be to produce action for change. Our understandings of ethical research led us to pause our negotiations for setting up two new projects, in kindergartens in Norway. By taking seriously our potential collaborators' concerns, we were alerted to how kindergartens were simultaneously seen as both the cause and the salvation for several issues. In media discussions, which often originated with the Minister for Education, there was a perception that there was a need for more learning, particularly of mathematics and language, to overcome difficulties that children, especially immigrant children, may have when they begin school. These discussions were often contradictory with kindergartens being placed in an invidious position of navigating these discussions for their work with children.

Keywords: *Ethical research, multilingual children, ICT, play, learning.*

Ethical research

In August 2016, we were in the early stages of setting up two projects involving dialogue between ourselves, as mathematics education researchers, and the parents and teachers of multilingual children in kindergartens in Norway. As described in the next section, our initial discussions with an organisation about one project made us stop and reconsider what we wanted to do and why. To do this, we clarified our ideas about the kind of research, ethical research, which we wanted to undertake. In this paper, we discuss how implementing its principles resulted in us investigating the shifting landscape of priorities in Norwegian kindergarten policies.

Our definition of ethical research includes a number of aspects, some of which are inspired by critical research. Critical research has a social justice aim and as such requires researchers to be comfortable with the ambiguity connected to working against oppression. Proposing bricolage as way of conceiving the range of research methodologies needed for conducting critical research, Kincheloe, McLaren, and Steinberg (2011) stated that "comfortable with the ambiguity, bricoleurs as critical researchers work to alleviate human suffering and injustice even though they possess no final blueprint alerting them as to how oppression takes place" (p. 173). We consider that ethical research involves more than evoking sympathy for participants. Like Harris (2013), we want to work with participants to produce "action for change" (p. 87). Nevertheless, we are aware that concerns about misinterpreting interactions and situations have led some researchers to withdraw from "action for change" research in case it produces undesirable outcomes, due to a lack of knowledge about the context in which they worked (Sultana, 2007).

Thus, we consider that to design research that would produce action for change, academic researchers need to negotiate the research process with participants (Potts & Brown, 2005). In accepting this requirement, we recognise that this may make us uncomfortable. Operationalising social justice as action for change means we must live with ambiguity associated with how data should be collected and analysed. Our uncomfortableness with this ambiguity is likely due to preconceptions that research

should provide definitive responses to issues; a reflection of the discourses that surround us as academics. It is also due to how we operate within the power relationships connected to being researchers, who tell the stories of participants (Etherington, 2007; Harris, 2013). Ethical research requires us to interrogate these power relationships and not assume we know the truth.

Therefore, reflecting on our decision-making in the research, both with and without our participants, is an important component in ethical research. Reflexivity has been promoted as important in that it makes transparent any dilemmas in the research.

> Reflexivity is … an ability to notice our responses to the world around us, to stories, and to other people and events, and to use that knowledge to inform and direct our actions, communications, and understandings (Rennie, 1998; Wosket, 1999). When we extend that skill [reflexivity] into the practice of reflexive research, we need to be aware of the personal, social, and cultural contexts in which we (and others) live and work and to understand how these affect our conduct, interpretations, and representations of research stories. (Etherington, 2007, p. 601)

In summary, ethical research is connected to action for change, but to achieve it we need to negotiate with participants what the research should be and how it should be conducted. This requires us to have a rich understanding of the context surrounding the site of the research. We must also accept uncertainty and be reflexive about our roles so that the power imbalance in our relationship with participants does not result in a covert control. We have a responsibility as researchers to find out about the relevant contexts and not just expect our collaborators to be the ones to inform us. We must also be aware that the negotiation can result in the research being "productive failure" (Harris, 2013, p. 89), rather than the change that we jointly want to work for.

Unease and the project proposal

In the negotiation of new projects with organisations that we had not previously worked with, our understandings of ethical research made us pause when some unease was shown. In the projects, we wanted to work with kindergartens teachers and parents of multilingual children. In one project, we hope to develop and trial playful mathematical apps that would encourage children to discuss them with kindergarten teachers in Norwegian and with their families in their shared languages. The change that we want to produce through having children engage with the apps is for them to develop both their Norwegian and home language(s) for discussing mathematical ideas.

However, in the initial meeting, the complexity, connected to combining ICT, through mathematical apps, with the development of children's mathematical register in more than one language as well as a request to involve immigrant parents, seemed to overwhelm those we talked to. The response was positive in that they felt the kindergartens would want to participate, but there was a constant stream of questions about what would happen and what the teachers would have to do. Although we tried to explain our aim of negotiating the project with the teachers and the parents, there was unease about why we did not have a clear plan for what we wanted (the teachers) to do. This unease made us reflect on the context of kindergartens in Norway, to determine what might provoke a need for certainty.

Our reflection indicated that the projects came at a time when those working with kindergartens face much uncertainty and it became important to identify the features of the shifting landscape which affect kindergartens teachers' work. We considered that an increased awareness of this landscape

would support us to be more respectful of the circumstances and improve our possibilities to negotiate with kindergarten teachers and parents about how the projects should be implemented.

The shifting landscape

In this section, we present our understandings of some of the features of the shifting landscape including discussions about changing the role of kindergartens as one of preparing children for school, through supporting children to learn better mathematics and Norwegian and by incorporating ICT into children's play. Each issue has been the subject of much debate over the last few years. Our investigation indicated that in discussions about kindergartens, teachers were often positioned both as responsible for the problems and simultaneously also the solvers of the very same problems.

Changes to curricula philosophy for early childhood

In Scandinavian kindergarten curricula, the focus has traditionally been on the whole child, emphasising their integration into society (Bennett, 2005). A revision of the Norwegian curriculum for kindergartens, the so-called Framework Plan (Kunnskapsdepartementet, 2011), that sets out their responsibilities has been ongoing for some years. However, in 2016 the Minister for Education rejected the draft, which followed the philosophy of play-based learning, proposed by contracted early childhood professionals. In particular, this delay to revising the Framework Plan seemed to result in our potential collaborators being uncertain and frustrated. However, the Minister had decided that his department would write the Framework Plan (Kunnskapsdepartementet, 2016a; Støbakk, 2016) in line with a white paper that he had commissioned about providing "better content" in kindergarten (Kunnskapsdepartementet, 2016b). Although this suggestion has received significant criticism from those working in the field, the Minister continues to talk about kindergartens needing to prepare children for school. As noted in some of the critiques (Bae, 2016), play – although in the title of the white paper (Kunnskapsdepartementet, 2016b) – is almost completely missing from the discussion with the attention being on what children are to "learn". This indicates a deliberate change to situating kindergartens' primary role as preparing children for school. This interpretation was reinforced with the revelation that Norway was to participate in the first round of PISA tests for 5 year olds starting in 2017/2018 (Moss et al., 2016). In Norway, five year olds attend kindergartens and comparing them on international tests will emphasise the importance of school knowledge. Mathematics will be one of the knowledge areas assessed in these International Early Learning Studies (Moss et al., 2016).

Mathematics and the "realfag" strategy

In a series of initiatives contributing to shifting the focus of kindergarten away from the social policy pedagogical tradition (Bennett, 2005), another report, specifically about improving mathematics and science subjects, "realfag", in kindergartens and schools, was commissioned by the Minister and released in August 2015 (Kunnskapsdepartementet, 2015). The Minister in justifying and promoting this policy had linked Norway's future financial well-being to a need for more focus on mathematics in kindergartens (Lange & Meaney, 2016). This prompted discussion about whether moving more towards a "readiness for school tradition" and away from the "social policy pedagogical tradition" is appropriate for Norwegian kindergartens. Like the white paper about better content in kindergartens (Kunnskapsdepartementet, 2016b), this report and its recommendations have been criticised by those working in the field even before it was published (see for example, Pettersvold & Østrem, 2014;

Schaanning, 2015). A question arises about what improving the content in kindergarten means when the current Framework Plan (Kunnskapsdepartementet, 2011) already contains goals for providing mathematical learning opportunities to children (Digranes, 2014). The implication was that kindergarten teachers were not doing enough to support children to learn the necessary mathematics knowledge for school. For kindergarten teachers and the administrative leadership, there remains uncertainty about how to implement the "realfag" strategy while they wait for the Framework Plan to be finalised. Although the outcomes are clearly connected to "improvement", perhaps assessed through tests of 5 year-olds, the lack of information for kindergartens about how to work with this report remains a source of frustration.

Multilingual children in Norwegian kindergartens

Alongside discussions about the role of mathematics within kindergartens, there have also been discussions about the children needing to learn "good" Norwegian language. These discussions are diverse and in some ways contradictory. Some of them refer to the white paper on better content in kindergarten (Kunnskapsdepartementet, 2016b), which includes the push by the Minister to introduce mandatory language testing of kindergarten children, such as in Fladberg (2015), and to the legislate requirements for Norwegian language skills of employees in kindergarten as noted by Haugsvær (2016). The Minister's justification for the testing was that a significant proportion of children begin school without good Norwegian skills (Svarstad, 2015). Although the suggestion for mandatory language testing was rejected by the parliament in June 2016, uncertainty about how kindergartens should work with children's language development remains (Fyen, 2016; Schaanning, 2016).

Connected to these discussions, although often implicitly, is the issue of immigrant children and their learning of Norwegian so that they would be ready for school (Redaksjonen, 2016). Children who have another language than Norwegian as their home language are given the same tests as those who have Norwegian as their home language. Unsurprisingly perhaps, the results generally indicate that multilingual children are not as competent as children who speak Norwegian at home. However, in this debate, the kind of language development seems to be implicitly about ensuring conversational language. Language to discuss mathematics is not specifically mentioned either in discussions about more mathematics in kindergartens or in discussions about improving language development.

Linked to the issue of multilingual children's Norwegian language skills is a long running debate about family payments that parents can use for children to attend kindergarten or to look after them at home (Rosa, 2007). Recently, attendance by immigrant children in kindergartens has increased (Barne-, ungdoms- og familiedirektoratet, 2016), providing them with increased opportunities to learn Norwegian. The discussion about insufficient Norwegian for school has been linked to children who are kept at home during the kindergarten years, although the Minister rarely acknowledges this. Still, there is some evidence that children may not be learning conversational Norwegian while in kindergartens. The responsibility for improving the situation lies with the municipalities which oversee kindergartens.

At the same time, there also has been criticism about the lack of effort by kindergartens to achieve the Framework Plan's (Kunnskapsdepartementet, 2011) requirement to develop all of the children's languages (Sundby, 2016). Nevertheless, it is acknowledged that it is difficult for kindergarten staff to do this if they are not be fluent in these other languages (Otterstad, 2016). Again, kindergartens are

situated as being responsible for not doing enough but with no clear pathway for how they could improve their possibilities for supporting children's home language skills. There is no discussion about using home languages for discussing mathematics.

ICT and kindergartens

ICT is an area that children in kindergarten are also supposed to have experiences with, according to the Framework Plan (Kunnskapsdepartementet, 2011). Yet the discussion about whether or how to incorporate ICT in kindergartens continues to circulate, partly because of the constant changing of hardware and software. For example, the rapid increase in the use of touch-screen devices by children at home (Hardersen & Guðmundsdóttir, 2012) has not been matched by their use in kindergartens (Bølgan, 2012). As well, research in information literacy skills connected to ICT has shown that older students who speak other home languages than Norwegian are likely to have less of these skills than those who speak Norwegian at home (Hatlevik & Guðmundsdóttir, 2013).

In seeming contradiction to the Framework Plan's requirements, the Minister has been critical of the unbridled enthusiasm for ICT in kindergartens and schools, suggesting that there is limited research evidence to show that ICT contributes to children's learning (Todal, 2015). During a visit to a kindergarten, he described his fondness for paper books over digital ones (Ruud, 2016). For kindergartens deciding how to use ICT with children, there are mixed messages about if and how they should integrate ICT into possible learning opportunities for children.

What did we learn from mapping the landscape?

Although as researchers we were aware of the debates raging around kindergartens, it was not until we investigated them that we understood how they may be affecting the possibilities our potential collaborators saw for negotiating with us. As teacher educators, we also face major changes to our working environment, initiated by the Minister of Education. However, our standing as academics, which provides us with recognition and discussions beyond our immediate working environment, perhaps made us blasé about the impact that uncertainty had on kindergartens' perceptions of what they could do. Kindergarten staff, even though many have a Bachelor degree, are often not given the same status as those working in universities or even schools by the general public. By investigating what was being discussed and in what ways, we better understand the uncertainty that kindergarten staff saw in how we presented the potential project to them.

The debates, around kindergartens and what their focus should be, situate kindergarten teachers and administrative leaders as being both responsible for the problems and also their solutions. As discussed in the previous section, kindergarten staff were being presented in the media debates as not preparing children well enough for school. Official reports situated them as not developing the children's, especially multilingual children's, language(s). They were also not providing children with the mathematical understandings that they needed to be successful at school and this was endangering Norway's economic well-being. Within these debates, kindergarten staff were positioned as not being competent, with kindergarten assistants' Norwegian language skills needing to be tested.

Simultaneously, the debates constantly shifted and changed, providing contradictions and no clear guidelines about what kindergarten staff should focus on and how they should implement any of the reports. Instead, they may have felt their competence was further being tested by whether they could

work out appropriate solutions to these issues. Having been judged as contributing to the problems, they are now being judged on whether they could become the kindergartens' saviours through finding solutions to those exact same problems. Being the focus of so much media attention, with limited possibilities for responding and positioning their work in positive ways, may have affected their willingness to engage with us, as outside researchers. It is not surprising that they seemed to want a specific plan for their participation in the proposed projects. Following someone else's plan not only would allow them to show they were working on solving the issues, but if the plan did not work then we would be responsible.

Yet ethical research demands that we negotiate with kindergarten staff and parents if action for change is to be achieved. Our investigations showed us that we needed to accept their concerns as genuine and be mindful about how we situated them in our negotiations. Recognising this shifting landscape provided some indication about how we could be respectful of their contexts. Still there remains significant ambiguity for us on how to conduct, not just the negotiation, but also the project itself.

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How Sámi teachers' development of a teaching unit influences their self-determination

Anne Birgitte Fyhn[1], Tamsin Meaney[2], Kristine Nystad[3] and Ylva Jannok Nutti[4]

[1]UiT- The Arctic University of Norway, Department of Education, Norway; anne.fyhn@uit.no

[2]Bergen University College, Norway; Tamsin.Jillian.Meaney@hib.no

[3]University of Oslo, Institute of Health and Society, Norway; kristine.nystad@gmail.com

[4]Sámi University of Applied Sciences, Guovdageaidnu, Norway; Ylva.Jannok.Nutti@samiskhs.no

Five teachers from a Sámi lower secondary school participated in two workshops on culturally-responsive mathematics teaching. During the first workshop, the teachers chose to focus on developing a unit about lávvu, the Sámi tent, to be taught between workshops. Their experiences are analysed with respect to Self-Determination Theory, which claims that all humans have a basic need for autonomy, competence and relatedness to others. The analysis of teachers' written notes reveals that the need for autonomy appeared as a need for inspiration and for courage. The need for competence concerned relating mathematics teaching to the two community resilience factors i) Sámi language competence and ii) traditional ecological knowledge. The need for relatedness to others was linked to Indigenous peoples, other teachers at their school, and teachers at other Sámi schools.

Keywords: Sámi, teacher, self-determination, indigenous, culturally-responsive teaching.

Introduction

This paper explores teachers' perspectives on culturally-responsive mathematics as it is imagined and utilized in the design and implementation of a teaching unit on the Sámi[1] tent, *lávvu*. This artefact, its design and its building, carries important connections to the Sámi people's intangible cultural heritage by embodying cultural traditions and ceremonies as well as rules for behaviour. To many Scandinavians, however, the *lávvu* is merely a tent; a cone-like building made with some poles that are covered by cloth. In modern Sámi societies, traditional knowledge of *lávvu* is not necessarily widespread as people use modern, factory-made *lávvut* with metal poles. Reindeer herding families use *lávvu* regularly and often are more familiar with traditional knowledge about *lávvu* than other Sámi. The younger Sámi generation consists of a variety of people with different interests.

Guovdageainnu nuoraidskuvla is the lower secondary school in the village Guovdageaidnu, Kautokeino, in Norway. North Sámi is mother tongue of more than 90 % of the students and it is the school's official language. The teaching is translated into Norwegian by an assistant teacher for students who do not understand Sámi well. The school follows the Sámi curriculum, which is equivalent to the national one. The school's teachers realised that *reahpen*, the north Sámi word for the smoke hole in the *lávvu*'s top, was considered a strange word by many grade 10 students. In order to increase students' cultural and mathematical knowledge, the teachers developed a culturally-responsive teaching unit about *lávvu*. The teaching unit was carried out in the period between two

1 The Sámi are an Indigenous people of the Arctic. They live in the northern parts of Norway, Sweden and Finland and on the Kola Peninsula in Russia. The Sámi is a heterogeneous group of people with different occupations.

workshops about culturally-responsive mathematics teaching. At the first workshop, the teachers planned the teaching, and, at the second workshop, they presented the outcomes. We consider this work to contribute to this group's self-determination as an Indigenous group.

Smith (1999/2006) highlights the importance of self-determination for Indigenous people, by describing it as the aim of a non-linear developmental process that departs from survival and recovery. We consider that self-determination is important in understanding Indigenous mathematics teachers' motivations for developing and implementing culturally-responsive teaching. Previous research about teachers' reflections about Sámification of school mathematics identifies several important issues. Jannok Nutti (2013) noted teachers' ability, drive and possibility, while Fyhn, Jannok Nutti, Nystad, Sara Eira and Hætta (2016b) describe relations between teachers' *autonomy* and their development. Fyhn, Jannok Nutti, Sara Eira, Børresen, Sandvik, and Hætta (2015) point to the importance of including teachers from other subjects, when the context for the teaching is related to their area. According to Kirmayer, Sehdev, Whitley, Dandenau, and Isaac (2009), self-determination also relates to resilience, as general discussions of identity tend to underemphasize the role of social action or collective agency in the production of well-being. Nystad, Spein, and Ingstad (2014) investigated a Sámi society in Northern Norway and identified community resilience factors including Sámi language competence, use of recreational and natural resources, and traditional ecological knowledge, such as reindeer-husbandry-related activities. These cultural factors appear to strengthen adolescents' ethnic identity and pride. Knowledge about *lávvu* and skills in how to raise a traditional *lávvu* are examples of traditional ecological knowledge in Sámi societies. Kirmayer et al. (2009) point out that resilience has a collective as well as an individual dimension.

Self-determination theory has provided empirical support for the proposition that all human beings have fundamental psychological needs to be *competent*, *autonomous* and *related to others* (Deci & Ryan, 2012). *Autonomy* refers to the perceived origin or source of one's own behaviour; it concerns acting from interest and integrated values. *Relatedness* is the psychological sense of being with others in a secure community. *Autonomy* is emphasized in traditional Sámi child rearing (Hoëm, 1976; Balto, 2005) and Balto (2005) highlights *autonomy* as a Sámi value. *Relatedness* to others is connected to holistically sharing and developing knowledge and so it is also considered an Indigenous value. The theoretical framework is constituted by the three categories *competent, autonomous* and *related to others*. Following Glaser (2001), we identified subcategories connected to each category by comparing incidents and named them using the teachers' own words. In this paper, we analyse five teachers' expectations and experiences of the two workshops. Our research question is, how does teachers' self-determination appear in their workshop notes?

Culturally-responsive teaching

Before discussing the workshops, we briefly describe culturally-responsive teaching which was the inspiration for the workshops. Gay (2013) described culturally-responsive teaching as "using the cultural knowledge, prior experiences, frames of reference, and performance styles of ethnically diverse students to make learning encounters more relevant to and effective for them" (pp. 49-50). Gay suggests that as part of culturally-responsive teaching, teachers conduct their own analyses of textbooks, the Internet and other sources. The investigation should include how different knowledge sources affect teaching and learning and reconstruct or replace existing presentations of issues and situations in the various resources with cultural knowledge and insights. This approach is in alignment

with Smith's (1999/2006) description of self-determination. Gay (2013) considered that interdisciplinary work with teachers of other subjects supported collaboration and provided different insights. Nevertheless, implementing culturally-responsive mathematics teaching needs to be done with care so that cultural artefacts are not simplified, to the detriment of both the culture and the mathematics. An example of simplification is to claim that the tipi, which is similar to the Sámi's *lávvu*, is a cone:

> That is surely wrong; the tipi is not a cone. Just look at a tipi with open eyes. It bulges here, sinks in there, has holes for people and smoke and bugs to pass, a floor made of dirt and grass, various smells and sounds and textures. There is a body of tradition and ceremony attached to the tipi, which is completely different from and rivals that of the cone. (Doolittle, 2006, p. 20)

According to Doolittle, there is a risk that Indigenous students who are presented with such oversimplifications feel that their culture has been appropriated by a powerful force for the purpose of leading them away from their culture. Thus, a teaching unit about *lávvu* has to respect the tradition and ceremony attached to it. Traditionally, a *lávvu* consists of two cloths that are wrapped around a set of poles and is a place for sleeping, working, relaxing, storytelling and even more (Nergård, 2006). It is easy to set up and take down and its permanent material, cloth and skins are transported when the family moves between living places. Other materials are gathered from the area where the *lávvu* is placed, making it local as well as mobile. There are rules for where to sit in the *lávvu* for parents, grown up children, workers and smaller children. In the old days, the innermost area was sacred and only the bear hunter returning from a successful hunt was allowed there (Petterson, 1905/1979). He entered from the back bringing the bear meat with him. Nowadays, people sleep anywhere and in modern *lávvu*, the floor is covered with carpets and stoves are used for cooking. Still, the tangible and the intangible cultural heritage remain important.

The workshops

Teachers from two Sámi schools participated in two two-day workshops, with six months in between. The workshop participants were a) teachers from the three subjects Sámi language, mathematics and *duodji*, Sámi handicraft at *Guovdageainnu nuoraidskuvla*, b) all teachers for grades 1-10 from a small Sámi school in another municipality, and c) some pre-service teachers from Sámi University College, who had a practicum at *Guovdageainnu nuoraidskuvla.* The teachers joined the workshops so they could contribute to the further development of culturally-responsive teaching in their schools. *Guovdageainnu nuoraidskuvla* had already started developing culturally-responsive mathematics teaching (Fyhn et al., 2015; Fyhn et al., 2016b) and the principal is one of the mathematics teachers. The two workshops included lectures and school-based group work. The group work was about the culturally-responsive mathematics teaching done in the period between the workshops. At the first workshop, the mathematics teachers who participated in the earlier project (Fyhn et al., 2015) presented their work. In addition, researchers presented theoretical perspectives connected to Indigenous mathematics education, mainly through examples from Sámi and Māori classrooms. The second workshop continued with theoretical perspectives and included an online lecture with two Indigenous mathematics teachers and researchers from New Zealand. At the first workshop, each school chose a theme for the culturally-responsive mathematics teaching and started the planning. The schools presented the results of their culturally-responsive mathematics teaching at the second workshop. *Guodvageainnu nuoraidskuvla* focused on *lávvu* and eight teachers from this school co-

authored a paper about their work (Fyhn, Sara Eira, Hætta, Juuso, Skum, Hætta, Sabbasen, Eira and Siri, 2016a).

The teaching unit about *lávvu*

During workshop one, the Sámi language teachers suggested to focus on *lávvu*, because many students did not know the names of central parts of the *lávvu*. The mathematics teachers agreed that *lávvu* would provide possibilities for teaching mathematics, among other things by having the students make a small *lávvu* model. Students could discuss different aspects of mathematics related to *lávvu*. Consistent with cultural symmetry (Trinick, Meaney, & Fairhall, 2016), the teachers designed the teaching unit so that it started with a history section that discussed *lávvu* and *goahti* (another common Sámi housing) and central concepts regarding these. The teachers highlighted the different parts of the *lávvu* construction and how each part functioned. Each part was connected to specific traditions and the students had to learn the North Sámi words for them. In this way, the teaching valorised the local culture, as recommended by Trinick et al. (2016) and Doolittle (2006).

The mathematical aspects of the unit focused on the three *válddahat*, the structural poles, the location of *árran*, the fireplace, and the size of the floor. The *válddahat* have a Y-shape in one end and are the first three poles raised. This triangular construction is common for Sámi frameworks; as constructions made by three sticks are stable and reliable (Fyhn et al., 2016a). Locating the *árran* can be done through eye estimation, which includes trial and error for those who are not skilled. *Árran* may also be located just below a *skerttet*, a special iron hook that hangs in a chain from the top of the *lávvu*. Locating *árran* can be connected respectively to a numerical approach or a geometrical approach, with both providing appropriate answers. The size of the floor depends on how many people are to stay in the *lávvu*; the steeper the walls are, the smaller the floor's area. In earlier times, people could determine from a distance how many people lived in a *lávvu*, based on the angle between the wall and the ground. The *lávvu* floor is covered with layers of *duorggat*, twigs in appropriate length that are cut from willow or birch. Eye measuring is used to estimate the amount of *duorggat* needed. The students used a trial and error approach to determine this, while skilled people fetch the correct amount first time.

The students raised a *lávvu* near the school. The teachers focused the students' attention on the three *válddahat.* The students also made a mini *lávvu*, which became a gift that the students enjoyed giving to an old people's home. The model's scale was 1:8. Afterwards the teachers regretted that they had chosen this scale, because the task would have required more mathematics if the students had to decide the scale themselves. Still the model proved mathematically challenging for the students, who had to choose materials and decide how to make everything in correct proportions.

Method

Five teachers from *Guovdageainnu nuoraidskuvla* participated in both workshops and their responses to the workshop are analysed in this paper. They work in a school where North Sámi is main language and were educated as Sámi teachers. The five teachers Bigga, Duiri, Vide, Sire and Aile are north Sámi native speakers and experienced teachers who teach two, three or four subjects each. Two of them teach *duodji*, four of them teach mathematics, and four teach Sámi language. The work between the workshops contributed to strong cooperation between the teachers in these three subjects. Sámi language and *duodji* are subjects that, among other things, aim to strengthen the students' cultural

identity. At the bequest of the researchers, the participants wrote about their expectations and experiences of the workshops at the beginning and end of each day. Fyhn et al. (2016b) studied relations between teachers' *autonomy* and their development of a culturally responsive mathematics exam. In this study, we chose to focus on more aspects of self-determination. In alignment with self-determination theory (Deci and Ryan, 2012), we analysed the teachers' writings in regard to a) being *competent*, b) being *autonomous* and c) being *related* to others. Designing and implementing a culturally-responsive teaching unit about *lávvu* requires the teachers to have the necessary *competence* about how to integrate cultural knowledge with mathematics teaching; this is an example of what Kirmayer et al. (2009) call community resilience. As well, the teachers need a capacity for and a desire to experience *autonomy*; that the work is regulated by themselves and that their integrity is kept through the work. When teachers from one school work together as a group, they are *related* to others and not alone in facing possible resistance or other difficulties in implementing a culturally-responsive teaching unit.

The teachers' experiences of self-determination during the workshops

The teachers' expectations and experiences are analysed with respect to the three issues *autonomy*, *competence* and *relatedness* to others (see Table 1). *Competence* was identified as the ability to include two community resilience factors i) Sámi language competence and ii) traditional ecological knowledge in the teaching of mathematics. *Relatedness to others* could be separated into three categories, relatedness to other teachers at their school, relatedness to teachers at other Sámi schools and relatedness to (teachers from) other Indigenous peoples. Before the workshops, the teachers' expectations mainly concerned their individual *autonomy* and *competence*, but during the workshops, most of their discussions of their experiences focused on *relatedness to others*. The analysis of the written notes reveals the teachers' need for *autonomy* manifested itself as a need for encouragement and for ideas or inspiration. These findings are in line with Fyhn et al (2016b).

Autonomy	Competence	Relatedness to others
Inspiration/ideas from others Becoming encouraged Awareness about competence	Include resilience factors: a) Sámi language and b) traditional ecological knowledge in mathematics teaching	Other teachers at their school Teachers at other Sámi schools Other Indigenous peoples

Table 1: Framework

The first morning, the teachers expressed their expectations towards the workshops. Sire and Aile referred to a need for supported *autonomy*, "I hope that I dare to do more interdisciplinary work", (Sire, expectation notes, March 2, 2015) and "Hope it motivates to more interdisciplinary work", (Aile, expectation notes, March 2, 2015). Vide, Aile and Duiri expected to hear about experiences with including resilience factors in mathematics teaching, "to get some ideas and hear about some experiences with culture-based mathematics", (Duiri, expectation notes, March 2, 2015). Aile expected ideas about how to connect different subjects, and Vide (expectation notes, March 2, 2015) wrote "To get input from other teachers about how to integrate more subjects in an interdisciplinary work where all subjects feel included". The teachers' references to interdisciplinary work are in line with Gay (2013), who points out that interdisciplinary work leads to collaboration, plus expectations

about knowledge. Interdisciplinary work in this setting means mathematics that treats Sámi traditional knowledge with dignity and respect. Nystad et al. (2013) identified traditional knowledge as a community resilience factor. Ability to integrate resilience factors was among Bigga and Sire's expectations. Bigga (expectation notes, March 2, 2015) expected to "be able to base more of the subject mathematics on culture".

In the experience notes, four of the teachers explicitly referred to a lecture about other Indigenous people, "We have learned about others' challenges, Indigenous thinking and perspectives", (Duiri, experience notes, March 2, 2015). This is categorized as *relatedness* to other Indigenous peoples, "we have learned that other Indigenous peoples have many things similar to us, the same challenges", (Sire, experience notes, March 3, 2015). Four of the five teachers had experiences that concerned their *relatedness* to other teachers at their school, like "the final part with concrete reflections and discussion/talk about *duodji*/mathematics at our school was very useful.", (Bigga, experience notes, March 2, 2015) and "good to focus on culturally based mathematics again, so that we can coordinate it in our school's plans", (Vide, experience notes, March 2, 2015). The second day of workshop one, the notes mainly concerned *relatedness* to other teachers at their school and to other Indigenous peoples, "the group work constitutes a basis for further work at our school. Informative to see that other Indigenous people have similar thoughts about this work. We see that they have similar challenges" (Aile, Sire, Duiri and Vide, experience notes, March 3, 2015). Bigga also noted that she experienced *relatedness* to teachers at the other Sámi school.

None of the teachers referred directly to being *competent*, but three of them made implicit references to this: "Alan Bishop's six fundamental activities makes us teachers more aware of our actions, teaching and thoughts about mathematics and language", (Duiri, experience notes, March 3, 2016), "I become more aware of my solid knowledge about Sámi culture. I can base more of my teaching on this knowledge… The theoretical part was more useful this time" (Sire, experience notes, March 2, 2016). Three of the teachers referred to supported *autonomy*, which was caused by the increased awareness about their *competence* and the fellow teachers' positive attitude and contributions to the workshop. These are examples of overlap between the basic needs *autonomy* and *competence*; the three basic needs do not constitute distinct categories.

The analysis of the second workshop's experiences mainly reveal *competence* and *relatedness* to others. Four teachers pointed at *competence*, Aile wrote, "the lecture about language and mathematics was very interesting, because I could see relations between Sámi language and mathematics", (experience notes, October 21, 2015). "I become more and more conscious about my own solid knowledge in Sámi culture, I can use this in my teaching", (Sire, experience notes, October 21, 2015). All five teachers pointed to the importance of *relatedness to others*. "The sessions where each school worked together, was very developing for us… The sharing of experiences was useful, between schools as well as within each school" (Vide, experience notes, October 21, 2015). Sire wrote, "It has been informative to learn about the international, and gives a wider spectre to think about other people's situations". The teachers had to start the last day one and a quarter hours earlier, in order to have a Skype meeting with two Māori mathematics teachers/researchers in New Zealand. The Skype meeting took place the day after the workshops' final conference dinner and the early morning attendance confirmed the participants' commitment to talking with the Māori teachers/researchers. They really looked forward to this meeting. The analysis reveals that three of the teachers experienced

relatedness to other Indigenous people from the Skype meeting: “This day has been useful in many ways … what Uenuku [Fairhall] said about the importance of how you teach mathematics … throw away the textbooks and teach mathematics at theme level :-)”, (Aile, experience notes, October 21, 2015).

Conclusion

The teachers expected increased *competence* and supported *autonomy* when they joined the workshops. They had no expectations regarding *relatedness* to others, but this seemed to become their most characteristic way of describing their experience. The analysis of the teachers’ needs for *autonomy*, *competence* and *relatedness* to others identified the ways in which these needs appeared. Subtypes of the three needs revealed information about the important factors that the teachers considered that they needed to succeed in developing their self-determination. Regarding *autonomy*, the teachers expected and experienced inspiration and being encouraged. They experienced *competence* in regard to relating mathematics teaching to the two community resilience factors i) Sámi language competence and ii) traditional ecological knowledge. *Relatedness* to others was linked to: Indigenous peoples; other teachers at their school; and teachers at other Sámi schools. The teachers’ notes also revealed that they would have benefitted from group work related to the introduced theory, but this was not fulfilled. They wanted and expected to learn more about how to integrate culture in their mathematics teaching; culturally responsive mathematics teaching.

The use of self-determination theory as a methodology for understanding teachers’ perceptions about culturally-responsive mathematics teaching reveals that the teachers’ development is influenced by several cooperating factors; inspiration and encouragement, working with theory and experiencing relatedness to other Sámi teachers as well as to other Indigenous people. Future workshops need to link culturally responsive mathematics teaching more closely to teachers’ group work.

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Class, 'ability' groups and mathematics in English secondary schools: A review

Colin Jackson

Independent academic, England; colinfjackson@hotmail.com

In an age where neoliberalism reigns, the predominance of 'ability' grouping as an organizational strategy in mathematics classrooms in England is now virtually unchallenged: it is seen as natural and 'common-sense' by the population at large and by the vast majority of mathematics teachers. This is despite the large volume of research which shows it has no effect on attainment overall but has a deleterious effect on the well-being of many children. 'Ability' grouping is a social justice issue as it always disadvantages someone. In this review, I examine how it continues to disadvantage working-class children in England.

Keywords: Ability, ability grouping, all-attainment.

Preamble: Class, schooling and the legitimation of inequality in a neoliberal age

Why does education, "increasingly positioned as the new panacea for the masses", lead to the majority of working-class students feeling "a sense of educational worthlessness?" (Reay, 2006, pp. 296–297). To answer this question we must first understand that since the mid-seventies, particularly in the UK and the USA, neoliberalism has been the dominant governmental discourse. Neoliberalism is a political ideology that seeks to reduce and limit the role of government in all areas including in the public sector, believing that markets result in greater efficiency and "inequality is a result of individuals' inadequacy" (Hursh, 2005, p. 4). Neoliberalism has appropriated the desire to want to "get on" translating it into "aspiration", a device that disguises the social and economic barriers that hinder the working class from doing so, shifting the responsibility for "people's opportunity to succeed or fail from the state onto individuals" while replacing "political concepts such as class, democracy, exploitation, solidarity, justice, dignity and rights" (Tyler & Bennett, 2015, p. 6). Thus students, lacking an alternative, come to accept as natural the unequal way that society is structured. This process, termed the legitimation of inequality by Bowles and Gintis (1976), leads those at both the top and bottom of society "to see themselves as largely responsible for their own places in it". Reay (2006) sees class as

> everywhere and nowhere, denied yet continually enacted […] while the privileged, for the most part, continue to […] ignore its relevance to lived experience. (p. 290)

The education system currently works in favour of the middle classes and is underpinned by their values (Zevenbergen, 2001). Reay (2006) says working-class students are frequently positioned as inadequate learners with inadequate cultural backgrounds: she says 'ability' grouping is used to "fix failure in the working classes while simultaneously fixing them in devalued educational spaces", making some students "feel stupid" (p. 298). It is 'ability' grouping which is the focus of this paper.

Introduction

In England, only around the top 60% (Ofsted, 2012) achieve a 'pass' grade at age 16. Working-class students are significantly over-represented in the other 40% (Gillborn & Youdell, 2000). Further, only 24% of white working-class boys on free school meals gain five good GCSEs (the public

examination at 16) including mathematics (Wigmore, 2016). This matters not least because, intentionally or otherwise, mathematics is used as a filter to many vocations and academic institutions (e.g., Stinson, 2004). One major barrier to achievement is putting students into 'ability' groups.

The aim of the study on which this paper is based was to explore the available research on 'ability' grouping and included reports on overall attainment and personal and social outcomes. In the main, the study focussed on research and reviews of research undertaken since the start of the eighties although important research studies from earlier periods were also included. The review included both quantitative and qualitative studies. The review was initially conducted using Google Scholar and the British Education Index database. The references of relevant articles were then scrutinised systematically for further pertinent references which were similarly scrutinised and followed up. The arguments and conclusions presented in this paper draw on the whole study; limitations of space mean that only a minority of sources can be cited.

Research on 'ability' grouping indicates that it has little, if any effect, on attainment overall but has long term detrimental effects in terms of personal and social outcomes (Nunes, Bryant, Sylva, & Barros, 2009; Boaler, 2005). Moreover, the social effects of 'ability' grouping "exact a social price, as ability levels largely overlap with socioeconomic differences" (Cahan, Linchevski, Ygra, & Danzinger, 1996, p. 30): It is inequitable and needs to be challenged. As Slavin (1990) argues, 'ability' grouping can be seen as an affront to basic ideas of democracy. 'Ability' grouping is a social justice issue because it always disadvantages somebody; and in England, amongst others, it disadvantages working-class children: this is what I explore here in the context of secondary mathematics.

Class and 'ability'

On arrival in secondary school, many middle-class children are already academically more advanced than working-class children as their education-conscious middle-class parents will have endeavoured to ensure their children have secured a place in the best performing junior schools (Lacey, 1970, p. 35). In addition, teachers' beliefs frequently lead to lower expectations of working-class children (Zevenbergen, 2003). Working-class students lack the social and cultural capital that the middle classes possess. This difference in capital legitimates the failure of working-class students with middle-class students' success being seen as the result of hard work or natural 'ability' rather than class-based inequalities (Bourdieu, 1992). Hence many working-class children start school at a disadvantage compared to many of their middle-class peers as they are less well equipped with the tools necessary to do well in school. One pattern of response to this is that of rebelling against a system that predisposes them to do badly, committing themselves to behaviour patterns which means that their work will stay poor (Lacey, 1970, p. 58).

One important effect of grouping by 'ability' is that middle-class children have minimal contact with those working-class children who are less well behaved (Ireson, Clark, and Hallam, 2002). Students who lack the social knowledge for what is seen to be appropriate behaviour (Zevenbergen, 2003) by teachers will tend to populate the lowest sets. Those who do succeed in making it into the higher 'ability' groups soon discover that to succeed in school they must conform to the accepted middle-class behaviour norms; failure to do so causes a descent into the lower attaining groups. Thus there is a self-correcting mechanism for dealing with children who do not conform.

Underlying the issue of 'ability' are issues of power and culture and hence whose ways of knowing are dominant. 'Ability' grouping is not a neutral disembodied organisational practice. 'Common sense' conceptions of 'ability' and intelligence are at the heart of schooling and the 'ability' discourse is part of an ideological battle defining children with lower socio-economic status (SES) as being expendable (Oakes, Wells, Jones, & Datnow, 1997): Attainment grouping serves purposes in schools other than that of teaching and learning. Schooling is designed to reproduce the current social, political and economic systems rather than to provide a meritocratic route to success in adult life (Oakes, 2005). Further the performativity regimes (Ball, 2003) imposed on schooling have created a climate whereby failing to conform to prevailing discourses carries huge risks to schools and to individual teachers. 'Ability' grouping measured by some form of assessment is seen as risk free and, in mathematics, is virtually unquestioned (Hallam and Ireson, 2003).

Contemporary English society assumes that middle-class values are superior to working-class values and hence the working classes need to 'aspire' to join the middle classes (Jones, 2011). The values that working-class children bring to school are neither recognised nor valued by schools while the abilities they bring to school are ignored at best and indeed are thought to be detrimental to a good education (Delpit, 2006). This feeds into the informal judgements about intellectual 'worth' noted above. In addition, in general, working-class students will not understand the mechanisms required to succeed in the curriculum as they will not have the cultural capital to 'play the game' that is involved in the learning of mathematics (Bourdieu & Passeron, 1990).

Student attainment and the idea of 'ability'

'Ability' is currently used as a proxy for intelligence (Wilkinson & Penney, 2014) and ideas that would normally be discarded are taken as 'common sense' when the discourse is about 'ability' rather than intelligence. Viewing 'ability' as innate has been a long established presumption, held to be true by the general population and by many teachers (Marks, 2016). This 'common sense' view leads directly to putting children in groups of the same predetermined 'ability' in order to teach them effectively (Francis et al., 2016). For many children this approach is damaging academically and socially; it is damaging nationally and is contrary to the stated aim of raising overall attainment.

Dweck (2000) has challenged this view of fixed ability, showing that a belief in growth mindsets, that progress is in large part down to effort and is not restricted to those with 'ability', enables students to make more progress and achieve higher. Grouping students heterogeneously is supported by a very large body of research which indicates that it improves educational outcomes (e.g., OECD, 2013). Despite this the current climate is unfavourable to all-attainment teaching. Indeed, it is often viewed as inimical to good teaching. The pressure on schools to conform to this view coupled with the tendency of teachers to replicate how they have been taught ensures that 'ability' grouping is almost universal in mathematics classrooms in England in contrast to much of the rest of the world.

'Ability' grouping sends a clear message that only some can do mathematics and that this is due to some type of 'natural ability' (Marks, 2016), a message some children (currently about a third) receive as early as age 4. Early research reviews (e.g., Sukhnandan & Lee, 1998) found that studies on 'ability' grouping produced few conclusive or consistent findings but recent research (OECD, 2013) indicates that where students are highly stratified, as in the case of setting, there is a wider range of achievement than when they are taught in heterogeneous groups. Hoffer (1992) reports that "the

conditions under which ['ability'] grouping benefits all students (or at least helps some and does not hurt any) do not generally exist" (p. 223).

Comparisons between countries may be misleading and in analysing the effects of 'ability' grouping it can be difficult to separate out factors. Nevertheless, research carried out in England and in the USA does show significant similarities: in the USA students in high tracks gain more than students in lower tracks (Slavin, 1990) while, in England, there is a consistent tendency for children of all 'ability' levels who are placed in lower sets to attain less than if they had been in heterogeneous groups (Bartholomew, 2001). A consensus is emerging that, whilst not necessarily raising the level of outcome for higher attainers, all-attainment teaching does not significantly supress it (Francis et al., 2016).

Allocation of children to 'ability' groups

Early grouping by 'ability' has long-term implications for children's educational opportunities (Boaler, 2005). Once a child is placed in a particular group it is very difficult to change because of differences in curriculum content and the pace of teaching (Wilkinson & Penney, 2014). The allocation of children to 'ability' groups is claimed to be objective with children being allocated on the basis of their prior performance; the process is portrayed as highly refined with children accurately allocated. However, in English secondary schools, although perceived 'ability' is found to be the main predictor of set, it is a relatively poor one. Schools have multiple reasons for the allocations, many informal and based on insubstantial evidence. Children with higher SES and/or ambitious middle-class parents are more likely to be assigned to higher sets. Children seen as disruptive or poorly behaved, the perception of which is linked to class (Bartholomew, 2001), are more likely to be in bottom sets (Muijs & Dunne, 2010). Boaler, Wiliam, and Brown (2000) found that working-class students in the UK tended to be placed in a lower group than would be expected on the basis of their attainment alone as a result of the school's desire not to alienate the most powerful (and highly valued) constituencies of parents (p. 130), a pattern also found in schools in the USA (Oakes, 2005).

'Ability': Beliefs and practices of teachers

In Britain, many teachers are philosophically opposed to mixed 'ability' and even where children are in mixed 'ability' classes the teachers practice in-class grouping (Marks, 2016). Oakes (2005) suggests that people unquestioningly continue the practice of 'ability' grouping because it is seen as being part of the 'natural' order of schools (p. 191). Lacey (1970) reports that the teachers gave the following reasons as justification for the introduction of streaming in *Hightown grammar*: It would make the teaching

> more efficient and […] facilitate the learning process for all […] [working in] the best interests of the individual pupil, even when relegating him to the bottom stream. [If he remained in the same group he would] either hold them up [higher attaining pupils] or […] become demoralised, and fall further behind. […] He would be able to proceed at a more suitable pace […]. (pp. 74–75)

Oakes (2005) evidences that the assumptions on which 'ability' grouping is based are unjustified, while teachers' perception that "teaching is easier when students are grouped homogeneously" may be because this is the classroom organisation they are used to. If they embraced the use of different organisational structures where the students cooperate they might similarly find teaching is easier in heterogeneous classes. Moreover, she says "these [classroom] differences are institutionally created

and perpetuated by tracking" (p. 194). Most of the benefits of 'ability' grouping are benefits for teachers and schools whereas most of the disadvantages concern the negative effect on students (Hallam and Ireson, 2003). Teachers treat children differently depending on their conception of their 'ability' (Bartholomew, 2001). Low attainers and high attainers who produce work of a similar standard find their work viewed quite differently (Marks, 2016). Higher attainers are constructed as well motivated, hardworking, well behaved and capable of independent working and thought whereas low attainers are constructed as poorly motivated, badly behaved (Wilkinson & Penney, 2013), incapable of independent working and thought and in need of repetitive tasks which require lots of practice (Watson & De Geest, 2005). In addition to this, there is a tendency for teachers assigned to high 'ability' groups to be both more competent and more motivated.

Hence the set a pupil is in can be crucial to their attainment. Students in top sets are expected to work faster covering work in more depth while pupils in low sets have a reduced curriculum where there is less discussion, more repetition and more structured work including merely copying off the board (e.g., Boaler et al., 2000) with lower attainers being deprived of role models of more successful learners (Hornby & Witte, 2014). Lower attainers find it more difficult to acquire 'basic' knowledge in sets compared to non-setted groups (Fuligni, Eccles, & Barber, 1995) while high level content may only be made available to high attainers (Cahan et al., 1996). Lower expectations of low attainers are communicated through a number of mechanisms. They are given easier work which they frequently repeat and the work they are given is broken down in smaller steps so they cannot make for themselves the connections needed to understand the mathematics they are doing. Their teachers talk *about* them differently and talk *to* them differently. They are described as being incapable of concentrating and teachers adopt a more authoritarian mode of talking to them (Watson & De Geest, 2005). Moreover, behaviour is constructed very differently in high attaining and low attaining groups. Bartholomew (2001) and Marks (2016), for example, report that teachers' focus on *learning* in high attaining groups while in low attaining groups they focus on *behaviour*.

'Ability' grouping: Concluding remarks

Four main conclusions emerge from the literature review. Summing up:

1. *'Ability' grouping remains a class issue* as working-class students are disproportionately placed in lower sets (Bartholomew, 2001) becoming demotivated and underachieving as a result. The preponderance of middle-class children in the upper sets show that grouping by 'ability' favours the middle class. In a socially just world all students would have *the opportunity to attain equally*, unrestrained by external factors such as perceived 'ability'. It is the case that when children from the working class have the same opportunities as middle-class children they can attain as highly (Boaler, 2005) though lower attaining students may need additional support so they can reach the higher expectations (Rubin & Nogura, 2004).
2. *The beliefs and practices of teachers* may be key to improving the outcomes for working-class students. They need training in order to teach all-attainment classes effectively. Teachers who hold conventional conceptions of 'ability' and intelligence may be the greatest obstacles to reform as they actively resist changes to the curriculum. Their beliefs can lead to resistance to change (Hynds, 2010) and they may enlist *the support of parents who are part of the dominant class* and who fear change will disadvantage their children (Oakes et al., 1997).

3. If ‘ability’ grouping worked as its supporters claim it works, then social class would be of no import, a child’s behaviour would be irrelevant and each child would be able to develop appropriately. *Allocation to ‘ability’ groups* would be commensurate with students’ current attainment and they would be constantly monitored and re-assigned to the correct group throughout their school careers with a mix of working-class and middle-class children reflecting the profile of the intake. However, as well as middle-class children having more economic capital they also have more cultural and social capital than working-class children. Middle-class children’s *understanding of the rules of the game* that is school is much more profound and they can use the rules much better to their advantage.
4. ‘Common sense’ conceptions of ‘ability’ are at the heart of schooling. A technicist approach to reform will not work as it assumes resistance to changing ‘ability’ grouping is rational. ‘Ability’ grouping is an ideological battlefield. Teachers in the main are, unsurprisingly, convinced by the powerful dominant discourse of individualisation accompanied by a natural ordering produced by ‘ability’. Alongside, powerful high-SES parents use issues of intelligence, ‘ability’ and merit to exercise power and control enabling them to secure high ‘ability’ groups for their often less than qualified children. A wholescale restructuring of school expectations and culture is required (Oakes et al., 1997) in order to succeed in providing a more just experience for working-class students in secondary mathematics classes in England.

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Content-related and social participation in inclusive mathematics education

Judith Jung and Marcus Schütte

Technical University of Dresden, Germany; judith.jung1@tu-dresden.de

In the 1980s, the WHO presented a differentiated model of disability. The central feature of disability according to this model is not the impairment but the resulting limitation in social participation. The UN Convention on the Rights of Persons with Disabilities (CRPD) goes a step further by demanding that persons with disabilities must have access to an inclusive school education together with persons without disabilities. From our point of view, not only social but also content-related participation is important for an inclusive school education. The following theoretical paper seeks to explore a theoretically grounded concept of participation in mathematics classes for a planned exploration of how these forms of participation are implemented in inclusive mathematics education in Germany.

Keywords: Inclusion, student participation, interaction.

Introduction

In Germany, most children first attend primary school from the age of five or six to nine or ten and are then separated by a supposedly achievement-based selection process into different school forms. This is a system that has long been criticised in several respects (Muñoz, 2007). Furthermore, from the mid-18th century a parallel branch of schooling for children with special educational needs was established alongside primary school and the four types of secondary school. The special needs school system continued developing into the 2000s, with different schools specialising in particular needs, e.g. social-emotional development, learning, language, vision, hearing, or mental development. In the 2009 school year in Germany, a total of 483,267 pupils with special educational needs went to school. Of these, 387,792 (80.24%) were schooled at special needs schools and 95,475 (19.76%) in regular schools (Sekretariat der Ständigen Konferenz der Kultusminister der Länder, 2016a).

With the EU's ratification of the UN Convention on the Rights of Persons with Disabilities, this praxis of the separated school system came under renewed criticism and also its legality came into question. The UN Convention requires that persons with disabilities should not be excluded from the general educational system because of their disability (CRPD, 2007, article 24). This legal right surpasses the mere freedom of choice to attend a regular school. According to the UN Convention, adequate arrangements have to be made within regular schools in order to ensure the educational success of each individual. Though the ratification of the Convention potentially necessitated little change in other countries, in Germany it opened a political and social discussion on structural changes to the school system. Since 2009, a policy of inclusion – in the sense of integrative schooling of pupils with special educational needs in the regular school system – has been implemented in the German states. Thus, in the 2015/16 school year, 322,518 (62.34%) out of the 517,384 pupils with special educational needs attended a special needs school and 194,866 (37.66%) a regular school. The ratification of the convention therefore appears to have led to a thorough implementation process, as the number of children with special educational needs in regular schools has doubled in only six years. However, looking at the figures on the background of demographic developments, which have seen a reduction in pupil numbers, the proportion of pupils with special educational needs in the total number of pupils

in Germany clearly rose between 2009 (6.17% of all pupils between Years 1 and 10) and the 2015/16 school year (7.1%). Thus, the proportion of pupils with special educational needs who are taught at special needs schools has barely fallen. Above all, children who were attending regular schools in any case are now more commonly being given special educational needs status (Sekretariat der Ständigen Konferenz der Kultusminister der Länder, 2016a, 2016b; Klemm, 2013).

So, it seems there is still a long way to go to achieve a school system that can be described as inclusive. We follow Katzenbach (2012) in holding that there is little difference between the terms integrative and inclusive in school life. However, there are some clear conceptual differences: the idea of integration depends on a categorisation of people. There are "normal" people and "the others", the non-disabled and the disabled who need to be brought together. The concept of inclusion, however, is based on the premise of diversity. Disability is only one characteristic among many, and school is a place where people with extremely different characteristics intermingle. Following UNESCO's (2005) understanding of the concept, inclusion can be understood as an ongoing process to find better ways of responding to diversity.

Towards providing an empirical base for this process of changing a traditionally strongly separated school system such as that in Germany, in respect to content-related – here mathematical – learning conditions, this paper discusses some theoretical considerations on mathematical learning and learning under inclusive conditions, and synergises these for a planned research project. According to Prediger, Bikner-Ahsbahs and Arzarello (2008), different networking strategies can be used to connect theories. The first strategy they mention, "having an understanding of the different theories", can be seen as the starting point for all other strategies, allowing them to be compared, combined or integrated in a further step. Therefore, in a first step we will present specific theories for mathematical learning and inclusion theories separately before we coordinate them with each other. This can be done because the theoretical concepts have consistent assumptions. The coordination creates a conceptual framework that helps in identifying the students' participation in inclusive settings and learning in mathematics.

Mathematics learning from an interactionistic perspective

For our understanding of content-related learning in school we refer to Miller's (1986) antagonistic differentiation between the research traditions of genetic interactionism and genetic individualism (Schütte & Krummheuer 2012). However, this is with the goal of bringing the two positions closer together. According to Miller (1986, p. 17), learning can either be anchored in the individual as a process of monologue, in the sense of genetic individualism[1], or be understood as a process of dialogue between individuals, in the sense of genetic interactionism[2]. Miller considers genetic interactionism, in contrast to genetic individualism, to have a more convincing empirical and theoretical base, at least in relation to learning processes in the early development of individuals,

[1] Genetic individualism is in the tradition of the later Piaget and Kohlberg (Miller, 1986, pp. 15 ff.).

[2] Genetic interactionism draws from the basic assumptions of sociological and psychological studies such as those of Durkheim, Mead, the early Piaget and Vygotski. These studies see social cooperation or interaction as fundamental to individual learning processes (Miller, 1986, pp. 15 ff.).

which he considers "fundamental"[3]. Above all, in primary school, where young learners come together, Miller sees learning processes taking place which are primarily collective and based on dialogue. Miller (1986, p. 223) describes individualised learning processes as "autonomous learning" and attributes them to the later development of the individual, locating these processes in moments of reflexive consolidation of things originally learned collectively. From a social-constructivist perspective, learning cannot be seen as a primarily internal cognitive restructuring process. Rather, it is a dualistic process which takes place both within the individual, in the sense of cognitive restructuring, and within interaction processes in which the person participates, which go before these restructurings (Sfard, 2008).

This kind of sociological or social-constructivist consideration of learning processes has in recent years gained increasing influence in the theoretical design of content-related learning, and has been taken up and further developed in mathematics education research (Lerman, 2000). Both nationally and internationally, mathematics has increasingly come to be seen as a cultural tool, constructed and mediated through language (Schütte, 2014). Since the mid-1980s interactionistic approaches of interpretive (classroom) research in mathematics education have engaged with the sociologically based social-constructivist perspective on learning processes (e. g. Bauersfeld, 1988; Krummheuer, 1992) using theories of symbolic interactionism (Blumer, 1969). With this kind of basic theoretical understanding of content-related learning the concept of collective argumentations gains central significance in the analysis of mathematical learning processes. According to Krummheuer and Brandt (2001), pupils are usually engaged in interaction processes in the classroom conversation, producing an argumentation in the totality of their actions. In this way, participation in a collective argumentation concerning statements about (mathematical) content, terms and/or methods creates the basic conditions for learning opportunities. This interplay of individual and social constituents is difficult to describe. If participation in collective argumentation provides orientation and convergence, then learning success can be seen as the improved coordination between individual attributions of meaning and the results of the interactive negotiation of meaning in the respective group. On an interactional level, this is manifested in an increasing adaption of the (verbal) acts of the learners to argumentations established collectively over the course of several interactional situations. The coordination of an individual's interpretations and actions can be reconstructed empirically as the increasingly autonomous adoption of steps of action within the collective argumentation. The learning of mathematics can thus be described as the "progress" of participation in mathematical collective argumentations. This idea of learning through participation can be linked back to the notion of equal opportunities for participation in educational institutions, according to the Convention on the Rights of Persons with Disabilities. Thus, the following will seek to explain how participation in learning processes in school can be understood from an inclusive-educational perspective.

Increasing participation as a goal of inclusion

Among the goals that are being set by the increasing implementation of the inclusion concept in

[3] These kinds of learning processes concern the development of "rationality, or rational knowledge structures" (Miller, 1986, p. 15).

teaching, ignoring for a moment the thoughts outlined above on the different understandings of the learning of mathematics, it can be noticed that providing all pupils with equal opportunities for participation in learning processes occupies a central position. This reflects UNESCO's principle whereby

> Inclusion is seen as a process of addressing and responding to the diversity of needs of all learners through increasing participation in learning, cultures and communities, and reducing exclusion within and from education. (UNESCO, 2005, p. 13)

This principle, formulated in the framework of the UNESCO "Guidelines for Inclusion", is reflected in the UN Convention on the Rights of Persons with Disabilities (CRPD), ratified by Germany in 2007. The Convention states:

> Persons with disabilities are not excluded from the general education system on the basis of disability. Persons with disabilities can access an inclusive, quality and free primary education and secondary education on an equal basis with others in the communities in which they live. (CRPD, 2007, article 24)

In addition, the Convention asserts that children/persons with disabilities should be supported within general education, according to their needs, to best progress their education. A "full and equal participation in education" (CRPD, 2007, article 24) should be made possible for them. Some years previously, Booth and Ainscow (2002) already published an "Index of Inclusion", which is intended as a tool to support an inclusive school development and contains a detailed description of how barriers to learning and participation for all learners can be dismantled. Here, too, the goal of a "greater participation of students in the cultures, curricula and communities of their schools" is cited (Booth & Ainscow, 2002, p. 2). In summary, according to the mentioned literature inclusion can be understood as an unending process of increasing learning and participation for all students (Booth & Ainscow, 2002; UNESCO, 2005) – and thus also as an ideal, which will never be fully realised but is already applied with the start of the process of increasing participation. However, it seems necessary to clarify at this point what is understood by full and equal opportunities of participation. Therefore, in the following the concept of participation in learning processes is developed. According to Booth and Ainscow (2002, p.3), participation can be understood as "learning alongside and in collaboration with others in shared learning experiences". They see participation as demanding active involvement in learning processes and the opportunity to express one's own learning experiences.

Adopting a broader definition developed to consider social structures, according to the German sociologist Bartelheimer (2008) five requirements can be distinguished for a sufficiently defined term of participation. These five requirements for a general social concept of participation can be transposed onto teaching processes in the classroom as follows. First, Bartelheimer states that participation is only to be understood as historically relative. Transposed onto the school situation, participation in processes that enable learning is to be understood only in relation to the given education system and fundamental features of the educational processes and paths that currently predominate. Secondly, participation is multidimensional and there are always different forms of participation. One approach to consider participation in teaching processes in various dimensions is provides by Roos (2014), who distinguishes a spatial, social and didactical/content-related dimension. Thirdly, participation does not describe a simple in or out; rather, there are always gradations of

unequal participation. Furthermore, participation appears as a dynamic concept, rather than a condition at a given moment. Lastly, Bartelheimer (2008) emphasises that participation is active, that is, it is striven for and realised through action and in social relationships. Summarising the last three points of Bartelheimer's participation concept for teaching processes, an exclusively quantitative description, that is, a definition of participation in the sense of "takes part/does not take part", appears inadequate. As far as possible then, the description of participation must be considered qualitatively over a longer time-frame, and might only be valid for the respective participating individuals within this time-frame, since this is a question of a dynamic concept that is constantly changing.

The underlying research project is concerned with describing scenarios from the process of progressing inclusion in German schools, and on this basis to design steps to further this process. To enable a description of this process the concept of participation in inclusive mathematics learning needs to be rendered more precisely, in order to open it to empirical description and analysis. For this purpose, in the next section the theoretical understanding of mathematics learning and the theoretical conception of participation in inclusive discourse, discussed individually above, are brought together.

A theoretical concept of participation in mathematics education

With the goal of describing the process of inclusion in mathematics learning in German schools using a participation-theoretical model, we take the above theoretical considerations on inclusion as a starting point and link these to our essential theoretical assumptions on mathematical learning. Taking Bartelheimer (2001) into account, it seems important to adopt a longer-term perspective on participation (cf. also Brandt, 2004). Although Krummheuer and Brandt (2001) go so far as to attempt an interactionistic theory of participation in mathematical learning, focusing also on individual learning using their system of categories, content-related learning and individual cognitive restructuring unfortunately seem to fall out of focus. Yet, our research is also and especially guided by looking at content-related learning. In this context, we are not trying to leave the basic interactionistic orientation. Instead we are trying to connect the ideas of learning and participation within interaction with individual learning. From an inclusion-theoretical perspective, realising inclusive teaching is a question of increasing participation. In order to make this useful for empirical investigations, Bartelheimer's (2008) more general participation-theoretical model is taken as a basis and combined with the participation-theoretical understanding of mathematics learning (e.g. Krummheuer, 1992; Sfard, 2008). To be able to describe participation in inclusive mathematics learning in Bartelheimer's perspective, one has to view it in relation to its historical context. The image of mathematics learning for all learners is taken to represent present conditions. Whereas at the end of the 1970s, national and international research on mathematics in primary schools was still focusing above all on the learning of skills and isolated concepts of a 'complete mathematics', with an emphasis on arithmetic, a shift took place around the mid-1980s. A new understanding of mathematical learning developed. According to this, children in school not only needed to acquire mathematical skills, but also to discover and understand the mathematical concepts behind them, and to argue and communicate with teachers and classmates using these concepts in order to ultimately be able to autonomously give reasoning for mathematical actions (Boyd & Bargerhuff, 2009). However, in adopting an orientation towards this image of mathematics learning we nevertheless acknowledge that it is (still) controversial for learners with special educational needs in special needs schools. The above-described changes in the understanding of mathematics learning can certainly be

seen also in the area of special needs education in mathematics (Sullivan, 2015). But these changes are the subject of heated debate and are taking place in the context of a (tried and tested) teaching tradition which is characterised by a reduction of learning content, an isolation of difficulties and a "small-steps" approach with specified solutions to problems (Boyd & Bargerhuff, 2009). In addition, Bartelheimer describes participation as an active process. Participation is pursued and achieved through social action and within social relationships. Therefore, it is observable in everyday mathematics teaching as active participation in classroom interactions. Bartelheimer also describes participation as multidimensional, which we have linked to Roos' (2014) spatial, social and didactical/content-related dimensions in the above theoretical section on inclusion. The spatial dimension of participation, according to Roos, relates fundamentally to how much time a student is spending in the same room as his or her classmates. However, in our perspective it also relates to the spatial configuration during time spent learning together in the mathematics classroom, for example in rotating through different tasks or group work. The social dimension focuses on social relationships (with fellow pupils, teachers and pedagogical staff) which emerge in mathematics teaching and which mediate to a great extent an increasingly autonomous participation in collective argumentations. The third dimension addresses participation in didactical/content-related negotiation. Didactical inclusion relates to pupils' participation in subject teaching, focusing on their engagement with the teaching approach and content, as well as any explanations or material supplied by teachers to support the learning process. For the purposes of analysis of didactical/content-related participation, the approaches developed in mathematics education for determining participation in collective argumentations (Krummheuer & Brandt, 2001) can be made use of. With reference to Goffmann (1981), Krummheuer and Brandt (2001) distinguish two types of involvement in a lesson: the active, verbally productive act, and the passive, receptive non-verbal act. The aim is to identify the type of authenticity, originality and responsibility of speakers, and to identify for recipients the type of non-active participation. This means that mathematical learning through active participation can be distinguished from learning through non-active participation, which explains "quiet" yet successful pupils. Learning situations become beneficial for learning, according to Krummheuer and Brandt (2001), when children participate increasingly in ways which permit a shifting from minor responsibility for content and form towards greater responsibility. In this way, participation in a collective argumentation concerning statements about (mathematical) content, terms and/or methods creates the basic conditions for mathematical learning opportunities in inclusive learning settings. Since this model remains more of a formal analysis of the content-related negotiation in the conversation, it will be complemented by the curricular concept of mathematical activities, developed by Bishop (1988), in order to approach also the mathematical content of the activity children participate in. Bishop (1988) differentiates six activities – counting, locating, measuring, designing, playing, explaining – which are used for the analysis of moments of subject-specific mathematical participation, following Brandt (in press) and Johannson (2015). In addition, Bartelheimer (2008) focuses on the principal dynamic, i.e. the changeability of participation over time, and the impossibility of a dichotomous categorisation of inside and outside, participation and non-participation. These considerations are in tune with an interactive understanding of mathematics learning and are taken into account within the research project by the theories and methodologies being used. This theoretical conception of participation will be used to address the empirical aspect; the theoretical conception will be further developed through the interrelationship between theory and

praxis, with the goal of enabling a description of participation processes in inclusive mathematics. This is while acknowledging that the goal of inclusion, which has the principle of egalitarian difference (Prengel, 2006) at its base, cannot be for all children to participate actively in class in the same way. The barriers to participation should be reduced for all children, and they should be given the opportunity for participation according to their abilities, so that they move forward in their mathematical learning process.

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The socio-politics of teacher explanation in mathematics education

David Kollosche

Goethe-Universität Frankfurt, Germany; kollosche@mathematik.uni-frankfurt.de

Despite efforts for a more student-centred teaching in mathematics education, data from interviewed German students suggest that teacher explanation is the most dominant form of introducing new knowledge and skills. From a Foucaultian standpoint and on the basis of the interview data, it is firstly argued that explanation belongs to an institutionalisation of mathematics education in which explanatory power is reserved for the teacher, leaving students with a passive role both towards learning and towards questioning mathematics as a discipline. It is secondly argued that such an organisation of teaching might be functional in identifying well-disciplined and fast-learning students through their achievements in mathematics. Thirdly, the point is made that the ignorance of research concerning the socio-political role of explanation is effective in the conservation of the socio-political functions of school mathematics.

Keywords: Mathematics instruction, theory practice relationship, teaching styles, student interview.

The role of explanation in pedagogical theory and practice

Under 'explanation' I understand the verbal and embodied communication of knowledge and skills from one person to another with the purpose of enabling the other to do something which the first person is already capable of. This understanding of the term is narrow, as in a wider sense, for example, explanatory texts, recorded speeches, videos and other media, where experts explain something to an anonymous audience, might be considered explanation as well. This paper has an even narrower focus on the explanations which mathematics teachers provide for their students, and it is mainly based on the situation in German schools and in the German research community.

Historically, explanation by the teacher had been a method central to any school teaching (Tenorth, 1988/2000). For example, the German philosopher Johann F. Herbart (1806/1897) developed a teaching methodology based on explanation, exercise, application and abstraction. Its popularity among both educational theorists and practitioners of the 19th century elucidates the traditional importance of explanation within the pedagogical discourse. The discussions around Herbart's pedagogy also stand exemplarily for the problems that modern pedagogy has developed with explanation. Claiming to follow Herbart's tradition, a group of educators, now referred to as the Herbartians, reduced Herbart's pedagogy to its methodological aspects and developed a strict teaching plan which was dominated by teacher presentations and copying by the student. The consequent critique of the Herbartian approach at the turn of the 20th century circled around the problems of the passive and obedient role of the learner, especially on the devastating effects on learning outcomes and democratic agency. Especially writers in the tradition of German *Reformpädagogik* such as Johannes Kühnel (1916/1950) considered the passivity that learners were introduced to as expressions and requirements of the civil obedience in the German Empire which allowed for the economic misery of the masses and the outbreak of the First World War in the first place. Ausubel (1968) argues that the fight against this passivity in learning has fuelled not only *Reformpädagogik* but many alternative pedagogical agendas up to that of discovery learning in his times, and that, over time, explanation has been increasingly denounced as a teaching method which

supports despotism, ignores the individuality of learners and denies them the benefits of self-regulated learning. Explanation, suspected to conflict with the aims of liberal education, has gained a negative connotation.

We run the risk that the condemnation of teacher explanation blocks the discussion of very different roles that explanation might have in teaching, reaching, for example, from introductory explanations followed by exercises over formalising explanations in the course of individual or collective explorations to summarising explanations at the end of learning activities, from whole-class talks to individual conversations, from short inputs to extensive presentations and so on. Recent developments in educational research however aim at a rehabilitation of explanation. Kathrin Krammer (2016, p. 76; all German quotes translated by D. K.) remarks in a teacher journal's special issue on "teaching":

> Many reform initiatives in the area of classroom development aim at the expansion and high-quality arrangement of self-regulated learning. Which meaning is yet assigned to teacher-centred, instructive phases – do they disappear, are they preserved, or are they rediscovered and altered?

In mathematics education research and educational policy, explanation as a teaching method is not a central field of study. The federal German educational standards for mathematics education in the grades 5 to 10 (KMK, 2003) may serve as an influential example of the discourse of educational policy. There, "explanation" is not mentioned once, nor are any other activities of the teacher. Instead, it is demanded that mathematics education provides "competences which students acquire in active involvement with manifold mathematical contents" and that it aims at "self-regulated learning" (p. 6). In the German academic discourse, recent introductions to mathematics education for prospective teachers (e.g., Bruder, Hefendehl-Hebeker, Schmidt-Thieme, & Weigand, 2015; Reiss & Hammer, 2013) do not even address how to *explain* knowledge and skills to students, and the only German book on teaching methods for mathematics education (Barzel, Büchter, & Leuders, 2007) presents 30 different methods but does not cover teacher explanations. The only recent studies in the German field discuss explanation from an epistemological (Wörn, 2014) and discursive (Erath, 2016) perspective, but could not be included in this study due to a lack of access to the publications.

The marginal position of explanation in mathematics education research and educational policy is confronted by the dominance of explanation as reported in empirical studies. The TIMSS video study (Stigler & Hiebert, 1999) compared the national teaching "scripts" of the USA, Germany and Japan, showing that both in the USA and in Germany, teachers usually introduce new knowledge by explanation. Recent empirical data, which will be presented here, propose that, at least in Germany, the situation has not changed. Despite continuing efforts in mathematics education research and educational policy to change the classroom culture towards forms where the teacher and her explanations play a less central role, interviewed 9th grade students from a variety of German schools report consistently that new contents are usually introduced by teacher explanations. Thereby, teacher explanations are not only political as they tend to establish distinct hierarchies concerning the distribution of knowledge, they are also political due to the tension between their dominance in school and their taboo in research, resulting in a structurally fostered unpreparedness of prospective teachers and a lack of support by research on this form of teaching. In this contribution, these issues are studied through an analysis of the subjectivities which students express in relation to teacher explanations in the mathematics classroom. The leading question is what role explanation plays in the development

of the students' subjectivities and where the socio-political dimensions of these forms of subjectivity may lie.

The student perspective

As part of a research seminar at the Universität Potsdam in 2016, master students orally interviewed 23 students from grade 8 to 10 in regular public schools in and around Berlin. The interviews were conducted in school rooms in private, recorded and transcribed. All students but two, who went to the same class, attended different schools. The semi-structured interviews focussed on the students' relationships to mathematics and included the prompt "Describe what a typical maths lesson looks like!" and question "How content are you with your maths teacher?", which appeared in the interview as items 2 and 3 of 12 stimuli in total. These item triggered answers which mentioned teacher explanations. Although explanation was not a topic that was explicitly addressed in the stimuli, we found it surprising that *all* students reported that their teachers usually explain new topics to the class. Only four students stated that other ways of introducing new topics, such as solving problems individually, in small groups or in whole-class conversations, were frequent, but in all cases these approaches were said to be followed by teacher explanations as well. We were also surprised that 19 students associated their confidence in their teachers with their qualities in explaining.

For my argument, it will prove important to discuss the ontological status of the students' reports and the epistemological approach taken in the analysis of the data. Here, I want to apply a Foucaultian view (Foucault, 1982, 1978/1991, 2011) to understand mathematics education as a disciplinary institution where teachers apply certain techniques for the conduct of the self and others in order to produce the expected behaviour in students, and where students, for their part, develop and enact certain technologies of the self in order to cope with these demands. The reports of school experiences and relations to mathematics cannot be understood as an objective account presented in a depersonalised language, but belong to distinct discourses around school mathematics, which are shaped by a shared knowledge of the actors. These discourses comprise values, interpretations and supposed truths whose paramount function is not to provide academic insights into any objectivity of the mathematics classroom, but to allow each individual to weave her experiences and relations into a meaningful web of explanations. Under these circumstances, each student's report should not be read as a mere account of a real experience, but as the expression of a permanent struggle to articulate experiences and relations which, from our point of view, are usually scarcely verbalised.

Given the incidental manner in which the topic of teacher explanation was touched in the interviews and the consequently low data base, this contribution will have to limit itself to the presentation, interpretation and discussion of a selected set of themes, and for that I chose to discuss the relations between teacher explanation, power relations in the classroom and the subjectivity of the learner.

Explanation and power relations

The central role which teacher explanation plays in all of the 23 interviews does not only provide insights in the unbroken dominance of a teaching method which large initiatives of pedagogues have fought against for decades, but first of all documents how students integrate the teacher into their narratives of success and failure in learning. Rebecca (all names changed while still indicating the original gender), a high-achieving 10th grader, describes her teacher as "really good", "the absolute burner", who "puts it across really well", "tries to adjust and can explain really well", and holds these

attributes responsible for the learning success of her and her classmates. On the downside, the teacher's explanation qualities are also considered the source of serious complication and failure:

Interviewer: And how content are you with your maths teacher generally?

Ingo: Huh, I would say it could be better. Well, I find, some things he doesn't explain well at all. Then at home, I have to sit down and look in my book. Yeah, he does not really explain it. So, when I hear what other classes tell, they have better teachers, they all understand.

Interviewer: Is there something you'd like to change in your mathematics classes? […]

Ingo: [If I were the teacher] I'd adapt myself to my students much more than my current maths teacher does. So, I'd go to them and ask if there's anything they don't understand, I'd do difficult exercises with them, those you need for exams […].

Apparently, students such as the 9th grader Ingo find their learning troubled by insufficient explanation. They also show awareness that the quality of explanation varies from teacher to teacher. Rebecca and Ingo follow a narrative in which their learning and achievement depend directly on the quality of their teachers' explanations. Ingo is not content when his teacher leaves him with difficult exercises after having explained the easy ones; he demands series of explanations which also cover the most difficult tasks. Simon, also a 9th grader, is even more explicit concerning these demands:

Simon: The teacher should, when he comes to the students, when he sees from the front that students have problems, then he should go to the students or the students to him and ask. The teacher should try to explain as simple as possible, so easy, perfectly easy, so that the student understands very quickly, so that he can go on with the exercises.

Patrick, another 9th grader, says that he was "actually very content" with his teacher, who "can explain well, so that we actually all understand", but later he adds that the difficulty of the contents has been increasing since primary school:

Patrick: I believe that what he does is actually really good, our teacher, but we, with us it's simply, no idea, that we simply don't understand when he tells something. And in front, well, there are a few of our students who understand and try to somehow explain it to the others, but that doesn't help either.

The position that teacher explanations hold in the narratives of the students has specific consequences for the power relations between teachers and learners. The dependency of Rebecca's and Ingo's learning on the quality of teacher explanation documents the monopoly of expertise which lies with the teacher. Especially, the students do not report any other promising sources for understanding such as textbook study, collaborative work or learning videos. In Patrick's case, the students of his class apparently attempted to support each other, but failed. Indeed, in the narratives the teacher is presented as the only agent the students can turn to in their struggle to understand. This narrative puts the teacher in the position of an exclusive 'knower' without whose expertise and goodwill no learning is possible, and thus it releases the students into passivity. The student, whose only hope is to be presented an understandable explanation, cannot do anything but wait for that explanation. Ingo's and Simon's cries for ever better explanations show the lack of alternatives they see.

From the perspective of traditional critique as brought forward already by the *Reformpädagogik*, we could argue that these experiences simply give empirical evidence that the traditional teaching methodology of explaining and exercising leads to passive and obedient students who are denied the flexibility and effectiveness of self-regulated learning and socialised into passive and obedient social agents. From the perspective of Foucauldian governmentality, we could add that the institution of mathematics education is successful both in channelling the conduct of the students into a form where their learning is totally dependent on the teacher, and in establishing a discourse in which this organisation of the learning of mathematics is considered inevitable. Here, it is interesting to note that both teachers and learners are constantly reproducing this organisation and narrative. At this point it is only possible to guess where the motivation for this behaviour come from: While the teacher may be led by the will to be the social centrepoint of the classroom collective, channelling all power on herself, the students might eventually enjoy their passivity. Ingo's reluctance to "have to sit down and look in my book" indicates that students may indeed resist to take a more active position in their learning. This resistance is connected to a constantly reinforced economy of learning in which students aim to "go on with the exercises" and pass "exams" with as little effort as possible.

Apart from the traditional critique of teacher explanation focussing on its consequences for learning and democratic agency, the exclusiveness of approaching mathematics through the teacher leads to a specific relation to the discipline of mathematics itself. In the reports of the students, mathematics is not presented as a discipline which can be approached and understood individually, but as a discipline whose understanding depends on the support of experts. The presentation of mathematics as a discipline which is only mastered by experts and cannot be fully understood by laymen, despite all efforts of specially trained teachers, adds to the construction of mathematics as an obscure, elitist and indisputable discourse which may be used as a tool of power throughout our society. Mechanisms leading to this image of mathematics have been identified before (Dowling, 1998; Skovsmose, 2005; Kollosche 2014), but to my knowledge they had not yet been associated with distinct styles of teaching.

Subjectivities of listening

Changing the focus from the teacher to the learner opens up a wide field of experiences of receiving explanations. Christian, a 9th grader, excels in some subjects but has problems in mathematics:

Interviewer: What do you think is different in maths; what's the reason you don't like it that much?

Christian: Well, I've never liked maths. Maths was never what I was good at, I always had my difficulties there. Though my parents think I'm somewhat lazy, which is true for the most part, it also gets more and more difficult and I seldom keep up with it, also because the teacher is bad at explaining.

Interviewer: How does a typical maths lesson look like at your place? Can you describe it?

Christian: Yeah well, the teacher comes in. Consequently, it's noisy of course, because she can't assert herself. In between, you're also getting distracted, and I'm no different, I admit I'm also getting distracted, doesn't let you work well, doesn't let you pay attention. […] Everything depends on paying attention deliberately.

It may be argued that it is the teacher's task to establish the quiet environment necessary for the students' understanding of any teacher explanation. But apart from the fact that such a narrative reproduces the active role of the teacher (who has to tame the students) and the passive role of the students (who have to be tamed), this narratives does not consider the subjectivities necessary to follow this form of teaching. "Paying attention deliberately" is a technique which students have to master, not only to follow teacher explanation individually, but to establish a fruitful learning environment in the classroom in the first place. Consequently, mathematics courses which build on teacher explanation give advantages to self-disciplined learners, especially when grouped together in socially segregated schools. The privileged school marks, which such advantages may result in, may then be taken as indicators for the self-discipline of an individual. In this sense, the pervasiveness of teacher explanation in mathematics education may have an underestimated economic function.

Yet of course, concentrated listening does not guaranty understanding. Students also have to be able to understand the presented contents in the pace in which the ideas are presented. Anna, a 9th grader, claims that her teacher's explanations are too fast for her to understand:

Anna: […] And I just find maths difficult, I don't understand it that fast. And of course, she [the teacher] does not have the patience for so many students to explain that to everybody separately. And some are simply faster in understanding concerning maths exercises, and I need a little longer and don't understand that fast.

Interviewer: […] If your teacher realises that several students put up their hands, will she then explain it again for all of you?

Anna: She is somewhat strange in this respect. She just says that she explains in a way that we all have to understand, and then we have to cope with the exercises somehow.

Anna realises that structural constraints in the organising of her teacher's approach hinder her to offer every student an understandable explanation. But instead of questioning the methodology of teaching altogether, some teachers succeed in hiding the problem. In Anna's case, the teacher's imperative that the students "have to understand" leaves the problem with the students, who do not seem to know how to cope with it. In the case of Emma, an 8th grader, the teacher asks the students to put up their hands if something is unclear, but "as we just know that she somehow cannot explain it properly", nobody would put up a hand.

In contrast to that, 9th grader Laura explains that her mathematics teacher has successfully taught her to indeed raise questions if something is unclear:

Laura: I had her in the fifth, sixth and seventh [grade]. I liked her. She was my favourite teacher. She's retired now. She has taught me to raise questions again and again, and that's it. Or to become more self-confident, because you don't know the others, you don't know the teachers. You still have some respect for them. […]

Interview: What do you believe the teacher could do against it [students not daring to ask]?

Laura: Oh god, that's difficult. He could pose questions, answer questions. But some don't really dare to ask. They have their private afternoon lessons, but actually this is also like a teacher. I think it's this collective. In class, you always have a position to fulfil. You are either the cool one or the somewhat quiet one or the class clown.

> […] You also notice that when fewer students are in class, the class is quieter and can work better. I believe, this also depends on the fact that you do not have to prove yourself and that you can rather concentrate on your stuff.

Laura's story documents that there are slight variations in the forms in which teacher teach through explanation, and that these variations can have severe consequences. In opposition to Emma's report of her classmates not asking in order to avoid further explanation, Laura has learned to demand further explanation if she is not confident with the explanation presented. Thus, her teacher enabled her to take a more active part in her learning and to add elements of conversation to teacher explanations. In addition to that, Laura outlines a sociological explanation for problems with explanatory phases. Exposed to the whole class, students may have an intense urge to fulfil their social role within the learning group, and that role might hinder them to engage in a lively discussion on mathematical contents.

Discussion

The findings presented first of all shed light upon black spots in mathematics education research. Firstly, the prevalence of teacher explanation shows that decades of academic and political initiatives aiming at changes in teaching and learning arrangements have hardly affected the reality of the mathematics classroom. Although several nation-wide and regional projects have focussed on introducing learning environments focusing on active learning in Germany, teacher explanations are still reported to be dominant in the mathematics classroom. Considering the apparent ineffectiveness of previous interventions, it would be useful to dedicate more research to the understanding of the didactical and social role of teacher explanation *before* any new interventions are planned. Secondly, in light of the central role of teacher explanation in the mathematics classroom, the marginality of the topic in mathematics education research leads to blind spots in our understanding of teaching practice. Especially the socio-political dimensions of teacher explanation, which might prove antagonistic to pedagogical ideals and nevertheless functional in a systemic sense of society, deserve further study. Deeper insights could lead the way to a teaching practice which incorporates teacher explanation without constructing the student as a passive subject to mathematics. Apart from that, it may be wise to critically prepare prospective teachers for the role that they apparently assume anyway, namely that of the explaining authority.

Further research should also focus on the psychology and the socio-politics of teacher explanation. Firstly, why would teachers contribute to the narrative that it would be possible to allow a large proportion of the students an understanding through central explanation, while counter-arguments are obvious in teaching practice and have been discussed in literature for decades? Why would students contribute to that narrative against all obstacles they experience in their learning and in spite of the passive role they have to assume in this learning arrangement? And secondly, how does teaching through explanation contribute to the narrative that the understanding and mastery of mathematics is reserved to higher authorities, who can share their knowledge and skills to the extent they wish and whose expertise has to be trusted in due to the lack of approachable alternatives?

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Polish parents' views on mathematics education in Swedish preschools

Dorota Lembrér

Western Norway University of Applied Sciences, Bergen, Norway; dorota.lembrer@hvl.no

This paper describes the results of a survey of Polish immigrant parents' views on mathematics education in preschool. In alignment with the Swedish preschool curriculum, the results show that parents viewed learning as being connected to play. The parents to a large extent see similar frequency of mathematical activities occurring at preschool and at home. Parents often commented on children's involvement in collaboration when playing. Swedish preschools' pedagogical practices about learning through play seem to have been adopted by parents. The findings suggest that parents, like children, can be socialized into the norms and values of Swedish preschools through their children's attendance of them.

Keywords: Immigrant parents, mathematics, preschool, socialization.

Introduction

Mathematics education, as a part of everyday life, is embedded within a variety of settings. These settings include institutions, such as schools and preschools, but also homes. Within each setting, there are structures and expectations which contribute to the development of norms and values (James, Jenks, & Prout, 1998). Therefore, Swedish preschool as being framed by institutional norms and values (Skolverket, 2011), must also be considered as shaping society's understandings about the world of children and their families. As James et al. (1998) state: "Childhood diversity considers the infinite variety of the social context in which children live, leading to a deconstruction of childhood's conventional, singular and reductive form" (p. 34). As a consequence of migration, culturally diverse societies can represent different views of what kind of childhood is available and thus the learning that children should receive in Swedish preschools. Therefore, newcomers' perspectives on mathematics education in preschool can contribute to a better understanding of the variety of children's childhoods, when those views are recognized as legitimate. As part of a wider project investigating this topic, I aim in this paper to provide insights into Polish parents' views about mathematics in Swedish preschools. In 2012 Polish immigrants were the third largest group of immigrants in Sweden (Statistics Sweden, 2012), yet their views on education have rarely been investigated. My research question is: How do Polish parents view mathematics activities in Swedish preschool and at home?

Socialization and parents' views on mathematics education

The theoretical framework for this study is socialisation, through which people who inhabit a society create it (James et al., 1998). Socialisation processes have two components, the production and reproduction of (1) norms and values and (2) skills and knowledge. Socialisation processes situate society members as needing to acquire relevant knowledge to sustain society over time and involve reproducing culture from one generation to another. As well, as the milieu in which a society operates changes, a need arises to produce new ideas and culture. Ebrahim (2011) stressed that social interactions contribute to the production and reproduction of rules and structures within a society.

Children acquire the understanding, skills and awareness of different mathematical concepts, through their experiences outside educational institutions (see for example, Brenner, 1998). This provides opportunities for children's own experiences to be the basis for developing their mathematical thinking in preschools (Lembrér & Meaney, 2015). In the Swedish preschool curriculum (Skolverket, 2011) four goals are related to mathematics: one is connected to content; while three require preschools to provide opportunities for children to develop mathematics skills, abilities and concepts. It also suggests that children should use their interests and experiences when acquiring mathematical knowledge and skills in preschool.

However, the institutional values and norms of Swedish preschools may constrain immigrant parents from being able to recognize experiences from their country of origin as being valued in the new setting (Lunneblad, & Johansson, 2012). This may have long term implications for families' involvement with educational institutions in new countries. For example, Giovannini and Vezzali (2011) focused on how contact between teachers and immigrant parents affected children in elementary schools. They found that parents' views about their role in the relationship with school can define boundaries of that role. The education institution has a role in changing those boundaries. In Goodall and Montgomery's (2014) study, parents' reflective involvement in their relationship with schools was strengthened by an acknowledgement of their contribution to children's learning. Yet language differences may impede these possibilities. Civil, Bratton and Quintos (2005) suggested that the dominant language spoken in school, can affect children's interest in identifying with their home language. As a result, children can have difficulties talking with their parents about their school work.

With an increase in immigration, there can be challenges around gaining the active participation of immigrant parents into the education system. Kaur (2010) suggested: "creating strong links between families and early childhood settings extends children's learning, fosters a sense of community and acknowledges the expertise of families" (p. 53). It has been found that when refugee and immigrant parents are included in education systems, there are academic benefits for their children (Krasteva, 2013). Yet, Whyte and Karabon (2016) found that relationships between home and school are often built on one–way communication, with information going from school to home. This creates boundaries between families and school. Wager and Whyte (2013) investigated how preschool teachers valued children's home experiences of mathematics. They found that preschool teachers act in two different ways to children's home mathematical experiences. The first involved only recognizing activities already familiar to the preschool teachers. In this way existing norms and values are recreated. The second integrates children's home experiences into new activities, providing opportunities to create new norms and values. Wager and Whyte's study raises a question about whose ideas are used in mathematical activities, also how these ideas are recognized as valuable. Understandings about socialization indicate that institutions need knowledge of immigrant parents' norms and values. Otherwise, opportunities offered by institutional settings and arrangements focus on integrating into the existing Swedish societal norms and values and not considering possibilities for creating new norms and values to match the changing milieu. Preschools have the opportunity be influenced by parental views, which could contribute to widening the variety of mathematics activities that can be based on children's own interests.

Method

The data were collected through a digital survey, consisting of 16 questions in August, 2016. The survey was provided in Polish and Swedish. Participants were identified through a snowballing approach (Cohen, Manion, & Morrison, 2000). First a Polish organization and an internet forum for Polish citizens living in Sweden were provided with a link to an anonymous online questionnaire. People who completed the digital survey were then asked to share the link with others. All participants had a Polish linguistic and cultural background and were immigrants to Sweden. They had children who had attended preschool in Poland and/or in Sweden. The participants were a convenience sample and cannot be considered representative of the population of Polish parents in Sweden. As such, the survey results can provide information (Coyne, 1997), which can be investigated in more in-depth studies at a later date. At the time of the survey, the participants had lived in Sweden between 2 and 19 years. 31 participants responded to the survey (1 male, 30 females), aged between 22 and 47 years. Participants were described in the study as: P1–P31.

The survey's questions were divided in three parts. The first part consisted of demographic information: gender; age; number of years living in Sweden; if participants' children attended preschool in Sweden (30 did, 1 did not); and if participants themselves had attended preschools in Poland (23 did, 8 did not). In the second set of questions, participants were asked to describe their experiences of learning mathematics, including their experiences of learning mathematics in preschools in Poland (Question 6). Question 7 asked about the experiences of both themselves and their children. Questions 8 and 9 asked about situations in which children could learn mathematics at home and at preschool in Sweden. The next question was about possible mathematical situations, in Polish preschools. However, only one participant had children attending preschool in Poland so this data were excluded from the analysis. The third part used multiple-choice questions (Questions 11 and 12) to investigate parents' perspectives of mathematics activities. Five activities were suggested as occurring at preschool and at home and these were based on what had been found in research by Bottle (1999) of observations, at home and outside home. In Bottle's research, parents talked about mathematics activities such as: number and counting; doing puzzles; making towers; putting things in and taking them out again and things like full, empty and half full. The multiple-choice responses in the questions about parents' perceptions of mathematical activities were: counting rhymes; jigsaw puzzles; counting things; playing with sand and water; and building with blocks. Participants were also asked to express what was important for them based on their own experiences of mathematics (Questions 6, 13 & 16) and to describe the importance they attached to their children learning language and mathematics (Questions 14 & 15). The final question was open-ended and provided parents with a possibility to share something about their children's learning of mathematics with teachers in preschools.

Two analyses of collected data were done. The initial quantitative analysis was of parents' views about mathematics activities at home and at preschool. The second, qualitative analysis used the socialisation components of production and reproduction of societal norms and values (James et al., 1998), to understand parents' views about mathematics in preschool. By analyzing the parents' responses, it was possible to identify the norms and values that parents held about mathematics in kindergarten and consider how these were related to production and reproduction.

Quantitative analysis

Asking about particular activities was seen as a way of concretizing what could be considered mathematics for the parents. The questions were "Which of these things do you do at home, which you think might help children learn mathematics?" and "Which of these things do your children do at preschool, which you think might help them learn mathematics?". The parents could provide more than one answer.

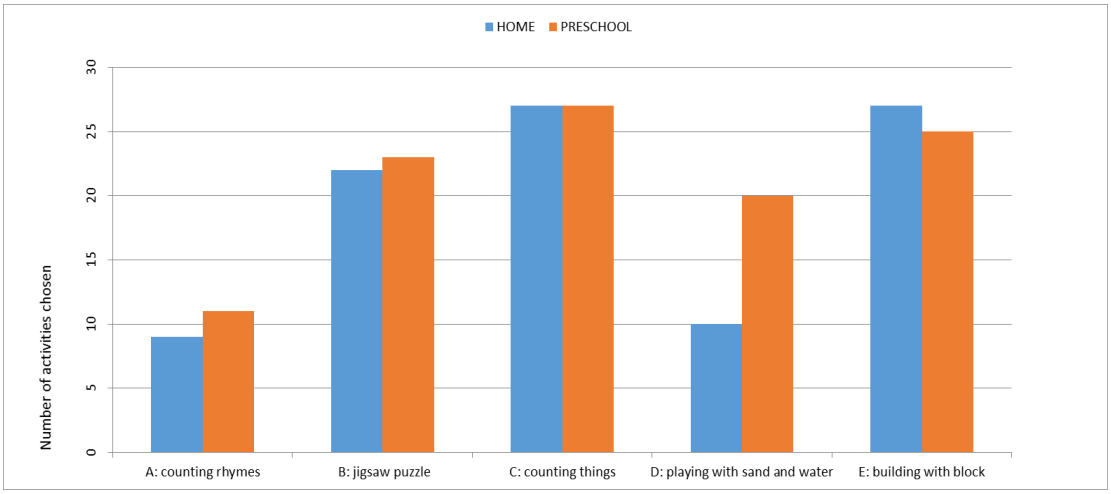

Figure 1: Frequency of home or preschool activities

Figure 1 shows which activities been chosen by parents as being present at home and at preschool. From most parents' perspective, the children engaged in similar types of activities at home and at preschool which would contribute to them learning mathematics. For example, 27 out of 31 participants considered that Activity C: counting things, was something that parents considered children would do at preschool and home and would support their learning of mathematics.

The only significant difference between perceptions of what kind of mathematics activities are present at preschool and home was for activity D: playing with sand and water. Parents did not chose this activity, as something their children engaged in at home. Although more research on this is needed, a simple explanation may be that they did not have the facilities at home for it. Similarly, none of the participants, when they responded to question about describing their own ideas about how and when children learn mathematics, suggested playing with sand and water.

Qualitative analysis

In this section, I describe parents' views about mathematics education for their children, as expressed in the survey. Analysis highlights why parents considered it is important for children to engage in mathematics activities in preschool and at home. The focus is on parents' perceptions of the institutional norms and values in Swedish preschool through their responses to the survey questions.

Parents' experiences, from their children attending Swedish preschool, seemed to have shaped their views about mathematics activities. This is clear in statements, such as the one made by P2, where mathematical experiences seemed to be transferred from preschool to home.

P2: Children learn basic shapes while playing. Shapes are used in different situations and aspects. My child comes home and continues asking us about different shapes "which is a shape of"? (in relation to various everyday objects).

It would seem that P2 in this statement is reproducing the norms and values about the importance of children knowing about shapes and about learning through play. Learning through play situates children as being active participants in the socialisation processes. It is unclear whether P2 had the same set of norms and values before her contact with Swedish preschools, but it is possible to claim that they are in alignment with the Swedish preschool curriculum (Skolverket, 2011).

The parents' reflections of their experiences from the institutional practices of the preschool in Poland, were evident in some of their answers. For example, P11 referred to ways of learning in preschool by saying:

P11: They (her children) attended (Polish preschools) and learnt exactly the same ways as I did at their age.

P11's perception was that Polish preschools had not changed in the generation since she had attended preschool. This suggests that she did not see Polish preschools as sites for producing new norms and values, just of reproducing the same ones across time.

The milieu in Swedish preschool was seen as a source of knowledge and learning of mathematics for children.

P21: It seems to me that the Swedish preschool have a big focus on mathematics. There is always enough mathematical activities, children usually have access to lots of toys/games which also are developing their mathematical skills.

The emphasis on preschool materials focusing on mathematics is viewed positively. P21 highlighted the mathematics that could be learnt through playing with toys at preschool. Opportunities for learning mathematics are in the environment and children have the possibility to explore them, as active agents reproducing societally-valued understandings about mathematics. However, it is unclear what the responsibility of a preschool teacher is, apart from making activities and toys available to children. It is unclear how the children learnt that what they are doing is mathematics.

P28 made reference to how everyday situations could support children's curiosity. Like many of the parents, points such as these were linked to the importance of children playing. The norm of Swedish preschool is for children to play and interact together. Play is *considered* as a stimulus for learning in interactions. Children's own knowledge can initiate interaction in play and, as Ebrahim (2011) stated, children can bring in imaginary characters during play. In this way, children produce and reproduce knowledge and understanding about their lives.

P28: Children learn through play with toys and friends in everyday situations. Children should be allowed to use their curiosity and discover new things so they can learn more easily.

A particular aspect of play that parents highlighted was that children could make their own decisions, situating them as active participant in the socialisation process. P13 expressed how much she valued this aspect of Swedish preschools.

P13: I like it here (in Sweden), that children have a lot of freedom in choosing and directing their play activities.

Responses such as this are in alignment with values expressed within the institutional practices of preschool (Skolverket, 2011). Children being active participants in activities, which they design or adapt, seems to be a shared value in Swedish society about what is normal for childhood. Children are socialised into this shared norm while engaging in these activities.

Parents' views of how children should learn mathematics through play is in alignment with the Swedish preschool curriculum (Skolverket, 2011). Parents mentioned play explicitly as an approach which was beneficial for learning generally, including the learning of mathematics.

P29: In preschool, play is the main form of learning. Children are enthusiastic and learn about the world around them through play. They should receive many interesting incentives in order to actively gain knowledge about the world in general, as well the mathematical world.

In responding to the final open-ended question in which they could share something about their children's learning of mathematics, the parents highlighted the importance of mathematics in everyday activities, such as: counting things; classifying objects; doing arithmetic; recognizing numerical symbols; building with Duplo. P30 reported that children gained a better understanding of mathematics while playing, but seemed to equate counting with mathematics.

P30: Through play children learn to count and get to know the numbers. I think that play is a good way to learn mathematics.

Counting objects during play, can be about making sense of existing knowledge. Awareness of this knowledge (counting) involves an active process of interpretation of this activity as mathematical. Similarly, P11 referred to everyday activities and emphasizing the value of learning mathematical terms and problem solving.

P11: Learning mathematics, vocabulary and mathematical concepts is necessary for children. They develop their abstract thinking, analyzing, reasoning and decision-making processes.

P11 emphasises what children need to learn, and why that is important. This indicates what this parent considers to be the valuable knowledge that children need to reproduce. What is interesting is that these practices reflect the goals and guideline in the curriculum, "develop their ability to distinguish, express, examine and use mathematical concepts and their interrelationships; develop their mathematical skill in putting forward and following reasoning" (Skolverket, 2011, p. 10). As in the first example in the results, it is unclear if this parent held these views of mathematics before coming to Sweden, but the fact that they are so closely aligned indicates that more research is needed to investigate this further.

These parents viewed play as a vehicle for learning and children's participation in mathematics activities as an active form of play. This is in alignment with the Swedish preschool curriculum (Skolverket, 2016), which state that play should be a stimulus for children learning and development. By expressing children's possibilities to make choices and decisions, parents show that they accept that children should be active participants in the socialisation processes. They also

had expectations that their children would have opportunities to engage in a variety of mathematics activities. They exemplify activities that promote the learning of knowledge of mathematics which is socially-valued. Within this perspective, socialisation appears as a process of recreating knowledge.

Conclusion

In this paper, I have presented the views of a set of Polish parents about the mathematics activities their children engage in at Swedish preschools. The findings suggest that, in alignment with thewedish preschool curriculum (Skolverket, 2011), the parents emphasized that young children's engagement in mathematical tasks in preschools should focus on learning through play. However, whether the parents' views had changed since they arrived in Sweden is unclear. It is also unclear if this group of parents can be seen as representatives of Polish parents in general, living in Sweden. Therefore, more research is needed to investigate if and how parents' views of mathematics in preschools are before and how opinions change once their children begin preschool in Sweden.

We need to gain a better understanding of how immigrant parents align with the pedagogical structures in the new country of resident. Much of the research based on work with immigrants' families (e.g. Civil et al., 2005; Giovannini & Vezzali, 2011), show gaps and struggles with parent's involvement with educational institutions. The group of Polish parents in this study show the opposite. This article gives a very brief indication of what is the potential for the group of immigrants from Poland. Their views are interrelated with their children attending preschool. They seem to interpret and adopt the Swedish preschool norms and values, through their children participation in preschool.

The educational structure, in which both children and parents adopt the norms and values of the society in their present country of residence, can be seen through processes of socialisation. Parents become learners of educational and pedagogical practices by using their experiences to recognize and work towards an understanding of the present. Socialisation is a complex and dynamic process with range of interconnected aspects operating simultaneously. Thus, more research is needed to understand the complexity of this process.

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The mathematics teacher's quasi-Darwinism: Problematizing mathematics education research

Alex Montecino

Aalborg University, Denmark; alexm@plan.aau.dk

This paper discusses the configuration of a quasi-Darwinian view of mathematics teachers, where the survival of the fittest is the cornerstone of a network of practices and discourses. It aims to contribute to the problematization of how mathematics education research and its discourses have effects of power in the fabrication of mathematics teachers' subjectivities, by unpacking naturalized truths of research – truths regarding a productive and successful mathematics teacher. It deploys a Foucault-inspired discourse analysis, and it argues how the research on the mathematics teacher becomes a practice that governs mathematics teacher's subjectivities through the enunciation of the desire subject, a productive, successful and effective teacher.

Keywords: The mathematics teacher, effects of power, quasi-Darwinism.

Introduction

Providing quality mathematics education has been a concern within the mathematics education community, research on education and international agencies. It is argued that the quality of education and the development of mathematical knowledge is essential for society and social development (Gellert, Hernández, & Chapman, 2013; OECD, 2010b). The idea is circulating that success in school mathematics is a prerequisite for personal and social success. Nowadays, it is considered that mathematics is a powerful mean to understand and control one's social and physical reality (OECD, 2010a), by being a tool and skill that helps people to undertake diverse tasks and problems of everyday life, and of their contexts (OECD, 2014b). However, according to OECD (2014a), modern societies valorize individuals not for what they know, but for what they can do with what they know, in other words, by their *mathematical literacy*:

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. (OECD, 2010b, p. 4)

Research in the field of mathematics education is seeking to modify reality – in the frame of social changes – through its findings, proposing rationalities, knowledge, and ways of improving education practices – for ensuring the quality of teaching and learning of mathematics. Furthermore, OECD (2014a) stated that "[h]igher educational achievement benefits both individuals and society, not only financially, but in the well-being with which it is also associated, such as better health outcomes and more civically engaged societies" (p. 104). In this fashion, the mathematics teachers become relevant, since they are considered as a central element in the establishment of quality education (Jong & Hodges, 2015; Luschei & Chudgar, 2015; OECD, 2005, 2014b). Several studies have argued on the relation between the quality of the mathematics teacher and the shaping of successful students (cf. Castro-Rodríguez, Pitta-Pantazi, Rico, & Gómez, 2016; da Ponte & Chapman, 2008; Hemmi & Ryve, 2015). Also, it is argued that the teacher is open to policy

influences, whereas factors regarding students and the classroom context are not open to the same policy influences, at least in the short run (OECD, 2005).

All the aforementioned, the circulating discourses around 'success' in mathematics, are positioning a way of thinking and understanding mathematics education through the configuration of valid methods of doing research and of arguing about the diverse issues involved in the teaching and learning of the mathematics. For example, "[a]ll research is built around a set of assumptions about the world and how it should be understood and studied" (Jablonka, Wagner, & Walshaw, 2013, p. 41). Thus, this paper problematizes mathematics education research and its effects of power on teachers' subjectivities and fabrication. A discourse analysis, inspired by Foucault's ideas, is deployed to unpack the naturalized truths and discursive formations about the effective and competitive mathematics teacher.

Movements to examine the mathematics teacher as a discourse formation

According to Pais and Valero (2012), mathematics education research produces languages and tools that shape what researchers see and say in the world of education and of mathematics education. Mathematics research as a field of inquiry is not an innocent or a neutral activity (Halai, Muzaffar, & Valero, 2016); it has been considered a "social institution which is inseparably linked to power" (Jurdak, Vithal, de Freitas, Gates, & Kollosche, 2016, p. 10). In this fashion, mathematics education, and also its practices, is considered to be political because it operates within governmentality techniques. Hence, by building on these techniques of government, this paper aims to understand how mathematics education research fabricates the mathematics teacher's subjectivity through regulatory practices embedded within naturalized truths. In other words, it addresses how research sees and talks about the mathematics teacher, by establishing regimes of power/knowledge.

According to Foucault (1972), "[w]e shall call discourse a group of statements in so far as they belong to the same discursive formation" (p. 117). Hence, discourse as a group of statements, provides a particular language and knowledge, assembling regimes of truths. Circulating discourses describe rules and enunciations of a particular body of knowledge from specific spatiotemporal conditions (Arribas-Ayllon & Walkerdine, 2008). This paper deploys a "research on research" (Pais & Valero, 2012) strategy built on Foucault's discourse analysis. This analytical strategy helps to unpack naturalized truths within research, that seek to generate a productive and successful mathematics teacher, and, at the same time, to trace the power effects on the fabrication of mathematics teachers' subjectivities. So, by problematizing the discourses, it is possible to understand research as a practice that governs subjectivities through the enunciation of the desired subject.

First, repeated statements about the 'must be' of the mathematics teacher are identified. Second, these statements are analyzed to trace their knowledge/power relationships, and their continuities and discontinuities amongst each other. It does this by analyzing published studies about teaching and learning of mathematics. The empirical materials consist of research about mathematics teachers released within the last four years of three journals: Journal of Mathematics Teacher Education, ZDM Mathematics Education, and Educational Studies in Mathematics.

Finally, it problematizes how research and its discourses have effects of power in the fabrication of mathematics teachers' subjectivities. It does this by portraying how certain rationality is circulating

within research in mathematics education. As will be described, such rationality promotes a quasi-Darwinism, in which the survival of the fittest and the idea of evolution are the cornerstone of a network of practices and discourses.

The mathematics teacher research and the survival of the fittest

In navigating through the discourses that are circulating about the mathematics teacher, amongst the materials analyzed, it is possible to identify some enunciations that are continuously repeated. By following a Foucaultian chain of thought, these particular enunciations constitute statements about how mathematics teachers are supposed to act and be within their practices, their 'must be'. Such statements respond to concerns raised by research in the field of mathematics education. For example, who is taken as valid for arguing about mathematics teachers, what does a mathematics teacher ought to do, and how to seek for the improvement of the teaching and learning of school mathematics. From the analysis, some discourses about the 'must be' of the mathematics teacher are configured as truths. These truths are advertised as desired features that teachers should have if they want to perform successfully, namely: a high knowledge (Fauskanger, 2015), an updated repertoire of techniques (Subramaniam, 2014), and a personality consistent with their practices – personal aspects such as beliefs or attitudes (Jacobson & Izsák, 2015).

These discourses are naturalized under a competition and comparison system of reason. International standardized testing – PISA and TIMMS –, and its reports are examples of how competition and comparison become part of society, by shaping social discussions, decisions, efforts, and initiatives. At the same time, through those tests' outcomes, diverse countries could monitor themselves to improve the weakest areas, since "[a]ll countries are seeking to improve their schools, and to respond better to higher social and economic expectations" (OECD, 2005). In this fashion, a variety of studies, that seek to improve the teaching of mathematics (see Boston, 2013; Lewis, 2016; Pang, 2016), are aimed to identify how mathematics teachers could achieve a successful practice by analyzing their students' achievement on national and international tests. But, as discussed elsewhere, what is taken, by research, as a successful practice leads to a system in which teachers compete against others teachers, against what is considered as a desired teacher, and, also, against themselves (Montecino & Valero, 2016). So, research discourse is raising comparison as a mean for knowing the characteristics of competent and effective teacher – the fittest teacher –, effective practices or successful experiences. Within these discourses, it is possible to see statements such as:

> By comparing and contrasting the practices of LS [Lesson Study] in mathematics in different countries, it will be possible to explicate the local theories of teaching and learning of mathematics, highlight educational values in each culture, and understand why and how these values support certain teacher development processes that are unique to the culture. (Huang & Shimizu, 2016, p. 394)

In the unpacking of naturalized truths of the analyzed materials, it is possible to see that some statements highlight mathematics teachers' deficits and flaws. These statements pay attention to what teachers need to improve in their lessons for increasing students' achievement (Spitzer, Phelps, Beyers, Johnson, & Sieminski, 2011). On one hand, by emphasizing that teachers need to achieve a higher expertise on school mathematical topics (e.g. Karakok, Soto-Johnson, & Dyben,

2015; Magiera, van den Kieboom, & Moyer, 2013). On the other hand, by focusing on the need for developing more effective teacher's practices (see Lee & Kim, 2016). This type of research acknowledges that mathematics teachers have a 'responsibility' for students' performances and, therefore, teachers ought to be highly trained. Alongside the statements about what needs to be improved, other statements exist that pay attention to what teachers lack, in other words, to skills that teachers are required to develop to reach what those studies perceive as 'successful professional development': on the one hand, studies regarding teachers' belief system (e.g. Conner, Edenfield, Gleason, & Ersoz, 2011; Cross Francis, 2015); on the other hand, studies regarding teachers' attitudes (e.g. Hannigan, Gill, & Leavy, 2013; Jong & Hodges, 2015).

According to some research, "[h]ow teachers perceive and adapt their roles will have great impact on overall classroom interactions, such as the teachers' questioning strategies or feedback patterns" (Lee & Kim, 2016, p. 366). This implies that teachers' decisions have an impact on students since it is believed that students' intellectual autonomy could be favored by teachers' practices (Goldsmith, Doerr, & Lewis, 2014). And so, the decisions made by the mathematics teacher have a high impact not only on students but also on their learning (Stockero & Zoest, 2013). This type of research shows that mathematics teachers should be constantly seeking to improve their professional development, practices, knowledge and skills not only for themselves but also for the sake of their students (Afamasaga-Fuata'i & Sooaemalelagi, 2014). Since professional development has been understood as a form of lifelong learning in which mathematics teachers are responsible for their own development and achievements, these types of statements, from a Foucaultian approach, are tracing the ways in which the mathematics teacher should become an effective and competitive teacher, through processes of self–regulation.

According to these studies, teachers should aim at improving, by themselves, diverse personal and technical aspects. Such aspects are supposed to encourage the development of a more effective and competent teacher, by recognizing their own deficits and flaws with the goal of overcoming them. This naturalized truth resonates not only within research but also amongst other discourses on education. For example, OECD (2012) states that effective teachers are a key to close achievement gaps between advantaged and disadvantaged students. And, therefore, the aim should be to (re)train and (re)shape teachers to become the desired effective teacher. In this regard, research is tracing a sort of 'evolutionary line' for mathematics teachers, in which at the end of the line rests the desired mathematic teacher. Teachers should evolve when they achieve the desired levels of knowledge and skills established by society, becoming the productive, successful and effective teacher. However, these desired levels are in constant movement, being redefined by new social interests, concerns, desires and demands as well as new mathematical knowledge that the modern citizen should have. This means that mathematics teachers have to govern themselves into a constant process of change, of (re)training and (re)shaping. As Deleuze (1992) asserts, currently nothing is considered to be finished; all is in a constant becoming.

The idea of the 'evolutionary line' helps to tell the narrative of the becoming of the mathematics teacher as the survival of the fittest, since research in the field highlights the features of the 'fittest subject'. This portrays that the survival of the fittest – the desired mathematics teacher – involves practices of self-regulation, but also of competition against other teachers, practices that could lead to the exclusion of certain teachers labeled as 'inferior subject', unproductive, unsuccessful, and

inefficient. For example, Lee and Kim (2016) have argued that mathematics teacher training programs "should include more specific investment in the effective use of classroom dialogue for learning" (p. 378), a 'fittest subject' should evolve in an effective classroom communicator whereas the 'inferior subject' will not evolve as a classroom communicator, and, will therefore be taken as ineffective. Consequently, the survival of the fittest governs the self and conducts mathematics teachers' practices towards the desire to evolve, (re)shaping the research about mathematics teachers within a system of reason rooted in a quasi-Darwinism, since it traces the paths for teachers to increase their abilities to survive, compete and evolve.

Quasi-Darwinism of mathematics education research and its effects of power

The analysis deployed has pointed to the existence of statements on the desired mathematics teacher, a self-regulated and evolved subject. These statements have been (re)producing certain truths about who the effective teacher is. For example, mathematics teachers should perceive themselves as responsible for others – i.e., their students' performances –, as promoters of social change – i.e., by closing achievement gaps –, and also, as responsible for themselves – i.e., tracing their professional development and learning the best possible way. These statements are building a quasi-Darwinian view of mathematics teachers; an 'evolutionary line' that is embedded within the above discourses and shapes the fabrication of the fittest subject.

The quasi-Darwinism (re)shapes mathematics teachers' ways of being and acting at a particular time and place, through discourses that are produced and reproduced under certain *regimes of power/knowledge* (Foucault, 1982). The naturalized truths are constituted, on the one hand, within a particular regime of knowledge, which delineates who is the one to discuss about the mathematics teacher and how, and in what way the knowledge regarding the teacher should be generated. On the other hand, within a regime of power which defines what understanding is meaningful to be studied – what discourses are taken valid regarding certain issues or aspect of the mathematics teacher – and which practices, knowledge and techniques should be targeted. Therefore, a quasi-Darwinian view (re)produces what the mathematics teacher should be – the becoming – towards the development of the 'human capital' (OECD, 2001). Human capital voices the value that subjects have in correlation with their knowledge, skills, education and preparation for the future, which translates into personal, social and economic well-being. Alongside, a quasi-Darwinian view (re)shapes a discourse aimed at optimizing the becoming of the teacher. Moreover, the research on mathematics teachers seeks to minimize all aspects that could lead to an 'inferior subject'. In order to be the fittest, teachers should engage in practices that turn them into accountable and measurable agents.

In this regard, it is possible to state that research in the field of mathematics education becomes a technology of the self (Foucault, 1997) that regulates mathematics teachers' conducts towards the shaping of the desired mathematics teacher. By promoting 'cultural thesis' (Popkewitz, 2008) about the desired mathematics teacher, the analyzed research has effects of power on teachers subjectivities, meaning how mathematics teachers understand themselves and their becoming. Only the 'fittest subject' is the one able to develop the skills and knowledge that society demands and requires, is the only one who can evolve in a 'superior subject'; subjects able to adapt themselves to the new social and professional demands. In other words, teachers are to evolve in subjects that

have the tools, skills, and knowledge to survive to all social changes and challenges; becoming a successful, effective, competent and fittest subject.

Thus, within the circulating discourses is configured a narrative in which if the mathematics teacher does not adapt or evolve, he/she is excluded or labeled as deficient. The teacher who survives through social changes and challenges is neither the knowledgeable teacher, nor the successful teacher, nor the most intelligent teacher; rather he/she is the most adaptable to change.

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Towards cultural responsiveness in mathematics education

Swapna Mukhopadhyay[1] and Brian Greer[2]

Portland State University, Oregon, USA;

[1]swapna@pdx.edu; [2]brian1060ne@yahoo.com

We describe ongoing work on the Culturally Responsive Elementary Mathematics Education (CREME) project, in which we work with teachers and students in two schools with contrasting student populations and communities. We present core principles of our emergent theoretical framework as we partner with the teachers and students to realize (in both senses of 'understand' and 'make happen') what culturally responsive elementary mathematics education might be. The backdrop for this activity is the educational/political arena within the United State, in particular in Oregon where we work, and in the schools themselves. We outline future development of CREME, and end with what we see as the implications of our experience with CREME for the ethical and political responsibilities of educational researchers.

Keywords: Culturally responsive, mathematics education, multiculturalism, educational politics, assessment.

Introduction

As an extension of the paper presented at CERME9 (Mukhopadhyay & Greer, 2015) we describe our continuing work on the *Culturally Responsive Elementary Mathematics Education* (CREME) project, at the core of which is a long-term exercise in teacher development, embedded in the schools and their communities.

We are not doing research in the narrow sense of stating research hypotheses, gathering data, conducting analysis. On the one hand, the paper provides a "thick description" (Geertz, 1973, pp. 5-6) of the evolution of a community of practice (Lave & Wenger, 1998), and, on the other, describes an effort to actualize a conception of culturally responsive elementary mathematics education (Greer, Mukhopadhyay, Powell, & Nelson-Barber, 2009) in a particular environment. As such, it can be characterized as raw material contributing to an emerging theory, elements of which we describe. As a development from the CERME9 paper, we devote more attention to educational politics within the United States, in particular the state of Oregon, making the argument that research cannot be considered politically neutral, and we raise meta-theoretical issues about the ethical and political responsibilities of researchers.

Principles and values of CREME

CREME is very much work-in-progress, but many influences are clear. Within mathematics specifically, we acknowledge Critical Mathematics Education (albeit fuzzily defined), and Ethnomathematics as major inspirations. Within critical education more broadly, we have worked to establish the centrality of mathematics (not always recognized) within multicultural and intercultural education (e.g., Greer & Mukhopadhyay, 2015).

Of particular relevance is the concept of "Funds of Knowledge" (Gonzalez, Moll, & Amanti, 2005) which is "based on a simple premise: People are competent, they have knowledge, and their life

experiences have given them that knowledge" (Gonzalez et al., 2005, p. ix). As an example, we studied Sandoval-Taylor's (2005) account of her development of a curriculum module for a second grade bilingual class composed mainly of Native American and Hispanic students, on the theme of local building construction. We later had the opportunity to talk about this work with her when she visited our class.

More recently, a related concept termed "Funds of Identity" has been developed (Esteban-Guitart & Moll, 2014), introduced by the authors thus (p. 31):

> We use the term funds of identity to refer to the historically accumulated, culturally developed, and socially distributed resources that are essential for a person's self-definition, self-expression, and self-understanding. Funds of knowledge – bodies of knowledge and skills that are essential for the well-being [sic] of an entire household – become funds of identity when people actively use them to define themselves.

This statement provides a concise rationale for the emphasis we put on identity work in CREME and in this paper.

From the above, it will be appreciated that CREME, both in terms of the development of a theoretical framework and as a form of political activism, is very much work-in-progress, guided by beliefs and values, such as a commitment to asset pedagogy and to valorizing diversity in all its forms (Mukhopadhyay & Greer, 2015). The very different natures of the two schools described below is illustrative of the diversity pointed out by Skovsmose (2012), relating not only to ethnicity, class, language, forms of life, and so on, but also to: variety of sites for learning mathematics (in contrast to the stereotype of the *prototype mathematics classroom* (p. 345); variety of forms of mathematics in action; variety of educational possibilities. The two schools represent two examples of the demographic and sociopolitical variety in schools within the US and, *mutatis mutandis*, anywhere else. Accordingly, the concept of culturally responsive mathematics education (Greer et al., 2009) must be adapted to the particular contexts of schools and communities. In the case of School A, one major focus of our work is support of cultural identity for the children who are recent immigrants, often as refugees. We have become acutely aware of the tensions for immigrants between the pressing issues of adapting to a new society and the practicalities of day-to-day existence, and the maintenance of cultural identity. (Other kinds of tensions for non-dominant groups in relation to dominant groups were discussed in Mukhopadhyay and Greer, 2015). In School B, two important aims are to enable the children who are White to become more aware of cultural diversity, and to model the use of mathematics as a tool for addressing issues of social justice. These contrasting foci are illustrated in activities being carried out at the two schools, as discussed below.

National political background

In Mukhopadhyay and Greer (2015), we sketched the political landscape for mathematics education globally and within the United States. No matter where you are, developments within the US are likely relevant because of the extent to which they directly or indirectly influence what is happening in your country. One point of this paper, accordingly, is to offer an example of how mathematics education and realpolitik interact within a particular context. Readers will recognize parallels with what is going on in their own contexts.

The importance of demographic changes within the United States is made clear by the fact that White children (as officially defined) now constitute less than 50% of the school population – while it is still the case that about 80% of the teachers are White. The legislation called *No Child Left Behind* that was passed in 2001 did address diversity among students in that it required the reporting of performance on standardized tests broken down by ethnicity, a major change welcomed by, for example, African-American organizations as a means to identify and then address inequity. However, the subsequent information on differences in standardized test scores (generally referred to as "achievement gaps", a term that connotes a blatantly deficit model) was instead used to label schools, teachers, and students as "failing" without providing resources to reduce the differences; instead, the data provided fodder for denigration of teachers and the furtherance of privatization (encrypted as "choice"). Now NCLB were has been replaced by the *Every Student Succeeds Act* (ESSA), a change that can be characterized as rebranding (Karp, 2016), with the continuing requirement to administer standardized tests to all students each year from Grade 3 to Grade 8. At the same time, there is a growing national movement against the excessive, inappropriate, and expensive use of standardized testing (Hagopian, 2014).

An overwhelming political fact is the increase in wealth and income inequality over recent decades in the United States and many other countries (OXFAM, 2017; Piketty, 2014). Education, far from the conception that it should act as a force to reduce such inequality through offering something like equality of opportunity, is now acting as an accelerator, as has been clearly shown in the United States by the statistical analysis carried out by Reordan (2011). In the period 1979-2009, the gap in academic achievement (as conventionally measured) between poor and rich children grew by about 40%.

For many reasons, the election of Donald Trump as president magnifies all of these issues, particularly given his choice as Secretary for Education of a billionaire supporter of the privatization of schools. All indications are that education will become, to an even greater extent, a driver of economic inequality and a generator of corporate wealth.

Local political background

One potentially positive aspect of ESSA is that it grants more power to states to make decisions. It remains to be seen whether this degree of autonomy will be sufficient to counteract federal policy under the new president.

In 2014, the Oregon Department of Education (ODE) issued a call for proposals for the *Culturally Responsive Pedagogy and Practices Grant* whose stated mission was to work towards formation of a culturally responsive teaching force in the state. CREME was the only project funded within this initiative focused on mathematics.

In June 2015, an extremely progressive and constructive statement on assessment, titled *A new path for Oregon* was published (Oregon Education Investment Board, Oregon Education Association, & Oregon Department of Education, 2015). This document represents the work of a very broad coalition that worked together over an extended period of time and consulted widely with teachers and others. It clearly shows the influence of Stiggins (2014), who acted as consultant.

From our perspective, *A new path for Oregon* is particularly notable in the attention it pays to "culturally responsive assessment". It is stated unequivocally (p. 8) that:

> A successful system of assessment should not simply highlight problems or generalize about groups; nor should it ignore conditions that influence performance. Instead, a successful system of assessment recognizes the myriad strengths of various learners within their respective communities and within the collaborative nature of the classroom. In addition, such a system is culturally responsive, and implemented by teachers who are assessment literate. (Emphasis added).

As just indicated, the report also highlights the concept of "assessment literacy", not just for teachers and students, but also for families, community members, educational officials and policy-makers, and – perhaps most importantly of all – politicians. The *Oregon Education Association* (teachers' union) is actively leading further efforts to develop quality assessment, and we anticipate working with them and other important actors in exposing and undoing the negative effects of excessive standardized testing, and in developing culturally responsive assessment. Mathematics is of particular importance in these efforts, given the prominence that it is accorded within standardized testing.

At the time of writing, a bill is being promoted in the Oregon Senate that would require educational state agencies to ensure that educators are providing culturally responsive education.

CREME schools, students, and teachers

The project is based in two urban schools in Portland, Oregon. Though less than three miles apart, the schools are contrasting in many respects. One (which will be referred to as School A) serves a very diverse population of children speaking more that twenty languages at home, many of whom are recent immigrants and came to the US as refugees, with indications that many have experienced trauma (Sottile, 2015). The other school (School B) is a public charter school within the Portland Public Schools system. Unlike many charter schools that are corporate, this school adheres to the original concept of charter schools as test beds of alternative approaches to education. It is based on a progressive set of principles such as constructivism and democratic education and serves mainly White children, not particularly affluent.

School A is required to follow the prescribed textbooks and other instructional materials adopted by the district, and adhere to strict testing requirements. As a result, as CREME teachers have testified, there is excessive emphasis on test preparation. Although the children are smart and creative, as evidenced in many of their open-ended projects, their performances in tests do not reflect that. And although the teachers from School A are kind, caring, and full of creative ideas they are compelled to follow the curriculum and testing regime thrust upon them.

For School B, not only is the demographic different, the curriculum and pedagogy are also in stark contrast. The teachers have intellectual freedom in designing their curricula, with minimal emphasis on performance on standardized tests. The teachers design and develop project-based curriculum spanning weeks at a time. Thus, while the students were learning how to respond to standardized test items in School A, their peers in School B were learning about election and democracy in a national program called Every Kid Votes 2016 (https://www.studiesweekly.com).

The participating teachers contribute to the richness of diversity of the project. They differ ethnically and linguistically, and in years of teaching experience. CREME is a teacher development project that differs in major respects from many endeavors labeled as such. It is not formulated as a group of researchers/academics proscribing, as experts, curricular content or teaching styles for teachers to follow. Rather, over two years and continuing, it is a collaboration founded on mutual respect wherein the mathematics educators propose ideas to the teachers, while the teachers and their students educate the academics about the realities of the circumstances in which they are teaching and learning. In the slow and organic development towards the idea of teachers as intellectuals, the teachers have had the opportunity to interact face-to-face and electronically with the advisors for the project, Marta Civil, Geneva Gay, and Danny Martin, and other scholar activists. And they are beginning to attend and present at conferences and to write papers.

What happens in CREME

We present here some examples of activities that we have collectively carried out (see Ford et al., (in press) for more details).

In School A, a very simple exercise with powerful effect has been to ask children to record, on paper and audio, numbers from 1 to 20 in their home language. Ongoing discussion was related to the large world map that hung in the room: In which part of the world is this language spoken? Where else is this language spoken? The list of languages spoken at home was long: Arabic, Bulgarian, English, Fulani, Kirundi, Korean, Oromo, Portuguese, Russian, Spanish, Sudanese, Swahili. (For a similar activity in Greece, see Chronaki, Mountzouri, Zaharaki, & Planas, 2016).

As a more general approach to identity work within an asset pedagogy, the children at both schools record, on strips of paper, lists of what they can do, including but not confined to school achievements. These lists are place in ‘talismans’, old medicine bottles that are decorated by the children. Recently, all the children and their teachers wrote autobiographical poems based on the prompt “I am from”, and drew self-portraits. A compilation has now been published as “We are from”, and the book was launched in the library at School A with children reading their poems and the self-portraits on display.

Another aspect of identity that we work on is that of being a potential college student. In School A, in particular, students tend not to have a clear notion of what college is like. Accordingly, we have organized field trips for students to Portland State University, during which children interacted with students and faculty in Architecture, Engineering, and Earth Sciences.

In School B, an example of teaching to make students aware of other cultures is that the teacher, having covered a standard account of the Lewis and Clark Expedition of 1804-1806 that opened up the West to colonizers, then presented an account from the perspective of the Native Americans. We have also connected teachers, through visits to the Portland Art Museum, to Native American artifacts that illustrate the richness of design that serves as a context for the study of geometrical concepts, in particular symmetry.

School B has a tradition of working through Storyline projects, an approach in which students, together with their teacher, explore a theme in depth through creating characters within that theme. Koopman (in press) describes such a project that began by students looking at the labels on their t-

shirts to see where they were made. The places were recorded with pins on a world map. Koopman reports that looking at the concentrations of pins in particular areas came as a revelation even to him. From there, the class collectively did extensive research about factories in the early 1900s where t-shirts were originally made. They used extensive arithmetic in calculating, based on historical data such as the prices of groceries and payscales at that time, the cost of living of a family in relation to the pay. They took on roles of workers and factory owner, and acted out situations of conflict. Koopman (in press) related the conflicts enacted by the students to his own experiences during a teacher strike when he was coerced by the school district to substitute for striking teachers. Later the principal of the school joined in the activity. The project culminated with research into contemporary conditions for workers in sweatshops in countries such as Bangladesh.

Another such project involved students building models of food carts (for which Portland is famous). Students created characters for the cart owners and operators, and devised detailed business models. They filled in actual forms relating to hygiene requirements, for example, for these simulated businesses. The teacher posted reviews on Yelp.

In these and similar projects, a great deal of arithmetic was done in context, illustrating how, in elementary school, students may:

(a) Be introduced to the conception of mathematics as a tool for interrogating sociopolitical issues, thus appreciating at an early age a sense of agency that will stand them in good stead as citizens.

(b) Consolidate computational and planning skills in complex contexts that afford relevance, interest, and motivation.

(c) Learn and practice mathematics integrated with other school subjects (science, obviously, but also art, social studies, language arts) – something that elementary teachers are in the best position to do.

Future of CREME

At the time of writing, we have submitted proposals for further funding to continue CREME, but it is a measure of the group solidarity that the teachers are keen to keep the project going even without funding. As we continue, we plan to pay more attention to explicit mathematical content, always within the framework of culturally responsive pedagogy (see Ford et al. (in press) for more details). We also plan to explore and expose cultural biases in assessment, including the more subtle ways in which they are manifested. In light of the extremely encouraging political developments within the state, we are developing links with important actors within educational politics in Oregon. And we will continue to foster identity, agency, and autonomy in the CREME teachers, and to help them develop into mentors of new teachers joining the project.

Final comments: Ethical and political responsibilities of researchers

The naive conception of research and researchers as politically neutral, providing systematic objective evidence for the guidance of state educational policy, if ever viable, is certainly not sustainable in the circumstances that pertain in the United States, as a neoliberal agenda is pursued – as, in the face of global corporatization and economization, is happening in many parts of the world (Spring, 2015). The experiences described in this paper makes clear that as educators we are

confronting issues of ethnic and cultural diversity, inequity, massive social engineering through the mechanisms of mass testing, the characterization of education as a market with huge profits to be made, the use of education as a means to maintain and even exacerbate economic inequality. The role of mathematics in all of these issues is particularly important, and we hold to a vision of mathematics education as providing people with tools for understanding and acting upon issues important in their lives and those of their families and communities. Along with mathematicians and mathematics teachers (D'Ambrosio, 2009), mathematics education researchers cannot absolve themselves from ethical and political responsibilities.

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Mathematics at the enterprise: Industry, university and school working together to facilitate learning

Hans Kristian Nilsen[1] and Anne Vegusdal[2]

[1]University of Agder, Department of Mathematical Sciences, Kristiansand, Norway
hans.k.nilsen@uia.no

[2]University of Agder, Department of Natural Sciences, Kristiansand, Norway
anne.vegusdal@uia.no

Based on a pilot study, this paper reflects upon how the industry, university and school can work together to facilitate learning of mathematics. Through a project called MathEUS (Mathematics at the Enterprise, University and School) a modern energy recovery enterprise, Returkraft, the University of Agder and a school collaborated to offer pupils (in 8th and 9th grade) an opportunity to engage with mathematics contextualized at an enterprise. Through comparing the 2015 edition with our re-design in 2016, we discuss the outcome of the project. In particular, we focus on the factors necessary for such a project to succeed. Findings suggest that mathematics contextualized at an enterprise as an isolated event has limited value. Further, findings seem to indicate that working together with the enterprise over time, by including a pre-project, strengthens pupils' experience of mathematics as a relevant subject.

Keywords: Diversity, relevance, authenticity, out-of-school contexts, enterprise.

Introduction

Diversity in mathematics education relates to political, cultural and linguistic aspects. In this paper, the issue of diversity will be addressed in terms of different sites where mathematics education potentially could take place. Meaney and Lange (2013) emphasize the importance of transitions between different contexts and that such transitions "can be a fairly minor issue for learners if they perceive similarities in what knowledge is valued and how learners and others should interact together and with the mathematical content" (p. 169). Throughout the decades, a growing body of research has been carried out, focusing on the learning of mathematics in different contexts (Masinglia, Davidenko, & Prus-Wisniowska, 1996). By linking mathematical tasks to an enterprise, we hope that the pupils experience mathematics as important, also outside the classroom, and in turn this influences their motivation for learning mathematics at school. In Norway, there has been an increased focus on how mathematics could be taught in ways that is perceived by pupils as relevant and more related to their own reality and experiences (Det Kongelige Kunnskapsdepartement, 2011). Diversity in teaching methods and arenas of learning has been important issues in this debate (Norges offentlige utredninger 2015, p. 8). Pupils in Norway are mixed together heterogeneously until grade 10 and with the curriculum focus being academic, offering pupils few perspectives on, for example, vocational and professional practice. In the attempt to diversify such experiences, we found that the processes going on at the energy recovery enterprise, Returkraft, bear the potential of visualizing mathematics at a working place. Further, *Mathematics at the Enterprise, University and School* (MathEUS), in some sense should be regarded as a response to the identified need of moving the teaching of mathematics in a more practical and vocationally-oriented direction. The last two years (2015 and 2016) funding from Regional Research Fund Agder enabled research to be carried out on the

MathEUS project. In this paper, we pose the following research questions: 1) *Which factors are of importance in the collaboration between school and industry for the pupils to experience the mathematics involved as relevant and authentic?* 2) *What might prevent the pupils for experiencing the project as relevant and authentic?* Our discussions focus on the differences in outcome between the first year and the redesigned project run in the second year.

Theoretical background

The idea that learning is *situated* within a certain context and practice was in many ways coined by Lave and Wenger (1991) and in mathematics education, research has shown that the environment in which activities takes place is of great importance for the outcome (Nunes, Schliemann, & Carraher, 1993). In the MathEUS project, mathematics is contextualized and linked to the activities at an energy recovery enterprise, Returkraft. *Situated* in this case alludes both to the physical environment at Returkraft (which surrounded the pupils while they were doing their activities) and to the mathematical content, which was consequently linked to the enterprise. By establishing this link between content and context, it is our aim that the mathematical tasks and activities provided appear more *relevant* and *authentic* to the pupils. When applying the concept of relevance, we draw on the work of Hernandez-Martinez and Vos (submitted) where the main point is that "it involves a judgment of value that is made by a person involved in the activity" (p. 26) and that the "judgment is connected to the motive or object that drives the activity" (p. 26). Nyabanyaba (1999) and Dalby (2014) point to the complexity of the relevance concept in mathematics by listing a number of different ways mathematics could be conceived of "relevant", depending on contextual, content-related and affective factors. Authentic has to do with the nature of the activity and has several somewhat distinct definitions. In the case of MathEUS, we find the definition from Gulikers, Bastiaens and Martens (2005) to be useful, "An authentic learning environment provides a context that reflects the way knowledge and skills will be used in real life" (p. 509). Here both the contextual aspect and the use value are emphasized, which is fully in line with the purpose of the MathEUS project. To achieve the goal of authentic tasks, considered relevant to the pupils, we aimed also to overcome some of the typical challenges that guided tours of enterprises often entail. Recent research, both national and international, shows that the "classical" excursion day, where pupils use a day to visit an enterprise in terms of a guided tour, is often disconnected to what they normally do at school and leads to some didactical challenges. The pupils easily become passive, engaging in artificial and low-quality tasks (like memorizing and duplicating information posters) due to the lack of preparatory and follow-up work (DeWitt & Storksdieck, 2008; Remmen & Frøyland, 2014).

Methods

In 2015, two schools were participating, each with one secondary class of 9th grade pupils (14-15 years old).

The implementation of the MathEUS project

The schools' visiting day at Returkraft took place in March on two different days, one day for each class. Before the day of visit, student teachers completing a master degree at the university had designed and prepared mathematical tasks for the pupils. After a guided tour at Returkraft, the student teachers constructed the tasks over a period of two weeks, and during this period staff from Returkraft was available for helping them with information concerning the enterprise. Teachers from the

participating schools were also available for commenting on their suggestions. The premises for these tasks were that they should in some way be linked to the enterprise at Returkraft. These tasks were given to the pupils on their day of visit, and they were given approximately two hours to work with the tasks in groups of four to six pupils. The tasks were solved in a 'classroom' at Returkraft. Following an evaluation of MathEUS as it had been carried out in 2015, we decided to make some changes in an attempt to improve the project (see also the section "results and discussion"). We wanted to strengthen the quality of the tasks so that the pupils experience them as more relevant. In addition we wanted the pupils' experience with Returkraft and the mathematics they engaged with to be more than just an isolated event. Aiming towards this, we invited the class to take part in a pre-project (lasting for six weeks), with a representative for Returkraft coming to their class early in the semester to initiate the following assignment: "How can we improve peoples' habits, when it comes to sorting their garbage for recycling?" To solve this assignment, the pupils developed questionnaires and went out to different geographical locations to interview people about their sorting habits. Due to the lack of responses during oral interviews the pupils posted the questionnaires on social media, and the number of response escalated rapidly. Their empirical data were treated in spread sheets, resulting in different diagrams, and the results were presented at the university in front of an audience consisting of both the student teachers and employees from Returkraft. In turn, this pre-project served as an important source for the student teachers when they designed the tasks for the pupils. In 2016, one secondary class of 8th grade pupils (13-14 years old) participate in the project.

Methods and data collection

We (the authors of this paper) were responsible for carrying out research and at the same time we planned the implementation of the project together with the participating schools and employees from Returkraft. We also provided the framework conditions for the student teachers' elaborations of the tasks. The pupils, the student teachers, the teachers (at the participating schools) and the employees at Returkraft all served as informants for our research. In this paper, we mainly focus on data from the pupils. Conducting research at the same time as we are responsible for carrying out the project situates us within the domain of *action research*. Bryman (2012) describes this as "an approach in which the action researcher and client collaborate in the diagnosis of a problem and in the development of a solution based on the diagnosis" (p. 709). We conceive of this definition to be in line with our agenda, where we use data from our experiences to discuss, develop and improve the forthcoming editions of the project.

To measure parts of the outcome of the MathEUS project, we developed a web-based questionnaire which the pupils should answer after the project was carried out. Except from minor changes and some reformulations, we used the same questionnaires both years of research, for the sake of comparison. The questions were mainly statements that the pupils were asked to scale from 1 to 6, based on their "degree of agreement" and in total there was 34 statements to be evaluated. Mainly these statements aimed to inform our research questions mentioned earlier in terms of measuring the pupils' own experience of their learning outcome, their motivation and beliefs and whether they consider the project meaningful and relevant. In addition, some open ended questions were posed where the pupils had the possibility to comment on these aspects. Since two school classes were involved in 2015, and just one in 2016, the sample size varied and in 2015 there was 44 respondents to the questionnaires, while 23 responded in 2016. But in 2016 we collected some additional data by

including a semi-structured interview with three of the pupils. These pupils were randomly selected among those who volunteered when we visited the class. In this interview, we went deeper into the pupils' own experiences related to the project, and to the content of the tasks and activities provided for them by the student teachers. Both in 2015 and in 2016, interviews involving representatives from the three collaborators (the enterprise, the university and the schools) were conducted, mainly focusing on the outcome of the project. Both years we made observations and voice recorded all the group work sessions at Returkraft. As part of our research, we also conducted interviews with some of the student teachers responsible for designing the tasks.

Results and discussion

Since the scale on our questionnaires ranges from 1 to 6, we sort the results in two halves for the sake of overview. The "low achieving" half ranges from 1 to 3 and the "high achieving" half ranges from 4 to 6. Figure one gives an overview of some of the questions we consider as most significant.

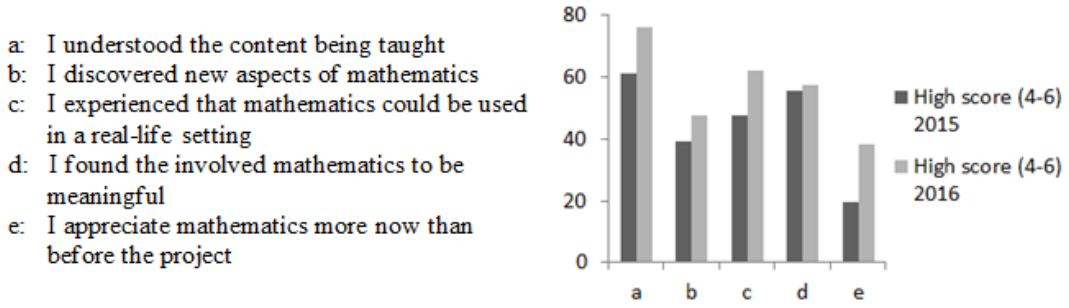


Figure 1: Pupils scaling (in per cent) in 2015 compared to 2016

One can observe from the first columns in the diagrams (1a) that the majority of the pupils both years felt that they understood the mathematics being taught. About half the pupils felt that they discovered new aspects of mathematics in the 2016 edition of MathEUS, a slight improvement compared to 2015 (1b). An improvement related to the experience of mathematics used in a real-life setting is visible (1c), while the experience of mathematics as being meaningful is almost unchanged (1d). Finally, there is a clear improvement on almost 20 per cent related to the appreciation of mathematics as a consequence of the MathEUS project (1e). Since the number of pupils forming the basis of these results is limited and varying one should be careful to draw strong conclusions only from this data set. But since statements in the questionnaires consequently were higher rated in 2016 compared to the year before, a positive trend could be suggested. In addition to the scaling of different statements, there was also a possibility for the pupils to add comments and give some additional justifications for their scaling. In 2015, about half the pupils wrote critical comments, especially related to the question of whether or whether not their expectations had been fulfilled. "I thought they would try to make this fun", "it was extremely boring, tiring and difficult maths" and "I thought we would learn more. The tasks were just like a typical maths tests", serve as examples of such comments. Despite that 43 out of 44 pupils in 2015 expressed themselves positively about learning mathematics at a working place, about half of the pupils responded in negative terms related to the question about the fulfilment of expectations, in line with the previous quotations. These comments mainly implied that a negative experience with the content of the tasks. Even though the tasks were designed with the purpose of being relevant, in terms of creating a link between school mathematics and the enterprise, the pupils did not seem to perceive this the same way. Dalby (2014) emphasizes that even though a realistic

context is provided, the impact on pupils can vary since "the context itself is often no more than a metaphor to illustrate an aspect of pure mathematics rather than authentic use of a scenario as a source of mathematics" (p. 90). In the aftermath of MathEUS in 2015, we saw that several of the student-designed tasks could represent examples of contextualized tasks, but with little authenticity.

Task 3 – payments

When the trucks arrive, the garbage is deposited in a big container. We know that each year about 130 000 tons of garbage arrives at Returkraft. Returkraft is paid for receiving our garbage. Juliette works at the counting department. Now you are going to help Juliette with the bill for Lillesand County. The prize is 1750 kroner for 1 ton of garbage.

a) How much do they have to pay for 2 t (tons) of garbage?
b) How much do they have to pay for 4 t (tons) of garbage?
c) What relations do you observe, between the numbers indicating the tons of garbage and the bill that the county receives?

Excerpt 1: Excerpt from task, 2015

The task above serves as an example of a task where the context was reduced to serve only as a metaphor, rather than an authentic use of Returkraft as a source of mathematics.

When revising MathEUS 2015, we wanted to improve especially on two aspects: 1) Make the content of the tasks more relevant and authentic for the pupils and 2) Engage the involved pupils and student teachers earlier in the process, so that the visit to Returkraft became something more than just an isolated event. In this process, we build on DeWitt and Storksdieck's (2008) "boarder visions on field-trips" (p. 181) and their research-based conclusions that field trips and out-of-school context ought to be embedded in teaching, more holistically. Even though their focus was on natural sciences, we found these ideas to be relevant for mathematics and in the 2016 edition of MathEUS we included a pre-project (as described in the "methods" section). Due to the pre-project, the student teachers designed tasks were also used earlier in the process.

From the web-based questionnaires, illustrated in figure 1, there are some indications that pupils' experiences in general were more positive in 2016 compared to 2015. Interviews with some of the pupils, also involved statements that could be interpreted as having to do with relevance and authenticity.

Interviewer: Earlier in the interview, you said something about "mathematics in practice"…was this something which characterized the tasks at Returkraft, or did you feel that you just as well could have done this at school?

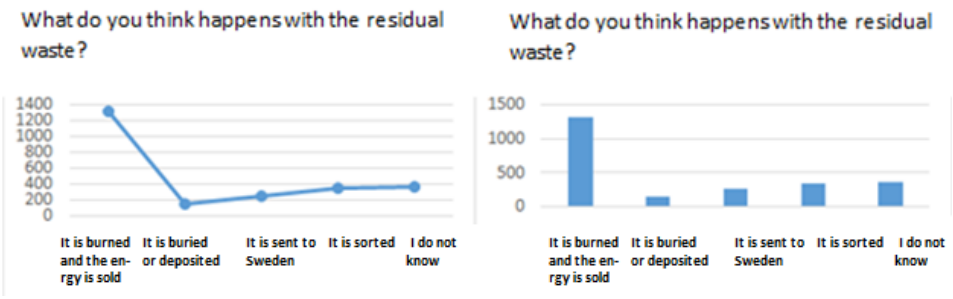
[…]

Pupil: Since we had been to Returkraft and the content was about Returkraft, and the numbers used were the same as we had learned about…the numbers weren't just made up, in a way.

This pupil emphasizes the experience of working with "real" tasks, in terms of pointing to a close connection between the context and the content of the tasks. Since the student teachers had followed their pre-project and had access to all their results and data, they were also able to build on pupils' own data from the pre-project when designing the tasks, and in this case provided a more authentic context.

Task 6 - diagrams

This example is from your survey. Discuss what type of diagram you want to use and why:



Excerpt 2: Excerpt from task, 2016

By using numbers from the pupils own pre-project, this task illustrates how one student teacher applied these data to discuss statistical issues. Based on observations from the pupils engaging with this task, we see this as an example of successfully creating a link between the context and the mathematics involved. Pupils judged this as "relevant" since an authentic connection between the mathematics involved and the context is provided in terms of linking the task to something familiar, namely their pre-project.

From this relatively brief analysis, we suggest that pupils' experience of relevance, and the authenticity of tasks provided in an out-of-school context not only depends on the tasks themselves, but also on the pupils' possibilities to *relate* to the content. For the context to "reflect the way knowledge and skills will be used in real life" (Gulikers et al., 2015, p. 509) a preparatory phase, were pupils are offered the possibility to become familiar with the context seemed to be advantageous.

Conclusions

Providing out-of-school contexts like excursions and field trips are, unfortunately, often being carried out as isolated, stand-alone events (DeWitt & Storksdieck, 2008; Remmen & Frøyland, 2014). There are few reasons to believe that this is different in mathematics. Our findings tend to show that if mathematics in out-of-school contexts is treated like stand-alone activities, the outcome when it comes to authenticity and pupils' experiences of relevance is limited. Pupils' experience of relevance changed in a positive way when a preparatory phase in terms of a pre-project was implemented. Nyabanyaba (1999) argues that pupils and teachers' conceptions of relevance often differ. In line with this, we suggest that even though mathematics in out-of-school contexts through excursions days at enterprises as stand-alone activities is carried out with good intentions from teachers, such activities ought to be more substantiated in teaching. Authenticity entails reflection on knowledge and how it could be used in real life (Gulikers et al. 2015) and for such a reflection to happen, Meaney and Lange (2013) point to the necessity of being able to make transitions between these different contexts. From our experience a pre-project substantiates the out-of-school context and better prepare the pupils for the subsequent visit to the enterprise. Hence, the connection between the mathematical content and the context provided are perceived by pupils as more relevant. "Making connections between mathematics and life that appear authentic and convincing for students" (Dalby, 2014, p. 91) serves as core criteria in the question of relevance and learning, also when it comes to the subject of mathematics.

Implications and possibilities

From a societal and political perspective, this paper addresses issues concerning the contextual nature of mathematics teaching and learning. One aspect is the pupils' outcome of experiencing mathematics in an out-of-school setting, but in a boarder perspective, MathEUS also contributes to strengthen the bond between school and industry, which according to Sjaastad (2013), is "too weak" (p. 16).

The results and reflections in the wake of the two previous editions of MathEUS serve as a basis for further developing our collaboration strategies. We found that by involving the enterprise at an early stage, pupils to a larger extent were able to make connections between the context and the mathematical content. The pre-project also entailed interdisciplinary activities, where mathematics became an important contributor and ICT was applied by the pupils as a crucial tool in the process of visualizing their findings. The interplay between mathematics and natural sciences in particular played a significant role in the pupils' activities, both during the pre-project and at their visiting day at Returkraft. The topic concerning recycling of garbage also links our project to local and global environmental issues. People's attitudes towards recycling of garbage, were a valuable outcome also for the enterprise, as Returkraft aims to reach out to the public with messages concerning the importance of such issues. These are all, from our point of view, valuable synergistic effects worth looking deeper into. Task-design is a topic of research on its own, and the process of developing tasks and activities within this particular context could also be worth mentioning as potential, forthcoming research.

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"No, it just didn't work": A teacher's reflections on all-attainment teaching

Hilary Povey[1] and Colin Jackson[2]

Sheffield Hallam University, England

[1] h.povey@shu.ac.uk; [2]colin.jackson@shu.ac.uk;

Setting – the practice by which learners are allocated to different classes on the basis of perceived ability – is a social justice issue. Despite overwhelming evidence that, overall, setting is educationally harmful and in discriminatory ways, the practice is almost universal in English secondary mathematics classrooms. To gain insight into this apparent contradiction, we offer the story of a single teacher's ultimate rejection of all-attainment teaching.

Keywords: Ability grouping, equity, ability thinking.

Introduction

In this paper we begin by arguing that setting by 'ability' is a social justice issue. Despite overwhelming evidence that, overall, setting is educationally harmful and in discriminatory ways, the practice is almost universal in English secondary mathematics classrooms. In order to understand this apparent contradiction, we offer the story of a single teacher who, early in his teaching career, embraced all-attainment teaching[1]; continued to think in fixed ability ways and therefore supposed that there should be differential teaching for different levels of 'ability'; found himself overwhelmed by such a task; and finally abandoned all-attainment teaching because "it just didn't work". We conclude with a brief discussion.

Setting and 'ability' thinking[2]

English education in terms of both policy and practice currently takes for granted hereditarian assumptions; and a discourse of ability is used very widely to place children in sets for mathematics in secondary schools (Wilkinson & Penney, 2014). The belief in fixed amounts of 'ability' and the consequent grouping of children according to how much they are perceived to 'have' is taken as natural and common sense (Francis et al., 2016). The idea that ability is a given and that only some students can be high achievers discourages many students (Boaler, 2005) and communicates and reinforces damaging fixed mindset beliefs (Boaler, 2013).

In almost all instances the methods used to allocate children to sets are claimed to be objective and based solely on their prior performance. However, in practice, in English secondary schools prior attainment is found to be a relatively poor predictor of set. A wide range of social factors come into play which privileges those with greater cultural power and systematically disadvantage others

[1] We use the vocabulary of "all attainment" rather than the more common "mixed ability" to avoid endorsing so-called "ability thinking" (see, for example, Boylan & Povey, 2014).

[2] In this section, we draw substantially on Jackson (2017).

(Muijs & Dunne, 2010; Bartholomew, 2003; Hallam & Parsons, 2013; Ireson, Clark, & Hallam, 2002; Macqueen, 2013; Wilkinson & Penney, 2014).

Teachers' expectations of children in lower sets tend to be low and these pupils are usually offered a restricted, narrow and instrumental curriculum which further inhibits performance. They are constructed as poorly motivated, badly behaved and incapable of independent working and independent thought and therefore in need of repetitive tasks which require lots of practice (De Geest & Watson, 2004). In contrast, those in the top set are constructed as well motivated, hardworking, well behaved and capable of independent working and independent thought and are given a more demanding curriculum and much richer opportunities to succeed (Bartholomew, 2003). Thus setting and ability thinking construct that to which they claim to be responding.

While 'ability' grouping has been shown to have little consistent effect on attainment (Francis et al., 2016), it is known that it has detrimental effects in terms of personal and social outcomes (Nunes, Bryant, Sylva, & Barros, 2009). The effect of setting continues into adulthood resulting in more limited horizons and stunting life opportunities (Boaler, 2005). Thus, as Slavin (1990) argues, 'ability' grouping can be seen as an affront to basic ideas of democracy. Involved here are issues of power and culture: 'ability' grouping is not just a neutral organisational practice. Oakes, Wells, Jones, and Datnow (1997) maintain that common sense conceptions of ability and intelligence are at the heart of schooling and, in regimes where neoliberalism holds sway, the ability discourse is part of an ideological battle defining children from lower social and economic status groups as expendable (Oakes, 2005). Further, the performativity regimes (Ball, 2003; Povey, Adams, & Everley, 2016) imposed on schooling have created a climate whereby failing to conform to the common sense view of the world carries huge risks to schools and to individual teachers; and grouping children by 'ability' as measured through some form of assessment, endorsed by policy makers, is seen as risk free.

A technicist approach to reform will therefore not work as it assumes resistance to changing 'ability' grouping is simply a rational choice by relatively free agents. We offer here a story of a single teacher, Jim, and his changing relationship to setting. (Pseudonyms are used throughout and some details have been changed to protect participant anonymity.) Before doing so, we consider very briefly the role of storying in the construction of knowledge.

Telling stories

We are telling this story about Jim, much of it in his own words, because we believe that stories help us understand more about the world. There is an "unavoidable moral urgency" (Clough, 2002) in stories which fits our purpose in this paper. Jerome Bruner (1986) wrote about two different kinds of knowledge: *paradigmatic knowledge* and *narrative knowledge*. Whilst the former is expressed through logical propositions, the latter is expressed through stories. He argues that it is characteristically human to think in stories and that they provide us with a way to make sense of experience. Stories imply, and attempt to lay bare, intentional states, that is, to offer insights into why we do what we do.

In constructing this story, it is, of course, our categories, concepts, constructs and so on which frame and shape the work. However, we have tried to stay as faithful as we can to Jim's own constructions, accounts and perspectives as far as we have been able to elicit and hear them. We

have also tried to offer sufficient detail to allow others to test out the trustworthiness or otherwise of the account and to judge, for example, whether the intentions suggested make sense.

Jim's story - or our story about Jim

Jim is a highly committed, very hard working teacher who has the interests of his students very much in the forefront of his thinking. On a personal level, he is open and his stance towards visitors to his school and department is always one of welcome. He has kept in touch with the university where he completed his initial teacher education and continues to work frequently and supportively with its current students. He agreed to be interviewed (with a close colleague). The interviews were recorded and transcribed. Working with the transcripts in variety of ways, we began to be compelled by Jim's story as honest, contradictory and telling about teachers' relationships to the issues of setting; we tell a version of this story below.

Jim's final teaching practice at McVee High had not been a happy one. He had clear ideas about how mathematics should be taught and wanted to create his own lessons and his own resources. He wanted the scope to try out different and novel approaches and to avoid the routine use of an indifferent textbook.

> I don't know what I was expecting. I didn't really enjoy working at that school at all and I was really glad to leave. The head of department didn't like me. He didn't like my teaching … He'd get a face on if I wanted to move the tables around, even just move them anywhere. He just wanted them where they were and if I didn't want to use a textbook he would have a face on about that as well. Like "Why are you not using that page?" – "Because I've made this instead". He didn't like that. It was *Lock Maths* and all you did was you started on page one and the scheme of work was just … go through the book. And if you didn't go through the book, then you were an idiot apparently. But that was how it was and it was just a waste … I didn't practice being a teacher at all. You'd practise administering "Do page 12."

Part of the way through Jim's initial teacher education course, his tutor, Barry, left in order to take up the post of head of mathematics at Broadbent School. Broadbent serves a large, white working class, social housing estate in an ex-industrial town with overall attainment below the national average. The mathematics department had had a chequered past and when Barry was appointed there were vacancies in the department. Barry and Jim kept in touch and Barry approached Jim to ask him to come and have a look round the school with a view to starting his teaching career there. After the visit Jim was offered a post at Broadbent School as a newly qualified teacher and accepted the offer.

> I didn't want to work in a posh school. I didn't want to do that … Like Our Lady's where the kids are all little robots. I didn't want to work there. I wanted to work in a bit more challenging area and I already knew Barry as well … I'd always said that I would start my career in a more challenging school and probably end in an easier school because I just wouldn't have the energy…

Broadbent offered Jim six week's work in the second half of the summer term preceding his permanent appointment in September so he could get to know the school and the pupils a little. It is clear that Jim was already confident about his mathematics teaching and keen to begin practising.

> It was intended I think that we were supposed to come and like just have a look about and observe and stuff, but I couldn't do that in the end because I was spending most of my time with a woman called Marion, who'd got a full-time maths timetable but she had no real maths qualification at all. She was an art teacher and I was just watching her teach all these lessons and just thought "I can't really let her do it because she's doing it wrong." So I just ended up teaching for six weeks … I just said "I'll do them for you and you can go and do something else." … She couldn't teach them. She was just teaching them drawing. They were drawing things and she would let them sit there and do nothing while she would like paint portraits of them and I was like no, we can't be having that.

Jim had wanted to be a secondary mathematics teacher for longer than he could remember and he looked inward to his own thoughts and backwards to his own experiences as a school pupil to frame and understand his practice. For him, Broadbent offered the freedom to develop in his own way as a practitioner, a freedom he highly valued, and one which was "quite liberating actually".

> I didn't enjoy going to university at all. I didn't even want to do anything there. I just hated the whole experience. And I didn't like going to college, didn't like doing my [school exams]. I just wanted to be a maths teacher and I just wanted to get there, so it was quite nice to get there and have your own classroom and then actually start teaching. I'd wanted to be a maths teacher since I was [a child]. So everything just seemed like in the way of trying to get there …

Thus, Jim did not respond to and make use of the mathematics education approaches and understandings offered to him by his university tutors during his initial teacher education. At a slightly later date, when offered a professional development opportunity linked to a local university, he asserted with confidence that he had "never read a book". This seemed important to him in constructing his way of describing himself in the world.

He had a complex and contradictory relationship to his school experiences of mathematics.

> All my maths teachers had been rubbish. Every last one … I wasn't really taught maths because I always followed the … [resource based] scheme of work … never did a teacher really stand at the front and say "This is how you do this."

Despite this, Jim had kept all his mathematics books from school "because I knew I was going to be a teacher" and he remembered working together as a whole class on investigations, material which he was continuing to use at Broadbent. Not only that, at school he had "just really enjoyed maths and always have". In the context of this paper, two things stand out about Jim's account of his school experiences. First, he had been taught in all-attainment groups using an individualised scheme and, despite his assertion that all his teachers were "rubbish", he said that "everybody did well because you had appropriate tasks". This "completely differentiated" approach seemed fundamentally to inform his thinking about all-attainment teaching. Second, he spoke about himself as having a fixed level of mathematical ability and he linked his understanding of his own competence as a mathematician entirely to external markers.

> I've never been like really good at it, but I just really enjoy doing it. I mean I only got a level 5 in my primary school SATs and I got a level 7 in my secondary school SATs and I got a B at GCSE. I got an E at A Level … [3]

This was echoed in the way Jim talked about the Broadbent pupils. Throughout the interview, the pupils were referred to by Jim in a variety of ways all of which seemed predicated on fixed ability thinking: "lower foundation type students"; "the very brightest students"; "ten kids that should definitely do high maths"; "their [SES] data … regardless of social context that is the grade they should get based on [results from primary school] … regardless of whether their mum's on drugs or they're on free school meals".

Coinciding with Jim's arrival at Broadbent, Barry introduced all-attainment teaching for the first year classes.

> We all knew what Barry was about … it's not like he kind of hides it under a bushel. He would say in meetings what was his kind of pedagogy and what he wanted to achieve.

But this claim seems to have related to using a more open and problem-solving approach rather than providing any sort of challenge to fixed ability thinking. Barry prepared packs of materials which were full of ideas that offered a more investigatory approach than the one with which the teachers were familiar organised around broad topics. When asked for an example, Jim said

> … the first half term … you would do a unit on triangles and you'd do a unit on cubes … and you could do them in whichever order you wanted to. [But] you didn't have to use any of it. You could use none of it, some of it, all of it, your own stuff … Some of the resources I didn't like so I didn't use them … [I used] a combination. We had textbooks, so sometimes I'd use those, sometimes I'd make my own and sometimes they'd do it off the board and sometimes … just find something on the internet and re-purpose something if you like.

Towards the end of the year, Barry asked his department if they would like to continue working in this way with the pupils during the following school year, thus extending his all-attainment project into the first two years of the school.

> Did we want to continue the kind of thematic approach? Did we want to continue the mixed ability approach? And we all said yes. We enjoyed it. We enjoyed doing it, so we said yeah.

However, for Jim, teaching all-attainment groups was synonymous with providing differentiated materials. On occasions he was able to make this work effectively for him and his class:

> If you really wanted to differentiate, particularly when we used to teach mixed ability and we were doing fractions … I just had the [levels of difficulty] on the board and they would just pick whichever one they wanted … most people just try and go for the one that's quite challenging. Some of them knew that there was no point in trying the level 8 one because they were a level 4

[3] These are all public examinations in the English school system. The curriculum and the associated SATs were structured into levels. Jim's results are mostly above average but not excellent. The final school leaving mathematics grade is lower than average for those who take the examination.

> kid or something, but they didn't go for the easy option. They went for an appropriate level one and I think they quite enjoyed it. They liked it … and I think they liked having the choice as well.

But overall the task of trying to provide differentiated materials across the attainment range, rather than adopting a fundamental pedagogy for attainment for all, proved overwhelming and undoable.

> My experience of [the second year groups] was at that point the difference between the highest and the lowest had increased dramatically and it was becoming a strain … They'd all made progress, but the higher ones had made more progress and so I was having to differentiate more and then do the same for my new first years … it was becoming very fraught and time consuming and I wasn't doing it as good as I could have… No, I wasn't teaching as well as I should have been teaching because I was spending too much time doing too much differentiation … I just couldn't do it effectively … there was just so much planning and I was kind of making do I think.

Jim did not give up easily and shortly afterwards when Barry had moved on and Jim was given responsibility for the department, he even extended the all-attainment teaching to a third year. However, and unsurprisingly, this did not last.

> The kids bottomed out, teachers were over stressed, over worked. I don't understand why I did it in the first place … I mean I can look back now and think "You stupid idiot!" I obviously already knew that it was really difficult to differentiate across two different year groups and it was a lot of planning, so I don't understand why I did it.

It is interesting to follow how Jim justified and explained the policy reversal when looking back several years later. The initial cohort of students who had had two years of all-attainment teaching – and experienced all the initial commitment and enthusiasm – had done remarkably well in both the high stakes, external tests they took, one at the end of their third year and one at the end of their fifth. The following year group was a much more challenging cohort and were problematic throughout the school. But the difficulties Jim and the department experienced were not seen in this light. Rather, they became the basis for a rejection of an all-attainment approach. And we see again the role that all-attainment teaching as individual differentiation played in making life impossible.

> It just didn't work. The kids weren't getting the grades or the marks or the levels, whatever, and behaviour was awful. No, it just didn't work ... you could physically see that there was more stress on teachers' faces because not only were you having to deal with challenging behaviour, but you were trying to deal with trying to get X to get a level 8 and Y to try and count up to 5 in the same class and it was too hard. It was too hard and it didn't work. It failed. Everybody was more than happy [to go back to setting] … The year after we taught just setted by ability and they got much higher results.

Jim is now firmly of the opinion that, at least in a school like Broadbent, there is no place for all-attainment teaching:

> I would just set them. I'm definitely now not a mixed ability fan in a challenging school. It's just too much.

Discussion

Our aims in this paper are modest. We do not expect stories like this to have any traction with policy makers and we very much welcome alternative approaches that may have the "requisite symbolic power" (Francis et al., 2016, p. 13) to do so. Here our purposes are rather different. Our intention has been to tell a story of a single teacher which illustrates how "powerful discursive productions of the 'obvious', 'real', and 'natural'" (Francis et al., 2016, p. 10) work in practice to shape this teacher's thinking about ability. Jim is striving to make sense within this discursive framework. He conceives the pupils as simply *being* such and such a level *person* in mathematics and so inherently needing a differentiated approach to learning: the pupil's essence determines within fairly narrow limits what she or he can do. With such a view, offering a more open curriculum in which the unpredictable is expected makes no sense and the task of all attainment teaching becomes simply unmanageable: Jim is led to validate practices with respect to pupil grouping that reinforce inequalities despite the honourable intentions to do otherwise.

If fixed hierarchies exist - of who can understand and achieve what in mathematics - and there is a predicted and predictable limit to what can be expected from any particular individual, as current policy technologies insist, then the possibility of creating a pedagogy where all can succeed, and where success is attributed to the learning community rather than to individuals, is precluded:

> the production of hierarchies of ability via a discourse of 'natural order' acts as a technology of privilege, and renders alternative accounts (including research evidence) unintelligible. (Francis et al., 2016, p. 12)

Knowledge, discursive practices and both deep and espoused beliefs all interact in complex and layered ways in shaping how we think and what we do. A two-fold argument follows from Jim's story. First, changing practice alone is unlikely to engender ways of being in the world that challenge established 'natural' hierarchies. Second, there is a need for research-informed, counter-hegemonic knowledge and understandings to be foregrounded, alongside curriculum innovation and the re-imagining of pedagogy, if the dominant and unjust practices of grouping by 'ability' are to be effectively countered in the countries in which they currently prevail.

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Diversity in an inclusive mathematics classroom: A student perspective

Helena Roos

Linnaeus University, Växjö, Sweden; Helena.Roos@lnu.se

This paper reports on a study exploring inclusion in mathematics education from a student perspective. The theoretical and analytical approach in the study is discourse analysis. The results presented in this paper are based on 8 interviews with students from lower secondary school and 4 observations of mathematics lessons. The teachers describe the students as students in special needs in mathematics (SEM). The results show that, from a student perspective, the teaching and learning of mathematics in an inclusive classroom is complex and diverse. At the same time, as these students are similar in that they are SEM-students, they are different when it comes to how they themselves want to be included in the mathematics. These differences regard both the organization and the content. Thus, diversity among students demands diversity in the mathematics education.

Keywords: Inclusion, diversity, equality, access to mathematics.

Introduction

A growing body of research in mathematics education is focusing on access and equity. This can be seen in some of the research books that have been published in recent years, for instance *Diversity in mathematics education: Towards inclusive practices* (Bishop, Tan & Barkatsas, 2015), *Towards equity in mathematics education* (Forgasz & Rivera, 2012), and *Mathematical literacy: Developing identities of inclusion* (Solomon, 2009). In addition to this, at school-level, the issue of the need to meet every student's needs in the mathematics classroom according to the preconditions and needs of each and every one has been recognized (Roos, 2015). This task, to be able to meet all students' needs and create opportunities to learn, is not at all an easy task. Some teachers even say that it is impossible to meet every student's needs in an inclusive classroom because of the diversity. On the other hand, one could look upon teaching diversity as Frederickson and Cline (2009) do, claiming that teaching is interesting because of the diversity among students, but it is only possible because of the similarities among students. This implies that even if there is diversity among students in our mathematics classroom the teaching does not have to be different for each and every student, but by being aware of the diversity as a teacher you can develop a sensitivity towards equality in the teaching. In that sense, you put the students' needs at the forefront in the explanations and tasks given.

In mathematics education one of the motivations to strive for inclusion and access to mathematics for all is that it is a human right to know mathematics to achieve participation in society (D'Ambrosio, 2010). Most often inclusion is discussed and researched from this ideological perspective even though it is often used as a method in schools. Hence, there is a need for studies investigating inclusion in practice, which this research aims at. This leads to the research questions of this paper, which are: how do students experience an inclusive mathematics classroom and when do they express having optimal opportunities to learn mathematics?

Diversity and Inclusion

What does diversity and inclusion in mathematics mean? It seems that when scholars talk about diversity in mathematics, they almost always speak of inclusion in the same breath (e.g., Bishop, Tan & Barkatsas, 2015), indicating the two notions are closely related. When investigating further, inclusion, if used as a tool in classrooms, can be seen as a way of meeting diversity, supporting all learners within a local community (Booth, Nes and Strømstad, 2004). Diversity on the other hand is not often defined, but used together with gender and culture (e.g., Forgasz & Rivera, 2012). It is also used together with specific subject area in mathematics and students' performances within this area (e.g., Hopkins & de Villiers, 2015); hence diversity is here connected to some kind of ability in mathematics. Accordingly, diversity can be connected to different things. In this paper diversity is connected to inclusion on the level of optimising students' performance in mathematics.

An inclusive classroom is in this paper defined as a classroom that is not grouped by ability but instead as a classroom in which students struggling with mathematics as well as students in need of more challenges in mathematics are taught working with similar tasks and the same mathematical content. Hence, diversity from an ability perspective is prioritized. This puts demands on the teacher, being aware of the diversity (Solomon, 2009), to have equitable instructional quality. This quality can be seen in the teachers' mathematical knowledge and their preparation for the teaching of mathematics as well as their beliefs about and skills in teaching diverse students (Allexsaht-Snider & Hart, 2001). So, in having equitable instructional quality, students' opportunities to learn mathematics might increase. But according to Rousseau and Powell (2005), there are factors that can work as barriers for increased opportunities to learn mathematics: large class sizes, high-stake-standardized tests, absenteeism and mobility of students and a lack of a high-quality curriculum. All these issues arise from a teaching or organisational perspective. Hence, it becomes important to also listen to the students' voices, enabling teachers to understand processes of exclusion and inclusion in the mathematics classroom like Solomon (2009) highlights.

One important issue arising when talking about the ability of individuals is labelling. There are many teachers that claim that in order to be able to meet the diversity of students they have to "label" them in some way. Though, Askew (2015) claims that meeting diversity does not imply that we have to label the students, because labelling might perpetuate exclusion instead of promoting inclusion. One way to meet diversity and create an inclusive classroom is to support cooperative learning (Askew, 2015) by building a sense of belonging and safety where diversity is valued (Reid & Valle, 2004). This implies that the teacher and the pedagogy the teacher uses in the classroom are really important to create this learning community. Then, as Liasidou (2012) points out, pedagogy is an important dimension of inclusion. All this implies that diversity and inclusion are intertwined, and if striving for inclusion one has to respect diversity (Booth, Nes & Strømstad, 2004). Accordingly, in this study the objective is to investigate diverse students' experience of an inclusive mathematics classroom.

The current study

The focus in this research is on students in special educational needs in mathematics (SEM-students). Special educational needs are here defined as a need of another teaching then the regular mathematics teaching. This is not unproblematic, since it signals a labelling of students in the

research. To be vigilant in terms of that issue, the teachers in the study have made the selection of students. In this research, inclusion in mathematics is investigated from two different SEM-directions: students struggling with mathematics, and students in need of more challenges in mathematics. This kind of selection is information-oriented and used to "obtain information on unusual cases which can be especially problematic or especially good in a more closely defined sense" (Flyvbjerg, 2006, p. 230).

An ethnographic approach is used, meaning the researcher participates directly in the social setting collecting data without meaning being imposed on the participants (Brewer, 2004). The ethnographic approach also offers in-depth study (Hammersley & Atkinson, 2007), which can be used to follow a process in a particular case, such as to be included in mathematics teaching and learning. An ethnographic study usually investigates people's actions and accounts in an everyday context. In this study, a Swedish lower secondary school (students are 13 to 16 years old) that sets out to work inclusively is observed. That is, the school strives to have all students in the classroom, even the students assumed to be in special educational need of any kind. To be able to meet all students' needs, they strive to have at least two teachers in the classroom for each lesson. The school is an urban school with about 500 students, located in the outskirts of the city.

One grade 7 and one grade 8 class were observed. The mathematics teachers in a discussion with the researcher selected the classes and the students. Grade 7 was selected based on the criteria that working inclusively might be new for them. Grade 8 was selected since they have been working inclusively for a year. Students from grade 8 and grade 7 that struggled with mathematics as well as students that needed more challenges were chosen and interviewed several times. Ethical considerations were made both before and during the process. Both the students and their guardians gave written consent. As a researcher, I reflected on what ways I affected the students and the research. Another aspect was that the students in the classes would be able to handle a researcher in the classroom.

Methodology

The approach used in this research is Discourse Analysis (DA) and the data consists of observations from two classes.

Discourse Analysis

Discourse Analysis is chosen as approach because of the power of DA to focus on language in interaction and language above or beyond the sentences (Gee, 2014a) and its explanatory power of social contexts and meaning making. The focus of DA is on language and text, what we actually can see, hear and read. In this study, ethnography was applied together with DA in order to make students' expressions of mathematics teaching and learning visible. DA and the ethnographic approach complement each other in this research; DA provides theoretical and analytical notions, while ethnography provides a way to conduct research.

In this paper, DA is used from the perspective of Gee (2014a, 2014b), since this focus is descriptive and I intend to describe how students want to participate in an inclusive mathematics classroom to be able to have optimal opportunities to learn. From Gee's perspective, DA covers all forms of interaction, both spoken and written, and he provides a toolkit for analysing this interaction. These

tools put focus on the communication and ask questions of the text. Hence, in this research, the toolkit is used as a methodological tool.

Gee (2014a, 2014b) also provides theoretical notions, such as *big and small discourses* (henceforth referred to as Discourse with capital D and discourse with lowercase d), where Discourse is looking at a wider context, social and political. Discourses are always embedded in many various social institutions at the same time, involving various sorts of properties and objects. For example, a Discourse can be "assessment in mathematics." Discourses are always language *plus* "other stuff" (Gee, 2014a, p. 52). This other stuff compromises actions, interactions, values, beliefs, symbols, objects, tools and places. Small d discourse is focused on language in use, what *stretches of languages* we can see in the conversations or stories we investigate (Gee, 2014a). In this research, big and small discourses will be the theoretical perspective. Hence, DA is used both as a theory and a tool and provides a set of methodological and theoretical lenses.

Procedure

During one semester (January to June 2016) I observed the two classes at the chosen lower secondary school. I was present at least one mathematics lesson each week for each class doing observations. After observations, I conducted interviews with the selected students. Since I had both ethical and organisational issues to take into consideration, the interviews were done when the students wanted to and had time, and the teachers allowed it (they did not want me to interview them when they had their ordinary lessons). The interviews took place in a room next to the classroom when the students had "class time" once a week. The interviews were based on the close in time observations; hence, they were situated and narrative. I asked questions about situations and tasks and showed photos of tasks on the blackboard. We also looked at tasks in their textbooks. The first and the last interview were based on a questionnaire about their mathematics education.

Data analysis

In this paper, eight interviews and four observations have been used in the analysis, two interviews with a student in grade 7 named Billy and two interviews each with three students in grade 8, Edward, Ronaldo and Jeff. The teachers perceive Ronaldo and Jeff as students in struggle with mathematics and Billy and Edward as students in need of more challenges in mathematics. In the interviews the students got questions about what they wanted from the teaching in mathematics, how they learned mathematics best and also got questions arising from the previous mathematics lesson, which were the four observed lessons. The observations were used as contextualisation for the interviews as well as for supporting identification of big Discourses. When analysing the data by asking questions to the text, both small and big discourses appeared. That is, while examining the text, I used Gee's toolkit by asking specific questions. Depending on the type of text, different questions were asked. For example, when using the subject tool, I asked, "What are they talking about here, and why?" When using the deictic tool, I asked, "What is pointed out in the text, and what is the listener assumed to already know?" When applying the fill-in tool, I asked, "What needs to be filled in to achieve clarity? What is not being said overtly, but is assumed to be known or inferable?" Then, stretches of language(s) appeared when finding answers to the questions, which signalled for small discourses. When adding analysis of the data from the observations, such as text on the blackboard and the actions of the teachers, big Discourses could be identified.

Result and analysis

In the analysed data, three themes, or using Gee's (2014b) terminology, three "stretches of languages" emerged. The first theme was about how students wanted to participate from an organisational perspective. The second theme was about tasks they did or did not like and the third theme was how the student wants a mathematics lesson to be like for an optimal learning opportunity.

Organisational aspects

In the interviews, stretches of languages about organisational aspects were showing. In the first interview Ronaldo says, "We are starting to *go outside [the classroom]* into small groups, like we did not do before, and it feels much better now. I am concentrating a lot better [in a small group] and like that." He also explained why it felt better to be in a small group: "It feels better actually, you get peace and quiet and then… like me… if they talk a little there [inside the classroom] I lose concentration right away and listen to what they [the other students] are saying; when it's smaller groups I am able to concentrate better and learn more." In Jeff's first interview he also highlights the possibility of being able to *go outside the classroom*: "… if it's a test or something I would rather be outside [the classroom] since I am more focused then." Edward also talks about the organisation, but within the classroom when he is discussing cooperation. "It is not very easy, since I have often come a long way, so I always explain to them, it never gets to a discussion for me… I mean, with somebody else, that we discuss and so on […]." When the researcher asks Edward if it is hard to discuss with everybody he says, "It depends on whom I am sitting next to." He also expresses that he does not *sit next to someone who can challenge* him, and says that he would like to do that more. "I think I would get more out of it."

Tasks

Other stretches of languages showed talk about tasks. Billy explicitly talks about his need of *more challenging tasks* in the classroom. "I like those [tasks] which are harder, those that challenge you." "[I would like to have] more challenges […] at the lessons." He explicitly talks about *problem solving tasks* as something challenging: "I like it when we have problem solving. You get to think for yourself and then talk to friends [about it]." Ronaldo also talks about *problem solving* but in a rather different way. "I hate problem solving tasks more than anything! I just cannot do it." He also describes why he does not like it: "It is hard with reading comprehension and like that, and to connect it with like the task and the text, it gets too much. It is often that kind of task I fail at on the tests." Jeff talks about tasks that he likes. These are tasks he knows "how to calculate and tasks that I understand." He mentions geometry tasks as tasks he likes. The type of tasks Edward likes the most is Algebra tasks. "To be able to find out all the variables, it is fun to figure out what it is." Here we can notice a difference to how Ronaldo thinks about algebra tasks when he states that he thinks that it is hard with "like all this with X and Y and everything … it is terrible."

A good mathematics lesson

A third stretch of language is the talk about what the students want from a mathematics lesson in order to be able to learn mathematics the best. Jeff states that what is most important for him in a mathematics lesson is "if they [the mathematics teachers] explain good … and thorough, if they write step by step." He also says if he knows what to do it "feels good, I know what to do and I get

on with it right away." Meaning, for him the *thoroughness and structure in the instructions* is important. Ronaldo likes to have a lesson when you "first work a little [by yourself] and then some 'going through'[1] and then you *work a little by yourself and then you do some group* work with those you sit with […]. It is more fun when you are in a group and cooperate." Edward on the other hand thinks that it is hard to cooperate and "extremely hard to get something from the others in the class." Instead, he like *lessons "with variation* so that you don't get tired." He also likes "a going through or something like that, gladly a game or something. You should [also] count a little by yourself I think." Billy thinks that the lessons are best "when you get to *explain to others* [students] how you have done it." He likes when "we have like problem solving." He also stresses that he wants "more challenges at the lessons."

Identified discourses and Discourse

The three themes appearing in the data – organisational aspects, tasks and a good mathematics lesson – can be interpreted as small discourses (Gee, 2014a) in the students' talk of their mathematics education. There are aspects in this talk from the different students that overlap, such as wanting to be in a small group sometimes, which both Jeff and Ronaldo stressed. Another aspect that overlaps is collaboration. Both Billy and Ronaldo highlighted collaboration, but Edward on the other hand felt that his peers did not challenge him in the discussions, but he said that he thought he would get more out of it (meaning the discussions) if he sat by a peer that had the possibility to challenge him. There are aspects that diverge, such as what type of task they want to have. Billy says he wants more problem solving and Ronaldo "hates" problem solving. Edward likes algebra tasks and Jeff likes geometry; hence there was no consensus on types of tasks or mathematical content they prefer. Another aspect that appears in the students' talk was the organisation of the lesson. To attend a structured lesson, with both explanations by the teacher and work by yourself or together with others seemed to fit them all. Jeff is more explicit about his wishes for structured lessons, indicating a need of structure to explain what to do. Even though all these small discourses contain diversities, they together indicate a big Discourse. When looking at the observation notes from the lessons you can identify support to this big Discourse, in terms of talking about tasks, organisational aspects such as "talk to your neighbour" or when the special teacher attending the classroom walks outside the classroom with some students; also, the type of task being addressed at the blackboard, and the way the teachers structure the explanations at the blackboard and the talk of what to include in an explanation. This big Discourse can be named "mathematics in school": what mathematics in school is or means for the students and what they want from the mathematics to be able to learn best.

Summary of results

The result shows both overlapping and diverging issues regarding how students experience an inclusive mathematics classroom and having optimal opportunities to learn mathematics. Regarding organisational aspects, Jeff and Ronaldo stressed the need of *being in a small group outside* the classroom from time to time even though the school promotes physical inclusion, and Billy and Edward highlighted *collaboration* in order to have opportunities to learn. Diverging aspects are type

[1] Going through is "genomgång" in Swedish, which is when the teacher is explaining something on the blackboard.

of tasks; for example, where Billy wants *challenging tasks* in the form of *problem solving*, Ronaldo hates problem solving. How the students want a mathematics lesson both diverges and overlaps: *thoroughness and structure in the instruction; work a little by yourself and then in a group; lessons with variation* and *explain to others* were expressed as good ways of learning.

Discussion

A diverse picture of how students want to participate in an inclusive mathematics classroom to be able to get optimal learning opportunities appears in this research. Although this is not unexpected, it is important to highlight this diversity and address the question of how the organisation and the education can support this diversity in order for students to be included in mathematics. This is not just only spatially included, but is also included in the teaching and learning of mathematics. The diversity of how students want to participate in the mathematics classroom that appears in this research stresses the need for diversity in mathematics education at school. One thing that is striking is that students expressed a need of being in a small group outside the classroom from time to time. The school promotes physical inclusion, and it did not seem to always benefit these students. You might say that diversity among students demands diversity in mathematics education. But, as Fredickson and Cline (2009) stress, even though students are different, teaching is only possible because students are similar in some ways. This research supports this, because even if the results showed diversity among the students (both within and between students that are perceived as students in need of more challenges in mathematics and students that are perceived as struggling students in mathematics), the results also showed similarities between the students. These similarities are something the teaching can take advantage of, in the organisation and planning of the mathematics education in order to get equitable instructional quality. However, barrier factors such as large class sizes, high-stake-standardized tests (Rousseau and Powell, 2005), etc. can be prohibitive to the work of equitable instructional quality. If the organisation is responsive to the diversity among students and is aware of barrier factors it might be dynamic and adjust accordingly to improve access to mathematics and an increased inclusion in mathematics.

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School mathematics education through the eyes of students in Ghana: Extrinsic and intrinsic valuing

Wee Tiong Seah[1], Ernest Kofi Davis[2] and Monica E. Carr[1]

[1]MGSE, The University of Melbourne, Australia; wt.seah@unimelb.edu.au

[2] Institute of Education, University of Cape Coast, Ghana

1256 students from 18 primary and secondary public schools across urban and rural settings in the Cape Coast Metropolis of Ghana responded to the "What I Find Important (in my mathematics learning)" questionnaire. The data analysed suggested that students in Ghana valued in their mathematics learning: achievement, relevance, fluency, authority, ICT, versatility, learning environment, strategies, feedback, communication, fun, connections, engagement, applications, and accuracy. The students' embracing of these attributes is explained by reflecting on the societal and pedagogical norms in Ghana. When compared to high performing economies in East Asia, it was found that most of the Ghanaian attributes represent extrinsic (versus intrinsic) valuing. Implications and suggestions for policy-making and for classroom teachers are provided.

Keywords: Values, Ghana, extrinsic/intrinsic valuing, East Asia, WIFI.

Mathematics education in Ghana

Students value attributes of mathematics learning (e.g. *practice* and *understanding*) differently, with implications for the quality of mathematics learning that takes place (Matthews, 2001). The extent to which a student values *understanding*, for instance, may influence how relational understanding may be preferred over instrumental understanding, the extent to which the development of algorithms is important, and indeed, the extent to which s/he is interested – and perseveres – in knowing how these algorithms or formulae came about. In other words, what and how much an attribute of (mathematics) learning and teaching is valued influences a student's development and/or application of cognitive knowledge and skills, as well as the maintenance of affective states.

Drawing from relevant seminal literature, Smith and Schwartz (1997) have argued that, while values are abstract concepts, they are not so abstract that they cannot motivate behavior. The ability to identify, review and facilitate/modify what students value in their mathematics learning should optimise the cognitive and affective-based pedagogical strategies that support learning.

This paper reports on Ghana's participation in a 19-country study on what students valued in their mathematics learning experiences. Focussing on this West African country, and analysing and interpreting the data collected there, was aimed at achieving an understanding of the Ghanaian mathematics education system, both in its own right and also through comparisons with other countries. This is especially significant, since Ghanaian students' mathematics performance has been low by global standards (Enu, Agyman & Nkum, 2015). For example, in the TIMSS 2011, Ghanaian eighth grade students ranked last amongst 45 participating countries in mathematics achievement (Mullis, Martin, Foy, & Arora, 2012).

Ghanaian students' transitions between school and out-of-school mathematics have not been without its issues. In the home context, the units of measurement of money and of capacity are different from the metric ones being taught in schools. The schools' language of instruction from

Grades 4 and above is also different from the languages used by students at home and outside in public. A further complexity when considering school mathematics education in the Ghanaian context is in the way students there experienced fractions differently in school and in out-of-school contexts. For example,

> the majority of the students were able to identify half in the out-of-school activity perhaps due to that fact that it is the only unit fraction that has local name (fã). However "fã" does not mean equal halves, it means about mid point. Thus three-fifths may also be categorised as "fã". Students' difficulty in naming the other units fractions may be due to the fact that in out-of-school setting they do not differentiate unit fractions. Thus with the exception of half which could mean about midpoint all the fraction are described as less than whole (sin). (Davis, Seah, & Bishop, 2009, p. 69)

Values in Mathematics Education

Values are "the principles and fundamental convictions which act as general guides to behaviour, the standards by which particular actions are judged as good or desirable" (Halstead & Taylor, 2000, p. 169). Essentially, then, values reflect what we think are important to us, and are thus distinct from beliefs, which reflect what we think are correct. Values can be viewed as a form of culturally-based tools with which we mediate our actions and behaviour in the learning process.

In the field of mathematics education, we adopted Seah and Andersson's (2015) definition that

> values are the convictions which an individual has internalised as being the things of importance and worth […]. Valuing provides the individual with the will and determination to maintain any course of action chosen in the learning and teaching of mathematics. They regulate the ways in which a learner's/teacher's cognitive skills and emotional dispositions are aligned to learning/teaching in any given educational context. (p. 169)

What are valued by the individual, as these are shaped and refined by life's experiences (including classroom learning experiences), impact on subsequent decisions and actions. They do so by affecting the ways the individual reasons and feels about the task or problem at hand. As the quote above suggests, this volitional force can be quite powerful, manifesting themselves in the form of will and determination.

When the individual interacts with others (e.g. teachers interacting with their students in the classroom), it is inevitable that there would be differences in what each person values. Such differences can potentially lead to conflicts, and one or more of the people involved will seek to negotiate and resolve these differences, achieving a level of cognitive harmony that is acceptable by most if not all involved.

In terms of the types of attributes of mathematics learning and teaching valued, Bishop (1996) had categorised these into mathematical values (i.e. regarding the mathematics discipline), mathematics educational values (i.e. regarding the pedagogy of mathematics), and general educational values (i.e. regarding the moral and civic virtues). Earlier, Bishop (1988) had conceptualised 3 pairs of complementary mathematical values, namely, *rationalism* and *objectism*, *control* and *progress*, and *openness* and *mystery*.

Prior to 2010, research of values and valuing in mathematics education had focussed on small-scale studies of what teachers valued (e.g. Chin & Lin, 2000). The setting up of the Third Wave Project in 2008 not only brought together a group of researchers internationally to support – and collaborate with – one another on research studies into valuing, but it also shifted attention to the examination of what students value in their mathematics learning (e.g. Seah & Wong, 2012).

On the other hand, much research related to PISA and TIMSS had been conducted by or with education systems which have performed relatively well in these tests, with relatively little research attention paid to mathematics education systems at the other end of the performance spectrum. Yet, the experiences of these countries can also serve as an important reflection on what (else) contribute to effective mathematics learning. As such, a study named "What I Find Important (in mathematics learning)" (WIFI) was designed to investigate what students in 18 different economies value when they were studying mathematics, Ghana being one of these economies. This paper reports on the Ghanaian data of the WIFI study, and how the findings address two of the research questions posed to guide the Ghanaian study, namely:

1. What did school students in Ghana find important when learning mathematics?

2. How might the valuing amongst the Ghanaian students be similar to or different from what their peers elsewhere in the world valued?

Methodology

The first research question suggested a need to 'map the scene' for Ghana (and indeed, for the other participating economies too). As such, the questionnaire survey method was adopted. The validated WIFI questionnaire has four sections. A Likert-type scoring format was used for the first 64 items in Section A, in which students were asked to indicate how important mathematics pedagogical activities such as small-group discussions (item 3), connecting mathematics to real-life (item 12), and mathematics homework (item 57) were to them. A five-point scoring system was used, ranging from absolutely important (1 point) to absolutely unimportant (5 points). Section B consisted of 10 continua dimensions, each related to two bipolar statements and respondents were asked to indicate along the continuum the extent to which their valuing leans towards one of the two statements. Section C consisted of four scenario-stimulated items; and Section D items asked for students' demographic data. The English language version of the WIFI questionnaire was administered, English being the medium of instruction in Ghana. In this paper, only the responses to Section A will be presented.

Student participants were sourced from public schools at the primary, junior high and senior high levels in the Cape Coast Metropolis of Ghana. Stratified random sampling procedure was used to select students from a mix of schools, by achievement levels and by rural versus urban settings. In all, 1256 research participants comprising 414 primary four, five and six pupils, 426 junior high school pupils and 416 senior high school students from 18 schools participated in the study.

In line with the data analysis conducted by the other 18 participating economies, a Principal Component Analysis (PCA) was performed.

Results

The data gathered from the 64 Likert-scale items of the WIFI questionnaire was cleaned prior to data analysis. They were first analysed to identify any missing values. The eleven missing responses identified out of the total possible 80,384 (i.e. 64 X 1256) was acceptable, and each of these was replaced with the value "9".

The Kaiser-Meyer-Olkin (KMO) (Kaiser, 1970) measure of sampling adequacy was 0.947 and Bartlett's test of sphericity (BTS) (Bartlett, 1950) was significant at the 0.001 level and so, factorability of the correlation matrix was assumed, which demonstrated that the identity matrix instrument was reliable and confirmed the usefulness of the principal component analysis.

Principal component analysis

A principal component analysis (PCA) with a varimax rotation and Kaiser normalization was used to examine the questionnaire items. The significance level was set at 0.05, while a cut-off criterion for component loadings of 0.45 was used in interpreting the solutions. Items that did not meet the criteria were eliminated. According to the cut-off criterion, 23 items were removed from the original 64. The analysis yielded 15 components with eigenvalues greater than one, which accounted for 52.73 % of the total variance. Each component can be considered to be an attribute that were valued by the students in Ghana, with the relevant questionnaire items regarded as describing the characteristics of the attribute. Accordingly, the three researchers discussed and agreed on the value labels for the 15 components based on the nature of the corresponding items.

The first component consisted of 17 items that together accounted for 13.31% of the total variance. Questionnaire items included in this component included "doing a lot of mathematics work" (item 37), "knowing the steps of the solution" (item 56), "knowing which formula to use" (item 58), and "understanding why my solution is incorrect or correct" (item 63). Guided by our Ghanaian collaborator's recommendation, we subsequently labelled this component as *achievement*.

The second component is made up of 6 items which together accounted for 6.64% of the total variance. The questionnaire items included "stories about mathematicians" (item 61), "explaining where rules / formulae came from" (item 40), "mystery of mathematics" (item 60), "stories about recent developments in mathematics" (item 18), and "using concrete materials to understand mathematics" (item 48). Given these items, we propose to name this component as *relevance*.

The third component is made up of 2 items which together accounted for 4.35% of the total variance. The questionnaire items were "explaining my solutions to the class" (item 19) and "practicing how to use maths formulae" (item 13). So, we named this component as *fluency*.

The fourth component is made up of 3 items which together accounted for 3.40% of the total variance. The questionnaire items were "learning maths with computer" (item 23), "learning maths with internet" (item 24) and "explaining by the teacher" (item 5). It was named *authority*.

The fifth component is made up of 2 items which together accounted for 3.04% of the total variance. The questionnaire items were "using calculator to check the answer" (item 22) and "using calculator to calculate" (item 4). Given these items, we named this component *ICT*.

The sixth component is made up of 2 items which together accounted for 2.75% of the total variance. The questionnaire items were ”looking for different possible answers” (item 16) and ”being lucky at getting the correct answer” (item 27). We named this component *versatility*.

The seventh component is made up of one item which accounted for 2.69% of the total variance, it being ”mathematics debate” (item 9). It has been named *learning environment*.

The eighth component is made up of 2 items which together accounted for 2.69% of the total variance. The questionnaire items were ”shortcuts to solving mathematics problems” (item 55) and ”given a formula to use” (item 38). Given these items, we named this component *strategies*.

The ninth component is made up of one item which accounted for 2.50% of the total variance. The questionnaire item was ”investigation” (item 1). We interpreted this component as *feedback*.

The tenth component is made up of one item which accounted for 2.22% of the total variance, which was ”outdoor mathematics activities” (item 34). We named this component *communication*.

The eleventh component is made up of one item which accounted for 2.00% of the total variance. The questionnaire item was ”mathematics games” (item 25). It was given the label *fun*.

The twelfth component is made up of one item which accounted for 1.92% of the total variance: ”relationship between maths concepts” (item 26). We named this component *connections*.

The thirteenth component is made up of one item which accounted for 1.80% of the total variance: ”stories about mathematics” (item 25). We named this component *engagement*.

The fourteenth component is made up of one item which accounted for 1.77% of the total variance: ”looking out for mathematics in real life” (item 39). We named it *applications*.

The fifteenth component is made up of one item which accounted for 1.66% of the total variance: ”getting the right answer” (item 50). Given this item, we named this component *accuracy*.

Discussion

1256 primary and secondary school students from 18 public schools located in both urban and rural areas of the Cape Coast Metropolis had responded to the WIFI questionnaire, thus allowing us to map the attributes of mathematics pedagogy that were valued by these students. The PCA has led to the identification of 15 attributes which the students valued in their mathematics learning in Ghanaian schools, explaining 52.73% of the total variance. These attributes are *achievement, relevance, fluency, authority, ICT, versatility, learning environment, strategies, feedback, communication, fun, connections, engagement, applications, accuracy.*

Most of the students in Ghanaian schools come from a farming background, where all available helping hands are needed on the farms especially during the harvesting periods. That the respondents of the WIFI questionnaire were still in school might explain why *achievement* was so highly valued by these students. For them and their families, it is thus not surprising that *relevance, applications, engagement* and *connections* of what is taught at school in relation to the knowledge and skills that are needed at home and in the farms are valued. Given the frequent use of expository teaching in schools (Enu et al., 2015), the students have probably learnt to value *authority*, *fluency* and *accuracy*. Yet, this dominant teaching style is not likely to meet the expectations of students

and their families if they have chosen to continue staying in school. Novel and effective learning styles will be important, and these are likely to involve the valuing of *ICT, versatility, learning environment, strategies, feedback, communication* and *fun*.

These 15 values may be compared with the attributes of mathematics learning that students in high performing PISA2012 economies which took part in the WIFI study valued (e.g. Zhang et al., 2016). Students in these high performing economies (all of whom are East Asian, since Finland did not participate in the WIFI study) valued *connections, understanding, communication* and *recall*. Though students in Ghana also valued *connections* and *communication*, they were less valued than at least 9 other attributes, such as *achievement, relevance* and *fluency*.

This distinction above had invoked in us the notions of intrinsic and extrinsic motivations. Emerging from the analysis we were reminded of Ryan and Deci's (2000) assertion that "intrinsically motivated behaviors […] are performed out of interest […] [whereas] extrinsically motivated behaviors […] are executed because they are instrumental to some separable consequence" (p. 65). In the context of our data here, we can interpret the top performing East Asian economies' valuing as being intrinsic to mathematics itself (*connections* and *understanding*, for examples, deepen the students' mathematics knowledge), and that the top values that were held by the students in Ghana to be more extrinsic to the mathematics discipline. Although *achievement, relevance, fluency* and *authority* were also attributes of mathematics learning and teaching, they were not so much about what was important about mathematics, but rather, what was important about what can be done with mathematics.

The contrast thus seems to be that of extrinsic versus intrinsic valuing. The top performing East Asian economies are located in places where mathematics study has traditionally been taken up for its own sake, and where problem solving and the study of proofs are regarded as tasks that maintain one's mental agility. Against this sort of tradition, then, it would not be surprising that East Asian students appreciated the structure and form of the discipline, and grew to value aspects of mathematics which reflect the nature of the discipline. On the other hand, education systems such as Ghana's might emphasise the utility function of the mathematics discipline, perhaps to satisfy the needs of local economies. Thus, the aspects of mathematics learning that are regarded as important would reflect this utility function and extrinsic valuing.

Given the large sampling size in this Ghanaian study, the findings above raised the question of the extent to which Ghanaian students' extrinsic valuing of mathematics and mathematics pedagogy might affect their mathematics performance. At the same time, how might the students' intrinsic valuing in places such as Shanghai, Hong Kong, Korea and Singapore be related to the high level of mathematics performance shown in TIMSS and PISA? To what extent might the attributes of mathematics education valued in the international assessment exercises be aligned with intrinsic valuing associated with the East Asian students?

Conclusion

Primary and secondary public-school students across both urban and rural settings in the Cape Coast Metropolis of Ghana valued *achievement, relevance, fluency, authority, ICT, versatility, learning environment, strategies, feedback, communication, fun, connections, engagement, applications,* and *accuracy* in mathematics learning. Comparing these against what students in high performing East

Asian countries valued, we propose that many of the attributes that were valued by Ghanaian students represented extrinsic valuing (versus intrinsic valuing in East Asia). Might the Ghanaian students' valuing of extrinsic attributes in part explain their relatively poorer performance in mathematics? Further analysis is being carried out, such as to investigate how the valuing differed according to student gender and school locations.

These and other related questions will be especially meaningful for Ghanaian policy-makers to consider. If extrinsic/intrinsic valuing is indeed a key variable of mathematical performance and achievement, the inculcation of intrinsic valuing amongst students would require strong and determined leadership at all levels of the society to model these values across the intended, implemented and attained curricula. In the meantime, the classroom teacher can be more mindful about espousing the intrinsic valuing of mathematics education. For example, teachers often do not sound very convincing to students that the content taught in class can be applied in life. Instead, it may be worthwhile for teachers to explain how the experience of learning mathematics might instill in students such attributes as *rationalism*. *openness* (see Bishop, 1988) and/or *understanding*. These are the very things which can be applied in life.

This knowledge should also be valuable to overseas (including European) researchers/experts who are involved with development work in Ghana, such as the British government's Transforming Teacher Education and Learning Project. Not only does it lead to a greater understanding of the local context, understanding what Ghanaian students value can also develop meaningful perspectives upon which culturally-appropriate and effective programs are designed and delivered.

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Integrating critical theory and practice in mathematics education

David Swanson[1] and Laura Black[2]

The University of Manchester, UK

[1]david.swanson@manchester.ac.uk; [2]laura.black@manchester.ac.uk

Recent years have seen many valid and important critiques of mathematics, mathematics education and mathematics education research (M, ME & MER). However, we also discern in some of these critiques a tendency toward one-sidedness and passivity. Unrelenting stress on the negatives of M, ME and MER can lead to a dismissal of the possibility of improving ME and a dismissal of those who attempt to do so. The separation of critical theory from critical practice which follows is then in danger of rendering critique sterile, becoming a mere pseudo-radicalism. As an alternative, we explore here the mutual relation between critical pedagogy and critiques of society, and the relationship between reform and more radical change, in wider society and ME. We argue that this analysis encourages a stress on joint activity, between individuals and organisations with a wide range of perspectives on what change is needed, to tackle the problems a critical perspective raises.

Keywords: Critical mathematics education, theory and practice.

Introduction

A growing body of research in mathematics education has explored critically the socio-political function of school mathematics in terms of its role in the mobilisation and production of capitalism (e.g., Kollosche, 2014; Pais, 2013, 2014; Swanson, 2016; Williams, 2012) leading to the reproduction of inequalities in education along the lines of social class, gender and ethnicity (e.g., Jorgensen & Niesche, 2008; Solomon, 2008; Black, 2004; Noyes, 2007). This work suggests that school mathematics (and associated qualifications) serves as a 'gatekeeper' in that it enables society and its dominant institutions (e.g., universities, employers etc.) to select and sort individuals under the rationale that 'mathematical ability' is a valued source of human capital. This critique is highly relevant to the current situation of mathematics education in England, where a new, more challenging curriculum has come into play since 2014 which serves to further substantiate the elitist position of mathematics in schools. At the same time, there is widespread concern in policy and in the mathematics education research community about achievement gaps (i.e. between rich and poor, or the most and least deprived) – a concern which has been said to further produce social inequalities – Gutiérrez (2008) refers to this as "gap gazing".

Whilst such critiques of mathematics and mathematics education are important and necessary to challenge dominant ideologies (including those pertaining to education more broadly), at the same time, we argue, there is a need to propose an ideologically grounded alternative. This paper presents a case for an alternative way forward by first looking at the potential mutual relation between critical pedagogy and critiques of society. We then examine the relationship between reform (i.e. improvement whilst remaining within the same overall framework) and more radical change, in terms of wider society, education and mathematics education. What follows from this exploration, we argue, is the need for activity which tackles the problems a critical perspective raises. In

particular, it suggests joint activity between those with a wide range of perspectives on what, and how far, things need to change. We then look briefly at the Stand Up for Education campaign in the UK, which brings together trade unionists, teachers, academics and parents, to show that what we outline here is not a purely abstract or ideal position. Spaces for the much required interrelation of critical theory and research with practice can and do exist, and we conclude by discussing why that matters for critical researchers.

What might a critical perspective on education look like?

We begin by looking at the relationship between critiques of education or society and critical mathematics pedagogy[1]. Arguably the most radical perspective here is to imagine and work towards a society beyond capitalism (e.g. Bowles & Gintis 1976; Counts, 1978; Freire, 2005) i.e. a change that involves a complete transformation of society. Discussions of what form education would take in such a society face certain limitations however. For instance, if we assume, as we should, that moving beyond capitalism entails the democratic collective control of society by the majority, then we who are so shaped by, and operate in, capitalist society are not best placed to either decide or predict what may happen. Nevertheless, we can speculate on how an alternative future education might work by taking the reverse of the features of capitalist education which are seen to lead to negative consequences today, for example the individual-competitive exam system which produces 'losers' who internalise their failure as objective qualities of themselves (for others, see Swanson, 2016). Then, we can combine these with the aspects of education that others have fought for (e.g. in school student strikes against corporal punishment, oppressive uniform policies or privatisation, see Lavalette & Cunningham, 2016) and at times, implemented (e.g., the banning of homework and exams in revolutionary Russia, see Karp, 2012).

Among the features we might expect to see are i) democratic collective control of education by teachers, students and other education workers, within the wider framework of its democratic shaping for society's needs; ii) much greater control by individuals over their own learning within that, but with an emphasis on social rather than individually competitive learning; iii) an end to exams and their production and reproduction of societal inequalities; iv) an equivalent end to the performativity culture of continual measurement to judge teachers and other education workers, v) an increase in societal resources (such that, for example, class sizes reduce to the levels seen only within private education in this society), and vi) a closer integration of education with wider life, reducing both the formal detachment of schooling from the world outside, and the artificial separation of subjects from one another.

The perspective above can loosely be termed the *revolutionary perspective* in education. We would define reformist perspectives, and these are far more common than revolutionary perspectives[2], as

[1] We use pedagogy here and throughout to mean teaching *and* learning combined due to the lack of an adequate single word in English.

[2] This is true even when reformist political organisations are weak or non-existent. However recent years have seen an important revival in reformist organisation with new parties such as Podemos in Spain, or around individuals in existing organisations, such as Corbyn in the UK and Sanders in the U.S.

those which may agree with some, many or even all of the elements above but which are accepting of greater limits as to how much things can change, for example limiting the possible changes to within one classroom, or to what is possible within capitalism. Although this covers an enormous range of possible beliefs (e.g., the free schools movement in the U.S., see Miller, 2002, or on a small scale, Boaler's work on reform pedagogy in the US, Boaler & Staples, 2008) and in Australia (Sullivan, Jorgensen, Boaler, & Lerman, 2013), in general we view such perspectives as radical and important. Fighting for fewer exams or less influence of exams on education clearly overlaps with fighting for no exams. We explore the general relationship between reform and revolution in a later section, but first we look at the relationship between the radical perspectives discussed so far and a particular type of reform, that of improving mathematics pedagogy.

The relationship between critical mathematics pedagogy and (active) critiques of society

We can see within critical mathematics pedagogy (in the broad sense of the term critical) parallels of many of the more general demands of radical educationalists. For example, we see pedagogies which aim to promote: a more active role for students in learning through open problem solving (e.g. Barron et al. 1998); teaching for understanding rather than for grades (e.g., Schoenfeld, 1988); an emphasis on dialogue and social learning (e.g., Lerman, 1996); and a more meaningful mathematics connected to the world outside of school and student experiences and concerns (e.g., Gainsburg, 2008). In doing this, pedagogy acts to counter some of the worst effects of capitalist education, even if it cannot overcome them fully. Here we argue, perhaps contra to some perspectives in critical MER, that it is worthwhile to subvert spaces, such as the classroom, as much as one can in these directions. Various forms of critical mathematics education which attempt to provide curricula and pedagogies which offer 'use value' to low status, disadvantaged or 'poor' learners and communities (e.g., Skovsmose & Greer, 2012; Gutstein, 2006) have much to offer. They can potentially challenge the 'gatekeeper' role of mathematics described above and can maybe transform the function of education, that is, rather than the learner serving the school/education system, education can begin to serve the community/learner (Williams & Choudry, 2016). Perhaps more importantly, attempts at developing critical thinking within the mathematics classroom have the potential to be generalised and transferred to other aspects of life, for example, to a pupil's future life in the workplace (see Black et al., 2010). The experience of critical thinking, of challenging everything, of weighing up arguments can assist in developing the confidence to do so elsewhere.

The possible connections between critical pedagogies and critical perspectives on society can work in the other direction too. The real limitations which schooling imposes on such pedagogy means that it is difficult to sustain critical educational activity if it is solely limited to the individual classroom. Teachers attempting to develop or sustain attempts within their classroom will come up against obstacles. For example, a head of department on a professional development course led by one of the authors was instructed by management to reverse pedagogical changes because students were now talking too much in class. However, arguably, the experience of these obstacles can make teachers open to looking beyond their immediate situation to help them achieve the changes they want. If teachers are connected to networks which challenge how schooling is generally organised and which also show sympathy for progressive forms of pedagogy, they may potentially move

towards engaging in critical activity outside the classroom, whether still directly related to pedagogy or beyond that. Such networks can also give teachers the confidence to persist with their efforts in their own classroom. (e.g., Volosinov, 1976, on the relationship between an individual's critical ideas and collective agency in such circumstances).

Taken together these points mean: Firstly, that it is in the interests of those who are critical of society to encourage meaningful activity in the classroom and to work alongside others who wish to do this, and, secondly, it is in the interests of those who want more meaningful activity in the classroom to work with those who have a critical perspective on society, precisely because they bring an understanding of the obstacles, and, usually, experience in organising networks to overcome these obstacles. A central task therefore for those who are critical of society and who work within mathematics education, is to help create, develop and shape organisational forms which encompass both these components.

Reform, revolution and the united front

The relationship between the particular reform, of developing more meaningful pedagogy in a classroom, and wider social struggles, rehearses similar arguments to that which can be made about the general relationship of reform to revolution. In general, reform and revolution are clearly different perspectives. As Luxemburg (1986) puts it:

> [T]hose who pronounce themselves in favour of the method of legislative reform in place of, and in contrast to, the conquest of political power and social revolution, do not really choose a more tranquil, calmer and slower road to the same goal, but a different goal. Instead of taking a stand for the establishment of a new society, they take a stand for the minor modification of the old society. (p. 56)

However, many of the elements key to a revolutionary strategy− for example, maximising active involvement and democratic control of movements; overcoming the division between purely economic and political struggles; attempting to connect up and generalise different struggles; developing an understanding of the interrelated nature of societal problems; and an emphasis that change comes from below, are not necessarily alien to those holding reformist ideas when they are engaged in struggles for particular demands (see, e.g., an account of the 2012 Chicago teachers' strike in Gutstein & Lipman, 2013). At the same time, revolutionaries are also in favour of reforms. First because they improve immediate circumstances, but also because it is through the struggle for reforms that people develop the consciousness and confidence required to transform society: "The struggle for reforms is its means; the social revolution, its goal" (Luxemburg, 1986, p. 5). This overlap in immediate situational objectives, and the potentially shared belief in activity to achieve them, can provide a basis for joint activity.

This joint activity between those who seek reform and those who aim for more fundamental change is central to a revolutionary approach and is termed the united front strategy. The strategy was explicitly formulated by the third congress of the communist international in 1922 as capitalism restabilised following the revolutionary wave around the end of the First World War. However, it has its roots in earlier practice. For example, during the Russian revolution of 1917, the unity of revolutionaries and reformists in repulsing Kornilov's attempted coup was central to the

development of the revolution, and the key organisational form of the revolution, workers councils or soviets, can be viewed similarly as a united front (see Trotsky, 1989).

For revolutionaries, there are two key aims of the united front strategy. The first is simply to increase the likelihood of success of the particular struggle through uniting the maximum number of people and organizations. Secondly, it aims to convince those involved in reform activity of the need for more radical change through i) joint experience of the benefits of revolutionary methods, ii) joint frustration at the limitations of reformist strategies, and iii) exposure to revolutionary ideas in ongoing dialogue. For these strategies to work, the unity and dialogue must be genuine of course, with the possibility of reformists winning revolutionaries to their strategy instead (see Trotsky, 1989). Although often from an alternative perspective, many with a reformist outlook on change equally see the importance of working together with others who hold different ideas to help achieve particular aims.

Implications for critical mathematics educators

Taken together, the arguments outlined so far imply the need for forms of organization which bring together various groups in mathematics education such as teachers, teacher educators, critical academics, parents, students and other education workers in common activity (a united front). This includes those who are particularly concerned with teaching and learning *and* those who are trade unionists; those who want to transform the world completely *and* those who just want to make things a little better. Through such activity radical mathematics educators can both assist in improving immediate circumstances in schools, classrooms etc., and also increase the numbers of those who see the necessity of more radical change (e.g., Gutstein & Lipman, 2013). We speculate that such an organization in relation to mathematics education is more likely to arise as part of, or emerging through, more general forms developed for the field of education as a whole. To illustrate that organisational forms such as this can exist, we now briefly describe the emergence of a network, local to the authors, which brings together the various forces described above.

Stand up for Education

The Stand up for Education campaign by National union of teachers (NUT) (2014) first emerged as a campaign launched by the National Union of Teachers (the largest teaching union in the UK), in the build up to the 2015 UK general election, to influence educational policy discussions and mobilise NUT members and others. Through that campaign, a network of academics supportive of the NUT's aims was formed called Reclaiming Schools. Together Reclaiming Schools and the NUT jointly published a collection of short articles from academics and researchers in support of the campaign (see NUT, 2015). The Reclaiming Schools network continued, with a website devoted to putting research in accessible form for teachers and others campaigning to improve education, and with occasional meetings in local areas which bring academics and teacher activists together and promote the website's activities. At one such meeting in Manchester, partly inspired by recent parent campaigns to remove their children from standardised testing (the Let Our Kids be Kids campaign, see https://letthekidsbekids.wordpress.com), the idea emerged for a local conference to be held which could pull together wider forces. NUT activists organised a follow up meeting, which included some parent groups and academics, to plan the conference. The primary aim of the conference was to share and develop understandings of key issues affecting schooling; to develop

and expand the different networks involved (parents, teachers, teacher educators and other academics), and to bring those networks together to promote mutual activity and campaigning. The conference (see https://www.facebook.com/standupforeducationmcr) united precisely the range of people that this paper has outlined, and, importantly, it discussed questions of organised activity, political issues *and* pedagogy. In future work we will discuss this movement in more detail, and in particular explore its potential in relation to critical mathematics pedagogy in particular. But we describe it briefly here to show that such networks can and do exist and are not merely an abstract desire of the authors.

Critical theory and critical practice

So far we have argued that i) critiques of ME and MER also require ideologically grounded alternatives; ii) both 'revolutionary' and 'reformist' alternatives exist; iii) critical pedagogy in mathematics (a particular reform) can be an integral part of both perspectives; and iv) this inter-relationship between reform and revolution is a general one. These last two together entail v) the importance, and possibility, of united front activity and organization within the field of mathematics education for all those who are to any extent critical of how things currently are, whether their initial motivation is teacher wages and conditions, less stressful exams for children, or more meaningful activity in the classroom.

We conclude with the particular relevance of the above for critical mathematics educational researchers. Marx argued that "Practice without theory is blind. Theory without practice is sterile", and this point is relevant for those who wish to criticise the world of mathematics education without attempting to change it. Arguably though, theory and practice always form an interrelated unity. No practice is uninformed by theory, (it may be unconscious of course). And no theory is unshaped by practice. The question for educational researchers is which practice shapes their theory – academic practice with its demands of publication and superficial novelty, or genuine critical practice and the needs of those trying to transform education. Critical theory detached from critical practice may provide useful insights, but ultimately its quality and usefulness will suffer from the separation. Uniting critical theory and critical practice, on the other hand, can enrich theory and research, and contribute to the development of the critical practice which can transform education.

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Cultural diversity as a resource or an obstacle for teaching practices in multicultural milieu: Experience of a training course for Italian teachers about Chinese Shuxue

Benedetto Di Paola[1] and Giovanni Giuseppe Nicosia[2]

[1]Dipartimento di Matematica e Informatica, UniPa, Palermo, Italy; benedetto.dipaola@unipa.it

[2]ISGEm, RSDDM, I.I.S. Aldini Valeriani Sirani, Bologna, Italy; gg.nicosia@gmail.com

Keywords: Culture, diversity, multicultural education, Chinese students, Shuxue.

Introduction

This contribution, examining the increasing presence at school of East-Asian students and the related didactical complexity rising from the contemporary presence of Western and non Western cultural heritages, presents an experience of a training course for teachers (from Pre-Primary to Secondary school grades) about the didactical problematic of teaching in a multicultural context (in particular with Chinese students).

Diversity as resource

According to Bishop's perspective, the notion of '*Multicultural Classroom*' has to be interrelated as the place where a variety of cultures come together and mixes, creating hybrid identities, epistemological and linguistic conflicts (real or potential), and a range of ways of valuing or devaluing mathematics as an academic subject (Bishop, 1988). In this sense, in many cases, working in a multicultural context is seen by teachers as an obstacle! *How to manage it? How to teach with different cultures and language's students?* These questions, for example, are often posed by western teachers of Chinese students. Their cultural reference system is in fact completely different from the western one (Bartolini Bussi, Di Paola, Martignone, Mellone, & Ramploud, 2016; Spagnolo & Di Paola, 2010) in language (as structure and as function in life), norms and social values and political references (Jullien, 2006). In the last ten years, researchers from all over the world have developed an ever increasing amount of work on the 'comparison' of student's performance in mathematics (Di Paola, Battaglia, & Fazio, 2016), especially between American and far-eastern countries' pupils such as Chinese, Japanese, and Korean ones (Leung, 2001). However, in Italy there is little research on this subject and all of them are aimed to deeply study the complexity of teaching/learning activities related to different cognitive processes emerging from different students' cultural references (Chinese in particular) in classrooms, assuming diversity as a resource for learning (Bartolini Bussi, Sun, & Ramploud, 2013; Spagnolo & Di Paola, 2010). A small amount of this research is focused of the training teacher.

A training teaching course as a need for mathematics multicultural classrooms

Following these assumptions, working in conjunction with other colleagues from East and West of *CTRAS* (*Classroom Teaching for All Students Research Working Group*), we designed a training 60 hours long course for Italian teachers (from Pre-Primary to Secondary school grades) which we implemented during the month of June 2016. It was organised almost entirely in small working groups, using activities with teachers and educators of different school grades (each one of them

chose independently to attend the training course). Aiming to analyse how diversity is seen through the trainees' eyes and trying to answer the research question posed before, we discussed with them some theoretical aspects related to their own epistemologies of the discipline. We then showed some experimental research conducted in Italian and Chinese classrooms, some Chinese textbooks strongly different from the Italian ones and finally some videotapes of some well implemented Italian teaching practices in multicultural contexts with Chinese students (K-12). According to our aim we discussed with trainees about the analogies and the differences coming out form these videotapes about the teaching/learning mathematic (*Shuxue*) in the two cultures. At the same time we tried to reflect with them on the possibility to 'use' this new knowledge to 'transform' the presence in classroom of 'other' cultures from an obstacle to a resource. All the proposed activities were videotaped and later carefully analysed. We can briefly say that the stimuli we offered permitted trainees to reflect on the complex situation of the simultaneous presence of Chinese and Italian students in their classrooms and to promote the desirable process of connecting different types of students solving strategies emerging from cultures.

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Reflections of Fröbel in discourses about mathematics in Scandinavian early childhood education and care

Andrea Eikset

Western Norway University of Applied Sciences; anei@hvl.no

Keywords: Fröbel, mathematics, kindergarten, paradigm shifts.

Finding reflections of Fröbel's ideas in Scandinavian ECEC

My research project is about how Friedrich Fröbel and his ideas have been used across time and place to present different perspectives about mathematics in early childhood education and care (ECEC). Fröbel continues to have an important role in discussions about ECEC in Scandinavia, and is often referred to when people talk about ECEC. In his writings, he discussed the inclusion of mathematics, through activities connected to his 'gifts' (Balke, 1995). In my project, I will identify how adaptions of his ideas (that I refer to as reflections) have been used to support ideological discussion about mathematics in kindergarten.

May (2016) used the metaphor of a swinging pendulum between two poles, to describe the history of early childhood education policy and pedagogy:

> Broadly the 'poles' characterise contesting paradigms of childhood: the child as nature whose holistic development is a natural process and who learns through play and discovery – construction. The child as a reproducer of knowledge, who as an empty vessel is filled with agreed knowledge, skills and cultural values – instruction. (p. 20)

Over time, this pendulum swings between the two sides, construction and instruction, because of economic, political or professional factors (May, 2016). Meaney (2014) highlights how different ideologies affect what content, such as mathematics, in kindergartens could be. "In the last 15 years, the focus of early childhood centres on supporting children to learn through play has been replaced in some countries by a more formal preparation for school" (Meaney, 2014, p. 1007). By analysing how key elements from Fröbel's theories, for example his 'gifts' or views of mathematical pedagogy, are reflected in discussions about ECEC I will be able to discuss where proponents of the reflections are situated on the pendulum swing. Identifying how Fröbel's ideas have been used in different ways in the past and across countries will provide understandings about how rhetorical devices are used to promote ideologies.

An example is how the Norwegian minister of education and research, Torbjørn Røe Isaksen, used Fröbel explicitly in a public debate in a Norwegian newspaper (Isaksen, 2014). Lange and Meaney (2016) argue that Isaksen uses Fröbel to promote mathematics in kindergarten as a common sense understanding which has always been a part of kindergarten. Thus, Isaksen uses a reflection of Fröbel and his mathematical 'gifts' to argue that having more mathematics in ECEC is a continuation of an existing tradition. By doing this, he tries to influence the trajectory of the swinging pendulum so that it swings more towards the pole of instruction and away from current kindergarten policy, which has been swinging more towards the constructive approach (Lange & Meaney, 2016).

Another example is from Sweden at the end of the nineteenth century, from a magazine for women about their home life. In this example, Fröbel is on one hand, used to argue for readiness for school and working life but on the other hand his toy gifts can make the child able to bring gifts to their family (Cristel, 1892). The author of the article reflects Fröbel, his gifts and view of mathematics, into the article in a different way than Isaksen did in 2014. Cristel (1892) argues that families can purchase the boxes of gifts "and in that way let the ideas of this high-minded pedagogue into their home" (p. 244, own translation). This again creates a different trajectory for the pendulum. The purpose of the gifts is presented here as a way of bringing the child closer to their family, and making them able to produce something by their own desire in their own lives. The pendulum here swings more towards the constructive pole.

The aim of my project is to look for reflections of Fröbel in Scandinavian kindergarten history and to identify how these reflections have been used to create momentum for the pendulum that swings in different directions. By studying the relationship between production, form and reception of a discourse, I can investigate the myriad of effects around the discourse in question (Fairclough, 2003). Initial investigations suggest that Fröbel is reflected in different ways, in order to argue for different agendas about how and why we should do mathematics in ECEC. As a mirror can reflect an image that is distorted or mostly correct, the reflections of Fröbel are more or less representative of his original views. However, this project is not about judging how valid the different reflections are. Rather by critically analysing the discourses on Fröbel and his ideas, I can identify factors that affect the paradigm shifts about mathematics in kindergarten.

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Collaborative learning in multigrade mathematics education at primary level

Rachel-Ann Friesen

Technische Universität Dresden, Faculty of Education, Dresden, Germany;

rachel-ann.friesen@tu-dresden.de

Keywords: Multigrade classes, collaborative learning, teacher training.

Background

In Saxony, predominantly in rural areas, social and political challenges are being posed by a declining number of pupils. To prevent especially younger pupils from having to travel long distances to school, organisational and curricular alterations are being considered. One suggestion is the merging of grades to establish multigraded classes. Multigrade education is also being discussed for its pedagogical advantages; for example, it can make the first phase of primary school more flexible by allowing pupils to remain within the same multigraded class for either one, two or three years. However, even in learning groups which are homogeneous in age, the differences between pupils' preconditions for learning can be up to four years (Hirt & Wälti, 2008). Yet, when making learning groups heterogeneous in age, the diversity of learning preconditions increases, and with it also the necessity for differentiation through diverse tasks with various levels of difficulty. For a subject like mathematics, which is strongly guided by a systematic course, this often means extremely individualised and separated learning (Nührenbörger, 2007). However, interaction is seen as foundational constituent of learning in early years (Miller, 1986; Schütte, 2009) and should not be neglected. More specifically, for mathematical learning to occur it is of great importance for the pupils to participate increasingly in collective reasoning within classroom interaction (Krummheuer, 2011). To enable individualised learning to take place in cooperation with others, *substantial learning environments* hold great potential both for mathematics education in general and specifically in multigrade mathematics education because they offer the opportunity for *natural differentiation*, which means that learners all work on the same task and the differentiation is not predetermined by the teacher but chosen by the pupils themselves (Scherer, 2013). The possibility for students to learn with and from others by communicating and helping each other is also seen as one of the pedagogical arguments for purposeful mixed-age grouping (Wagener, 2014).

Project

The question this research project wants to address is how collaborative learning takes place in multigrade mathematics education at primary level. Therefore, firstly student teachers at the TU Dresden are asked about their experiences with multigrade education by using a questionnaire. Then, an interview study with individual teachers who teach in multigraded classes in Saxony will be conducted in order to identify their attitudes and concepts concerning multigrade mathematics education. Based on these empirical results, concepts and learning environments for multigraded learning within mathematics education for grades 1–6 will be developed in cooperation with teachers and student teachers in order to prepare them for the future challenges they will face when teaching mathematics in multigraded classes. Later these learning environments will be applied with pupils

in practice to test the suitability of the tasks. The collaborative processes the pupils use to solve the tasks will be filmed, transcribed and then analysed from an interactionist perspective (Krummheuer, 2011) to reconstruct constituent characteristics of successful multigrade mathematics education. This process will be accompanied by the development of seminars for student teacher concerning collaborative learning in multigrade mathematics education.

Initial results

The initial results of a questionnaire with student teachers at the TU Dresden show that less then 40% have experienced multigrade education in some form (e.g. during an internship, while being a student themselves). Even though they are all able to reflect about possible advantages and disadvantages of multigrade education, many of them say that they would feel overwhelmed, unsure or not prepared to teach in a multigraded class during their traineeship. These results need to be assessed in more detail but they emphasize the necessity of training teachers more specifically for multigrade education.

Acknowledgement

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Supporting young children's mathematical register learning in two languages: ICT possibilities

Troels Lange[1] and Tamsin Meaney[2]

Western Norway University of Applied Science; [1]trl@hvl.no; [2]tme@hvl.no

Keywords: ICT, kindergarten, bilingual mathematical language development, parents.

Language as a resource

We have recently begun a project where we assume that digital games/apps have the possibilities to utilise family and kindergarten language resources to support multilingual children's development of mathematical registers. In so doing, we adopt the perspective of "language as a resource" in which languages act as pedagogical resources in the learning of mathematics (Planas & Civil, 2013). Developing children's fluency in their home language as well as Norwegian is an aim of the Norwegian Framework Plan, the early childhood curriculum (Kunnskapsdepartementet, 2011). Research suggests that in regard to learning and using content knowledge such as mathematics, multilingual children benefit from developing both their home language(s) and the majority language for discussing abstract ideas (May, Hill, & Tiakiwai, 2004). However, when children speak a different language at home to the one in their kindergarten, it is difficult for teachers who do not speak the children's other languages to know how to provide support for developing the home language. Parents on the other hand have the language resources for developing the mathematics register in the children's home languages, although their contribution to their children's engagement with mathematics is often under-rated (Civil, Bratton, & Quintos, 2005).

Digital games as prompts for mathematical language learning at home

There is little research about using apps/games to prompt dialogue about mathematics among young children, however what there is suggests that affordances of digital games/apps can promote discussions (Lange & Meaney, 2013; Lembrér & Meaney, 2016; Palmér & Ebbelind, 2013). Plowman, Stephen, and McPake (2010) found that children's engagement with digital tools at home were richer than those in kindergartens, partly because the children asked more questions and could learn from watching other family members using the devices.

In order to find out what digital games/apps were being used in multilingual families and to gain some understanding from parents of multilingual children (aged 1-5 years) about what they considered to be the mathematical register demands and affordances of the apps/games, we set up an online survey, using a snowballing sampling method. The survey had 8 questions that asked about the age of the children, the languages that they spoke and the digital games/apps that they played with.

56 parents identified the features of the digital games/apps that made them attractive to their 74 children and what they would include in a digital game/app if they were designing one. The features of the digital games that the parents would include were not the same ones as they recognised that their children valued. Digital games were recognised as providing children with opportunities to use aspects of the mathematics register. Although puzzles was frequently cited as a digital game that the children played, this did not seem to result in the features of shapes being discussed to the same

degree as other mathematical ideas (about half the number of parents considered their children talked about this as compared with attributes for classifying things).

The parents identified that being successful or being surprised prompted children to talk about the digital game. Having features in new digital games which would produce such feelings in children appears likely to develop discussion with adults about mathematical ideas. As the language of the game can influence the children's use of language, it seems best not to include any language so as not to restrict the choice of language.

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Bangladeshi rural secondary madrasa (Islamic school) children's participation in higher mathematics optional course: A close look through gender lens

K.M. Nabiul Alam

UCL Institute of Education, University of London, United Kingdom; nabiul40@yahoo.co.uk

Keywords: Bangladesh, gender, higher mathematics, influence, rural madrasas.

Background

Compared to boys, a small number of girls study higher mathematics optional course in Bangladeshi rural secondary madrasas beyond the requirement of compulsory mathematics. The overall participation rate of girls in higher mathematics is also low when compared to other optional subjects.

Research questions

1. How do Bangladeshi rural secondary madrasas influence children's participation or non-participation in higher mathematics optional course?
2. How do these influences vary with respect to children's gender?

Research design

Quantitative data have been collected from 500 children of grades 9 and 10 across eighteen rural madrasas using a Bangla translated version of the Fennema-Sherman Mathematics Attitudes Scales (Fennema & Sherman, 1976). Qualitative data have been collected from three case studies using focus group discussions with fifty children and semi-structured interviews with three principals, five math teachers and fifteen parents. Nine mathematics lessons have also been observed to capture teacher-student interactions and their engagement. Purposive sampling process, rather than recruiting a representative sample, was used in order to reflect the aspect of diversity in the population and to carry out comparisons among different groups.

Conceptual framework

Following conceptual framework is adapted from Ker (2016) to present multiple levels of factors for mathematics participation and achievement (in Table 1). Selection of most of the factors is based on the suggestions from educational effectiveness literature. From the economic point of view, educational effectiveness is a production process of schools or madrasas, which is a transformation of inputs to outputs. The inputs include school resources, students' characteristics and instructional hours. The process includes school factors, teacher factors and student factors as shown in Table 1. The outputs can be measured by the number of children participating in higher mathematics course and their achievement in the subject. Therefore, madrasa and classroom characteristics, student's background and home environmental support, time allocated for each subject, and teacher's instruction to engage the students are very crucial to achieve the popularity of an academic course like higher mathematics.

Table 1: A multilevel conceptual framework for mathematics participation (and achievement)

School level →	**School climate** -School academic climate -School discipline & Safety	**School Resources/ Instructional Hours** -Instruction affected by math resource shortage -Total instructional hours per week/year -Computer availability -School composition by student background		
Teacher level →	**Teacher Preparation** -Years of teaching experience -Career satisfaction -Confidence in teaching maths	**Instruction** -Collaboration to improve teaching -Instructions to engage students -Math instructional hours per week	**Climate/Facility** -Emphasis on academic success -Safe and orderly school -Working conditions	
Student level →	**Student Background** -Gender -Parents' highest education level -Parents' profession	**Environmental support** -Home educational resources -Number of home study support	**Student's school experience** -Students engage in math lesson -Students bullied at school	**Motivation/ Attitude** -Students like learning maths -Students value learning maths -Student's confidence in maths

Themes identified in preliminary analysis

- There is a shortage of good professionally committed teachers for higher mathematics. Most of the mathematics teachers are engaged with private tuition as an extra earning source. Children have no other option than taking private tuition as the support from madrasa for both general and higher mathematics is inadequate.
- Parents' religious belief is one of the reasons for sending their children to madrasas instead of usual schools. Parents wish some of their children to become an Imam, a teacher in a madrasa or an Islamic scholar. They want their daughters to learn Quran reading and hadith, pray Salah regularly and get married with a man of their choice as soon as possible. In doing so, studying higher mathematics is not in their priority list as their daughters and even sons can pass other subjects easily.
- Children seem to have confusions and frustrations about the value of their madrasa education compared to the education from general schools as there are local discourses that government is less interested about their madrasa qualifications.

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Culturally-inspired mathematics and science education

Shipra Sachdeva[1], Per-Odd Eggen[2] and Halvor Hoveid[3]

[1]Norwegian University of Science and Technology, Program for Teacher Education, Trondheim, Norway; [1]shipra.sachdeva@ntnu.no; [2]perodde@ntnu.no; [3]halvor.hoveid@plu.ntnu.no

Keywords: STEM education, culturally relevant education, student centered curriculum, intercultural communication, student diversity.

Background

The relation between culture and mathematics education has been evident since long but taking advantage of pupils' cultural identity to increase students' engagement has not been focused much in Norway. Culture is closely interwoven with children's routine. It can provide a gateway to connect mathematics and science education to their daily life experiences. Culture, in this project, is not mainly or only centered towards pupils' ethnicity but will employ their personal cultural identity (interests, leisure activities, hobbies etc.) as a stimulating source for designing the teaching activities. Due to increased immigration, many different cultures have been introduced in Norwegian schools. Students having various cultural backgrounds may feel alienated in classroom activities and find it difficult to follow current classroom teaching. School leaders in Norway are also familiar with these increasing hetrogenous cultural issues and thus have began accepting and implementing culturally-relevant changes in their respective school systems (Jacobson, Johnson, Ylimaki, & Jacobson, 2013).

The LOCUMS (Local Cultures for Understanding Mathematics and Science) project aims at finding out how students' engagement in mathematics and science gets affected when they themselves are asked to influence the starting point of teaching-learning process. Initiating their education process by first asking the students to provide relavant input about *their own culture,* for example, likes, interests and leisure activites to the teachers and researchers, they are assumed to get an autonomy, control and a part of responsibility of what *they want* to learn. Their desires will plausibly serve as an inspiration to the educators for designing the teaching-learning activities, and also as a possibility for the pupils to bring forward their own cultural identity in the class. Ascribed to the fact that commencement of teaching-learning situation would be derived by students' interests and activities they want to learn more about, we consciously use the term *culturally-inspired mathematics and science education* to describe the attention of this project. This notion allows us to bring in culturally-responsive mathematics and science teaching in the form of interactive cooperation between teacher and students in Norwegian multicultural classrooms.

Theoretical framework

The aim of my project is to study changes happening in the knowledge gaining process of pupils as a result of engaging them in culturally-inspired activities surrounded by intercultural context (students work in groups). Therefore, we find socio-cultural and cultural-historical activity theories to be relevant. In addition, we will draw on research related to ethnomathematics, culturally-relevant pedagogy and instruction (Ladson-Billings, 1994; Rajagopal, 2011), and culturally-responsive teaching (Gay, 2010), as they suggest an educational approach that advocates valuing students' cultural background and prior experiences in the same way as socio-cultural theory. These themes favor cherishing cultural diversity present in the class to enrich the socio-cultural surroundings of

diverse students so that they can learn effectively. Nevertheless, it does not mean that the teacher should teach in a "black" or "asian" way, but the level of educational activities should be reachable from and meaningful for students' personal level of understanding. Simultaneously, involving their interests in planning the lessons can motivate them to learn using their own culture. Therefore, we belive that this literature would enrich and facilitate our project to enhance the learning experiences of diverse children in Norwegian classrooms.

Methodology

Being a problem driven pragmatic and empirical research, the aims to be fulfilled during the course of this project are justified by using a combination of design-based and action-based research methodology. The first student projects were designed and finalised with the cooperation of teachers, school leaders and students and, as the participators would be the practitioners as well, it shapes itself as action research. An iterative design cycle (3 repetitions) would be employed for each of the three planned student projects we plan to work out throughout the project. Until now, we have collected data from the first iteration of our initial student project.

Plan of action

Some of the classroom interventions are planned to be executed in a multicultural school in Trondheim. In the first trial, pupils' input on their aspirations was collected through questionnaires. Accordingly, teachers and researchers planned and implemented the new teaching-learning strategies. The classroom activities were audio and video recorded. Teachers and researchers will now review, analyze and reflect those recordings to find and correct up eventual shortcomings in the first attempt. Further, two following iterations would adopt similar pattern as the first one and each part of data collected would be observed both before and after each student project to capture the changes in students' engagement, participation and learning. Data analysis will be carried out in the light of socio-cultural and activity theories.

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Being in the diversity and going through the infinity

Cássia Aline Schuck[1] and Cláudia Regina Flores[2]

[1]*Federal Institute Catarinense*, Blumenau, Brazil; cassia.schuck@blumenau.ifc.edu.br

[2]*Federal University of Santa Catarina*, Florianópolis, Brazil; claudia.flores@ufsc.br

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Introduction

The debate on visual culture raises questions that have been stressed in the educational research, such as the role of the image in the process of subject-formation and the visuality as a form of knowledge. Visual Culture is an interdisciplinary field that combines arts, philosophy, anthropology, and cultural studies. The main point of this field is the question of visuality, which means focusing on the relationship between the seen and the seer. In short, the target of Visual Culture Studies is not an object at all, but it "consists of things we can see or whose existence is motivated by their visibility; things that have a particular visuality or visual quality that addresses the social constituencies interacting with them" (Bal, 2003). Thus, the term of visuality has become an important keyword for this field (Mirzoeff, 2006), because it refers to the vision and visuality are both socially and historically constructed.

With regards to mathematics education research, Flores (2012) has proposed this perspective of visuality to investigate how images, particularly paintings, affect and are affected by Cartesian perspectivalism visual practices, as well as to grasp the fabrication of a certain mathematical rationality to look and to think within the realms of culture. The author has considered

> the term 'visuality' instead of 'visualization', because the former leads to a deconstruction of the founding principles of sense of vision and perception. In contrast, visualization is understood as a process of construction and transformation of mental images, whereas visuality is the sum of discourses that inform how we see. Thus, while the latter is concerned with learning geometry's concepts and visual skills, visuality discusses visual practices in the context of history and culture. (p. 7060).

Thus, at stake is a kind of mathematical thinking presents in contemporary school practices and that is provoked notably itself through images of arts (Flores, 2016).

Going through infinity

In response to this, we have been examining historically the artistic practices of infinity considering, particularly, those involving the discussions of perspective in renaissance painting. Furthermore, we have analysed modern paintings exhibiting characteristics of a classical system of visuality, which means using concepts of harmony, symmetry, parallelism and perspective. On a whole, we have found not only that our way of looking is shaped within an already built field of techniques and discourses, but also that mathematics ideas play an important role in the constitution of how we think.

Being in the diversity

In the methodological path of cartography, Schuck (2015) has developed an intervention plan with alcohol and drugs dependents in treatment on a Psychosocial Care Center in Florianópolis, Brazil, in 2014. Four workshops were conducted and each one was centered on images suggesting the infinity idea. By focusing on discourse analyses, interpreting multi-faceted narrative, looks and affections by the subjects involved, it was possible to highlight the effects of looking at infinity rather than 'truth' *per se*. For instance, seen from the perspective of the people involved, the ways of looking at infinity deal with reflections situated between the mystical and the emotional experiences such as nothingness, emptiness, freedom. In this regard, we call attention to two points: "mathematical mind-set" is not the sole result of a schooling regime; and subject's visual subjectivities emerge in the entanglement of the individual in discursive formations. In order to discuss this, we, in the poster, displayed some information concerning on the visualization and visuality, the cartography as research methodology, the workshops itself, and the discourses about infinity staged by the people involved in the intervention mentioned above.

Following Foucault's assumptions, we consider that by a better understanding of how a certain kind of mathematical visualization has been constituted within the sociocultural practices might contribute not only teacher education but also mathematics learning practices. In sum, we bring under our attention that both mathematical rationality and visualization are staged and configured within the diverse sociocultural practices.

Acknowledgment

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TWG11: Comparative studies in mathematics education

Introduction to the papers of TWG11: Comparative studies in mathematics education

Paul Andrews, Eva Jablonka, Miroslawa Sajka and Constantinos Xenofontos

As with earlier CERMEs, TWG 11 adopted an eclectic perspective in its interpretation of comparison as referring to any study that documents, analyses, contrasts or juxtaposes cross-cultural or cross-contextual similarities and differences across all aspects and levels of mathematics education. In this way the TWG aimed to encourage critical but supportive discussions around a number of predetermined and emergent themes.

A recurrent but very productive aspect of this working group has been the relatively small number of paper presentations. This year ten papers and three posters created space not only for colleagues to share their research in detail but allowed participants to engage in lengthy and inclusive discussions on the nature of comparative mathematics education research and the means by which it can be meaningfully and rigorously undertaken. As is shown below, the different paper contributions fell naturally into four themes, each reflecting significant substantive and methodological variation.

Firstly, two papers, with very different foci and methodological conceptualisations, framed a discussion on the importance of identifying an appropriate unit of analysis. This issue, while of importance in all research, is particularly foregrounded in comparative research. One of the two papers, Sajka's eye-tracking examination of different response groups' visual attention to the statement of a mathematical problem showed that such groups respond in culturally conditioned ways. The second of these two papers, Clarke, Mesiti, Cao and Novotna's study of the culturally-located variation in the vocabulary of the typical school mathematics classroom, foregrounded the importance of understanding that much mathematic didactics vocabulary may be culturally unique. Importantly, both studies exposed hitherto unconsidered demands with respect to what is actually being analysed.

A second theme could be found in the three papers construed as having a tacit focus on beliefs. Firstly, Andrews and Xenofontos presented a quantitative analysis of Cypriot and Greek initial teacher education students' understanding of a hypothetical solution to a non-arithmetical linear equation. They found considerable emphases on rote solution methods although Cypriot students were more articulate on the matter than their Greek colleague. Secondly, Koljonen's case study of one Swedish teacher's deployment of the teacher guide associated with a translated Finnish textbook showed how teachers' responses to such materials, irrespective of any instructional material they contain, are interpreted by culturally-determined expectations of what is appropriate. Thirdly, Nosrati and Andrews interviewed Norwegian and Swedish upper secondary students about their experiences of school mathematics and found a dominant perception that the purpose of school mathematics is to prepare students for a successful economic engagement with the world. All three studies highlighted well the extent to which mathematics classroom participants' perspectives are informed by culturally-located beliefs.

A third theme, also reflected in three papers, concerned mathematics teacher knowledge. In the first of these, Kingji-Kastrati, Sajka and Vula used an extant test to examine Kosovar and Polish teacher education students' knowledge of fractions. In the second, Tchoshanov, Quinones, Shakirova,

Ibragimova and Shakirova used a battery of TIMSS-derived test items to examine differences in US and Russian lower secondary teachers' content knowledge. Both studies highlighted substantial differences in the content knowledge of the groups under scrutiny. Finally, Xeonfontos and Andrews examined Cypriot and Greek students' didactical explanations of the same hypothetical solution to a non-arithmetical linear equation as discussed above. In this case, Cypriot students' explanations were more didactically robust than their Greek colleagues. All three papers confirmed the extent to which mathematics content knowledge is a not the culturally independent body of knowledge assumed by international studies of student knowledge.

The fourth theme drew on two papers framed by the anthropological theory of didactics (ATD). Firstly, Modeste and Rafalska's drew on ATD's of didactical transposition, or the transformation of academic knowledge to that knowledge taught, to highlight differences and similarities in the presentation of algorithmics in the curriculum materials of Ukraine and France. Secondly, Asami-Johansson, Attorps and Laine exploited ATD's concept of praxeology, which provides the methods for solving a domain of problems (praxis) and a structure (logos) on the discourse those methods, to compare the practices of case study teacher educators in Japan, Finland and Sweden. The two studies, in addition to highlighting substantial differences between the cultural groups under scrutiny, showed how different elements of ATD can be productively employed in cross-cultural studies.

Finally, three posters, each with different foci and methodologies, were presented. Haara, Bolstad and Jenssen, as preparation for a later Norwegian study, presented a research review on mathematical literacy in school. Istúriz, González-Ruiz, Diego-Mantecón, Recio, Búa, Blanco, González and Polo reported on an Erasmus project in which students in different countries develop activities to integrate art into STEM activities. Finally, Tesfamicael, Botten and Lundeby presented a comparative analysis of Norwegian and Ethiopian textbooks presentation of relations and functions.

Overall, the papers and posters presented to the group reflected not only cultural diversity but also methodological pluralism. For example, studies included those that were informed by a priori theoretical assertions and those that were not. There were equal numbers of qualitative and quantitative studies focused on a range of aspects of children's and teacher education students' learning of mathematics. All studies confirmed the extent to which mathematics and its teaching and learning are culturally normative.

Beginning teachers' perspectives on linear equations: A pilot quantitative comparison of Greek and Cypriot students

Paul Andrews[1] and Constantinos Xenofontos[2]

[1]Stockholm University, Sweden; paul.andrews@mnd.su.se

[2]University of Nicosia, Cyprus; xenofontos.c@unic.ac.cy

In this paper we examine aspects of beginning primary teachers' understanding of linear equations. First-year teacher education students on a programme in Cyprus (12 Greek and 21 Cypriot) were shown a solution to the equation $x + 5 = 4x - 1$ comprising four rows of mathematically correct algebra but no commentary. They were asked to explain, in writing, the solution to α friend who had missed the lesson in which such equation solving processes had been taught. Analyses found almost all students, irrespective of nationality, writing about knowns and unknowns before offering a 'change the side and change the sign' rule. However, a major difference was that Cypriot students' accounts typically included an objective for the equation solving process, which the Greek students' did not.

Keywords: Linear equations, comparative research, Cyprus, Greece, teacher education.

Introduction

The topic of linear equations occupies an important position in students' learning. It "stands on the border between mathematics as concrete and inductive and mathematics as abstract and deductive", offering "one of the first authentic opportunities for them to connect their understanding of arithmetic to the symbolism of mathematics" (Andrews & Sayers, 2012, p.476). Yet, it is a difficult topic to teach well, because when learning arithmetic, learners typically come to see the equals sign as an instruction to operate (Kaput et al., 2007). This operational perspective (McNeil & Alibali, 2005) creates few problems with respect to arithmetical equations, with the unknown in one expression, because it supports a process of operation reversal (Herscovics & Linchevski, 1994; Kieran, 1992). However, non-arithmetical equations - the unknown in both expressions - requires a relational (Kieran, 1992) understanding of the equals sign as an assertion of equality between two expressions (Alibali et al., 2007; Filloy & Rojano, 1989) in order that they can operate on the unknown as an entity. In short, students find equation solving problematic because operational perspectives on expressions like $3x+1$ prevent their being construed as objects subject to, in relational terms, operations themselves (Kieran, 2004). Furthermore, to compound students' difficulties, teachers' practices frequently collude in the maintenance of an operational perspective on the equals sign (Haimes, 1996; Harel et al., 2008; Stephens, 2008), highlighting a need to evaluate the equations-related understandings that beginning teachers bring to their courses. This reflects the aim of this pilot study and an atypical approach, which is described below.

Perspectives on the teaching and learning of linear equations

Typically, research on equation solving has focused on approaches to the solution of non-arithmetical equations, not least because their solution poses few conceptual difficulties. In this respect, the most widely criticised is redistribution, a rote-learned, *change the side, change the sign* procedure (Nogueira de Lima & Tall, 2008), focused on transposing the equation so that the

unknown finishes on the left-hand side and a value on the right (Filloy & Rojano, 1989). The unknown's arbitrary leftwards movement perpetuates operational conceptions of the equals sign and fails to support students' understanding that such movement does not change the equation's equality (Capraro & Joffrion, 2006). Other approaches, like trial and improvement, support an understanding of the relational nature of the equals sign and the role of the unknown in context (Knuth et al., 2005). However, while it may be an appropriate initial strategy in a teaching sequence, it is inefficient and does not support the learning of general equation solving strategies (Filloy & Rojano, 1989). Other approaches(see Dickinson and Eade, 2004; Fong & Chong, 1995), present the equation as two rows of mathematical objects, one laid on top of the other, as in the representation of $2x + 10 = 4x + 2$ shown in Figure 1. Here, the authors claim, students can see easily how it reduces to $2x + 2 = 10$; an equation is amenable to an operation reversal procedure.

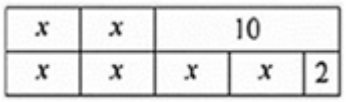

Figure 1: A representation of $2x+ 10 = 4x + 2$

However, while such approaches be procedurally helpful and support students' understanding of equations as manipulable objects, they may hinder students' understanding of the invariance of the solution. Moreover, despite Dickinson and Eade's (2004) optimistic arguments otherwise, they remain problematic with respect to negative coefficients (Marschall & Andrews, 2015).

Finally, in studies of teachers' unprompted approaches to equation solving, the balance scale has been the most widely reported, being the approach of choice in case studies from, for example, Canada (Haimes, 1996), Finland, Flanders and Hungary (Andrews & Sayers, 2012), New Zealand (Anthony & Burgess, 2014) and Poland (Marschall & Andrews, 2015). Here the solver manipulates, through addition or subtraction, weights on scale pans, while keeping the scales in balance. Its advocates argue that it helps students see the equation as a whole and not an instruction to operate (Warren & Cooper, 2005). Moreover, it supports an understanding of the need to do the same to both sides (Anthony & Burgess, 2014) and underpins the symbolic foundations of later algebraic formalisms (Andrews, 2003). Systematic attempts to evaluate the balance's efficacy have shown that it helps students to understand the principles of equations, solve non-arithmetical equations with understanding, particularly from the perspective of doing the same thing to both sides (Araya et al., 2010; Warren & Cooper, 2005) and facilitates students' acquisition of an appropriate vocabulary (Vlassis, 2002). Its critics argue that it cannot represent negatives in anything but a contrived way (Pirie & Martin, 1997), a criticism supported studies showing teachers simulating the tying of helium filled balloons to scales to counter the weight of objects in the scale pans (Anthony & Burgess, 2014).

In this paper we explore how students following an initial primary teacher education programme in Cyprus construe non-arithmetical linear equations. Due to Greek being the language of instruction, the programme includes both Cypriot and a high proportion of Greek students.

Methods

Shortly after the start of their course and before they had been exposed to university mathematics teaching, students were shown a solution to the equation $x + 5 = 4x – 1$ and asked to write a short

account of how they would explain it to someone who had missed the lesson in which it was introduced. The solution, with no additional narrative, was presented as follows

$$\begin{aligned} x + 5 &= 4x - 1 \\ 5 &= 3x - 1 \\ 6 &= 3x \\ 2 &= x \end{aligned}$$

A non-arithmetical equation was used for several reasons. Firstly, it could not be solved by means of a reversal of operations. Secondly, it should expose, in ways that an arithmetical equation could not, the underlying equations-related conceptions students bring to their courses. Thirdly, it would uncover the extent and depth of their equations-related procedural knowledge as, at each step, they would need not only to interpret and explain the solver's hidden thinking but decide what would need to be made explicit to the unknown learner. It was believed that this would be a more effective means of uncovering students' familiarity and understanding of the topic than a conventional test item and expose any pedagogical predisposition they bring to their course, as a result of their previous schooling.

Analysis

Students' transcripts were subjected to a constant comparison analysis (Fram, 2013), whereby a transcript was read and re-read to identify different equations-related understandings. This was followed by the next transcript being read and re-read in order to find further evidence of the original codes and any new ones not seen in the first. If new categories were found then the earlier transcript was re-read in case they had been previously missed. This process continued for all 33 transcripts, a number typically thought sufficient to achieve categorical saturation, and yielded seven categories of understanding, which can be seen in Table 1.

With respect to demonstrating the emergence of these categories, we turn to Ekaterini, a female Cypriot student, who wrote that

> To solve this exercise we have to first set apart the known from the unknown numbers. The known numbers are the ones that don't include a letter, as for example, 5 and -1. Unknown numbers are the letters or the numbers that are accompanied by a letter, for example, 4*x* and *x*.

Ekaterini's comment about separating the known from the unknown implies an implicit objective for equation solving; in essence, solving an equation entails precisely that. In the same sentence can also be seen evidence of her awareness of the unknown and its role in equation solving. In the second and third sentences she goes further and defines an unknown. She then wrote (her parentheses):

> To separate the two, the known numbers should be on one side of the equation, for example, on the left side, as we solved it in the class, while the unknown numbers should be on the other side. Later, I added 5 + 1 (whose sign has changed because it moved to the other side of the equation) and subtracted 4*x* – *x* (again the sign changed because *x* has moved to the other side of the equation and when there is x or y alone this means 1*x* or 1*y*). Finally, I reached 6=*3*x and so I divided 6 by 3 so that *x* is equal to 2.

Throughout this paragraph runs a rote procedure for equation solving invoking two simple instructions. The first is that knowns must be moved to one side of the equation and unknowns to the other. The second is that when an object moves from one side of an equation to the other its sign changes. Finally, having achieved this objective, an understanding of the role of inverse operations is invoked to divide the total of the knowns by the coefficient of the unknowns. Within this procedure, as with many of the students' suggestions, is evidence of flexibility in that it does not matter to which side of the equation which type of object travels, implying that it may be a matter of convenience.

Articulating an awareness of the unknown	31
Defining what is meant by an unknown	15
Offering an implicit objective for equation solving	18
Offering an explicit objective for equation solving	1
Offering a rote procedure for equation solving	29
Articulating an understanding of the role of inverse operations	26
Offering an unspecified process for equation solving	4

Table 1: The seven categories of understanding elicited from the data

With respect to the remaining two categories, only one student offered an explicit objective for equation solving. In this case, Irene, a Cypriot female, wrote that the "question in this equation is to find 'x' and what value it has". Finally, four students wrote of an unspecified procedure. For example, Moira, a Greek female wrote, "I would tell the student that we separate the known from the unknown numbers and then make the calculations". From her comment we inferred three perspectives on equation solving; an implicit objective tied to separating the known from the unknown, an awareness of the unknown and an unspecified procedure. The seven categories of response, along with their respective frequencies, can be seen in Table 1.

Results

The figures of Tables 2 and 3 show which of the seven categories identified by the constant comparison analysis were found in the accounts of the Greek and Cypriot students respectively. From these can be inferred both similarities and differences. With respect to the former, several similarities were identified. Firstly, with a single exception in each country, students' explanations showed an explicit awareness of the unknown. Secondly, a very high proportion of students from both countries - only four students did not - offered a rote procedure for solving the equation and of these, all focused on the mantra, change the side change the sign. Thirdly, only five students did not write in ways indicative of their understanding inverse operations, although this was typically seen with respect to explaining how the solution is reached from the point where $6=3x$, as in Chloe's comment that "after reaching 6=3x, we divided 6 by 3 so that x will be by itself". Fourthly, around half of all students in each group defined what they meant by an unknown

.

Greek student	A	B	C	D	E	F	G	H	I	J	K	L	12
Aware of unknowns	1	1	1	1	1	1	1	1	1		1	1	11
Defines unknowns		1			1	1			1		1	1	6
Implicit objectives						1					1		2
Rote procedure	1	1	1		1	1			1	1	1	1	9
Inverse operations		1	1		1	1	1			1		1	7
Unspecified process				1			1	1					3
Explicit objectives													0
	2	4	3	2	4	5	3	2	3	2	4	4	*3.17*

Table 2: Distribution of Greek students' equations-related insights

With respect to differences, only two Greek students offered any sense of objective, albeit implicit, for the equation solving process compared with 17 of the Cypriot, of which only one, student F, offered an explicit goal.

Cypriot student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	21
Aware of unknowns	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	1	1	1	1	20
Defines unknowns	1		1	1				1							1	1	1		1		1	9
Implicit objectives	1	1	1	1				1	1		1	1	1	1	1		1	1	1	1	1	16
Rote procedure	1	1	1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	20
Inverse operations	1	1		1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	1	19
Unspecified process				1																		1
Explicit objectives						1																1
	5	4	4	5	3	4	3	5	3	3	4	3	4	4	5	4	5	4	5	4	5	*4.1*

Table 3: Distribution of Cypriot students' equations-related insights

The figures of Tables 2 and 3 show also that Greek students' data yielded fewer codes per student, 3.17, than their Cypriot counterparts mean of 4.10. Indeed, at the upper end of the spectrum the accounts of seven Cypriot students, A, D, H, O, Q, S and U, yielded five categories of response, compared with that of just one, F, Greek student. At the lower end, the accounts of four Greek students, A, D, H and J, yielded only two categories each, compared with zero Cypriot students. These differences were statistically significant in two ways. Firstly, a t-test showed differences in the mean number of codes were unlikely to be due to chance ($t = 2.95$, $p = 0.006$). Secondly, a chi-square test performed on the data in Table 4 confirmed that variation in the number of codes yielded by each of the Greek students and Cypriot students respectively were unlikely to be due to chance ($\chi 2 = 9.15$, $df = 3$, $p = 0.027$). This difference in the codes, we argue is likely to be a consequence of differences in how the two systems introduced their students to linear equations; a possibility warranted by, for example, evidence that Cypriot students tended to specify objectives in their accounts in ways that their Greek colleagues typically did not. Finally, of the eight students whose accounts yielded five codes, seven yielded the same five. That is, they indicated an awareness of and defined the unknown; they offered implicit objectives and a rote procedure alongside an awareness of inverse operations related to division. In short, the most complete responses were typically the same.

	Number of codes per student				
	2	3	4	5	
Greek	4	3	4	1	12
Cypriot	0	5	9	7	21
	4	8	13	8	33

Table 5: Distribution of total number of codes per student per country

Discussion

The aim of this pilot study, the quantitative analyses for which are presented in this paper, was to explore the equations-related understanding primary teacher education students bring to their courses and, in so doing, evaluate the effectiveness of a simple to implement tool for later comparative use. In this instance comparison was made possible by the fact that courses in Cyprus are taught in Greek, making them accessible to Greek students. The results are methodologically encouraging but, acknowledging the fact that all respondents were prospective teachers, mathematically worrying, albeit with some qualifying strengths.

Methodological encouragement stems from the evidence that students responded positively to the invitation and produced written accounts sufficient to expose their perspectives on or conceptions of linear equations. It was also encouraging that the tool was able to discriminate between the two cultural groups in its highlighting similarities and differences in students' accounts that, we infer, reflect systemic differences in the ways in which linear equations had been experienced by these two sets of students as learners of school mathematics. In short, the tool proved fit-for-purpose.

The mathematical disappointment derives in part from the lack of any evidence of students holding a relational (Kieran, 1992) conception of the equals sign, in that nothing said by any student indicated an understanding of the equals sign as an assertion of equality between two expressions (Alibali et al., 2007; Filloy & Rojano, 1989). Mathematical disappointment also derives from the very high proportion of students in both countries who seemed to construe equation solving as a rote process of 'change the side and change the sign'. That is, the majority of students appeared to have a procedural rather than a conceptual perspective on equations in which symbols are moved around "with a kind of additional 'magic' to get the correct solution" (Nogueira de Lima & Tall, 2008, p.4). Indeed, even those students whose accounts yielded the most categories of response presented procedural perspectives with implicit objectives and rote procedures. However, in contrast with international research showing the balance as teachers' preferred representation (Andrews & Sayers, 2012; Anthony & Burgess, 2014; Haimes, 1996; Marschall & Andrews, 2015), no reference to the balance was made by any student. Indeed, with the exception of their implicit awareness of inverse operations, which we discuss below, nothing written by any student indicated a narrative based on performing the same action to both sides of the equation.

Despite the negatives, there were some interesting positives. The majority of students, particularly the Cypriot, indicated an awareness of the role of inverse operations. In every case this occurred at the point in the solution where the step connecting $6=3x$ to $2 = x$ was discussed. Here, students indicated, albeit implicitly, an awareness that division was the inverse operation to invoke, insights that seem to confound the mechanical procedure of 'change the side, change the sign'. Also, despite the highly procedural nature of their accounts, only two students, one from each country, did not

demonstrate an awareness of the unknown. That is, students not only used an appropriate vocabulary but were generally aware of the function of the unknown in the equation solving process. Indeed, around half of all students from both groups offered a definition, typically implicitly, of the unknown, as in Carissa's account in which she wrote, "I separated the known from the unknown numbers. Unknown x + 5 (known) = unknown 4x -1 (known)".

Finally, Cypriot students' accounts yielded more response categories than their Greek colleagues, typically due to their tendency to offer objectives, again implicit, which their Greek colleagues did not. We speculate that such a difference may reflect cultural teaching norms; while the evidence of these students' accounts indicates the outcome of procedural teaching this particular finding suggests that Cypriot teachers may warrant their procedural approaches to a topic in ways that their Greek colleagues may not. In sum, the responses from both sets of students indicated deep-set procedural perspectives on linear equations. It was clear that they had understood the task in that all their explanations were valid but, with both the Cypriot and the Greek curricula advocating that students learn a relational mathematics, the implications of this study for teacher education are profound.

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Comparing the practices of primary school mathematics teacher education

Case studies from Japan, Finland and Sweden

Yukiko Asami-Johansson[1], Iiris Attorps[1] and Anu Laine[2]

[1]University of Gävle, Sweden; yuoasn@hig.se, ias@hig.se

[2]University of Helsinki, Finland; anu.laine@helsinki.fi

In this study, we have observed three different teacher educators' lessons, concerning area determination of polygons in primary school teacher training courses in Japan, Finland and Sweden. The aim of this paper is to investigate the main elements of the lessons and to compare the differences between the countries. We focus on how the teacher educators relate the didactic construction of the lessons for prospective teachers to the school mathematical and didactical organisations by applying Chevallard's anthropological theory of the didactic (ATD). The analysis shows how the curricula and the different traditions of teaching practice in each country influence the mathematical and didactical construction of the lessons.

Keywords: Teacher education, anthropological theory of the didactic, praxeologies, Japan, Finland, Sweden.

Introduction

The notion of *didactic divide* is introduced by Bergsten and Grevholm (2004) to illuminate the problematics within the teacher education in Sweden. They refer to Kilpatrick, Swafford and Findell (2001) stating that teacher education needs to provide opportunities for prospective teachers to *connect* different kind of knowledge, and if such connection is not realized, one may say there is a didactic divide between disciplinary and pedagogical knowledge of mathematics. Bergsten and Grevholm also illustrate the point of issue in teacher education citing Ball and Bass's argument; "teacher education across the 20th century has consistently been severed by a persistent divide between subject matter knowledge and pedagogy", a gap that "fragments teacher education by fragmenting teaching" (Ball and Bass, 2000 in Bergsten and Grevholm, 2004, pp. 125-126).

As an attempt to elucidate the phenomenon above within the teacher education in Sweden, we studied the lesson structures of mathematics education at the primary school teacher education programs, using a comparative study. For comparison, we chose Finland which had significantly high result in mathematical literacy among Scandinavian countries (OECD, 2013); and Japan, where teaching culture in mathematics seems to be more shared, compared to the US and Europe (Winsløw, 2012).

The aim of this study is to investigate and compare lessons of three countries' teacher education programs, type of teaching methods courses concerning area determination for teaching in school.

Theoretical framework and research questions

Chevallard proposed to study the mathematical knowledge in an institutional context; learning mathematics is extended to any other human activity and gives rise to the anthropological theory of

the didactic (ATD). There, mathematics learning is modelled as the construction of praxeologies (Bosch & Gascón, 2007) within social institutions. A praxeology provides both methods for the solution of a domain of problems (praxis) and a structure (the logos) for the discourse regarding the methods and their relations to broader settings. Hence the praxis part includes *types of tasks* (**T**) and a *technique* (**τ**) to solve the task type and the logos part includes a *technology* (***θ***) which justifies the techniques and a *theory* (**Θ**), which further justifies the technology. The form of a praxeology is determined by a mutual interaction of a mathematical organisation (MO) and a didactical organisation (DO). The MO describes mathematical activities of the praxeology, and the DO describes the activities to support the learning or teaching of the MO.

In the case of lessons of a teaching methods course in teacher education, types of tasks of the DO are usually to make prospective teachers (PTs) to learn the content knowledge and its teaching approaches. Teacher educators' DOs promote the PTs to learn how to construct the praxeologies (the MO and DO) of their future lessons. We call the didactical praxeology of teacher education as the DO_{TE} and school mathematical and didactical praxeologies encapsulated under the DO_{TE} as MO_{SCH} and DO_{SCH}.

$$\boxed{\mathbf{DO_{TE}}}$$

$$\updownarrow$$

$$\boxed{\mathbf{(MO_{SCH} \leftrightarrow DO_{SCH})}}$$

The praxeologies ($MO_{SCH} \leftrightarrow DO_{SCH}$) demonstrated in the lessons are thus *encapsulated* in the teacher educators' DO_{TE}.

To realize the aim of this paper, we addressed three research questions: 1.What are the main elements of each teacher educator's didactical praxeologies in their lessons? In particular, (how) do they relate the didactic praxeologies of the lessons to school mathematical and didactical praxeologies relation to the area determination? 2. What are the main differences between the three lessons, concerning the research question 1? 3. What could be the wider explanations for those differences?

In order to find out the answers to the research questions 1 and 2, it was necessary to identify the components of both the praxis and logos of the school praxeology ($MO_{SCH} \leftrightarrow DO_{SCH}$) presented in the lessons. Subsequently, we describe how the teacher educators' DO_{TE}s encapsulate the ($MO_{SCH} \leftrightarrow DO_{SCH}$) in their lessons.

To identify mainly the praxis part of the DO_{TE} and the part of the encapsulated school praxeologies, video recordings were made; "Quantity and Measurement" by Mr. Matsui (Japan, with 53 PTs), "Area of Polygons" by Mrs. Laine (Finland, with 34 PTs) and "Area and Perimeter" by Mrs. Nilsson (Sweden, with 20 PTs). The teacher educator in Finland is the third author of this paper. The names of the educators in Japan and Sweden are pseudonyms.

The questionnaire consists of eight questions and we consider that some of the questions may support to identify different components of the DO_{TE} and ($MO_{SCH} \leftrightarrow DO_{SCH}$). For instance, Q1: "What do you intend the prospective teachers (PTs) to learn on this content (e.g. area of polygons)?" is related to identify the components of the logos part of the MO_{SCH} and what is prioritized in the DO_{TE}. Also, Q4: "Which kind of difficulties connected with teaching the content" and Question as

Q7: "What are your teaching procedures and particular reasons for using these to engage your teaching?" gives an indication to identify the technology ($\boldsymbol{\theta}$) and the theory ($\boldsymbol{\Theta}$) of the DO_{TE}. Studying the answers to those questions is also relevant to the research question 3, since the Q4 and Q 7 particularly may give us the picture of what kind of conditions and constraints in each context *form* the praxeologies in each country.

Further, we studied each country's curricula and textbooks concerning the chapter of measurement to reinforce the investigation concerning the research question 3. We selected the chapters concerning measurement, since the lessons in all three countries more or less deal with the introducing the area determination of rectangles using arbitrary objects

Results and analysis

Curriculum concerning measurement in each country

In the Japanese guideline for the curriculum for grades one to six (MEXT, 2008), the determination of length, area and volume is described in an own chapter *Quantity and Measurements*, between the chapters of *Arithmetic* and *Geometry*. The contents for each grade are described in detail with concrete teaching proposals. As guidelines for teaching methods, it is stressed to build on pupils' previously learned knowledge and their various ways of solving problems. The introduction of the chapter consists of four phases; direct comparison, indirect comparison, comparison using arbitrary objects, comparison using standard units. This order is clearly followed by Japanese textbooks (Miyakawa, 2010).

The content regarding quantities, units and measurement are shortly described in the chapter *Geometry and Measurements* in the Finnish curriculum (Finnish National Agency for Education, 2014) and in the chapter *Geometry* in the Swedish curriculum (Skolverket, 2011). These curricula do not give any practical guidelines for teaching the contents. In Sweden, textbooks are not controlled by the ministry. The presentations of those contents in the textbooks for grades 1-3 are often placed in sections covering Arithmetic (e.g. Brorsson, 2013), although the Swedish curriculum introduces them in Geometry. Unlike the Japanese curriculum, the four phases of the introduction of the concept of measurements are not known in Sweden, some textbooks introduce direct comparison and comparison using standard units at the same time (ibid.). Also, the problem that corresponds to the indirect comparison is not addressed in most textbooks[1]. Comparing these two contexts, we might state that the Japanese curriculum does not give much space for different interpretations of its contents. It provides a suggestion of a uniform teaching approach for textbook authors and the users. We assume the reason that many Swedish textbook authors locate the section of measurements in the domain of arithmetic, is to enable a natural connection between area calculations and the basic arithmetical operations. It indicates that different textbooks provide different teaching approaches in Sweden.

Lesson observation "Quantity and Measurement" in Japan

Mr. Matsui is the lecturer of the course "Arithmetic Education" in a state university located in the middle part of Japan. He has worked as a mathematics teacher in lower secondary school for 14

[1] We have not completed the investigation of Finnish textbooks yet.

years and as teacher educator at universities for 12 years. He explains the four phases in the process of pupils learning about measurement by referring to the curriculum guidelines and clarifies those different comparison methods for the class. Thereafter, he discusses how the above mentioned four phases are treated in digital textbooks for grades one to five. The second half of the class is spent to experience the structured problem solving approach. This approach emphasises learner's active participation in mathematical activities, challenging problems and collective reflections (Stigler & Hiebert, 1999). Mr. Matsui lets the PTs find out several different methods for the determination of the area of parallelograms aiming to teach pupils of grade five. Four PTs draw pictures and explain their different solutions on the blackboard (see Figure 1). Mr. Matsui points out the different kinds of "shifts" used by the PTs, and categorises them in: using the sum of the squares (Figure 1 in the middle), "*parallel translation*" (top left), "*rotation*" (bottom left), "*same area transformation*" (top left and bottom left) and "*double area transformation*" (top right). Then, he explains the formula for the area of parallelogram as height times length since the geometric transformations shows that the width (or height) and length of the parallelograms corresponds to those of rectangles. In the same way, he gives a final problem to find out methods for determining the area of trapezoids, using same didactical approach, and concludes the formula for the area of trapezoid; (a + b) h/2.

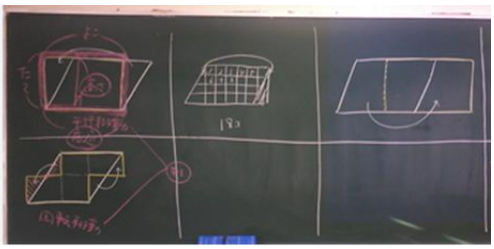


Figure 1: The solutions of the prospective teachers

Findings: The (T) of the DO_{TE} in this episode is to let the PTs experience what the MO_{SCH} and the DO_{SCH} of "determination of area of a parallelogram and trapezoid" can look like. There, (T) of the DO_{SCH} is to encourage the pupils to find out different solving methods of determination of area of a parallelogram and trapezoid to lead to establish the formula. To anticipate how pupils in grade five would solve area determination, Mr. Matsui makes the PTs participate in an exemplary lesson using the structured problem solving approach. He let them follow up one of the ($\boldsymbol{\tau}$) of the DO_{SCH} – whole-class discussions, where the ($\boldsymbol{\theta}$) of the DO_{SCH} – applying the statement of pupils' previous experienced MO_{SCH}s – is demonstrated. Those components of the DO_{SCH} promote to construct a praxeology where several local MO_{SCH}s from the previous to the forthcoming grades are connected. The Guidelines is both the technique ($\boldsymbol{\tau}$) and the technology ($\boldsymbol{\theta}$) of the DO_{TE}, since it suggests different kind of fundamental didactical approaches; e.g. using pupils' previous knowledge from grades one to six, and the use of divers didactical terms of the measurements (direct, indirect, arbitrary comparisons, standard units) and area determination (e.g. "same area transformation").

Lesson observation "Area of Polygons" in Finland

The observed lesson is a workshop using manipulatives in the course "Didactics of Mathematics" for prospective teachers for grades one to six in a state university located in southern Finland. Mrs. Laine, the lecturer of the course, has worked as mathematics teacher in primary and lower secondary school for 5 years, thereafter, as teacher educator for 16 years. Previously she explained

classification of mathematical figures (e.g. set of squares belong to set of rectangles, and set of rectangles belong to set of parallelograms…), line symmetry and rotational symmetry, perimeter and the area of polygons, property of circle, concept of scale. Today, the PTs move between six different tables to work practically with above mentioned concepts. The PTs work in groups using a compendium giving them instructions how to demonstrate those mathematical concepts practically for pupils. The compendium is written by Mrs. Laine and she also moves between the tables to give advices to the PTs on to how solve the tasks the compendium suggests. In this paper, we focus on the workshops "Area of Polygons" and "Area and perimeter".

In accordance with the description in the compendium, one PT in a group plays the "teacher role". As it is prescribed in the compendium, the "teacher" explains how to calculate the area of rectangles by using grid paper with squares of $1cm^2$. PT1 reads the text in the compendium and explains that the sum of the squares is equal to the area of the rectangle. In the compendium, it is emphasised that teachers shall promote pupils to use an inductive way of working/learning. It means, letting pupils experience how to calculate the area of different types of rectangles, and have them find out the formula "height times length". The next task is to find out the formula for the area of a parallelogram. The compendium describes the method of parallel translation (however, these didactical terms like *parallel translation* and *same area transformation* were not observed in the lesson) and explains that the same formula as for rectangles can be applied. PT2 explains this method by drawing the figures for their colleagues. In the same way, the PTs explain to each other the method of area determination of triangles, by reflecting the instruction in the compendium: "make a parallelogram by drawing two similar triangles and let pupils notice that area of the one of the triangle is the area of the half parallelogram".

The next task "Area and perimeter" is to make different kinds of quadrangles with area 12 cm^2. Ms. Laine encourages the PTs to make even irregular quadrangles with the same area. The PTs test to make several different shapes of quadrangles and eventually notice that the perimeter do not need to be the same even the area is same. Ms. Laine then asks the group that how a figure does look like in order to have big perimeter. PT3 makes a long slim rectangle and shows it to others. Then PT4 wonders and asks Mrs. Laine, "why does it work in that way? Are there any rules to be able to describe?" then Mrs. Laine answers, "it has to do with the inductive way of working in lower grades. We can derive understanding toward this phenomenon by working with many single cases in the lower grades. That is good enough on these levels (lower grades)".

Findings: In the first episode, the task (**T**) of the DO_{TE} is to let the prospective teachers learn "inductive way of teaching/learning" to make school pupils find out the formula of area of a rectangle. There, the description from the compendium with exercises (workshop) and role-play are the DO_{TE} ($\boldsymbol{\tau}$) to let the PTs to experience the praxeology of the school lessons ($MO_{SCH} \leftrightarrow DO_{SCH}$). Using the compendium and the statement of (not mathematical) induction is the DO_{TE} ($\boldsymbol{\theta}$) to justify the praxis of the DO_{TE}. The compendium describes directly ($MO_{SCH} \leftrightarrow DO_{SCH}$) where e.g. the MO_{SCH} ($\boldsymbol{\tau}$) is figures, counting of the grids and multiplication, also the ($\boldsymbol{\theta}$) are standard units and commutative property of multiplication. Consequently, the DO_{SCH} **($\boldsymbol{\tau}$)** is to let pupils try to count different kind of rectangles' area to find out the formula by their own. Here, the use of the inductive way of thinking is an essential element of the didactic technology of both the DO_{TE} and DO_{SCH} about the teaching of concept of area determination.

In the last episode, PT4 wants to know the theory level of the MO_{SCH} regarding the area and perimeter. However, Mrs. Laine's DO_{TE} (**τ)** consistently aims to inform the prospective teachers the (**θ**) of the DO_{SCH} –"derive the understanding toward this phenomenon through many single cases" and do not aim to create a technological discourse of the MO_{SCH}.

Lesson observation "Area and Perimeter" in Sweden

The course Mathematics and Learning for Primary School, Grades 4-6 Teachers II, Geometry, in a state university located in middle of Sweden, treats the knowledge in mathematics and mathematical education in relation to the current Swedish curriculum. The lecturer Mrs. Nilsson has worked as a mathematics teacher in grades 4-7 in 13 years and as teacher educator in 12 years. To begin with the lesson, she asks the PTs to write down what are their "own perceptions of the area". Then she gives five group-exercises concerning area and perimeter. The sixth exercise consists of determining the area of different polygons by using Geo-board. Mrs. Nilsson demonstrates a method for area-determination of an isosceles triangle by using a rubber band around the triangle. She divides the rectangle into two squares which are in turn divided into two halves. Half of the area of the squares is subtracted from the each side. Now, the PTs ponder the method for area-determination of another isosceles triangle in groups.

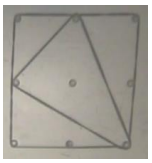
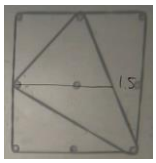
2a  **2b**  **2c** 

Figure 2a: an isosceles triangle. 2b: with an auxiliary line. 2c: PT6's figure

Mrs. Nilsson then demonstrates PT5's solution where the same method is applied as the one she explained. (See Figures 2a & 2b). 4 – 1 – 1 – ½ = 1½ (area units). Then PT6 asks if he can apply the formula of the area determination for a triangle. He explains; first, dividing the original triangle into two triangles with the base of 1.5 length units (see Figure 2c), and then adding the area of the two triangles. This gives the area, $(1.5 \cdot 1)/2 + (1.5 \cdot 1)/2 = 0.75 + 0.75 = 1.5$ (area units). Some of the PTs express that they do not grasp directly how it works. Then Mrs. Nilsson comments "one (a pupil) can understand (this method) if he/she has more mathematical skills".

Findings: The (**T)** of the DO_{TE} is making the PTs to learn a teaching method regarding area determination of isosceles triangles with manipulatives, where the MO_{SCH} (**τ**) is the division of figures and subtraction of area. However, the DO_{TE} (**τ**) and DO_{SCH} (**τ**) – using Geo-board – ensures actually several another mathematical techniques than Mrs. Nilsson has planned to apply. This caused a breaking of a didactical contract (Brousseau, 1999) when the PT6 proposed another technique. Mrs. Nilsson's intention was to train the PTs' algorithmic skills with one technique. She let PT6 explain his alternative technique, nevertheless, did not validate it. Her intention was not to discuss the viability of different mathematical techniques for the grade five class but to establish a certain technique which is possible for all prospective teachers to manage. The didactic theory of the DO_{TE} is difficult to identify from the observation.

The summarized answers to the questionnaires

Mr. Matsui states that the PTs should learn area of polygons can be determined in various ways by using pupil's previously learned knowledge. He stresses also that the prospective teachers should be

able to use some mathematical terms; the terms describe the various methods for area determination. He mentions also that the PTs should know the flow of the problem solving closely.

Ms. Laine's intention in this lesson is, each method of area determination of polygons area is related to each other; the area of parallelogram is based on rectangles, and the area of triangles is based on parallelograms. She emphasizes the importance of the application of inductive ways of working to find a general result, by examining a number of specific examples. She describes her PTs' fragmental knowledge of the formulas for area determination. During her lesson, she often discusses pupils' misconceptions of area and perimeter to let the PTs realise their own misconceptions of this content.

Mrs. Nilsson remarks her PTs' difficulties and limitations concerning geometrical figures. Some of them have learnt formula for area determination superficially and sometimes incorrectly. Also their perception, "geometry is a difficult subject" blocks their learning process. Furthermore, the PTs have not developed mathematical terms that allow them to explain their solutions. To deal with these difficulties, she uses manipulatives to give them concrete ideas of different mathematical concepts and train to establish their own interpretation of the concepts. To train their mathematical communication skills, she uses group discussions with workshops.

Final discussion

The detailed Japanese curriculum with sufficient specifications about the teaching approach, the tradition of the structured problem solving and the textbooks adopting the same teaching approaches – all these factors contribute to give practical hints about how to design the lessons with epistemologically connected praxeologies to a Japanese teacher educator. It becomes explicit for the prospective teachers how to construct mathematics lessons in which alternative techniques are assessed and a technological discourse is taking place. Also, as it is described in Mr. Matsui's answer to the questionnaire, a didactical terminology that describes mathematical techniques such as "same area transformation" is *collectively* used. It leads to the knowledge of the $MO_{SCH} \leftrightarrow DO_{SCH}$ being institutionalized in the community of teacher educators and prospective teachers.

In Finland, the explicit didactical theory of the DO_{TE} (the compendium applying the inductive way of thinking) supports the PTs to learn the knowledge of the $MO_{SCH} \leftrightarrow DO_{SCH}$. However, the analysis from the lesson observation indicates a limitation of the compendium as the ($\boldsymbol{\theta}$) of the DO_{TE} to mediate the theory level of the $MO_{SCH} \leftrightarrow DO_{SCH}$. Even though the prospective teachers are interested in learning more about the theory level of the MO, the workshop with the compendium lacks a function to give them a space for the discussions to institutionalize the theory block of the praxeology. According to the questionnaire, Mrs. Laine's DO_{TE} aims to stimulate prospective teachers' cognitive learning. Hence we might state that the institutional conditions which forms her DO_{TE} are actually originated from a pedagogical level. Thus the compendium gives single techniques in the MO_{SCH} to let the PTs to *visit* the praxeology of the school lessons.

In the case of Sweden, a didactical theory of the DO_{TE} is not clearly distinguished neither from the observation nor the results from the questionnaire. The lack of shared knowledge of the DO_{TE} indicates that praxis part of the DO_{TE} – presentation of how to construct the $MO_{SCH} \leftrightarrow DO_{SCH}$ is *individually* designed by teacher educators in Sweden. The result from the questionnaire shows that Mrs. Nilsson's focus is definitely on the pedagogy. The PTs' fragmented mathematical knowledge

and their anxiety for applying mathematics strongly influence her teaching strategies. Similarly to the Finnish case, neither the Swedish curriculum nor the customs of the lessons with manipulatives help the teacher educators to encapsulate the lesson sequences with complex praxeologies.

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The lexicon project: Examining the consequences for international comparative research of pedagogical naming systems from different cultures

David Clarke[1], Carmel Mesiti[2], Yiming Cao[3], & Jarmila Novotna[4]

[1]University of Melbourne, Australia; d.clarke@unimelb.edu.au

[2]University of Melbourne, Australia; cmesiti@unimelb.edu.au

[3]Beijing Normal University, China; caoym@bnu.edu.cn

[4]Charles University, Prague, Czech Republic; jarmila.novotna@pedf.cuni.cz

Use of English as the international language of educational research can mask the nuanced meanings of constructs that researchers working in languages other than English originally employed in framing their practice and their theories. Cross-cultural comparisons are framed in terms of constructs expressed in the language of publication, usually English. Attention has been drawn to the significance of the resulting validity-comparability compromise (Clarke, 2013). The Lexicon Project investigates the pedagogical naming systems used by educators in nine countries (eight languages). Drawing on examples from the Australian, Chinese and Czech lexicons, this paper outlines the project's research design and addresses the implications of distinctive lexical features for comparative classroom research between communities employing different lexicons to describe the phenomena of middle school mathematics classrooms.

Keywords: Professional language, mathematics education, international comparison.

Introduction

The Lexicon Project involves research teams from Australia, Chile, China, Czech Republic, Finland, France, Germany, Japan and the USA. The project aims to document the naming systems (lexicons) employed by different communities speaking different languages to describe the phenomena of the mathematics classroom. Such lexicons consist of words of locally agreed meaning in a single language that collectively accord to lexical norms and conventions characteristic of the language community (mathematics educators) of the particular country.

The theoretical position adopted by this project is that our experience of the world, our engagement in socio-cultural practices, and our reflection on those experiences and practices are mediated and shaped by available language. The Sapir-Whorf hypothesis suggests that our lived experience is mediated significantly by our capacity to name and categorize our world.

> We see and hear . . . very largely as we do because the language habits of our community predispose certain choices of interpretation (Sapir, 1949, p. 162).

Marton and Tsui (2004) suggest that categories "not only express the social structure but also create the need for people to conform to the behavior associated with these categories" (p. 28). In this project we examine this normative role of language in relation to classroom practice and research.

While a professional language of teaching practice seems lacking in the USA (Lampert, 2000), such a language seems to be well-established among educators in China and Japan (Fan et al., 2004;

Fernandez & Yoshida, 2004). Our interactions with classroom settings, whether as learners, teachers, or researchers, are significantly mediated by our capacity to name what we see and experience (Clarke et al., 2016). Speakers of one language have access to terms, and therefore to perceptive possibilities, that may not be available to speakers of another language. This has implications for international comparative research.

Any claim that researchers speaking different languages are analyzing "the same classroom," even when working from the same video records, can be usefully contested. In two published translations of Vygotsky, we find the Russian term, "obuchenie," represented as "instruction" in one translation and "learning" in another. This is not merely a problem of mistranslation. The term refers to an activity in which teachers and students are jointly participant for which there is *no equivalent term* in English. The term offers a conceptualisation of classroom practice with profound implications for the theorization of classroom teaching/learning. Recognition of these implications is not afforded in English. Educational research increasingly employs English as the primary language through which theory is developed and disseminated. It is essential to recognise the constructs that other cultures have employed in conceptualising their practice and examine the consequences for research and for theory of those distinctive terms (and the designated constructs) that might otherwise be ignored by an international community restricted to communication in English.

Research design

In the Lexicon Project, local teams of researchers and experienced teachers in all nine countries viewed a common set of video records of one eighth-grade mathematics lesson from each participating country. Specific lessons were selected by each country team for the diversity of activities displayed rather than for their representativeness. The purpose of this activity was to stimulate identification of those terms in the local language of each team that constituted the national pedagogical lexicon with respect to the teaching of middle school mathematics (ages 11 to 14). It is via the medium of these terms that teachers plan, engage with, discuss, and reflect upon the mathematics classroom. It was assumed that the vocabulary available to researchers in each country included most terms employed by teachers, but that the researchers' vocabulary would include other terms not necessarily locally derived, being translations or literal appropriations of terms generated by other educational communities. For that reason, because of the focus on the local language, it was the teachers' lexicon that was the principal focus of investigation in each country.

The key prompt used by all teams was: "What do you see that you can name?" Once a term was identified and endorsed by the local team, a consensus description was constructed of the specific classroom phenomenon to which the term referred and examples and non-examples identified to maximize effective communication of the term's meaning and classroom referent. These descriptions, examples and non-examples were crucial to the communication between teams of the meanings of terms that originated in the local (predominantly non-English) language.

But teams were not restricted to only those phenomena visible in the video material. For example, where observation of one type of classroom activity reminded the observer/s of another activity type not evident in any of the video material, that term was included in the lexicon, together with a description, examples and non-examples. It was anticipated that these terms would describe classroom practices, both structures (such as organizational patterns or activity sequences) as well as

specific activities observable in the middle school mathematics classroom. The function of the video material was fundamentally catalytic, stimulating recall of the names of classroom phenomena present and absent in the classrooms filmed. However, the video material also assisted communication within and between the different teams, clarifying the meaning of terms.

Local team consensus was required for the inclusion of a term in the lexicon and in problematic cases authority was accorded to classroom experience and the team members' capacity to argue that the term was in current use by teachers. The essential point was to record single words or short phrases that were consistently and widely used by teachers within that country with a consistent and agreed meaning. Subsequently, a process of local and then national validation was pursued to refine and ratify each lexicon. The means by which this validation process was undertaken varied from country to country, but basically involved inviting a national cross section of mathematics educators to comment on the adequacy, accuracy, and clarity of the constructed lexicon for that country.

Of course, such lexicons are continually evolving and a process of regular updating is anticipated. The international project team takes particular interest in the studying the connections between terms within a given lexicon and the consequent clusters of related terms that provide the structure for each country's lexicon. Both teachers and researchers were involved in the identification of these connections. The Chinese example below illustrates one approach to the identification of such structures. Comparison of the emphases evident within each country's lexicon reveals distinctive features of the different countries' mathematics pedagogy and priorities of classroom practice.

Lexicon selection for the purposes of comparison and contrast

In this paper, one English speaking and two non-English speaking communities have been chosen to provide contrasting examples of the language that educators in Australia, China and the Czech Republic employ to describe the objects and events of the middle school mathematics classrooms in their countries. Structural aspects of the lexicons suggest underlying pedagogical principles or associations that shape the ways in which middle school mathematics teachers function and interact within the mathematics classrooms of that country. The lexicons also offer insight into the language available to researchers in each country, by which they study, classify, analyze, conjecture and theorize about the practices and the affordances of the mathematics classrooms of their country.

The project identified both similarities and differences in the national lexicons, revealing significant differences in the way teachers and researchers from each country perceive the classrooms that are the focus of their professional activity. These differences raise the question of the extent to which the international community of mathematics teachers and researchers can meaningfully and productively share the wisdom of long-established pedagogical traditions of practice, where these are encrypted in the naming systems by which each community identifies those classroom activities that it considers to be significant. Discussion is provided of: (i) the implications for comparison of the lexicons, and (ii) the implications of the lexicons for other comparative classroom research. These two purposes are conceptually distinct but connected.

In this paper, English is used to describe the content and structure of both the Chinese and Czech lexicons. This reflects the underlying purpose and challenge of the Lexicon Project: to identify and make accessible to the international community the pedagogical principles and distinctions encrypted in different lexicons. Examples from the original language are cited for purposes of clarification. For

example, some terms can be approximated in English (e.g., "Teacher Feedback" adequately names 教师反馈 which is "jiào shī fǎn kuì" in Chinese pinyin[1]) but there are those that have no simple equivalent English term or phrase but can only be represented in pinyin and an extended English description (e.g., 课堂生成 which in pinyin is "kè táng shēng chéng" and which refers to "when the teacher makes full instructional use of an unexpected event beyond the intended plan for the lesson"). Similarly, the Czech term "S cílem objevit" (literally, "with the aim to discover") refers to the occasion when "by solving the problem students discover something new." Examples are used, if needed, in the discussion that follows. The three lexicons: Australian, Chinese and Czech are used to illustrate respectively the methodological processes of Lexicon Identification, Structure and Interpretation.

The Australian lexicon: Generic rather than discipline-specific terms

The Australian National Lexicon consists of 63 terms that are familiar and in widespread use (e.g., *Assigning Homework, Rephrasing, Worked Example*). A description was constructed for each term, together with both examples and non-examples of the use of the term. Because of the role of video in stimulating the recognition of terms, many terms can also be illustrated with video examples.

In consultation with practicing teachers, the lexical items were organized in five categories: Administration (8 terms); Assessment (11 terms); Classroom Management (6 terms), Learning Strategies (27 terms) and Teaching Strategies (50 terms). A lexical item appeared in more than one category if the Australian team decided on the basis of teacher advice that there was a strong association with each category.

One feature of the Australian National Lexicon is that none of the 63 terms identifies a practice unique to the mathematics classroom. The terms all refer to general pedagogical practices. Also worthy of note is the prevalence of 'gerunds' (a verb form that also functions as a noun; "teaching" and "learning" are relevant examples) in the Australian National Lexicon. This duality provides both advantage and disadvantage: "learning" as a noun is explicitly the product of the activity of "learning" in a way that objectified "knowledge" is not, but this duality can also result in less precise communication due to the inherent ambiguity over what is being referred to: process or product. The duality of simultaneously invoking both object and activity is not available in some languages, highlighting the affordances of particular languages and the difficulties of translation.

The generic character of the Australian Lexicon content suggests that the lexicon might also be applicable to other school settings besides the mathematics classroom.

Integrating forms of connection to structure the Chinese lexicon

In the Chinese Lexicon, 126 terms were identified as being used by Chinese middle school mathematics teachers in describing their classrooms. Within the lexicon, every term is related to some other terms, which makes the teacher's language an organic entirety. The challenge for the Chinese

[1] Pinyin is a phonetic rendering of Chinese characters using the Latin alphabet employed in English and four basic tonal annotations.

team was to clarify the structure of teacher's pedagogical language as encompassed in these 126 terms. A two-step process was employed by the Chinese team to do this.

Step One: Three types of connections were identified between lexical items: Hierarchical, Coincident and Sequential.

Hierarchical

Level One: In the first level, the terms can be divided according to whether the term referred to Teachers, Students or to Teacher-student Interactions.

Level Two: Terms within the category "Teachers" could be divided into: Classroom Management, Demonstration, Questioning, Feedback, Summarizing, Explanation, and Tutoring.

The category "Students" included: Classroom Management, Demonstration, Questioning, Feedback, Summarizing, Doing Exercise, Collaborative Studying, Self-learning and Listening.

The category "Teacher-student Interaction" had no sub-structure at this level.

Level Three (only one example can be shown for reasons of space): "Classroom Management" included: Teaching Affairs Management and Order Management.

Apart from the hierarchical links, it was clear that some activities can happen at the same time while others occur in a sequence. This provided two additional mechanisms for the clustering of terms.

Coincident

This category refers to terms used in a teacher's pedagogical language that refer to activities that can happen at the same time. For example, Group Report and Student Listening — when a group is reporting their findings or answers, the other students must be listening carefully in the class.

Sequential

Teacher Questioning and Student Answering are an example of a pair of tasks that are intrinsically sequential — when the teacher asks a question of the class, this action is typically followed by an individual answering or the class answering together.

Step Two: Using the three types of connections, it was possible to organize the terms in the Chinese Lexicon into a structured array. Experienced teachers were recruited from high achieving schools in different parts of China to identify the connections. In this way, regional variations in interpretation and association could be identified. This illustrates the value of the lexicon in helping to identify regional pedagogical variations in a country as large as China, while also highlighting common pedagogical elements.

Differences between professional language communities: Teachers and researchers in the Czech Republic

As for all the country lexicons, the Czech lexicon does more than simply describe the current pedagogical vocabulary of practising Czech mathematics teachers. It should also be considered as a way to understand the Czech "culture of education" by providing examples that illustrate how it is possible to think about education. The use of pedagogical terms varies according to the groups of users (authors in different fields of pedagogy, teachers, etc.). One purpose of the Czech lexicon at the national level is to provide teacher education with a tool for triggering and framing discussion among pre-service students and practising teachers to facilitate better understanding of lesson structure and classroom practice.

The terms of the Czech Lexicon were classified using the following categories: Classroom Management; Introductory Communication; Explanation of New Topic; Revision of Previously Taught Topic; Solving of a Problem; Checking Individual Work; Institutionalisation; Summary; Non-mathematical Social Interaction; Assessment; Concluding the Lesson; Individual Consultation with a Pupil. The Czech lexicon is highly stratified. There are several sub-categories in each of these themes. One distinctive feature is the prevalence of student-oriented terms that reflect the importance attached within Czech education to the teacher-student relationship. For example, "shrnují na pokyn učitele" (student invited to recapitulate teacher's instruction) and "vysvětlují na pokyn učitele" (student invited to explain teacher's instruction) are distinctively student actions.

When constructing the Czech lexicon, the research team identified particular characteristics. One characteristic concerned the difference between how the language was used and understood by different target groups. Practising teachers used few technical pedagogical terms and communicated mostly using words from the language of everyday life. Pre- and in-service teachers were more likely to make use of terms drawn from their lessons on mathematical didactics. For example, "Heuristický rozhovor" is an academic term meaning "heuristic dialogue," but a practising teacher would be more likely to say, "řízená diskuse" meaning "guided discussion" to refer to the same classroom phenomenon. The Czech team's concern was how to combine the two ways of using the Czech lexicon – as a tool describing the structure of Czech lessons and highlighting important parts of lessons and as a tool to facilitate discussion between different groups. This dilemma is accentuated by the relative paucity of Czech technical terms not only in the domain of didactics of mathematics but also in pedagogy. The Czech example illustrates the different uses of the lexicon for teachers, student-teachers and educational researchers and identifies the potential for the Czech lexicon to serve as a catalytic focus for discussion between these different communities.

Discussion: Implications for comparison

The key steps in lexicon construction of Identification, Structure and Interpretation have been illustrated with examples from the Australian, Chinese and Czech lexicons respectively. Each step offers insights into the pedagogical history encrypted in each lexicon and the potential value of the lexicon to the teaching and research communities in each country. In this section, we explore the question of comparison. Two forms of comparison warrant discussion: within-project comparison of the separate lexicons for the purpose of gaining insight into the pedagogical principles of each language community encrypted in the professional lexicon of middle school mathematics teachers; and, second, the broader implications for international comparative classroom research of the documented differences in how the phenomena of the middle school mathematics classroom are conceptualized within each language community. Each of these is discussed separately below.

Comparing lexicons: Constructs as boundary objects

The primary consideration in making comparison between any two lexicons is the mediating construct that forms the basis of comparison (Clarke, 2015). For example, comparison might be made between the *agency* accorded in one lexicon to the teacher or to the students. That is, what proportion of lexical terms refer to teacher actions and what to student actions (cf "Level One" in the Chinese lexicon), and what is the nature of the actions in each case: initiating or reactive (for example). The Australian lexicon makes a comparable distinction between Teaching Strategies and Learning

Strategies. In this case, "agency" provides the boundary object by which the two lexicons might be compared. Clearly, "agency" need not be a term situated in either lexicon. Instead, it represents an organizing construct with comparable conceptual legitimacy within each lexicon. As such, it constitutes an acceptable boundary object for the purposes of comparison of the lexicons. Other boundary objects might name categories of lexical items, such as: *assessment* or *management*. The requirement for legitimate comparison would be that the organizing construct (say, assessment) has local validity within each lexicon as designating a cluster of lexical terms and cross-lexicon validity in characterizing conceptually the same shared attribute for each lexical cluster being compared. Comparison of the Australian and Czech lexicons is possible on this basis, with respect to either of the mediating constructs: agency or assessment.

Comparative research: Validity-comparability compromise

The documentation of the separate lexicons has the potential to heighten the legitimacy of comparative classroom research being undertaken across two communities. For example, application of a measure of *participation* to the comparative analysis of classroom data from two countries is problematic, unless it can be demonstrated that "participation" has the same cultural relevance within the pedagogical practices of each community. However, if participation is treated not as the basis for an imposed metric, but as a boundary object, then the question can be asked, "What forms of participation are legitimized within the lexicons of the two countries whose classrooms are being compared?" Analysis of the separate lexicons to identify those terms that characterize forms of participation in classroom practices would reveal both similar and different types of participatory activity. For example, choral response has been documented as a frequent form of participatory activity in mathematics classrooms in China and Korea, but not in classrooms in Australia and Japan. Reciprocally, student-student talk is a common form of participatory activity in mathematics classrooms in Australia and Japan but not in China and Korea (Clarke, Xu & Wan, 2013). Any comparison of student participation in classroom activity in these four countries can be undertaken with much greater validity, where attention is given to important distinctions between forms of participation as these are facilitated and named by teachers in each of the countries whose classroom practices are being compared. Utilization of the lexicons from each country to identify legitimate points of comparison would heighten both validity and comparability (Clarke, 2013).

Conclusion

The construction of national lexicons representing the naming systems employed by educators using different languages to "name what they see" in the middle school mathematics classroom represents the starting point for the deconstruction of pedagogical histories and norms of practice enshrined in the languages by which classroom phenomena are described, studied and theorized in different countries. The documentation of these lexicons has significant practical value to each participating community and to the international community of mathematics education practitioners and teacher educators. The focus of this paper, however, has been on the implications of such lexicons for the legitimacy of international comparative research and on the use of any named construct as a boundary object for the purposes of comparative research analyses. It is intended that the lexicons serve as tools to interrogate, enhance and advance comparative classroom research internationally.

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A comparison of Kosovar and Polish pre-service teachers' knowledge of fractions

Jeta Kingji-Kastrati[1], Miroslawa Sajka[2] and Eda Vula[3]

[1]University of Ljubljana, Faculty of Education, Slovenia; jetakin@gmail.com

[2]Pedagogical University of Cracow, Faculty of Mathematics, Physics and Technical Science, Poland; msajka@up.krakow.pl

[3]University of Pristina, Faculty of Education, Kosovo; eda.vula@uni-pr.edu

The aim of this study was to examine Kosovar and Polish pre-service teachers' knowledge of fractions. Thirty-three Kosovar and thirty-five Polish pre-service teachers participated in the study. They were asked to complete a fractions knowledge test, which was adapted from Cramer, Post, and del Mas's (2002) study. The results identified substantial differences between Kosovar and Polish pre-service teachers' knowledge of fractions. The differences between these two groups of preservice teachers' knowledge of fractions appeared to be a consequence of the number of courses related to mathematics, the number of hours of lectures in mathematics during their studies and the structures of the programmes in both universities.

Keywords: Pre-service teachers, fractions, mathematical knowledge.

Introduction

Fractions present one of the most important and, at the same time, most complex mathematical concepts of the elementary school curriculum (Behr, Post, Harel, & Lesh, 1993; Charalambous & Pitta-Pantazi, 2007) with applications to many other areas of mathematics. Researchers internationally have shown that in the context of primary school education, fractions are one of the most problematic concepts for both pupils and teachers (Cramer, Behr, Post & Lesh, 1997; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981). The NCTM (2000) points out that secondary school students should possess an in-depth knowledge of fractions and be able to use them appropriately in the process of problem solving. Misconceptions that students have about fractions, both in terms of fractions (as numbers) and how to operate with fractions, relate particularly to the way fractions are represented and how they are taught (Barmby, Harries, Higgins, & Suggate, 2009). These difficulties begin in elementary school (Empson & Levi, 2011; Moss & Case, 1999), continue through secondary school (Smith, 2002) and high school and very often even at the level of higher education (Orpwood, Schollen, Leek, Marinelli-Henriques, & Assiri, 2011).

There are few doubts that a deep understanding of fractions, both conceptually and procedurally, and the skills to solve word problems with fractions are necessary for mathematics teachers to teach effectively. They should not only experts from the point of view of mathematical content knowledge but they also need to know how to teach this content, considering their position as experts from a pedagogical content knowledge point of view.

However, many studies have shown that teachers also have difficulties in understanding fractions and operations with them (Ball, 1990) and have stressed that teachers' knowledge was not satisfactory (Charalambous & Pitta-Pantazi, 2007; Lin, Becker, Byun, & Ko, 2013). Therefore,

closer attention must be paid to the future teachers, in order to help them improve children's procedural and conceptual knowledge generally and fractions specifically. Such gaps in knowledge directly influence the learning of fractions by students (Charalambous & Hill, 2012; Hill, Rowan, & Ball, 2005). When working with fractions they encounter difficulties with respect to both teaching and explaining them, which poses a constant challenge for the teaching community of mathematical education. Therefore, it is important to pay closer attention to the subject matter knowledge of pre-service teachers (Wilson, 2010), as well as pre-service teachers training concerning the issue. In this study, therefore, we focus on Kosovar and Polish pre-service teachers' knowledge of fractions. We tested their conceptual and procedural knowledge of fractions, and their ability to explain their solution strategies to a variety of tasks, both standard and word problem.

Theoretical background of the study

The concept of fractions has been examined and discussed by many authors. Many of them have identified and discussed different ways of presenting the fractions (Behr et al., 1993; Kieren, 1976). Different theoretical models have been proposed for understanding of fractions (Behr et al, 1993; Charalambous & Pitta-Pantazi, 2007). Kieren (1976) was the first to propose that fractions should not appear only as a single concept model, recommending that they should be conceptualized as a set of interrelated sub-constructs: part-whole, ratio, operator, quotient, and measure. The presentation of fractions as part of a whole refers to division of an amount or a group of discrete objects into equal parts and comparing these pieces with the total value of quantity (Vula et. al., 2015). The ratio sub-construct represents the interconnection between two quantities and usually appears as a:b or a/b. This way of presenting the fractions expresses a relationship between the two quantities. The operator is a function that transforms segments, figures or numbers (Behr et al., 1993). Fractions as a quotient appear as a result of division of two integers and measure construct identifies fractions as numbers or associating fractions with the measure assigned to some interval (Kieren, 1976). To be able to teach fractions successfully, teachers require a comprehensive understanding of these different conceptualizations, the interconnection between them and a battery of teaching approaches (Behr et al., 1993).

Purpose and research questions

The main purpose of the research was to compare Kosovar and Polish pre-service elementary teacher's knowledge of fractions. In so doing the study aimed to (a) assess and compare pre-service teacher's knowledge of the different fractions' concepts and their didactical representations and (b) analyze the ways in which pre-service teachers from both countries explain their rationale for a procedure.

1. What are the main differences between Kosovar and Polish pre-service teacher's knowledge of fractions?
2. Are there differences in strategies used by Kosovar and Polish pre-service teachers?

Method

Participants

The data were collected from 68 pre-service elementary teachers from the University of Pristina in Kosovo (N=33) and the Pedagogical University of Cracow in Poland (N=35). In Kosovo, elementary school teachers for grades 1-5 are generalist teachers. Consequently, all pre-service elementary teachers are trained in all school subjects, including mathematics. The elementary Bachelor's degree program is a 4-year qualification. Three courses of elementary mathematics are taught in the first and second year of study (in total 514 hours) and the course on teaching mathematics (in total 178 hour) is taught in the last year of the study program.

Teachers in Poland are generalist only for grades 1-3 of elementary school, and to teach for the next three grades they must specialize in a chosen subject. In order to teach mathematics from grades 4 through 6 they should achieve Bachelor's degree in mathematics with a teaching specialization (3 years). They attend many theoretical courses in pure mathematics (in total 1239 hours) as well as courses designed for the teaching specialization (in total 893 hours) and others. Although, there is no subject like elementary mathematics, its main themes are discussed and practiced within the course named Didactics of Mathematics.

Procedure

Participants were asked to complete the fractions' knowledge test, developed and administrated to measure their performance of fractions' knowledge. The items of the test were used in previous studies (Cramer et. al. 2002; Charalambous and Pitta-Pantazi 2007; Lin, et al. 2013). The test was divided in three subsets of tasks and time for its completion was not limited.

Methodology

The first subset of tasks, drawing on Kieren's (1976) model, includes fraction-related problems focused on pre-service teachers understanding of fractions as parts of a part-whole (tasks 1-2), ratio (task 3), quotient (task 4), operator (task 5) and measure (tasks 6-7). The second subset addresses how pre-service teachers explain the process of solving fractions problems. The third subset focused on fractions-related word problems and analyzed according to Vula's (2006) model.

Results

First research question: What are the main differences between Kosovar and Polish pre-service teacher's knowledge of fractions?

Figure 1 shows the success rates, as percentages, of the two groups of pre-service teachers on each of task, with the Kosovar results being shown in the left-hand column and the Polish results in the right-hand column for each task. It can be seen, for example, that 100% of the Polish participants solved task 5 correctly (all with justification), while only 60.61% of the Kosovo participants solved this task correctly (although only 14 gave the justification). We also found notable differences on tasks 11, 13, 14. More than 60% of the Kosovar pre-service teachers failed to solve task 11, while 66% of the Polish participants solved it correctly. Similar differences were identified on tasks 13 and 14. In particular, the most significant differences were found with respect to tasks 7 (measure)

and 15 (solving word problem with 3 steps). While more than half of all Polish students solved them correctly, 65.70% and 54.30% respectively, not one Kosovar students solved either.



Figure 1: General success rate results

Second research question: Are there differences in strategies used by Kosovar and Polish pre-service teachers?

The figures of Table 1 show the number of participants who answered the different tasks correctly with justification, without justification or incorrectly.

Kind of answer **(Kosovo; Poland)**	Whole-part (Task 1)	Whole-part discrete (Task 2)	Ratio (Task 3)	Quotient (Task 4)	Operator (Task 5)	Measure (Task 6)	Measure (Task 7)
Correct with justification	(15; 28)	(26;30)	(23; 34)	(31; 34)	(14; 35)	(23; 34)	(0, 23)
Without justification	(10; 3)	(4; 4)	(6; 0)	(0; 0)	(12; 0)	-	(0, 0)
Incorrect	(8; 4)	(3; 1)	(4; 1)	(2; 1)	(7; 0)	(10; 1)	(33, 12)

Table 1: Number of answers in both groups respectively in pairs (Kosovo, Poland)

The majority of participants from both countries provided the correct answer to task 1, although 8 students from Kosovo (Figure 2) and 4 from Poland (Figure 3) provided incorrect answers when asked to present a fraction as a part of an 'irregular' unit, assuming that the 'whole' would be a circle (the examples presented are chosen randomly).

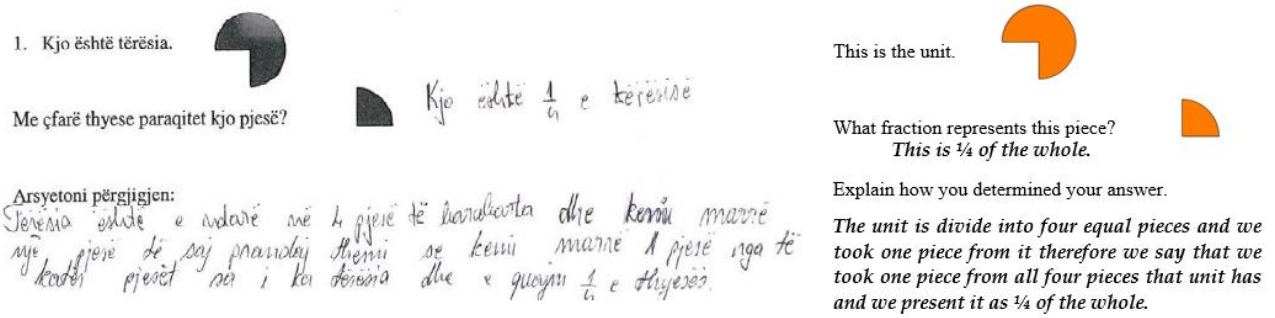
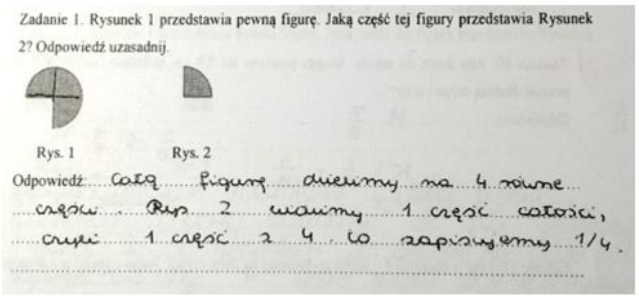


Figure 2: Task 1 sample answer, Kosovo


Zadanie 1. Rysunek 1 przedstawia pewną figurę. Jaką część tej figury przedstawia Rysunek 2? Odpowiedź uzasadnij.

Rys. 1 Rys. 2

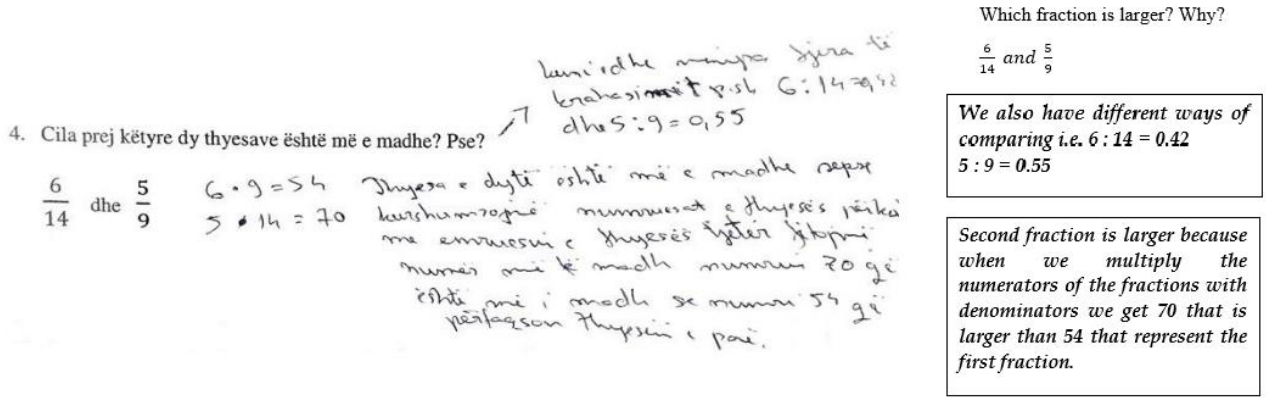
Odpowiedź: Całą figurę dzielimy na 4 równe części. Rys 2 mamy 1 część całości, czyli 1 część z 4 to zapisujemy 1/4.

Explain how you determined your answer.

We devide a whole figure into 4 equal pieces. In Fig. 2 we can see one piece of the whole, so 1 piece out of 4. We write this as $\frac{1}{4}$.

Figure 3: Task 1 sample answer, Poland

For solving task 4 (quotient), different strategies were used by Kosovar and Polish pre-service teachers. Almost all Kosovar participants who completed the task correctly, converted fractions to decimal numbers or used "butterfly method" (cross multiplication) and explained which fraction is bigger. The Figure 4 provides an example how 1 student used two strategies for solving the task. In the first part it is shown the cross multiplication strategy and then explained why the second fraction is bigger. The second strategy used by this student is by converting fractions to decimal numbers.


4. Cila prej këtyre dy thyesave është më e madhe? Pse?

$\frac{6}{14}$ dhe $\frac{5}{9}$

6 · 9 = 54

5 · 14 = 70

Which fraction is larger? Why?

$\frac{6}{14}$ *and* $\frac{5}{9}$

We also have different ways of comparing i.e. 6 : 14 = 0.42
5 : 9 = 0.55

Second fraction is larger because when we multiply the numerators of the fractions with denominators we get 70 that is larger than 54 that represent the first fraction.

Figure 4: Task 4 sample answer, Kosovo

The majority of Polish participants, 27 people, found a correct solution using the strategy of transferring given fractions into a common counter (3 people, common counter: 30) or common denominator (24 people), which was 126 (12 people) or 63 (11 people) and one person only described the method (without calculation). The second strategy used by 6 people was comparing the given fractions to ½ and noticing that the first is smaller and the second bigger than ½. Only two people used the method of solving the task by converting fractions to decimal numbers, and one of them only described the method but did not apply this (getting 0 points for that task).

Strategies in group	Common denominator	Estimation (decimals)	Estimation (comparing to ½)
Kosovo	2	24	0
Poland	33	3	1

Table 2: Distribution between strategies used

For solving task 5 three strategies were used from participants: strategy of common denominator, estimation strategy in the form of decimals and strategy of the comparison to half. Two participants from Kosovo and the great majority of 33 participants from Poland solved the task using a common

denominator strategy, while 24 participants from Kosovo and 3 participants from Poland used an estimation strategy in the form of decimals. Table 2 shows the distribution of the strategies used by both groups. Two Polish participants used two strategies, providing apart from common denominator the estimation strategy - one in decimal form and the other comparing to ½.

While in Poland only 2 students did not answer task 7, all of Kosovo students failed to solve given task (Figure 5). 23 Polish students calculated the distance between adjacent points marked on the number line (Figure 6).

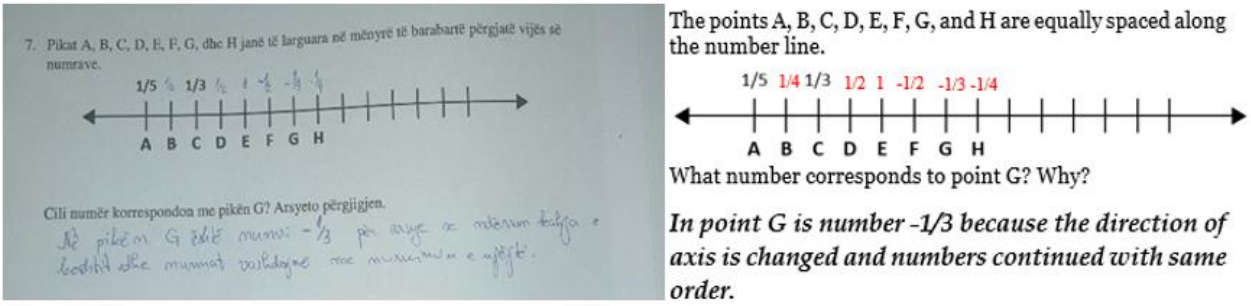


Figure 5: Task 7 sample answer, Kosovo

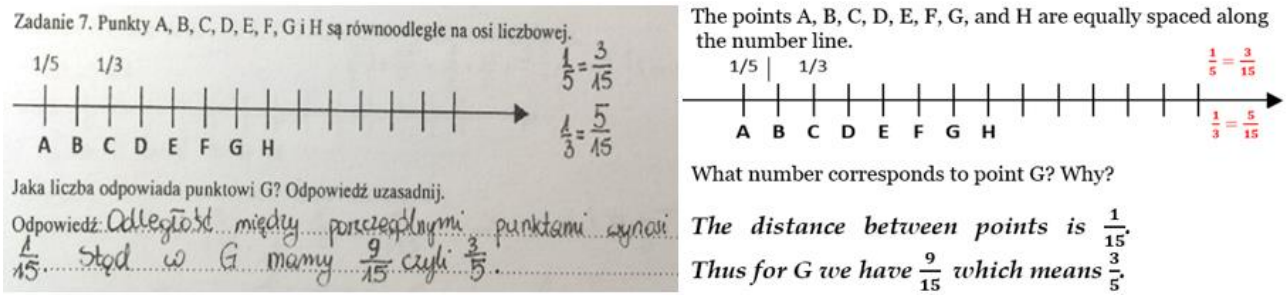


Figure 6: Task 7 sample answer, Poland

Discussion

The purpose of this study was to (a) assess and compare Kosovar and Polish pre-service teacher's knowledge of fractions and their representations and (b) analyze how they explain the process of solving fractions. To answer these questions, we analyzed test tasks completed by pre-service teachers from the two countries. In so doing, we identified substantial differences between Kosovar and Polish pre-service teachers' knowledge of fractions, with Polish students being more procedurally successful on every task and typically able to offer better explanations. These differences were found across all fraction conceptualizations as discussed by Kieren (1976).

There may to be several reasons for these differences. For example, the number of hours that students receive in mathematics courses and teaching methods in mathematics are much higher in Poland than in Kosovo. Also, because pre-service teachers in Kosovo are being trained to be generalist teachers from grade 1 to 5, they may not receive sufficient hours dedicated to mathematics and didactics in order to fulfill their needs for mathematics knowledge of pre-service teachers (Wilson, 2010). In addition, differences in the curricula and the ways in which fractions are represented in elementary mathematics textbooks (Vula et al., 2015), both sources on which pre-service teachers draw during teaching practice, and their own previous learning as school students

may also explain why Kosovar students performed less satisfactorily than their Polish colleagues. Therefore, the findings of this study should act as a springboard for further research into and discussion of how elementary teachers are prepared for their professional responsibilities. In this respect we argue that it is important for the instructors of pre-service teachers' mathematics courses to provide adequate opportunities for their students to develop a knowledge of fractions that better prepares them for their future roles as teachers of children.

In light of the above, our future aims are to investigate further not only Kosovar and Polish pre-service teachers' knowledge of fractions but also those of pre-service elementary teachers in other European countries.

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Finnish teaching materials in the hands of a Swedish teacher: The telling case of Cecilia

Tuula Koljonen

Mälardalen University, Sweden; tuula.koljonen@mdh.se.

A common perception in Sweden is that the best teachers do not rely on ready-made teaching materials. The position taken in this paper, building on socio-cultural theory, assumes that teacher materials can support teachers. Although there is an emerging body of research focusing on teachers' use of teaching materials, cross-cultural studies on this are scarce. The current study addresses this gap by offering unique insight into how a Swedish teacher makes use of teaching materials originally from Finland but slightly adapted to the Swedish context. Based on teacher interviews and classroom observations, I studied how the teacher planned for and enacted lessons. Findings indicate that she fits the material to her pre-existing practice and, thus, does not follow the material's original intentions. The results are compared with previous results on materials and their use, and finally some implications for Swedish mathematics education are presented.

Keywords: Cultural scripts, educational context, primary school, teaching materials, telling case.

Introduction

Finland is known as a country with good learning outcomes in mathematics (e.g. OECD, 2013). Furthermore, since the 1980s the country has had a tradition of producing exhaustive teacher guides (TGs) in collaboration with teachers, teacher educators and other experts (Niemi, 2012). These two factors have likely increased the interest in applying commercially produced Finnish teaching materials, such as textbooks and TGs, in Sweden as well as other countries, such as Italy. Applying new teaching material from Finland in Sweden could be achievable, as there are many similarities between the school systems in the two countries – for instance, the inclusive nine-year compulsory basic education with no special tracking, and the national core curriculum that provides an overall outline for school education. In addition, primary school teachers in both countries often teach all subjects, and are free to choose which teaching materials to use. There are also differences between the countries' educational systems; e.g., Swedish teachers at primary school level are seldom subject specialists while Finnish teachers are well educated, but also the widespread negative talk in Sweden concerning the use of ready-made teaching materials (Bergqvist et al., 2010), which does not occur in Finland. Swedish teachers' orientation toward ready-made teaching materials most certainly affects how they engage with and use them (cf. Stein, Remillard & Smith, 2007).

Swedish teachers seldom use TGs in planning and enacting mathematical instruction (Jablonka & Johansson, 2010). Instead, they rely mostly on the student textbook as their main source, a common feature of teachers in many parts of the world (Remillard, 2005; Stein et al., 2007). Also, Swedish compulsory school teachers have for the last two decades often organized individualized teaching, whereby students work individually in different areas and at their own pace (Bergqvist et al., 2010; Remillard, Van Steenbrugge & Bergqvist, 2016). The fact that students are taught largely according to the structure of their textbook has also resulted in less variation in teaching (Jablonka & Johansson, 2010). However, Finnish teachers, especially primary school teachers in mathematics, trust and use

commercially produced TGs extensively, and there are indications that Finnish teachers often organize whole-class teaching and use teaching methods other than individual seatwork. This has consequently led to a classroom practice that is different from the Swedish one (Jablonka & Johansson, 2010; Pehkonen, Ahtee & Lavonen, 2007). While there is growing interest in adopting and implementing mathematics materials in a new educational context, we know little about how imported mathematics materials are used or how they may influence classroom practice.

Therefore, in this paper I aim to investigate the interplay between a Swedish teacher and the written curriculum as represented in the suggested lesson plans in a TG – *Favorit Matematik* (*FM*), originally from Finland. Moreover, I intend to show how this interplay may impact on enacted teaching. Teaching is viewed as a cultural activity, and cultural activities are represented in cultural scripts (cf. Stigler & Hiebert, 1999) and are consistent with the stable web of beliefs and assumptions within a cultural group. Scripts provide a background for interpreting behaviors; however, they do not describe, determine or predict the behavior of individual teachers (Stigler & Hiebert, 1999). Both teacher and teaching materials are significant participants and are situated in a socio-cultural context, a specific educational context. Through this, they both play a role in mediating that interplay, which is shaped by historical, social and cultural factors (Brown, 2009; Remillard, 2005). Since cultural scripts are deeply rooted in practices and are hard to see from within a given culture, I opted for a case study approach, allowing for a deep analysis. I therefore anticipate that a study on the use of teaching material from one culture by a teacher from another culture will advance our understanding of the cultural scripts in both cultures, and of the participatory relationship between teacher and teaching material.

I have previously analyzed TGs from four Finnish textbook series in mathematics, and found that their structure, form and content were relatively homogeneous (Koljonen, 2014). In Koljonen, Ryve and Hemmi (under review), we captured what kind of mathematics classroom the Finnish guides promote. Recurrent cultural scripts of the classroom practices were found, comprising: keeping the class around a specific topic; keeping the teachers and students active; clear lesson goals are vital features; different recurrent activities; concrete material; and embedded differentiation. Due to these findings and the different classroom practices in Sweden and Finland, it is of interest to investigate a Swedish teacher's interplay with Finnish teaching material as a way to compare the written and the enacted curricula grounded in two different cultural platforms. The research questions guiding this study are: 1) *How does a Swedish primary school teacher, locally regarded as competent, interact with a Finnish teacher guide while planning and implementing teaching?* 2) *How does this interaction influence the classroom practice?*

Methodology

This study is part of a larger cross-cultural project examining the interplay between Swedish and Finnish teachers using the same mathematics teaching materials. The data for this project are comprised of semi-structured interviews with four primary mathematics teachers from each country. The interview questions cover seven themes: teacher's education; teacher's experience; school settings; classroom culture; beliefs about mathematics and its teaching; TGs; and planning of lessons. Additionally, three consecutive mathematics lessons per teacher were videotaped. When videotaping during the lessons, I used two cameras: one teacher camera that captured the teacher's actions and talk, and one whole-class camera focused on the students' actions and talk. I conducted and

transcribed the audio-recorded interviews (50-110 minutes) and the videotaped lessons (40-60 minutes). *FM* (Asikainen, Nyrhinen, Rokka & Vehmas, 2015) includes references to the Swedish national core curriculum (Lgr 11), but does not describe how the lesson goals actually serve to prepare students to meet the curriculum goals. Earlier studies (Koljonen, 2014) revealed that *FM* lacks educative support (cf. Brown, 2009) for teachers as well. For example, the rationales behind its suggested lesson activities are rarely explicitly discussed, which is a critical component in teacher learning. Each lesson (4 pages) in *FM* has a similar structure, both visually and content-related; for instance, clear recurrent headings located in the same place on every page, and a variety of optional activities presented for each lesson. The activities are all linked to the central content and the lesson objectives, from which the teacher is to choose appropriate activities for their practice.

As a starting point in the larger cross-cultural project, I selected one of the Swedish teachers, Cecilia (fictitious name), to exemplify a single case as this approach offers possibilities for deeper theoretical insights that would otherwise go unseen (Andrews, In press). Cecilia graduated in 2010 as a compulsory school teacher (F-6), and was prepared to teach all other subjects besides mathematics as well. Thus, she is not a mathematical subject specialist. However, one of the criteria for selecting the teachers was that they were regarded as locally competent (cf. Clarke, 2006). Among the other teachers, Cecilia is recognized and esteemed for her locally defined 'teaching competence' and has been nominated by the school's principal and the municipality and is thus regarded as a local subject specialist. At the time this study was conducted, Cecilia was in her third year of teaching with FM. She teaches children in Grade 3; her 24 students come from a constrained socio-economic area, with mostly non-native speaking families. Cecilia volunteered to participate, knowing the study was on *FM* and its use.

Data analysis

Teaching is viewed as a cultural activity (Stigler & Hiebert, 1999) and a design activity, whereby teachers craft instruction, and do so with different degrees of artifact appropriation: offloading, adapting and improvising (Brown, 2009). Cecilia's interaction with TGs is characterized through these three analytical constructs. *Offloading* emerges when a teacher follows material and assigns a great degree of authority to the teaching material. That is, the agency for the delivery of content lies in the material. *Adapting*, on the other hand, occurs when a teacher reflects when elaborating with the material. Here the agency is embedded in both the material and the teacher. *Improvising*, finally, relates to when a teacher does not closely follow the material. That is, the agency lies with the teacher as she relies on her own strategies for instruction, with minimal reliance on the material. The relationship is further characterized as participatory or non-participatory (cf. Remillard, 2005). When the teacher regularly and deliberately uses the material, and also looks at it critically, this provides an intimacy between teacher and material and is thus categorized as a *participatory relationship*. Meanwhile, if the teacher's use of the material is more tacit and sporadic, it will lack intimacy and is thus categorized as a *non-participatory relationship*. My intention is to characterize Cecilia's interaction with the material in use (*FM*) and to compare the written curriculum in *FM* and Cecilia's enacted curriculum. I do not intend to evaluate which degree of interaction (offloading, adapting or improvising) or the relationship (participatory or not) is better than the other. However, I acknowledge that comparison and evaluation are intertwined (cf. Jablonka, 2015). Below, I present

Cecilia as the telling case through some merged snapshots of from both interview and classroom recordings.

The telling case of Cecilia

During her interview Cecilia said that it is a waste of time making a written plan, because "if a lot of the students don't understand today's lesson, we would have to repeat it tomorrow and then my intended plan would crash if I'd written it down" (Cecilia, 9 Nov 2015). On the one hand, such comments indicate that Cecilia, as a locally competent teacher, trusts her ability to deliver the mathematical content with appropriate strategies for instruction. On the other hand, it can also be due to a lack of time that she does not write her plans, as she stressed that the ongoing national professional development program, Matematiklyftet, takes time away from all the things she has to do. Cecilia stated that she starts her planning for the introduction phase not by using the TG but the student textbook:

> I turn to the current page in the textbook and see that the next passage is about multiplication by 9. Immediately, I have an idea about which strategies I want the students to know, and notice that the book is using the same strategy as me... but I prefer to create my own [instructions] using my own language. (Cecilia, 9 Nov 2015)

Cecilia's prospective mental plan is partially consistent with the textbook's plan. However, here the agency stays with Cecilia, since she claims to have her own mental plan. Her use of and interaction with the textbook could be understood through the Swedish teachers' context, in which they are not accustomed to using TGs in planning and teaching. In addition, the minimal support provided for how TGs may be used may compel Swedish teachers to turn to the textbooks instead. This and several other similar excerpts led me to infer that Cecilia is influenced by Swedish culture, as she states that she "prefers to create her own instructions", reflecting a generally held perception of Swedish teacher competence. During the interview, Cecilia mentioned, due to lack of time, that she occasionally glances at the TG to get a skeleton plan for the lesson. She then looks at its "introduction box", which suggests how to introduce the lesson's topic on the board. Hence, from the interview I infer that she perceives the material as worth having in the classroom but not necessary for planning. I infer that she improvises when planning, and that the agency lies with her. I further infer that she has a more tacit than close relationship with the material, especially since she seldom uses the TG and hardly ever reflects on the material or its impact on the context.

The video data reveal that, while mobilizing the teaching, Cecilia sequenced her lessons into four distinct episodes. The allocated lesson time of her lessons consists of: the introduction phase, taking approximately 22% (~10 min) of the lesson time; what to work on in the student textbook; the students' individual seatwork, taking approximately 56% (~25 min) of the lesson time; and the closure of the lesson. Cecilia always starts her lessons by showing a strategy or method in the introduction phase that is applicable to that day's pages in the student textbook, and by referring to textbook: "Hey, listen! Last Thursday we went over page 90 in the textbook, as we used these hands [pointing to the cut-out hands in fabric on the blackboard] as one strategy for multiplying by 9. Today we'll revise it" (Cecilia's L1, 9 Nov 2015). Here, Cecilia is simultaneously showing the textbook pages they have been working with. This revision is not included in the TG's suggested lesson plan, and no elaboration or reflection is revealed. But this could also be due to the evaluation at the last

lesson closure. Hence, I infer that Cecilia is improvising. Nevertheless, the video data frequently show that the delivery of the content is based on the material, as she offloads the agency to the textbook as she follows the textbook pages, lesson by lesson. Cecilia is very firm during the interview that the cut-out fingers she refers to are not from the TG but were instead an idea that simply came to her. During Lesson 1, she first shows two examples of the old strategy before introducing a new strategy for multiplying by 9: "Hey guys, listen! At the top of page 91, it says 'Multiply and write in the table'… This is a different strategy… So, let's try this too! Ehh, *they want us to think like this*... Can you give me a multiplication from the 9 table, Ali?" (Cecilia's L1, 9 Nov 2015). This extract illustrates Cecilia turning to a rather close offloading to the student textbook – especially when she says *they want us to think like this*. The textbook lacks a description of how to deal with this task, and the fact that Cecilia does not explain to the students how "to think" indicates that she has not elaborated on this task beforehand. I infer that she shows this task since it is included on that day's lesson pages in order to prepare the students for their individual seatwork. However, the TG offers some explanation, and recommends that they fill in this table together in the whole-class setting, which Cecilia has missed since she does not read the TG carefully or regularly. This displays that her relationship with the TG is rather tacit. The video data further reveal that after the introduction Cecilia always tells the students which pages to work with during the individual seatwork. She does this through the material's website and the SmartBoard, where she shows the students the pages. She also writes the pages on the whiteboard. This procedure is not stated in the material, which confirms that Cecilia is improvising and maintaining the agency. The following is an example of how she transitions the students into their individual textbook work: "I think most of you managed to do both pages 90 and 91, and possibly also 92 or 93. On page 94 it says 'We rehearse', and these two pages are the goal of today's lesson" (Cecilia's L1, 9 Nov 2015). This extract additionally confirms that Cecilia is closely offloading to the textbook, as she assigns a great degree of authority to it. At no time does she present the lesson objectives, which are clearly visible in the TG. Instead, she mentions that the lesson goal is to do pages 94 and 95. Neither does she use the different recurrent activities or concrete materials included in the TG during any of these lessons.

The video data further reveal that, during the individual seatwork, some students are working on other pages than the ones Cecilia had announced before they started working individually, and some are even working in a textbook for Grade 2. This is not in line with the material's intention, as it offers embedded differentiation instead. As shown in the video data, Cecilia always closes the lessons with a blind evaluation to determine whether she can move on or if rehearsal is necessary.

> Now, close your eyes and answer YES to my question by raising your hand. If your answer to my question is NO … leave your hand on the table […] 'I feel confident about the strategy of using my fingers to multiply by 9' Okay, those of you who have your hands up can put them down. 'I still think it feels a little hard to use this strategy, using my fingers to multiply by 9' Thanks! ... 'I feel pleased with what I did during the mathematics lesson today' Great! (Cecilia's L1, 9 Nov 2015)

This extract confirms that Cecilia does not just say she wants all her students to be *on track* but that she also checks this before ending the lesson. In so doing, she is checking their understanding of the "old" strategy for multiplying by 9, but not whether they understand the "new" strategy she has introduced, or the objectives displayed in the guide. Two of the questions are related to the

mathematics, whereas the last is connected to students' individual seatwork. There is no support in the material for how to end the lessons, so Cecilia trusts to experience and improvises the evaluation. Hence, from the video data I deem that she uses the textbook for support for the students' individual seatwork but not for her actions or events when mobilizing the teaching. The video data show that Cecilia improvises, but does not critically reflect on the material or its impact on the educational context, or make any changes in relation to the material. In addition, at several points, the interview and video data show collectively that Cecilia largely offloads the agency to the textbook and uses it on an ad hoc basis. Cecilia's use of the TG is minimal. Thus, this settles her weak interaction with the material; i.e., having a non-participatory relationship with it.

Discussion and conclusion

In this paper, I present the telling case of Cecilia, a Swedish teacher, locally regarded as competent. The aim is to reveal her interaction with an imported TG from Finland when placed in her specific context. Thus, the material is sited in a new educational context. In the analysis I compare the written Finnish TG with Cecilia's actual classroom practice. The analysis is therefore combined with in-depth descriptions and snapshots of events, and is thus in line with the telling case (cf. Andrews, In press) as an attempt to make visible how she interacts with the Finnish material and how this interaction may have affected her classroom practice.

First, how does Cecilia interact with the Finnish material? My analysis revealed that Cecilia uses the student textbook when teaching, and that she offloads agency to the textbook. This interaction is categorized as non-participatory since it lacks intimacy. Her interaction with the TG is even weaker, and more sporadic and tacit than with the textbook, and is thus also non-participatory. When she trusts in her own knowledge and experience, improvising occurs, especially in regard to the repetition at the beginning of the introduction phase and the closure of the lesson with the blind evaluation. No adaptation was observed, since no equally embedded agency was found. Cecilia says she creates her lesson plans mentally. However, her focus is not on the entire lesson, since she only prepares the introduction phase. Even though she has chosen *FM* due to her judgment of its overall good quality, she does not seek support for teacher learning or to enhance the variety in her lessons through its recurrent activities. *Second, is Cecilia's classroom practice affected by her interaction with the Finnish material?* My analysis revealed that Cecilia's enacted classroom practice mirrors the "typical Swedish" practice, with short introductions and then individual seatwork most of the time (cf., e.g., Remillard et al., 2016). Cecilia does not keep the students together around a specific mathematical topic by using the embedded differentiation, and no concrete materials are used during these three lessons. No objectives are stated, either. These are all important parts of the cultural scripts found in Finnish TGs (Koljonen et al., under review). Thus, Cecilia's classroom practice is in contrast to those promoted by the Finnish TG. I deem that Cecilia's practice is marginally affected by her relationship with the material. This may be because it does not offer enough support for how to use it, or explain its intentions, therefore forcing Cecilia to follow the common norms of Swedish classroom practice; as well as the fact that it is challenging for teachers to change their teaching (Stein et al., 2007; Stigler & Hiebert, 1999). Further studies are needed to capture the essence of the Swedish classroom practice when using imported material.

My conclusion is that the use of the originally Finnish material has not had the intended impact on the practices as promoted by the guides. Instead, Cecilia uses and confirms her preexisting culture

rather than the intended one as in the Finnish TG (cf. Davis, Janssen & Van Driel, 2016; Stein et al., 2007; Stigler & Hiebert, 1999). One possible implication of this is that it may be hard to implement material from other educational contexts, even if it is quite similar and is assumed to change or even improve the quality of teaching. Yet without targeted support for how to use new material it is hard, even if a teacher is regarded as competent, to independently conduct changed or improved teaching and simultaneously maintain or gain pedagogical autonomy. This is especially important since the Finnish material lacks educative support and, thus, is not regarded as educative material (Hemmi, Krzywacki & Koljonen, 2017; Koljonen, 2014). I argue that this requires that teacher materials be included in professional development programs, as previously argued for by Ball and Cohen (1996), in order to proficiently convey and highlight the principles of the materials and adjust them to the new context that is underpinned by the social and cultural practice. It remains to be seen whether subsequent case studies of the other teachers in the larger project reveal whether the above-mentioned tentative conclusions hold for the larger data set.

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Algorithmics in secondary school: A comparative study between Ukraine And France

Simon Modeste[1] and Maryna Rafalska[2]

[1]Université de Montpellier, IMAG – UMR CNRS 5149; simon.modeste@umontpellier.fr

[2]National Pedagogical Dragomanov University; m.v.rafalska@npu.edu.ua

This article is focused on the teaching and learning of Algorithmics, a discipline at the intersection of Informatics and Mathematics. We focus on the didactic transposition of Algorithmics in secondary school in France and Ukraine. Based on epistemological and didactical frameworks, we identify the general characteristics of the approaches in the two countries, taking into account the organization of the content and the national contexts (in the course of Mathematics in France and in the course of Informatics in Ukraine). Our results give perspectives on understanding the place that Algorithmics can hold in the teaching and learning of Mathematics and Informatics.

Keywords: Algorithm, algorithmics, didactic transposition, curricula comparison, France, Ukraine.

Introduction

Mathematics and Informatics[1], as disciplines, have strong links. On one hand, Mathematics gives theoretical basis and instruments to Informatics, and on the other hand, Informatics enriches Mathematics with new objects and problems, and brought some changes in the mathematical activity. Many disciplines (Discrete Mathematics, Algorithmics, etc.) developed at their interface. Nowadays, there is an international movement towards including Informatics and these subjects in secondary education. Algorithmics is more and more present in secondary school in many countries. It can be involved in programs of Mathematics (as in France) or Informatics (as in Ukraine). This situation questions the goals of teaching and learning Algorithmics in secondary school, the contents of the curricula as well as the approaches and activities proposed to pupils. To contribute to the study of these issues, we propose a comparative study regarding the concept of algorithm and the contents of Algorithmics in secondary school in France and Ukraine.

Algorithm is a central notion in Mathematics and Informatics. Algorithm in "classic" Mathematics is generally used with the meaning of a general effective procedure for solving problems. Since the origins of Mathematics, algorithms have been designed and used for solving problems (arithmetic operations with decimal numbers, solving equations, etc.). The use of computers and programming languages brought a new point of view on the notion of *algorithm* referring to the formalization of procedures, the automation of computations and the problem of data treatment. In this context, the notions of finiteness, iteration and recurrence play an important role (Chabert, 1999, pp. 6–7).

As a reference, we retained the following definition of an algorithm given by Modeste (2012, p. 25) and based on the academic literature on the subject: "a procedure for solving a problem which, in a finite number of constructive, effective, not ambiguous steps, gives a result for any instance of the

[1] We will use the term "Informatics", a more faithful translation from the French and Ukrainian than Computer Science.

problem". We will delimit the discipline Algorithmics as the field that deals with algorithms, the problems they can solve, their design, their use, their analysis and their comparison.

Research questions, theoretical framework and methodology

To formalize our problematic, we have formulated the following research questions:

- What is the place of Algorithmics in secondary school in France and Ukraine?
- In what way the learning contents related to Algorithmics are organized?
- What types of activities in Algorithmics are proposed to pupils in both countries?
- What common points and differences appear? How can they be interpreted?

Our study is based on the concept of didactic transposition (Chevallard, 1985) developed in the Anthropological Theory of the Didactic (Bosch & Gascón, 2014). The didactic transposition describes the processes of transformation between academic knowledge, the knowledge to be taught and the taught knowledge. It takes a step back from the curriculum and the content actually taught, to understand the role and the influence of the different institutions involved in these processes.

In order to analyse the content devoted to Algorithmics in the curricula, to measure their distance to the academic knowledge, and to develop our comparative study, we lean on an epistemological framework, in accordance with a classical methodology in Didactics of Sciences and Mathematics (Artigue, 1990). Concerning the concept of algorithm and Algorithmics we used the epistemological framework developed by Modeste (2012; Modeste & Ouvrier-Buffet, 2011). The epistemological model distinguishes five fundamental aspects of the concept of algorithm: ***problem*** aspect (the fact that an algorithm is a tool to solve all instances of a problem, the notions of input and output); ***effectiveness*** aspect (all elements referring to the fact that an algorithm solves problems effectively: the notion of operator; the finiteness of instructions and executions, etc.); ***complexity*** aspect (all elements referring to the notion of complexity of algorithms and problems); ***proof*** aspect (referring to the links between algorithm and proof); ***theoretical models*** aspect (referring to the theoretical works in Logic and Informatics concerning the concept of algorithm). Among these aspects, *problem* and *effectiveness* refer to algorithm as a tool, whereas *proof, complexity* and *theoretical models* refer to algorithm as an object. This tool-object dialectic of the concept of algorithm will be useful to understand the points of view in different institutions.

In this study, we analysed the official instructions, the official resources for teachers and particular textbooks (we focused on the knowledge to be taught, but our analysis of textbooks also informs about the taught knowledge). For this purpose (addressing the didactic transposition in a given institution), it is not sufficient to study the discourse about algorithms: it is essential to examine the algorithms selected by the institution, their representations and the activities involving these algorithms: this can reveal differences (and even contradictions) with the general discourse. To do this, for each resource, we answered to the following questions:

- What kind of definition of algorithm is proposed?
- What algorithms (or types of algorithms) have been selected?
- What representations of algorithm are used?
- What tasks (or types of tasks) in Algorithmics are proposed?

Answering these questions brings elements that permit to address our principal question:

- What aspects of algorithm are presented (according to the five fundamental aspects)? Do this aspect relate to algorithm as a tool or as an object?

Our corpus of resources is described in Table 1. In the following, we present the results of our analyses and the answers to the research questions. As we cannot provide all details in this paper, we present only the main results of the comparison. Before that, it is necessary to give an overview of the situation in France and Ukraine, regarding the teaching of Algorithmics.

Resources	France	Ukraine
Official instructions and documents	Official program of Mathematics for middle school; Official curricula of Mathematics for high school: grade 10, grades 11 and 12 (all paths); Official accompanying resources in Algorithmics for grade 10; Official program for the ISN option.	Official program of Informatics for middle school (5-9 grades); Official programs of Informatics for high school (grades 10 and 11): standard, academic, professional and advanced levels.
Textbooks	Three collections of Mathematics textbooks (Indice, Bordas ; Math'x, Didier ; Transmath, Nathan) for grades 10, 11 and 12 (scientific, economic and humanities paths) ; Textbooks for middle school are not available yet.	One collection of textbooks in Informatics (Ryvkind J.Ya. et al., 2011) for grade 11, standard and academic levels, part 'Algorithmics'; two collections of textbooks in Informatics (Morze N.V. et al., 2014, 2015, Ryvkind J.Ya. et al., 2014, 2015) for grades 6 and 7, part 'Algorithmics'.

Table 1: Analysed resources.

Presentation of the contexts and evolutions of curricula in Ukraine and France

Situation in Ukraine (organization, history and recent evolutions)

The Ukrainian school system consists of primary school (grades 1-4, age of pupils – 6-9 years), middle school (grades 5-9, 10-15 years) and high school (grades 10-11, 16-17 years). In high school pupils make a choice between general or professional-oriented paths. Informatics has been taught since 1985 in secondary school in USSR. From the beginning, Algorithmics was a part of it. At that time, the course was mainly dedicated to writing algorithms in pseudo-code and executing them manually. Only two programming languages were used: Rapira (especially elaborated in USSR for teaching) and Basic. The major part of the first manual of Informatics in USSR is devoted to solving algorithmic problems. An algorithm is defined as “a clear and precise instruction destined to an operator to carry out a sequence of actions in order to reach a goal or solve a problem”. In this manual, the notion of *operator* plays a central role. It is also underlined that an operator executes an algorithm formally, i.e. it can carry out operations one by one in defined order without understanding the goal. A *scheme for solving a problem with a computer* is presented: in brief, it consists of modelling the problem, constructing an algorithm, writing it in a programming language, executing it, and analysing the results. Many tasks require constructing algorithms for solving mathematical and physical problems with a computer, such as Horners' method or the approximation of the area under the graph of a positive function.

Since 1985, Informatics has always been a mandatory subject in high school. In the programs published in 2008, Informatics can be taught at the following levels: standard (for general and humanitarian paths), academic (for science path), professional and advanced (for informatics and mathematics paths). The difference is the number of hours of Informatics per week (between 1 and

5) and the contents. As a part of the subject, Algorithmics is studied at every level with a total amount of hours that varies a lot: Standard (5), Academic (28), Professional (175), Advanced (191). In 2013, Informatics also became a mandatory subject in primary (from grade 2) and middle school. Thus, at the moment, Algorithmics is also present in middle school (from grade 6).

Situation in France (organization, history and recent evolutions)

The secondary French school system consists of middle school (*collège*, grades 6-9, 11-15 years) and high school (*lycée*, grades 10-12, 16-18 years). In high school, professional, technical and general orientations are proposed, and the general orientation is divided into humanities, economic and scientific paths[2]. In this study, we will concentrate on the general orientation. Recently, many reforms happened in the French curricula, the last one was in 2016 and concerned middle school. Gueudet, Bueno-Ravel, Modeste and Trouche (to appear) give more details about the evolution of French mathematics curricula, including Algorithmics. Informatics appeared for the first time in the French curricula in the 1980's (Baron & Bruillard, 2011), with an introduction to Programming and Algorithmics in high school. It disappeared in the 1990's, replaced by the use of computer tools and new technologies. Recently, Informatics came back in secondary school. In 2012, an optional course was created in grade 12 in the scientific path (ISN: "Informatics and digital sciences", 2h/week), and in 2015, an optional course appeared in grade 10 ("Informatics and Digital Creation", 1h30/week). Starting from 2016, Informatics will also be taught in middle school (principally in grades 7-9) in the mathematics and technology classes. Few years before that (from 2009 for grade 10 to 2012 for grade 12), some contents of Algorithmics were introduced in the curricula of Mathematics in high school (Modeste & Ouvrier-Buffet, 2011).

Comparison at high school level (grades 10-12, ages 15-18)

Algorithmics in high school in Ukraine

Through all grades and levels, a common approach to Algorithmics can be identified. It includes the presentation of *the steps for solving problems using a computer* and the role of algorithms in this process, distinguishing algorithms from programs, with an emphasis on the notion of *operator*. The activities involve various representations of algorithms (common language, flowchart and program). In the analysed textbooks the term *algorithm* is defined as a finite sequence of instructions that determines what operations and in which order to carry out for obtaining a goal. In this definition, as well as in the description of the properties of an algorithm (discreteness, certitude, feasibility, finiteness, effectiveness) given explicitly at all levels, we can identify the aspect *effectiveness*. The *problem* aspect is expressed in the property of "generality" of an algorithm, which says that an algorithm applies to a set of similar problems, which have the same question and solving procedure and differs only by the values of initial data. At the same time, the specific term "instance algorithm" is used in the textbooks to define an algorithm that solves only one case of a problem. Most part of the proposed instance algorithms are: algorithms of daily life (e.g., preparing meal), algorithms from others disciplines (e.g., geometrical constructions), algorithms implementing a

[2] For details about French system: http://eduscol.education.fr/cid66998/eduscol-the-portal-for-education-players.html

strategy (e.g., the wolf, goat, and cabbage problem). The expected competence is to represent the solution of a problem in an algorithmic way (as a sequence of instructions) rather than to solve it.

At **standard level**, many tasks concern the construction of algorithms for a given operator. The activity is centred on identifying the system of commands of an operator and writing an algorithm using only these commands. Although, most of the problems are quite easy and the goal of the tasks is to find the best strategy and present it in the required form. This concerns principally instance algorithms and refers to the *effectiveness* aspect. Generic algorithms (algorithms with many instances) mostly relate to solving mathematical problems (solving equations, evaluating the area of a polygons.) and computations (evaluation of simple expression). Pupils construct algorithms, describe them in the requested form (principally flowcharts) and execute them manually.

At **academic level**, Algorithmics is based on Delphi, an object-oriented programming language: pupils get accustomed to Algorithmics by learning the instructions of Delphi. 65% of the tasks are about writing programs, executing and modifying them, 19% of tasks are devoted to object-oriented programming. Algorithms are mostly verified by testing the programs. *Generic algorithms* are more present than *instance algorithms*. Most of them concern computations (e.g., evaluation of simple expressions) (33 %) and solving mathematical problems (e.g., solving equations, primality test) (31%). A bit less tasks concern the computation of sequences, products, sums (11%) and data treatment (e.g., sorting, searching in an array) (17 %). The textbook includes many tasks where an algorithm is only used for formulating a procedure before programming it. One can also found tasks where algorithms plays the role of *tool* for problem solving (e.g., finding the divisors of an integer). In this case, the focus is more on the construction of algorithms than on writing and debugging programs. Although we found many generic algorithms, only the *effectiveness* and *problem* aspects are strongly present. The *complexity* aspect is only evoked concerning the binary search algorithm.

At **professional** and **advanced levels**, an algorithm is presented not only as *tool* but also as an *object*. At **professional** level we found many "rich" algorithms, such as recursive algorithms, algorithms on graphs, algorithms of treatment of stacks and lists, etc. Expected competences for pupils are not only to understand some algorithms and write programs, but also to analyse algorithms' efficiency and compare them. The *complexity* aspect is also evoked. At **advanced** level the *theoretical models* aspect is present (e.g., NP-complete problems). Expected competences concern the abilities to choose an algorithm appropriate to a problem, to compare algorithms according to their complexity, to analyse and compare algorithms. Algorithm is present as an object.

Algorithmics in high school in France

Programs of Mathematics for high school, for all levels, contain the same Algorithmics part, with a list of expectations for the end of high school: pupils must be able to "describe algorithms in natural and symbolic languages", to "carry out some of them using a spreadsheet or a small program written in a calculator or a software" and "interpret more complex algorithms". It is mentioned that "algorithmics has a natural place in all the mathematical subjects". Pupils must learn elementary instructions, conditional instructions and loops. At each level, few specific algorithms are mandatory, e.g. plotting a curve (grade 10); solving equations of the type *f(x)=0;* simulating random experiments (grade 12, scientific path). Most of the algorithms in the programs deal with sequences, numerical methods and simulations in probability and statistics. Algebra and geometry are just

mentioned as fields for algorithmic activities. Discrete Mathematics have a very little presence. The priority seems to be given to the implementation of algorithms in a programming language.

The **accompanying resource for grade 10** – that seems to have driven the approach to Algorithmics in high school (Modeste, 2012) – does not define the term *algorithm*, and does not even distinguish it from the term *program*. The activities are focused on language and rigorous expression of operations, and often aim at writing programs. It results in a confusion between *program* and *algorithm* that indicates a focus on the *effectiveness* aspect. A specific language to describe algorithms is implicitly developed, mixing pseudo-code and technical programming constraints – that we called *paper-program* (Modeste, 2012). Many instance algorithms are present, which confirms a confusion between writing algorithms and describing step-by-step operations.

In the studied textbooks for all levels, we can see the strong influence of this accompanying resource. Most algorithms are described as "paper-programs" before being implemented (generally as immediate translation). Many exercises deal with interpreting, writing or translating algorithms in a given language. Algorithm is only shown as a tool, even the problem aspect has little presence. In the program for the **ISN** option, the approach differs. The program explicitly defines the notion of algorithm and mentions that it must be distinguished from the notion of program. Algorithmics is presented as a branch of Informatics and algorithms are not restricted to programming. The concept of algorithm appears as a tool and as an object (*complexity* and *proof* aspects are present).

Comparison in high school

In Ukraine as in France, the effectiveness aspect is central. In all levels, algorithm is used as a tool, but the *problem* aspect is more developed in Ukraine. Algorithm is treated as an *object* only at professional and advanced levels in Ukraine, and in the ISN option in France. We could have expected them to appear in the French scientific paths but it is not the case. In Ukraine, the approach to Algorithmics seems to be guided by the development of algorithmic thinking whereas in France the focus is on the programming and language skills. It appears clearly in the texbooks: in Ukraine, the concept of algorithm is defined and a list of its properties is given, whereas, in France, an algorithm is defined by the language that describe it. Although in France, Algorithmics is taught in the Mathematics class, the focus on programming seems to be stronger than in Ukraine for standard level, where programming is not required and the focus is on elementary algorithmic thinking. In Ukraine, two features can be highlighted, probably inherited from historical context of teaching of Informatics in USSR: significant role of the scheme of problem solving (presumably influenced by the problem-solving theories) and the emphasis put on the notion of operator. In France, in Mathematics, algorithms are used to solve mathematical problems and are considered as a mean to deal with the mathematical concepts. The important presence of programs for simulations in probability or for embodying properties of mathematical objects attests to this point of view. The approach developed in ISN, in France, is close to the approach proposed at professional and advanced levels in Ukraine. They involve advanced concepts and aim at developing advanced algorithmic thinking, but we suspect a difference between the programs and the taught knowledge.

Comparison at middle school level (grades 6-9, ages 11-15)

Algorithmics in middle school in Ukraine

In the programs for grades 6 and 7, the part devoted to Algorithmics is similar to the program for standard level of high school. Although the program declares programming as one of the pupils' activities, it does not specify any programming language to use. In the textbooks for grades 6 and 7, an algorithm is defined as a finite sequence of instructions to be carried out for solving a problem. As we can see, in the given definition the effectiveness and problem aspects are on the first plan. At the same time, the fact that an algorithm solves all instances of a problem is not presented. The main part of proposed problems concerns the construction of instance algorithms for different operators. Both textbooks propose to program in Scratch. Pupils' activity is focused on developing programs and projects, using this programming environment. In grades 8 and 9, the Algorithmics part of the program is devoted to object-oriented programming. In grade 8, the notion of variable and different types of data are introduced. In grade 9, search algorithms in arrays are studied. The expected competences of pupils refer mostly to writing, modifying and debugging programs. The aspects related to algorithm as an object are not present.

Algorithmics in middle school in France

In 2016, Informatics appeared in middle school. Algorithmics contents essentially appeared in the Mathematics course, in the cycle 4 (grades 7-9), in the theme "Algorithmics and Programming". Textbooks for this reform were not available at the time of the study, so we only analysed the programs. One general competence guides the program: "write, elaborate and execute a simple program". Then, more specific competences are listed (decomposing a problem into sub-problems, designing a program to solve a problem; writing programs driven by events; and writing parallel programs) and contents are specified: notions of algorithm and program; variables in Informatics; event-driven action, sequence of instructions, loops, conditional instructions; exchanged messages between objects. These contents are strongly oriented towards programming and, even if it is not declared, the software Scratch must be used to teach these notions. The chosen approach implies developing projects and games (in order to develop pupils' reasoning) and does not focus only on mathematical concepts. Effectiveness aspect of algorithm is present and problem aspect is more notable than in the high school curricula. Contrary to the approach proposed in Mathematics in high school, Algorithmics is introduced by programming (independently from mathematical contents).

Comparison in middle school

In France and in Ukraine, in middle school, algorithm is presented as a tool. The *effectiveness* aspect is dominant. Although the *problem* aspect is mentioned, the role of algorithms for problem solving and the place for generic algorithms are not clear. At this level, the *complexity* and *proof* aspects are not proposed. Contrary to the curricula for high school, there are more similarities between the two curricula for middle school. Particularly, the notions of algorithm and program are distinguished; the introduction to Algorithmics includes event-driven programming in Scratch, and objects and variables are introduced later. The approach is based on solving concrete problems and developing projects in Informatics and in other disciplines. This could be explained by the influence of an international movement towards the teaching of Informatics in primary and middle school. Nevertheless, there are also important differences. In France, the most part of the proposed projects

are in Mathematics (maybe because it will be taught by Mathematics teachers, not well trained yet in Informatics), and there is a strong focus on programming (in a different way from high school). In Ukraine, the notion of *operator* is still highlighted, representations of algorithms with schemes and flowcharts are requested and many examples of algorithms are taken from everyday life. This is directly inherited from the didactic transposition proposed in the 1980's and today in high school.

Conclusions and perspectives

On the one hand, this comparative analysis of the didactic transposition of the concept of algorithm and Algorithmics in secondary school in France and Ukraine brings out differences that reveal the impact of institutions, traditions and historical contexts on the curricula. The comparison of two contexts where Algorithmics is not a part of the same course (Mathematics versus Informatics) shows the influence of these disciplines on the contents, on the points of view on Algorithmics and on the algorithmic activity. On the other hand, in the two countries, we see general orientations in middle school that seem to be part of an international movement towards the teaching and learning of Informatics. This study contributes to understand and improve curricula, by taking into account the points of view of Mathematics and Informatics on Algorithmics. It gives perspectives to study the development of algorithmic thinking, and the teaching and learning of Algorithmics' concepts.

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Ten years of mathematics education: Preparing for the supermarket?

Mona Nosrati[1] and Paul Andrews[2]

[1] Norwegian Centre for Mathematics Education, Norwegian University of Science and Technology Norway; mona.nosrati@matematikksenteret.no

[2]Stockholm University, Sweden; paul.andrews@mnd.su.se

In this paper we investigate Norwegian and Swedish upper secondary students' perspectives on the purpose of school mathematics. Students were group interviewed in various schools in both Norway and Sweden and the video recordings of those interviews fully transcribed. In each country transcriptions were subjected to a constant comparison analytical process that, coincidentally, yielded the same two dominant themes that we report here. Firstly, all students spoke about how learning mathematics facilitates being able to manage shopping, personal finances and other functional aspects of the real-world. Secondly, students spoke about how learning mathematics would facilitate their getting a job, including some Swedish students who saw mathematical success as a high status qualification preferred by employers.

Keywords: Purpose of school mathematics, Norway, Sweden, student perspectives.

Introduction

Despite evidence that large scale assessments of mathematics achievement like the Organisation for Economic Cooperation and Development's (OECD) Programme of International Student Assessment (PISA) have prompted curricular changes in many OECD member countries (Breakspear, 2012), the nature of school mathematics is not an unproblematic given and varies considerably cross-culturally (Andrews, 2016). Indeed, all curricula, which are based on a culture's conception of an ideal person (Cummings, 1999), are rooted in substantially more than what can be inferred from official documents. Consequently, the ways in which teachers conceptualise and present mathematics varies from one cultural context to another, inevitably influencing the beliefs that students form about mathematics and its purpose (Cobb, 1985).

Students' beliefs about mathematics and its teaching have been the focus of much research for more than four decades, following Erlwanger's (1973) case study of Benny, a twelve year-old student who had come to see mathematics as invented rules, each for a particular type of problem, that work like magic. Benny's beliefs about mathematics effectively laid the ground for the field, not only from the perspective of what students believe about mathematics and its teaching but also the ways in which teachers' beliefs and practices are implicated in the construction of such beliefs. In short, the significance of this work draws on the premise that students' beliefs are informed by, and inform how they respond to, their mathematics-related learning opportunities (Erlwanger, 1973; Callejo and Vila 2009). In this regard, "beliefs constitute, for the believer, current knowledge about the world" (Cobb, 1986, p.4).

Over the succeeding years, research into students' mathematics-related beliefs has addressed a variety of themes, which have been well summarised by Op 't Eynde and his colleagues (Op 't Eynde et al., 2002; 2006), which they present under three broad headings; beliefs about mathematics

education, beliefs about the self as a mathematician and beliefs about the mathematics class context. Within these three broad headings, which space prevents us from discussing in depth, are located most aspects of mainstream beliefs-related research. However, a strand that has received less explicit attention the literature, particularly in the Scandinavian context, has been the beliefs students hold about why they are compelled to learn mathematics for so many years.

The purpose of school mathematics

In general terms, the reasons why children spend so many years studying mathematics are not always clear, although typically they seem to be tied to ensuring either the social, economic and political mobility of the individual citizen or the nation's economic growth through preparing the next generation of employees (Schoenfeld & Pearson, 2009). Such perspectives are philosophically rooted in either personal emancipation or the reproduction of the existing social order respectively. That being said, while much philosophical research has addressed the general aims of education, "very little sustained work of this kind appears to have been carried out in mathematics education", to the extent that "there is sufficient disagreement, lack of clarity and modesty to warrant further enquiry" (Huckstep, 2000, p. 8). It is on this that we next focus, confirming Huckstep's concerns.

Various scholars have considered the purpose of school mathematics. For example, Niss (1996) discusses the justification and the goals of mathematics education from both theoretical and historical perspectives. Davis (2001) adopts a largely philosophical standpoint in suggesting three distinct purposes, which he describes as the teacherly, the rhetorical and the hermeneutic. The teacherly purpose for teaching mathematics derives from a set of largely forgotten reasons tied to the belief that "knowledge of mathematics is necessary for every citizen of today's world" (p. 18). The rhetorical purpose derives from an "explanatory fantasy that is currently preferred to organize and structure experience" (p. 19). The hermeneutic purpose lies in beliefs that "there are moral and ethical imperatives that operate in the human and in the more-than-human realms" (p. 21). In a slightly more prosaic manner, Noyes (2007) writes of six broad purposes; mathematics for the academy, for employment, for general education, for citizenship, for the information age and, finally, critical mathematics education. Ernest (2016), in a not unrelated manner, describes seven purposes; functional numeracy, work-related knowledge, advanced specialist knowledge, problem posing and solving, mathematical confidence, social empowerment through mathematics and, finally, appreciation of mathematics itself. Finally, Watson (2004), in a manner that seems to summarise both Ernest and Noyes and mirror Davis (2001), proposes three broad purposes; mathematics as a set of useful skills and procedures, a support for the burgeoning mathematics-related professional, and a tool that facilitates successful societal participation. In sum, as Huckstep (2000) indicated, mathematics is taught for various reasons tied to cultural norms and values.

Interestingly, despite research discussing the purpose of mathematics from an outsider perspective, few studies have engaged with the insider; what do recipients of mathematics teaching think is the purpose of what they are taught? In this respect, it is salient to note that Brown, McNamara, Hanley and Jones (1999), when investigating beginning primary teachers' understanding of mathematics and its teaching, found that "a sense of bafflement about the purpose of school mathematics permeated many accounts" (Brown et al., 1999, p. 305). That is, these students had completed school and still had no idea as to the purpose of school mathematics. In the light of such uncertainty and acknowledging the lack of research in the field, particularly in the context of Scandinavia, we

examine the beliefs about the purpose of school mathematics of both Norwegian and Swedish upper secondary students. As in Brown et al.'s (1999) study, these students are close to the end of their school careers and should have formed clear ideas as to the purpose of mathematics.

The study and its methods

The data on which this paper is based derive from a comparative interview study of upper secondary students in Norway and Sweden. In broad terms the study set out to explore students' perspectives on their many years of school mathematics. Students were interviewed in pairs or threes, and the interviews were structured around the following four questions, with follow-up questions where appropriate:

1. How would you describe a typical mathematics lesson at school?
2. What do you think is the purpose of compulsory school mathematics?
3. What do you think mathematics as a subject has to offer to those who engage with it?
4. If you could say something about the nature of mathematics education to those in charge of the educational system, what would it be?

It is the answers to question 2 that we will be focusing on in this paper.

The Norwegian data derived from 17 interviews involving 42 students from three schools. Two schools, one in Oslo and one in Trondheim, were high-achieving academic schools, while the third was a relatively low-achieving vocational school in Oslo. The Swedish data derived from 18 interviews involving 50 students from four schools. These schools, from various parts of Stockholm, all offered a range of vocational and academic tracks. Consequently, we make no claims about schools' representativeness nor do we seek to generalise. All participants were fully aware of the purpose of the research and of their rights to withdrawal.

Interviews, undertaken at a time chosen by the students, were video recorded on laptop computers, a decision justified in four ways. Firstly, video, especially when participants talk over each other, simplifies transcriptions. Secondly, video captures non-verbal communication. Thirdly, due to their classroom ubiquity, laptops were expected to create less disruption than tripod-mounted video cameras. Fourthly, laptops record data directly to their hard-drives, simplifying data storage and analysis. All interviews were transcribed and, in each country, subjected to a constant comparison analysis whereby each transcript was read and categories of response identified. With each new category, previously read episodes were re-read to determine whether the new category applied to them also. The two data sets were analysed separately to ensure the cultural integrity of the findings.

Results

The data from both countries yielded a variety of themes. However, two closely related themes dominated the analyses in both contexts and it is these we turn to in this paper. These were related to being able to function in the real world and getting a job, and were present in every single interview in one form or other. Below we give a more detailed description of the two themes as well as exemplifying interview extracts from both academic (A) and vocational (V) students. Other

themes, relating to mental training, appreciation, and uselessness were also present in the data (though much less so). However, it is beyond the scope of this paper to discuss those here.

Mathematics enables one to function in the real world: Norway

A large number of students emphasised the importance of mathematics in daily life, which typically seemed to consist of going to the supermarket or calculating one's salary and taxes. In respect of the former, Line's (A) comment was typical; "maths is a lot about everyday life… you wouldn´t be able to go to the store and buy goods if you didn´t have some maths". Similarly, Robin (V) commented that "when you are going to buy food and stuff, you need to know how much things cost and stuff".

With respect to both shopping and income, Tania's (A) comments were not atypical. She said that

> You need it in daily life and… you´re in the shop, you want to buy something, and then you must add… sum the price of the things you buy and stuff… like if you work, and if you want to know how much you earn and you want to know your net income... if you know it yourself you can double check if the person who does it has done it right and stuff.

While Ruben (V) added that "you can calculate numbers and stuff… it helps you on in life... if you work you can calculate your salary and stuff", before adding that "there is a lot of maths that you don´t need, but you need at least a part of it". Others offered less specific statements regarding 'daily life'. For example, Amalie (A) commented that she hadn't "thought that much about it" before adding that "you do at least have to know some maths… like in daily life, then maths is useful". Interestingly, she then offered the observation that "when we are at high-school level then I don´t exactly think that all the maths I learn will be useful".

In sum, the Norwegian students were confident that knowing mathematics would support their functioning in the real world, although there were occasions when the manner of this support was vague and imprecise. It was also interesting to see Amalie's and Ruben's comments that while mathematics was a real-world support, much of what they learned was unnecessary in this respect.

Mathematics enables one to function in the real world: Sweden

In every Swedish interview students spoke about how they saw mathematics as preparing them to face a world beyond school, mathematics as supporting their real-world functionality. Within this utilitarian strand two major themes were identified. The first concerned personal finance and the management of money. Here, almost all students spoke about understanding interest, as with Jacob's (V) comment that "percentages in terms of interest and loans and things like… that's very good to know because you might not really figure out in your head how much you can spend". In similar vein Pedram (A) said that "amortization…, interest rates, interest costs, to be able to figure it out, it is important". Others spoke more generally about the management of personal finances. For example, Mark (V) commented about the need to manage "larger sums, as well as your salary", while Kenneth, in the same vocational track interview, added that "it becomes much easier to make financial plans… if one has several years of mathematics". Finally, several students spoke of the need to avoid overextension, as in Göte's (A) concerns with respect to "SMS loans and stuff, there are many who do not know how much you lose there" and Omar's (A) worries that "there are too many adults today who do not really understand interest and how it works".

The second major theme relating to being able to function in the real world related to the mundane world of everyday shopping. Comments typical of others were like those of Alice (A), who said that "it's something you use in everyday life, it's typical when you go shopping, it's just the basics", or Mark (V), who said that "it's always good to have a base in maths. So, if you go shopping and have a hundred, then you cannot buy for more than a hundred". Others offered a slightly different perspective. For example, Hanna (A) raised the overall mathematical expectation of such transactions by commenting that "if you've gone shopping and there is a discount and know how to do that or how to pay what you're gonna pay in. It is like useful to use maths like this", while André (V) added that "you always look at the prices and compare them to the prices in another store".

Knowing mathematics enables one to get a job: Norway

A number of students spoke about how they saw their learning of mathematics in relation to their future careers. In most cases students spoke hypothetically as few seemed to have considered their own particular career aspirations. For example, Sarah (A) said that

> I think that maybe we need to spend as much time on maths as we do because we don´t know who ends up as engineers or who maybe works with something that you maybe don´t need a lot of maths for... you can´t know in advance, so everyone has to learn everything... so that you can be what you want.

In similar vein, Kristine (A) commented, after acknowledging "that is a bit of a difficult question", that

> it depends a lot on what you plan to do after high school... if you have planned to take a fairly high education then it is clear that there is a lot of maths; if you, for example, are going to be a chemist or a physicist or... medical studies too... or generally medicine, if you are to calculate dosages for a pain killer it is a bit bad to get a really huge dose… but if you plan to work at Mc Donalds for example, then it is not so important.

The comments of both students indicate an understanding that different professions will require different levels of mathematical competence. However, neither of them sees mathematics as an important component of education for everyone for its own sake. However, having observed that working in McDonalds does not require much by way of mathematical competence, Kristine added that "but how do we know from the start who will be a physicist and who will work at McDonalds?"

Other students noted that they do mathematics for as long as they do because it is a formal entry requirement for higher education (even if that higher education is unrelated to mathematics). In this respect, Andreas (V), in a comment typical of others, noted that "actually it is an entrance requirement for the college I plan to go to later... but apart from that I don´t see any reason to know such advanced maths". In such a comment, and those of Sarah and Kristine, lies a tacit concession that everyone has to spend twelve years learning mathematics *just in case* it might be useful, a conclusion summarised well by Gina (A), who said, "I think maybe you don´t notice the importance until later".

In a related manner, Malin (A) spoke of how mathematics supports the learning of other subjects. She commented that

> you need to use maths a lot in other subjects, so you have it in chemistry and physics and stuff... so maths is a very important tool… so it is not... if you look at the maths as a subject... just a subject, then it can be a lot of numbers and a bit like... but if you put it in the context of other science subjects, then it is the most important tool you have.

Knowing mathematics enables one to get a job: Sweden

Very few students did not, during the course of their interviews, refer to the ways in which learning mathematics would enhance their employment prospects. Within the Swedish data two major subthemes were identified. The first of these concerned students' beliefs about the generic ways in which mathematical knowledge would prepare them for work. At its most basic, according to Andreas (V), "any job requires quite some math knowledge, in all cases, plus or minus, often times. … and that applies to the majority of all jobs". More specifically, as in the comments of Thomas (V), "you get better chance of getting a better job with like higher level of mathematics", while Alice (A) believed that "when one has read maths, one can study further and get … more opportunities for the future, one can say, more career choices". Others still, saw beyond their first job, as with Kenneth's (V) comment that "even when you think you have found out what you want to be, you might want to switch careers later and then maybe you need more knowledge in maths". From the perspective of particular career routes, Dennis (V) commented that "as an electrician you need so this current, voltage, and everything like that, it requires maths for", while Roxana (A) suggested that "if you're going to become an engineer for example you're gonna read a lot of mathematics".

The second, slightly cynical, theme found students discussing how being mathematically qualified gave them an advantage over those who were not. For example, Ted (V) commented that "I think that many who have not read maths will not get a job, for the employer will not hire those who cannot work in the same way", while Jacob felt that the mathematically qualified "is more knowledgeable, that way he can solve the problem the less skilled cannot; he can see the big picture of things better". Hanna's (A) view was that such advantages stemmed from the fact that

> it might be like easy for that (mathematically well-qualified) person to think logically, to think outside the box and that kind of stuff. And the other (less well-qualified) person might have a few more problems with that, maybe… and probably it's easier for the person that's really good at math to excel in whatever they want to do… which is harder for the other person

Most Swedish interviews also contained some reference to the ways in which mathematics was seen as a service tool to other forms of activity. Typically, these focused on the natural sciences, as in the comments of Adam (V), who said, "you use math well in other subjects like physics", Mikaela (A), who believed that "I know … in chemistry or physics … I need maths. So I think to learn the physics or those things you need to learn, to know the maths first. So you can build on it" and Julio (V), who argued that "maths is the grounding for several subjects, so if you remove maths, it's not just maths skills that will be worse, but then it is physics, chemistry and yes science subjects just disappear, so maths is an important part". Others' comments were more general, as in Winston's comment that mathematics "works well with other subjects that also need the maths, like they complement each other", while Max commented that "sports is a subject where you use it (mathematics)".

Discussion

In this paper we set out to uncover what Norwegian and Swedish upper secondary students believed was the purpose of school mathematics, particularly as it has been an ever-present compulsory part of their school careers. The results seem much removed from mathematics as a problem posing and solving discipline (Ernest, 2016) for which teaching aims to "lead students to appreciate the power and beauty of mathematical thought (Dreyfus and Eisenberg, 1986, p.2). Indeed, the two dominant themes, mathematics as a support for functioning in the real world and mathematics as an entry into employment, are as utilitarian as it is possible to be, showing no connection to mathematics as a cultural artefact to be appreciated as would be art, music or literature. Moreover, the themes were not utilitarian in the sense that students saw themselves as having been educated for citizenship (Noyes, 2007), made socially empowered (Ernest, 2016) or inducted into societal participation (Watson, 2004). They were utilitarian solely from the perspective of personal advancement. There was no evidence that these students, many of whom were expecting to go to university and study mathematics-related subjects, saw mathematics as part of the "moral and ethical imperatives that operate in the human and in the more-than-human realms" (Davis, 2001, p. 21).

While such findings may be disappointing, we acknowledge that these students' perspectives have not emerged by chance but from individually unique experiences of mathematics. These ten- and eleven-year experiences, located in different countries and schools, will necessarily have influenced the formation of individuals' beliefs about the nature of mathematics and its purpose (Cobb, 1985), and yet their collective voice was close on deafening. So, are there any explanations? Well, the literature is not particularly expansive on such matters, although a recent case study may offer some insight. In their study of largely disaffected Swedish upper secondary school students, Andersson, Valero and Meaney (2015) motivated their students by means of tasks involving, inter alia, percentages related to personal economics. This approach, which was received positively by the students concerned, raises at least two important questions. The first is whether such an approach legitimates students' perception that school mathematics is about preparing them for the real world. The second is whether students' positive reactions were a consequence of the task meeting their expectations of school mathematics. In short, which came first, the chicken or the egg?

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Visual attention while reading a multiple choice task by academics and students: A comparative eye-tracking approach

Miroslawa Sajka

Institute of Mathematics; Pedagogical University of Cracow, Poland; msajka@up.krakow.pl

This study exploited eye-tracking technology to analyse the visual attention, while engaging with a multiple-choice mathematical task, of 103 participants with different levels of expertise and experience, including academics, university students and secondary school students. The majority of participants, irrespective of experience or prior knowledge, skipped or did not process all the information provided by the task. Important differences were discerned between how the academics and the two student groups attended to the different areas of the task's presentation, highlighting the different problem-solving approaches of the experienced and the inexperienced. Moreover, a 'hit ratio' parameter allowed the identification of those participants who did not look at these important areas of the task. The research highlights methodological advantages and disadvantages of using eye-tracking and different ways of data analysis.

Keywords: Eye-tracking, multiple choice task, comparative study.

Introduction

The use of eye-tracking for research in mathematics education is an emergent field, albeit still relatively rare. For example, during the ICME-13 congress, only 4 of 1952 papers and 533 posters, contained the phrase eye-tracking or its variants in their title. Most mathematical education–related eye-tracking research has been undertaken in laboratories, with classroom-based studies using head-mounted eye-tracking device are rare (e.g. Garcia & Hannula, 2015; Hannula 2016). Examining visual attention not only shows where and how gaze is directed, but also constitutes a basis for further analysis of problem solving, reasoning, attention and mental images. (Just & Carpenter, 1976; Zelinsky & Sheinberg, 1995; Ball et al. 2003; Yoon & Narayanan, 2004). The measurement of the eyes' fixations can provide reliable and sensitive insights into otherwise unavailable cognitive processes (e.g. Sosnowski, 1993).

From the perspective of mathematical problem solving, researchers have used eye-tracking to distinguish between the behaviors of experts and novices with respect to the solving of linear equations (Susac et al., 2014), geometry problems (Epelboim & Suppes, 2001) and their interpretation of mathematical representations (Andrà et al., 2009), interpreting graphs (Wcisło et al., 2014), perception of Cartesian coordinates (Krichevets et al., 2014) as well as perception of a number line (Shvarts et al., 2015).

Students need to learn how to read mathematical problems. Thus, in-depth knowledge of strategies for reading mathematical problems has important didactical consequences, although more needs to be known about how participants of different levels of education approach unfamiliar problems. In this paper I report an eye-tracking study of approaches to the reading and solving of a multiple choice graphical problem concerning motion. The relatively large sample of the current study, more than 100 participants, is novel as the majority of previous eye-tracking studies in mathematics education have been case studies with small samples of fewer than 20 people.

Methods

The primary aim of the study was to use eye-tracking to investigate whether different groups of participants' visual attention while reading and analyzing a mathematical problem varied with experience and problem-solving expertise. The first group, academics (A), comprised one professor of physics and three academics in mathematics, physics and computer science respectively. The second group, university students (U), comprised 75 university students at different stages of their courses in computer science, physics, mathematics and biology. The third group comprised 24 high-achieving school students (ages 17-18 years) (S) who attended a so-called *university class* with an extended curriculum in mathematics and physics. Thus, study participants included people with widely differing experiences of mathematics. All participants were guaranteed anonymity and could withdraw from the study at any time.

Task and apparatus

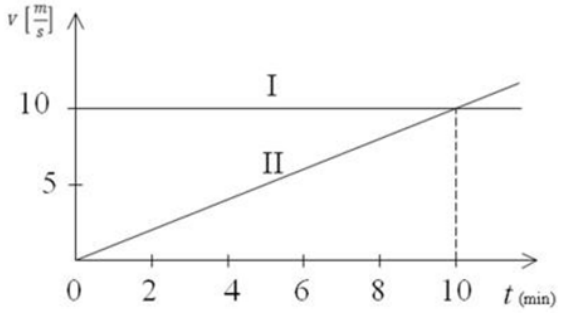
Respondents were invited to solve a multiple-choice mathematical task (proposed by Prof. W. Blasiak) concerning the interpretation of a motion graph. The Polish mathematics curriculum introduces the interpretation of graphs of functions to students in grade seven (lower secondary school). Later, students aged 14-15 years old learn the concept of a function and use its representations. During both mathematics and physics classes they also learn the relationships between average speed, displacement and time as well as constant acceleration. Thus, the mathematical subject matter knowledge of every participant should have been sufficient to solve the problem. However, the task, shown in Figure 1, is complex, caused mainly by the simultaneous presentation of two time-velocity graphs. In addition, participants were asked to identify incorrect statement(s), what is not a standard request. Academics were invited to identify any incorrect statements, of which there were two, A and E. For the other participants the task was simplified to one incorrect statement A.

Statement A is incorrect due to its assumption that the graphs represent trajectories. Statements B, C, D are correct. Answer B (and E) concerns the distance driven by the vehicles. Statement B can be verified in an elementary way, based only on the analysis of speed values: vehicle (I) moves in the time span 0-10 min. with higher speed than vehicle (II), so its displacement is greater. Statement C concerns only the interpretation of values of the functions for the argument t = 10 min. Statement D essentially requires a basic understanding of acceleration, although knowledge of the monotonicity of linear functions would be sufficient. Statement E is the most sophisticated, being true only at t = 10 min. One can notice it comparing the area of respective figures: rectangle and triangle bounded by the graphs of the functions and *x*-axis (graphical interpretation of the distance at the motion graph). Concerning version for academics (see Fig. 1) for all positive $t \neq 10$ statement E is incorrect, thus the general statement is also incorrect. The difficulty connected with statement E was specifically included to make a more challenging task for the academics. Other participants were asked to verify the statement for t = 10 minutes, therefore statement E for them was correct.

To record participants' eye movements, the Eyetracker Hi-Speed 1250 with iView X™ was used. The sampling rate was set to 500 Hz, monocular. The movements of the left eyeball were examined for every participant and the data obtained processed by the BeGaze software. The 13-point calibrations were accepted with an angular accuracy of less than 0.5º. All respondents sat at a distance

of 50 cm from a 22-inch monitor. The duration of the experiment was not limited. Participants' eye movement data, question responses and mouse clicking were recorded by Experiment Center 3.1. Additionally, respondents were asked to orally confirm their selected answers.

The motion graph illustrates the dependence of speed (v) in time (t) for two vehicles (I, II). Which of the following statements are incorrect?



A	B	C	D	E
Vehicle (II) caught vehicle (I) after 10 min.	In the timespan of 0-10 min the greater distance was driven by vehicle (I).	The speed of both vehicles was the same for t = 10 min.	Vehicle (II) moved with greater acceleration.	The distance driven by vehicle (I) is twice longer than the distance driven by vehicle (II).

Figure 1: The task translated from Polish

Results

The analyses were undertaken in several stages.

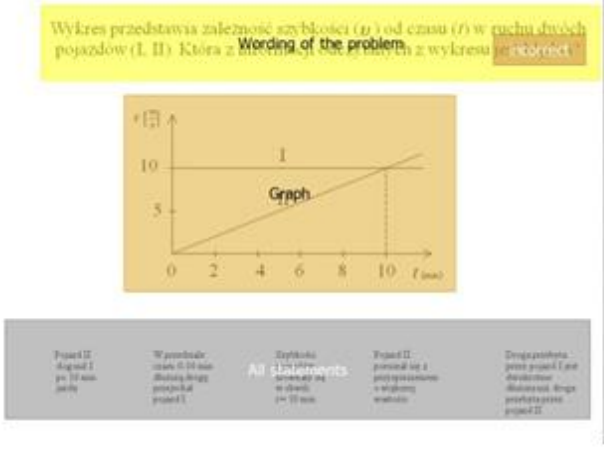
Firstly, the figures of Figure 2 show the problem solving results for each group. It is interesting to note that while both the university and school students were not particularly successful, identifying many correct statements as wrong and not identifying the incorrect statement, the academics, with their amended statement E, failed to identify it as incorrect.

Selected answers	A	B	C	D	E
A (4)	4	0	0	0	0
U (75)	24 (32%)	19 (25%)	8 (11%)	9 (12%)	15 (20%)
S (24)	8 (33%)	5 (21%)	3 (13%)	0 (0%)	8 (33%)

Figure 2: Results of the task for each group

Secondly, with respect to the eye-tracking analyses, participants' eye movements were studied initially by five broad areas of interest (AOIs), as shown in the left hand side of Figure 3. These were the words comprising the task formulation (Wording), the key word within that text (Incorrect), the graphical diagram (Graph), the statements A – E (Statements) and, finally, the remainder of the slide (White Space). The right hand side of Figure 3 (right) shows the average data for each AOI for each

group: the percentage *dwell time*, number of *revisits*, *fixation* time, *fixation count* and so called *hit ratio*, which informs how many participants looked at the AOI.



Group	Eyetracking average data	Areas of Interest (AOIs)				
		Wording	Incorrect	Graph	State-ments	White Space
A	Dwell Time [%]	24,5	1,8	23,8	43,5	1,2
	Revisits	1,8	1,8	8,3	6,8	0,5
	Average Fixation Time [ms]	202,9	217,7	236,4	282,5	65,1
	Average Fixation Count	44,3	4,0	38,0	61,5	2,5
	Hit ratio	4/4	4/4	4/4	4/4	2/4
U	Dwell Time [%]	16,4	1,4	22,1	43,9	3,0
	Revisits	3,8	1,9	12,2	12,1	5,3
	Average Fixation Time [ms]	195,5	191,5	203,4	281,9	171,8
	Average Fixation Count	37,5	3,5	51,1	79,2	8,5
	Hit ratio	75/75	68/75	75/75	75/75	74/75
S	Dwell Time [%]	14,5	1,7	24,3	44,4	2,2
	Revisits	3,3	1,7	11,3	12,3	5,0
	Average Fixation Time [ms]	194,3	204,0	209,0	301,9	163,8
	Average Fixation Count	33,2	3,6	58,5	81,0	7,4
	Hit ratio	24/24	23/24	24/24	24/24	23/24

Figure 3: Definitions of broad AOIs (left) and their eye-tracking data in defined groups (right)

The data presented in Figure 3 (and Figure 5) should be viewed with caution because of the calculation of average data. That said, some interesting differences can be discerned between the academics and the two student group. Firstly, academics responded to 'Wording' differently from the two students groups. They spent 24.5% of their total time on task dwelling on the words of the problem, in comparison to 14.5% (S) and 16.4% (U). Academics achieved a greater number of fixations (44,3) on 'Wording' than the student groups (33.2 (S), 37.5 (U)), indicating that the academics' attended more to the words of the problem than students' in either group. Academics addressed 'Wording' more attentively than did the students because their average number of revisits, 1.8, was around half of that of the two students' groups (3.3 (S) and 3.8 (U) respectively). Secondly, the reverse seemed to be true for 'Graph'; the average number of fixations for academics was lower (38) than for in either group (58.5 (S) and 51.1 (U). Thirdly, Academics' attention to 'Statements' was lower than that of the students (61.5 (A), 81 (S) and 79.5 (U)). Fourthly, academics' dwell time was uniformly distributed between 'Wording' and 'Graph' (24.5% and 23.8% respectively) which was not the case for students. Fifthly, students revisited 'Statements' twice as many times as the academics. Sixthly, students made more than ten times as many revisits as the academics to 'White Space' and spent three times as much time on it, indicating that their visual attention was more located outside the defined parts of the slide. Finally, the *hit ratio* for showed that eight students (1 S and 7 U) failed to look at the AOI "incorrect", prompting one to ask, what question were they answering?

In sum, academics not only spent a higher proportion of their time on the wording of the task than the participants of either student group but also made fewer revisits. It can also be seen that the academics made fewer fixations on and made fewer revisits to the graphs, the statements and the white space than did the participants of either of the other two groups, indicating that the time they spent on the words benefitted their problem solving.

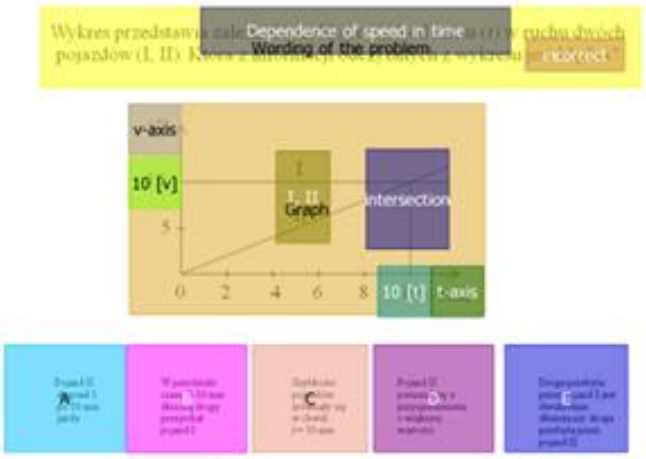


Figure 4: Detailed definitions of AOIs

The third stage of the analysis was to examine more specific AOIs, which are shown in Figure 4. Ten additional areas were associated with task those characteristics that should be analyzed for the problem to be solved. These concerned the phrase "dependence of speed in time" (Dependence), the two axes (v-axis, t-axis), the graphs' labels (I, II), the intersection of the graphs (Intersection) and the Statement A – E, separately. The chosen average eye-tracking data are presented in Figure 5.

	A	U	S	A	U	S	A	U	S	A	U	S	A	U	S
	Dependence			**v-axis**			**10[v]**			**t-axis**			**10[t]**		
Dwell time[%] (average)	5,4	5,9	5,3	3,2	1,8	2,1	1,5	1,3	1,3	0,7	1,4	1,2	0,9	1,3	1,5
Hit ratio	4/4	73/75	23/24	4/4	66/75	23/24	4/4	64/75	22/24	4/4	73/75	21/24	4/4	65/75	23/24
Revisits (average)	2,8	3,5	3,3	2,8	2,6	2,8	2,8	2,6	3	6,8	4,4	1,9	1,8	2,6	3,3
Revisitors	4/4	69/73	20/23	4/4	54/66	19/23	3/4	52/64	20/22	4/4	71/73	13/21	4/4	53/65	17/23
Average fixation [ms]	210,7	193,5	180,7	315,8	232,7	262,8	254,5	176,4	159,6	230,9	222,7	211,2	141,2	155,3	165,4
Fixation count (average)	14	13,5	13	6,3	3,9	4,6	4,3	3,4	4,1	8,8	6,4	2,8	3,5	3,6	4,7
	Statement A			**Statement B**			**Statement C**			**Statement D**			**Statement E**		
Dwell time[%] (average)	11	10,3	11,4	11,4	10,6	11,5	7,3	7,1	6,3	9,7	7,1	5,9	9,4	7,6	8,5
Hit ratio	4/4	74/75	24/24	4/4	74/75	23/24	4/4	74/75	23/24	4/4	71/75	23/24	4/4	65/75	19/24
Revisits (average)	3,3	3,7	3,4	3,8	4,8	4,3	2,5	3,7	3	2	3,6	2,9	1,3	2,7	3,5
Revisitors	4/4	68/74	24/24	4/4	71/74	22/23	4/4	64/74	17/23	3/4	64/71	21/23	4/4	46/65	17/19
Average fixation [ms]	329,6	297,8	328,9	233,3	252,7	261,5	292,3	266	284,6	317,2	272,3	278,8	284,8	231,9	219,8
Fixation count (average)	18,5	16,2	17,4	27	20,6	22,7	14,8	13,7	12,2	17,3	13,5	11,2	18,5	14,9	17,5

Figure 5: Eye-tracking average data in groups for chosen detailed AOIs presented in Figure 4.

The figures of Figure 5 show both similarities and differences. It is interesting to note, for example, that the times spent by all three groups on the five statements were similar. However, the academics focused proportionally more time on the v-axis and less on the t-axis than either student group, which is interesting as their number of revisits to the v-axis was comparable to the students but much greater with respect to the t-axis. It is also interesting to note that academics, despite their failure to identify the incorrectness of the statement, made fewer revisits to statement E than either student group. In such an instance, it is possible that the academics had drawn an over-confident conclusion. Finally, the hit ratio shows that while all the academics attended to all ten AOIs, students' attention was complete only one occasion, when all school students fixed on statement A.

Discussion and summary

In general, the data indicate that academics were better focused on the task formulation than either group of students. Not only did they spend a higher proportion of their time on the wording of the task but made fewer revisits. They read each word carefully and did not revisit the same places as often as students. Academics also made fewer fixations on and made fewer revisits to the graphs, the

statements and the white space than the participants of either of the other two groups, indicating that the time they spent on understanding and interpreting the problem benefitted their problem solving. Unsurprisingly, being academics, they understood what they read and knew where to look to analyse a problem. They were more competent than students in their graphical interpretation, being able to understand the key elements of the graph. In related vein, their visual attention was distributed uniformly across both the problem statement and the graph, and they were not distracted by irrelevant parts of the slide. Thus, while the conclusions are not surprising, the data highlight well differences in the ways in which experts and novices address mathematical problems.

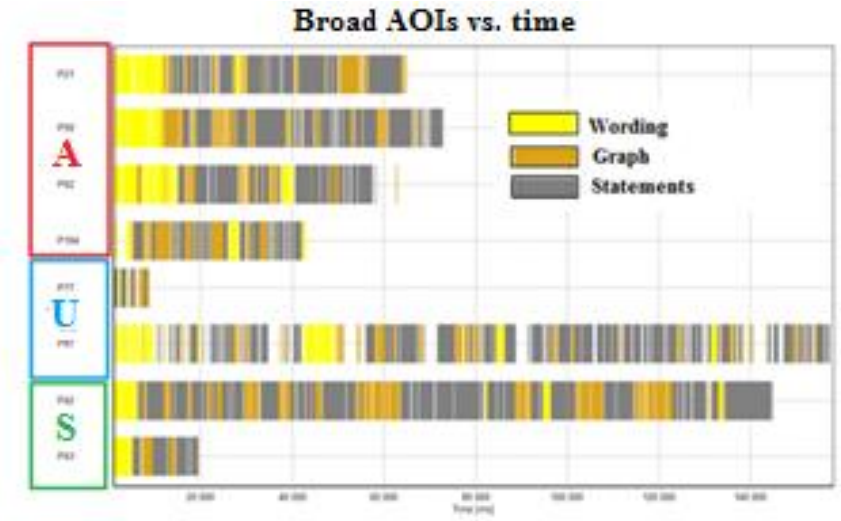


Figure 6: Sequence charts for all academics and two students per group

As indicated above, the data presented here are the averages for each group on each AOI. Individually, there was an interesting variation within each group of participants. For example, Figure 6 shows the complete sequence of eye-tracking activity with respect to the broad AOIs for all four academics and the two extremes, the longest and the shortest, from each student group. It can be seen clearly that there was a tendency towards an academic homogeneity, particularly the first three who spent considerable amounts of time on the task wording. However, the variation within each student group was considerable. Indeed, the academics typically spent around a minute on the task, while university students' time on task ranged from around seven seconds to two minutes and 40 seconds. In sum, while the study above has been helpful in showing some differences between expert and novice problem solvers, individual variation has highlighted the need for further research into how eye-tracking can support our understanding of effective problem solving strategies.

This paper has offered one analysis, based on average behaviours, focused on how eye-tracking data can support our understanding of the problem solving process. However, this averaging process with the atypically large numbers of participants may have masked individual differences, which themselves may prove insightful. Thus, focusing on individuals may be a fruitful direction as it may identify different factor implicated in either problem solving success or failure. For example, Figure 7 shows the variety of the participants' visual attention, presenting so called *scan paths* for an academic (left), a U student (middle) and an S student (right) respectively.

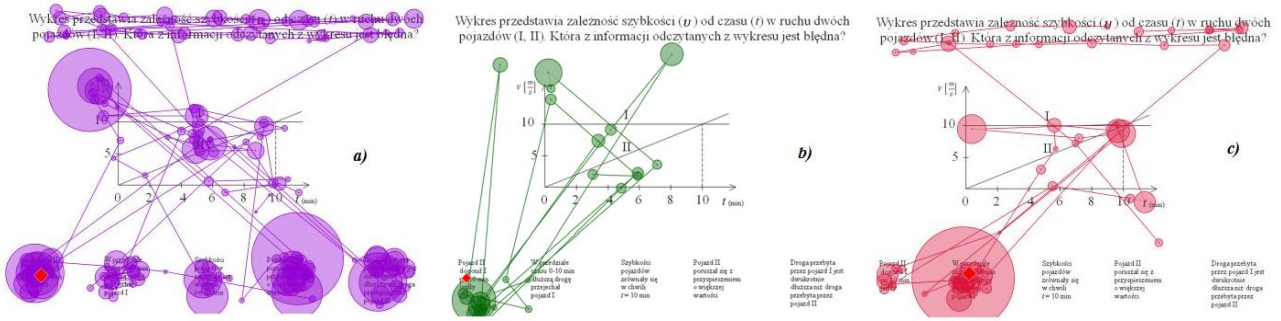


Figure 7: Scan paths of a chosen participant form each groups: A (a), U (b) and S (c).

Finally, it was interesting that none of academics questioned the correctness of Statement E. Here we faced the limitation of pure eye-tracking methodology – mixed methods with interviews or written questionnaires may have exposed the reasons. Thus, several months after the research, academics were asked to think again and solve the task, without eye-tracking. While solving the task again they confirmed answer A, read again all the other statements and eliminated them, treating again as correct. When asked why statement E is correct, they answered that the displacement can be counted as the area under the graph. They indicated t=10 min. After further questions they were surprised that "t=10 min" was not written explicitly in the statement. The previous statements for t=10 min together with the graph and dashed line indicating the point (10, 10) caused their certainty that the statement E was also formulated for t=10 min. That was their tacit assumption, which could be provoked by the three factors. One of academics mentioned about the routine of solving multiple choice tasks, usually with the only one correct answer, therefore his inquiring mind was asleep after finding the correct answer A.

Acknowledgment

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A cross-national study of lower secondary mathematics teachers' content knowledge in the USA and Russia

Mourat Tchoshanov[1], Maria Cruz Quinones[2], Kadriya Shakirova[3], Elena Ibragimova[3] and Liliana Shakirova[3]

[1]University of Texas at El Paso, USA; mouratt@utep.edu

[2]Universidad Autónoma de Cuidad Juárez, Mexico; titacq@hotmail.com

[3]Kazan Federal University, Russian Federation; esandakova@mail.ru

This study presents a quantitative analysis of middle school mathematics teachers' content knowledge in two countries. The sample comprises lower secondary mathematics teachers from the US (grades 6-9, N=102) and Russia (grades 5-9, N=97). The instrument was designed to assess teacher content knowledge based on the cognitive domains of knowing, applying, and reasoning, as well as addressing the lower secondary mathematics topics of number, algebra, geometry, data and chance. Results indicate significant differences in teacher knowledge between the countries in content as well as in cognitive domains. The study results may inform the field on priorities placed on lower secondary mathematics teachers' knowledge in USA and Russia.

Keywords: Comparative studies, teacher knowledge, lower secondary mathematics.

Introduction

The motivation for the study is based on the 8th-grade mathematics portion of the TIMSS-2011 results (Mullis et al. 2012). We identified two countries ranked closely to each other: Russia - in the 6th position and the USA – in the 9th position. At the same time, a difference in the US and Russian students' performance was revealing: the average score of Russian students in the content domain was 539 and of the US students 509, with Russian students gaining higher scores on Number (534 vs. 514), Algebra (556 vs. 512) and Geometry (533 vs. 485) whereas US students outscored Russian students in the domain of Data and Chance (527 vs. 511). Russian students also outperformed the US students in each cognitive domain: Knowing (548 vs. 519) Applying (538 vs. 503), and Reasoning (531 vs. 503). These data triggered the following question: to what extent does US and Russian lower secondary mathematics teachers' knowledge differ by content and cognitive domains?

Cross-national studies of teacher knowledge

Conducting cross-national studies allow comparing, sharing, and learning about issues in an international context which in turn helps researchers understand their own context, teaching practice, teacher knowledge, and student learning (Stigler & Perry, 1988). During the last decade, the number of cross-national studies on teacher education is increasing in order to understand differences in student performance on international tests such as TIMSS, PISA (Wang & Lin, 2005). Scholars have addressed these differences focusing on characteristics such as teachers' perceptions of effective mathematics teaching (Cai, Ding, & Wang, 2013), teacher knowledge (Tatto & Senk, 2011; Tchoshanov et al., 2017), among others.

Few cross-national studies focused on teacher knowledge. A large-scale study conducted by the University of Michigan examined the mathematical content and pedagogical content knowledge of pre-service teachers from 17 countries including USA and Russia (Tatto & Senk, 2011). The nature of mathematics teacher knowledge, conceptual representation, and curriculum materials were examined by Ma (1999) to explain differences in students´ performance in the U.S. and China. An, Kulm, and Wu (2004) studied the PCK of middle school teachers in the U.S. and China. They found that mathematical PCK differs among the countries since Chinese teachers emphasize developing procedural and conceptual knowledge through traditional teaching practices while their counterparts in the U.S. focus on promoting creativity and inquiry through activities designed to develop students' understanding of mathematical concepts. Sorto et al. (2009) administered a survey that measured teachers' content knowledge in Costa Rica and Panama and found that teachers in both countries focus more on knowing rules and procedures than on making connections and reasoning.

In the last several decades, the field of mathematics education is expanding its knowledge-base in understanding the role of teacher characteristics in student learning and achievement. The major shift in the field had happened with Shulman's (1986) work on teacher knowledge that proposed an alternative approach to the educational production function perspective, which was concerned with examining proxies of teacher knowledge such as coursework/certification and its impact on student achievement (Charalambous & Pitta-Pantazi, 2016). Research on teacher knowledge initiated by work of Shulman (1986) has focused on teacher knowledge as a major predictor of student learning and achievement. Recently, the field benefited from numerous studies (Hill, Ball, & Schilling, 2008; Baumert et al., 2010) that substantially advanced the conceptualization of teacher knowledge and its association with student performance.

Following up on this conceptualization, some scholars (Izsak, Jacobson, & de Araujo, 2012) examined different facets of teacher knowledge without explicitly emphasizing its connection to student learning. Other scholars stressed the importance of the kind of knowledge a teacher possesses because it impacts his/her teaching (Steinberg, Haymore, and Marks, 1985). Another line of research (e.g., Baumert et al, 2010; Hill, Ball, & Schilling, 2008; Tchoshanov, 2011) specifically targets the effects of different types of teachers' knowledge on student achievement. Additionally, scholars have advanced the field by examining teacher knowledge in variety of domains including Number Sense (Ma, 1999; Izsac, Jacobson, & de Araujo, 2012), Algebra (McCrory et al., 2012), Geometry and Measurement (Nason, Chalmers, & Yeh, 2012), and Statistics (Groth & Bergner, 2006). However, the field lacks cross-national research that provides a comprehensive analysis of the various facets of teacher knowledge (including content and cognitive domains) and its connection to student performance.

Methodology

The proposed study is based on the assessment framework used by TIMSS (Mullis et al. 2012). In this section, we will describe the study participants, the instrument as well as data collection and data analysis procedures.

Participants

The sample of this study consisted of lower secondary mathematics teachers from the US (grades 6-9, N=102) and Russia (grades 5-9, N=97). The US teacher-participants were selected from urban

public middle schools in the Southwestern part of the country. Teacher sample demographic information was self-reported by participating teachers. In terms of gender distribution, 55% of teacher participants were females and 45% - males. Most of the US participants (64%) had 1-5 years of teaching experience. Additionally, 62% of the teacher sample received their teaching certificate through traditional teacher preparation programs and 38% of participating teachers were certified through alternative programs. The Russian teacher-participants were selected from urban public secondary schools in the Volga region. Russian participating teachers had attained a secondary mathematics teacher preparation Specialist's degree, which allowed them to teach in secondary schools (grades 5-11). The majority of participating teachers were females (89%). The sample was composed of 78% of teachers who have more than 10 years of teaching experience.

Instrument

The instrument used in this study was the Teacher Content Knowledge Survey which was developed using TIMSS framework (Mullis et al. 2012). It was designed to assess teacher content knowledge based on the three cognitive domains: Knowing, Applying, and Reasoning. The TCKS survey consisted of 33 multiple-choice items addressing main objectives of lower secondary mathematics curriculum: Number, Algebra, Geometry, Data and Chance. The instrument was piloted for construct and content validity as well as checked for the reliability. The alpha coefficient technique was utilized to evaluate the reliability of the teacher content knowledge survey. "The value of the coefficient of .839 suggests that the items comprising the TCKS are internally consistent" (Tchoshanov, 2011, p. 148). Examples of the TCKS items in Algebra domain across different cognitive types (Knowing, Applying, and Reasoning) are presented below.

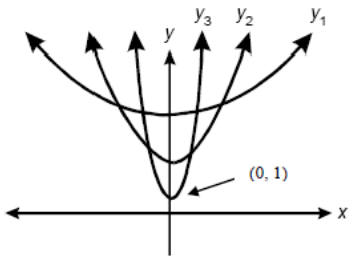

Figure 1. Diagram to the TCKS item in Algebra domain

Use the diagram above (see Figure 1) to answer the questions that follow.

1. Knowing

Which of the following equations best describes the function y_3?

A. $y = ax^2 + bx + c$

B. $y = ax^2 + bx + 1$

C. $y = ax^2 + 1$

D. $y = x^2 + 1$.

2. Applying

The function y_3 is translated 4 units left and 7 units down. Which of the following equations best describes the new function?

A. $y = ax^2 + 11x + 28$

B. $y = ax^2 + 4x + 7$

C. $y = ax^2 + 8ax + c$

D. $y = x^2 + 28x + 11$.

3. Reasoning

The diagram shows a family of functions in the form $y = ax^2 + bx + c$. Which of the following statements best describes the changes in the values of the coefficients as the graphs transform from y_1, to y_2, to y_3?

A. a is increasing, b = 0, and c is increasing

B. a is increasing, b = 0, and c is decreasing

C. a is decreasing, b is increasing, and c = 0

D. a is decreasing, b is decreasing, and c = 0.

Data Collection and analysis

Each teacher was given 90 min to complete the survey. In correspondence with the research question, data analysis was performed using non-parametric techniques (chi-square). This statistic was selected to measure the variance between independent groups of the same (not normal) distribution with arbitrary sample sizes of each group. The selection of this test was also based on the ranked nature of data for content and cognitive domains of teacher knowledge and student performance.

Results

In this section, we first analyze teacher knowledge data by content domain, then we examine teacher data by cognitive domain, and finally we discuss parallels between student and teacher performance within and between countries.

Content Domain	Mean	SE	SD	Conf. level (95%)
Number	623	20.3129	205.1512	40.296
Algebra	563	23.2356	234.6679	46.093
Geometry	514	25.4349	256.8802	50.456
Data and Chance	593	20.9738	211.8252	41.606

Table 1. US teachers´ means scores by content domain

The results reported on teacher content knowledge show that the US teachers' highest mean score was obtained on Number domain – 623 and lowest on Geometry domain - 514 (see Table 1).

Russian teachers' highest mean score was obtained on Algebra domain – 728 and lowest on Data and Chance domain – 387 (see Table 2).

Content Domain	Mean	SE	SD	Conf. Level (95%)
Number	656	106.5819	319.7456	23.873
Algebra	728	82.8841	248.6523	30.648
Geometry	586	72.7004	218.1013	45.505
Data and Chance	387	125.0891	306.4044	35.844

Table 2. Russian teachers´ means scores by content domain

Moreover, we found that in the cognitive domain the US teachers' highest mean score was obtained, as expected, on Knowing – 734 and lowest on Reasoning - 495 (see Table 3).

Cognitive Domain	Mean	SE	SD	Conf. level (95%)
Knowing	734	19.7673	197.6733	39.2226
Applying	505	20.7101	207.1015	41.0934
Reasoning	495	23.8130	238.1303	47.2502

Table 3. US teachers´ means scores by cognitive domain

Russian teachers' highest mean score was obtained, as expected, on Knowing domain – 760 and lowest, unexpectedly, on Applying domain - 504 (see Table 4).

Cognitive Domain	Mean	SE	SD	Conf. level (95%)
Knowing	760	14.2486	135.1745	28.3117
Applying	504	12.7961	121.3950	25.4257
Reasoning	593	17.7406	168.3028	35.2503

Table 4. Russian teachers´ means scores by cognitive domain

Moreover, we identified that there is no significant difference between Russian and US teachers' knowledge on Number and Geometry domains (Chi-square 0.347 $p>.05$ and Chi-square 1.293 $p>.05$) (see Table 5).

Content Domain	Number	Algebra	Geometry	Data and Chance
Russia	656	728	586	387
USA	623	563	514	593
Chi-square (df=1)	0.347	6.311*	1.293	8.003**

Table 5. Russian and US teachers' knowledge by content domain (* $p<.05$, ** $p<.01$)

However, there is a statistically significant difference between Russian and US teachers' knowledge on Algebra domain (in favor of Russian teachers; Chi-square 6.311 $p<.05$) and Data and Chance domain (in favor of US teachers; Chi-square 8.003 $p<.05$) (see Table 5). This finding closely parallels the US and Russian students' performance on TIMSS on Algebra domain (in favor of Russian students) and Data and Chance domain (in favor of US students).

Also, this study reported that there is no significant difference between Russian and US teachers' knowledge on Knowing and Applying cognitive domains (Chi-square 1.707 p>.05 and Chi-square 0.008 p>.05) whereas there is a statistically significant difference on Reasoning domain (in favor of Russian teachers; Chi-square 19.117 p<.05) (see Table 6).

Cognitive Domain	Knowing	Applying	Reasoning
Russia	760	504	593
USA	734	505	495
Chi-square (df=1)	1.707	0.008	19.117**

Table 6. Russian and US teachers' knowledge by cognitive domain (* p<.05, **p<.01)

This finding parallels the US and Russian students' performance on TIMSS' cognitive domain.

Discussion and conclusion

This study confirms the differences between Russian and the U.S. lower secondary in-service teachers' knowledge in the content domain as it was reported by the TEDS-M study that was focused on pre-service teachers (Tatto & Senk, 2011). At the same time, this study expands the examination of in-service teachers' knowledge to the cognitive domain.

Teacher preparation could be considered as the main factor contributing to the differences between Russian and US teachers' knowledge. Overall, there is a tangible difference in secondary teacher preparation curriculum between the two countries: in average, Russia offers about 240 credit hours in teacher preparation programs compare to 120 credits in the USA. Furthermore, cross-national curriculum analysis shows that Russian lower secondary mathematics teachers have more extensive content preparation compare to their American counterparts. A number of contact hours for mathematical content knowledge, as well as pedagogical content knowledge and specialized mathematics knowledge offered at selected teacher preparation programs (e.g., the University of Texas at El Paso, USA and Kazan Federal University, Russia) in two countries, are presented in table 7.

Country	Mathematics Content Knowledge	Pedagogical Content Knowledge	Specialized Mathematics Knowledge
Russia	1857	278	380
United States	442	72	87

Table 7. Contact hours in Mathematics related disciplines in teacher education programs in Russia and United States

Numbers depicted in the table are compatible with the findings of the TEDS-M study (Tatto & Senk, 2011). Close examination of secondary teacher preparation curriculum in Russia shows that more emphasis is placed on an analytic and algebraic component of mathematics curriculum and less emphasis - on statistic and probability component compare to the US curriculum. Moreover, item analysis of standardized tests for the lower secondary schools in USA and Russia revealed the difference in selection and composition of algebra problems as well as problems related to data and chance in the test: while in Russia more emphasis is placed on algebraic problems and less emphasis on data and chance problems, in the USA – the emphasis is equally distributed among algebraic problems and data and chance problems. We observed another noticeable difference in the role of

proof in the academic mathematics component of the teacher preparation program which could explain the difference in the reasoning domain of the teacher knowledge: Russian curriculum places a heavy emphasis on proof across the mathematics coursework including school mathematics whereas the US curriculum uses proof in selected mathematics courses primarily in academic mathematics coursework.

We are cognizant of the limitations concerning the convenient sampling technique that influences generalizability of the study results. Moreover, there is no cluster matching between teachers participating in the study and students tested in TIMSS. However, the study main results suggest that student performance on international tests could be explained by teacher knowledge. The study also presents opportunities for comparing, sharing, and learning about issues in cross-national context in US and Russian teacher education, training, and development. Moreover, the reported cross-national study on teacher knowledge may inform the field on priorities placed on lower secondary mathematics teachers' knowledge in USA and Russia by content and cognitive domains.

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Explanations as tools for evaluating content knowledge for teaching: A cross-national pilot study in Cyprus and Greece

Constantinos Xenofontos[1] and Paul Andrews[2]

[1]University of Nicosia, Cyprus; xenofontos.c@unic.ac.cy
[2]Stockholm University, Sweden; paul.andrews@mnd.su.se

In this paper, as a proxy for their mathematical knowledge for teaching, we examine Greek-Cypriot (n=21) and Greek (n=12) first-year undergraduate teacher education students' written explanations regarding a linear equations-related scenario. Qualitative analyses identified four broad themes permeating most accounts, irrespective of nationality, which were interpretable as either disciplinary explanations or instructional explanations. The themes concerned (a) definition of unknowns, (b) inverse operations, (c) presentation of objectives, and (d) rote procedure. The analyses indicated the weakness of students' explanation, whether from a disciplinary or an instructional perspective, which was independent of their country of origin.

Keywords: Mathematical knowledge for teaching, instructional explanations, disciplinary explanations, prospective teachers.

Introduction

The question regarding what kind of knowledge is needed to teach mathematics has preoccupied a number of scholars, while, at the same time, the association of mathematics teacher knowledge with instructional quality and student learning is considered to be complex (Charalambous & Pitta-Pantazi, 2016). Over the last years, several conceptualizations of this kind of knowledge have been proposed. Drawing upon Shulman's (1986) seven tentative categories for teacher knowledge, Ball, Thames, and Phelps (2008), for instance, developed the *mathematics knowledge for teaching* framework, which divides *subject matter knowledge* and *pedagogical content knowledge* into three further sub-categories each. From a different perspective, grounded in data-driven analyses of video-taped lessons by prospective elementary teachers, Rowland, Huckstep, and Thwaites (2005) proposed the *knowledge quartet*, a framework comprising four units, namely, foundation, transformation, connection, and contingency. These units, the authors claim, can be found in every mathematics lesson and can be used for evaluating teacher knowledge during the course of teaching. More recently, Davis and Renert (2013) have written on *profound understanding of emergent mathematics*, arguing that teachers' knowledge could be productively interpreted as a complex evolving form which is tacit and is better understood as a learnable disposition than a domain to be mastered.

Several of scholars (i.e. Copur-Gencturk & Lubienski, 2013; Phelps & Howell, 2016; Hill, Schilling, & Ball, 2004) have urged for the development of effective measures of the knowledge required for teaching mathematics. Various approaches to this task have been have been undertaken (Charalambous & Pitta-Pantazi, 2016), including, paper-and-pencil tests, the use of lesson videos inviting teachers to critique and/or predict, as well as inviting post-instruction reflections on both lesson plans and the actual realization of the plan. Adaptations of such approaches, however, should be carried out with careful sensitivity to the cultural context, as teacher knowledge, of both in-service (Andrews & Sayers, 2012) and pre-service teachers (Xenofontos, 2014), is conditioned by

cultural expectations with respect to what mathematics is valued and how it is presented. For example, Andrews (2003) has highlighted how some cultures construe competent mathematics teachers as those who ensure their students complete a large number of tasks every lesson, while others see competence in the completion of a few.

An important element of a teacher's didactical repertoire is the explanation. Explanations, which typically draw on students' prior knowledge (Leinhardt & Steele 2005), are manifestations of teachers' depth of content knowledge (Inoue, 2009). Of Leinhardt's (2001) four classifications of explanation, two are particularly relevant here; disciplinary explanations are domain specific and conform to the epistemological expectations of the relevant discourse, and instructional explanations, which are intended to teach some aspect of a particular subject matter to others, are "jointly built through a coherent discourse surrounding a task or text that involves the whole class and the teacher working together" (Leinhardt, 2001, 340). Therefore, in accordance with prior research that explanations may be a useful tool for measuring prospective teachers' knowledge (see, for example, Charalambous, Hill, & Ball, 2011; Inoue, 2009), we present the results of a pilot study focused on the use of a single task as a means of evaluating Greek and Cypriot beginning teachers' didactically-related mathematical understanding. The extent to which we consider our tool as effective is discussed in another paper of this TWG (see Andrews & Xenofontos, 2017). Here, we focus on the identification of similarities and/or differences between the two cohorts' written explanations, which, we believe could offer better insights into the mathematical knowledge beginner teachers bring to teacher education and how we, as teacher educators, could build on or deconstruct this knowledge.

The study

Participants were first-year undergraduate students reading for a degree in elementary education at a private university in Cyprus. With Greek being the language of instruction, the programme includes both Greek-Cypriot and Greek students. Data collection took place before students had experienced any university instruction and involved 21 Greek-Cypriot and 12 Greek students. Students were shown, with no additional text, the solution to the equation presented below, and asked to write a short account indicating how they would explain it to someone who had missed the lesson in which it had been taught. Such a task has the advantage that it allows for both disciplinary and instructional explanations (Leinhardt, 2001).

$$\begin{aligned} x + 5 &= 4x - 1 \\ 5 &= 3x - 1 \\ 6 &= 3x \\ 2 &= x \end{aligned}$$

Data were analysed as single set by means of a *constant comparison process* (Fram, 2013). These analyses, as reported in Andrews and Xenofontos (2017, TWG11), yielded seven themes with respect to how participants explained the solution. Of these seven, four themes were present in most accounts and are here considered in depth. In the following, we present the results of this process, discussing each theme in relation to first the Cypriot and then the Greek students. All names are pseudonyms.

Results

Definition of unknowns

A number of Cypriot students wrote statements interpreted as explicit definitions of the unknown. For example, Ekaterini wrote that the "known numbers are the ones that don't include a letter, as for example, 5 and -1. Unknown numbers are the letters or the numbers that are accompanied by a letter, for example, 4*x* and *x*". In similar vein, Carissa wrote, "I separated the known from the unknown numbers: Unknown x + 5 (known) = unknown 4x - 1 (known)". Both comments, we argue, are structurally equivalent in their assertions that unknowns are represented by letters and the knowns by numbers. Other students' definitions tended to the implicit. For example, Hermione wrote that "the first thing to do is to separate x from the numbers. In other words, to set apart the known from the unknown so that we can find what *x* is". Likewise, Alexandra commented that their teacher had given "two expressions, each with an unknown (*x*) and which are equal to each other. She asked us to find out what the value of *x* is". While neither student explained explicitly what they meant by the unknown, their comments present a tacit understanding of the unknown as the number represented by the letter *x* that has to be found.

In ways similar to their Cypriot colleagues, Greek students offered either explicit or implicit definitions of the unknown. With respect to the former, Nina wrote that "we separate the known from the unknown numbers: known: 5, -1 and unknown: *x*, 4*x*", while, in a slightly more fluid manner, Paraskevi commented that "I separate the known from the unknown. The known numbers are all numbers, unknown is whatever is a letter, for example, *x, y, z, w*". Students explicitly distinguished between *x*, as a representation of the missing number, and numerical values separated from the unknowns. With respect to the latter, Callisto, having written of the need to separate knowns from unknowns, wrote that "all the 'x' are taken to the right and whenever 'x' is on the left part of the equation, you change the sign. On the left part, the known numbers enter while the unknown enter the right". Similarly, Daphne commented that "the first unknown x is moved to the second part of the equation but instead of a plus we make it a minus". In both cases, students offer instructions in which the x terms are afforded a treatment that distinguishes them from numerical terms in ways that indicate their understanding of the nature of the unknown and its significance in the equation solving process.

Presentation of objectives

The accounts of almost all Cypriot students included some statement regarding the objectives of equation solving. In most cases this was implicit and typically represented in statements concerning the separation of knowns from unknowns. The briefest such statements were as in Carissa's "I separated the known from the unknown numbers" and Ioanna's "we set apart the known and unknown numbers". In such statements can be seen an understanding that the identification of the unknown was the objective. Even when such students wrote longer statements, the message was the same, as in Chloe's "we separated known and unknown numbers, that is, 5 and 1 and 4*x* and *x*". Here, the distinction between Chloe's and the other two students' statements is that she incorporated an implicit definition of the unknown, but offered no more than them with respect to her objectives. Other Cypriot students offered accounts with a slightly less implicit objective in that they also discussed the separating of the knowns from the unknowns but in so doing explicitly mentioned the

role of *x*, which Chloe did not. For example, Medea wrote that the "first step is to set apart the known and the unknown numbers, that is, to bring '*x*' to one side and the numbers to the other", while Stamatia commented that "we separated the known from the unknown numbers, that is, the '*x*' and the 'simple' numbers.

Four students wrote statements indicative of an explicit objective. Two of these were brief, as in Stefania's "the aim is to find which number equals x" and Irene's "the question in this equation is to find 'x' and what value it has". The other two students wrote longer statements, as with Hermione's "the first thing to do is to separate *x* from the numbers. In other words, to set apart the known from the unknown so that we can find what *x* is". In all three statements, albeit expressed differently, can be seen an expression concerning the identification of the unknown, x. Finally, of the four students who offered an explicit objective, Alexandra's statement also offered the only evidence of a relational understanding of the equals sign. She wrote that "the teacher gave us two expressions, each with an unknown (x) and which are equal to each other. She asked us to find out what the value of x is".

For the Greek students, the same three categories of response were identified. For example, with respect to implicit goals embedded in statements about separating the knowns from the unknowns, Pantelis wrote that "we part the known from unknown numbers", while Moira wrote that "I would tell the student that we separate the known from the unknown numbers and then make the calculations". As far as the second category is concerned, whereby students offered implicit objectives alongside an explicit mention of the role of *x*, Callisto commented that "we separate the known from the unknown numbers. For assistance, you can underline the 'x' from the numbers", while Paraskevi wrote that "I separate the known from the unknown. The known numbers are all numbers, unknown is whatever is a letter, for example, x, y, x, w". Finally, one Greek student offered an explicit objective. In this respect Panorea wrote that she "would explain to the student that it's an equation whereby we are trying to find the unknown x. The data we have are the numbers. After, I would explain how to find x".

Rote procedure

Few Cypriot students did not offer a rote rule for solving the equation and of those that did all alluded, either explicitly or implicitly, to the redistributive 'change the side and change the sign'. With respect to those who discussed the rule explicitly, one of the more detailed accounts was offered by Irene, who wrote that

> Whatever moves to the other side changes 'sign'. The 'sign' is (+) or (-). We have x + 5 = 4x - 1 and want to separate 'x' from the numbers, so we have to say 5 = 3x - 1. Why? 1x went to the right part and changed signs becoming -1x (since x is now 1x) therefore 4x - 1x = 3x (4 -1 = 3). 'x' in common. Now we have to justify 6 = 3x. We have 5 = 3x - 1. Since (-1) is a number, it must shift to the left part of the equation and become plus, that is 5+1= 6. (It was -1 and since crossing over the = it becomes plus). So 6 = 3x.

These, and other, students seemed both confident and clear as to the process involved in solving the equation. They offered rules whereby objects were moved from one side of the equals sign to the other along with a change of sign. Of course, the direction of such movement was determined by the solution presented to them but in no case did students offer any justification for the changing of the

sign. Interestingly, and unique among students, Elina offered a no less unwarranted but general account. She wrote that

> At first we get an equation with letters and numbers. The first thing to do is to move all numbers to one side and all letters on the other. If a number has a minus (-) sign or plus (+) once they are moved to the other side their sign changes; in other words, from minus (-) it becomes plus (+). That's how we deal with similar calculations.

Several Cypriot students offered accounts, typically short, in which the movement and the changing of signs were implicit. For example, Hermione wrote that "[a]s you can see x and $4x$ have been joined and the same is true of 5 and 1. Therefore, x has become $3x$ and then $5 + 1$ is 6". In these, and other, cases neither the movement across the equals sign nor the changing of the sign were made explicit. Our interpretation is that students were familiar with the process and saw such properties as givens rather than something in need of either explanation or warrant.

As with their Cypriot colleagues, Greek students typically offered a 'change the side change the sign' rule. Also, as before, the extent to which this was presented explicitly varied. For example, with respect to an explicit account, Daphne wrote that

> Since we have two unknowns on both parts of the equation, the first unknown x is moved to the second part of the equation but instead of a plus we make it a minus. Then we subtract it from $4x$ to get $3x$. After getting -5 in the second part, we move it to the first, therefore 1 is not a minus but plus. Then we add 5 and 1 to get 6.

In their accounts can be seen an understanding of the 'change the side change the sign' rule for solving equations. In neither case, however, can be seen evidence of a relational understanding of the equals sign as would be represented in statements justifying the described actions. There was no sense, for example, that students saw these actions as a consequence of adding or subtracting equivalent amounts from each side. In addition, as with the Cypriot students, one student offered a generalised account of the same process. In this respect, Callisto wrote that "[a]ll the 'x' are taken to the right and whenever 'x' is on the left part of the equation, you change the sign. On the left part, the known numbers enter while the unknown enter the right. Afterwards, we add the known to the known and the unknown with the unknown".

However, the majority of Greek students offered implicit summaries of the solution with which they had been presented. Typical of others, Panorea wrote that she would "shift the given numbers to one side and the unknown, x (or 3x) on the other side (e.g., 3x) and that way I would find the unknown x". Similarly, Paraskevi wrote that "[a]t this stage, we carefully observe when to change (+) or (-). When they move from the 1st part to the 2nd, the sign changes from let's say (+) to (-) and vice versa. Once we reach this point 6=3x". In such statements can be seen different but equally incomplete accounts of the rote rule. For example, Panoreas's account highlighted the bringing together of the knowns and unknowns respectively, leaving the reader to infer the changing of the sign, while Paraskevi's emphasised the changing of the signs at the expense of her detailing the changing of the sides of like objects. In other words, both left significant elements for their readers to infer.

The inverse operation

In almost all cases, having articulated some sense of a rote rule, students from both countries invoked an operation reversal to explain the final line of the solution, whereby 6 = 3*x* became 2 = *x*. In this respect, eleven of the 21 Cypriot students offered, albeit implicitly, an understanding of the inverse operation necessary to transform the line 6 = 3x to 2 = x. Typical of these were the comments of Chloe, who wrote that "after reaching 6 = 3x, we divided 6 by 3 so that x will be by itself. I found x to be 2" and Hermione, who wrote "then I divide 6 by 3 and x is equal to 2". Of these eleven students, two students offered more extended but no less implicit suggestions, as in Stamatia's comment that "we take number 6 and equate it '=' with 3x. Then we divide 6 by 3, i.e. 6/3 equal to x1, "6/3 = x. 6/3 makes 2 and so x1 is equal to 2, 'x = 2' ". In these cases, students' comments about dividing by three seemed to us to reflect, at least implicitly, a recognition of the structural significance of the unknown's coefficient that necessitated division.

A further seven students offered accounts indicative of a better developed understanding of the role of the coefficient and its function with respect to inverse operations. For some this could be seen in the ways they reiterated earlier comments about isolating the unknown. This was seen in, for example, the writing of Perikles, who added that "arriving at 6 = 3x I divided 3x by 3 to get x alone, and 6 on the other side with the 3 before the unknown. So the answer is x = 2. Others in this group were more explicitly aware of the coefficient and its significance, as in Medea's comments that "once the basic calculation was made, we divide both parts by the unknown's coefficient and then get the result" or Vassiliki's "I take the number in front of the x and divide both sides of the equation by it". In such comments, focused on the coefficient of the unknown, can be seen.

The Greek students' comments could be categorised similarly, although a smaller proportion, three of the eleven, offered entirely implicit statements. Of these, typical was Crino's "we have reached 6 = 3x and consequently, we divide 6 by 3 to get the result". Seven students, also a higher proportion than with the Cypriot cohort, presented accounts with an explicit reference to the coefficient, as with Ivy's comment that he (the invisible teacher) "divided by the coefficient of the unknown 3 to get 2 = x" and Paraskevi's note that "once we reach this point 6 = 3x, I must divide by the coefficient of the unknown to see what the value of x is, that is to say, to divide both the 1st part and the 2nd by 3".

Interestingly, two of these seven students offered accounts, albeit whose intentions were clear, that employed an incorrect vocabulary, as with Daphne's "finally, we divide 3 by the fraction" or Nikoleta's "we divide it by the denominator of *x*, namely, 3 and we arrive at the final result that 2 = *x*". In such cases students' intended meaning was clear.

Discussion

Our analyses provide substantial insights, both encouraging and discouraging, into the conceptualisation of mathematics these beginning teachers bring to their teacher education programme. In this respect, both sets of students appeared procedurally competent, recognising the equation for what it is and typically understanding how it had been solved. However, from a disciplinary perspective (Leinhardt, 2001), students' explanations showed very little awareness of the epistemological underpinning of mathematical knowledge, almost without exception warranting their chosen procedures on the basis of their personal authority as teachers. Moreover, from the perspective of research on the solving of equations, students indicated no relational understanding

of the equals sign, a prerequisite for learners confident solving of equations of the form above with the unknown on both sides (Alibali et al., 2007; Filloy & Rojano, 1989). From an instructional perspective (Leinhardt, 2001), students' explanations typically alluded to a presentation of objectives and the importance of the unknown and its role in equation solving. However, the presentation of such a rule would typically allow little opportunity for learners to understand the reasoning behind it. Interestingly, although their rote rules never alluded to operations performed on both sides, many students were aware of inverse operations in relation to the unknown's coefficient. In sum, and drawing on Skemps's (1976) distinctions, our data showed that both Cypriot and Greek students were locked into an instrumental rather than relational understanding of mathematics, perspectives which their respective curricular traditions would not have encouraged. Finally, when framed against the mathematical knowledge for teaching literature, it seems that at this stage of their careers both sets of students showed evidence of Rowland et al.'s (2005) foundation knowledge, albeit problematic, but, as yet, none of transformation, connection or contingency. Moreover, if the goal of teacher education is to facilitate beginning teachers' profound understanding of emergent mathematics (Davis & Renert, 2013) then much is still to be done.

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Research on mathematical literacy in schools - Aim, approach and attention

Frode Olav Haara[1], Oda Heidi Bolstad[1] and Eirik S. Jenssen[1]

[1]Western Norway University of Applied Sciences, Faculty of Teacher Education and Sports, Sogndal, Norway;

frode.olav.haara@hvl.no, oda.heidi.bolstad@hvl.no, eirik.jenssen@hvl.no

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Research in mathematical literacy has different emphases; whether the teaching and learning of mathematical literacy or the synthesis of results from worldwide studies that emphasises mathematical literacy (e.g. the PISA 2003 studies) to focus on the curricular implications. However, a comprehensive review of the current empirical research on the area is missing. The poster presentation focuses on what research brings attention to in empirical studies where mathematical literacy is highlighted. These are both quantitative and qualitative based projects, and include for instance articles with data collection where mathematical literacy is emphasised in the development of tools for the data collection or in what is being measured, and articles focusing more on teachers' implementation of mathematical literacy in teaching and learning. The poster also focuses on what research finds to be implications for teaching and future research on mathematical literacy.

The poster is based on our work on a review article, submitted to an international journal, that aims to address the research on mathematical literacy in primary and lower secondary school, by bringing together, comparing and synthesizing the diverse body of current research, emphasise implications for research on the area, and point to necessary areas for research to come. Through the poster we aim to inform about our review on research on mathematical literacy in school. The poster is used to present a systematic review of recent empirical studies through comparing, analysing, and discussing the body of articles in relation to the following key questions:

1. *What methodologies have been used to examine emphasis on mathematical literacy in primary and lower secondary school?*
2. *How is mathematical literacy conceptualised?*
3. *What is the focus of attention in research on mathematical literacy?*
4. *What are the implications for primary and lower secondary school teaching, and recommendations for future research on mathematical literacy?*

We have applied methods well-known for review articles that aim to identify a state of the art within a field of research on school related issues, and to make suggestions for further research within the area at hand (e.g., Beltman, Mansfield & Price, 2011). This counts for the identification of parameters for the review, search in data bases based on the identified parameters, and selection of publications to form the basis of the review and analysis. The body of articles for the review consists of 28 articles that fulfil the selection criterions used. The studies were conducted worldwide, but with a clear majority of European and Asian contributions.

Our findings show that the research is dominated by quantitative approaches, and do not focus on what goes on in the classroom. It focuses on the outcome of what goes on in school. The lack of identified attention to qualitative research on teaching for mathematical literacy seems to be due to four main reasons. The decision to apply existing data from PISA studies for more quantitative analyses on mathematical literacy related areas, application or even exploitation of the mathematical literacy concept in studies that are not directly focused on mathematical literacy, and tension between policy documents and practice in school on one hand, and tension between learning achievements and mathematical literacy on the other hand. This leads to lack of attention to best-practice projects. Hence, it seems that research in the future to a larger extent might report research based results on what schools and teachers ought to do in order to teach for mathematical literacy. Several of the articles use data from PISA test results, and are therefore obliged to acknowledge the prevailing OECD definition at the time of testing, because the attention to mathematical literacy in the PISA tests is based on this definition. In addition, some of the articles reviewed connect subject matter theories within mathematics education with the concept of mathematical literacy. A common factor for these articles is their interest regarding the teaching of mathematical literacy in school. Regarding implications for primary and lower secondary school teaching and further research on mathematical literacy, three main challenges were identified: both researchers and teachers are uncertain about how to develop students' mathematical literacy, specific attempts to work directly with mathematical literacy through mathematics alone have not been successful, and teaching for mathematical literacy appears to require non-traditional methods for teaching mathematics.

Furthermore, the subject of mathematical literacy is given extensive attention both at political and societal levels (OECD, 2009) and within mathematics education research. In fact, Sfard (2014, p. 141) urges the research community to address this issue: "The question of how to teach for mathematical literacy must be theoretically and empirically studied. When we consider the urgency of the issue, we should make sure that such research is given high priority." The approach to such a quest seems to be through increased emphasis on qualitative research, for instance, through studies of best-practice and research projects involving practising teachers. Therefore, the research community's attention needs to shift from nurturing data and findings that highlight student results on mathematical literacy tests to research on what to do in order to improve the students' opportunities to develop mathematical literacy. A starting point for such a shift in focus could be to examine how mathematical literacy is understood, facilitated and experienced in schools.

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Kids Inspiring Kids for STEAM (KIKS)

Maitane P. Istúriz[1], Ignacio González-Ruiz[1], Jose M. Diego-Mantecón[1], Tomás Recio[1], José B. Búa[2], Teresa F. Blanco[2], María J. González[1] and Irene Polo[1].

[1]University of Cantabria, Faculty of Science, Santander, Spain; perezim@unican.es; gruizi@unican.es; diegojm@unican.es; tomas.recio@unican.es; mariaj.gonzalez@unican.es; irene.polo@unican.es

[2]University of Santiago de Compostela, Faculty of Education, Santiago de Compostela, Spain; jbenitobua@gmail.com; teref.blanco@usc.es;

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STEAM

STEM (Science, Technology, Engineering and Mathematics) is an educational approach based on the interdisciplinarity and applicability of scientific and mathematical knowledge to technology and engineering. STEAM integrates Art into STEM in order to promote children's creativity (Fenyvesi, Téglási and Szilágyi, 2014). In many European countries, the number of graduates in science, maths, technology and engineering areas is clearly insufficient for the needs of their companies and industries. To stimulate students' interest in these areas and art, the European Union has dedicated a lot of resources and effort, developing a large number of projects for pre-university classrooms. For a review, see Rocard, Csermely, Walwerg-Henriksson and Hemmo (2007).

KIKS project

KIKS, Kids Inspiring Kids for STEAM is a European Erasmus+ Project, which involves four European institutions: Metropolitan University of Budapest (Hungary), STEM Team East (Cambridge, United Kingdom), University of Jyväskylä (Finland) and University of Cantabria (Spain). The project started in March 2016 and its main aim is to promote secondary education students' interest on the STEAM areas, by developing activities and presenting them to other students locally and internationally. Many students and teachers do not enjoy or have confidence in maths and STEM: they have anxiety even maths/technophobia and drop it as soon as they can. So we seek to promote the creativity and motivation for learning of these less confident students, working interdisciplinary, using technology, and fostering communication and the transfer of ideas/knowledge across cultures. From a research point of view, KIKS aims to compare cross-culturally the elaboration and resolution of STEAM activities at secondary education level.

Development of activities

Students, in teams of fives and led by at least one teacher, are asked to elaborated STEAM activities or projects under the following approach: How would you get your schoolmate to love Maths? The activities or projects can emerge from a teacher, a pupil, or a KIKS coordinator's idea. Once the idea emerges, it is developed into an activity or project. It should involve different STEAM areas, but its duration and degree of difficulty can vary according to teams' availability. Once an activity is elaborated, the team presents it to their local homologous (in face to face events) and to their international homologous (through video conferences). Schools from different countries are invited

to participate in the project, at the moment we have more than 25 participant schools from different countries and backgrounds.

Products to be developed by the students

Each participant team has to elaborate a written document, an explanatory video, and a presentation of its work. (1) The written document (Word Doc or Power Point) has to include a presentation of the team members, and a description of the activity with the main results and the material used. (2) The edition of the video has to include the practical or technical aspects of activity, which are difficult to explain on paper. For example, the manipulative construction of objects, the use of measurement tools, etc. All the products have to be developed in the English language. The limited scope of this paper does not allow us to include here examples of the activities already developed by our teams, but they can be found at our website (http://www.kiks.unican.es/en/actividades/).

KIKS support

KIKS provides support to the teams through different platforms including Goggle Drive, YouTube, Facebook and a Website (www.kiks.unican.es). The Google Drive and Facebook platforms function as storages of information— where teachers and coordinators can exchange ideas— as well as repositories of documents elaborated by the teams. The YouTube Canal works as repository of videos, and the Website provides different and meaningful information about the ongoing process of the project. Apart from the above, KIKS provides support to the teams proposing activities, helping in aspects related to the English language, and furnishing technical support for video edition, online connections, etc.

Evaluation

Parallel to practical work of the project, we are undertaken a research study aiming to evaluate the strengths and weakness of KIKS. Firstly, this research aims to assess cross-culturally teachers' and students' perceptions about STEAM. Secondly, we aim to characterise the STEAM activities elaborated by the teams, according to the cognitive (competences, capacities, skills) and motivational (attitudes, emotions) dimensions they may develop in the learners. In short we seek to evaluate the impact of STEAM activities in the learning process. Tools for evaluating these two dimensions are currently under construction.

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The teaching and learning of relations and functions: A comparative study of Norwegian and Ethiopian textbooks

Solomon A. Tesfamicael and Geir H. Botten and Øyvind A. Lundeby

Faculty of Teacher and Interpreter Education (FLT), Norwegian University of Science and Technology (NTNU), Norway; Solomon.a.tesfamicael@ntnu.no; geir.h.botten@ntnu.no Øyvind.a.lundeby@ntnu.no

Keywords: Relations, functions, textbooks, comparative studies.

Introduction

Comparative studies in mathematics include studies that document, analyze, contrast or juxtapose similarities and differences across all aspects and levels of mathematics education (Jablonka and Andrews, 2012). In this project, which is in an early phase, we intend to carry out such a study at the cross-national level. The rationale behind it is to identify similarities and differences between mathematics education in Norway and Ethiopia, to reflect on their practices in the light of international wisdom (Clarke, 2003), and to lay down grounds for further intervention studies, for example for the Norwegian NORHED project that will start in 2017. (https://www.norad.no/en/front/funding/norhed/news/).

We have started comparing textbooks, since textbooks are the main resources used in mathematics classrooms (Pepin, 2010) in many countries including Ethiopia and Norway. Textbooks are tools, or instruments, that facilitate the daily work of teachers. They also contribute to the field of mathematics by preserving and transmitting skills and knowledge (Rezat, 2008a). In general, Mathematics is a subject that has long been associated with textbooks and curriculum materials (Remillard, 2005). Therefore, it is important to look at textbooks as a source of comparison.

At this stage, emphasis is given to the teaching and learning of relations and functions as presented in the textbooks, partly due to students' difficulties with learning these topics (Denbel, 2015). As teacher educators working with students preparing to work in primary and middle school in Norway, we have also observed that many student teachers struggle to grasp these concepts and hence to teach them.

Method

In this study we selected six textbooks in Norway and the one textbook from Ethiopia from lower secondary level which covers the concepts of relation and function. At this early phase of the study, definitions, examples, representations, exercises and problems, activities, group works, contexts and level of abstractions in the textbooks are being identified and compared.

Findings

As mentioned above, the purpose of this poster is to communicate the beginning of our project, which will enable constructive sharing of knowledge and experience about the teaching and learning of mathematics between the two countries, and we hope with the international mathematics education community in the coming years. We report our findings to date as follows.

Among the selected textbooks, only two of them (one text from Norway and the textbook from Ethiopia) address the concept of relations directly by providing definitions, domain and range of relations and different representations, examples and exercises, without including the topic of function. The other Norwegian textbooks deal with the topic of function by taking for granted that students understand the concept of 'relation' in mathematics. Most of the books follow the teaching of functions by giving context-based definitions and examples, beginning with proportional relationships of variables, building to linear and then quadratic functions.

In The Ethiopian textbook (M9) the definitions of relations and functions are provided in terms of subsets of a Cartesian product of two sets. Examples and problems are consistently abstract and unrelated to any real context. In contrast, we find no single definition and representation in the respective Norwegian textbooks. In addition, the Norwegian textbooks include many real life related contexts that are accessible by the students, and they are full of different representations (graphs, symbols, words, tables and physical figures) for both concepts. With reference to this topic, the textbook M9 has a higher level of abstraction than its Norwegian counterpart. Symbolic and graphic representations are present in M9, but it is devoid of contexts and real life related examples and problems.

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TWG12: History in mathematics education

Introduction to the papers of TWG12: History in mathematics education

Renaud Chorlay[1]; Kathy Clark[2]; Katalin Gosztonyi[3]; Snezana Lawrence[4]

[1]Uni. Paris Diderot & ESPE de l'Académie de Paris, France; renaud.chorlay@espe-paris.fr

[2]Florida State University, USA

[3]Eötvös Loránd University, Budapest, Hungary

[4]Anglia Ruskin University, UK

History of mathematics in mathematics education continues to receive much attention. However, empirical research and coherent theoretical/conceptual frameworks within this area have emerged relatively recently. The purpose of this TWG is to provide a forum to approach mathematics education in connection with history and epistemology dedicated primarily to theory and research on all aspects of the role, effect, and efficacy of history and epistemology as elements in mathematics education.

TWG12 welcomes both empirical and theoretical research papers, and poster proposals related to one or more of the following issues:

1. Design and/or assessment of teaching/learning materials using the history of mathematics, preferably with conclusions based on empirical data; all levels can be considered, from early-age mathematics to tertiary education and teacher training.
2. Surveys on the existing uses of history or epistemology in curricula, textbooks, and/or classrooms in primary, secondary, and tertiary levels;
3. History of mathematics education;
4. Relationships between, on the one hand frameworks for and empirical studies on history in mathematics education and, on the other hand, theories, frameworks and studies in other parts of mathematics education research.

Even though the creation of this TWG is fairly recent – it started in CERME6 (2009) – it has deeper institutional roots within the maths education research community. Indeed, the *HPM* study-group (*History and Pedagogy of Mathematics*) was created at the 1972 ICME conference; it has been organizing satellite conferences to the ICME meetings since 1984, and has several active regional branches (HPM-Americas, European Summer Universities). In CERME10, 16 papers and 2 posters were presented in TWG12, for a total of 23 participants affiliated to this group, covering a large range of European countries (from Ireland to Russia) and beyond (Brazil, Mexico, the U.S.). A short survey showed this TWG attracts newcomers to the CERME community from the HPM community, since 9 participants were CERME first-timers, yet only two had never attended any HPM-related event.

Before going into any details, it should be stressed that this TWG has four general but distinctive features which give these meetings their specific flavour. Firstly, its topic lies at the intersection of

different fields of research – maths education research and history of mathematics – which requires versatility and methodological vigilance (Fried, 2001; Chorlay & Hosson, 2016). Secondly, the strength of the historical and the HPM community varies greatly among countries, and these meetings play a crucial role for researchers working in relative isolation, and with difficult access to resources in the field. Thirdly, the scope of TWG covers both history *in* mathematics education and history *of* mathematics education, which are two significantly different research topics (TSG 24 and 25 in ICME13); connecting the two lines of investigations is a constant challenge. Fourthly, since the topic of TWG12 is neither specific to one level of the educational system (from primary education to teacher-training) nor to any single mathematical topic (be it fraction concepts, algebra, proof, etc.), the work in TWG12 intersects that of most other TWGs. This time, intersection with TWG1 (Proof and argumentation), TWG18 (Teacher education), and TWG22 (Resources and task design) was significant. It should be noted that, for this edition, there was little intersection with what was covered in TWG8 (Affects and mathematical thinking), TWG10 (Diversity and maths education), in spite of the fact that it is not uncommon for outsiders of the HPM research community – among which most policy-makers and curriculum-designers – to ascribe such goals to the historical perspective in teaching.

These four features made this meeting not only useful but also challenging and exciting. As the final discussion made clear, the general feeling among the participants was that one of the main outcomes of this meeting is that we actually *learned* a lot from the one another, both from their papers and from the lively discussions. Let us now highlight some of the significant feature of the 2017 conference.

For quite some time it has been stressed that more attention should be paid to the actual effects of the use of historical sources, either in the classroom or in teacher training (Chorlay, 2016; Jankvist, 2009). This year, at least two papers contributed to this line of research. For example, the *Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources* (TRIUMPHS) project is a five-year project funded by the National Science Foundation in the United States, which will create and test 25 full-length Primary Source Projects (PSPs) and 30 one-day "mini-PSPs." Each PSP is designed to cover its topic in about the same number of course days as mathematics classes would otherwise. With PSPs, rather than learning a set of ideas, definitions, and theorems from a modern textbook, students learn directly from the original work of mathematicians such as Leonhard Euler, Augustin-Louis Cauchy, or Georg Cantor. The project includes an extensive "research with evaluation" study, which will seek to address several evaluation and research questions and enable both formative and summative evaluation of the project activities. Data sources to inform the research are pre- and post-course surveys (of both students and instructors), post-PSP surveys, student interviews, student PSP work samples, video captures of selected classroom instruction and audio captures of selected small group student work, and instructor post-implementation reports. By the end of the project it is expected that some 50 instructors and over 1000 students will participate in undergraduate mathematics classrooms where PSPs are used.

On a smaller scale, Areti Panaoura studies the manifold difficulties faced by an "ordinary" teacher attempting to use a textbook activity on Egyptian multiplication. It raises many questions for our research community to investigate further: as to the level and nature – mathematical, didactical,

historical – of expertise required from the teacher; as to our (as researchers and teacher-trainers) criteria for assessing such teaching sessions; as to the relevant theoretical frameworks for the description and analysis of teacher-practice (in particular the use of pedagogical documents). Along with these questions, it shows the importance of leaving our comfort-zone, a zone in which the teaching sessions are implemented by the researcher who designed them or by teachers with a significant experience in the field.

As is customary in HPM-related meetings, a large number of papers carry out detailed content analysis. Let us restrict ourselves to those dealing with numbers and early-algebra: Antonio Oller-Marcén and Vicente Meavilla describe forms of argumentation about equations of the $ax^n = bx^m$ type in a 16th century Spanish treatise, and endeavour to make sense of what we would consider to be errors or flaws; Chorlay studies arguments justifying the rule for fraction multiplication in a Chinese treatise from the Han dynasty and compares them with arguments found in today's textbooks; Maria Sanz and Bernardo Gómez devise a structural classification of sharing problems on the basis of a large historical sample, and complement this classification by showing the variety of methods – both arithmetic and algebraic – for solving them. Coming from a perspective of history of education, Rui Candeias discusses in details a pedagogical approach to operation on decimals, in a context which combines proportionality and magnitudes. Although this line of investigation may seem to be very content-oriented, its connections to didactical questions – be they theoretical or more applied – are manifold. First, it is hardly necessary to say that content-analysis is a central part of *a priori* analysis, and that – on a par with a purely mathematical analysis – investigations into the history of mathematical knowledge and mathematical practices provide key background data. Second, the work presented in some of papers is explicitly described as a first phase in a larger research project focusing either on learners (Sanz-Gómez) or teachers (Chorlay). Third, the content-analyses presented in these papers contribute to the general theoretical discussion on some important didactical concepts, such as "epistemological obstacle" (Oller-Marcèn) or "generic example" (Chorlay).

As far as history of education is concerned, let us highlight the contribution of Katalin Gostonyi. Her comparative study of the works of mathematics educators T. Varga (in Hungary) and G. Brousseau (in France) shed light on the origins of theoretical frameworks which are still very much alive in mathematics education research.

Finally, the paper of Liz de Freitas contributes to the ongoing work in the HPM community from a new perspective. Being a philosopher of mathematics, she draws on both her personal research – in the continental tradition of the philosophy of mathematics and mathematical practice – and her experience in the training of maths teachers to suggest a large number of research questions which are relevant for the historian and the maths education researcher alike. Here, we briefly mention two such issues which we feel would be worth investigating further. A first series of questions bears on diagrams: the way they are drawn and read; their cognitive impact and their epistemological significance; the historical evolution of the meta-rules governing the use of diagrams, in themselves and in their relations to other elements of mathematical texts. A second series of questions bears on the image of mathematics maths teachers have, its impact on their teaching, and the way teacher-training modules may impact this image. Investigating this second series of questions could bridge the gap between the maths education community and the science education community, a

community in which research on the Nature of Science (NOS) is a central research topic (see for instance Abd-El-Khalick, 2013).

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Arithmetical problems in primary school: Ideas that circulated in São Paulo/Brazil in the end of the 19th century

Luciane de Fatima Bertini[1] and Andreia Fernandes de Souza[2]

[1]Universidade Federal de São Paulo, Brazil; lfbertini@gmail.com

[2]Universidade Federal de São Paulo, Brazil; deianandes@hotmail.com

This article aims to analyze teaching programs, textbooks and journals of education in order to better understand the ideas that circulated in São Paulo, Brazil, at the end of the 19th century about the conceptualization and the utilization of problems for the teaching of mathematics in these documents. The research was conducted through a culture-historical approach (Chartier, 2002), which compels the researcher to have a questioning attitude towards the object of study. Thus, it was possible to observe that both the teaching program and the textbook valued the utilization of problems because both connect the concept of problem with activities presented through narratives in order for students to apply previous knowledge. On the other hand, the theme did not have the same value in the journal of education because it discussed that topic in only one article, which introduced the concept of problem as a synonym for a type of exercise related to calculations.

Keywords: Arithmetical problems, primary school, intuitive method.

Introduction

In the state of São Paulo, Brazil, the end of the 19th century was marked by events that would bring significant changes to primary school over the next decades and that would also be a reference for other states. In the last decade of that century the foundations of school organization – time, space and curriculum structuring - were established. In addition, *grupos escolares*, which represented the idea of a modern and quality urban school, began to be introduced in the state of São Paulo in 1893 (Souza, 2009).

Since that was a period whose proposals and determination impacted primary school nationwide over the next decades, research on the history of mathematics education has great interest in understanding the ideas that circulated in that period regarding the teaching of mathematics.

This paper aims to contribute to such understanding as it intends to analyze proposals for the utilization of problems in the teaching of mathematics by using textbooks[1], journals of education[2] and teaching programs[3] from that period as sources.

[1] Choppin (2009) observed the existence of several expressions to name school books. Names vary according to context, use or style. In this paper the expression "textbooks" is used to name all publications written in order to be used in Brazilian primary school classrooms.

[2] Publications that compiled articles written by intellectuals and teachers on themes related to teaching.

[3] Publications that provided guidance on school organization, content and methodology.

Previous research done by the authors of this paper (Bertini, 2016a; Bertini, 2016b; Souza, 2016a; Souza, 2016b) shows the presence of arithmetical problems, or of proposals for their application, in several documents from different periods and with different goals. In this paper, the authors aim to conduct an analysis of these documents in order to better understand the ideas that circulated in São Paulo, Brazil, in the end of the 19th century about the use of problems for the teaching of arithmetic.

In this study, the definition of *problem* is not presented a priori because that term is understood in different ways according to the historical period in question. Thus, one of the objectives of this research work is to identify what concept of *problem* the documents contain.

Methodological and theoretical framework

To develop the historical production in this study, all analyses took into account the historical moments and spaces in which the documents were produced as well as the interests involved in this production. Thus, we take a culture-historical approach as proposed by Chartier (2002), which not only encourages researchers to carry out descriptive work, but also compels them to raise questions about the documents studied in order to identify how a specific reality arises and establishes itself according to the place and time in which it emerges. According to De Certeau (2001, p. 35), a historian's work is more connected to finding *meaning* and *purpose* than to simply *narrating* facts.

From that perspective, the notion of *appropriation* is crucial for a significant historical production because it proposes the existence of creative invention in the process of reception (Chartier, 2002, p. 136). Teaching programs, textbooks and journals of education will be analyzed based on that notion because, although in different manners, they all *appropriated* the ideas and determinations that circulated about the utilization of problems to teach arithmetic. Besides, throughout the process of creative invention, the authors will work to understand which ideas and determinations are produced in the documents about this utilization.

Moreover, the notion of purpose, presented by Chervel (1990), will be harnessed once it is related to the options made for teaching. The historical study that involves the use of arithmetical problems in primary school will encompass the understanding of the purposes of their use. The authors will be guided by two questions when studying the documents: How were problems harnessed in these sources? Why were they suggested that way?

It is important to emphasize that, according to Valdemarin (2004), in the 19th century the Brazilian educational context was influenced by ideas which arose from the intuitive method. She sustains that proposals for school activities included presenting a variety of objects to the senses so that ideas would be formed as a result of a rational, concrete and active teaching style. Two ideas are presented as paramount in the proposals that followed the intuitive method: first, the notion that observation leads to reasoning; second, the belief that work prepares individuals for the future (Valdemarin, 2001).

The proposal for educational renewal opposed the abstract and little useful character that teaching had had so far and comprised a new teaching method (the intuitive method). It started to be introduced

in Europe, through Pestalozzi's[4] e Frëbel's[5] elaborations, and in the USA, where the work "Primary object lessons", by Calkins,[6] was first published, in 1861. Brazil was inserted in this refreshing effort in the 1880's, when the country began to adopt the new intuitive teaching method. That movement was influenced by foreign ideas but it was also an attempt to meet political demands in the country due to the end of the Empire (Valdemarin, 2001, p.159).

Dialogues with these productions on the History of Brazilian Education are considered necessary for harnessing the theoretical concepts here presented as an option for the construction of a historical narrative because it will contribute to the understanding of the context in which the documents were produced.

Teaching programs, journals of education and textbooks

The first teaching program in the state of São Paulo, as determined in Decree 248 of July 26th 1894, provided school management guidance and teaching instructions such as school organization, materials, students' attendance, reports, disciplinary procedures, school-year calendar, curricular content and methodology. The Decree was signed by Bernardino de Campos and Cesário Motta Junior, who were respectively the President of the State of São Paulo and Secretary of the Interior.

Nevertheless, besides being determined by law, all those instructions needed to be spread among both active teachers and future teachers. In that sense, journals of education were presumably a tool to transmit models of work that would help teachers appropriate the new educational proposals. In the last decade of the 19th century, the *paulista* journal "A Eschola Pública" was sponsored by the government and its editorial staff was composed of teachers, principals and school inspectors.

In addition to teaching programs and journals of education, textbooks were another tool used to guide teachers' work because they presented proposals of activities to be done by students in the classroom, as well as proposals for school organization (ordering of contents, quantity and style of activities).

Finally, in this analysis we will articulate these different documents in order to generate understanding of the ideas that circulated in São Paulo, Brazil, in the end of the 19th century regarding the utilization of problems for the teaching of arithmetic.

- **The program of 1894**

Teaching programs are part of the norms that integrate school culture and help us understand it. However, we know that changes and innovations proposed by governments are the result of political disputes, which prevents them from happening naturally and passively (Souza, 2009, p.83- 84).

In the *paulista* program of 1894 there is guidance on choosing a methodology:

> Article 9 – Lessons on subjects of any course year need be more empirical and concrete than theoretical and abstract, and should be conducted in order for children's faculties to be developed

[4] Johann Heinrich Pestalozzi (1746-1827), Swiss educator.

[5] Friedrich Wihelm August Frëbel (1782-1852), German educator.

[6] Norman Alisson Calkins (1822-1885), American educator. This work was translated/adapted to Portuguese by Rui Barbosa in 1886.

> in a gradual and harmonic manner. Article 10 – The teacher need aim, especially, to develop the faculty of observation by applying intuitive processes for this purpose. (São Paulo, 1894)

The expressions "lessons... more empirical and concrete" and "intuitive processes" remind us of the educational trend that was disseminated at that time, i.e., the intuitive method.

It is possible to notice that in the program some subjects are not present in every school year. Arithmetic, however, remains in all grades/years, with contents that are graded according to their difficulty level.

Besides providing the list of subjects, the program included "more and more detailed prescriptions coming from teaching administration departments". The 1894 program was extensive, according to testimonials by inspectors and principals of *grupos escolares*. Contents related to reading, writing, calligraphy and arithmetic were considered essential by teachers. On the other hand, the ones related to geography, history and science were of secondary importance (Souza, 2009, p. 84).

For the teaching of arithmetic, the term *problem* appears in the content list in the following expressions: "Supplementary studies: problems and practical questions", "Easy problems", "Problems", "Supplementary assignments: problems, practical questions".

It is important to emphasize that the term *problem* is present from the second year on, always in the end of the list of contents. As years/grades advance, the words "easy" and "practical questions" join the term *problem* in expressions.

- **Journal "A Eschola Pública"**

Journals of education are extremely rich and varied sources (Monarcha, 2004). The author reveals a chronological list of journals of education of São Paulo, which includes three titles that were published in the last decade of the 19th century. Out of those three publications, the UFSC Digital Repository[7] has two: "Revista do Jardim da Infância" and "A Eschola Pública", respectively *Kindergarten Journal* and *Public School*, in free translation. As this paper aims to analyze primary school, we will focus on articles published in "A Eschola Pública". This journal of education was first published in 1893. Its early stage, which lasted until 1894, comprises eleven issues. The second publishing stage started in 1896, with its final issue published in the following year.

This journal is the result of a council formed by people who had graduated at Escola Normal da Capital[8] (*Capital's Normal School*, in free translation) and actively took part in political and cultural movements at the time. After that publishing period, many of its council members held important offices in the government, for example Oscar Thompson, who was named General Manager of State Public Education (*Diretor Geral da Instrução Pública do Estado*), and Arnaldo Oliveira Barreto, who was a teacher at *Escola-Modelo do Carmo* (Carmo Model School, in free translation), in 1894, and the inspector of the associated schools of São Paulo.

[7] Database fed by GHEMAT researchers, where theses, dissertations, articles, textbooks, education journals and students' notebooks are available. Available in https://repositorio.ufsc.br/handle/123456789/1769

[8] First Normal School in the state of São Paulo, called Caetano de Campos today. It was founded in 1894 and prepared future teachers.

According to Monarcha (2004), the publication had 21 issues. Nevertheless, only 18 issues are available at the UFSC Repository, whose digital database is fed by the GHEMAT researchers. All over the 18 issues, only 15 articles feature the teaching of arithmetic and only one, which was written by Arnaldo Oliveira Barreto in 1897, referred to the term *problem*.

In addition to being a teacher at Escola-Modelo do Carmo, the first *grupo escolar* in São Paulo, in 1894, Arnaldo Oliveira Barreto (1869-1925) reorganized Grupo Escolar de Lorena, São Paulo. In the period between 1902 and 1904, he was editor-in-chief of Revista de Ensino (*Journal of Teaching*, in free translation). He was also part of Sociedade de Educação de São Paulo (*Education Society of São Paulo*). Throughout his career, he produced several books, articles and guidebooks.

In an article published in March 1897[9], the author provides suggestions regarding the order of the work a teacher needs to perform in the classroom: "write the exercises", "students with their arms crossed", "distribute all necessary material", "ring the bell for work to start", "divide your board" and copy "all problems and then do them" (Barreto, 1897, p.38).

In this article, the author talks about the utilization of calculations involving all four fundamental operations whose results are not higher than 20. Some examples of calculations presented in the article are as follows: 3 + 2 =, 4 + 3 =, 6 ÷ 3 =, 4 – 2 =, 6 x 2 = (p. 40). He also suggests two different correction procedures: individual and collective. For the collective correction, he recommends that each student reads his/her problem:

- Three plus two makes five.
- Four plus three makes seven.
- Two and two are four.
- A six has two threes, etc. (Barreto, 1897, p.39)

To refer to the proposed calculations, the author alternates between the term "exercise" and the term "problems", as we can observe in the final part of his article: "provide daily variation in the exercises, which is most convenient, and I emphasize that the problems should always be about the four fundamental operations" (Barreto, 1897, p. 39).

- **"School arithmetic"**

In the period between the end of the 19th century and the beginning of the 20th century there was a "close relation between the public primary expansion and the publishing expansion in the state of São Paulo" (Razzini, 2004, p. 1), which was originated with the establishment of the Republic in 1889. The textbook was set as one of the necessary aspects for implementing the proposal of *grupos escolares*.

Costa (2011), in a study about arithmetic textbooks in *grupos escolares* in São Paulo, states that Ramon Roca Dordal's work titled "Arithmetica escolar – exercícios e problemas para escolas primárias, famílias e collegios" (*School Arithmetic – exercises and problems for primary schools, families and other schools*, in free translation), which is composed of six books, circulated in public schools in the state of São Paulo in the end of the 19th century and beginning of the 20th century. The

[9] Available in https://repositorio.ufsc.br/xmlui/handle/123456789/126750

first four parts of his work were analyzed in this paper. However, after an extensive search, the two final parts could not be obtained.

In addition to the title, a highlight to the fact that the work is composed of exercises and problems is also present in the introduction done by the author himself. Although there is no clear indication of what is understood by "exercise" or "problem", the order of the activities featured in each lesson is pretty similar and seems to focus on the following: introduction of the rule, exercises and problems. Therefore, the term "problems" seems to be associated with a narrative of a daily life situation.

Lesson XI from the second book, for example, introduces the rule "When the sum does not provide exact tens, one should write the exceeding units in the sum and the tens should join the following order" (p. 11), followed by operations (Figure 1).

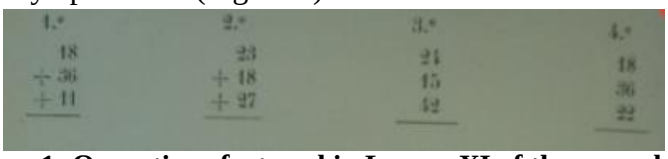

Figure 1: Operations featured in Lesson XI of the second book.

Finally, problems are presented as a final part of each lesson:

> 5th – A traveler has covered 25 leagues on a train, 14 leagues on a horse and 44 by ship; how many leagues has he covered?
>
> 6th – From Santos to São Paulo there are 16 leagues, from São Paulo to Jundiaí there are 12 and from Jundiaí to Campinas there are 9; how many leagues are there between Santos and Campinas?

Another point that Roca Dordal highlighted about the use of problems was that it was necessary to use easy problems, which should compose short lessons, so that children would remain interested and attentive, which is paramount for the success of the teaching process.

The first three books feature all problems that involve the addition operation and the fourth contains problems which are all related to subtraction. The situations featured in the problems have as backgrounds mostly children's everyday life situations (school, shopping, etc) and also adult life situations (distance between two cities, populations, etc).

Conclusion: Ideas that circulated

From observing the three sources selected for this study, it is possible to characterize some ideas that circulated in the state of São Paulo, Brazil, in the end of the 19th century about the use of problems in the teaching of arithmetic.

In that historical period the use of problems in classrooms to teach arithmetic was highly valued, which is characterized by the presence of problems in almost every teaching program for primary school (except for the first year/grade) and also by the presence of the term *problems* on the cover of Ramon Roca Dordal's work. Textbook covers somehow try to introduce some of the work contained inside so that it is more likely to be bought, adopted or used. Therefore, the presence of the term *problem* on the cover seems to endorse the notion that it was valued by those who would make use of it or recommend it (institutions or teachers).

Despite the high regard observed in the teaching program and in the textbook under analysis, discussions over the utilization of problems do not seem to have been a serious debate topic in the

articles featured in the journal that circulated in São Paulo at that time. The teaching of arithmetic was the theme of fifteen journal articles but the term *problem* was referred to in only one of them. Still that one article did not discuss effectively the understanding of what a problem actually is. In that article, the term *problem* was applied as a synonym for exercise and referred to the calculations proposed using numbers and operation signs.

Nevertheless, a different understanding is expressed both in the teaching program and in the textbook. In these two documents, even though there is no clear exposition of what a problem is, they seem to be more related to proposals of activities based on narratives, which somehow approach daily life topics, just like the situations contained in the textbooks by Ramon Roca Dordal. The use of the expression "practical questions" alongside the term *problem* in the teaching program may also be related to everyday life or to the utilization of objects to be presented to the senses for the formation of ideas, once the program represents ideas based on the intuitive method.

The recommendation to use easy problems is one more idea which is highlighted both in the teaching program and in Ramon Roca Dordal's textbook. This aspect is clearly revealed in the way that the documents are written and also in how their utilization is suggested. In the teaching program, for example, there is no proposal for the use of problems in the first year/grade, which suggests that first children need to acquire knowledge of the four operations, know their signs and be able to perform calculations with objects or numbers before being able to make use of this knowledge to solve problems. Likewise, in the book by Ramon Roca Dordal, problems are introduced only in some of the lessons, which happens after children have exercised their knowledge of the operations. One needs to have knowledge of how an operation is done to be able to apply that knowledge in problem solving, which points to a suggestion that the purpose of the problems was the application of previously acquired knowledge. This interpretation is reinforced when we observe that all problems featured in the book are related to the topics contained in the lesson in which the problems are shown or in previous lessons.

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Mathematics in the initial pre-service education of primary school teachers in Portugal: Analysis of Gabriel Gonçalves proposal for the concept of unit and its application in solving problems with decimals

Rui Candeias

Agrupamento de Escolas Terras de Larus, UIED/FCT Portugal; rp.candeias@campus.fct.unl.pt

This study aims to analyze Gabriel Gonçalves didactic proposal for the initial approach of rational numbers in primary education, published in the 1974 edition of Didactic of Calculation (Didática do Cálculo). At a time where modern mathematical ideas began to influence mathematics education in the early years, it is important to understand how mathematics was addressed in the pre-service education of teachers of this school level. Among the themes approached in this textbook, the choice of rational numbers was due to the many difficulties that primary students usually have with this subject matter, as well as the teachers in teaching them. The study was conducted with a historical perspective based on documents analysis. The proposal of Gabriel Gonçalves emphasizes the initiation of rational numbers through decimals instead of fractions and the different types of problems that should be presented to students in the initiation to decimals.

Keywords: History of mathematics education, elementary school mathematics, decimals

Introduction

This paper is part of a broader work that aims to characterize, in a historical perspective, mathematics in the initial pre-service education of primary school teachers in Portugal.On the one hand, from the point of view of the centrally issued legislation, on the other hand fromthe analysis of textbooks for teachers, which constitute a point of view closer to the practices. This is a pioneering study that will focus the analysis of the teacher training textbooks on their approach to non-negative rational numbers. It is intended to select a set of representative authors from different periods, between 1844 and 1986, and examine how the proposed initiation to non-negative rational numbers in the early years of schooling evolved. In this article, I selected a representative author of the beginning of the 1970s, whose proposal of approach to rational numbers will be described.

This paper analyzes the didactic proposal of Gabriel Goncalves, former teacher of the Primary Teacher Training School of Porto (Escola do Magistério Primário do Porto), for the initial approach to rational numbers presented in the textbook for teachers of the Special Didactic (Didática Especial) discipline of initial pre-service education of primary school teachers, entitled Didactic of Calculation, 1974. The main questions in this paper are: What is the initial proposal for the teaching of non-negative rational numbers presented by this author? What kind of representations are privileged? What kind of teaching materials are referred to? What importance does this author give to the definition of unity? What are the most used contexts for the display of decimal numbers? What kind of problems does the author propose to students?

According to Chartier (1990), the pedagogy textbooks for teachers constitute a source for the history of teacher professionalization, and are a sample of what constitutes teachers professional knowledge. Pintassilgo (2006) considers these textbooks for teachers a major instrument of innovation and

control, legitimizing certain ideas and practices, and simultaneously withdrawing this legitimacy to others. Moreover they are important resources in building a school culture and as guides in classroom and students management, as well as in the professional teacher development.

The importance of research in the context of the history of mathematics teaching is not limited to the knowledge of the past. Chervel (1990) points out that, through the historical observation, we can bring this disciplinary models and operating rules whose knowledge and exploitation may be useful in discussions about teaching today. In this sense, Matos (2007) states that knowledge of the past may allow an action more grounded in the present. In this perspective, it is important to see how it was done the training of primary school teachers, in a mathematical topic as the rational numbers, at a moment when the Modern Mathematics Movement begins to emerge, favorable to an active construction of knowledge and the use of structured materials in mathematics teaching. The operationalization of the analysis of the work of this author was conducted as an historical study based on the collection and selection of sources as defined by McCullough (2004). At this stage of the work, the document was analyzed essentially in a descriptive way, trying to organize a scheme with the topics to be analyzed (Creswell, 2012).

Pre-service education of primary school teachers during the New State regime (Estado Novo)

In Portugal, the military dictatorship implanted in 1926, and later the New State regime that followed, changed the pre-service education of primary school teachers (Pintassilgo, 2012). In 1930, still in the transition from military dictatorship to the New State regime, the Normal Primary Schools (Escolas Normais Primárias) were replaced by Primary Teachers Training Schools (Escolas do Magistério Primário) involving a radical change in school organization, the curriculum framework and, later on the syllabi, in 1943.

With the restructuring of the course in 1942, and the syllabi published in 1943, the mathematical content of the programs of the Primary Teachers Training Schools came to be centered on teaching and methodological dimensions of the primary content. In these 1943 syllabi, did not exist any discipline with mathematical content. The reformulation of syllabi in 1960 reinforced the discussion on teaching methodologies. These syllabi had no disciplines with mathematical scientific content, situation that remained until 1975 syllabi. It is in the context of these 1960 syllabi that the Didactic of Calculation, analyzed in this paper, was published.

Decimal numbers in the teachers textbook of pre-service education of primary school teachers: Gabriel Gonçalves' approach in *Didática do Cálculo*

Didatics of Calculation (Didática do Cálculo), composed of two volumes published in 1972 and in 1974 by Porto Editora, is part of a set of textbooks for teachers written from the 1960s to serve as a support to the discipline of Special Didactics B of the courses of pre-service education of primary school teachers. According to the author, Gabriel Gonçalves, former professor of the Primary Teachers Training School of Porto and inspector-advisor at the time of the edition of the manual, Didactics of Calculation was mainly intended for students-masters of the Primary Teachers Training Schools although it could also be used by all of those who were interested in education issues.

From chapter VII to chapter XIII of the second volume of Didactic of Calculation, Gonçalves (1974) addresses the teaching of decimals. Due to space limitations, this article will not address the entirety

of this author's proposal. This paper focuses on chapter VII which deals with general aspects of the teaching of decimals and chapter XIII dedicated to the concept of unit and its application to problems, which are important aspects of the teaching of these numbers.

The chapter VII, entitled “Preparation of the study of decimals; measurements with linear units already known. Writing and reading of representative numbers of these measurements; using the decimal point” is organized into three main sections: 1) Goals; 2) General considerations and 3) Direction of Learning. In the goals, the author begins by pointing out that the aim is that the child expands his knowledge of decimal number system, extending it to tenths, hundredths and thousandths. These concepts would appear as an extension of the base ten numbering system.

Gonçalves (1974) presents subsequently some general considerations about the teaching of decimals and fractions, starting by putting the question if the teaching of rational numbers should be done with decimals instead of common fractions. On this issue, Gonçalves (1974) presents two opposing trends. On the one hand, quotes methodologists[1] that, according to Gonçalves (1974), claim that one should start with decimal fractions in its decimal representation, because it would be "as a continuation of the study of decimal number system, but with numbers lower than the unit "(p. 38). He presents the submultiples of length measurements, capacity and weight, as examples stating that:

> Each of these units contains ten units of the next lower order. So, we can operate in the written calculation as if they were natural numbers. And as the calculation with decimal is much easier than with fractions, it will be with the decimal fractions, in the form of decimal representation, which should start. "(Gonçalves, 1974, p. 38).

On the other hand, he presents the opinion of methodologists[2] who claim that we should begin the study by the common fractions, of which the decimal fractions would be only a case. Then he forwards the arguments of these authors, stating that:

> The calculation becomes more intuitive and rational: the half, third, fourth, ..., are easier to understand than the tenth, the hundredth, ... The calculation of common fractions prepare better for the decimal than the contrary. (Gonçalves, 1974, p. 38).

Given these two divergent trends, Gonçalves (1974) refers that he will follow the first, as it was prescribed in primary syllabi at the time[3] that is, starting with decimals representation. In section 3 of this chapter, called the Direction of learning, Gonçalves (1974) shows what stood as the teaching of decimals in the primary education syllabi of the time. This topic was considered "the greatest obstacle to overcome in the 3rd grade" (p. 39). According to the primary syllabi, the approach to decimals should be made from the length measurements, placing students in situations that were necessary to measure with meter and decimeter. These measurements would express numbers in what was called mixed decimals, numbers with a whole part, that after a decimal point had a decimal part. After working with these mixed decimals, students should verify that the rules used with whole numbers also applied to decimal numbers, "the numbers continue to have an absolute value and a position

[1] On this subject, Gonçalves (1974) quotes methodologists like Büttner, Tank or Pikel, but does not identify the works of reference of these authors.
[2] On this subject, Gonçalves (1974) quotes methodologists like Böhme or Hentshel, but does not identify the works of reference of these authors.
[3] At the time were in effect the syllabi approved in Decree No. 23,485, Government Daily, 167, 16.07.1968, 1019-36.

value." (p. 39). After performing this work, situations that could lead them from mixed decimals for simple decimal numbers should be offered to students.

Gonçalves (1974) establishes a relationship between the perspective in the previous two chapters of his manual, which addressed the metric system, with measurements only with positive integers, in the final part of his general considerations. In this chapter, he proposes to address measurements, using the decimal notation, with the decimal point.

Gonçalves (1974) continues the chapter with section 3. Direction of Learning, with the suggestion of some techniques and activities for the introduction of the concept of the tenth, starting from decimeter, and the notion of the hundredth and thousandth, from notions of centimeter and millimeter. The author proposes the introduction of the tenth in eleven steps (table 1):

1) Measurements, in which the meter is used a whole number of times;	2) Measurement expressed a whole number of times in meters and decimeters (ex.: the picture measures 1 m and 2 dm);
3) In the measurements, identification of the entire unit, the meter, and the tenth of the entire unit, the decimeter. Representation in conventional manner, with the decimal point and the identification of the position value.	4) Identification that the rule that governed the whole numbers also apply in decimal numbers: "In a number, any digit at the right of other is order units ten times smaller than the first one" (p. 30);
5) Measurements that result in mixed decimal representation. Registration in tables;	6) Measurements that result just with decimal part. Lead students to understand that the zero to the left of the decimal point is the absence of whole units;
7) Exercise that does not exceed the unit Ex.: 3 dm + 2 dm = 5 dm 0,3 m + 0,2 m = 0,5 m	8) Exercises which form exactly the unit Ex.: 5 dm + 5 dm = 10 dm 0,5 m + 0,5 m = 1,0 m = 1 m
9) Exercises that exceed the unit Ex.: 4 dm + 5 dm + 3 dm = 12 dm 0,4 m + 0,5 m + 0,3 m = 1,2 m = 1 m + 0,2 m	10) Exercises Ex.: 0,4 m = 0,1 m + 0,1 m + 0,1 m + 0,1 m = 0,2 m + 0,2 m = = 0,3 m + 0,1 m
11) Application and verification exercises.	

Table 1. Proposal for an approach to decimal (Gonçalves, 1974)

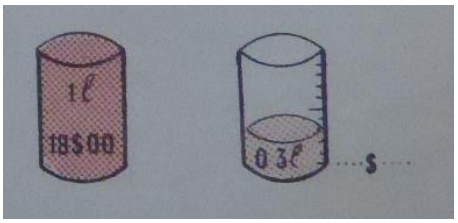
Chapter XIII, titled the "Expansion of the Unit Concept. Its application to solve problems with decimals" is divided into two sections: 1. Goals and general considerations and 2. Preparatory exercises. In a footnote at the beginning of this chapter, Gonçalves (1974) draws attention to the possible application in problems with common fractions[4]. The first section provides some general

[4] The footnote with reference to common fractions is placed in the text by the author, because in later chapters, when it comes to addressing the common fractions, will present the same kind of problems. However, we will not address common fractions in this paper.

considerations about the unit and its nature. Gonçalves (1974) begins by distinguishing the single units of the 1st or 2nd order units, such as ten or hundred, or units designated as decimals, as 0.1; 0.01. Also distinguishes other composite units as the dozen or the quarter of a hundred or other sets as a basket of oranges that can be considered as a whole. For Gonçalves (1974), this expansion of the concept of unit "is the basis of an important branch, allowing you to easily solve questions that otherwise would be too complex" (p. 79) and therefore should be developed in children. Gonçalves (1974) points out that many problems with decimals have the following expressions: "the amount corresponding to the unit; the fraction[5]; the amount corresponding to that fraction or else their counterparts, the value of the unit (the whole); the fraction; the value of the part corresponding to the fraction "(p. 79). He points out that being given two of the above items is always possible to find the third, and stresses that this implies the possibility of formulating three groups of problems: 1) given the amount corresponding to the whole and the fraction, find the value of the part corresponding to that fraction; 2) given the fraction and the amount corresponding to that fraction, find the value of the whole; 3) given the value of the whole and the value of a part of the unit, find the fraction which corresponds to that part.

The author presented several examples considered similar, for the first group of problems. The first problem, with a context of capacity measures, is to find the amount corresponding to the respective unit. For this type of problem is presented a resolution, first find the value of the decimal unit 0.1, and then multiplying by the number of times it is repeated, in this case, multiplying by three.

> 1) Each liter of olive oil costs 18$00. How much will cost 0.3l of that olive oil?
>
> The problem can be solved, first finding the decimal value of each unit (18$00:10 =1$80) and then multiplying it by number of decimal units (1$80x3= 5$40). (Gonçalves, 1974, p. 79)

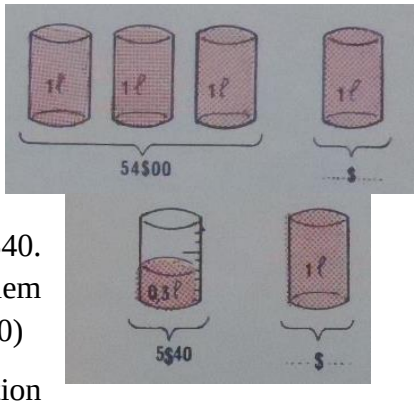


Soon afterwards a second example is shown. It is also an iterative situation that leads to the meaning of multiplication and can be considered a counterpart of the first: "2) Each liter of olive oil costs 18$00. How much will cost 3 l of the same olive oil "(p. 80). The resolution is the multiplication of 18$00 by 3, 18$00x3=54$00. Gonçalves (1974) believes that after children observe the resolution of the second problem they will eventually realize that the action in the first problem is also multiplicative, noting that multiplying by 0.3 is the same as dividing by 10 and multiply the result by 3. He points out that the child will also conclude that" *given the value of the unit, to know the amount* (whether higher or lower than the unit), *the action is multiplicative*."(p. 80, italics in original). For this first group of problems are still presented other examples. According to Gonçalves (1974), the intention is to extend the concept of unit, to the concept of the whole.

[5] Gonçalves (1974) uses the term “fraction”, in the sense of part of a whole and not in the sense of fraction representation of the rational number.

In the second group of problems titled “given the fraction and the amount corresponding to that fraction, find the value of the whole” several problems are presented. The first two are in the context of capacity measures and the author intended that should be solved by analogy.

1) They bought three liters of olive oil for 54$00. How much did cost one liter?



2) Were purchased three deciliters (0.3 l) of olive oil for 5$40. How much did cost one liter? The meaning of the first problem is clearly partitive (54$00:3=18$00). (Gonçalves, 1974, p. 80)

The first problem is associated with a partitive situation and the solution involves a division, 54$00:3=18$00. From this, the student should recognize that the second problem presents a similar situation, inferring that if you know the value of a certain amount, to know the unit, the meaning is to divide. For the second problem, another kind of solution is suggested: first determine the price of each tenth and then multiplying this result by 3. Other examples of similar problems are then presented.

In the third group of problems, “given the value of the whole and the value of a part of the unit, find the fraction which corresponds to that part”, are initially presented two problems.

1) With 54$00, which portion of olive oil can we buy, whose price is 18$00 per liter?

2) With 5$40, which portion of olive oil can we buy, whose price is 18$00 per liter? (Gonçalves, 1974, p. 82)

The first problem is considered to be the division quotative meaning, and the resolution proposed is 54$00:18$00=3. The second problem is also framed in a similar reasoning and therefore should be solved similarly 5$40: 18$00 = 0.3. To Gonçalves (1974), these two problems comprise the quotative meaning. Gonçalves (1974) points out that "*the fraction is given to us by the relationship (or ratio) between the value of the quantity and value of the unit*." (p. 82, italics in original). It points out that learning these problems should not only be supported by the memorization of rules, or repetition, without be a prior understanding. Gonçalves (1974) considers that this understanding of work was previously done when they worked multiplication and division of decimals as a generalization of the basic rules of these operations with whole numbers.

In section two of this chapter, entitled preparatory exercises, are suggested three different types of problems, 1. Recognize (or find) the fraction; 2. Find the value of the fraction; 3. Find the value of the unit (or the whole) which corresponds to the sorts of problems previously shown. Gonçalves (1974) begins by highlighting that for the understanding of the basics for learning problem solving, should be practiced some sensory exercises, called concrete phase, such as manipulation, paper folding, drawing, of which the preparatory exercises were examples.

Concluding remarks

In chapter VII of the manual in analysis, the first dedicated to the teaching of decimals, Gonçalves (1974) starts by discussing where to begin the study of rational numbers, by the decimals or by

fractions. In his work, Gonçalves (1974) follows the indication of the primary school official syllabi of that time and primarily addresses the rational numbers by its decimal representation. Gonçalves (1974) also refers to arguments of different authors. Regarding the teaching of rational numbers, Brousseau, Brousseau and Warfield (2007) also present a discussion on the best way to introduce them to students considering that it is not necessary to know fractions to learn decimals. Rather, decimals can be understood at once as a decimal number, supported by the decimal measuring system, allowing that all practical measurement problems can be solved more easily. They consider that this solution has many advantages for teaching, especially in countries where children are already familiar to the use of metric measurements.

The different syllabi of mathematics discipline for primary education in Portugal also seem to reflect this discussion. In the 1960s two syllabi to this level were in force. In both cases the rational numbers were discussed in the third grade from decimal numerals, with the use of linear measurements. The fractions were worked only in the fourth grade, but only the concept of fraction. In the syllabi for the school year 1974/1975 the introduction of rational numbers was still made with decimal representation and working with fractions was no longer part of the primary syllabi, happening the same in 1975 syllabi. However, in the 1978 syllabi, the chapter devoted to rational numbers deals first with the fractions and then the decimal representation. In the 1980 syllabi, rational numbers were again addressed exclusively by decimal representation. In 1990, the official syllabi for primary education began the work with rational numbers in the second grade, with the fractions, but only had an applied operator to a discrete set. Afterwards and until 5th grade rational numbers were worked out just with decimal representation.

Gonçalves (1974) distinguishes different types of units, referred to as units of "various kinds". He defines the single unit, but ten and hundred are first and second order units, that means they are composed units. Other composite units are also presented, called set-unit as the dozen or a quarter of a hundred. Monteiro and Pinto (2009) highlighted the different types of unit as one of a major difficulty in the study of fractions in the early years.

This unit concept is considered by Gonçalves (1974) as essential for solving problems with decimals, because of that definition results the possibility of grouping the problems into three distinct groups: 1) find the value corresponding to a part of the designated fraction; 2) find the value of the whole giving a part; 3) find the part of a whole.

In the presenting of the problems, Gonçalves (1974) emphasizes the symbolic representation, but also presents some problems, and the respective resolutions, illustrated with pictorial models. However, in the second section of chapter XIII, he highlights the importance of concrete phase, suggesting the use of sensory exercises using the manipulation, paper folding and drawing. However, no structured didactic materials are referred.

Synthesizing, in this proposal of Gonçalves the importance attributed to the work with the decimals stands out by the affinity with the calculation with the natural numbers that the students previously worked. This option is based on the curriculum that was in force in Portugal at the time, but methodologists from other countries who support this option were also mentioned. As the works of the mentioned methodologists are not mentioned, it is not possible at this moment to attest to the influence they had on the proposal of this portuguese author. This is a discussion that continues today

as it is possible to see in the works of Brousseau et al. (2007) and the successive changes in the approach to rational numbers presented in the portuguese curricula.

The work with different units and the importance given to the definition of the unit it is also relevant in Gonçalves work. This definition of unit allows Gonçalves to systematize the type of problems to be presented to the students in the beginning of the learning of this numerical set. The most used contexts in these problems are those of measurement (length, weight).

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Mathematical analysis of informal arguments: A case-study in teacher-training context

Renaud Chorlay

LDAR, Uni. Paris Diderot, France & ESPE de l'Académie de Paris, France

renaud.chorlay@espe-paris.fr

We will endeavour to show that some historical documents which are probably too difficult to be used in the classroom can nevertheless be fruitfully used in teacher-training, in order to provide teachers with tools for the analysis of informal or semi-formal justifications. We will analyze an extract from the Nine Chapters, so as to spell out tool for the analysis of three school documents bearing on the multiplication of decimal or general fractions.

Keywords: History of mathematics, fractions, teacher-training, argumentation, generic examples.

Introduction

We will endeavour to show that some historical documents which are probably too difficult to be used in the classroom can nevertheless be fruitfully used in teacher-training, in order to provide teachers with tools for the analysis of informal or semi-formal justifications. We will analyze an extract from the *Nine Chapters*, so as to spell out tool for the analysis of three school documents bearing on the multiplication of decimal or general fractions.

A historical example: Multiplying fractions in the *Nine Chapters*

The *Nine Chapters on the Mathematical Art* (九章算術) is a Classic that was compiled during the Han Dynasty (206 BC – 220 AD). It contains an organized list of problems with general procedures to solve them, yet with no attempts at justification. In 263 AD, scholar Liu Hui wrote an extensive commentary in which endeavoured to justify all the procedures, and identify key subprocedures of general scope. We will comment on the passage in chapter one which deals with the multiplication of fractions, on the basis of the critical edition and French translation (Chemla & Shuchun, 2004)[1].

> Problem 1.19. Now given a [rectangular] field, 4/7 *bu* in breadth and 3/5 *bu* in length. Tell: what is the area? Answer: 12/35 *bu*.
>
> (…)[2] Procedure for the multiplication of parts: multiply the denominators to make up the divisor; multiply the numerators to make up the dividend. Divide the dividend by the divisor.
>
> (…) [Liu Hui]: In all cases when a dividend does not fill a divisor, then they are called denominator and numerator. If there are parts, expand the corresponding dividend by multiplication, then, when the divisor is filled, the division yields an integer. If, moreover, one multiplies something by the numerator, the denominator must therefore divide in return [*baochu*]. To divide in return is to "divide the dividend by the divisor". Now, "the numerators are multiplied one by the other", hence both

[1] An independent English translation is also available (Crossley, Lun & Shen, 1999). As far as this passage is concerned, the two translations differ significantly; our interpretation is based on Chemla's translation and interpretation.

[2] I edited out two similar examples (with proper fractions), as well as Li Chunfeng's commentary.

denominators have to divide in return. Whence the multiplication of the two denominators, and the division at one go [by their product].

The mere length and width of a rectangular field leave no room for a general explanation. Let someone ask: "20 horses are valued at 12 *jin* of gold. Now 20 horses are sold and the proceeds are shared by 35 persons. How much does each one get?" The answer is 12/35 *jin*. To solve this by the procedure for dividing into parts, take 12 *jin* of gold as dividend, and 35 persons as divisor. Now, change [the problem] to: "5 horses are valued at 3 *jin* of gold; now 4 horses are sold, and [the proceeds] are shared by 7 persons. How much does each one get?" Answer: each one gets 12/35 of *jin*. To solve this, you need to homogenize these quantities (*shu*) of people and of gold; it is then completely similar to the first problem, that of dividing into parts. If so, "multiply the numerators to make up the dividend" does homogenize this quantity of gold; "multiply the numerators to make up the dividend" does homogenize this quantity of people. Equalizing the denominators yields 20, but that plays no part: we just need the homogenized. Moreover, when 5 horses are valued at 3 *jin* of gold, these are the *lü* in whole numbers. If we express them with parts, 1 horse is valued at 3/5 *jin* of gold. Let 7 people sell 4 horses, 1 person does sell 4/7 of a horse. (…) As far as expression is concerned, that's different; yet as far as quantities are concerned, the three procedures boil down to the same.

Figure 1: (Chemla & Shuchun, 2004, 169-171). Free translation: R. Chorlay

Even if the procedure in quite easy to understand – and familiar to the reader – Liu Hui's commentary is probably hard to decipher. Even without a clear understanding of the details, one can probably see at least two things: the goal of Liu is to justify the general procedure of the classic; he gives two different justifications, one which does not rely on a semantic context nor on specific numbers, and one which does.

Let us first focus on the first justification. Liu first distinguishes between two cases: a basic case in which the fractions represent whole numbers; in this case, the procedure for multiplication is already known. Of course, the goal is here to justify the new procedure – in the case where the divisions have non-zero remainders – on the basis of the basic procedure. The word "procedure" is important, since what is to be justified is the validity of an algorithm. More precisely, two sequences of calculations are to be compared:

Sequence 1: divide *a* by *b*; divide *c* by *d*; *multiply* the two quotients.

Sequence 2: multiply *a* by *c*; multiply *b* by *d*; divide the first product by the second product.

If the two sequences relied on operations defined previously and independently, the goal would have been to prove the universal equivalence of these two algorithms (that is, same entries yield the same output). However, if one is not to beg the question, the first sequence actually contains the undefined operation *multiply*, since the final multiplication may involve a multiplication of fractions[3]. So the goal here is rather to justify that sequence 2 (which involves only well defined operations on integers)

[3] When the remainder in the Euclidean division is non-zero, the quotient, the remainder and the dividend were read as a mixed number. For instance, the division of 7 by 3 has quotient 2 and remainder 1, so its result is 2 and 1/3.

should be universally equivalent to sequence 1, thereby implicitly defining "*multiply*" in terms of "multiply."

To perform this, Liu uses terms which are specific to the commentary: whereas the Classic uses "divide" and "multiply", Liu uses "expand by multiplication", and "divide in return" (*baochu*); these terms point to the role of these operations. However, it is important to know that throughout the *Nine Chapters*, "*baochu*" denotes a division whose role is to compensate for a multiplication by the same number, so that these two steps of the procedure do not affect the final output of the sequence of operations. On this basis, one can see that Liu's justification involves a third sequence of operations:

Sequence 3:
1. multiply *a* by *b*, so that the first factor becomes *ab* / b, which is equal to *a*
2. multiply *c* by *d*, so that the second factor becomes *cd* / *d*, which is equal to *c*
3. multiply the two factors, which yields *ac*
4. divide by *bd*, which is the product of the two denominators

This sequence is not to be *actually* performed, it is mentioned for justificatory purposes only. Since step 4 is a *baochu*-division, it compensates steps 1 and 2. The line of argumentation can thus be reconstructed as: sequence 1 (which is yet undefined) should be universally equivalent to sequence 3, and sequence 3 boils down to sequence 2.

This line of argumentation may seem far-fetched. However, it is quite close to the following contemporary proof of the following theorem: the only map *f* from $\mathbb{Q} \times \mathbb{Q}^*$ which is $\mathbb{Z}$-bilinear and which is a prolongation of integer multiplication (i.e. if *m* and *n* are integers, then $f(m,n) = m \times n$), is $f\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a \times c}{b \times d}$ (where *a*, *b*, *c*, *d* denote integers, with $bd \neq 0$). The proof goes as follows:

$$(b \times d) \times f\left(\frac{a}{b}, \frac{c}{d}\right) = f\left(b \times \frac{a}{b}, d \times \frac{c}{d}\right) \text{ by } \mathbb{Z}\text{-bilinearity}$$

$$= f(a,c) \text{ by a Lemma}^4 \text{ which relies only on addition in } \mathbb{Q}$$

$$= (a \times c) \quad \text{(prolongation requirement)}$$

Diving both sides by (*bd*) completes the proof. One could argue that, although Liu Hui states the prolongation requirement quite explicitly, never does he say anything resembling the bilinearity requirement. Even if this requirement is not explicit, it should be stressed that the bilinearity of the operation being defined (i.e. *multiply*) can be rephrased as follows: multiplying either of the two factors by and integer *n* should multiply the product by *n*; which is exactly the property which justifies the role of the *baochu* division.

The second part of Liu Hui's commentary clearly follows another line of argumentation. The first part does not rely on a semantic context of interpretation, nor does it use specific numbers. The second part, however, depends on a semantic context which Liu Hui introduces out of the blue after dismissing the context provided by the Classic (rectangular areas), and uses the numbers from Problem 19.

In the second part, several in-context problems are mentioned, and their relationships discussed.

[4] $b \times \frac{a}{b} = \frac{a}{b} + \cdots + \frac{a}{b}$ (with *b* terms) $= \frac{ba}{b} = a$ (assuming $b \in \mathbb{N}^*$).

Pb. 1: 20 horses are valued at 12 *jin* of gold [for all 20 horses]. Now 20 horses are sold and the proceeds are shared by 35 persons. How much does each one get?

Pb. 2: 5 horses are valued at 3 *jin* of gold [for all 5]. Now 4 horses are sold and [the proceeds] are shared by 7 persons. How much does each one get?

Problems 1 and 2 can be connected by a series of problems. Starting from Problem 2, on can consider:

Pb. 2': 5x4 horses are valued at 3x4 *jin* of gold. Now 4 horses are sold and [the proceeds] are shared by 7 persons. How much does each one get?

Although Pb2 and 2' differ from a semantic viewpoint, they are equivalent in the following sense: the numerical answers to Pb 2 and 2' are equal, since the second parts are the same, and the answers depends only on the ratio between horses and *jin* of gold. The argument would hold for any factor, it so happens that "4" is more relevant than others for what follows[5]. This is quite explicit in the text, since throughout the *Nine Chapters* the technical term *lü* denotes either a ratio, or numbers which are to be considered up to multiplication by a common factor. The relationship between Pb2 and Pb 2' is the same as that between Pb2' and Pb2'':

Pb. 2'': 5x4 horses are valued at 3x4 *jin* of gold. Now 4x5 horses are sold and [the proceeds] are shared by 7x5 persons. How much does each one get?

Now, Pb 2'' is the same as Pb 1, so their numerical answers are the same: 12/35.

But there is another way to solve Pb 2. The numerical answer to Pb 2 is the same as that of

Pb 2''': 1 horse is valued at 3/5 *jin* of gold. Now 4/7 horses are sold and [the proceeds] are shared by 1 person. How much does this person one get?

We want to define multiplication of fractions so that the following property of integer multiplication remains valid: total value = unit value times quantity. As a consequence, the product $\frac{3}{5} \times \frac{4}{7}$ should be the numerical answer to Pb 2''', hence to Pb 2 just as well, hence equal to 12/35.

A more formal summary would go as follows (we use $\otimes$ to denote to multiplication of fractions, to be defined here in terms of $\times$):

$$\frac{3}{5} \otimes \frac{4}{7} = \frac{3 \times 4}{5 \times 4} \otimes \frac{4 \times 5}{7 \times 5} = \frac{3 \times 4}{\cancel{5 \times 4}} \otimes \frac{\cancel{4 \times 5}}{7 \times 5} = \frac{3 \times 4}{7 \times 5}$$

As for the extra-steps mentioned in the first argument, none of these intermediary operations are to be performed when actually multiplying fractions; they are here for justificatory purposes only. This formal summary does not do justice to the nature of the argumentation, since it is the context only which justifies the equalities: dependence on ratios only (first equality); then reduction to a problem involving only two integer data instead of four, and which can be solved by ordinary division. Just as well, the fact that any given fraction can be freely replaced by any equivalent fraction can be justified

[5] We will not comment on the use of technical terms such as "equalizing" and "homogenizing", which Liu Hui introduced in the context of the addition of fractions: $\frac{a}{b} + \frac{c}{d} = \frac{(ad)+(bc) \quad \leftarrow \text{this is what you get by "homogenizing"}}{(bd) \quad \leftarrow \text{this is what you get by "equalizing"}}$.

by the context; for instance, the unit price depends only on the ratio between the number of horses and their total value.

Tools for the analysis of arguments in today's textbooks

Reflecting on the *Nine Chapters* from a teacher-training perspective

The obscurity of the Chinese text probably makes it impossible to use in the classroom. In this paper, we would like to show how the mathematical analysis of this text that we carried out in the first part can help raise the awareness of prospective teachers as to several general features of argumentation in rather informal contexts (this is our Hypothesis #1); and even supply them with tools which are useful in everyday teaching, when they are to decide how to (somehow) justify some mathematical definition or property (this is our Hypothesis #2)[6].

The analysis of the *Nine Chapters* drew our attention to several facts, some of which bear specifically on fraction multiplication, and some of which are of a more general scope:

- There are several ways to justify the rule for the multiplication of fractions.
- The use of letters is not the only way to express a general line of argumentation, as Liu Hui's first argument illustrates.
- The rule for fraction multiplication is not only an equality between *formulae*; it can also be seen as the equivalence of two different algorithms (one with two divisions followed by a multiplication, and one with two multiplications followed by a division). This equivalence of algorithm can be established using general arguments within the algorithmic framework (i.e. arguments *about* algorithms), as opposed to the rewriting of formulae within an algebraic framework.
- Argumentation is not limited to properties or theorems: the choice of a definition (here: for the multiplication fractions) can also be justified. With fraction multiplication we are dealing with a case domain extension: some notion was already defined for a given class of objects; the to-be-defined notion has to apply to a class of objects that encompasses the first class, and to coincide with the former notion on the first class. In our case, a very weak form of justification would be: this definition of fraction multiplication boils down to ordinary multiplication (of integers) should the fractions denote integers (this could be called the *control case*). The *Nine Chapters* suggests a stronger kind of justification: the extended definition of "multiply" should preserve some structural properties of the former definition, properties that we value. For instance: bilinearity; or the validity of the formula "total value = unit value times quantity". On the other hand, we are ready to give up on some properties, such as: the product of two numbers is greater than or equal to either of them. To some extent, when justify the choice of a definition, these required properties play the part that hypotheses play in the proof of a theorem.
- In some cases, from a mathematical viewpoint, these requirements completely characterize the new notion. In these cases, whether or not this uniqueness property should be proved in

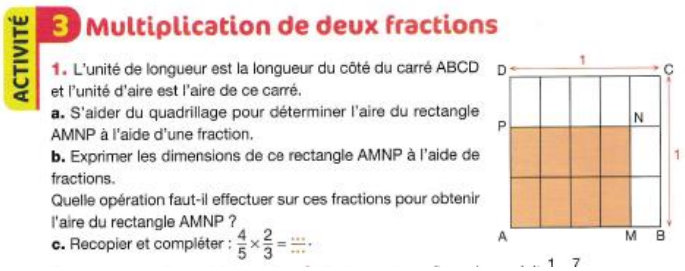
[6] The decisions as to *how* to justify a given definition or property is closely related to the decision as to whether or not it *should* be, and to the knowledge of ways to justify it (*can* it be defined, or not, in a given mathematical context?). We will not have time to discuss the interplay of these three aspects in this short paper.

the classroom calls for another decision. Even if the teacher thinks the uniqueness aspect cannot / need not be made explicit, the *Nine Chapters* suggest another lead. Making explicit the properties that we want to extend also has a *heuristic* value, suggesting the path to *a* definition of the extended notion.

- Liu Hui's second line of argumentation also suggests many fruitful leads, which we can only mention in passing for lack of space:
    - A rather unusual form of argument: proving that two formulae or two algorithms are universally equal / equivalent by exhibiting a class of in-context problems for which the answer is unique – for any given set of initial data –, and for which – for reasons that can be spelled out – both formulae or algorithms work out the solution. Of course, this form of argumentation – that can be called *semantic immersion* – raises questions as to the context-dependence of the argument. For instance, multiplication plays a part in the context of commercial transactions and in the context of area calculation, and nothing proves that the fact that the relevant operations coincide in both contexts when the data are integers will still hold with all rational data.
    - Liu Hui's second argument is not only in-context (a horse-deal), it is also an arguments which clearly claims full generality while dwelling on a single specific case, that of 3/5 times 4/7. This raises the well-kown but deep questions pertaining to the notion of *generic* example (Balachef, 1987, 157). Let us mention two such questions: (1) from a mathematical viewpoint, which conditions guarantee the minimal level of non-specificity that an example should enjoy to be potentially seen as generic (for instance: dealing with 1/5, 7/5, or 15/5 instead of 3/5 would probably make this case less generic)? (2) The *genericity* of a case is not a mathematical property but is a multilayered property of the relationship between it and the students; it involves pragmatic aspects (students should be able to *adapt* the reasoning to other cases), and epistemological aspects (students should *regard* this case as a mere representative of all cases, thus providing some form a general argument); what are the general conditions for a didactical genesis of this relationship?

Field-work

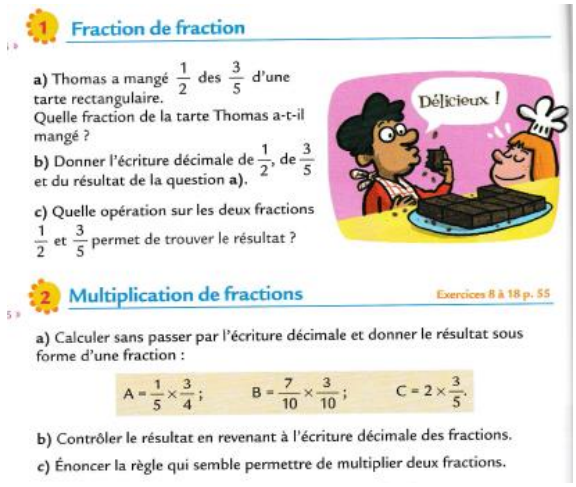
We shall end with some applied work. The following three documents are extracts from contemporary French textbooks or standard teacher-training documents, for students in the second year of middle-school (Figures 2 and 3) or in the last year of primary school (Figure 4). In cases 2 and 3 they are meant for students who are somewhat familiar with fractions and their addition, but with no knowledge of multiplication. As to Figure 4, students are familiar with integer multiplication, and with multiplication and division of decimals by powers of 10. These extracts are what most French textbooks call "introductory activities", in which students are to experience a new notion from a hands-on, guided but non-dogmatic approach. We suggest these documents be analyzed in TWG12 in order to test Hypotheses #1 and #2.



ACTIVITÉ 3 Multiplication de deux fractions

1. L'unité de longueur est la longueur du côté du carré ABCD et l'unité d'aire est l'aire de ce carré.
a. S'aider du quadrillage pour déterminer l'aire du rectangle AMNP à l'aide d'une fraction.
b. Exprimer les dimensions de ce rectangle AMNP à l'aide de fractions.
Quelle opération faut-il effectuer sur ces fractions pour obtenir l'aire du rectangle AMNP ?
c. Recopier et compléter : $\frac{4}{5} \times \frac{2}{3} = \frac{\dots}{\dots}$.

Figure 2: Two ways to work out a rectangular area (Malaval, 2014, 69)



1 Fraction de fraction

a) Thomas a mangé $\frac{1}{2}$ des $\frac{3}{5}$ d'une tarte rectangulaire. Quelle fraction de la tarte Thomas a-t-il mangé ?
b) Donner l'écriture décimale de $\frac{1}{2}$, de $\frac{3}{5}$ et du résultat de la question a).
c) Quelle opération sur les deux fractions $\frac{1}{2}$ et $\frac{3}{5}$ permet de trouver le résultat ?

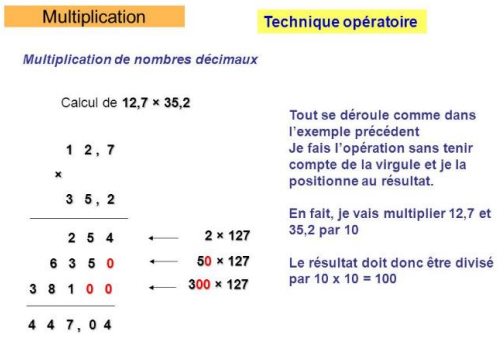
2 Multiplication de fractions Exercices 8 à 18 p. 55

a) Calculer sans passer par l'écriture décimale et donner le résultat sous forme d'une fraction :

$A = \frac{1}{5} \times \frac{3}{4}$; $B = \frac{7}{10} \times \frac{3}{10}$; $C = 2 \times \frac{3}{5}$.

b) Contrôler le résultat en revenant à l'écriture décimale des fractions.
c) Énoncer la règle qui semble permettre de multiplier deux fractions.

Figure 3: A justification in terms of combination of operators, and a conjecture relying on the control-case of decimal fractions (Mante, 2006, 50)



Multiplication — Technique opératoire

Multiplication de nombres décimaux

Calcul de **12,7 × 35,2**

```
  1 2 , 7
×
  3 5 , 2
-----------
    2 5 4     ←  2 × 127
  6 3 5 0     ←  50 × 127
3 8 1 0 0     ←  300 × 127
-----------
4 4 7 , 0 4
```

Tout se déroule comme dans l'exemple précédent
Je fais l'opération sans tenir compte de la virgule et je la positionne au résultat.

En fait, je vais multiplier 12,7 et 35,2 par 10

Le résultat doit donc être divisé par 10 x 10 = 100

Figure 4: Generalizing multiplication from integers to decimals using bilinearity[7]

Conclusion

The goal of this short paper was to contribute to the ongoing reflection on the use of original historical sources in the classroom or in teacher-training; a reflection which bears both on the goals of this use, and on the relevant ways to use such documents.

Within this general field of research, the main features of this specific contribution are: we dealt with teacher-training only, since we do not think this excerpt from the *Nine Chapters* can be used with

[7] Document retrieved from a teacher-training slideshow: http://docslide.fr/documents/techniques-operatoires-cycles-2-et-3-multiplication-jean-luc-despretz-cpc-landivisiau-avril-2010-lacquisition-des-mecanismes-en-mathematiques-est-toujours.html (slide 22). Accessed September 14, 2016.

students who learn to multiply fractions; we endeavoured to show that even for such documents, they can be used in teacher-training for other purposes than simply teaching some history of mathematics, or enriching teacher's image of mathematics. As teachers, it is or will be part of their everyday work to analyze/assess teaching documents and make decisions on the basis of this analysis/assessment. Among these documents, some will be of an argumentative nature: sometimes in the form of pretty formal and academic proofs; more often than not, these documents will explicitly display or rather implicitly point to other forms of argumentation. We think the conceptual analysis – an analysis which has a mathematical component and an epistemological component, and which is to be distinguished from both the historical analysis and the didactical analysis – of some historical documents can give teacher-trainers opportunities to make explicit some tools which can be of constant use in the analysis and assessment of argumentative documents.

In this paper, we dealt with pretty informal arguments and elementary mathematics. However, we do not think this general scheme is specific to such contexts, since this work is a continuation of our CERME8 paper (Chorlay, 2013), in which a similar approach was used in a formal and rather advanced mathematical context (proof of the relationship between the variations of a function and the sign of its derivative).

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Primary source projects in an undergraduate mathematics classroom: A pilot case in a topology course

Kathleen Clark[1], Daniel Otero[2] and Nicholas A. Scoville [3]

[1]Florida State University, USA; kclark@fsu.edu

[2]Xavier University, USA; otero@xavier.edu

[3]Ursinus College, USA; nscoville@ursinus.edu

Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) is a five-year, seven-institution collaborative project funded by the US National Science Foundation to design, develop, test and evaluate curricular materials for teaching standard topics in the undergraduate mathematics curriculum in the United States via the use of primary historical sources. Three short projects designed for use in a topology course are described, together with elements of a pilot study that collected data from students in the course to evaluate changes in attitudes toward mathematics and its study.

Keywords: Instructional materials, primary historical sources, topology, worldviews.

Introduction

There are numerous motivations and benefits for incorporating the use of primary sources into undergraduate mathematics teaching. Primary among the cited motivations is that providing students experience with reading texts in which the genetic development of a topic is presented gives them an opportunity to expand their mathematical education in such a way that includes both traditional and modern methods of the discipline (Fried, 2014; Laubenbacher et al., 1994). Another motivation is that using original sources in the teaching of mathematics makes it possible to contextualize the mathematics in ways that many textbook treatments do not afford. That is, original sources place particular mathematical ideas in the context and setting of the investigations in which the author was engaged at the time. As a result, the problems with which the author was struggling, and the motivations for solving them, are often more clearly and naturally described, and more compelling than traditional textbook expositions. Exposing the original motivations behind the development of esoteric mathematical concepts may be especially critical for placing the subject "within the larger mathematical world," in the hope of making it more accessible to students (Scoville, 2012). Furthermore, primary texts seldom contain the specialized vocabulary that comes with later formalism, promoting access to the ideas by students with a wider range of backgrounds than is achieved with more standard presentations.

Many mathematics instructors interested in bringing the history of mathematics to the classroom question the use of primary historical sources in light of the increased availability of high quality secondary historical sources (e.g., Katz, 2009). Such resources may suffice to help students reap some of the benefits of original works; however, they carry their own difficulties. For example, there is a risk of placing too much emphasis on the history of mathematics per se, as opposed to using that history to support the learning of mathematics. Other key differences between using primary and secondary sources are described by Jankvist (2009):

> When using secondary sources, the students are exposed to a given historian's presentation and, possibly, interpretation of history, and they must make their choices based on this (Furinghetti, 2007, p. 136). When reading original sources, the students must, on the other hand, perform their own interpretation of what actually took place, why a certain mathematician developed a theory in one way or another, whether or not what is written is true, what internal and/or external forces drove the development of the work, etc. … The extent to which original sources are being used does, of course, have an impact on what the students learn: what students may gain from just "sniffing" at a few picks from an original source and what they might learn from being exposed to systematic readings of original sources are immensely different things. (p. 250)

In this final statement, regarding "what students may gain from just 'sniffing' at" selections from an original source, Jankvist points to the primary focus of the *Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources* (TRIUMPHS) project.

The TRIUMPHS project

In 2015, a seven-institution collaborative project to design, test, and evaluate curricular materials for teaching standard topics in the university mathematics curriculum in the United States via the use of primary historical sources was funded by the National Science Foundation. The TRIUMPHS project seeks to help students learn and develop a deeper interest in, and appreciation and understanding of, these mathematical concepts by crafting educational materials in the form of Primary Source Projects (PSPs) based on original historical sources written by mathematicians involved in the discovery and development of the topics being studied. These PSPs contain excerpts from one or several historical sources, a discussion of the mathematical significance of each selection, and student exercises designed to illuminate the mathematical concepts that form the focus of the sources. PSPs are meant to guide students in their explorations of these original texts in order to promote their own understanding of those ideas. TRIUMPHS plans to work with mathematics faculty and graduate students from over forty institutions in the United States who will participate in the development and testing process of these PSPs. As part of this five-year project, impacts of the materials and approaches to implementing them will be investigated in terms of teaching, student learning, and departmental and institutional change.

Organization of the paper

The remainder of the paper is organized into four sections. First, we describe the three "mini-PSPs" implemented by the third author during his topology course in spring of 2016. Next, we present a broad overview of one of the components of research we are conducting during the five-year project, and then discuss a small subset of the data collected from five student participants. The paper concludes with a discussion of the implications from this small data set from the pilot year of the project, and a review of next directions planned for the research.

Primary source projects in topology: The case of three mini-projects

The third author developed and implemented three "mini-PSPs" during the Spring 2016 semester at Ursinus College, a small liberal arts school outside Philadelphia, Pennsylvania. His experience using primary source materials in the classroom began in 2012 when he introduced them into a discrete mathematics course. His current interest in using primary sources in the classroom involves teaching topology. Three "mini-PSPs," each taking up two 50-minute class periods, were written for use in an

Introduction to Point-Set Topology course, an upper-division elective course for mathematics majors and a designated "capstone" course intended to provide an experience for mathematics students to test and apply previously acquired mathematical knowledge and skills. The author taught this course twice before in a standard lecture-based style. A short description of each of the three projects is provided below.

The course was introduced with a mini-PSP titled "Topology from Analysis" that investigates a paper by Georg Cantor (1872) in which he considered a problem in Fourier series, namely, if a function has a Fourier series expansion, when is such an expansion uniquely determined? Cantor proved in a previous paper that two convergent trigonometric series with equal sums have the same coefficients (uniqueness theorem), even if – for a finite number of values of the variable – the series either fail to converge, or converge toward different sums. Could this theorem be extended to certain infinite sets of points? After reading Cantor's statement of the problem, the students explore simple examples of infinite sets where such an extension is possible. They investigate what properties these examples have that allow for such an extension. The desirable properties ultimately prompted Cantor to define concepts like limit points, derived sets, point sets, and iterated derived sets. The students were then able to use these concepts to prove a more general theorem. Even though these concepts were used to prove a result in analysis concerning Fourier series, they are naturally topological concepts. Hence the project helped to connect analysis and topology, thereby motivating new definitions through the need to clarify concepts, rather than introducing them as standard jargon.

The second mini-PSP focused on the topological concept of connectedness. It again considers a work of Cantor (1883) and his study of the continuum. Students follow Cantor's musings concerning the best way to define such a concept. After he defined a perfect point set based on derived sets, he investigated whether the property of perfectness is sufficient to characterize a continuum. The students wrestle with this question, and eventually observe that such a definition will not suffice. Cantor then proposed an additional property, connectedness, which he defined using a metric. After reading Cantor's definition of connectedness, students examine a work of Jordan (1883, pp. 24-27) which viewed connectedness in terms of separation of point sets into components. This introduces the students to a new conceptual perspective for the same notion considered by Cantor. Next, students read from a paper by Schoenflies (1904, pp. 209) in which it was proven that connectedness is a topological invariant. To do this, Schoenflies required a definition of connectedness that does not appeal to a metric, and is therefore purely topological. Finally, students read from a work of Lennes (1911, pp. 287, 303) in which he attempted to give a proof of the Jordan Curve Theorem and gave yet another definition of connectedness. Students are then asked to show that Lennes' definition is equivalent to the definition used today.

The final mini-PSP used in the course studies excerpts from a paper of John Henry Smith (1874) on discontinuous but integrable functions. Smith intended to provide a counterexample to a "theorem" of Henkel. He constructed an integrable function that is "very badly" discontinuous. Students are led through Smith's work involving the definition of the concept of nowhere dense set in order to construct such functions. As in the first mini-PSP, students examine a problem that motivates the need for a new mathematical concept by abstracting the essence of particular examples in order to capture the essential properties that a set must satisfy to prove a result. Ultimately, this project

culminates in Smith's construction of a generalized Cantor set. A function that is continuous except on a generalized Cantor set is then seen to provide a counterexample to Henkel's claim.

A major benefit of these mini-PSPs is that they naturally induce sophisticated discussions about the mathematics by students. During classroom implementation, students were observed to be carefully and thoughtfully working to understand concepts, answer questions, and pose their own questions and conjectures. One such instance occurred when students began to question Schoenflies' definition of connectedness, without prompting from the instructor. The question was raised as to what Schoenflies meant by the phrase "... can be decomposed ..." One student suggested that he meant a partition, but it was soon realized that such an interpretation would be too general. Others then began to modify the partition idea to make it work. This sort of high-level engagement had not occurred in any previous topology course taught by the instructor.

Description of the research

The TRIUMPHS project includes an evaluation-with-research (EwR) study, designed to provide both formative and summative evaluation of the key project activities and defined goals for each. In the original EwR study design, we designated three project components for which we would conduct extensive research and evaluation, and designated these as "student change," "faculty expertise," and "development cycle." Here we only describe a small piece of the "student change" component of the study.

Rationale and research questions

For decades much of the research literature on the impact of the history of mathematics on students, particularly at the secondary level or post-secondary level, was focused on students' attitudes (e.g., Marshall, 2000; McBride & Rollins, 1977). There was scant focus on the use of primary sources as a classroom tool in the early work in the field of history in mathematics education. However, more recent work on the use of primary sources has been done in countries such as Denmark (e.g., Kjeldsen & Blomhøj, 2012) and Brazil (e.g., Bernardes & Roque, 2016), while such research has not yet been conducted with student populations in the United States. Thus, we are committed to investigating the ways in which mathematics students respond to concepts within the undergraduate curriculum that are taught via primary sources. To this end, we developed several research questions, of which we provide a subset here:

1. As a result of students' work with or study of a PSP, what changes do students report in their attitudes and beliefs about learning mathematics?

2. As a result of engaging with PSPs, what do students report as challenges and benefits of learning from primary sources?

3. What is the dominant mathematical worldview reported by students on pre- and post-course surveys? And, does academic major (or gender, or race or other attribute) make a difference in the reported worldviews?

Pilot study: The case of a topology course

This work took place in the first year (pilot year) of the project, which was focused on developing instruments and refining the questions that were formulated in the original grant proposal. The only

data sources available for analysis were student pre- and post-course surveys, student work samples (from the three mini-PSPs), and instructor surveys and post-implementation reports. Data were collected from four undergraduate mathematics courses during the pilot year of the TRIUMPHS project: two courses in Fall 2015 (geometry; analysis) and two in Spring 2016 (abstract algebra; topology). In the topology course, the three mini-PSPs described above were implemented and tested for the first time by the third author. In this paper we report data that inform the first, second, and fourth research questions listed above for the five of eight students enrolled in that course who consented to participate in the research and for whom we obtained a complete set of data.

Exploring student responses: Research questions 1 and 2

Our initial pre- and post-course surveys (to which students responded before instruction on any PSP occurred and again at the end of the course) only contained one pair of open-ended questions asking students to identify what they enjoyed most and least about studying mathematics:

> What do you enjoy most about studying mathematics. (Explain briefly.)
>
> What do you enjoy least about studying mathematics. (Explain briefly.)

For this small group of students, pre- and post-course responses were mostly stable for this pair of questions. This could be a function of the fact that all five students were in the final two years of their undergraduate program. However, we discuss two interesting pre-/post-course survey pairs of responses below.

First, in response to the second prompt, Student 2 stated on the pre-course survey that she was "not fond of professors/texts that state a definition/theorem/idea without giving any hint to how that conclusion was derived, either historically or through proof/explanation." By the end of the course, however, her response no longer referred to lacking historical grounding or a thorough proof or explanation. Instead, her response focused on the fact that "it is very easy to go through math classes and fall behind. If you don't understand one topic, the class often moves on without you…"

A similar change in identifying what he enjoyed least about studying mathematics occurred with Student 3. On the pre-course survey, Student 3 stated that he least enjoyed "how everything is given to you but never comes with an explanation of where the math is coming from." However, in response to the same item on the post-course survey, Student 3 only commented that he disliked having to remember definitions and equations. It is possible that for Students 2 and 3 that the historical sources related directly to the formal content of the course satisfied their initial "sore spot" with regard to what they previously enjoyed least about studying mathematics. Since there was no effort in the pilot year to ask for student clarification (e.g., via follow up interviews), we cannot link the change in the sample responses presented here as resulting from students' engagement with the PSPs. However, the responses signal potential interesting outcomes and we have modified our pre- and post-course surveys and have added post-PSP surveys to further investigate this potential influence.

We also asked students to describe their experience using the mini-PSPs in the topology course, as a way to explore the benefits and obstacles they identified. The student responses were encouraging, and in important ways their responses point to the underlying effect of engaging with materials that provide an opportunity to understand the evolution of a mathematical concept. Here, we provide a sample of three of the students' descriptions:

Student 1 (Mathematics major): "We used these sources to learn about topics such as connectedness as we thought about origins of the idea and read about how the definitions changed the longer it was studied."

Student 2 (Computer Science major): "Each PSP was an interesting addition to class, and it was unique to be able to see the process of mathematical ideals through the eyes of the various mathematicians."

Student 5 (Physics & Mathematics major): "I enjoyed the document on definitions of connectedness, because it showed how mathematics is really done and how it takes time to accurately articulate an idea. I also liked the first document because it helped motivate topological ideas and illustrate the connection between topology and analysis."

Exploring student responses: Research question 3

To address the third research question, we included a subset of 20 items from Törner (1998) on the pre- and post-course surveys. Törner and his colleagues surveyed 310 German secondary mathematics teachers in order to identify attitudes about mathematics that captured "the essence of mathematics," where they defined "mathematics as a field and not as a subject taught in school" (p. 119). In doing so, they identified attitudes relating to four aspects of a mathematical worldview: scheme, formalism, process, and application. Tables 1 and 2 report the pre- and post-survey means of students' responses for these items. Item responses ranged from 1 to 5, where higher values indicate stronger association with that particular worldview. The predominant worldview (in bold in tables 1 and 2) of the Spring 2016 topology students on both the pre- and post-course survey was the process view.[1]

Mathematical worldview	Student 1 (female; 4th year[2])	Student 2 (female; 4th year)	Student 3 (male; 3rd year)	Student 4 (male; 3rd year)	Student 5 (male; 4th year)
Schema	3.4	3.2	3.4	**4.2**	1.6
Formal	3.8	3.4	3.8	4	4.4
Process	**4**	**4.6**	**5**	3.6	**4.8**
Application	3.6	3.4	3.6	3.4	3.4

Table 1: Mathematical worldview of Topology students, Spring 2016 sample (pre-survey)

Mathematical worldview	Student 1	Student 2	Student 3	Student 4	Student 5
Schema	3.4	2.6	4.2	3.6	1.4
Formal	**4.8**	3.4	4	3.8	4
Process	4.2	**4.4**	**5**	**4**	**4.4**
Application	3.8	3.4	4.4	3.8	3.4

Table 2: Mathematical worldview of Topology students, Spring 2016 sample (post-survey)

[1] Törner (1998) described each of the four aspects. For example, the process aspect characterizes mathematics "as a process and as an activity in thinking about problems and gaining knowledge" (p. 122); whereas the schema aspect represents a view of mathematics as "a "tool-box and bundle of formulas" and an idea oriented with algorithm and schemes" (p. 123).

[2] In the college and university system in the United States, typical undergraduate degrees are four years in duration.

Törner (1998) reported on the relations among these four aspects, stating, "the formalism and schem[a] scale represent both aspects of the static view of mathematics as a system and intercorrelate highly" (p. 125). However, these two aspects of a static paradigm "correlate with the process scale in a significantly negative way" (p. 125), which is a finding that appears to hold for several students in our sample. For example, where there are lower means on the scheme and formal aspects (e.g., Student 2 pre/post; Student 3 post; Student 4 post), higher mean values occurred on the process aspect.

Discussion

This paper highlights the promise of robust investigations and implications that may result from the EwR efforts of the TRIUMPHS project. The project's pilot year enabled us to trial student and instructor survey instruments and data collection procedures. We chose the topology course as a case because of the particular nature of the PSPs ("mini" as opposed to full-length), the students' mathematical maturity (juniors and seniors), and the expectation that many of the courses participating in TRIUMPHS will also likely have small enrollments. As a result of the pilot year, we have significantly modified our survey instruments and research questions, and our aggregate student population has now increased. As we move forward, our research plans include conducting multiple analyses to identify trends in students' mathematical worldviews (within courses, across courses, and disaggregated by other demographic characteristics). We have also developed and incorporated post-PSP surveys and are currently developing and piloting multiple interview protocols.

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A course in the philosophy of mathematics for future high school mathematics teachers

Elizabeth de Freitas

Manchester Metropolitan University, Manchester, UK, l.de-freitas@mmu.ac.uk

In this paper, I discuss a course in the philosophy of mathematics designed to help future high school mathematics teachers develop an understanding of philosophical questions about mathematics. Throughout the course, our discussions link core philosophical questions to particular theories of mathematics teaching and learning. Thus the course mixes traditional philosophy of mathematics with the study of everyday embodied mathematical habits, offering a kind of descriptive and synthetic approach more often associated with continental traditions of philosophy. This paper describes some of the key theoretical themes and issues, and argues that such courses offer future teachers important insights into the nature of mathematics.

Keywords: Philosophy of mathematics, ontology, epistemology, teacher education.

Introduction

Philosophical questions about mathematics open up discussions about why we have the mathematics we have, inviting consideration of how mathematics is embodied in particular material practices. This paper discusses a course designed for pre-service mathematics teachers with the aim of diversifying their image of mathematics, and enhancing their skills at philosophically analyzing mathematical behavior. Throughout the course, our discussions link core philosophical questions to historical developments in mathematics (and philosophy) and to particular pedagogical approaches to mathematics education. Thus the course mixes historical perspectives with traditional philosophy of mathematics and with the study of everyday embodied mathematical habits. The focus on embodied material practices in mathematics – past and present – is more often associated with phenomenology and other continental traditions of philosophy, rather than the analytic tradition one typically finds in the philosophy of mathematics. Such a mixture of analytic and continental philosophy is extremely challenging, in part because analytic approaches have been typically concerned with foundational questions that do not always translate into studies of everyday mathematical practice. The course is modeled on what Corfield (2003) calls *descriptive epistemology* insofar as it entails interpreting mathematical activity for how it reflects certain philosophical assumptions about the nature of mathematics, following earlier efforts by authors such as Lakatos and Kitcher to remake the philosophy of mathematics into the study of mathematical practice, while incorporating and expanding approaches to the foundational concerns.

Students enroll in the course often not knowing anything about philosophy, let alone the philosophy of mathematics. They are usually in their fifth year of a combined bachelor degree in mathematics and master's degree in education, and are just beginning to visit high school mathematics classrooms and practice-teach. I open the course with a set of statements that Brown (2008) calls the common "image" of mathematics. Students are then exposed to the following set of core questions. These core questions drive many of the discussions and course assignments.

1. Can a diagram function as a mathematical proof?

2. What is the nature of proof? How has mathematical proof changed over history?
3. Is there such a thing as mathematical intuition? Where is it? Is it innate?
4. Is mathematics indispensable to science? Could science work without math?
5. What should be the relationship between logic and mathematics?
6. What is the status of axioms? Are they grounded in reality?
7. Are mathematical propositions *necessarily* true (or false) (rather than culturally or contingently true/false)?
8. Is mathematics a language (a system of symbolic signs that are immaterial and not part of the physical world)?
9. Can we speak about actual infinity (or just potential infinity)? How has the concept of infinity influenced the development of mathematics?
10. What is the role of the body in doing mathematics? How is mathematical knowledge embodied?
11. Is mathematics discovered or invented? What are the ontological implications of your answer?
12. Is mathematics objective and certain (rather than subjective and open to revision)?

It's absolutely essential that before tackling these questions, we unpack the difference between epistemological and ontological concerns (knowing versus being). Of course the two concerns are always entangled, but students need to identify the distinctive contribution of each. The need to keep these two concerns separate while understanding their relationship helps considerably as they go on to formulate arguments to support their positions on the core questions. Perhaps because these are education students, they seem more at home with epistemological questions (How do we come to know the concept of number?) and are initially baffled by ontological questions (What is number?). I have learned to motivate the latter by suggesting that such questions will be of huge interest to their future students. I suggest they treat the typical student query "how is this relevant?" as a philosophical question, and that they consider these queries akin to the very questions posed by philosophers of mathematics. Indeed, these bored students are asking, in their own way, the philosophical question: "Why mathematics?". Rather than offer the usual unconvincing answer, such as "because you can use it to do …." teachers might explore with their students the various schools of thought that developed as a means of answering this very question. They could, I suggest, introduce one of the core questions into their lessons, as a motivator for all those astute students who have raised this central concern about the relevance of mathematics. In other words, rather than dwelling on the pragmatic nature of mathematics, I suggest we direct attention to the more *speculative* aspect of mathematical activity.

The course uses historical case studies to help students diversify their image of mathematics. Pairs of students research and present a 10-minute slideshow on a discussion topic each week, focusing on what are deemed controversies. Sample topics are: Zeno's paradoxes and the parallel postulate. As the instructor, I introduce links between the topics and the core philosophical questions that structure the course. For instance, a presentation about the parallel postulate might simply recount attempts to prove it and mention developments of non-Euclidean geometry without linking these developments to our readings about Kant and his claims about the *a priori* synthetic nature of geometry. Further links need to be raised that help the students grasp how this topic is related to questions about the certainty and objectivity of mathematics, and its relation to science. With little to no training in history or philosophy, these students tend to present their topics without consideration of social context or

cultural ramifications. These brief presentations feed into their later more substantial assignment to compose a philosophical paper, arguing in support of a position on one of the key questions listed above. It has always intrigued me that these students, despite being immersed in mathematics, a field known for its careful deductive methods, struggle so much in composing a formal philosophical argument. Many of these students confess to having selected mathematics because they don't enjoy reading and writing. However, I feel strongly that, as future teachers, they need to become excellent communicators, and I treat the course as an opportunity to build that skill as well. I have designed guidelines to help them structure their assignments, and I work with various draft versions of their papers to help them improve this skill.

General philosophical themes

The distinction between ontology and epistemology helps us narrow in on students' assumptions about mathematical activity, as we discuss how Platonism and other schools of thought consist of an ontological claim *and* an epistemological claim. In this we follow Bostock (2009) who effectively differentiates these kinds of claims for different schools of thought. The ontological questions are the most difficult for the students to comprehend. We ask: In what sense can universals (redness or beauty or triangles or numbers) be said to "exist"? This, as Bostock reminds us, is a question about the ontological status of universals. Most students don't quite know how to engage with this question, although they are more than ready to grant universality (generality) to geometric figures or arithmetic entities like numbers. They tend to think of this generality as cross-cultural, and confuse it with the question of truth value. My task is to tease out questions about truth from metaphysical questions about being. I offer them some choices: If universals do exist, do they exist outside the mind, or simply as mental entities? If they exist outside the mind, are they corporeal or incorporeal? If they exist outside the mind, do they exist in the things that are perceptible by the senses or are they separate (or independent) from them? To further support and scaffold their exploration of these questions, I offer three schools of thought, each with a different answer to these questions, and I ask the students to decide who they most identify with. I am really forcing their hand in this, in that I hope to show them that these three responses do not actually exhaust the possible answers to the ontological question. In the next section, I discuss how new directions in the philosophy of mathematics offer different choices. But the choices first given, drawn from those used by Bostock (2009), are simplifications so that they can begin to engage in debate. As in all such sorting and labeling, we can query whether a particular mathematician or philosopher is a good example of a particular philosophical paradigm (for instance Hilbert is egregiously characterized as a nominalist in this list), and I am careful to tell the students that they will debate these issues later, after reading more primary texts:

•The *realist* (Plato, Frege, Godel) claims that universals exist outside the mind and are independent of all human thought.

•The *conceptualist* (Descartes, Kant) claims that they exist in the mind and that they are created by the mind. Some claim that we create these universals based on sense perception and some say they are innate and do not require perceptual stimulation.

•The *nominalist* (Hilbert, Field) claims that they do not exist outside of language. Some claim that the words and symbols we use are mere shorthand for longer ways of expressing the same idea and some claim that statements with such terms are simply untrue in the sense that they refer to nothing.

The assignment of the names to the schools is imperfect, but it works as a starting point. One of the difficulties in starting with the main schools of thought, and then trying to tease out the subtle differences and ways in which these philosophers' claims are not perfectly aligned with the school, is that the students are not yet ready to delve deeply into these historical subtleties. For instance, it might seem a travesty to put Descartes and Kant together, since Kant pushed past Descartes' claim that mathematical truths are innate, clear, and distinct ideas, so that he might attend to the synthetic nature of mathematical judgment. According to Kant, space and time are the mind's contribution to experience. Space and time are the "form" of experience, a form imposed by us on the raw data of experience. Historians of philosophy usually oppose Descartes (the rationalist) against Locke and Hume (empiricists), and posit Kant as the reconciler. Bostock (2009), however, claims that Locke, Hume and Descartes, despite their differences, share the same beliefs about the ontological status of mathematical objects (they are ideas or mental entities), and differ in how they think we acquire these mathematical ideas. One might then associate Kant with this approach as well, since, as Brown (2008) suggests, according to Kant, "Our *a priori* knowledge of geometric truths stems from the fact that space is our own creation."(p.119) Similarly, arithmetic for Kant is connected to time and the fact that time is also a form we impose on the world. This conceptualist approach seems to have saturated many of the later treatments of the philosophy of mathematics, seeping into the realist and nominalist camps as well. Brown indicates that Frege (a Platonist) embraced Kant's view on geometry, Hilbert (the formalist or nominalist) embraced Kant's view on arithmetic, and even Russell (the logicist) can be characterized as Kantian. One might also argue that the conceptualist approach has saturated theories of learning, and has become full-fledged in cognitive psychology and its dominant image of learning. This image assumes that learning entails an acquiring of a set of cognitive 'schemas' and assumes that brains are the seat of reason. Pre-service teachers need to be aware of this history so that they might become empowered to identify and critique the theories of learning that structure the curriculum policy they are meant to adopt in their classrooms.

Diagrams and the body

Questions about the status of diagrams in proofs are easy for students to connect with, and link directly to the opening readings by and about Plato. Students are drawn to the compelling distinction that Plato draws between the physical world and the realm of mathematics. We discuss the theory of ideal forms, and how Plato was motivated by the gap between the ideas we can conceive and the physical world around us. Some students see in the proposal of an ideal realm a way of reconciling their belief in the universality of mathematics with the messiness of learning, but more often than not they are drawn to a conceptualist approach, perhaps Kantian, whereby mathematics is considered a cognitive invention that aligns with the physical universe. Thus they tend to ascribe to the human mind a consciousness or intuition or faculty that is capable of bringing together the ideal forms (triangles, numbers, etc) that are unchanging and eternal (the realm of being or essence) with the physical realm (the realm of becoming or change). We discuss how there is a strong dualism (between mind and body) at work in this approach, and how this dualism plays out in different pedagogies. The vast majority of pre-service teachers split mind from body, arguing that we grasp the ideal forms only

through mental reflection, while we understand the physical world through the senses, just as Plato might say. Most of the contemporary philosophy of mathematics we read in the course questions the validity of this dualism, and we discuss the main criticisms of Platonism that were formulated centuries ago.

Diagrams figure prominently in this discussion, as they have, since Plato, if not before, bridged the dualism in ways that trouble its claim to a clean distinction (de Freitas, 2012). In small groups, the students are given a set of visual proofs, and asked "What does this diagram prove?" I use this question to provoke debate, as it gets to the heart of concerns about what constitutes a legitimate proof in mathematics. We discuss to what extent the diagram might function as a proof of an arithmetic statement.

We situate the question historically, by discussing readings by Plato (Meno, Theatetus, Rupublic). Although the students tend to find Socrates overbearing in the Meno, they begin to grasp how the Socratic method emerges from a particular set of philosophical assumptions about the nature of mathematical diagrams and concepts. We compare this method to the kind of questioning sequences they see in their observations in classrooms. For Plato, geometrical knowledge is obtained by pure thought and divorced from sensory observation, which seems to go against what many of the students experience in mathematics classrooms. This is when they become somewhat unhappy with their Platonism. As Brown (2008) explains, Plato considered the diagram as merely a heuristic to help us "access" the pure forms of mathematics. Plato is critical of the geometers who work with diagrams and are led astray by the visual *images* of mathematical ideals. Plato is rather disparaging of all this talk of diagrams and gestures and activity, chiding the geometers for using material verbs to talk about mathematical actions:

> Don't you also know that they use visible forms besides and make their arguments about them, not thinking about them but about those others that they are like? They make the arguments for the sake of the square itself and the diagonal itself, not for the sake of the diagonal they draw, and likewise with the rest. These things themselves that they mold and draw, of which there are shadows and images in water, they now use as images, seeking to see those things themselves, that one can see in no other way that with thought. (Plato, Book VI, 510d, p. 191)

Here, true apperception is achieved only through rational discernment ("thought"), rather than empirical investigation or what Kant will call synthetic reason. For Plato, geometers use diagrams and visual forms to speak about *ideal* forms, "seeking to see those things themselves" when only "thought" in its pure disembodied capacity can access such ideal forms. In the course, we discuss the consequences of Platonist and conceptualist approaches that deny or demote the significance of the material activity of doing mathematics and prize instead only the mental or cognitive reasoning faculty. We begin to read contemporary theories of embodied cognition that attack this approach philosophically (Lakoff, G. and Núñez, R., 2000; Nemirovsky et al, 2009; Roth 2010). The students begin to grasp how diagramming (and other embodied activities) are not merely heuristic but rather necessary for thinking mathematically. We discuss what it might mean for thinking to occur in and through this activity rather than independent of it.

The readings in this section of the course range across phenomenology, focusing on the role of the body in learning mathematics. We begin to consider how recent work in embodied mathematics might

engender a new philosophy of mathematics. Although the work of Lakoff and Núñez is still 'conceptualist' in how it treats the body as the container of the mind (a dualism inherent to their approach), there are other scholars who attempt to move even further into a monist philosophy of mathematics (Nemirovsky & Ferrara, 2009; Roth, 2010; Stevens, 2012). For instance, de Freitas and Sinclair (2014) push past the phenomenology framework, seeking a more post-humanist approach that attends more generally to the diverse material forces at work, and less exclusively on the human body's individuated capacities.

The ontological turn

The pre-service teachers are shown how much of the philosophy of mathematics since the nineteenth century has been contending with the Kantian assertion that mathematical truths are *a priori* and *synthetic*. Kant claimed that if a proposition is thought as (1) necessary and (2) universal then it is an *a priori* truth; and if a judgment of truth requires one to engage with the phenomenal world in some fashion, it is a *synthetic* judgment rather than an analytic one. Mathematical truth, according to Kant, is both synthetic and a priori. How can such knowledge be possible? This is a perennial question in the philosophy of mathematics, the question as to how *pure* mathematics is possible as an activity in this messy world (Hacking, 2013). In other words, how can we grasp universal and necessary truths by using our material bodies to determine whether they are true? Hacking claims that one has to look closely at applications of mathematics if one is to address – or contest – this question of purity.

Corfield (2003) argues that the philosophy of mathematics has spent far too much time on the foundational ideas of the 1880-1930 period, and neglected the thinking and doing of "real" mathematicians both before and after that period. Corfield believes that a philosophy of mathematics should "concern itself with what leading mathematicians of their day have achieved, how their styles of reasoning evolve, how they justify the course along which they steer their programmes, what constitute obstacles to these programmes, how they come to view a domain as worthy of study and how their ideas shape and are shaped by the concerns of physicists and other scientists."(p.10) He names this approach *descriptive epistemology* and defines it as the "philosophical analysis of the workings of a knowledge-acquiring practice." (p. 233) Imre Lakatos (1976) is often taken as inspiration in this kind of approach to the philosophy of mathematics. He examined the process of meaning-making in mathematics, by studying the historical evolution of concepts and procedures, and offering insight into the form of deliberation that characterized creativity in the work of mathematicians. He was interested less in the so-called foundational issues in mathematics, and more in the empirical and material making of mathematics, an approach he called "critical fallibilism":

> It will take more than the paradoxes and Gödel's results to prompt philosophers to take the empirical aspects of mathematics seriously, and to elaborate a philosophy of critical fallibilism, which takes inspiration not from the so-called foundations but from the growth of mathematical knowledge. (Lakatos, 1978, p. 42)

Hersh (1997) characterizes Lakatos as a philosopher of mathematics who was committed to studying the social and "humanist" aspects of doing mathematics. For Hersh, Lakatos was a humanist because he celebrated the specificity of informal reasoning found in the work of mathematicians, rather than or in addition to the generality of its truth claims. For Lakatos, these examples of informal reasoning are not simply unfinished formal proofs, in which the pertinent axioms and logical rules of inference

are suppressed, but rather a significantly different mode of inquiry, a non-axiomatic argument that has its own trajectory and its own becoming.

Despite the significance of this more humanist perspective on the philosophy of mathematics, which values the study of informal and unfinished mathematical activity by experts, we still lack philosophical insight into the experiences of those who, for the most part, do mathematics from an outsider or fringe position, like most students. Although recent moves in the philosophy of mathematics – like Corfield – have insisted that we look more closely at the *practice* of contemporary mathematics to build a philosophy of mathematics, these scholars are still entirely focused on extremely accomplished mathematicians, and remain focused on the epistemological dimension of that activity. In other words, *they are still concerned with how mathematicians determine the truth of their mathematical propositions*. But this is not the only issue! Despite the more expansive study of mathematical activity, Corfield's approach of "descriptive epistemology" is, as the name suggests, directed at mathematician's epistemology or theory of knowledge production.

A focus on *knowledge production* confines one to attend to particular aspects of activity – indeed, this focus explains why scholars like Corfield look exclusively at accomplished mathematics. In order to grapple philosophically with mathematical activity more broadly - be it expert or novice, animal or human, revolutionary or controlling, conceptual or algorithmic – one needs to consider mathematics not merely as a knowledge production activity. Contemporary philosophers like Zalamea (2014) and Châtelet (2000) and Deleuze (1994) lend support in this venture, as they grant mathematics more ontological import, although continuing the focus on high-stakes achievement. These scholars track how mathematics operates in the world as both an expression of human cultures (perhaps as knowledge), but also as a kind of *worlding* in itself. In other words, mathematics is an activity both pragmatic and speculative that makes and mutates possible worlds. As part of what many have called the "ontological turn" in the humanities, this speculative work ("worlding") informs a contemporary shift in the philosophy of mathematics, towards an emphasis on "mathematics as ontology", the latter refrain capturing Alain Badiou's attempt to position mathematics *within* philosophy, but not merely as logic in drag.

The ontological turn and related developments in philosophy are reshaping the way we think about all material-cultural practices, let alone mathematics. The course aims to move students through conventional idealism (best formulated in Plato) through conceptualism (best formulated in Kant) through phenomenology (best formulated in Merleau-Ponty) to a more post-humanist perspective that dethrones the human subject as the central orchestrator of his/her mathematical participation. It is near impossible to move students through these radical shifts in one course, but one can begin to problematize the landscape and trouble assumptions about mathematics.

Conclusion

This course aims to help pre-service teachers develop a nuanced appreciation for the philosophy of mathematics, so that they might begin to critique the intellectualist and conceptualist model of mathematics teaching and learning.

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Variational strategies on the study of the existence and uniqueness theorem for ordinary differential equations

Rodolfo Fallas-Soto[1] and Ricardo Cantoral[2]

[1]Cinvestav, Mathematics Education Department, México City, Mexico; rfallass@cinvestav.mx

[2] Cinvestav, Mathemathics Education Department, México City, Mexico; rcantor@cinvestav.mx

In this paper, we study historical proofs of the existence and uniqueness theorem for the differential equation $y' = f(x, y)$. We analyze original works that played a part in the problematization of this concept, thus offering a rational reconstruction of its genesis which sheds light on its meaning. We will use results from the socioepistemologic theory to show that variational strategies are efficient in the analysis of the proof. We believe this epistemological analysis may help in the future for pedagogical designs.

Keywords: Socioepistemology, variation, existence, uniqueness, differential equation.

Introduction

We present the results of an ongoing research on the existence and uniqueness theorem for first order ordinary differential equations. The socioepistemological theory, theoretical basis of the research, faces, from a systemic view, a possible reconstruction of its meaning due to the variational strategies in its development, without limiting itself to a chronological reproduction of the contributions of the mathematical works, nor a reinterpretation of what we actually know of the theorem.

Variational strategies involved the construction of the mathematical notions related to the theorem mentioned above are analyzed. These strategies are part of a study program called "Variational Thinking and Language" that the socioepistemological program develops. From the study a different use of the actual knowledge is recognized, that is considered a potential element to start changing the relation with said knowledge from the objects to the actual practices (Fallas-Soto, 2015).

This notion of "from the objects to the actual practices" talks about constructing a new interpretation of the object (the theorem in this case) based on practices (Cantoral, 2013), with the notion of starting with the *problematization of knowledge*, from the Socioepistemology, finding the meanings of said knowledge at the moment of actual use. Then, the problematization from this view, consists in performing a double study whose elements, *historize* (historical reconstruction of knowledge) and *dialectize* (coordination of mathematical notions, examples, counterexamples and conceptions and misconceptions), are the base to study the evolution of the theorem throughout history and thus to analyze how its associated mathematical notions plays in order to construct the theorem from the point of what is known today.

Therefore, the research problem is linked to the *meaning* in mathematics, this because the way how the existence and uniqueness theorem is presented on textbooks does not appears to be deducted from the practices properly, nor also from the mathematization. Then, the problem, which are the principles that give meaning to the notions of existence and uniqueness as specific characteristics of the solution's nature? The hypothesis of this research work is to assume that the construction of the theorem can be a prediction model with the study of variation. It can be said that there's a common

thread that organizes the conditions for the existence and uniqueness, and is concected with the prediction idea that was the basis for the theorem along with its hypothesis.

The research focus for this research was to *mean the existence and uniqueness theorem* from a particular problematization of mathematical knowledge.

Methodology

The phases that describe the research are briefly described below: we study the textbooks used in the teaching of Differential Equations, particularly we analyze the demonstration. By showing that there is no problematization of this mathematical knowledge and only remains as a test of hypothesis, it is decided to study the genesis of this knowledge and its evolution. We offer a rational reconstruction of the main arguments used by mathematicians and in the part of conclusions we make a comparison of the origin of the theorem with the current didactic treatment.

A rational reconstruction for the existence and uniqueness for differential equations, in a first stage, a bibliographical research is presented with some of the mathematical works of the time that helped on the construction of this knowledge. These are:

- (Cauchy & Moigno, 1844), *"Leçons de Calcul Différentiel et de Calcul Intégral"*
- (Lipschitz, 1880), *"Lehrbuch der Analysis"*
- (Lipschitz, 1868) *"Disamina della possibilità d' integrare completamente un dato sistema di equazioni differenziali ordinarie"*
- (Peano, 1973 – new-edition) *"Sull' integrabilità delle equazioni differenziali di primo ordine".*
- (Picard, 1886) *"Cours d' Analyse"*

This proposal there will be reported the results obtained in the analysis of Cauchy & Moigno (1844) and Lipschitz (1880) papers, all of them for the study of the variational strategies. These five books were chosen because of Picard and Peano appear in the current textbooks contributing to the theorem. Then Picard (1886) in his work refers to Cauchy and Moigno in addition to Lipschitz.

On a second stage, a documental analysis, the elements used on the mathematical works for the study of the existence and uniqueness theorem are reconstructed. This gives contributions to generate implementation strategies that propitiate the construction of the theorem based on a pragmatic evolution of the practices. To obtain the conclusions of tis work, we perform a confrontation between the mathematical works (also used for teaching) that arose at the end of the XIX century and at the beginning of the XX century, with ideas from the textbooks of XX century and nowadays.

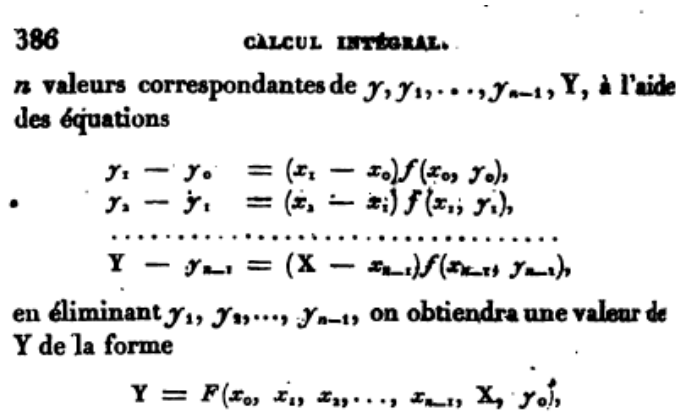
With respect to the variational strategies presents on this theorem, we are based on the PylVar (That its acronym in Spanish means Thinking and Variational Language) program that has been used throughout the years by some authors (Caballero, 2012), (Cabrera, 2009), (Cantoral, 2004; 2013a), (Cantoral & Farfan, 1998). This approach seeks to show the construction of mathematical knowledge from the study of motion, change and the variation of physical or natural phenomena. Our research work does not mention the modeling of phenomena related to the theorem of existence and uniqueness, however in the study of change is how the differential equations are born and we continue to deepen in current research. These works have permitted to explicit the variational strategies (practices on the study of change) that the Pylvar program describes, and that are discussed in Caballero (2012):

- Comparison: Associated to the action of establishing differences between states.
- Serialization: It is associated with the action of establishing relations between successive states. To study the changes to determine a certain pattern.
- Estimation: Starting from other knowlegde of changing states, proposing states for a short term.
- Prediction: The action of being able to determinate after analyzing some states to deduct posterior states. It means to anticipate to a certain rational state.

Results

We do not offer a mathematical proof but to show arguments that helped the mathematicians in formalizing this mathematical knowledge.

Cauchy & Moigno (1844) consider the differential equation $y' = f(x, y)$ with the hypothesis that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions with the initial condition (x_0, y_0). The authors prove the convergence of the sequence of points obtained by the fractions method.



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n valeurs correspondantes de $y, y_1, \ldots, y_{n-1}, Y$, à l'aide des équations

$$\begin{aligned} y_1 - y_0 &= (x_1 - x_0) f(x_0, y_0), \\ y_2 - y_1 &= (x_2 - x_1) f(x_1, y_1), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y - y_{n-1} &= (X - x_{n-1}) f(x_{n-1}, y_{n-1}), \end{aligned}$$

en éliminant $y_1, y_2, \ldots, y_{n-1}$, on obtiendra une valeur de Y de la forme

$$Y = F(x_0, x_1, x_2, \ldots, x_{n-1}, X, y_0),$$

Figure 1: Fractions method utilized by Cauchy & Moingo, page 386 of the lesson 26 of the work Leçons de Calcul Differentiel et de Calcul Intégral.

This method (this method of approximation was expounded by Euler in 1768 in his *Institutionum Calculi Integralis*) consists on determining successive states that depend on the preceding state parting on the study of the linearity and small variations that can be taken with the differential equation (as the rate of change that determines the slope of the line) and the initial condition (point by which the line passes). This procedure is shown in the following figures.

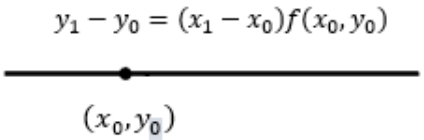

Figure 2: First iteration of the fractions method.

$$y_2 - y_0 = (x_1 - x_0)f(x_0, y_0) + (x_2 - x_1)f\big(x_1, y_0 + (x_1 - x_0)f(x_0, y_0)\big)$$

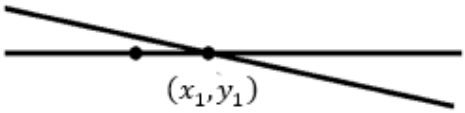

Figure 3: Second iteration of the fractions method

Therefore, the process is generalized and it is approximated that $y - y_0 = \pm\Theta A(x_n - x_0)$where $A$is an average of the $f(x_n, y_n)$and Θa value between 0 and 1. Then, the authors generalize the process and stating that $y_n$is equal to

$$y_n = y_0 + (x_n - x_0)f\big(x_0 + \theta(x_n - x_0), y_0 \pm \Theta A(x_n - x_0)\big)$$

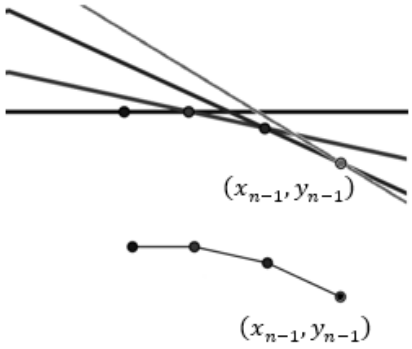

Figure 4: N-th iteration obtaining a numerical approximation of the solution.

From which $y_n$corresponds, practically, to the same value that $y_0$if the difference $x_n - x_0$is small. In other words, if $x_n \to x_0$then $y_n \to y_0$.

Besides, convergence is studied in the following form as part of a stability of the system. If a small increment $\varsigma_0$is added to y_0, then $y_n$will have an increment ς_n. In order for it to converge, this last increment has to be as small as ς_0. This seeks the stability of the function $y$to guarantee the existence. Again, the small variation plays a fundamental role to compare states and thus determine a local prediction on each iteration to determine a final global prediction (estimation) of the system.

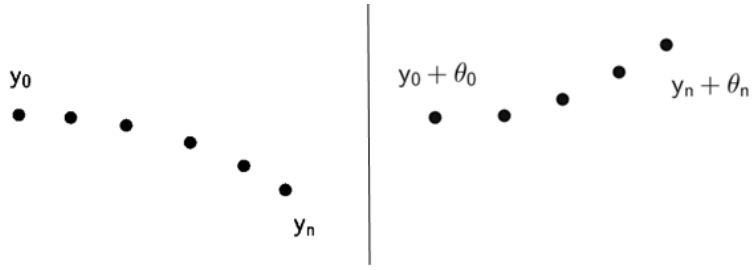

Figure 5: Case where the increment of $y_n$changes considerably with respect to the increment of y_0.

This result is studied even further by Lipchitz (1880) when working with systems of equations and the uniqueness of the solution. By looking at figure 05, what is really happening is that from the first

iteration two tangent lines are obtained that correspond, respectively, to each of the solutions. Because of this, on the point (x_1, y_1), and another one close to it, say (x_1, η_1), with $\eta_1 = y_1 + \theta_1$such that $\eta_1 - y_1$is close to zero, two tangent lines are determined, given by

$$y - y_1 = (x - x_1)f(x_1, y_1)$$

and

$$y - \eta_1 = (x - x_1)f(x_1, \eta_1)$$

Which are depicted in the following representation

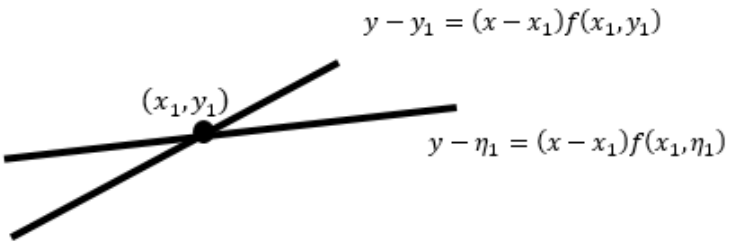

Figure 6: The existence of two solutions for the equation.

Then, if we study the difference between these two lines, we see that is the same that subtracting their two respective slopes, this because $\eta_1$is a value that is close to y_1. Notice that the distance between the two lines is determined by

$$|f(x_1, \eta_1) - f(x_1, y_1)|$$

That is why the Lipschitz condition plays a very important role, due to the fact that this difference would be bounded by

$$|f(x_1, \eta_1) - f(x_1, y_1)| < M|\eta_1 - y_1|$$

where the constant $M$is the bound of $\frac{\partial f}{\partial y}$ that not necessarily is a continuous function. Therefore, if this condition holds, we would have that $|f(x_1, \eta_1) - f(x_1, y_1)| = 0$, being a unique solution to the equation. If the Lipschitz condition does not hold, we cannot guarantee uniqueness, it can or cannot exists.

Conclusions

This study, from the socioepistemological point of view, broadens the knowledge on the Existence and Uniqueness Theorem for Ordinary Differential Equations, but most of all it shows the kind of practices (variational strategies) that play a role in the justification of both existence and uniqueness. The fractions method, absent in most textbooks on Differential Equations, is present in textbooks on numerical methods. This is the second time this phenomenon arises. A similar situation arose in the prediction based on Taylor series. This findings, are worth mentioning, are derived from an appropriate problematization of the mathematical knowledge. Two things that gave genesis to this problem were the looking for a formalization on the proof of the theorem and also the determination of the minimum quantity of hypothesis that guarantee the existence and uniqueness of the solution.

Additionally, the initial questions referred to the inverse tangent problem and the different examples present in the mathematical discourse for school were answered, but these time with the support different resources: variational, numerical, analytical and visual. All these was obtained thanks to a

documental analysis that was based on original mathematical works. On the other hand, it was possible to discuss other constructs, such as convergence, the Lipschitz condition and the continuity of the functions $f(x, y)$ and $\frac{\partial f}{\partial y}$, with respect to y.

Besides, the variational strategies on the construction of this theorem were:

- Comparison: The states that correspond to the numerical solution of the differential equation are compared. Besides, the solution is compared before and after of the small variation to determine its uniqueness.
- Serialization: When finding a relation between a state and another, starting with the initial conditions, in a lot of cases are possible to obtain an analytical solution to the equation, while in other cases it is only possible to predict the value that the solution will take in the next state (numerical solution), both cases with the support of the study of the patterns between one state and the other.
- Estimation: When knowing the initial values and unknowing the next value that determines the numerical solution of the equation, is when the linear approximation enters to determine the next value or state.
- Prediction: This theorem corresponds to a predictive model, it is utilized to predict the existence of the solution, and with the small variation we are certain of the convergence and uniqueness of the solution.

When performing a study such as this one, it is possible to study the rupture between most contemporary textbooks against the works reported on the mathematical studies of the past. Besides, it can be observed that some rationalities arise that will help to design teaching and learning activities by means of the use of teaching variables or control variables to modify, keeping in mind the present construction of these meanings. Reconstructing these meanings helped us to understand other problems related with differential equations, such as the stability of a system of differential equations, and to construct other visual interpretations.

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Understanding didactical conceptions through their history: A comparison of Brousseau's and Varga's experimentations

Katalin Gosztonyi

Eötvös Loránd University, Budapest, Hungary; katalin.gosztonyi@gmail.com

In this article, I compare two experimental projects led in the 1960s and 1970s by Tamás Varga in Hungary and by Guy Brousseau in France, concerning the teaching of combinatorics and probabilities. I attempt to show that this comparison can contribute to a better understanding of their theoretical reflection on the teaching of mathematics and the dependence of their didactical conceptions on the particular historical context in which these reflections were realized.

Keywords: History of mathematics education, didactical theories, New Math, probability teaching, abstraction.

Introduction

In the Hungarian mathematics education community, the "complex mathematics education" reform led by Tamás Varga during the 1960s and 1970s is considered as one of the most important milestones of the history of mathematics education in Hungary. Varga's conception is viewed as a representative of the so called "discovery based" tradition of mathematics education. However, this conception was never developed in a theoretical level: it can be understood from the documents of the reform (as the curriculum, the textbooks, teacher's handbooks etc.) and from some articles of Varga, but most of the time, these texts present his approach on very concrete examples and give only limited, indirect access to the conceptual basis of his didactical conception.

The research presented in this article is part of a more complex work, aiming to analyse Varga's conception with a combined, historical and didactical approach, and in comparison with the French reforms and mathematics educational movements of the same period (Gosztonyi, 2015a). This analysis contains three levels: the study of the historical context, of the epistemological background (the conceptions about the nature of the mathematics in the background of both reforms) and a didactical analysis of the reforms' documents.[1] For this third part, French didactical theories were used, Brousseau's *Theory of Didactical Situations* (TDS) (Brousseau, 1998) among others. The TDS proved to be an expedient theory to reveal characteristics of Varga's conception; however, some difficulties of the analysis let to think there were also some important differences between Varga's and Brousseau's didactical approach.

These theoretical questions led to consider Brousseau's work not only as a theoretical framework for my research, but also as a *research object*: actually, Brousseau himself was one of the actors of the French math education scene of the 1960s and 1970s. He led experimentations during this period, and he developed his didactical theory in relationship with these experimentations, in the very particular historical context of the French reform movement. So, a historical analysis of Brousseau's projects

[1] (Gosztonyi, 2016b) gives a short summary of this work, showing how the difference of the epistemological background, represented by influent groups of mathematicians in both of these countries, can explain, at least partly, the differences of the two reforms.

of the period could contribute to the comparison of the two countries' reform movements and to the understanding of Varga's project.

One more reason explains my interest for Brousseau's experimentations. One of the main domains of Varga's activity concerns the teaching of combinatorics and probability: so I was interested to do comparative analysis partly on these domains. At the same time, these domains are missing from the French obligatory education of the period which makes the comparison obviously difficult between the two countries. However, during the 1970s, these subjects appear in France in certain experimental projects, and one of the most developed examples is Brousseau's experiment on teaching probabilities, described in a retrospective article (Brousseau, Brousseau & Warfield, 2002).

In the following, I will resume the comparison of Brousseau's experiment with Varga's teaching projects and experiments, and I attempt to show how this comparison helps to understand the authors' didactical conceptions.

Some elements of the historical context

The international discourse on mathematics education during the 1960s and 1970s is dominated by the New Math movement and the debates following the introduction of New Math related reforms in several countries. France is one of the leading countries of the international reform movement, with the "mathématiques modernes" reform introduced in 1969/70. Varga declares also being influenced by the international movement: he starts his experimentations in 1963, inspired by Z. P. Dienes' lectures in Hungary in 1960 and a UNESCO conference organized in Budapest in 1962, and follows the reforms of several countries during the following decades. His experimentations concerning the primary and later also the middle-school are progressively spread in the country, and give the basis of a new official curriculum in 1978.

Although Brousseau is not among the main actors of the French curricular reform movement, his early experiments can also be interpreted in the context of the reform. In France, various experimentations on primary and secondary school mathematics education were encouraged and supported during the 1960s and 1970s. For example the system of the IREMs[2] was progressively created from 1968, with the mission to form in-service teachers to the implementation of the reform, but also to continue experimentations. During the 1970s, the IREMs became also centers of critical debates on the reform, and principal sources of the emergence of didactical research in France. Brousseau's early work fits clearly in this context: he led experiences since the 1960s, and his projects led in the Jules Michelet experimental primary school from the beginning of the 1970s became the basis of the development of his later theory. The experimentations on the teaching of probability described in his retrospective article were also led in 1973 and 1974 in this school. [3]

Brousseau's and Varga's teaching projects and experimentations

The reasons to teach combinatorics and probability

In France and Hungary, combinatorics and probability are not present in primary and middle-school curricula before the New Math period. So, the experiments concerning these domains attempt to

[2] Insittut de Recherche sur l'Enseignement des Mathématiques

[3] For more details and references on the historical background of the two projects, see (Gosztonyi, 2015a, part I.)

introduce new elements of curricula or introduce them much earlier than in precedent cases. Varga explains in several papers why he found important to introduce these domains very early during mathematics education. According to him, these domains contribute to the diversity of mathematical subjects treated in school and help to implement dialectic relationships between mathematical domains – which is one of the main principles of his reform curriculum's structure. He says combinatorics and probabilities can be studied on the basis of very concrete, material experiences and so can give a good example of the process of mathematical abstraction very early, using a diversity of representational tools, and without the need of complex theoretical background. These domains can also give occasion to many playful activities. More particularly the teaching of probabilities represents a "different kind" of mathematics: the mathematics of uncertain things, which is especially important to describe real world phenomena. Furthermore, in the frame of tasks related to probabilities, and especially through estimations, student's autonomous thinking and the expression of students' various opinion can be easily encouraged. The quotation below shows also that this last question is closely related to more general pedagogical considerations concerning the teacher's and student's role in the learning process or students' education to democracy.

> My own view is that estimating, guessing, predicting, mentally representing, the future and expressing our opinion about it is a human ability which should play a greater role in education than it does now. All these activities [...] get kids personally involved in learning. […]
>
> Reasons must be strong, maybe not unrelated to sentences from "a child should not have an opinion" to "a child should have not will." If this is a correct conjecture, then the issue is a more general one about education, not necessarily school education or school math in particular. (Varga, 1982, p. 30)

All the arguments resumed above are important characteristics of Varga's conception on mathematics and its teaching (Gosztonyi, 2016a, b). Thus, one can understand that the emphasis made on the teaching of combinatorics and probabilities is closely related to the general educational goals of Varga's reform, and these domains' curricula and tasks can serve as efficient examples to understand the realization of these goals in Varga's reform.

In Brousseau's examined experiment, there is no question of a general curricular reform: probabilities are not included in the official curriculum of primary school, and Brousseau's experimentation is realized in the frame of general activities ("activités d'éveil") and not during the mathematics lessons. However, Brousseau underlines also some general arguments to explain the interest of teaching probabilities. The idea of "another kind of mathematics" appears also in his work (Brousseau, Brousseau & Warfield, 2002, p. 397). Beyond that, students' autonomous work and the repartition of responsibilities between the teacher and the students is in the center of his theoretical thinking, and as he admit in the quoted paper (e.g. p. 384, p. 411), the experiments on teaching probabilities played a crucial role in the development of his didactical theory.

Some tasks of Varga on probabilities

In his articles, Varga describes several probabilistic tasks developed in the frame of his experimentations (see e.g. Varga 1970, 1982). In the case of the subtraction game,

> Each kid draws boxes for the digits [of two two-digit numbers] the way they are used in writing subtractions. The goal is to make the difference as great as possible. They can fill the boxes in any order, but only with random numbers produced by rolling dice. (Varga, 1982, p. 28)

In another game, the "game with three disks" (Varga, 1970, p. 424), three discs are in a box, one of them has a cross in each side, one has a cross on one side only and one is blank in both sides. The teacher draws a disk at random and shows one of its faces at random to the students. They have to guess the other face.

In each of these situations, there is a competition between students, which motivates the development of a strategy. The comparison of these strategies, their test during further experiments contributes to the development of probabilistic thinking. According to Varga's description, frequentist approach, based on repeated experiments and the observation of relative frequencies is alternated with a classical approach to probabilities, based on logical arguments concerning models with equally probable events. However, the frequentist approach is rarely developed in deep details: it seems more to serve as an experimental basis to develop classical models which are then confirmed by logical and combinatorial arguments.

In the situations described by Varga, teacher and students are in permanent dialogue: Varga gives several examples how to guide these dialogues in order to help the development of mathematical notions, but to give also important autonomy to the students in this process.

Beyond the description of tasks and problem situations, Varga describes also the conception of long-term teaching processes. Even if some texts help to understand the long-term conception of probability-teaching, the clearest descriptions on the construction of long-term teaching processes concern the domain of combinatorics. We study this in the next section.

Varga's series of problems to teach combinatorics

In Varga's conception, ordered series of problems play a crucial role in the construction of long-term teaching processes (Gosztonyi, 2015b). A particularly interesting example concerning combinatorics is described in the first grade teacher's handbook associated to Varga's reform (C. Neményi et al., 1978, pp. 243–258). It is interesting not only because it is particularly clearly structured (compared to other parts of the handbook), but also because it gives quite explicit explanation about the principles of ordering problems. This ordering is not completely given in advance: teachers have to conceive (and reconceive) it depending on the particularities of the class, and on the reactions of the students.

The series contains activities with different materials: the students build towers with coloured cubes; thread beads, draw flags or build houses with three parts of different colour; write 'words' (letter series) with a given number of characters or play music with a given number of notes. One organizing principle is the variety of experiences, apparently fare from each-other, and stimulating a diversity of senses. But there is also certain progressiveness in their order, namely in the level of abstraction: starting from the manipulation of physical objects, through drawing and until the manipulation of symbols.

The activities with one material follow also a progressiveness, which is explained in detail on the example of building towers: after a free game with the material (elements with, for example three different colours), students are asked to build towers with a given height; then different towers with

the same height; and then come the question of the number of possibilities. The handbook suggests variations of the mathematically important variables as the number of levels or colours, but also the mathematically neutral elements as the type of the material or the colours of the elements. It suggest also some additional restrictions (as for example two neighbouring levels cannot have the same colour; or a given colour can have on the top level, etc.).

A similar process is described for different materials. The handbook indicates the analogies between the corresponding phases of these different activities, and explains also their differences which make them problems of different nature in the eyes of students: for example, in the case of building houses, the order of the elements is not immediately given, contrarily to the building of towers or the colouring of (three-stripe) flags. Students have to recognize progressively the links and the analogies between these different problems: that is what will lead to a progressive generalization of methods and solutions.

The process is planned for several years: the object at the first grade is mostly the collection of experiences in structuring possible cases and in looking for the number of all possibilities, in concrete situations. The systematic variation of conditions and the formulation of rules for calculating the number of cases come some years later. In a later article, Varga (1982) describes in detail a process on the long-term, based on the example of building coloured towers: there he also explains how the progressive construction of representation tools (as tables and trees) leads in his conception to creating proofs and general formulas in combinatorics.

Brousseau's experimentation

In his experimentation on the teaching of probabilities, Brousseau also conceives a long and coherent teaching process, although the structure of this process is quite different from those of Varga, as we will see below. The whole process emerges from one situation: the teacher fills three opaque sacks with black and white balls, five in each sacks but the proportion of blacks and withes is unknown. The students have to find out the exact composition of each sack by drawing balls one by one from the sacks. The process goes through several phases that the authors describe as follows:

i. "An introduction to hypothesis testing" (5 sessions)
ii. "Modelling and experimenting" (3 sessions)
iii. "Graphic representation of long series" (8 sessions)
iv. "Convergence and statistical decision" (4 sessions)
v. "Decision intervals" (5 sessions)
vi. "Events and their probabilities" (7 sessions)

After a number of drawings and hypotheses made on the compositions, they decide to model the sacks with transparent bottles where they put five balls in different compositions, in order to compare the outcomes of these bottles with those of the sacks. Students work almost autonomously during the first two phases, only with some "regulative" interventions of the teacher: that's what Brousseau calls later an *adidactic situation*. The teacher intervenes more directly during the third phase, in order to stabilise the method of students' experimentation. A difficulty emerge in this phase, as even after long series of drawings, students are still not able to prove a decision between the possible compositions and they start to loose motivation. The teacher refuses to open the sacks but suggest proving the compositions by elaborating a methodology which allows students to do predictions. That is what

happens during the fourth and fifth phases using computers for simulations and different tools and methods of representation (as tables and graphics).

The described process represents a typically frequentist type of approach to probabilities, with the use of complex statistical methods and tools, and with a long process of repeated experimentations. The first five phases emerge from one initial problem. The classical approach as well as other problem situations appear in the sixth phase but, considered more conventional as the precedent ones, this phase is not developed in the article (p. 405)

A comparison of Brousseau's and Varga's projects

Comparing Brousseau's and Varga's teaching projects briefly described above, one can observe some common points but also several interesting differences.

Although both frequentist and classical approach appears in both of the authors' projects, Brousseau puts more emphasis on the first one while Varga emphasize more the second one. This difference can be explained by different things, as by the coherence with other parts of the curricula (links with statistics on one hand, with logic and combinatorics, privileged domains of Varga on the other; the use of decimal numbers, privileged by the French curriculum of the period, or the use of fractions, emphasized in Varga's curriculum), but also by some pedagogical questions.

One of the common motivations to introduce probabilities into the primary school mathematics education is, in both of the cases, the opportunity offered by this domain to work with students guessing and estimations, and develop mathematical thinking with an important responsibility provided to students during the learning process. However, the repartition of responsibilities between teacher and students does not happen in the same way by the two authors. For Brousseau, this repartition has to be provided by the alternation of *adidactic phases*, where students work fundamentally autonomously and teacher does not intervene on the level of mathematical knowledge, and phases of *institutionalization*, where teacher intervenes essentially in order to transform students' context-dependent discoveries into stable, decontextualized and institutionally accepted knowledge. As Brousseau underlines, the experiments on teaching probabilities contributed essentially to develop these key-notions of his later theory. In Varga's case, although students' responsibility in the learning process plays also a key role, the teaching situations are more based on a permanent dialogue between the teacher and the class: the teacher acts as an experimented guide to develop progressively collective knowledge.

In both cases, we can see sophistically constructed long-term teaching processes. But the structure of these processes is quite different. Brousseau's experiment is developed starting from one problem situation. This is something that he calls later *fundamental situation*: a problem situation which is rich enough to lead to the emergence of a whole theme. The process goes through the alternation of adidactic situations and institutional phases. According to him, the transmission of knowledge constructed by students in the context of a particular situation is not possible without the decontextualisation realized during the institutionalization (Brousseau, Brousseau, & Warfield, 2002, p. 407). Indeed, in the examined experiment, other problem situations appear only in the last sixth phase. Varga's approach is quite different to the construction of long-term teaching processes. He suggests starting from a diversity of problem-situations, with a big variety of contexts. The

construction of *series of problems* offer occasions to recognize similitudes, analogies between these different problems, and leads to a progressive generalization of the solutions.

Behind this difference of the two authors' conception on long-term teaching processes we can recognize a fundamental difference concerning their conception on the mathematical abstraction. As for Brousseau, abstraction is *decontextualisation*, for Varga, the abstraction process does not mean the elimination of the context, more a *progressive generalization* on the basis of a diversity of contexts.

Conclusion and discussion

The comparison of Brousseau's and Varga's experimentations led in the 1970s showed several common points and also some important differences. This comparison can help to understand better the conceptions of the two authors on mathematics and its teaching, and also the relationship of these conceptions to their special context. Both of the two authors' projects inscribe clearly in the context of the international New Math movement, with the reforms of the content of the curricula, the debates on the nature of the mathematics and on the psychology of mathematics education. For both of them, Piaget's constructivist theory represents an important reference, but they are both critical with it: the alternation of the adidactical situations with institutionalization in Brousseau's case, and the dialogic relationship between the teacher and the student in Varga's case can be interpreted as two different answers to the limits of constructivism considered by these authors. Moreover, Brousseau's notion of *adidactical situation* seems to be related to the discourses about the French reform, more precisely to the notion of *situation* and its role in pedagogical practices and students' learning processes (Artigue & Houdement, 2007). Varga's choices seem also to respond to several elements of local discourses in mathematics education as well as in pedagogy and psychology (Gosztonyi 2015a).

An interesting relationship can also be observed with the dominant epistemological background of each countries reform movement, concerning the author's conception on mathematical abstraction: Brousseau's decontextualisation approach seems to be related to the "bourbakian" conception on abstract mathematical notions, as Varga's ideas on the progressive generalization can be find in the writings of several Hungarian mathematicians supporting his reform (Gosztonyi, 2015a, part II).

This case study reveals how a didactical theory, like Brousseau's one, may depend on the particular context in which it emerges. Thus, further studies on the history of didactical theories are susceptible to contribute to their understanding, and enrich also projects connecting and comparing didactical theories as the "Networking Theories" project (Bikner-Ahsbahs & Prediger, 2014).

The comparative analysis presented above contributed importantly to the original objectives of my research, namely to a reconstitution of Varga's missing didactical theory. But it can also contribute to recent didactical discourse in several further ways, as to the research on the teaching of probabilities or to the recent reflections on the notion of mathematical abstraction.

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Historical methods for drawing anaglyphs in geometry teaching

Rainer Kaenders and Ysette Weiss

Rheinische Friedrich-Wilhelms-Universität Bonn, Germany: R.Kaenders@uni-bonn.de

Johannes Gutenberg-Universität Mainz, Germany: weisspid@uni-mainz.de

The historical anaglyphic method was in use for more than hundred years to create spatial illusions of mathematical objects and for technical constructions. While algebraisation is predominant at school, students lack experience in understanding the causalities of technical tools. Modern technical devices rarely allow for direct investigations on underlying technical principles. Here we use the historic anaglyphic method to enable the students to produce high-standard 3D illusions just by using coloured pencils, a ruler and glasses with colour filters. We developed an approach to the anaglyphic method that uses nothing but similarity and especially surpasses projective geometry. The presented approach relates plane and spatial geometry, and can be grasped by all students that have some understanding of the similarity of triangles.

Keywords: Anaglyph, binocular geometry, historical methods of visualization, 3D-representations.

Introduction

Concept development in high school mathematics – in particular A-level subjects in many countries – is characterized by predominant algebraisation. This seems to match successful and effective heuristics, attitudes and problem-solving strategies in modern everyday lived experience. Out of school it fosters the learning of pattern recognition, algorithmic procedures and trial and error. The progressive digitalisation of most fields of experience and actions of contemporary students seems to make the search for causalities and functional principals superfluous and unnecessary. Modern technical devices rarely for allow direct investigations on underlying technical principles. However, approaches and attitudes aiming only for the use and not the understanding of technical devices involve the risk of simple manipulation of the user and restrict creative developments of the tools by their users. In order to help our students to become autonomous, mature individuals, we need in mathematics (and also as a foundation for technical sciences) teaching designs where students start to wonder: How does it work? How can I accomplish it on my own?

In the following we present materials for discovery learning, supporting the described educational goals. The design of teaching and learning materials allows for the implementation of the history of engineering and technical drawing in different ways and with different aims. A historical investigation using original sources can be conducted as an introductory part of a workshop on 3D presentations. Historical investigations can also be undertaken as a part of an individual student's presentations after the workshop on anaglyphs and binocular geometry. We start with a short historical introduction into anaglyphs as well as into the literature and original sources, which are applicable by the students at school or by student teachers in teacher education.

We then give a short summery of how binocular geometry can be related to the mathematics curriculum and discuss a textbook presentation of the topic. The following section deals in detail with

the geometrical properties of central projections, which enable the drawing of anaglyphs without using three-dimensional analytic geometry.

We tried out the developed set-up and the material in four workshops at an international Kangaroo Camp, each with 15-20 high school students between the ages of 15-18, and in a day course at the Hausdorff Center with local high school students in Bonn, with a lecture and problem sessions with about 60 high school students.

Historical background

Sir Charles Wheatstone invented the earliest type of stereoscope in 1838 (Wheatstone, 1838). However, as David Brewster writes in (Brewster, 1856, p.27) a certain Mr Elliot was led to the study of binocular vision … as early as 1823. Wheatstone and Elliot used mirrors, whereas Brewster invented lenses. The mathematician Wilhelm Rollmann (1853) invented the anaglyph stereographic method. The Greek word *Anaglyph* is derived from $\acute{\alpha}\nu\acute{\alpha}$(aná), meaning "on" or "on each other" and $\gamma\lambda\acute{\upsilon}\varphi\omega$(glýphō) "to carve", "engrave", or "represent". In his method, two pictures in the mutually different colours blue and yellow are superposed onto each other. The observer in Rollmann's method separates these pictures by using glasses with colour filters, i.e. a red glass for the left eye and a blue one for the right eye.[1] All cited sources are digitalised and available online. The descriptions of the methods are mainly verbal. The few calculations can be performed and understood using elementary middle school mathematics. This makes the cited literature a good and suitable source for historical investigations in the classroom.

Rather than being a goal as such, the anaglyph method was already used by Rollmann to illustrate mathematical facts and insights in a three-dimensional fashion. Another impressive example of this kind of illustration is the Imre Pal's beautiful book (1961). Although the books of Rollmann and Pale describe the three-dimensional model of spatial seeing, nevertheless they do not instruct how to draw two-dimensional anaglyph stereographic pictures. For the latter we designed a special workshop, which can be related to schoolbook exercises in analytic geometry (Körner et al., 2010, p. 92) or can serve independently for project work.

Anaglyphs in teaching

Binocular geometry is not a canonical topic for mathematics lessons. So we were rather surprised to find a set of exercises related to the anaglyph method as well as well as materials for teacher training courses. The following excerpts in Figure 1 are taken from a German school textbook for grade 11 (one year before A-levels). The book is one of the five most widely used textbooks in high schools in Germany. The excerpts stem from a chapter in analytic geometry dealing which the calculation of intersections of lines and planes given in Cartesian coordinates. In addition to perspective drawings of a house and bowling pins, some general properties of perspective drawings are given (left). In preparation for the work with perspective mappings, the textbook authors posed the exercise to compute the image of the shadow of a cube illuminated by a lamp given in Cartesian coordinates. The shadow has to be calculated in a given plane using the formulas for intersection points of lines

[1]In our case, we use a red glass for the left eye and a blue or better turquoise one for the right eye.

with planes. The green image (right) is the cube shadow in the given plane. The red image is the result for the same calculation but with a slightly shifted illuminant. The students are asked to make red-green glasses themselves and to look at the picture through these glasses. Among the textbook materials for computer-based learning, there are 3D dynamic geometry applets with the green and the red calculated images corresponding to different dynamic light sources. Even though the exercises deal with a geometrical context – perspective drawings – the proposed approach is purely algebraic.

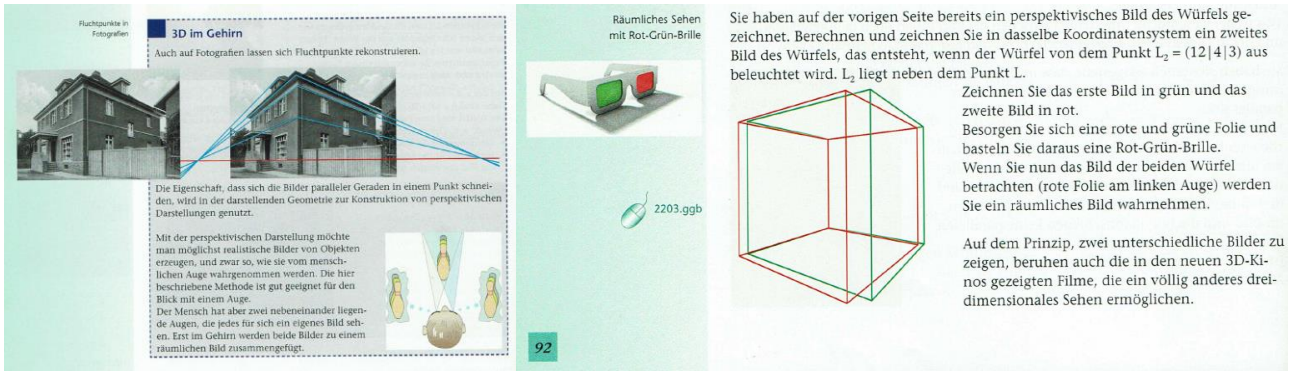

Figure 1: Erecting a stick of thumb size by lifting a point.

The pictures are visualizations of the results of algebraic calculations. In (Färber, 2016) the author proposes designing the excursion on the anaglyph stereographic method as discovery learning in groups. Nevertheless, the lesson planning only involves algebraic manipulations. As we already discussed in (Kaenders & Weiss, 2016), the high degree of abstraction and technical complexity of algebraic symbolic language gives students few opportunities to question the underlying rules, to introduce their own situated notations and notions reflecting their individual understanding of a problem and its context or to develop their own mathematical questions. To deal with this problem we develop a geometrical context for discovery learning of the anaglyph stereographic method.

A simple approach: Lifting plane figures

When we considered using the historical anaglyph method for teaching, we expected a couple of difficulties. First, we were convinced that we would have to give a quick introduction to projective geometry and then apply it to the anaglyph method. We were then surprised to discover an approach to the fabrication of such binocular illusions that does not use projective geometry at all. It is the technique of *lifting a point*. Well understood, it allows not only for the lifting of points but also the lifting of any figure from the plane as long as it is supposed to become an illusion of a congruent figure in a plane parallel to the initial plane. We give a description of the course of action in the workshops.

Practical preparation

Before the students are given the task of creating their own anaglyphs, we show them some exemplars. By looking through the red-green-glasses they begin to get a feeling for what such a painting could look like and which type of effects are generally possible. It turned out to be a helpful practical hint to let the student put their fingertip on the spatial spot where they expect the figure to be. A few students do not succeed in recognizing the intended illusion. The reason for this might be problems like shifted eyesight, where one eye has a stronger visual faculty than the other, or a red green deficiency. Nevertheless, these students could successfully participate in the workshop.

Erecting a little stick

The first exercise is to erect virtually a little stick of thumb size to an illusion that appears to be an orthogonal stick on the paper. In a second step, we can also let the stick appear slightly levitated. In order to find such a representation of the desired illusion, the students can turn the question around: the stick is given and we seek the red-green drawing on our paper. If we put a stick (like a pencil) orthogonally on the paper, we can conceive the red and the green drawing as the shadow we obtain when we imagine our eyes to be light sources.

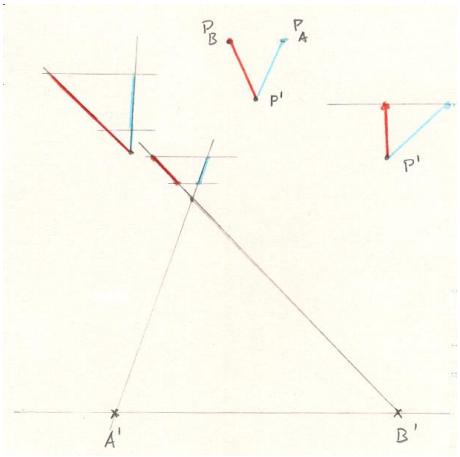

Figure 2: Erecting a stick of thumb size by lifting a point.

Almost all students draw a short red and a short green line segment that produce two lines that intersect at the point that is ought to be the orthogonal projection of the stick on the paper. By discovering this, two questions arise:

- What is the angle between the red and the green line segment? The students relate it to the position of the eyes and some conjecture that the lines prolong to the pedals of the eye points.
- How far do we have to draw the line segments for a perfect illusion? After having explored the situation, the students give two conjectures: The ratio of the red and green line segment is the same as the ratio of line segments between the pedal points of the eyes and the point where the stick touches the paper. They indicate that because of similarity, both assertions are equivalent.

Analysis of the exercise

To analyse the situation, we assume the position of the eyes A and B on fixed height h over the table. We assume the eyes A and B to be parallel to the table plane and to have a distance of 7 cm, which is about the average eye distance for adult human beings. When we project the eyes A and B orthogonally onto the table, we obtain two pedal points A' and B'. We consider an arbitrary point P' in the plane. Now we want to find points P_A and P_B in the plane with the effect that they create the illusion of a point P in space, that *lifts* our point P' to some height h.

In Figure 3 we see that this illusion arises when the lines AP_A and BP_B cross in the point P. Then ABP forms a plane. Then the three planes $A'B'BA$, $A'B'P$ and ABP have three intersection lines, two of which are parallel. Then the third one is parallel as well. The reason is what one can call the *Theorem of the Tent*: Given three planes E_1,E_2 and E_3 that intersect in three lines $g_{12} = E_1 \cap E_2$ and$g_{23} = E_2 \cap E_3$ as well as $g_{13} = E_1 \cap E_3$. When two of these lines are parallel to each other, the third is

parallel to both as well. The proof of this proposition is an occasion to show the efficiency of the set theoretic language: Assume that the lines $g_{12} = E_1 \cap E_2$ and $g_{23} = E_2 \cap E_3$ are not parallel; they then intersect in a point P, since both lie in a plane. Then $\{P\} = g_{12} \cap g_{23} = E_1 \cap E_2 \cap E_3$ and $g_{23} = E_2 \cap E_3$ intersect in P as well.

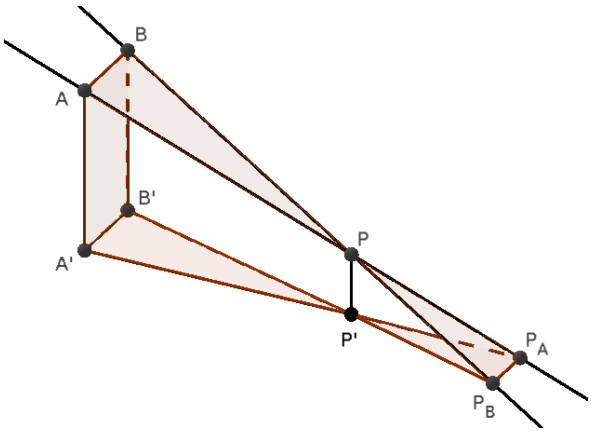

Figure 3: The basic principle of point lifting.

Hence, we can *lift* the point P' to an illusionist point P, when we draw the lines $A'P'$ and $B'P'$ and end up at points P_A and P_B, such that P_AP_B is parallel to $A'B'$. In order to find out how far the point will be lifted, we use the point of view of similarity as the students uttered it. We consider one of the two triangles $A'P_AA$ or likewise $B'P_BB$ (see Figure 4). We especially want to understand the relation between the height h and the distance $d \coloneqq P_AP_B$.

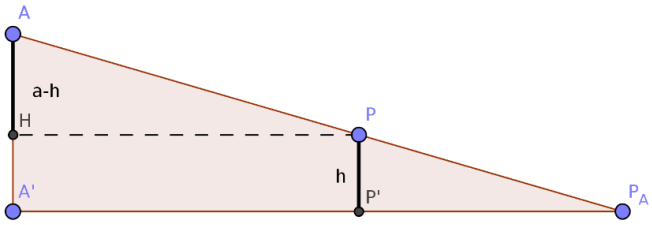

Figure 4: One of the triangles of the basic figure.

In Figure 4 we read off the following ratios: $\frac{a}{h} = \frac{A'P_A}{A'P'}$ and $\frac{a-h}{h} = \frac{AP}{PP_A}$. Combining this with the ratio between the triangles ABP and PP_AP_B, we conclude $\frac{7}{d} = \frac{AP}{PP_A} = \frac{a-h}{h}$. Thus $d = \frac{7h}{a-h}$ or$h = \frac{ad}{7+h}$.

Note the remarkable fact that the distance d does not depend on the special position of P'. Hence, we have one method to lift not just one point but also a whole figure to a certain fixed height h. For instance, we can lift a square by lifting its vertices on the same height and then connect the corresponding red and green points. If we do that twice, we can construct the illusion of spatial box.

Similarity as key concept

Finally, we want to understand how we can lift a figure that does not consist of line segments, e.g. a circle. For this, we need to understand how to lift an arbitrary figure to a fixed height h. For this fixed h, we consider the map of the plane to itself, that maps P' to P_A and likewise the map that

maps P' to P_B. We know already the answer, since $\frac{A'P'}{A'P_A} = \frac{a}{h}$. Therefore, both are central dilations with the factor$\frac{a}{h}$, one with centre A', and the other with centre B'.

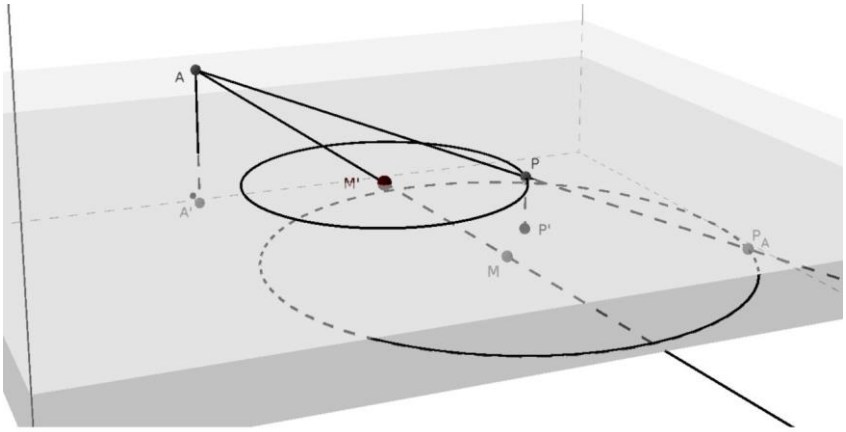

Figure 5: Central dilation with factor $\frac{a}{h}$.

These considerations on similarity can be used to construct tasks for instructional scaffolding as well as materials for explorative learning.

Observations during the workshops and development of research questions

In our workshops, we developed most of the tasks and drawings together with the students on the blackboard. This gave us the possibility of choosing between small-step guiding tasks and rather open activity-oriented tasks corresponding to the work of the students. The groups in the four workshops at the international Kangaroo Camp were very inhomogeneous regarding their English language skills as well as their mathematical preparation. These groups had students interested in mathematics but without any training for competitions or mathematical extracurricular experiences and participants of the International Mathematical Olympics. None of the participants had ever dealt with binocular geometry. In spite of their age (15-18 years old) there was no problem to get the students to draw pictures with crayons. It became also evident that the problem 'How to draw an anagram' is extremely suitable for inhomogeneous groups. The students worked in small groups and were quickly fascinated by their own experiments and pictures. Our objective that was reached in all four workshops was to get the students to search for the underlying principles of constructions and to produce their own drawings by using the principles. As soon as the students are acquainted with the technique of point lifting, there are many possible projects to tackle.

The day course in Bonn was organized differently. The workshops were hold by mathematics teacher students. We gave introductions into anaglyphs first for the tutors than for the students, attended the different workshops and moderated the presentations of the results. For the day course in Bonn, we prepared a script for the tutors with problems they were supposed to solve. Before the day course there were several meetings were the tutors could asked questions and discuss the concept. There was a noticeable difference between tutors who tried to grasp on their own the concept of drawing anaglyphs by lifting points and curves using the script more to look up some of the details and tutors who first read the script and tried to solve the problems by using the methods described there. The first type of tutors led their workshops in a more explorative experimental way; some of the second type tutors had inserted into their workshops small lectures on the basic of the script. The students

asked questions from different perspectives: from a phenomenological perspective (What are conditions for two points to create the illusion of a floating point?), from the perspective of geometrical invariants (Which properties do not depend on the position of the centre of the projection?), from the perspective of geometric transformations (How to place the figure to support its three-dimensional illusion?).

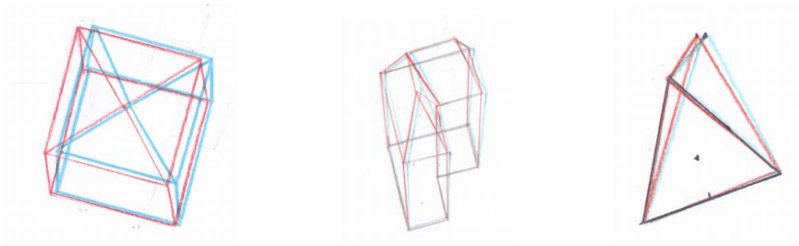

Figure 6: Examples of student products: a box with concave top, a house, a tetrahedron.

Nevertheless, the students of all groups of the day course were very motivated to understand how to draw a 3D picture, experimented with pencils, lifted points and produced their own 3D images (Fig.6). During our reconsiderations of the first repeated workshops, we tried to describe the atmosphere when the students started to construct central projections and calculate distances. At the beginning - may be because of our technical explorations into the past - we called it the mind of engineers and inventors. During the next workshops non the less we realized that a substantial part of workshops the students were engaged with thought experiments: projections through transparent tables, cutting up spheres, building houses out of cubes and tetrahedrons, rotating trefoil knots. This led us to think about the role of thought experiments in physic and mathematic lessons. May be it was so easy to inspire so different students for experiments because there were not real experiments? Students meet nowadays in their everyday life not often people who both are said to be cool and decompose technical devises in order to understand basic functional principles. Choosing advanced courses in science for A-levels does not imply one has grown up with a soldering iron or a chemistry box.

One could think that our students good performance and experiences in virtual worlds could give the historical extremely important thought experiment a new place in physics lessons in order to develop interest in functional principles even if there are only very limited prior technical experiences. But at least in German physic lessons, it seems to be rather the other way around: Many computer based visualisations and experiments transform the very nature of the thought experiment and replace it by a confirmation experiment of a virtual programmed reality (as it is also done in the earlier discussed mathematics textbook task using the applet to compute the anaglyphs).

May be it became rather a task for mathematics educators to integrate thought experiments and virtual engineering into mathematics teaching to support the perspective: How does it work?

Conclusion

The fact that the participants in our workshops were able to pose independent research questions in geometrical terms with geometrical meaning is for us an indication of their development of a geometrical language and concept. The students gave their proofs by construction and in different geometrical notions using invariants and similarity mappings. We were especially impressed by the ability to switch between plane and spatial perspectives on the stereographic pictures developed by the participants during the workshops.

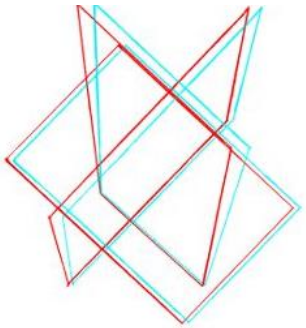

Figure 7: Borromean rings made of three golden rectangles.

Acknowledgment

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Teaching kinematics using mathematics history

Alfredo Martínez, Olimpia Figueras and François Pluvinage

Centro de Investigación y de Estudios Avanzados del IPN, México

alfymago@hotmail.com; figuerao@cinvestav.mx; fpluvinage@cinvestav.mx

This article describes a proposal for teaching kinematics using mathematics history; recognizing that history, mathematics, physics and language share similar knowledge structures that must be articulated to achieve teaching-learning processes. To account for this, we discuss how mathematics and physics has historically contributed to describe motion representations. A learning unit is described based on the experimental technique that probably Galileo used to measure times and distances and is not well known by textbooks' authors. This experimentation with intervention tries to interrogate students' alternative ideas. We intended to provoke a conceptual change using several mathematical and physical representations that have been constructed throughout history and have allowed describing the nature in a better way. A qualitative study of the experience in a physics class was carried out. Forty students aged about 13-years-old from a public secondary school in Mexico City participated in the study. The learning unit described in this article was the first stage of a broader study constituted by five stages. It was important to work with students to establish the relationship between the variables involved in the description of uniformly accelerated motion, so as the ratio between the distances travelled and the time elapsed.

Keywords: Conceptual change, alternative conceptions, acceleration, uniformly accelerated motion, mathematics and physics representations

Introduction

The role of the history of mathematics in teaching and learning mathematics has been discussed in several workshops (Fauvel & Van Maanen, 1997). However, Fried (2007) has pointed out a major problem: Where do teachers find time to teach the history of mathematics? It isn't easy to answer this question, but given that both history of mathematics and mathematics themselves embrace genuine ways of knowing, it may be possible to make a didactic design that combines mathematics with valuable elements of history of mathematics without investing too much time.

It is important to recognize that mathematics, history and language are ways of knowing related to semiotic systems that are used to understand or interpret any kind of knowledge. Therefore, we consider the problem presented above as our own, that's why we propose an approach that has been tested with 13-year-old high school students, to give an example with the intention to contributing to answer the previous question.

Acceleration concept is an example of the relationship between our ways to understand nature and to interpret it. It is well known, since ancient times, that interpretations had been made to explain acceleration as a physical phenomenon without being able to describe it fully. Unfortunately, the nature interpretations that human beings do are determined by common sense, this situation causes obstacles to describe it correctly in Bachelard and Viennot's sense (cited respectively in Jankvist (2009), and Bastién, Mora & Sánchez, 2013). Obstacles to understand sciences have been called in several ways associated with different epistemological origins, however, authors of this paper agree

with Halloun & Hestenes (1985), Hierrezuelo & Montero (1989/2006), Laburú & Carvalho (1992), Duit & Treagust (1998), and others who have contributed to define obstacles as alternative conceptions. Ideas that are related to common sense, which are persistent, shared by many persons and can be adapted and modified in such a way that one can believe that those ideas are adequate to explain the reality of the physical world.

Some ideas in science history have been considered satisfactory because they explained, at least in part, a field of reality, but others, much more powerful, creative and audacious, have emerged to break paradigms and challenge common sense. Such are the ideas of those who have not been satisfied with the accepted description of reality at a certain historical moment.

The so-called scientific revolutions allowed the advance of science and a changing in our understanding of the universe. An analogy of these scientific revolutions is found in the processes of teaching and learning, as well as alternative conceptions that are compared, but if those ideas cannot be used to explain reality, they lose their validity. Anomalies then arise, which will open the way to new explanations, often more complex, but those ideas could be used to explain a situation apparently in a better way. This process has been called conceptual change (Posner, Strike, Hewson & Gertzog, 1982).

If ideas are images and representations of reality, then in so far as representations are tested, the validity of the ideas is proved. Mathematics makes possible the testing of diverse representations of distinct levels of complexity, so concept formation can be achieved in an increasingly structured way (Pozo & Flores, 2007).

Considering that it is necessary to challenge the ideas of the common sense, Galileo made experiments on the acceleration leaving aside the causes of its origins. Galileo in his *Discorsi e Dimostrazioni Matematiche* (Hawking, 2013) explained some experiments and his description about uniformly accelerated motion, but he did not explain his efforts and trials to obtain satisfactory results or how he accurately measured time or displacement. Drake (1975) researched these problems by reviewing Galileo's notes and pen strokes to discover clues as to how he did his research.

This article deals with the description of a didactic intervention using a teaching experiment in which Galileo’s technique is used to describe uniformly accelerated motion. The didactic activities allow the students to interrogate some alternative conceptions, such as the idea that, in a uniformly accelerated motion equal distances are travelled in equal times (Laburú & Carvalho, 1992). Relationships between experiments have also been established, data organized in tables and qualitative diagrams to propose a mathematical expression that allows to relate distances and times; with the intention to promote a conceptual change.

Historical development of the kinematical concept

The first ideas about motion came from Aristotle who in his dialogue On Philosophy uses the term *proton kinoun* as the first motor, the cause of every motion in the Universe (Düring, 1990, p.188). According to Aristotle, the velocity of a moving object is directly proportional to the thrust force and inversely proportional to the media resistance (Düring, 1990, p.477).

It was not until the 14th century, when mathematicians from Oxford (most of whom came from Merton College), took up again the study of motion of bodies. This was mostly done in the decade between 1330 and 1340, as explained by Farmaki, Klaudatos & Paschos (2004).

William Heytesbury (1313-1400), Richard Swineshead (1340-1354), and John Dumbleton (1310-1349), mathematicians and logicians of Merton College at Oxford, known as "Calculators", introduced the idea of functional relationships in attempt to describe magnitudes with quantitative measurable features. They defined several kinds of motion, proposed theorems concerning motion and proved them mathematically, using Euclidean geometry. Swineshead defined uniform motion, and Heytesbury the uniformly accelerated motion (Farmaki, et al., 2004, p. 506).

Nicole Oresme in 1362, in the *Configurationibus qualitatum* represented the variations of qualities (see Figure 1), such as velocity and time, by means of geometrical figures, in which the line AB represents the time and the perpendicular lines the speed or its increasing value (Farmaki et al., 2004, p. 507).

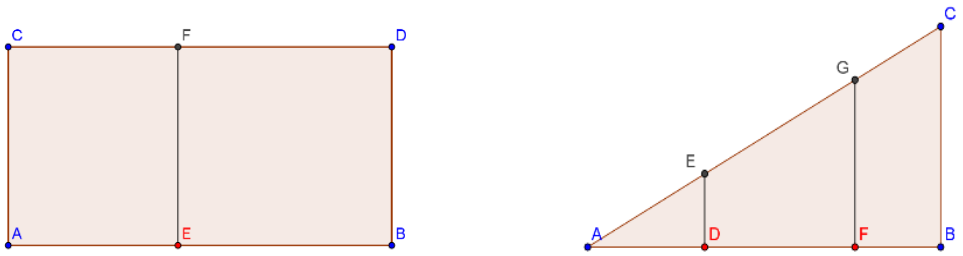


Figure 1: Oresme representation of a qualitative magnitude
(Adapted from Farmaki et al, 2004)

There were many efforts to solve the problem of the unequal velocities that appear in a uniformly accelerated motion, however they were not concretized. It was not until Galileo's deduction presented in the *Discorsi*, where he artfully simplifies the free fall of a ball rolling down along a plane with a few degrees of inclination.

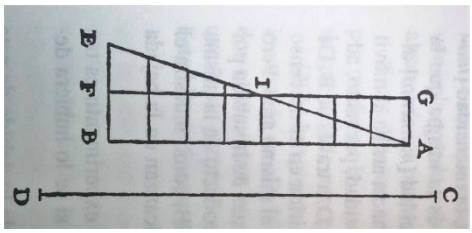


Figure 2: Theorem I, Proposition I, Third journey in
Discorsi e dimostrazioni matematiche (Hawking, 2013, p. 468)

In Theorem I, Proposition I, Third Day of the Discorsi, Galileo states the relationship between velocity and time, using schemes such as those used by Oresme, but replacing uniform motion with uniformly accelerated motion (Hawking, 2013). In Figure 2, time is represented as a quality by the line AB and velocities are represented as intensities by perpendicular lines.

The schemas used by Oresme and Galileo suggests that the relation between time and velocity could be drawn in a Cartesian coordinate system, but in Galileo's scheme there is also a line

representing an external magnitude, the distance. In the Proposition II, third day of *Discorsi*, Galileo explains the proportional relationship between time and distance travelled in a uniformly accelerated motion, including the relation between time and velocity. But the method reported by him to measure time was not considered practical, in accordance with Drake, who estimated that Galileo achieved more accuracy with another method than would have been possible with the clepsydra. So, there is a controversy about how Galileo found these results. Galileo just wrote:

> For the measurement of time, we employed a large vessel of water placed in an elevated position; to the bottom of this vessel was soldered a pipe of small diameter giving a thin jet of water, which we collected in a small glass during the time of each descent, whether for the whole length of the channel or for a part of its length; the water thus collected was weighed, after each descent, on a very accurate balance; the differences and ratios of these weights gave us the differences and ratios of the times, and this with such accuracy that although the operation was repeated many, many times, there was no appreciable discrepancy in the results (Galileo Galilei, 1633/1914, p.179).

Stillman Drake's proposal

In "The Role of Music in Galileo's Experiments", Drake (1975) described an experiment that probably took place in 1604. The experiment was part of a series of investigations that led Galileo to obtain finally the correct rule of times-squared being proportional to the distance an object falls from rest during the time elapsed. Ben Rose reconstructed the experiment according to specifications supplied by Drake (see Figure 3).

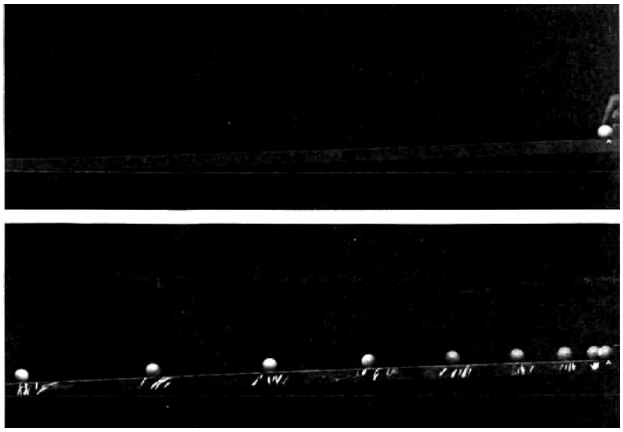


Figure 3: Reconstruction of Drake's experiment (Drake, 1975)

The objective of this modernized test was to measure as precisely as possible the distances travelled from rest by a ball rolling down on an inclined plane taken at the end of eight equal time intervals. The grooved inclined plane used in the reconstruction was 6 ½ feet long (198.2 cm) and was set at an angle of 1.7 degrees. The time intervals were established at a tempo almost two notes per second. At one note the ball was released, and the positions of the ball at subsequent notes were marked with a chalk; for comparison the exact 0.55 second positions were also captured by multiple-flash photographs. A rubber band was then put around the plane at each chalk mark. The positions of the rubber bands were adjusted so that the audible bump made by the ball in passing each band would always coincide exactly with a note; the bumps were visualized by leaving the camera shutter open during an entire run. Finally, the distances between pairs of adjacent bands were measured. The

ratios of the successive intervals were found to agree closely with a set of figures logged by Galileo (Drake, 1975).

Methodology

In Mexico as in other countries, the role of textbooks is very important for the teaching-learning process from primary school to the first university-level courses. The first Physics course is offered in secondary education to 13-year-old students, where conceptual approach prevails. We decided to work with a group of forty students who attend a high school physics class in Mexico City to make a qualitative study of their reactions to a teaching sequence proposal. This group had no teacher, so the learning unit did not conflict with the daily activities related to the ongoing study of curriculum contents.

After having done an analysis of a Mexican textbook (Gutiérrez, Pérez & Medel, 2012) of the secondary school on the proposed activities to study the topic of acceleration, it was decided to make a design of a learning unit complemented by a physical experimentation. First, the students had to perceive with their senses how the ball falls that is rolling down on an inclined plane and, as it rolls, rings bells placed at equal distances.

Students were asked how distances were covered by the rolling ball between successive rings of the bells, how time between each ringing of the bell varied, and to make a personal description of the rhythm they had heard. They modified the height of the inclined plane and the distances between bells to find out if they perceive changes in the phenomenon.

At the end of this experiment, students wrote their observations in a notebook. The following student's activity was designed with had the purpose of recalling the meaning of proportionality in a context of a recipe to make a three-layered cake. They had to find the proportion of the ingredients with respect to the amount of cake that had to be prepared. Students were asked how they found the proportion and what the proportionality constant for each quantity was. The amounts of ingredients provided were given in kilograms.

Finally, students were asked to carry out a different measuring process to find the times and distances travelled by a ball that rolls down on an inclined plane. The instructions for assembling the device in the laboratory were given to the students. The teacher previously tested the assembly to control the time in which distances were covered by the rolling ball. The material used consisted of an aluminium rail, a steel ball, a universal holder, burette clamps, bells and clips.

The learning unit took place in a week. All sessions were video recorded. Students wrote their answers in a booklet guide prepared by the researcher, those answers were later analysed.

Description of physical experimentation

The material was placed on working tables in the laboratory for each team. The teacher had a digital metronome on the computer connected to an amplifier so that all students could hear loudly the rate of 60 beats per second.

The experiment started when the teacher activated the metronome; then the students had to synchronize the beat of the metronome with their mental count. A student in each team then had to drop the iron ball from the top of the aluminium rail upon hearing a beat (of the metronome) and

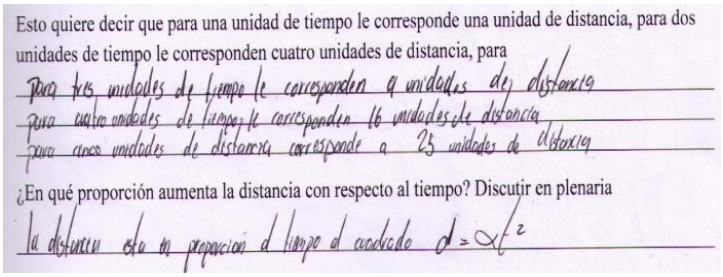
stop it upon hearing the next beat. The first distance should be considered as a unit. This action is repeated several times to determine the distance travelled in a second, which has to be marked each time with a permanent marker pen. The process continued in basically the same manner; students must drop the iron ball from the beginning of the rail, just when hearing a beat and stop it after having heard two beats making a mark on the rail. The same has to be done with three, four and more beats.

Once the distances travelled per second by the rolling ball were determined, students had to put a bell in each marked position. To verify that the travelled distances were correct, they dropped the iron ball while the metronome marked the time, so that at each beat of the metronome a corresponding bell rang.

Results

With the help of the written guide, students took notes on what they had observed. In the first activity, 17 out of 40 students answered that the distances travelled between successive bells were equal while the time intervals were getting shorter and also described that the ringing of the bells during the journey of the ball were getting faster and faster. In the activity to review the proportionality issue only three out of 40 students could find the factor to determine the correct amount of ingredients for each cake.

A small exercise was included, which consisted in completing some arithmetic progressions, specifically a succession of integers, odd, even, and finally squares. The purpose of this exercise was to serve as a heuristic to conjecture that in a uniformly accelerated motion the distance travelled increases as the square of the time intervals do.


Esto quiere decir que para una unidad de tiempo le corresponde una unidad de distancia, para dos unidades de tiempo le corresponden cuatro unidades de distancia, para

Para tres unidades de tiempo le corresponden 9 unidades de distancia
Para cuatro unidades de tiempo le corresponden 16 unidades de distancia
Para cinco unidades de distancia corresponde a 25 unidades de distancia

¿En qué proporción aumenta la distancia con respecto al tiempo? Discutir en plenaria

la distancia esta en proporcion al tiempo al cuadrado $d = \propto t^2$

This means that for a unit of time corresponds a unit of distance, for two units of time corresponds four units of distance, for . . .

For three units of time correspond to 9 units of distance.
For four units of time correspond to 16 units of distance
For five units of time correspond 25 units of distance

In what proportion does the distance increase with respect to time? To discuss in plenary with the wholegroup.
The distance is in proportion to square time $d = \propto t^2$

Figure 4: Relationship between distances and time: a student's conclusion

Then with the intention of analysing the information provided by the previous activity, the teacher asked the students in a plenary session, what relationship was observed between elapsed time and distances travelled. Students answered for each unit of time elapsed, that the distance travelled increased as the square time intervals, thus establishing a proportional relation between both magnitudes associated with uniformly accelerated motion.

The teacher introduced for the first time the algebraic symbol ($\propto$) to denote the correspondence rule which is used in physics when the proportionality constant is not fully determined. Finally, it was found that 16 out of 40 students gave incomplete arguments about time measurement and distance travelled, but 24 of 40 answered that the rhythm of time was constant while distance increased faster and faster as square of the time intervals (see Figure 4).

Conclusions

The learning unit described in this paper was important as a starting point of a larger study involving several phases of experimentation, because it allows a perceptual approach to uniformly accelerated motion.

The activity provides a simple way to engage students with the task of describing uniformly accelerated motion. They associated the problem with Galileo's experiments and were able to identify how nature is scrutinized in order to describe and understand it.

Experimentation allowed students to be challenged by the alternative conception which assumes that in a uniformly accelerated motion the velocity is constant.

This procedure avoids the problem of synchronization with several timers placed simultaneously at equal distances, an impractical measure option that is often suggested in textbooks. Another advantage is that students can listen to a metronome carefully and measure time mentally, as is done in music.

The students realised some difficulties Galileo may have faced in order to measure time and determine their relationship with the distances travelled, providing both experimentation and the learning unit design, a learning context that is much more meaningful than any anecdotal knowledge.

Finding the proportionality relationship between elapsed time and distance travelled may be enough as a first approach to describe a uniformly accelerated motion for being used when necessary to determine more precisely the proportionality constant of a uniformly accelerated motion.

The results shown in this research give an account of how the obstacles attested by history for the understanding of scientific concepts help overcome the obstacles that students face.

Our didactic proposal was enriched with the development of ideas in the history of mathematics and physics by adapting them to provide a different way of accessing to physics and mathematics knowledge. In this didactic design, we consider the genesis of the concept of uniformly accelerated motion throughout history as an example of the inductive thinking that human being does in the construction of their own knowledge, so it has been tested with students, getting encouraging results.

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The concept of infinity – different meanings through the centuries

Regina D. Möller[1] and Peter Collignon [2]

[1]University of Erfurt, Germany; regina.moeller@uni-erfurt.de

[2]University of Erfurt, Germany; peter.collignon@uni-erfurt.de

The concept of infinity and its use is one with different meanings through the centuries within various contexts reflecting mathematical historical development. This development is scarcely clear to pupils during school time and rarely stressed in teacher education although it offers a lot of potential to understand mathematics. The field of arithmetic is one example in which to study infinity within a range that student teachers are able to understand and that is useful for and in their future teaching. The paper focusses on the potential of this concept within the arithmetic field using an original article of Cantor and on examples, also from Hilbert, that stress different counting methods and various illustrations of infinity.

Keywords: Infinity, countable set, denumerable set, very small/large numbers.

Introduction

Like many other mathematical concepts ideas of infinity developed and diversified through the centuries. Well known is its appearance within the Elements of Euclid within the ninth chapter: There are more prime numbers than any given number of prime numbers, cf. Euclid (1997, IX, §20). This statement answers the question, if there is an end of prime numbers. The proof uses the fact that the product of arbitrary prime numbers added by 1 has a new prime number in its prime factorization. That is, to conceptualize the concept of infinity of prime numbers Euclid uses the idea that there is always another one apart from the given ones which has a somewhat operative aspect.

Also within geometric contexts there are ideas about infinity regarding the extensions of space, planes and lines as well as the number of points on lines and figures. There are considerations of the behavior of parallel lines regarding infinity, like for example the parallel axiom states.

Already these examples show that there exist different meanings of infinity depending on the various mathematical objects. Also, the meanings are verbalized without formulization. Doubtless does the use of very large numbers within everyday experiences come close to a sense of infinity. For example, the addition of two numbers, one of which is very much larger than the other, form a sum that does not differ very much from the large number in terms of the relative error concept.

Infinity and the use of it deserves a closer look because terms like “infinite”, “endless” and “unlimited” have a colloquial meaning that sometimes provides a reasonable starting point for the understanding of abstract patterns and sometimes not. In some instances infinity incorporates the imagination of a very large extension, endlessness is for many just a word for a very large but still finite set of objects or a very large extension like the ocean.

First-year students often show a vague notion of the term “infinity”, an observation that was done also by Woerner (2013). She even points out that a thorough understanding of infinity is neither a goal nor is it a step towards understanding mathematics. Dötschel (2011) even finds that the

understanding of infinity does not vary much between teacher students and pupils of secondary level. In school they learned the lemniscate symbol "∞" and used it associated with the limit of sequences, series and – at the best – the differential quotient. At least in principle they know that it is not allowed to use this symbol like a number or a variable in all respects (like it was used partly in the 18th century), but one can observe a high degree of uncertainty. For instance, what does it mean when the differential quotient is interpreted in terms like "zero divided by zero"? And what is the outcome? Standard lectures like the ones for Analysis and Linear Algebra usually do not change the perspective and are continued with refined conceptions of limits and the definition of concepts like that of an infinitely dimensional space. Concerning the cardinality of sets there remains often the sketchy notation $|A| = \infty$ or the like.

Precisely because counting, the determination of a number, is a fundamental concept at all school levels, it is surprising that the issue of cardinality seems to be somewhat neglected. Nevertheless, it is not too difficult to provide a base of knowledge due to Georg Cantor. In many cases original mathematical treatises are definitely unsuitable within teacher education, but there are notable exceptions. One of these is the article of Cantor (1895), which is understandable in large parts. The reason for this is that students are able to acquire understanding with supporting examples on the side of the teacher which will be shown later in the paragraph.

Our paper describes the used examples during the seminar held in the summer term 2016 together with some details around the concept of infinity.

Various ways towards an understanding of infinity

The mathematical education of future elementary math teachers at the University of Erfurt includes among mathematical survey lectures a seminar on the basic principles of arithmetic and algebra. One of the goals of this seminar is to improve the student's understanding of (natural) numbers and their properties since this field will constitute one of the bases of their future teaching. During the seminar students are often encountering the question in how far they have an understanding of the set of natural numbers being infinite. The upcoming discussions circle around the question how one could find out "what is true". And the discussion ends with the question why. The phenomenon of infinite many natural numbers very often brought about an astonished attitude on the part of the students who pondered about the reasoning of Aristotle: There will be always one number that can be added.

In our seminar there was a focus on aspects of numbers, especially of the natural numbers. Apart from the Peano axioms, the aspect of ways of counting provides a reliable foundation, especially since the cardinalities of sets may be included. This approach promotes a formalization due to the concept of bijective mappings, which is of value in its own. The natural numbers can be regarded as cardinal numbers of finite sets. But what do students know about the cardinality of the set of the natural numbers? Being asked, students claim "infinite" and denote the lemniscate symbol which they know from limits of sequences or functions. In interviews they show an obvious uncertainty about the arithmetical behavior of the object "infinite". In case of doubt they often suggest treating it like a "usual" number.

We feel responsible for giving a brief insight into the cultural heritage of the different approaches dealing with "infinity". In particular we seek to show with what kind of caution Euclid, and Cantor

too, got closer to proper descriptions, depending on the actual contexts. Especially we are interested in giving insight into the richness of mathematical thoughts and ideas: "It is possible to regard the history of the foundations mathematics as a progressive enlarging of the mathematical universe to include more and more infinities" (Rucker, 1982, p. 2). With regard to Cantor, we know that "... soon obtained a number of interesting results about actually infinite sets, most notably the result that the set of points on the real line constitutes a higher infinity than the set of all natural numbers. That is, Cantor was able to show that infinity is not an all or nothing concept: there are degrees of infinity." (Rucker, 1982, p. 9)

There are a lot of ways how to understand mathematical statements. Some point out their proof, some stress their genetic development, some point out their formal argument. Our actual understanding of infinity allows us to give statements like: the set of natural numbers or the set of natural numbers between 0 and 100. The first is an infinite set, the second is a finite one. Stressing the idea of a potential infinity which we could not grasp as a solid concept, we help ourselves by a stepwise approach knowing that we will never succeed. This very constructive standpoint or procedure permits a very simple activity and that is adding one, again and again: |, | |, | | |, … In this manner one can distinguish finite and infinite sets. For the first set the procedure ends with a certain number, for the second one there is no certain last number and it becomes clear that the procedure never ends. In both cases the counting is mathematically a 1-1-correspondence.

The different meanings of the term infinity show the richness of mathematics and its historical development. Needless to say that mathematical history does not develop in a regular and uniform way (Dieudonné, 1985, p. 16). Some epoch does not show any development in a field, in some there is a continuous change because of new developments. The fact that we use the word infinity the way we do with numbers goes back to Cantor (1895). It was the upcoming of new ideas, e.g. the idea of a set that changed the understanding of infinity.

How very much different this meaning is in contrast to the “old” Greek meaning shows when student teachers learn about it in their first mathematical lectures: infinity is hard to grasp and the use of it shows that school mathematics does not at all build a proper foundation. Because of its lack it is even more important to build a solid understanding during mathematical studies especially for student teachers as there are many potential links to basic notions of counting in their future teaching.

German mathematical education often refers to three basic experiences (“Grunderfahrungen“), by Winter (1996):

- perceiving phenomena of nature, society and culture;
- knowing (and appreciating) mathematical issues, represented by language, symbols, images and formulas;
- acquiring heuristic competencies.

We like to refer to Winter (1996) because he stresses a connection between everyday life experiences, heuristics and beginning formalization. In order to get aware of basic experiences and deepen the understanding there is a strategy necessary that gets students involved. Kattou et al. (2009) points out:

> In particular, academic programs offered to teachers should include mathematical knowledge regarding to infinity in combination with instructional approaches related to the concept. A proposed teaching approach could include the following steps: presentation with several typical tasks aimed at uncovering teachers' intuitions about the concept, discussion about infinity's applications in real life, introduction of the formal definition of infinity and the two aspects- potential and actual- and attempt to distinguish them in examples. (Kattou et al.2009).

Within this this context we pinpoint the following aspects:

1. The notion of infinity changed its meaning through the centuries. In the late 19th century the notion of aleph 0, aleph 1 and so forth came up.
2. The way infinity appears in mathematical textbooks follows the idea of Freudenthal's anti-didactical inversion. It is common to introduce infinity by using the lemniscate symbol, mostly just informing about it. The mathematical developments are neglected.
3. Some examples of infinite sets can be solved with simple steps used with finite sets. This presents an approach with a low barrier to student teachers and enhances their understanding of infinity.

Our didactical approach is influenced by Vollrath (1987) who proposed a phase model showing the process of understanding mathematical concepts:

$$\text{intuitive and content-related} \rightarrow \text{formal / integrated} \rightarrow \text{critical}$$

We therefore stress the finding of variations of standard examples and of counting strategies on the side of the students. The integration of Cantor's text provides some formalism and fostered discussions about the historical circumstances which were not controversial, that is the conflict between Cantor and Kronecker e.g.

The following paragraph presents examples that proved useful within elementary school teacher education.

Methods of counting

In many cases it turns out difficult to provide an original text to students with the expectation of an adequate comprehension. But just mathematical topics that lead to very fundamental issues may prove appropriate in order to their connection with intuition and imagination. Cantor (1895) develops a concept of elementary set theory, which includes transfinite cardinalities and their arithmetic properties. During teaching it became evident, that important parts of this text are quite understandable and can be an opportunity to discuss an historical treatise and express own reflections.

Before we start investigating into various ways of counting infinite sets we observe that there is no uniform definition of the concept infinity. The word occurs as an adjective to characterize sets especially. We concentrate therefore on the arithmetic field.

The following sections present a couple of examples that may foster the understanding of infinite sets.

Counting as one-to-one correspondence

The concept of a set, as introduced from Cantor (1895), surely fits into these frameworks. To him we owe the so-called "naïve" definition of a set.

By an "aggregate" we are to understand any collection into a whole *M* of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of *M*.

A counting or numerating of a finite set M with exactly *n* elements means, that every number 1, 2, 3, …, *n* is assigned to exactly one element of the set. This is linked to the concepts of maps and functions and more over bijectivity.

The set ℚ is countably infinite

This follows out of a scheme in which every positive rationale number shows one time and is arranged like this:

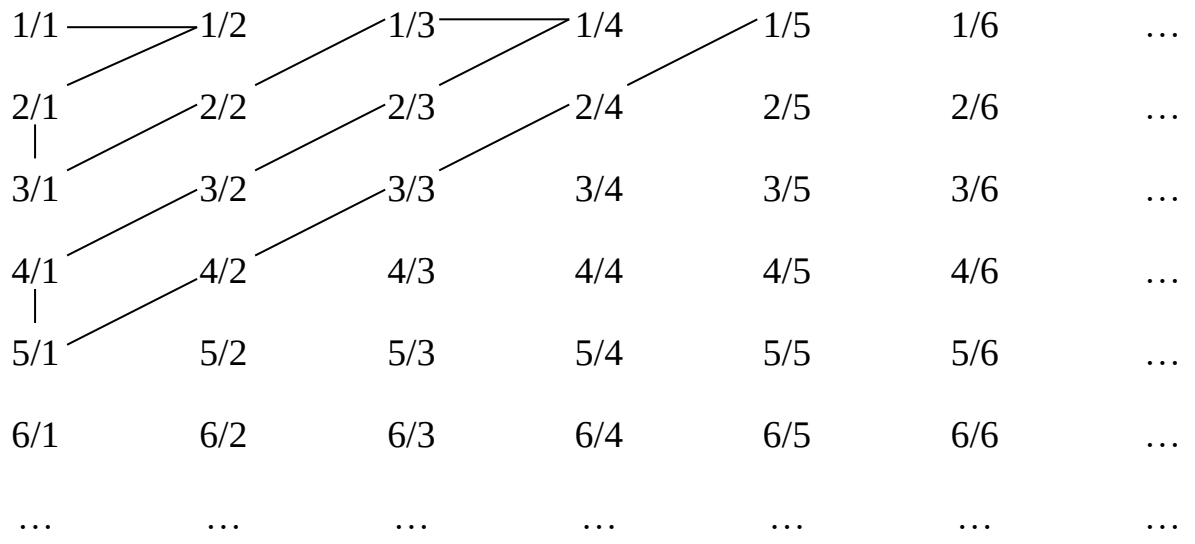

Table 1: Cantor's first diagonal method

The way the scheme is counted goes back to Cantor and is called the "diagonal method".

Following the presentation of the scheme students were invited to vary it and write down their proposals. Are there other suitable paths? What do they have in common? Furthermore, how could repetitions of numbers be avoided? In the scheme above every positive rational number is repeated infinitely. Does this cause problems? What options do we have to be represented by a reduced table of fractions? Is this already an indicative of the countability of even "larger" sets? After all most students could design various methods for counting even all rational numbers, for instance by designing spiral paths or the like.

The set ℝ is uncountable

Suppose that there is an enumeration

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...$$

of the interval [0, 1[, which is a subset of $\mathbb{R}$, and the numbers are represented in the decimal system, i.e.

$$\alpha_1 = 0,\underline{\alpha_{11}}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}.....$$

$$\alpha_2 = 0,\alpha_{21}\underline{\alpha_{22}}\alpha_{23}\alpha_{24}\alpha_{25}.....$$

$$\alpha_3 = 0,\alpha_{31}\alpha_{32}\underline{\alpha_{33}}\alpha_{34}\alpha_{35}.....$$

$$\alpha_4 = 0,\alpha_{41}\alpha_{42}\alpha_{43}\underline{\alpha_{44}}\alpha_{45}.....$$

.....

with digits $\alpha_{i1}, \alpha_{i2}, \alpha_{i3},... \in \{0, 1, ..., 9\}$ for every positive natural numbers *i*.

Now one can define a number $\beta = 0,\beta_1\beta_2\beta_3... \in [0, 1[$ such that

$$\beta_i = \begin{cases} 1, & if \ \alpha_{ii} \neq 1 \\ 7, & if \ \alpha_{ii} = 1 \end{cases}$$

for all positive integers *i*. Obviously, the representation of β possesses at least one decimal digit that differs from α_i, namely, the *i*-th digit. Therefore β cannot occur in the enumeration above, which is inevitably incomplete. Now, if the given interval is already uncountable, then all the more the real numbers are. This scheme originates from Cantor, too, and is called the second diagonal method.

To promote an adequate understanding, students did vary this scheme in a written form, also regarding other *b*-adic representations. At this point, the fundamental significance of place value systems in general is to be clarified. During teaching lessons students were encouraged to replace the digits by other symbols such as letters or notes from sheet music etc., and it has become clear, that the relating interpretations ("the entity of 'texts' is uncountable") can foster an adequate understanding in the sense that students are able to make a transfer.

Hilbert's Hotel

The cardinality of the set of the natural numbers is denoted by $\aleph_0$. In set theory several properties of this first transfinite cardinality are elaborated, as there are

$$1 + \aleph_0 = \aleph_0,$$

$$n + \aleph_0 = \aleph_0$$

for any $n \in \mathbb{N}$, as well as

$$2\aleph_0 = \aleph_0$$

and

$$n\aleph_0 = \aleph_0,$$

again for any natural number *n*. To illustrate this, the thought experiment of Hilbert's hotel is helpful: Suppose that there is a hotel with an unlimited number of single rooms, which are numbered according to the natural numbers. The hotel is fully occupied and one other person is knocking on the door. Will the hotel be able to accommodate this person, too? In the classical version each present guest moves up to the room that is numbered one greater as yet. In this way the

first room (numbered by 0 or 1) becomes available and no one has to leave the hotel. The situation is very similar if two or a finite number of new guests ask to come in: The present guests move up in the rooms that are numbered *n* greater than now.

A bit more challenging is the arrival of a "Hilbertian bus" with an infinite number of passengers, named or numbered due to the natural numbers. In this case a constant moving up will not be successful. But the past guests could double their initial room number, and every passenger gets an oddly numbered room. This is not the only option available, students should contribute alternatives. More general, if there are two or a finite number *n* of "Hilbertian busses", one can multiply every original room number with $n + 1$ and assign the passengers of the first bus those rooms, which numbers are congruent $n + 1$ modulo 1. The occupants of the second bus move into the rooms that are numbered by natural numbers congruent $n + 1$ modulo 2 and so on. Students are expected to formulate a proper mapping rule and to come up with their own ideas relating alternatives.

Where is the border line? Even a "Hilbertian bus-fleet" of infinite number of "Hilbertian busses" numbered according the natural numbers, is still not able to overstrain the hotel. For example, one can assign a double index to every passenger due to his bus number and his seat number within this bus. Now Cantor does the work by applying his first diagonal method to this matrix structure. Also here students could consider a formula or a formal description of an algorithm.

Students varied the above solutions in several and diverse ways. For example, prime numbers were used and alternating methods of simultaneous counting. Of course, the most important task is the clarification of the impossibility of lodging an uncountable amount of recent arrivals.

The above considerations go along with the equation

$$\aleph_0 + \aleph_0 + \aleph_0 + \ldots = \aleph_0,$$

where the number of the summands on the left side is countable.

The given examples above, which were part of the studies of our futures teachers, have certain potential to support understanding.

Conclusions with respect to understanding the concept infinity

We referred to Cantor especially when we stressed the 1-1-coresspondance (or mapping) and some insights of arithmetic rules including infinity. Since all examples are rather basic but initially unknown to most of the students they gained competencies with counting and the notion of bijectivity. It is important to realize that the arithmetic rules known from the basic arithmetic operations may vary, depending on the context. Another example, but in a different relationship is the "double distributivity" in case of unions and intersections of sets. The phenomenon "infinity" holds in itself ambiguities which contradict common sense at first glance. It is of great educational value to become acquainted with some of them, namely in two respects: in terms of general education, which should be a concern of mathematics education and for the purpose of educating "good" teachers. The well-educated primary teacher is then in the position to react properly when children ask smart questions or questions that show insight but do not use proper wording. Pupils occasionally may achieve even philosophical significance – so long as the teacher recognizes its meaning.

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Analyzing some algebraic mistakes from a XVI century Spanish text and observing their persistence among present 10th grade students

Antonio M. Oller-Marcén[1] and Vicente Meavilla[2]

[1]Centro Universitario de la Defensa de Zaragoza, Spain; oller@unizar.es

[2]Universidad de Zaragoza, Spain; vmeavill@hotmail.com

The use of symbolic language and the resolution of equations are of great curricular importance and often cause difficulties to our students who find several obstacles. These obstacles can be of different nature and epistemological obstacles can sometimes be traced back to ancient mathematical texts. In this paper, we will focus on one of the first texts with algebraic content that was published in Spanish, the Arithmetica practica, y specvlatiua by Juan Pérez de Moya. In particular, we will present the mistakes made by the author when solving equations of the form $ax^m = bx^n$ *and we will analyze and give explanations for some of them. Furthermore, we will observe how 10th grade students face these types of equations and if some of the old arguments are still present among our students.*

Keywords: Algebra, mistakes, obstacles, history of mathematics education, 16th century.

Introduction and objective

Obstacles to learning can be (Brousseau, 1983) of different types according to their origin. They can be ontogenetic, didactical, epistemological or cultural. An obstacle is epistemological if it is independent from the teaching practice. Brousseau suggested (Chorlay & de Hosson, 2016) that a distinctive characteristic of epistemological obstacles is their appearance in mathematics "from the past". It might be true that in ancient texts we barely find traces of the errors, difficulties and failures associated to mathematical creation (Cid, 2000). However, in some kind of texts, devoted to teaching or to the introduction and dissemination of knowledge it is sometime possible to find non-trivial errors[1].

The use of symbolic language and the resolution of equations are of great curricular importance and often cause difficulties to our students (Socas, 2010). Consequently, it can be of interest analyze ancient algebraic texts searching for possible mistakes or misconceptions.

The first algebraic texts written in Spanish appeared during the second half of XVI century. The first one was the *Libro primero, de Arithmetica Algebratica* (Aurel, 1552) written by a German living in Valencia (Puig & Fernández, 2013). The first text with algebraic content written by a Spanish author was (Meavilla, 2005) the *Arithmetica practica, y specvlatiua* (Pérez de Moya, 1562).

In this work will identify some mistakes in Pérez de Moya book. For some of them we will try to give plausible explanations and furthermore, we will see how present 10th grade students face the type of equations where those errors arise.

[1] By non-trivial errors, we mean errors that imply cognitive difficulties of some degree; i.e., we do not take into account small arithmetical mistakes, typos, etc. that are also usually found in this kind of texts.

Juan Pérez de Moya mistakes

In his book, Pérez de Moya discusses the possible solutions of equations of the form $ax^m = bx^n$ by considering four different cases. In table 1 we summarize the information given by the author in his text:

Case 1	$a = b$ and $m = n$	Unique solution, x=1
Case 2	$a \neq b$ and $m = n$	No solution
Case 3	$a = b$ and $m \neq n$	Infinite solutions
Case 4	$a \neq b$ y $m \neq n$	Unique solution (not given)

Of course, this information is clearly incorrect from a modern mathematical point of view. It is interesting to point out that the same mistakes can be found in different text printed during the same period. For instance, in the *Arithmetica* by Rocha (1565) we find the exact same errors, while in Marco Aurel's previously mentioned work we can find a fragment where the author considers cases 1 and 2 above in the same way as Pérez de Moya.

This fact can of course be explained because many of these authors used the same sources and they read each other. However, it is noteworthy than none of them noticed or corrected them. This leads us to the hypotheses that they did not see them as incorrect and some deeper explanation for them must exist. In this work, we will focus on cases 1 and 2.

Analyzing case 2

In this paper, we are going to focus of case 2 above. From our modern point of view it is rather straightforward that this type of equations have the unique solution $x = 0$. In any case, straightforward as it is, we can try to make some kind of aprioristic analysis of how a hypothetical solver can face one such equation.

Consider, for instance, that we want to solve the equation $4x = 2x$. We can conceive, at least, the following possibilities:

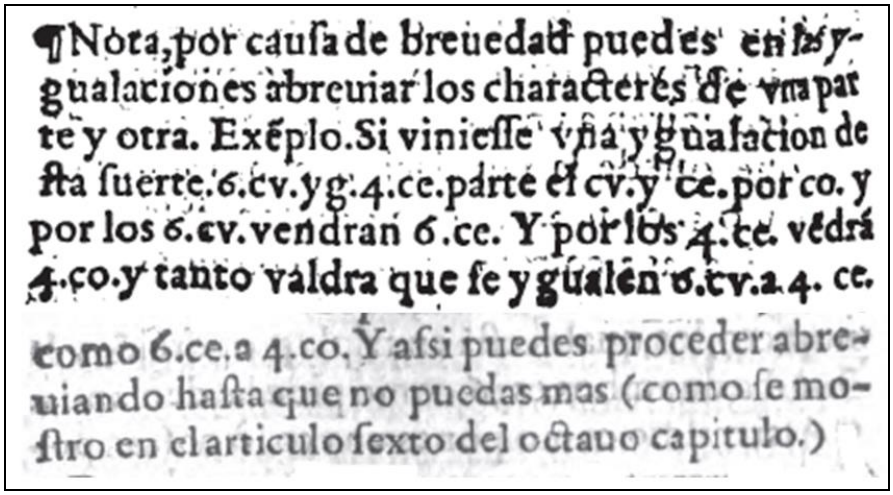
1. The "canonically correct" procedure. Transform the equation into $2x = 0$ subtracting 2x from both sides of the equation and then divide by 2 to obtain the unique solution $x = 0$.
2. A procedure which is "syntactically correct" but "semantically incorrect". Divide both sides by x to obtain that $4 = 2$ and conclude that there is no solution to the original equation.
3. A "materialistic procedure". Four objects of some kind cannot be equal to two objects of the same kind. Hence, the equation has no solutions.

Of course, at that time it was natural not to consider zero as a number so it is reasonable that a XVI century algebraist thought that this equation has no solution at all. Hence, from this point of view, the answer given by Pérez de Moya would not be a mistake at all.

Anyhow, it can be of interest to identify which of the three previous possibilities, if any, was chosen by XVI century algebraists.

In principle, point 1 above was well-known at that time. In fact, this was the procedure used by Pérez de Moya throughout his work as soon as the considered equation had a constant term. On the other hand, Pérez de Moya also used point 2 in his work. For instance (see Figure 1) we can read

(pp. 546-547): "*if you were given an equation like* $6x^3 = 4x^2$*, divide* x^3 *and* x^2 *by* x […] *and the equation* $6x^3 = 4x^2$ *will be the same as* $6x^2 = 4x$*. And thus you will continue until you can no more*".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como 6.ce.a 4.co. Y aſsi puedes proceder abre-
uiando haſta que no puedas mas (como ſe mo-
ſtro en el articulo ſexto del octauo capitulo.)

Figure 1: Dividing by *x* during XVI century

But, in spite of his constant use of point 1 and 2 above throughout his algebraic work, Pérez de Moya did not use any of those arguments when facing equations described in case 2. Instead, he appeals to that "materialistic procedure". In particular, we read (p. 544): "*if 3x were equated to 4x or* $5x^5$ *to* $2x^5$*, in such case, those equations will be impossible and they cannot be done because two reals cannot be the same as three, provided they have the same value*". Moreover, this exact same idea, with nearly the same example can be found in Marco Aurel's work (fol. 78 v): "*three ducats are not worth the same as four ducats since ducat have one only value*" (Figure 2).

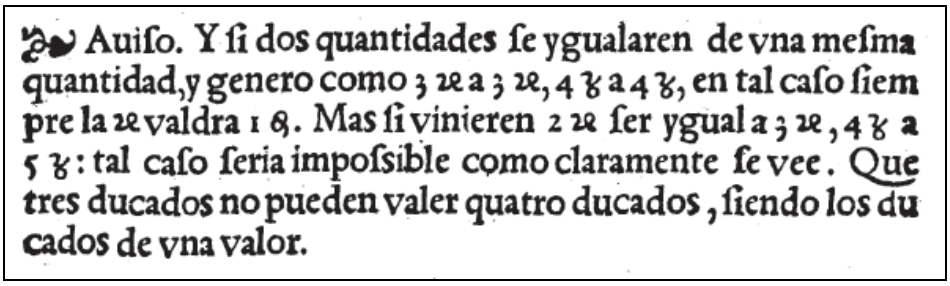
Auiſo. Y ſi dos quantidades ſe ygualaren de vna meſma
quantidad,y genero como 3 ꝛ a 3 ꝛ, 4 ℔ a 4 ℔, en tal caſo ſiem
pre la ꝛ valdra 1 8. Mas ſi vinieren 2 ꝛ ſer ygual a 3 ꝛ, 4 ℔ a
5 ℔: tal caſo ſeria impoſsible como claramente ſe vee. Que
tres ducados no pueden valer quatro ducados, ſiendo los du
cados de vna valor.

Figure 2: Materialistic argument in Marco Aurel's work

As we already pointed out, the first possibility led to the solution x = 0. It is very likely that these authors conceived '0' just like a figure and not like quantity. Since '0' is not a quantity, it cannot be the solution of the equation. Hence, a reasoning like the described in the point 1 above was flawed and could not be used.

Regarding the second possibility, we have seen that Pérez de Moya in fact used this technique. Nevertheless, he is quite imprecise saying that it must be used "*until you can no more*". It is possible that, since in this case this technique leads to non-sense expressions, was also considered flawed and abandoned.

Therefore, from this point of view, the third possibility was the only hope to give an answer to this type of equations and so they used it. As we will see, this idea is still present among our students and it has an interesting explanation (Booth, 1984). The use of this argument implies that the

unknown is conceived as an object itself and not as the representation of a number. Clearly Pérez de Moya and Marco Aurel have this in mind when they talk about "*two reals*" or "*four ducats*" when referring to $2x$ and $4x$, respectively.

Back to case 1

After the previous analysis, we have a more or less clear idea of why XVI century authors could not manage correctly equations of the form $ax^m = bx^m$ with $a \neq b$. Moreover, we understand why they said that they had no solution. Then, we might want to apply a similar reasoning to equations of the form $ax^m = ax^m$; i.e., to case 1 above.

These equations, from a modern point of view, have – trivially – infinite solutions. They are, in fact, what some people call identities. Pérez de Moya, Marco Aurel and other authors provide $x = 1$ as the only solution and, unlike in the previous case, they provide no argument supporting this claim. If we try to apply the three aforementioned possibilities to the case of, say, $2x = 2x$ we would get:

1. $0 = 0$.
2. $2 = 2$.
3. 2 euros are always equal to 2 euros.

All three cases lead to some kind of tautology. To some kind of essential identity of an object with itself as opposed to an accidental identity of the form $2x = 3x^2$. When dealing with aspects regarding these topics we can turn to Aristotle's *Metaphysics* which suggests a genealogy or rational grounding for this answer. In particular, in the chapter IX from Book V we read (Taylor, 1801, p. 122): "*But some things are said to be the same essentially, in the same manner as things which are essentially one. For things of which the matter is one, either in species or number, are said to be the same*". Thus, when facing an essential identity "*the matter is one*" and the answer is the unity as our XVI century authors claim. In any case, as to the historical question of the impact of Aristotle's *Metaphysics* on our XVI century algebraists, this paper is no place to discuss it.

Dealing with cases 1 and 2 today

We worked with 57 students of 10th grade during a 50 minutes class session. By 10th grade, Spanish students should be completely familiar with algebraic language and notation (introduced in 7th grade). They have not systematically solved polynomial equations of degree higher that two, but they know how to use techniques such as taking common factor, etc. We designed a questionnaire (Table 1) that included the four cases treated by Pérez de Moya in his work. In particular, items (1) and (2) corresponded to the cases 1 and 2 discussed above.

Solve the following equations:
(1) $x + 1 = 1 + x$
(2) $x = 2x$
(3) $4x^7 = 4x^5$
(4) $8x^2 = x^5$

Table 1: Questionnaire

Among other aspects, we wanted to analyze how the students faced identities like the one presented in item (1) and equations like item (2) that lead to the materialistic procedure described above.

Item 1

For this equation, Pérez de Moya proposed $x = 1$ as the only solution. The answers given by the students can be classified according to the following categories (Table 2):

- NS: The student says that there is no solution.
- US: The student says that there is a unique solution.
- IS: The student says that there are infinite solutions.
- N: The student does not give an answer.

NS	16 (28%)
US	5 (8.7%)
IS	19 (33.3%)
NDS	17 (29.8%)

Table 2: Answers for item 1

It is clear that answers from the categories NS and US are incorrect. Moreover, 15 answers from the category IS are also incorrect. Thus, only four (7%) students gave a correct answer to this item. From these correct answers, two are worth mentioning:

1. "*Infinite solutions. Because the order of the factors does not change the result and if you sum one to a number is the same as if sum the same number to one*".
2. "*It has infinite solutions, because any value that you give to the x is the same. For instance, $x = 5$, $5 + 1 = 1 + 5$, $6 = 6$*".

The first answer shows that the student has observed the structure (Linchevski & Livneh, 1999) of the algebraic expression and has identified it with the commutative property of addition. In the second answer, we see that the student has the idea that the solution of an equation is a number that leads to an identity when x is substituted by this number.

Among the students in the category US, only one gave $x = 1$ as the unique solution to the proposed equation. His answer was the following:

$$x + 1 = 1 + x \rightarrow x - x = 1 - 1 \rightarrow x = 1$$

The most plausible interpretation for this answer is that the student compared both sides of the second equality and assigned to the symbol on the left hand side (x), the corresponding symbol on the right hand side (1). Of course, we could not expect any XVI century-like reasoning.

In addition to this answer, some other mistakes were found that are worth mentioning:

1. $0x = 0$; $x = 0/0$. "*It has no solution. Dividing by zero gives an irreal* [sic] *number*".
2. $x - x = 1 - 1$; $0x = 0$. "*It has no solution because it gives* $0x = 0$, *that is, there is no*".
3. $x - x = 1 - 1$; $0x = 0$; $x = 0$. "*No solution*".
4. $x - x = 1 - 1$; $x = 0$. "*Infinitely many solutions*".
5. $1 - 1 = x - x \Rightarrow 0 = 0 \Rightarrow x = 0$.
6. $x + 1 = 1 + x \Rightarrow x - x = 1 - 1$. "*No solution, that is, infinite solutions, because if we find zero, it means that there can be many solutions*".

The first mistake comes from the use of the "canonical" procedure to solve linear equations. As we pointed out before, this procedure lead to "non-standard" situations, the student cannot manage. The second mistake arises when the student tries to give a meaning to the expression 0x. The student understands this expression as "*there is no* [*x*]". Since there are no *x*, there is no value to assign to it. Finally, the last four mistakes show different ways to face the expression 0 = 0. Some are wrong, some are arbitrary and the last one shows hay the student fails in remembering what he is supposed to say when he gets 0 = 0.

Item 2

Pérez de Moya states that this equation has no solution. The answers given by the students can be classified according to the following categories (Table 3):

- CS: The student correctly solves the equation.
- WS: The student gives a wrong answer but with a unique solution.
- NS: The student is unable to give a numerical solution (either correct or incorrect).
- N: the students gives the wright answer without any explanation.

CS	18 (31.6%)
WS	20 (35%)
NS	15 (26.3%)
N	4 (7%)

Table 3: Answers for item 2

In this case, we mainly focus on the categories NS and CS, especially on the last one. Students belonging to the category WS usually make mistakes when operating and manipulating algebraic expressions. The following steps give a rather paradigmatic example:

$$x = 2x \; ; \; x / x = 2 \; ; \; x = 2.$$

Regarding those students not providing a numerical solution, it is noteworthy that one of them gave an answer that essentially reproduces the XVI century reasoning: "*No solution. x cannot be equal to 2x*". The most interesting other mistakes that we have found in this item are:

1. $x - 2x = 0$; $-x = 0$. "*It has no solution because x cannot be negative*".
2. $0 = 2x - x$; $0 = x$. "*No solution*".
3. $2 = x / x$; $2 = 1$.

The first two answers show a good knowledge of the algorithmic procedure used to solve a linear equation but both students fail in the last step, which involves some kind of interpretation. The third answer again involves dividing by the unknown.

In this case, 18 students provided a correct answer. One of these answers, in some sense, completes and corrects the original XVI century mistake: "$x = 0$. *Because x cannot be equal to* $2x$ *unless the solution is* 0".

Some final comments

Leaving apart problems regarding algebraic manipulation and notation, we find a main difference between the student who correctly face the analyzed situations and those who do not. In the case of item 1, for instance, we see that most of the correct answers involve a clear idea of the notion of

solution to an equation, while wrong answers always involve the mechanical and algorithmical search of the solution using the steps of some kind of canonical procedure. This thoughtless application of a procedure implies that the student is not usually able to manage non-standard situations like 0 = 0 or 0/0. Then they sometimes try to use some memoristic knowledge or just do not know how to give an answer. This algorithmical conception also leads very often to divide by the unknown, even if this assumes that *x* is not 0. Most of the students work at a syntactical level and they simply do not care about the meaning of the symbols and the operations among them.

Consequently, we think that the mechanical manipulations and rules to solve equations, if presented, should never be the starting point of our teaching. Rather, the solution of equations should be introduced starting from particular problematic situations. After all, algebra was initially a method to solve some arithmetical problems involving unknown quantities. It might be possible that such an introduction implied what we have called "materialistic procedures", which can also lead to mistakes as we have seen, but at least statements like "*x cannot be equal to 2x*" can be the basis for interesting discussions in the classroom.

Finally, in the light of our results, it seems clear that we should devote plenty of time to work with equations leading to identities and to expressions of the form 0 = 0 and 0/0. They will consistently appear throughout the mathematical life of our students and a good understanding of the meaning and implications of these expressions will be a great benefit for them.

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Multiple perspectives on working with original mathematical sources from the Edward Worth Library, Dublin

Maurice OReilly

CASTeL, Dublin City University, Ireland; maurice.oreilly@dcu.ie

In the spring semester of 2016, the author worked with twenty undergraduate (BA) students on original sources from the Edward Worth Library (1733), in Dublin. The end goal was to produce an online exhibition of the mathematical works from that library. The approach to this collaborative work is given, set in the context of a more general framework of the pedagogical value of working with original sources for teaching the history of mathematics. Examples of feedback from the students are given, as is an outline of how the exhibition itself was eventually shaped. The conclusion reflects on the learning gained by all involved in the collaboration.

Keywords: History of mathematics, original sources, cooperative learning

Introduction

This paper outlines the process and product of working, with twenty students and two other collaborators, on the mathematical works in the Edward Worth Library (EWL). This collection opens a window on two centuries of the development of mathematics in England from the 1530s to the 1730s, a period of momentous cultural change in Ireland – with the expansion and consolidation of English political power and the emergence of an Anglo-Irish Protestant elite – and scientific blossoming, culminating in the work of Isaac Newton and his circle, with very strong influences from continental Europe, especially from France and the Netherlands. The work, on original sources from EWL, of the author with his students and the librarian is outlined, as is the subsequent work in preparing an online mathematical exhibition, launched in November 2016.

The Edward Worth Library and my encounter with it

The Edward Worth Library was founded in 1733 by its benefactor, Edward Worth (1676-1733), a medic who had a passion for fine books and who left some 4500 volumes to Dr Steevens' Hospital, in Dublin. Unusual in Ireland for this period, about one third of his collection comprise texts of medical and scientific interest. Of these, in turn, about 9% are mathematical. The collection is housed in a single room with glazed cabinets designed specifically for that purpose. The books themselves are in impeccable condition due, in no small way, to the fact that they were used very little during the 283-year history of the library (Mc Cormack, 2005).

I have taught a one-semester module on the history of mathematics every two years since 2008. A few weeks after completing teaching such a module in 2012, I visited EWL on 15th June, for the first time, in the company of Professor William (Bill) F. McComas, visiting Fulbright Professor (to Dublin City University, DCU) from the University of Arkansas. Combining his interest in the history of science and mine in the history of mathematics, EWL gave the impression of an Aladdin's cave of early modern European learning. We were enthusiastically welcomed by the librarian, Dr Elizabethanne Boran. With a permanent staff of one (the librarian herself) and some visiting scholars and interns from time to time, it is challenging to open the doors to the public, although small specialist conferences are accommodated as much as possible. The librarian has overseen the

publication of online exhibitions that enabled the treasures of the library to be appreciated by a wider readership (Edward Worth Library, n.d.).

On 6th May 2014, the librarian welcomed the first group of my students (undergraduates in their second of third year of a three-year BA) to visit EWL. She had put on display works I had chosen by Viète, Harriot, Descartes, L'Hospital, Wallis, Ward, Huygens and Newton, dating from 1615 to 1732. The reaction of the students to viewing the original sources was very positive, however it was too late in the academic year for this cohort of students to engage with these original sources in any meaningful way. Their reaction, together with my own growing familiarity with scholarship in the HPM community on the importance of working with original sources, prompted me to seek an opportunity to work with EWL in a more significant way in 2016. The librarian was keen to extend the scope of the online exhibitions at EWL, and it seemed realistic to build a mathematical exhibition around the work of the next cohort of students (twenty, including two visiting from the USA). Necessary preparatory work on my part was to become familiar with the online EWL catalogue and derive from it a (provisional) list of the mathematical works in the collection. From this list (of 109 works), I identified for which ones the author had a biographical entry in MacTutor (O'Connor & Robertson, 2016), and found online versions of about a third of the texts, using EROMM, for example. Searching for so many original sources was time-consuming, yet very rewarding; it was an exercise I had previously no reason to undertake on such a scale. At this stage, I considered myself ready to lead the students in the EWL Project. This paper gives an overview of how the project itself was conceived and implemented.

Original sources and teaching History of Mathematics

Much is said in the literature about the use of original sources in teaching the history of mathematics, and this is not the place to provide a comprehensive overview. Kiernan identifies the challenge facing both teacher/lecturer and students:

> The task of presenting original works of mathematics to a class of undergraduates may seem like a daunting task. It requires much preparation. How does one become confident that one can do this? Well, you must remember that you are not alone in this task. Attending any conference with themes on the history of mathematics will remind you that there are many others who have the same interest as you. (Kiernan, 2010, p. 412)

Because I, the author, am also the lecturer and so a crucial player in the work described in this paper, it is appropriate to describe, in the first person, how I engaged with the tasks involved. Glaubitz is quite clear that working with original sources is quite a different kind of work from that which students usually think of when they 'do' mathematics:

> Of course, on the part of the teacher it involves quite some preparation, but this is always the case when you are going to try something new. What is more, the study of original sources requires teachers and students to be prepared to dive into some strange and unknown realm of thinking, to appreciate cultural and historical contexts and – last but not least – to deal competently with written text that is more extensive than the word problems they are used to in mathematics. (Glaubitz, 2010, p. 351)

If working with original sources is, indeed, 'something new', it "[provides] context, motivation and direction for students' mathematical endeavours" (Barnett, 2012, p. 336). Thus, such work has a strong potential to open up new perspectives for students:

> By reading historical sources students can be acquainted with episodes of past mathematics where other meta-discursive rules governed the discourse. (Kjeldsen, 2010, p. 52)

Even when students do not make explicit reference to unfamiliar 'meta-discursive rules', appreciation of such rules can often be noticed indirectly in the 'surprise' they express in their reflections on working with original sources.

Designing the EWL project

The EWL Project was designed as an integral part of a 5-credit (ECTS) module on the History of Mathematics (having a strong, but not exclusive, emphasis on the development of algebra). Credit for this module comprised two main components: 70% for a terminal examination and 30% for continuous assessment (CA). Roughly one third of the examination and 17 marks of the 30 allocated to CA were associated with the EWL Project. Thus, in effect, a substantial 40% of the total credit for the module was assigned to this project. What were the salient aspects of the project and how was it presented to the students?

The semester began on 1st February (2016) and it was important to arrange a visit to EWL early in the semester to allow the students to become familiar with material that would be very foreign to their experience. In advance of the visit, I prepared a one-page overview (with web links, **indicated in bold**, below):

> It is a great privilege to be invited to view a selection of the mathematical works at EWL of which there are about 109 in total. These span a period of almost two centuries, from Cuthbert Tunstall's *De arte supputandi libri quatuor* (1538) to Isaac Newton's posthumous *Arithmetica universalis* (1732), published the year before the establishment of EWL and the death of its founder.
>
> According to the entry in MacTutor, **Tunstall** (1474-1559) had a prominent career as a diplomat (at the court of Charles V in Aachen, for example) and bishop (eventually of Durham). His *De arte supputandi* (1522) was the first printed work published in England devoted entirely to mathematics. This was not an original work, but was based on Luca Pacioli's *Suma* (1494).
>
> Newton (1643-1727), arguably the greatest English mathematician, worked in a broad range of disciplines including theology, optics, mechanics, algebra and (especially) the calculus. EWL has no fewer than 16 works attributed to Newton. There are several other works in EWL by others associated with Newton (such as Barrow, 'sGravesande and van Musschenbroek), all of whom (along with Newton himself, of course) are presented in the EWL online **exhibition on Newton**.
>
> The 109 books are written in four languages: English (21), French (15), Latin (64) and a combination of Greek and Latin (9). The authors of 74 of these (and some of the authors, such as Newton, have more than one book in EWL) have **biographical entries in MacTutor**. EWL

> expects to prepare an online exhibition on Mathematics at EWL in summer 2016. You are invited to contribute to this by writing a page on one of the mathematical works in EWL.
>
> In advance of the visit to EWL on 25th February, take a look at one of the seven 'big' exhibitions already online at EWL (**Newton**, **Botany**, **Alchemy & Chemistry**, **Infectious Diseases**, **Astronomy**, **Dr Steevens' Hospital** and **Aldines**) or one of the eight smaller exhibitions (e.g. **Looking at the Moon** or **Surgery**). Think about the following:
>
> 1. How do you find navigating these webpages?
> 2. What features of the exhibition (you chose) do you find attractive?
> 3. What do you find exciting/daunting about the prospect of writing a page for the Mathematics at EWL exhibition?

This page set the scene for the project. Its subsequent development required much attention to detail, taking on board, to a greater or lesser extent, the reactions of the students as observed in their submitted work, contributions to a forum (in a virtual learning environment), and queries by email or verbally.

On the day of the visit, 16 of the twenty students made it, along with a colleague (Fionnán Howard) who played a crucial role later in editing work for the online exhibition. The students were given hard copy of the following:

- A list of the twelve works of which nine were put on display by the librarian
- A page of text or an illustration from ten of these
- Chapter 3 ("How are mathematical ideas disseminated?") from Stedall's The History of Mathematics – A very short introduction (Stedall, 2012)

Students were asked for their reflections on visit, and responded enthusiastically and on a wide range of aspects. Different perspectives on the library itself included (with student identifiers given in [square brackets]):

> On arrival to the Edward Worth Library (EWL) it was just how I would have pictured it. Not only had the books all been perfectly preserved but so had the library itself. [8]
>
> I expected the library to be quite big and look similar to any other library. My expectations were quite incorrect. [15]
>
> I felt as if I went back in time [12]

In each case, the student's surprise, even sometimes delight, is evident. On the nine books put display, comments included:

> I was surprised that we were allowed to view some texts up close given their iconic meaning. [16]
>
> The books themselves were written in a number of different languages, mainly Latin and French, with only a small amount in English, which again highlights how things have changed. [10]
>
> The most astonishing thing I saw was the quotient rule in a book from 1696. It was bizarre seeing it in such and old context. [2]

There is a real sense here of situating a rule, familiar from school, now in the 'exotic' period of the calculus textbook (L'Hospital, 1696). In her introduction the librarian had made a very reasonable conjecture on the relative plainness of the binding of almost all of Worth's mathematical books, and one student picked up on this:

> Worth didn't seem to pick these books for their binding or appearance alone, but their content [7]

Some students were thoughtful, as if for the first time, about how mathematics emerges and is communicated:

> It quickly became apparent to me that to study any subject during this time period you would need a huge amount of devotion and intellect. You would need to be fluent in various languages to even access three quarters of the books, you would need to travel to find the books and you would need an in-depth knowledge of the subject to comprehend them. [2]

> I was curious about the various languages used in the books and wondered whether every mathematician needed to be fluent in order to understand and broaden their mathematical knowledge. [6]

> The visit gave me a much better and clearer insight into the HoM module, it put into context how the maths we do and formulas we take for granted did not just fall from the sky, but were sought after and achieved by great minds. [12]

Other students articulated hermeneutical sensibilities about the foundations of mathematics:

> Seeing some of the diagrams and equations that were printed in the math books we viewed (though in a different language and confusing at times) made me very aware of how long math has been around. That may sound childish at first, but it was extremely eye-opening to realize that though we may now solve problems using different methods, the foundational concepts and ideas are similar. [9]

> I suppose because these books were collected such a long time ago, we assume that they had a complete different idea of maths, but in fact this is where our understandings stemmed from. The visit to the Edward Worth library made me completely believe and understand that. [15]

Others again drew attention to the language in which the works were written:

> I was also reminded how much I take it for granted that English is the principal language of science today. [7]

> I was taken aback by how easily I could comprehend what was in the books even though they were written in different languages. [18]

Overall the response to viewing original sources close up and in the 'intense' atmosphere of EWL was strong and clearly expressed. After giving these initial informal impressions of their visit, students were asked to work on the detail of the EWL Project in two phases. The first phase was introduced as follows:

> The overall aim of this project is to produce material that will contribute to an online exhibition of the Mathematical works at EWL. Specifically, you are asked to choose one mathematical book from the collection and explore it in whatever way you can.

Several approaches that students might have adopted were suggested, without being prescriptive. These included (along with many others): finding out about the author (from the web), about the historical context in which they lived and about who influenced their mathematical development. Students were encouraged to make plenty of rough notes, to be explicit about why they had chosen a particular work, and to relate it to their prior knowledge and readings. Moreover, the librarian sent a message of encouragement with some notes on the availability of English translations of books in other languages using the British Library's English Short Title Catalogue (ESTC). She also recommended paying attention to the preface of their chosen book (if available in English) so as to get an idea of the context of the book and what the author considered important, and to identifying the most striking illustrations in the book (if any).

The four students who could not make the visit had the opportunity to read the feedback of those who did, and compose questions for them. In this way, they were integrated into the work of the EWL Project alongside their peers.

Not surprisingly, some books were chosen by more than one student: seven chose Wilkins (1691), three chose L'Hospital (1696) and 2 chose Tunstall (1538). Having reviewed the work of phase one and given feedback, I nudged students to choose unique books, so that, in the end, 19 distinct books were 'chosen' (with two opting for Wilkins).

In phase two, students were required to review and polish there work, taking my feedback into account, reviewing features of the existing EWL online exhibitions and distilling their work to produce an engaging, informative and reliable piece of between 800 and 1200 words.

Later, in a short 'capstone' exercise, students were asked to identify the two readings they found most helpful for insights into the history of algebra, and give reasons why? Many of them chose readings related to their work on the EWL Project, indicating its strong significance in their view.

Preparing the online mathematics exhibition

Once the students' work was complete, the next challenge was to devise the structure for the online exhibition. The librarian, Elizabethanne Boran, had much experience in this matter and, together, we decided to use the headings (Arithmetic, Algebra, Geometry, Conic Sections and Infinities) in Ward's *The Young Mathematicians Guide* (1719) as categories for five sections, and to augment these by four more, namely Probability, Applications, Notation and Communities. These nine sections were then to be introduced by an opening section entitled 'What is Mathematics?' A decision was made to incorporate the students' project work to the greatest extent possible (within an appropriate editorial rubric). Each of these thematic sections included links to the works of featured mathematicians. A full list of these mathematicians is given in Table 1 (presented chronologically, with reference to the life of Edward Worth himself).

Number represented in EWL who died (or were born) in a particular period	List of those chosen for the EWL Exhibition
7 died before 1000	1. Euclid of Alexandria 2. Archimedes of Syracuse 3. Apollonius of Perga
16 more died before 1640	4. Tunstall (1474–1559) 5. Commandino* (1506–1575) 6. Cardano (1501–1576) 7. Viète (1540–1603) 8. van Ceulen (1540–1610) 9. Clavius (1538–1612) 10. Harriot (1560–1621) 11. Marolois (1572–1627)
13 more died before 1678 (before Worth's birth in 1676, roughly)	12. Descartes (1596–1662) 13. Pascal (1623–1662) 14. Wilkins (1614–1672) 15. Barrow* (1630–1677)
of the remaining 37:	
16 were born before 1640 (and were living during Worth's lifetime, but were much his senior)	16. Kersey (1616–1690) 17. Wallis (1616–1703) 18. Brouncker (1620–1684) 19. Dechales (1621–1678) 20. Leybourn (1626–1716) 21. Huygens (1629–1695)
15 more were born before 1676 (and were contemporaries of Worth, and senior to him)	22. Ozanam (1640–1717) 23. Ward (1648–1730) 24. Fontenelle (1657–1757) 25. Gregory* (1659–1708) 26. L'Hospital (1661–1704)
6 were born after 1676 (and were younger than Worth)	27. Montmort (1678–1719) 28. Steell (1690–1726)

Table 1: List of those chosen for the EWL Exhibition

In this table, three (indicated by *) are included for their editions of Euclid. The exhibition itself can be seen online (OReilly, et al., 2016).

Conclusion

This paper has outlined the design and implementation of collaborative work within a module dedicated to the history of mathematics, drawing significantly on original sources. The result of this work has led to the production of an artefact external to the module, the online mathematical

exhibition at EWL. It is evident that much was learned by all those involved, students, librarian and lecturers, giving clear testimony to the value of engaging deeply with original sources, in this case the mathematical works collected by the medic, Edward Worth, in the late 17th and early 18th centuries. It is hoped that many will view the exhibition and thereby enjoy the work of those who created it.

Acknowledgment

I acknowledge the work of collaborators, Dr Elizabethanne Boran, Fionnán Howard and the twenty BA students (named on the online exhibition). I would like to dedicate this paper to Professor Jan van Maanen who was the first to alert me to the wealth of insight to be enjoyed through working with original sources.

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Inquiry-based teaching approach in mathematics by using the history of mathematics: A case study

Areti Panaoura

Frederick University, Cyprus; pre.pm@frederick.ac.cy

The use of the history of mathematics during an inquiry-based teaching approach is expected to multiply the positive effects on students' learning. The present work investigated a "typical" teacher's difficulties while trying to use the history of mathematics as a teaching tool during inquiry-based teaching activities. Two examples which were presented in the textbook of the 5th grade of primary education were used to observe the teaching practices. Results indicated that the teacher had difficulties in understanding how students could investigate a mathematical concept by integrating the history of mathematics and how the study of the history would enable them to construct the new-acquired knowledge. The respective knowledge of the domain seemed to be a prerequisite in order to be able to use the history of mathematics fluently and flexibly in a learning environment which asked students to explore or investigate the mathematical concepts.

Keywords: Case-study, inquiry-based teaching approach, history of mathematics.

Theoretical background

Mathematics education aimed to develop pupils' abilities to think logically, critically and creatively by recognizing that mathematics permeates the world around and by recognizing the power and the beauty of mathematics. We believe that a central key for those aims is the appreciation of the multicultural and the historical perspective of mathematics which faced the tendency to understand it as a formal science which has already been discovered. Using authentic problems from the history of mathematics provides experiences for students to actively engage in classroom discourse (Gulikers & Blom, 2001), and to realize the role of the construction of the science of mathematics.

Since 2009, in the context of the CERME we have the appearance of the specific group which discusses the role of using the history of mathematics, the theoretical framework, the teaching practices and the respective learning results. In a meta-analysis, Butuner (2015) included 56 researches in Turkey and abroad in order to reveal the influence of using the history of mathematics on success. By the same way numerous articles have been published in scientific journals and many conferences have been done, without exhausting the discussion on how to use in a more productive way the history of mathematics in order to fulfill the aims of mathematics education. Recently there was a special thematic issue of the Menon Journal of Educational Research about the use of the history of mathematics in mathematics education. The emphasis concentrated on the educators' experiences, beliefs and practices on using the study of the historical aspects of many different concepts for the teaching of mathematics in different ages. There are many studies on the level of higher education (e.g. Weng-Kin, 2008) and on the level of secondary education (e.g. Kaygin, et al., 2011, Lim & Chapman, 2015) and fewer about the primary education.

The present work joined the use of the history of mathematics at a specific grade in primary education, with the aim of using the inquiry-based approach as a teaching method which was supposed to enable students explore and investigate the new mathematical concepts. At the Curriculum of Mathematics which was constructed in 2011 for the primary education in Cyprus, the

use of the history of mathematics was suggested in order to develop students' positive beliefs about mathematics and the usual use of inquiry-based teaching was proposed as the main teaching approach. The two central concepts for the inquiry-based teaching approach which were proposed were "investigation" and "exploration". A case study of a "typical teacher" was used in order to investigate the two specific research aims: a) to examine his knowledge and beliefs on using the history of mathematics in an inquiry-based framework and b) to reveal the teaching practices which are used and the teaching difficulties which are faced during the implementation of the innovation.

The history of mathematics as a teaching tool

In 2000 the International Commission on Mathematics Instruction has set up a study on the role of the history of mathematics in the teaching and learning of mathematics. The main intention was to study the role of the history of mathematics in relation to the teaching and learning of mathematics to the teacher training. Jankvist (2009) explains the use of the history of mathematics both as a tool and as a goal and suggests that introducing the history of mathematics in school curricula enhances learners' motivation, promotes favoured attitudes, and draws attention to possible obstacles faced in the generation and evolution of mathematical concepts. As a pedagogical tool it can serve as a guide to understand the difficulties students may encounter as they learn a particular mathematical topic (Haverhal & Roscoe, 2010). History of mathematics enables teachers to present to their students how mathematical ideas develop and to guide them appreciate mathematics as a creative disciplinary activity. Schubring and colleagues (2000) also posit that programs based on the history of mathematics could increase self-confidence in working with mathematical tasks and develop learners' ability to apply mathematical methods. A journey through the history of mathematics can also enable learners to construct mathematical meanings and support new conceptions about mathematics by changing learners' existing beliefs and attitudes (Dubey & Singh, 2013).

Jahnke (2000) suggests three general ideas which are suited for describing the special effects of studying a source on the teaching of mathematics: (a) the notion of replacement according to which mathematics is seen as an intellectual activity, (b) the notion of reorientation according to which history reminds us that the mathematical concepts were invented and (c) the notion of cultural understanding. As Siu (1997) claims, using the history of mathematics in the classroom does not necessarily increase students' cognitive performance, but "it can make learning mathematics a meaningful and lively experience, so that learning will come easier and will go deep" (p. 8). As Panasuk and Horton (2013) underline the learning of mathematics can be facilitated by studying the cultural significance of mathematics and understanding that "in the earliest stages of invention, many of the mathematical concepts were extremely difficult to define, understand and accept for even the most gifted mathematicians" (p.38).

Although the mathematics teachers in the study by Lit and Wong (2001) were very supportive in using history in their teaching, Siu (1997), in an invited talk given at the working conference of the 10th ICMI study on the role of mathematics in mathematics education, offered a list of thirteen reasons why a school teacher hesitates to make use of the history of mathematics in classroom teaching such as "I have no time for it in class", "Students don't like it", "There is a lack of teacher training on it", "Students do not have enough general knowledge on culture to appreciate it", etc.

The inquiry-based teaching approach

The inquiry-based approach in mathematics education is supposed to promote engagement and ownership and a "human view" of science as knowledge in the making (Savelsbergh et al., 2016). It requires teachers to use pedagogical methods which actively engage students in developing conceptual understanding of mathematical concepts (Chapman, 2011). The challenge for educational systems is to enable its teachers to adopt the values of the inquiry-based pedagogy. The scientific journal of *ZDM* in Mathematics Education has published a special issue in 2013 with nine papers focusing on inquiry-based mathematics education and their implementations, indicating that many questions remain unanswered.

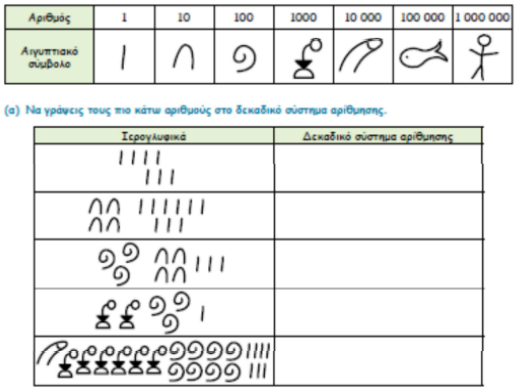
Teachers need to develop their ability to foster student decision-making by balancing support and independence in thinking and working (NCTM, 2000). Classroom management is a crucial aspect of instructional quality (Taut & Rakoczy, 2016). Chin and Lin (2013) claim that there are obstacles and difficulties such as: (i) teachers did not experience inquiry-based learning in mathematics in their own school years, (ii) they do not have complete understanding of the inquiry-based teaching, (iii) there are practical constraints such as that the allocated teaching hours are not enough, (iv) the influence of teaching for success in tests.

Maab and Artique (2013) examine the implementation of the inquiry-based approach and look at its implementation through resources and professional development. They indicate that there is a need to promote a widespread uptake of inquiry-based approach in day to day teaching. One of the main emphases of the new proposed teaching model of Mathematics in the centralized educational system of Cyprus which is presented at the New Curriculum (NCM, 2011), is the use of "exploration" and "investigation" of mathematical ideas, as two dimensions of the inquiry-based teaching and learning approach. The whole idea is to introduce a mathematical concept by using an inquiry-based activity through which the teacher asks students to express their ideas and arguments, to communicate by using the language of mathematics. The emphasis is on using authentic and open-ended problem solving activities without only one correct answer and by respecting the value of inter-individuality.

Methodology

The emphasis of the present study was to examine the teaching practices used during the implementation of the inquiry-based activities by using the history of mathematics in authentic classroom situations. We chose to observe two lessons where the use of the history of mathematics was proposed by the textbook, at the 5th grade of primary education. We are referred to a centralized educational system where the Curriculum, the textbooks and the teaching materials are proposed by the Ministry of Education. A "typical" teacher was chosen after the first phase of the study which is not presented at the present paper. The criterion for the selection was his medium performance concerning his knowledge and beliefs about using the history of mathematics and the inquiry-based teaching approach in mathematics. He took part at a first phase of the project which collected data about teachers' knowledge and beliefs (details about the questionnaire are presented at Panaoura, 2016). We aimed to make the link between what he might say during an interview and what he actually did during the teaching. By using the case-study approach we emphasized the analysis of the teaching conditions in real-life classroom situations and the interpretation he proposed during a

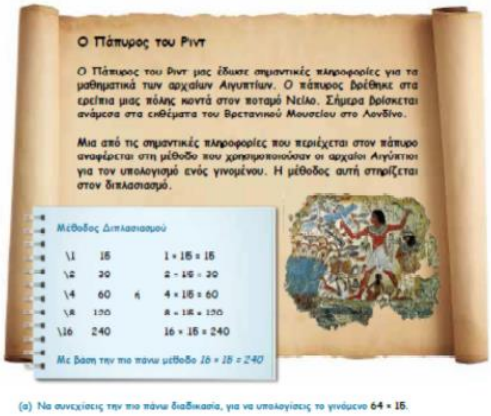
follow up interview. Firstly the teacher at the 5th grade was observed by the researcher and then semi-structured interviews were conducted in order to discuss the lessons. The lessons were chosen because an activity of using the history of mathematics for introducing a concept during an investigation was suggested by the school textbooks. The proposed activities are presented at the Figure 1.



Egyptians used the hieroglyphs in 3000BC which included 7 different symbols in order to represent the numbers. Write the numbers in the decimal numbering system.

A follow up task asks them to compare the two systems and write their comments

Unit 3, page 73



The Reed's papyrus gave us important information about the mathematics of the ancient Egyptians. One of them is the method of multiplication by using the doubling method. After studying the method, apply it in order to find out the result of 64X15 and then use the distributive property in order to find out the 13X15.

Unit 3, page 100

Figure 1: The activities as presented at the textbook (in Greek and in translation)

A protocol for the observation was constructed and used in order to concentrate the observer's attention on: a) teacher's guidelines at the introduction of the activity and his interventions while students were working and b) teachers' feedback on students' difficulties and mistakes. The semi-structured interviews with the teacher were concentrated on the practices he used and the difficulties he faced.

Results

The teacher's observation enabled us to concentrate our attention on the teaching practices he followed in order to use the inquiry-based approach during the teaching of numbers and operations, by using a historical perspective.

In the first case the teacher asked students to study the page, then they had to write few numbers by using the hieroglyphs and finally they were asked to transform other numbers into the decimal arithmetic system. After they presented a few numbers, their teacher asked them to discuss with the members of their group the similarities and differences of the two systems. The specific activity lasted for 10 minutes and then a whole class discussion was conducted. Teacher insisted by posing questions to guide them understand the limitations of the ancient Egyptians' numeric system. Many correct answers were given by the students and only one unexpected question was posed by a girl:

"Today in Egypt people use these symbols or something which remind them the attempts of their progenitors?" The teacher explained why the ancient systems were not survived by repeating arguments which were presented previously by the students, such as the complexity of the symbols. Nevertheless he admitted that he was not able to answer whether there is something in Egypt today which is related with the specific system. He continued by showing his clock and the roman symbols on it, he explained that there were residues of arithmetic systems and symbols which were used in the past. He then asked students voluntarily to look in their free time for more information about the arithmetic system of the ancient Egyptians in order to be able to answer their classmate's question in three to four days. As he admitted during the interview there were some students who tried to find out more information about the numeric systems. They had not found anything about Egyptians; however they discover the Babylonians' impact on the way of measuring the time and the Latin numbers on buildings such as the German Parliament.

In the second case, it was the use of the ancient Egyptians' algorithm of multiplication. The teacher asked students to study individually the method which was presented and applied it at the multiplication 35X17. Few students were not able to continue after the 32X17. One of them continued by writing 3X17 and then she added the two products. Teacher said that it was a wrong solution because "Egyptians did not know how to find 3X17". The follow up dialogue is interesting:

Student: How is it possible to know 2X17, 4X17, 32X17 and they didn't know 3X17?

Teacher: They knew only to double the product.

Student: Why they did that?

Teacher: It was their algorithm.

Student: But the guideline at the book asked to use the distributive property to find the product. I had used it, 32X17 and 3X17.

Teacher: It is right today, but not for the ancient Egyptians.

Student: They were not clever.

It is obvious that the student did not understand that the method of the Egyptians depended on the property according to which when a factor is duplicated the whole product is duplicated and she was not able to understand why this method was easier for them rather than the algorithm which is used today. However it is important that she understood the use of the distributive property in mathematics. Actually this was the objective of the specific course and probably the teacher did not know that the history of mathematics was proposed in the specific case in order to enable students investigate and understand the use of the distributive property in multiplication. When the teacher was asked about the teaching aim and the respective learning aim he said:

Teacher: The history is used in order to understand that mathematics was created by humans.

Researcher: Yes, but they could understand this at the previous lessons, with the arithmetic systems.

Teacher: Here they can understand that complicated processes were used as well.

Researcher: Which was the impact of those processes on the development of mathematics?

Teacher: I don't know. However it is important for humans to study their past.

Researcher: Do you know which were the ancient Egyptians' occupations and where did they use mathematics?

Teacher: No, I am not sure, probably for their transactions.

The teacher used only naive teaching arguments for studying the history of mathematics without understanding that students by investigating the way the arithmetic properties were used, they could understand the use of those properties in order to simplify the used processes. He seemed to not have adequate knowledge about the cultural, political and economic framework of using the specific processes in order to be able to judge their utility.

Discussion

Teachers will continue to be expected to actively engage students in inquiry-based experiences. At the same time most of the Curriculum will continue to ask teachers to use the history of mathematics as a teaching tool in order to enable students to understand the continuity and the development of mathematics in respect to the cultural circumstances. The current study provided evidence that although probably a teacher may express positive beliefs about the importance of the history of mathematics for the introduction or the understanding of mathematical concepts, he or she may face serious difficulties in implementing an inquiry-based teaching approach. Teachers needs experiences during their school life or even during their pre-service training in order to be convinced for the results of the inquiry-based learning and the positive results of exploring and investigating the mathematical concepts through a historical perspective.

The historical approach is supposed to encourage and enable students to regard mathematics as an intellectual process and an on-going activity of individuals (Grugnetti & Rogers, 2000). The prerequisite is to enable them to understand how mathematics thinking and applications developed in different cultures, in response to the needs and thinking of different societies. As it is obvious from the present qualitative study, there are fundamental problems in the implementation of this objective in relation to other main objectives such as the use of the inquiry-based teaching approach. In the case of the New Curriculum in the educational system of Cyprus the history of mathematics is proposed to be used as a tool in teaching the students topics or concepts within the curriculum (Jankvist & Kjeldsen, 2011).

The present study is just a part of a project which investigates the use of the inquiry–based approach. Much more research has to be developed in order to relate the teachers' knowledge and beliefs about the use of the history of mathematics with their beliefs and knowledge about the inquiry-based approach in different grades Teachers' knowledge and beliefs are the official targets of educational reform (Uwe, Espinoza & Barbe, 2013). Emphasis has to be given on studying further teachers' difficulties in implementing the inquiry-based teaching approach in general and in the case of using the history of mathematics in particular, by examining the results of intervention programs in real classroom actions, with an emphasis on facing the teachers' difficulties

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Classification and resolution of the descriptive historical fraction problems[1]

Maria T. Sanz Garcia[1] and Bernardo Gómez[2]

[1] Departamento de Didáctica de la Matemática Universidad de Valencia, Spain
m.teresa.sanz@uv.es

[2] Departamento de Didáctica de la Matemática Universidad de Valencia, Spain
Bernardo.gomez@uv.es

This paper reports on a study about word fraction problems. These kind of problems have been transmitted by the school tradition. Nowadays, they have disappeared because the education model has changed. A classification of the problems is given and the historical resolution methods are presented here because they may be useful in creating knowledge for teaching.

Keywords: Descriptive, historical, fraction problems, resolution methods.

Introduction

In textbooks, there are a variety of descriptive word problems. Swetz (2012) mints *descriptive* and it refers a story or pseudorealistic situation that is not meant to address any practical real situation (some of them are known as recreation or puzzle problems).

Until recently, these problems were used as an essential part of the teaching of mathematics. However, the educational model and the design of mathematics textbooks began to change. Then, many problems disappeared from the current books, because the confidence in the educational power of these problems had declined.

Nowadays, descriptive problems have emerged with renewed interest because the curriculum proposals consider problem solving as a core competency in the development of arithmetical and algebraic thinking. In this context, they become relevant because the historical record of the development of mathematical ideas and methods is useful to produced knowledge valuable for teaching. Particularly, to support an alternative pedagogical approach different to those that use problems as sub-product of other specific content learnings (as an exercise and practice).

When we have reviewed descriptive problems that appear in textbooks, we have found them under different headings which are related with Methods and Rules or with Context, Actions and Agents. For example,

- Methods and Rules: Rule of Three, Compound Proportion (or Double Rule of Three), Conjoined Proportion, Fellowship, Allocation, Interest, False Position (single and double), proportional distribution, etc.

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- Contexts: Fountains or Pipes filling or Holes Emptying the Cistern, Inheritance, Clock Problems, God Greet You or Heap problems, Passing Through Tax passes, Water in wine, Division of Casks, etc.

- Agents: Ass and Mule, Hound and Hare, Hundred Fowls, Couriers or Mobiles, The Epitaph of Diophantus, Animals eating a Sheep, Men Buy a Horse, Men Find a Purse, Lazy Workers, Posthumous Twins, Dishonest Butler, Apple-sellers', Snail Climbing out of Well, Broken Bamboo, Monkey and Coconuts, A travelling Merchant, Lotus, etc.

- Actions: Overtaking and Meeting, Giving and Taking problems, Selling Different Amounts at the Same Prices (to yield the same amount), Co-operative work, etc.

This way of showing problems is the result of historical motivation and it does not allow an overall vision of the problems. For example, there is usually not a single resolution method for the same problem. Also, problems with different appearance may have the same characteristics or structure. This is the reason why the problems can be not organized according to the context, the actions or the agents, or even method of resolution.

Our goal in this research is to build the classification of descriptive fractions problems, with a criterion that contributes clarity and generality, for an overall view of them. At the same time, we want to recover the different resolution methods, as authors have been reflected in textbooks.

The remainder of the paper is organized as follows. First, sample problems are presented and their classification is explained. Also, the relation between quantities is shown with a generic statement. Second, different resolution methods are shown. Finally, the conclusion of the research and some suggestions for future research are proposed.

The problems

The problems, which are studied in this research, seem similar because they are multi-step fraction problems.

> *Dying man.* A dying man gave 6000 escudos to distribute in this way: the half will be given to the monastery of the Jacobites; the third part will be given to the convent of San Agustin; the fourth of the escudos to the monastery of the Friars Minor; and the fifth will be given to the order of the Carmelites". Question: if the whole is 6000 escudos, how many escudos will each monastery have? (Silíceo, 1996, p. 266)
>
> *The Cloth*. A certain man buys 4 pieces of cloth for 80 bezants. He buys the first for a certain Price, and he buys another for $\frac{2}{3}$ the price of the first. He truly buys the third for $\frac{3}{4}$ the price of the second. Moreover, the fourth he buys for $\frac{4}{5}$ the price of the third. It is sought how much each piece is worth. (Sigler, 2002, pp. 274-275)
>
> *Lotuses.* From a bunch of lotuses, $\frac{1}{3}$ are offered to Lord Siva $\frac{1}{5}$ th to Lord Visnu, $\frac{1}{6}$ th to the Sun, $\frac{1}{4}$ th to the goddess. The remaining 6 were ofered to the guru. Find quickly the number of lotuses in the bunch (Bhāskarācārya, 2001, pp. 57-58, ex. 3)

> *The Eggs.* A country woman carrying eggs to a garrison, where she had three guards to pass, sold at the first, half the number she had and half an egg more; at the second, the half of what remained and half an egg more; and the third, the half of the remainder and half an egg more when she arrived at the market place, she had three dozen still to sell. How was this possible without breaking any of the eggs? (Ozanam, 1884, pp. 207-208)

Nevertheless, they presented two main differences. First, the whole may be known or not. The whole is the amount that will be split up in parts. Second, the different parts of the whole are interrelated or not.

The Dying Man is a problem in which the known whole is split up in unrelated parts, i.e., we know the whole (*T*) and for example we have two parts: $p_1 = \alpha_1 T$ and $p_2 = \alpha_2 T$, where $\alpha_1 + \alpha_2 \geq 1$ and $\alpha_i \in \mathbb{Q}\ with\ i \in (1,2)$. We want to know each part.

The Cloth, is a problem in which a known whole is split up in related parts, i.e., the known whole (*T*) is divided in, for example, three parts, $p_1 = a_1 + \alpha_1 p_1$ and $p_2 = a_2 + \alpha_2 p_2$, where $a_i\ and\ \alpha_i \in \mathbb{Q}\ with\ i \in (1,2)$. We want to know each part.

Lotuses are problems in which the unknown whole is split up in unrelated parts, i.e., we do not know the whole (*T*) and it is divided in three parts: $p_1 = \alpha_1 T$, $p_2 = \alpha_2 T$ and $p_3 = A$ where $\alpha_i \in \mathbb{Q}\ with\ i \in (1,2)$ and *A* is a known quantity. We want to know the total amount.

The Eggs, are problems in which the unknown whole is split up in related parts, i.e., we do not know the whole (*T*) and we want divided it in, for example, three parts: $p_1 = a_1 + \alpha_1 T$, and p_2 is a_2 and $\alpha_2\ of\ the\ remaining$, the third part is known *A*, where $a_i, \alpha_i, A \in \mathbb{Q}\ with\ i \in (1,2)$. We want to know the total amount.

These differences permitted us to classify the problems and show a global vision about descriptive word fraction problems (Figure 1).

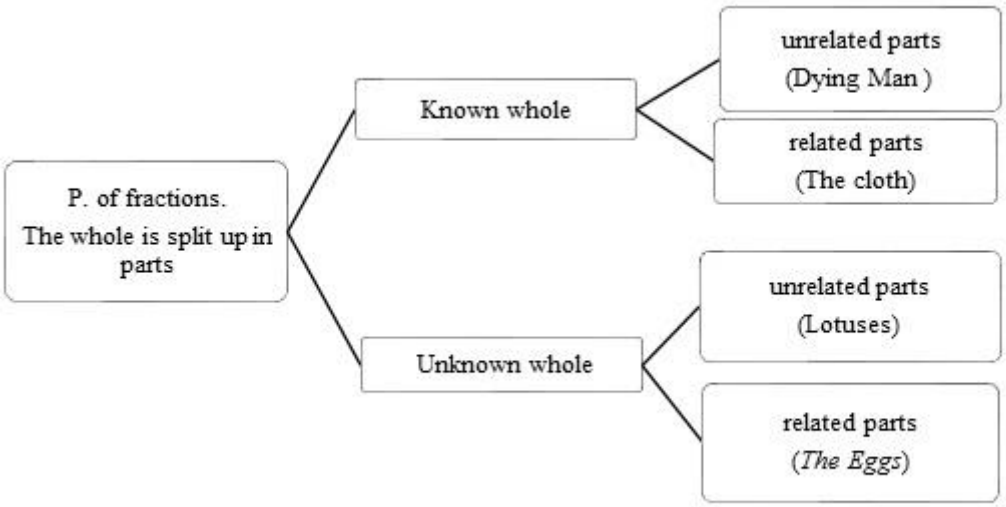

Figure 1: The classification of the problems

Note that it is possible to which the draft of the classification according to the relations between the parts: additive (one part is the sum of another or others), multiplicative (one part is a multiple or a fraction of another or others), combinations of both, …but the research focuses on a classification without subcategories because of the allowed extension.

Resolution methods

Different Textbooks, from different periods or historical moments, has been reviewed to select problems. The resolution methods observed are the following.

Method of inversion

The method of inversion method is explained by Colebrooke (1817, p. 21) in these words, "To investigate a quantity, one being given, make the divisor a multiplicator; and the multiplier, a divisor; the square, a root; and the root, a square; turn the negative into positive; and the positive into negative."

In this method, the problems are solved by the inverse operations, i.e., if the problem name the product, the division is used to solve, or in the case of addition, the subtraction is used. Note that, the resolution process begins with the last operation, and continues up to the first operation indicated in the problem definition.

The next problem of "unknown whole and related parts" shows the method:

> *The Eggs solution.* It would appear, on the first view, that this problem is impossible, for how can half an egg be sold without breaking any? The possibility of it however will be evident when it is considered, that by taking the greater half of an odd number, we take the exact the half $+\frac{1}{2}$. It will be found therefore that the woman, before she passed the last guard, had 73 eggs remaining, for by selling 37 of them at that guard, which is the half $+\frac{1}{2}$, she would have 36 remaining. In like manner, before she came to the second guard she had 147; and before she came to the first, 195 (Ozanam, 1884, pp. 207-208).

False position

False position is related with the algorithmic process where an assumed value is chosen. The operations are done with this number so the result is not correct because the value is not the real value. Then, a rule of three or a proportion is done to obtain the correct result[2].

We can see this method in a lot of fraction descriptive problems, for example in this known whole and related parts problem,

[2] The conditions of the statement can be modeled with a first-degree equation with one unknown*:* $b = ax$. The rule commands that the equation be solved by giving an assumed value to the unknown $x = x_1$, which gives rise to the error b_1, $b_1 = ax_1$. From these two equalities: $b = ax$ and $b_1 = ax_1$, we obtain the ratio $= \frac{b_1}{x_1}$, from which the value of x is followed.

The Cloth solution. Put the first piece worth 60 bezants (False position), because 60 is the least common multiple of 5 and 4 and 3. Therefore, if the first is worth 60 bezants, then the second is worth $\frac{2}{3}$ of it, 40 bezants and the third worth 30 bezants, that is $\frac{3}{4}$ of the price of the second. The fourth worth 24 bezants, that is $\frac{4}{5}$ of 30. Then you add 60, 40, 30 and 24, i.e., sales prices of the four pieces; They are 154 and should be 80; says, got 60 for the price of the first piece and 154 bezants result that the sum of the four pieces; How much will I put to the sum of the parts it is 80 bezants? Multiply 60 by 80; and there will be 4,800 which is divided with the rule by 154, i.e., 1/2 0/7 0/11; the ratio is 6/7 1/1131 bezants. And this is the value of the first piece. Also in order to obtain the price of the second, multiply 40 by 80, then divide again by 1/2 0/7 0/11; the ratio is 20 4/7 8/11 the price of the second piece. Also, to know the price of the third, it multiplies 30 by 80, and divide by 1/2 0/7 0/11; the ratio is 3/7 6/11 15 bezants; in the end, the price of the fourth, multiply 24 by 80, and divide by 1/2 0/7 0/11; the ratio is 1/7 5/11 12 bezants, and you realize that in each of the four products is canceled 1/2. (Sigler, 2002, pp. 274-275)

Or in these other two problems of "unknown whole and unrelated parts":

The Tree solution. Because the least common denominator of $\frac{1}{4}$ and $\frac{1}{3}$ is 12, you see that the tree is divisible into 12 equal parts; three plus four parts are 7 parts, and 21 palms; therefore as the 7 is to the 21, so proportionally the 12 is to the length of the tree. And because the four numbers are proportional, the product of the first times the fourth is equal to the second by the third; therefore if you multiply the second 21 by times the third 12, and you divide by the first number, namely by the 7, then the quotient will be 36 for the fourth unknown number, namely for the length of the tree; or because the 21 is triple the 7, you take triple the 12, and you will have similarly 36 (Sigler, 2002, p. 269).

Lotus solution. Suppose the total number of lotuses is 1. Then the number of lotuses left is $1-\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{4}\right)=1-\frac{20+12+10+15}{60}=1-\frac{57}{60}=1-\frac{19}{29}=\frac{1}{20}$. So [if $\frac{1}{20}th$ is 6] the total number of lotuses is $\frac{6\times1}{\frac{1}{20}}=120$ (Bhāskarācārya, 2001, pp. 57-58, ex. 3).

In both examples, the value of one part is known and the other parts are fractions of the whole unknown. If *T* is the unknown whole and *A* is the known part, in these problems $T=p_1+p_2+p_3$, where, $p_1=\alpha_1 T;\quad p_2=\alpha_2 T$, where $\alpha_i \epsilon \mathbb{Q}\, with\, i \epsilon (1,2)$. Consequently, we add the fractions $\alpha_1+\alpha_2$, and an equation is drawn between the difference of this sum with *T* and the known value . To solve this equation is arithmetically run as in the false position. For this, one can proceed by avoiding the fractions, by taking an assumed value for *T* that is a multiple of the denominators of the fractions (*The Tree*); Or, taking as value the whole unit (*Lotuses*), and then with a rule of three.

Direct method

In this resolution method, the arithmetic operations with fractions are performed as the statement says. We present a "known whole and unrelated parts" problems to show this method:

*Dying Man Solution***.** For Jacobite the half of 6000 escudos; for St. Augustine a third of the 6000 escudos, i.e., 2000; for Friars Minor the fourth part 6000 escudos that is 1500; and for Carmelites the fifth of the 6000 escudos, i.e., 1200. All of these parts add up 7700, but it is not possible

because the man only has 6000 escudos. The divisor is considered 7700, and the multiplier is the money that would must be distributed, i.e., 6000. Then, each part is multiplied by this ratio. Therefore, if the part of the Jacobites is multiplied by the multiplier and is divided by the divisor, they get 2337 escudo 23 duodenos and 2 turonos and 1400/7700 parts of turon. Then, this is the amount that corresponds to the Monastery of the Jacobites. In other cases, we proceed similar and get the amount corresponding to each of them. Note that 1escudo = 35duodenos; 1duodeno = 12turonos. (Silíceo, 1996, p. 266)

In *Dying Man*, once the arithmetic operations with fractions are performed directly as the statement says, is needed an unequal distribution. *T* is distributed according to the rates $\alpha_1 : \alpha_2 : \alpha_3$. Then the required distribution of *T* is: $p_1 = \frac{T}{\alpha_1+\alpha_2+\alpha_3}\alpha_1;\ p_2 = \frac{T}{\alpha_1+\alpha_2+\alpha_3}\alpha_2;\ p_3 = \frac{T}{\alpha_1+\alpha_2+\alpha_3}\alpha_3.$

The direct method appears in the "known whole and related parts" problems too. In this kind of problems, the parts of the whole are considered in different ways. The next examples illustrate it,

A walker. A certain man walking in the street saw other men coming towards him, and he said to them: "O that there were so many [more] of you as you are [now]; and then half of half of this [were added]; and then half of this number [were added], and again, a half of [this] half. Then, along with me, you would number 100 [men]." How many men were first seen by the man? (Hadley & Singmaster, 1992)

Solution. We can suppose that the half of half is a part; then the half would be two parts; The group of men would be four parts and the others are four parts. Therefore, if you divide 99 into eleven parts, the result will be half of the half, then the solution is 36 men in the group. Testing: 36+36+18+9+10100. (Sánchez Pérez, 1949, p. 58)

Day laborers. Three laborers charged for one hour 610 pesetas. The oldest earned 1/8 more than the median and this 1/5 more than the youngest. How many pesetas are each?

Solution. Supposing that the youngest earned 1, the median would earn $1 + \frac{1}{5} = 1.5$, and the oldest would earn $1.2 + \frac{1.2}{8} = 1.2 + 0.15 = 1.35$. If we divide 610 in proportional parts 1:1.2:1.35, the youngest earn $\frac{640}{3.55} = 171.83$ pesetas… (Solís y Miguel, 1893, p. 52)

The hours. You know the hours, could you say me how many hour have passed since this morning? There remain twice the two thirds of the hours that have already passed. (Jacobs, 1863, p. 42)

Solution. You must divide the length of the day in 12 parts, the question is divided this number in two parts, such that the $\frac{4}{3}$ of the first are equal to the second, it is $\frac{51}{7}$, consequently, for the rest of the day, 6 hours and $\frac{6}{7}$. (Ozanam, 1844, p. 192)

The work. In one work, 25 men, 12 women and 30 children are employed. A woman's salary is $\frac{2}{3}$ of a man's salary, and a child earns $\frac{3}{4}$ of a woman salary's. The work has cost 403.20 pesetas. What is the salary of each?

Solution. The 12 women earn as much as $12 \cdot \frac{2}{3} = 8$ men. A child earns $\frac{3}{4}$ of a woman's salary, i.e. or $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$ of a man's salary. The 30 children earn $30 \cdot \frac{1}{2} = 15$ men. The 403.20 pesetas are

the salary of 25 + 8 + 15 = 48 men. Each man earns $\frac{403.20}{48} = 8.40$ pesetas. Each woman: $8.40 \cdot \frac{2}{3} = 5.60$ pesetas. Each child: $\frac{8.40}{2} = 4.20$ pesetas. (Aritmética razonada, 1940, pp. 151 & 660)

Algebraic method

This method is where the problem is reduced a list of quantities and the relation between them gives rise to an equation (Puig, 2003). The equation is obtained as a result of matching two algebraic expressions that represent the same amount.

The next example of "known whole and related parts" illustrated this resolution method (Aurel used the cossic sign, not *x*)

Three men want to share 100 ducats. Three men want to share 100 ducats. The first person has $\frac{2}{5}$ of the second person; and if the ducats of third person are divided by the ducats of the first person, the number is$4\frac{5}{6}$. So, how many ducats are there?

Solution. Three men want to share 100 ducats. The first person has $\frac{2}{5}$ of the second person; and if the ducats of third person are divided by the ducats of the first person, the number is $4\frac{5}{6}$. So, how many ducats are there? Solution. I suppose the second person have 1x ducats so the first person have $\frac{2}{5}$ *x*; and the third will have the remainder, $100 - 1\ \frac{2}{5}\ x$. The ducats of the third person are divided by the ducats of the first person and the result is $\frac{100-1\frac{2}{5}x}{\frac{2}{5}x}$, it is $4\frac{5}{6}$. Reduce the equation, $100\ -1\ \frac{2}{5}\ x = \frac{29}{15}x$. Calculating, $1x$ is equal to 30, then the first and third person have 12 and 58 ducats, respectively (Aurel, 1552, fol. 102v, ex 88; Aurel used the cossic sign, not *x*).

Conclusion

With this research, we try to use the historic descriptive problems to produce useful information to the mathematics education.

Firstly, the descriptive word fraction problems, which are in the ancient textbooks, have been studied. Secondly, a classification of them is built and is focused in the whole and the parts. With this, four different types of problems have been obtained and a generic statement of each problem is proposed. Thirdly, some resolution methods have been explained and analyzed: method of inversion, false position, direct method and algebraic method. Note that this is a classification that supports subcategories, and they are been studying in this moment.

This research allows us to offer students a range of methods to choose the most appropriate method in each problem to solve it correctly.

In a future, we will design a questionnaire to carry out a cognitive analysis to obtain the students achievement in this kind of problems. That is, an empirical analysis that will try to find out the difficulties faced by students in each of the determined types. The difficulties will be related with the essential elements studied in this research.

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Mathematics at the Royal Danish Military Academy of 1830

Asger Senbergs

Roskilde Katedralskole, Roskilde, Denmark; asger@senbergs.dk

This paper takes its starting point in the question concerning why mathematics was chosen as the most important subject in the education of Denmark's new officers when the Royal Military Academy was founded in 1830. To answer this question, source material from the period is researched. From the source material, three educational aims in the mathematical education can be derived: mathematics as a goal in itself; mathematics as a tool; and the mathematical method. These educational goals can be seen as a desire to educate officers in three skills: theoretical knowledge; vocational knowledge; and general education. Together these two sequences of each of the three elements are connected in pairs, thereby creating a link between the officers to be educated and mathematics as a taught subject. It is concluded that mathematics was chosen as the most important subject, because it supported the needs an officer educated at the Academy was required to possess.

Keywords: Military schools; officer's mathematics education; societal elite.

The Royal Danish Military Academy

In 1830 a new military education was created in Denmark – an education that was very different compared to similar educations that existed in Denmark at the time and even very different compared to the modern officer's education in Denmark. This new education was called The Royal Military Academy, and it played a central role in the history of Denmark due to the establishment being an expression of the emerging nationalism that was forming in the country. It also marked the start of a professionalization of the military, which consequently formed a new social class: the educated officers. With this institutionalization of military training there was a desire to organize the knowledge believed necessary to become an officer. This created a need for both skilled teachers and teaching material of a certain quality that could support this new institution.

In particular a need for educated mathematicians arose, since mathematics was chosen as the main subject for the entire education regardless of which field of study, the students were subjected to. At the Royal Military Academy officers could study the following fields: general staff officer; engineer officer; artillery and rocket officer; and road officers. For all four fields of study, mathematics was the subject that had the highest number of modules throughout the 4-year education. The education was split into two classes – referred to as the youngest class and the oldest class – both having a duration of two years. Students in the youngest class followed courses classified as being "purely scientific", which included mathematics, chemistry, physics and languages. In the oldest class the students had courses classified as the "applied" subjects, which included war history, mine teachings, and field maneuvers.

The split mentioned above seems consistent with the general view of mathematics as a teaching subject at the time, i.e. in Denmark as well other countries in Europe mathematics was viewed as an entity made up by two parts: a pure and an applied (Kragh et al., 2005, pp. 297–298). Pure mathematics was perceived as the theoretical part, which was considered beneficial to the ability to think logically and connect structures in a certain way – important aspects of what was described as

the "general education". The applied part of mathematics was perceived as a tool for solving various problems, often in a very practical context.

Out of all the subjects taught at the academy, mathematics made up half (49%) of the total number of lessons taught during the youngest class. In total during the four years, mathematics made up one fourth (25%) of the lessons taught making it by far the largest single subject taught during the entire education. Not only was mathematics the largest single subject, it was also taught at a very high level in particular in comparison to mathematical knowledge of the average population of Denmark at the time. The students at The Royal Danish Military Academy were taught descriptive geometry, mathematical analytics and rational mechanics, which consisted of both complicated proofs and assignments varying from very abstract situations to practical use. Even compared to the current officers' education in Denmark at The Royal Danish Defense College, this prioritization of mathematics is notable. An article brought in a Danish military magazine present the fact that 40% of all permanent military employees have difficulties even with 9th grade level mathematics suggesting that the mathematical knowledge of today's military education has been somehow downsized (Frovin, 2014, p. 16).

In a central source material book from 1855, which is concerns the creation and subsequent operation of the Royal Danish Military Academy, the following is stated:

> The Danish academy is modelled on the French educational institutions. […] the organization, the teaching methods (first theory taught by a teacher and then repetition), the exams, which are supervised by other officials than the teachers – these main points with minor details are directly copied from the French educational institutions, only with smaller modifications which are caused by the economic situation and the smaller scale of the Danish educational system. (Caroc, Bjerring, Reich, & Købke, 1855, p. 31, my translation)

This strong inspiration was most likely due to the fact that several of the officers in the planning commission of the Royal Military Academy in Denmark had themselves studied at the Ecole Polytechnique. A key difference between the Ecole Polytechnique, created in 1974, and the Royal Military Academy was however that the Royal Military Academy did not train civil engineers, which was the case at the Ecole Polytechnique.

The prioritization of mathematics in a military context, as in the case of the establishment of the Royal Danish Military Academy is not only a Danish phenomenon, it is closely linked to similar trends which started years before in central Europe with France as a starting point (Bradley, 1976, p. 165). The French Revolution and the European wars throughout the 1700s led to developments in both weapon technology and construction work. This again increased the need for better educated mathematicians and hence teaching in more advanced mathematics.

Still, the question remains, how mathematics was considered to help cadets become better soldiers? According to Karp and Schubring (2014) not much research has been carried out on mathematical education in a military context. In fact, they describe how this field is only just beginning to take shape alongside the methodology. Nevertheless this relatively new research field can contribute to our historical knowledge about the development of a society, because mathematical education have often been prioritized in historical periods, where society was in need of engineers, technical and scientific personal (Karp & Schubring, 2014, pp. 9-10). Hence, an understanding of how the mathematical education was constructed at military academies can provide us with an indication of

what kind of work tasks the military personal was meant to fulfil, thereby also explaining the general perception of mathematics at the given time. This sums up the focus of this paper, which may be condensed into the following research question: *Why was mathematics chosen as the most important subject, when the Royal Danish Military Academy was founded in 1830 as the new national officer's education program?*

Methodical approach

The answer to the above question shall be based on a carefully selected sample of source material from that period alongside a variety of secondary literature, which will form the historical background for the source material. The historical source material falls in two categories: books concerning administrative aspects; and teaching materials.

In the first category, two very central works from the period are used. The first book is *Plan to the Royal Military Academy* (my translation) and was written by a committee in 1830. The committee consisted of eight officers from the Danish army. Lieutenant general Franz Christopher Bülow oversaw this committee, and he was also the official author of the work. This book formed the official establishment of the Royal Military Academy and described all relevant matters for the creation and operation of the academy. The content covered everything from the overall structure and economic frame to detailed descriptions of the teaching subjects and lesson tables for all fields of study spanning over all four years. This book makes up the main source, since it contains several descriptions about why mathematics is important and what role mathematics was to play in various connections. The second central book is *Overview over the education in the special corps before 1930 and the Royal Military Academy's operation from its creation to 1855* (my translation), written by three officers and a professor from the University of Copenhagen: F. C. V. Caroc, V. J. Bjerring, C. E. Reich and J. P. Købke. This book was released on the occasion of the Academies 25-year anniversary. The book makes up an important piece of source material, because it is one of the only available books from the period that describes the development of military education and does so in a critical manner.

In the second category, the teaching materials, there are seven books in total. All seven are teaching materials in mathematics: three of these are textbooks in mathematical analysis; two are textbooks in descriptive geometry; and the last two are figure books in descriptive geometry the purpose of which were to serve as support for the textbooks. One textbook by Bendz (1831) was in particular used to support the conclusions about the role of mathematics at the Royal Military Academy. The book is in mathematical analysis describing differential and integral calculations, and an analysis shows that it contains practically no tasks related to extra-mathematical circumstances, which supports the interpretation of the text by Bülow (1830).

To set the historical frame for the Royal Military Academy, several books have been used, of which *Danish history of Natural science, Volume II* edited by Helge Kragh, and *Private Schools through 200 years, volume I* edited by Christian Larsen, are some of the more important books. These works provide an overall understanding of the educational level of young men from the higher levels of society in Denmark, who would be the students at the Royal Military Academy.

In order to set the European historical frame, both regarding military education in general and mathematics as a teaching subject and as a research profession in particular, several articles and books have been used. Here it is important to mention: Boyer (1968); Barnett (2015); and Bradley (1975). These works provide the study presented in this paper with background knowledge on how the

development of the Royal Military Academy might have been influenced by similar institutions in other European countries, in particular France.

Mathematics as the cornerstone

In the main source (Bülow, 1830) there are several descriptions and explanations about the various teaching subjects which were taught at the Royal Military Academy. The description of mathematics is rather interesting in this context. The following quote from the introduction became the starting point for the research described in this paper:

> Mathematics, especially its analytical part, including rational mechanics, is considered the cornerstone in education as a whole. Thus it should give the students the means to understand a series of other important subjects. (Bülow, 1830, p. 38, my translation)

Through careful readings of this material several other interesting quotes was found and eventually they came to form the analysis. Through this archival based analysis I found evidence to suggest that the commission, who planned and designed the Royal Military Academy, were subject to three central educational aims, all having mathematics as the cornerstone: mathematics as a goal in itself; mathematics as a tool; and mathematics as a method. These aims were never explicitly uttered in the source material and could therefore make up a separate question for discussion. In the present analysis, however, they came to structure the reading of the source material. In the following I present carefully chosen excerpts from the source material in order to support the finding of the three underlying educational aims of the commission.

Mathematics as a goal in itself

There are several quotes that put emphasis on the importance of "mathematics as a goal in itself" without mentioning a specific purpose of the mathematical knowledge, which might lead one to think that there was another purpose besides the more direct usage of mathematics. A central quote on this is from the descriptions of the teaching subject of physics, where the proper way of learning is explained:

> The dual way to realization; the experimental and the mathematical, which here make up the right teaching method, which also has an exceedingly significant influence on the entire development of the mind; because the mind gets used to neither neglecting the simplicity of the real world nor making us lose ourselves in these details, thereby forgetting the general principles. (Bülow, 1830, pp. 39–40, my translation)

Here mathematics is mentioned as one of the two ways to true realization, whereas an experimental or practical way is mentioned as the other. In fact we touch upon a very central aspect. As mentioned earlier, several publications state that mathematics in this historical period is seen as a unity consisting of two parts, namely pure and applied mathematics. The distinction between these two is noticeable in the way the mathematical subjects were structured during the education at the academy. For example, descriptive geometry was divided into a theoretical and an applied part, with a vast difference between what kind of theory and assignments the students were to do. The following quote is a description of what kinds of assignments the students should be able to accomplish during a lecture in the theoretical part of descriptive geometry.

> For a given curve, draw a tangent, which has a horizontal projection that is parallel to a given line. (Bülow, 1830, p. 76, my translation)

This assignment is purely mathematical without any trace of an extra-mathematical scenario, which was very important since the purely mathematical way was thought to be one of two parts in the learning process. This central point can be further validated by looking at the teaching material used in descriptive geometry, which are two textbooks both authored by L. S. Kellner. The first book (Kellner, 1830a) – which conisted of only one booklet – described a theoretical approch to geometry, whereas the second book – consisting of six booklets – described different ways to use the given theory in a wider range of subjects, spanning how to construct different buildings to how shadows fall on objects (Kellner, 1830b). From reading the source material it seems clear that students were believed be able to understand the basic sciences to the fullest before they could apply them in other connections. Since mathematics was a subject that had its usage in other scientific disciplines and a large part of the applied disciplines, mathematics appears to be a subject that students should learn and master before applying it in extra-mathematical connections.

However, the mathematical curriculum exceeds the need in other subjects, which indicates that the students were meant to learn beyond the point of what they were actually going to use in these subjects. This suggests that mathematical knowledge was an educational goal in itself, because it was thought to be beneficial in various ways.

Mathematics as a tool

The second identified educational aim was that it was essential that the students learned how to apply mathematical equations and a logical approach to problems, because it was a required tool in both the purely scientific and applied subjects taught in the youngest and oldest class at the academy. Especially in the applied subjects mathematical knowledge was necessary, since these applied subjects were the ones to make a student capable of fulfilling his later duties in a specific position within the military.

For that reason it was important that the students knew how to use mathematics as a tool, which according to the source material forms the experimental way to realization. This practical use of mathematics can be seen in a description of an assignment from descriptive geometries in the applied part, which the following quote illustrates.

> Explain the various timber connections in roof- and stair constructions. (Bülow, 1830, p. 79, my translation)

This example forms a clear contrast to the other assignments from descriptive geometry in the theoretical part, where there is no connection to the real world. In this short assignment, the student should be able to construct very specific structures according to a set of requirements.

This vast span from theoretical knowledge to the ability to use it practically is closely linked to the term “general education”, which is commonly referred to by Bülow (1830). Hence, this term is vital for this analysis, and needs to be explained in this paper as well. The term “general education” (in Danish “almendannelse” similar to the German “Allgemeinbildung” or “Bildung”) was a common term in the Danish society in the beginning of the 1800s, and the exact meaning of the term was greatly influenced by the first Danish professor in pedagogy, L. C. Sander. The “general education” term was also seen as a unity with two parts: the general part and the professional part. The

professional part was a person's ability to carry out a certain profession or job, which required some specific skills and knowledge. The general part was a person's ability to engage in and contribute to different levels of the society, which required overall knowledge about culture, history, behavior, science, manners and language (Slagstad, Korsgaard, & Løvlie, 2003, pp. 9, 245). Both of these rather different parts should be in place before the general education was complete. And this was one of the central goals at the Royal Military Academy. Hence, in order for the students to gain this general education, they should be educated to be good soldiers and loyal officers with insight in every relevant subject in their field.

The mathematical method

This brings us to the last identified educational aim, namely the mathematical method. The mathematical method is mentioned several times in *The Plan to the Royal Military Academy* and was described as something that benefitted the general ability to think and to conclude logically, not only in a mathematical context but also in other circumstances. Actually, it was believed that once a student had acquired the mathematical method, this student would be able to transfer this knowledge and approach to other fields of study and everyday scenarios (which we of course know from mathematics education research often is not the case).

This idea becomes clear through the following quote, which is taken from a paragraph that describes the Danish language as a subject, taught at the Royal Military Academy.

> The incessant desire for clarity and definitiveness of expression leads to clarity and definitiveness in concepts of the person, whose thinking has become sharpened by the mathematical study of the natural sciences. Thought and language are in close contact with each other; they constantly excite and control one another. (Bülow, 1830, p. 41, my translation)

Not only is the mathematical study seen as something that sharpens the minds of the students, which makes them more precise in their choice of words, it is actually seen as something that makes the students more likely to draw the right conclusion based on logical assumptions. This is further supported by the following quote.

> The verdict is always most correct, when it can be constructed with numbers, or, and even more perfect, with specific algebraic symbols that simultaneously provide idea and form, the common and the special, the abstract and the concrete. (Bülow, 1830, p. 38, my translation)

The above suggests that the mathematical method played several important roles in the education at the Royal Military Academy. Not only was mathematics a central subject in order to acquire the other subjects taught at the academy, it was also believed to develop the mind and the communication of those who mastered the mathematical method to its fullest extent.

The officer: Gentleman and soldier

In the following, I compare the above described three educational aims, which were condensed from the source material, and connect them with the three levels of knowledge that might make up some of the main reason why mathematics was chosen in the first place. These three levels of knowledge could be described as: theoretical knowledge; practical knowledge; and general education.

Theoretical knowledge: It is the idea that an officer must be trained in the basic sciences that are used in various parts of his work as a soldier. Courses in this type of knowledge are taught at the youngest

class, which is certainly not a coincidence, but part of a basic philosophy that these basic sciences must be mastered in full, before the student will be able to apply the knowledge in other contexts. This explains why several of the mathematical topics were not connected to any applications. The point was to learn the theoretical basis without any applied context.

Practical knowledge: This type of knowledge is gained through the courses taught at the oldest class, because these subjects are related to the different fields of study, i.e. the "applied" subjects. For example, it was in the oldest class that artillery and rocket students had training in artillery, mine teaching and statistics, as well as practical exercises in artillery shooting and marching. This tendency is present in all fields of study. Many of the courses for students in the youngest class served as basis knowledge for the applied subjects in the oldest class.

General education: Here the underlying idea is that an officer in addition to knowing his profession to his fingertips also should be a well-educated gentleman. That meant that an officer trained at the Royal Military Academy should be a person with a broad insight into the culture, history and language of Denmark as well as other European countries. The students should also have a wide knowledge of the natural sciences and their mutual connections, but also with sensible ethics, a good appearance, attitude and solid behavior.

It may seem odd that an officer was taught about Danish literature, since this does not immediately suggest itself as a useful subject for an officer. But at the time an officer was also a person who should be able to be part of Denmark's highest layer of society. In fact, officer was one of the best educated professions in all of Denmark at the time. Therefore, it was indeed expected that the officers were well-educated in many subjects and trained in the spirit of the time.

Conclusion

My starting point was a question about why mathematics was chosen as the cornerstone subject, when the new officer training institution in Denmark, the Royal Military Academy, was founded.

In this paper the reasons for choosing mathematics as the main subject are examined through source materials and mathematical teaching materials from the academy and the given period. The analysis of these contemporary sources was structured around three educational aims, which were supported by a close reading of the source material linked to other similar institutions of the time, e.g. the Ecole Polytechnique. These educational aims could be thought to show three distinct levels of knowledge, which were desirable for a recent graduate student to possess from the Royal Military Academy, i.e. theoretical knowledge, practical knowledge, and general education.

Besides these two sequences being findings in their own right of the present study, yet a finding is that they belong together in pairs. That is to say, the aim of mathematics as a goal in itself was to support students' development of theoretical knowledge; the aim of mathematics as a tool that of practical knowledge; and the aim of acquiring the mathematical method served a purpose of general education. Hence, it seems fair to say that mathematics was chosen as the cornerstone for the education at the Royal Danish Military Academy, because it supported the objective – namely to educate good officers.

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Design research with history in mathematics education

Bjørn Smestad

Oslo and Akershus University College of Applied Sciences, Norway; bjorn.smestad@hioa.no

This paper describes a research project analysing design research projects with history of mathematics. As a background, the theory of design research is invoked. For the purpose of this paper, preliminary analyses of three publications have been made. In later phases, interviews will supplement text analyses to enable a discussion on both explicit and implicit considerations involved when designing materials with history of mathematics in mathematics education.

Keywords: Design research, history of mathematics, mathematics education.

Introduction

The study of how history of mathematics (HM) can contribute to mathematics education has been ongoing for a long time. From time to time, major efforts have been made to design materials for teaching mathematics with history of mathematics. Parallel to this, design research has emerged as an area of study in its own right. The purpose of this research project is to use the insights that recent literature of design research provides to study how design have been done with HM.

Design research

In a recent ICMI Study on task design in mathematics education, Kieran, Doorman and Ohtani (2015) outlines the history of "design-related work" in mathematics education. Design efforts have had many forms and names, but I will take as my starting point Malcolm Swan's encyclopaedia article on design research (Swan, 2014). He defines design research in this way:

> [Design research] is a formative approach to research, in which a product or process (or "tool") is envisaged, designed, developed, and refined through cycles of enactment, observation, analysis, and redesign, with systematic feedback from end users. In education, such tools might, for example, include innovative teaching methods, materials, professional development programs, and/or assessment tasks. Educational theory is used to inform the design and refinement of the tools and is itself refined during the research process. Its goals are to create innovative tools for others to use, describe and explain how these tools function, account for the range of implementations that occur, and develop principles and theories that may guide future designs. Ultimately, the goal is *transformative*; we seek to create new teaching and learning possibilities and study their impact on teachers, children, and other end users. (Swan, 2014, p. 148)

I choose to lean on Swan's definition and use the phrase "design research" here. Others use "task design" for similar efforts – task design is not to be understood as merely designing tasks:

> […] designing a task or task sequence in isolation from consideration of the design of the instructional culture in which the task is to be integrated may be of quite limited value – somewhat analogous to expecting a bird to fly with just one wing. (Kieran et al., 2015, p. 61)

Based on a selection from the literature on design research (including task design), I will focus on four dimensions: the goal; theories, values and design principles; testing; and end result.

First, what is the goal of the project? As seen in the quote above, the goal is to improve something (a product, process or tool) – for instance it could be to create materials (based on history of mathematics) that will improve how geometry is taught.

Design can be seen as an art or as a science (Kieran et al., 2015, p. 62) or as both. Seen as an art, creativity is an important factor, seen as a science, design will be based on previous theories. In addition, values will always play a role: "the frames and principles used in task design are intimately related to aims of mathematics education" (Kieran et al., 2015, p. 65). The role of theories is debatable, for instance Burkhardt (2013) stresses that "strong theories" are often overestimated in education, and argues for "phenomenological theories for specific areas" (p. 233). Kieran et al. (2015) distinguish between three levels of theoretical frames: Grand Theoretical Frames (such as the constructivist), Intermediate-Level Frames (such as the Theory of Didactical Situations) and Domain-Specific Frames (such as theoretical frames concerning specific parts of mathematics). Based on such theories, as well as on values, design principles are often developed for the research project. Thus, the second dimension is which theories, values and design principles are involved.

Kieran et al. (2015) also makes a further distinction: "*Design as implementation* focuses attention on the process by which a designed sequence is integrated into the classroom environment and subsequently is progressively refined, whereas *design as intention* addresses specifically the initial formulation of the design" (p. 28). In design as implementation, the testing in "cycles of enactment, observation and redesign" (Swan, 2014, p. 148) has a key role. Moreover, Burkhardt (2013) states that there is a "crucial difference" between exploration of teaching possibilities by a researcher and testing "what can be achieved in practice by typical teachers with available levels of support" (p. 207). He claims that impact on practice at least requires involving "typical teachers" in testing. The third dimension is therefore what the role of testing is in the project, who is doing it, in what way and in how many cycles.

The fourth dimension concerns the results of the project. The result may be the designed product that can be used by others. Often, local theories about the designed material are also developed:

> Potential users of a curriculum should know what conditions are necessary for its successful implementation, so they can make sure the conditions are in place […]. It is the development team's job to discover and provide this information in the later stages of development and from use in the field. (Burkhardt & Schoenfeld, 2003, pp. 6–7)

In addition, researchers also want to contribute to more general theories of mathematics education: "Design experiments […] are about improving both theory and practice" (Schoenfeld, 2014, p. 404)

For several reasons, design research studies do not always conform to the definition. For financial reasons, testing cycles are often reduced to a minimum. There are also political factors; theories seem to be valued more than practical solutions. When publishing or applying for grants, theoretical results may be stressed more than design results. However, this may be changing, as signalled by the introduction of the ICMI Emma Castelnuovo Award – an award "for excellence in the practice of mathematics education". Burkhardt (2013) contrasts the situation in education with medicine, where the development of new medicines and treatments are valued as much as new theories.

Kieran et al (2015) concluded with a note that "knowledge about design grows in the community as design principles are explicitly described, discussed, and refined" (p. 73–74). This is exactly my motivation for looking at how design is conducted when history of mathematics is concerned.

History of mathematics in mathematics education

Jankvist (2009) shows that the literature on history of mathematics in mathematics education was for a long time dominated by "publications advocating […] for history in mathematics education" and "publications describing either concrete uses by teachers or developments of teaching materials" (p. 13). Some publications in the latter category can be seen as small design research studies, but were mostly based solely on reflections by the teacher-researcher. By adding systematic testing and data collection, the projects can become empirical studies on the "effectiveness" of history of mathematics. Recently, there have been a number of these, and they are often focusing on generating theory rather than the development/design of material (although design nonetheless plays an important part). Alternatively, putting more weight on the development part, they can become fully-fledged design research studies. There does exist a small number of large-scale design research projects, for instance the Historical Modules project (Katz & Michalowicz, 2005).

Research questions

The main research question of this study is: What are characteristics of the design projects that include history of mathematics?

The design research perspectives are used to analyse the projects to shed light on what is considered important by researchers and the community. I will base the analyses on the four dimensions discussed above: the goal; theories, values and design principles; testing; and result.

As not all these dimensions are likely to be described explicitly in written articles about the projects, there are two sub questions: a) How is this process presented in writing? b) What considerations are involved which are not explicitly included in the written results?

In addition, this project may give suggestions on ways in which the literature on design research can contribute to HPM design projects and vice versa.

Methods

The project has three phases. In the first phase (reported in this article), I analyse three publications describing efforts in designing materials for teaching mathematics with history. The analysis is twofold, the texts are analysed in accordance with the categories of the design research literature, and also to find additional considerations not included in the design research literature that I have surveyed. The first phase can be regarded as a "pilot" to see if the approach seems worthwhile. In the second phase, a more thorough literature review is done and more texts are included, whereupon a more thorough analysis is done. In the third phase, interviews are conducted with researchers from selected design projects to identify considerations absent in the published texts.

For the first phase, three texts were chosen: Weng (2008), Barnett, Lodder, Pengelley, Pivkina, & Ranjan (2012) and Jankvist (2009). They were chosen because they are different in scope, target group and context, and could therefore be expected to provide diversity. Two of them are not design

research studies on the face of it, thus the analysis can give me a clue as to whether including such other design-related studies in my analyses are worthwhile.

Preliminary results

Weng: Using history of mathematics in Singapore

Weng (2008) gives an overview of the use of history of mathematics in Singapore, while section 6 of the article describes "an action-research based case study" in which the author developed and gave a course using history of mathematics.

Goal: The stated goal of the study was "integrating history of mathematics into the teaching and learning of mathematics" and "investigate whether such a methodology help the students develop (or even enhance) a positive attitude" (p. 18). The article also includes a ten-page appendix giving examples from several projects, suggesting that the examples are assigned a value of their own.

Theory, design principles, values: The article refers to several potential effects of employing history of mathematics, but advocates the use of history of mathematics "to inculcate positive attitudes of the learner, as well as the teachers, towards mathematics" (p. 3). Weng proposes a "didactical framework", based on the thought that "the learner must make *intellectual leaps*" while mankind make "*historical leaps"*. "[The] relationship between the mechanisms which are responsible for each of these leaps" (p. 13) is important. The intellectual leaps should be identified, "psychogenetical mechanisms" to help should be found, historical mechanisms associated to these should be identified, and historical points found which the historical mechanisms were employed to tackle. Identifying the historical points is called "sourcing", and concerns searching the literature and discussing with colleagues.

In the appendix, seven kinds of "implementation methods" are given – these could perhaps be seen as seven sets of design principles. The seven are historical snippets; primary sources; worksheets; historical packages and enrichment programmes; experimental activities using ancient instruments and artefact; outdoor experiences; integration into modes of assessment.

Testing: The course was taught (once) by the researcher himself. The article includes results from students' and teacher's logs and a student survey.

Results: The stated result of the case study was that the historical approach was effective concerning belief and perseverance. However, as mentioned earlier, some of the materials created are given as examples in the appendix, and there are also examples of "evaluations" connected to the concrete examples: "[…] students appeared motivated since this approach *replaced* the usual, *re-orientated* their mathematical perspective and promoted *cultural understanding*." (p. 35)

There is no discussion of which contexts the examples given could be suitable in, but there is discussion on the Singaporean context, including data on teachers' attitudes and a lament on lack of teacher training in history of mathematics, lack of curriculum time and lack of assessment rubrics. This could perhaps be helpful for others to see whether their context is similar to the Singaporean.

Barnett et al.: Designing student projects via primary historical sources

Goal: The project described in Barnett et al. (2012) builds upon an earlier design research project (a "pilot program") in which "over a dozen historical projects for student work in courses in discreet

mathematics, graph theory, combinatorics, logic, and computer science" (p. 189) were developed. In the new project, "additional projects based on primary sources are being developed, tested, evaluated, revised and published" (p. 189). The goal was thus to develop these resources, with the aim "to recover motivation for studying particular core topics by teaching and learning these topics *directly* from a primary historical source of scientific significance" (p. 190). The article was written while the authors were in the second year of the four-year project.

Theory, design principles, values: The article does not give an overview of the theory it is building on, instead just stating that "Much has already been written about teaching with primary historical sources", and then referring to chapter 9 of the 10th ICMI Study. Some design principles are given:

> each historical project is built around primary source material which serves either as an introduction to a core topic in the curriculum, or as supplementary material to a textbook treatment of that topic. Through guided reading of the selected primary source material and by completing a sequence of activities based on these excerpts, students explore the science of the original discovery and develop their own understanding of the subject. Each project also provides a discussion of the historical exigency of the piece and a few biographical comments about the author to place the source in context. (p. 190)

In addition, fifteen "pedagogical goals guiding the development" are given. They include "students' verbal and deductive skills", "moving from verbal descriptions […] to precise mathematical formulations", "the organizing concept behind a procedure", "understanding of the present-day paradigm [and] standards", "attention to subtleties", "students' ability to equally participate", "offer diverse approaches", "provide a point of departure for students' work", "more authentic (versus routine) student proof efforts", "a human vision of science and of mathematics", "a framework for the subject", "a dynamical vision of the evolution of mathematics"; "greater understanding of its roots" and "engender cognitive dissonance (dépaysement)" (p. 190).

Testing: The testing is done "by faculty at twenty other institutions" (p. 189), but no more detail is given in the article on the procedure, number of iterations and so on.

Results: The projects are published online at http://www.cs.nmsu.edu/historical-projects/, including "notes to the instructor" and comments from users of the projects. The article includes some experiences from the implementations (p. 199–200), including some comments from students and some possible ways of using the materials. This approach to using history to teach mathematics "is effective in promoting students' understanding of the present-day paradigm of the subject" (p. 200).

Jankvist: Using history as a 'goal' in mathematics education

Jankvist (2009) is a dissertation, and therefore has more room for (and demand of) a clear theoretical underpinning than the articles. Moreover, Jankvist's project is not design research as such – to the contrary, the project is an empirical research study whose stated goals are to answer general questions, with materials only as "a byproduct" (p. 8). The three research questions are

> RQ1. In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of 'history as a goal'?

> RQ2. In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and 'ensured' through a 'modules approach'?
>
> RQ3. In what way may teaching modules focusing on the use of 'history as a goal' give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs? (p. 45)

However, for two reasons it makes sense to regard this project as having a design research project at its core. Firstly, his way of answering his research questions is by designing and testing two modules. Secondly, the developed materials are interesting results in their own right, as evidenced by their being published in full (Jankvist, 2008a, 2008b). Thus, in this analysis, I will look at the design parts of Jankvist (2009) as an example of design research.

Goal: The goal of the design research part follows directly from RQ1–3; to design teaching modules engaging students in meta-issue discussions and reflections of mathematics and its history, anchored in in-issues, changing students' beliefs about mathematics in the process.

Theory, design principles, values: The theories are treated systematically and in detail. First, he gives his categorization of the whys and hows. Then, he discusses Meta-Issues (inner and outer driving forces; pure and applied mathematics; epistemic objects and epistemic techniques; discovery versus invention; multiple developments), In-Issues (in particular Sfard's theory of commognition) and Student Beliefs (stressing students' beliefs about mathematics as a discipline and the role of reflection in changing beliefs). As Meta-Issues and In-Issues are part of what the students are supposed to reflect on, a thorough theoretical treatment of them is of particular relevance.

Design principles are not treated as systematically; they are found throughout the dissertation:

> Obviously, original sources have to be chosen with great care, depending on the educational level in question, in order to make sure that the students have a realistic chance of actually working with them. (p. 33)
>
> The historical cases chosen for [a modules approach] should […] be exemplary, e.g. in such a way that they embrace as many general topics and issues related to the history and historiography of mathematics as possible. (p. 89)
>
> [cases should be chosen] for which the in-issues could be built up in front of the eyes of the students in parallel with the explaining of the related meta-issues. (p. 94)

Other design principles are "using modern notation in the presentation of the mathematical in-issues" (p. 95), "setting the text of the teaching material with two different fonts; one for in-issues […] and one for meta-issues" (p. 95), and "Following their group discussions, the groups were to write essays on the topics in question and hand these in" (p. 95).

Jankvist also offers some of his "personal viewpoints", such as that it is "important to provide students with a 'picture' of what mathematics in time and space is" and that "one must have some kind of understanding of the involved mathematics also" (p. 7).

Testing: The actual teaching was done by "a typical upper secondary mathematics teacher" (p. 96), being "coached" by the researcher (p. 115), but no teacher's manual was written (p. 95). There was

just one cycle, but the testing of the first module led to some changes in the second module. Most importantly, "[instead] of the introductory essay assignments, so-called historical exercises were introduced" (p. 157). Moreover, discussions with the teacher also led to at least one change, in that the researcher agreed to discuss the final essays with the class (p. 127).

An immense amount of data was collected: videos of the teaching and of focus group discussions, interviews with teachers and students, lots of hand-ins, including essays, and several questionnaires.

Results: The modules have been published, but not (as far as I know) in a new version informed by the results of the testing, although there are examples of details that were "ill-suited" (p. 150) and examples of new ideas; including role play (p. 202) or using the wording "on the shelves" (p. 203).

There is no attempt in the dissertation to describe conditions necessary for using the modules, except that "In other countries with different types of curricula, the possibilities for doing this may be somewhat limited" (p. 108). It is pointed out that although this "typical" teacher was coached by the researcher, she felt she lacked historical knowledge (p. 275). This makes it doubtful that other teachers with the same level of confidence would use the modules on their own – suggesting that having teachers collaborating with researchers to develop materials might be better (p. 304).

On the questions that the dissertation set out to answer, however, there are ample answers: Students were able to have discussions on meta-issues, anchored in in-issues. The essay assignments "appear to be a suitable setting for having the students engage in meta-issues" (p. 201). "[S]ome of the effects of choosing a newer history over an older one appear to be that it may be easier to relate to" (p. 281). Changes in students' beliefs/views were observed.

Preliminary discussion

The three publications include the theoretical background to very different degrees – probably partly because of context and space restrictions. Therefore, I will not discuss this in detail here. Design principles, however, are detailed in all three publications. Some of these concern the parts of history to be chosen. In Weng's case, specific "historical points" are found that will help students make "intellectual leaps", while in Jankvist's case, the historical cases should be "exemplary", but without concern for whether the mathematics covered is already a central part of the curriculum. Barnett et al., on the other hand, does not discuss the choices of topics but seem to choose topics already central to the curriculum.

All three projects include testing to some degree, although they include different levels of detail. While Barnett et al. explicitly states that testing will be used to revise the materials, Jankvist gives examples of revisions that could be made but he does not make them. None of the publications give very detailed (testing-based) pointers on what "conditions are necessary for its successful implementation", to quote Burkhardt and Schoenfeld. However, both Jankvist and Weng are concerned about the teachers' attitudes and knowledge, raising the question of whether the materials could be used by "average" teachers at all, without significant support.

For two of the publications, the materials produced are not presented as the main result of the studies. If this emerges as a pattern, it would be interesting to investigate whether this is because of the authors' opinions or because of external factors such as the expected format of research texts.

Conclusion of the first phase ("pilot")

The first phase of this project establishes that there are significant differences in the goals, theoretical underpinnings, design principles, testing and results in the three chosen texts. Bringing such differences into the foreground may contribute to a discussion which can, in turn, benefit future design research projects.

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"Sickened by set theory?" – About *New Math* in German primary schools

Tanja Hamann

University of Hildesheim, Germany; hamann@imai.uni-hildesheim.de

Keywords: New Math, primary, Germany, set theory, reform.

Purpose and question of the study

In the Federal Republic of Germany, public remembers so-called *New Math* mostly as a curious episode during which set theory has preceded numbers in primary mathematics education, a reform that has been controversially discussed and finally taken back, having been branded a failure. Thus, a closer look at the history of this reform is required.

As in any educational reform, changes concerned all different components of the educational system. Keitel (1980, p. 449) summarizes three levels that are crucial for the course of any reform, namely a subject's educational scientific community, educational administration and curricula alongside their implementation in textbooks. Neglecting practical classroom experience, this list stucks to a more theoretical view, whereas Fend (2008) constitutes a multi-layer model of educational system upon the levels administration (politics and curriculum), school (teachers) and lesson (pupils and parents), putting more emphasis on practice and social reception. In both concepts textbook production is seen as being closely linked to curricular decisions, nevertheless it is also stated that – particularly within the German *New Math* movement – schoolbooks served as "instruments of innovation" inside the classroom (Keitel, Otte, & Seeger, 1980, p. 73). For these reasons, the project is based on a multi-layer model of schooling comprising the following levels: scientific theories of education, curricula, textbooks and implementation in classroom. Former analysis of the reform in Germany has mostly emphasized on curricular aspects (Damerow, 1977; Keitel, 1980; Zumpe, 1984), has mainly been focused on secondary education and dates back from when *New Math* curricula were still mandatory, causing those accounts to criticize the concepts and implementations rather than bringing them in line with long-term historical development.

From this derives the main purpose of the project, which is to describe ideas and concepts (with regard to aims, contents, principles, methodical suggestions) leading to the reform of mathematics education in West German primary schools and compare them to how they were implemented into methodical concepts. Three different 1st grade textbooks and associated teacher handbooks from the 1960s and 1970s were chosen as exemplary key sources: *alef* by Bauersfeld et. al., which developed from the country's only long-term classroom project (*Frankfurter Projekt*), *Wir lernen Mathematik* by Neunzig & Sorger, which was the first *New Math* textbook for primary level being published and which is explicitly based on the concept of Z. P. Dienes, and *Mathematik für die Grundschule* by Fricke & Besuden, which originated from a former numeracy textbook based on the operative principle and therefore on the results by J. Piaget.

Results

It shows that the courses differ when it comes to basic curricular decisions as well as to the relation between mathematics, sets and arithmetic. Bauersfeld et. al. have created a non-linear course that is

based on fundamental logical (relations, sets, transformations) and geometrical concepts. Here, arithmetic is founded on mathematics; sets serve as one example of a basic mathematical notion. Fricke & Besuden subordinate all their subject matter decisions to the learning of operative thinking. Thus, mathematics becomes a means for this, and arithmetic as well as set theory, logic and geometry serve as examples for mathematical thinking. Neunzig & Sorger's course, however, is aiming at a mathematical foundation of arithmetic, which is solely based on sets.

Especially the latter, which largely narrowed the original idea – namely to replace pure arithmetic by propaedeutic mathematics from the start – to teaching set theory in advance of numbers and numeracy, was widely disseminated and thus influential for the course of reform. The question occurs what might have caused this development. One result of further investigation of concepts of numeracy education up to the 1960s at the German *Volksschule* is that such an approach could be brought in line with long-term tradition of German primary arithmetic education.

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The role of genetic approach and history of mathematics in works of Russian mathematics educators (1850-1950)

Ildar Safuanov

Moscow City University, Russia; safuanov@yahoo.com

Keywords: Russian mathematics education, history, genetic approach.

Introduction

The aim of this poster is to present the study of the history of the development of genetic and, in particular, historic-genetic approach in works of Russian mathematics educators of the second half of the 19th century and first half of the 20th century. It is important because the genetic approach and the history of mathematics are used in mathematics teaching nowadays (e.g., Fauvel & van Maanen, 2000). The role of genetic approach and history of mathematics for mathematics teaching is important today and it has been investigated by many authors (e.g., Furinghetti & Radford, 2008). The role of genetic approach and history of mathematics in works of Russian mathematics educators also needs a systematic research. In our study (work in progress), we use books and articles written by Russian mathematics educators of the past, notes of their lectures, and proceedings of Congresses of mathematics teachers held in the beginning of the 20th century.

Genetic and historic-genetic approach to the teaching of mathematics in works of Russian mathematics educators

Progressive Russian mathematics educators developed ways of using history and genetic approach in mathematics teaching since the middle of the 19th century.

Russian mathematics educator Petr Semyonovich Guryev (1807-1884) was acquainted with works of F.W.A. Diesterweg. As early as in the middle of the 19th century he insisted on the use of elements of genetic and concentric (Safuanov, 1999) teaching. He wrote (Guryev, 1857, p. 176):

> It is necessary that theory would develop like concentric circles… It is more appropriate and in accordance with the progress of development, that the definitions should be given at the end rather than in the beginning of the teaching of elementary arithmetic and geometry.

Viktor Viktorovich Bobynin (1849-1919), in his report on the 7th All-Russia congress of scientists and doctors he explicitly argued that it is necessary to use genetic method in geometry teaching and, in particular, to base the methods of teaching on the history of mathematics (Bobynin, 1886, p. 31).

At the end of the 19th century and in the beginning of the 20th century, one of the leading figures in the development of Russian mathematics education was Semyon Shokhor-Trotsky, author of mathematical textbooks for elementary mathematics. He invented the "Method of expedient tasks" which was essentially similar to genetic method. He wrote in one of his methodical guidebooks for mathematics teachers (Shokhor-Trotsky, 1935, p. 9):

> The true method consists in that we should put a child in conditions at which human mind started inventing arithmetic, we should make him a witness of that invention. But it is not sufficient today. Nowadays we should aim at putting a child in conditions at which she/he would become not only a witness but rather the active participant of that invention.

Most interesting was the development of didactic ideas of Nikolai Izvolsky. Beginning with the method of "combination of mathematical representations" (Reports of the First Russian Congress of Mathematics Teachers, 1913, pp. 148-157), he eventually came to the profound expression of the genetic approach (not reduced to the historical one) to geometry teaching: "…A view of geometry as a system of investigations aiming at finding answers to the consequently arising questions" (Izvolsky, 1924, p. 9). He elaborated an original version of the genetic approach.

Conclusion

Thus, we see that in Russian mathematics education many researchers, mathematicians, as well as mathematics educators, strongly contributed to the development of the genetic teaching of mathematics. N. Izvolsky articulated the idea of indirect genetic teaching prior to O. Toeplitz. We think that ideas of Russian mathematics educators, especially those of S. Shohor-Trotsky and N. Izvolski, would be useful for the further elaboration and improvement of the genetic approach in future. The development of ideas of Russian mathematics educators will be presented in the poster via the use of images and schemes.

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TWG13: Early years mathematics

Introduction to the papers of TWG13: Early years mathematics

Ingvald Erfjord[1], Christiane Benz[2], Esther Levenson[3] and Bozena Maj-Tatsis[4]

[1]University of Agder, Norway; ingvald.erfjord@uia.no,
[2]University of Education Karlsruhe, Germany; benz@ph-karlsruhe.de,
[3]Tel Aviv University, Israel levenso@post.tau.ac.il;
[4]University of Rzeszow, Poland; bmaj@ur.edu.pl

Introduction

The working group on Early Years Mathematics was established at CERME 6. The aim of this working group has always been to share scholarly research concerning mathematics for children aged 3-8. In this age group, the transitions within preschool, and from preschool to the early grades of primary school are important areas of attention. The working group on early years mathematics has also considered that in different countries preschool education has different objectives and children in different countries begin primary school at different ages (e.g., in Sweden there is one preschool curriculum for children aged 1-6 followed by primary school, and in UK there are different curricula for with nursery, preschool and primary school: 0-2, 3-4 and 5-8 years). These differences have always stimulated fruitful discussions and CERME 10 was no exception.

The 15 papers were each allocated 30 minutes of attention during the TWG 13-sessions; 12 minutes for presentation, 8 minutes where a different participant prepared a response or query, and 10 minutes for open discussion. The five posters were allocated 15 minutes each for a brief presentation and open discussion. One 30-minute slot was reserved for group work where participants from different countries were grouped and discussed quite openly topics of common interest (such as, e.g., the role of aesthetics in mathematics and the use of manipulatives in mathematics), and another 30-minute slot was reserved for discussing the plenary by Lieven Verschaffel on children's number sense. It was very fortunate to have him as a participant in TWG13..The final 90-minute session on Saturday was used to collaboratively prepare the report for Sunday, which together with what was presented and discussed in the other sessions, formed the basis for this introduction.

One of the Friday sessions included a 60-minute presentation and discussion of the Early Years Mathematics chapter draft for the soon to be published ERME book, where one of the editors of the book (Kenneth Ruthven) was present. Participants had been asked to read and send their questions and comments to the chapter in advance of the conference, and these inputs, as well as comments during the session, contributed important input to the ongoing discussion in the group.

The number of participants in TWG13 has been quite stable over the past five conferences, and each time the group has benefitted from a quite broad attendance when it comes to the area of Europe. Twenty-nine participants from 11 different countries attended the working group in Dublin (see the table below).

Country	Number of papers	Number of posters	Number of Participants
Belgium			1
Denmark			1
Germany	1	1	3
Greece	2		2
Israel	1		1
Norway	3	2	8
Poland	1	0	1
Portugal	0	1	1
Sweden	3	1	4
Spain	2	0	4
UK	2	0	4
TOTAL	15	5	29

General summary of presentations

Seventeen of the 20 papers and posters concerned preschool children aged 3-6 years, preschool/kindergarten teachers' professional knowledge and competence development or teaching materials for activities in preschool. The remaining three contributions focused on children aged 6-7 years, which in most countries means Grade 1 in school. Compared to previous years, there was an increased attention given to preschool compared to early schooling.

Broad attention was given to the roles of preschool teachers' competencies in promoting early years mathematics learning. Vanegas, Gimenez and Samuel studied school mathematics narratives in early childhood teacher education, focusing on how the future preschool teachers recognized the potential of two professionally designed geometrical tasks to promote mathematical processes. Palmér and Björklund considered how preschool teachers characterize their own mathematics teaching in terms of design and content, while Hundeland, Erfjord and Carlsen analysed preschool teachers' orchestration of researcher-designed mathematical activities. They used this analysis to discuss what can be learned about teachers' knowledge by adopting the Knowledge Quartet by Rowland and colleagues for kindergarten level. Tirosh, Tsamir, Levenson and Barkai studied preschool teachers' variations when implementing a patterning task, where they investigated the impact of the various implementations on children's success in extending repeating patterns. Gifford, Griffiths and Back presented a theoretical paper concerning the use of manipulatives with young children, while Skoumpourdi presented a framework for designing inquiry-based activities for early childhood mathematics. Silva, Costa and Domingos offered an nterdisciplinary approach to linking science and mathematics in early mathematics. A new issue presented and discussed during our WG was

mathematics for special needs children during the early years. Gasca, Clemente and Colella outlined and discussed the design of mathematical instructional activities to foster achievements in mathematics for children with Trisomy 21 (Down's Syndrome). Peter-Koop and Lüken considered the role of inclusive compared to exclusive settings for the learning of mathematics for children with special needs.

In total, eight papers and posters focused on the learning and development of mathematics for preschool children. The paper by Tzekaki and Papadopoulou was based on a teaching intervention for developing generalisation in early childhood. Björklund's paper considered the importance of adults challenging two year old toddlers' evolving concepts of numbers in their play. Her paper stimulated a discussion as to whether 3 years of age should be the lower limit of attention for our group as it was in the past. Sundström and Levenson reported on an exploratory study of young children's aesthetic development in the context of mathematical explanation, while Pettersen and Volden presented a poster considering the use of maps in kindergartens and children's development of spatial orientation and navigations. Bjørnebye reported from a multi-case embodied design study on early learning of numbers, while the paper by Schöner and Benz studied preschoolers' perception and use of structures in sets, adopting eye-tracking as a tool. Breive studied preschool children's argumentation as part of an inquiry approach in reflection symmetry and Rinvold investigated children's learning of numerocity. Both Breive and Rinvold adopted Radford's theory of knowledge objectification to kindergarten settings.

Regarding the first year of primary school, Maj-Tatsis and Swoboda reported on an intervention study on first grade school children's ability in noticing and using regularities in three-dimensional geometrical objects while using playing blocks. Van Bommel and Palmer focused on 6 years old children's representation of the semi-concrete and semi-abstract, as connections between concrete (objects) and abstract (signs) representations. Finally, Thoules presented a self-study action research study on the role of gestures in supporting mathematical communication for first grade students with language delays.

Main characteristics and issues in focus

The group had ongoing discussions about the importance of fostering children's early development of mathematics for later success. As such, the group experienced presentations and discussions of several intervention and exploratory studies, teacher development and ways of fostering children's mathematics learning. The role of play and the term "playful learning" was given great attention. This proposed term takes into account that distinguishing between play and learning at this age-level makes little sense. The argument offered was that an activity designed, planned and guided by the teacher may indeed promote learning among children, but can be considered as play by the children.

As usual, the group presented a mixture of small scale studies (case study design), large scale studies and several pilot studies for larger studies. These different studies contributed valuable insights. We also experienced a broad range of theory in discussions of papers and posters, including the theory of objectification, variation theory, socio-cultural theories, the use of Clements and Sarama's framework, the core knowledge system, as well as development of conceptual frameworks. The theoretical discussions of the group were supported by Lieven Verschaffel's perspective and contributions, generating fruitful discussions of theory use and development.

The most prominent “news” for our group came from presentations discussing the following issues: ways in which special needs children can engage with mathematics, toddlers and mathematics, and young children’s appreciation for the aesthetical dimensions of mathematics. The group was also introduced for the first time to the possibility of using eye tracking as a tool in early years mathematics research. We encourage researchers to continue with their important work regarding early years mathematics and hope to see new insights in the next CERME meeting.

Aspects of numbers challenged in toddlers' play and interaction

Camilla Björklund

University of Gothenburg, Sweden; camilla.bjorklund@ped.gu.se

The aim of this pilot study is to explore when and how toddlers discern aspects of number in exploratory play and communication. Data for analysis consists of video-observations of 23 1–3-year-olds' mathematical activities in preschool, revealing occurrences where toddlers' conceptions of number are challenged. A qualitative analysis informed by Variation theory of learning provides important clues for early mathematics education: toddlers encounter many aspects of number on a daily basis, but their conceptions are rarely challenged. This study will work as a basis for intervention and further studies of the possibility to enhance early mathematics education.

Keywords: Early childhood education, mathematics, number concept, toddlers.

Introduction

There is a diversity of pedagogical approaches and theoretical understanding of young children learning mathematics. Palmér and Björklund (2016) discuss in a recent review of contemporary preschool mathematics research that what children are actually offered to learn in preschool differs to a great extent, even though there is a consensus among the research community that early math matters. Consensus is however not reached when it comes to what and how mathematics should be taught. Some argue that basic understanding, which some children already have acquired, is enough and preschool should offer a space for children to use and master these skills. Others find it necessary to challenge children's knowledge and set goals for learning that the majority of the children in a group have not yet mastered (Claesson, Engel, & Curran, 2014).

Many researchers have made efforts to describe developmental trajectories in mathematics learning and what to expect from children at different ages or in what order skills develop in general. Consequently, there is a large body of research departing from different theoretical approaches (see Baroody, Lai & Mix, 2006; Sarama & Clements, 2009 for overviews). A contemporary theory – Variation theory of learning (Marton, 2015) – proposes a bold conjecture that the understanding of whole numbers originates from experiences of different aspects of number, rather than a predictable development trajectory. Focus is here shifted from descriptions of children's competences and learning trajectories towards the content to be learnt, and particularly what it takes to learn that content. Based on this conjecture, this study directs attention to what aspects of numbers are discerned by toddlers in play and interaction in preschool. The theoretical framework furthermore provides analytical tools for interpreting *why* children express different ways of experiencing numbers and what is possible to learn in different interaction. The study will therefore have impact on education and further research. There are many aspects of mathematics that could be made object for this study, but in this particular pilot project the interest is demarcated to number concept.

Aspects of number as learning object in the early years

The concept of number is complex and research shows that many competences seem to be necessary for children, to develop their numerical reasoning and arithmetical skills. By the time children turn three years, most of them already know the beginning of the sequence of counting words and are

acquainted with nursery rhymes with number words. To develop an advanced understanding of the different meanings of number in different contexts is however a process that continues until children are about eight years old, according to Fuson (1992). In the early years, the sequence of counting words is best described as a continuing string of words that lack numerical meaning. During their preschool years, children's understanding of number words includes also a numerical meaning, but how this process happens is an issue raised by Wynn already 25 years ago (Wynn, 1992). There is of today no consensus how this development occurs or what seems critical for learning to handle numbers in arithmetic tasks. Some (c.f. Carpenter & Moser, 1982; Fuson, 1992) claim that counting is the foundation for arithmetic competence, while others argue for the necessity of perceiving the part-whole relationship of numbers (Baroody & Tiilikainen, 2003). One way of finding a path to understand the development of number sense and arithmetic skills could be to direct more attention to the features of numbers and how numbers may be perceived in different ways, as proposed by Marton (2015). A controlled study by Benoit, Lehalle and Jouen (2004) has for example shown that the perception of numbers as parts and whole simultaneously, supports children in acquiring the meaning of small numbers. This builds on the ability to perceive numbers as exact sets of items without counting, referred to as "subitizing" (Kaufman et al., 1949). Subitizing is limited to small amounts of three or four, but the ability can be enhanced to larger numbers, so called conceptual subitizing (Clements, 1999). When items are arranged in ways that make them possible to recognize as patterns or collections of sets, it is easier to compare and estimate both exact number and larger magnitude. However, the sequence of counting words and the meaning of numbers as descriptions of sets of objects (a sense of "manyness") probably has to be considered as two necessary aspects of numbers that children presumably learn during their first years.

Even though young children seem to have the ability to perceive exact number, there is no guarantee that all children make use of this ability. Studies (Hannula, Mattinen & Lehtinen, 2005) show that far from all three-year-olds take initiatives to focus on numerical features of their surroundings or use number words in routine situations and play. Spontaneous attention to number among young children has nevertheless shown to predict later mathematical achievements (Hannula-Sormunen, Lehtinen & Räsänen, 2015) and this can be enhanced by social interaction and directed attention towards number relations, with long-lasting effects.

From earlier research we have reasons to believe that certain aspects of number are necessary to discern in order to develop number sense and arithmetic skills. Neuman (1987) argued, based on a study of 6-7-yearolds that children who suffer from math difficulties cannot experience the first ten natural numbers as magnitudes nor the relations between them. This lack hinders them in perceiving that 7 may be a part of 9 and the difference between them is 2. Most children develop strategies to deal with numbers and arithmetic tasks, but those who cannot discern the part-whole relationship have to turn to cumbersome strategies of counting up and down and even double-counting. Neuman further described how children's strategies for problem solving are related to their conception of numbers and what aspects of numbers that the children can differentiate. The following aspects are known from the vast literature in the field, confirmed as aspects necessary for arithmetic computation in Neuman's study: a) numbers can be *represented in different ways*, b) numbers constitute a *part-part-whole relationship*, c) number words refer to *cardinality* (or manyness), and d) number words refer to *ordinality* (or sequence)**.**

In an on-going project (Björklund et al., 2016) we found these aspects present in some, but not all, 5-yearolds conceptions of number. It is thereby of interest to further investigate how younger children perceive numbers. Thus, the question raised in this pilot project is: What aspects of number are discerned by the youngest preschoolers and how are their conceptions of number challenged?

Theoretical framework

The theoretical framework used in this study is Variation theory of learning (Marton, 2015). The theory states that variation is central for learning and patterns of variation constitute necessary conditions for learning (Marton & Tsui, 2004). When learning new concepts, there is in particular one pattern of variation that is essential – contrast. By contrast means that what is to be learnt is held constant but some critical aspect of the learning object has to be contrasted to enable discernment. In other words, to learn the meaning of "five", sets of five (items constituting the sets) may vary (five flowers, five dogs, five children) but the "fiveness" will appear only when *five* in a set is contrasted with for example *four* in another set. Only then, can the numerical meaning of five be discovered and thereafter generalized to the different sets of five. Generalization, differentiation of aspects that are potentially necessary to discern in order to better understand the concept, and simultaneous attention to several aspects are the other patterns of variation that the framework holds as necessary for learning (Marton, 2015). This theoretical framework provides tools for analyzing empirical data to describe the process of learning by focusing on how necessary aspects of the learning object are presented to the learner. For example, the child needs to differentiate that numbers have both ordinal and cardinal meaning to be able to operate with numbers in arithmetic tasks, but if the child has not discerned numbers' cardinality they cannot make use of numbers as a representation for sets of objects.

Children's conceptions of a phenomenon constitute, according to this theoretical framework, of those aspects that the child can discern in a particular situation (either due to earlier experiences of similar phenomena or due to what the situation offers the child to experience). Conceptions can thereby be interpreted as the way a child encounters a phenomenon. Lack of discerned aspects is thereby also expressed in children's use of numbers in problem solving (Neuman, 2013) and in particular when they are encouraged to communicate their understanding of numbers to peer or adults.

Methods

The aim of this study was to get an overview of toddlers' conception of number and in what ways their conceptions of number are challenged. This is done through an analysis using Variation theory of learning as analytical tool, which directs attention to the content of learning and in particular how a phenomenon (number in this case) is perceived by the children in a specific situation. The unit of analysis is interaction that provides opportunities to discern necessary aspects of number.

The participating children attend two randomly selected preschools in Finland, where they take part of a pedagogical practice with their peers in both planned activities and self-initiated play. 23 toddlers altogether (aged 13 months – 3 years 9 months) are observed in their common activities in these preschools. The data consists of 45 hours of video-documentations originally collected for a larger study with a broader mathematics focus (see Björklund, 2007). Of these, episodes where

number concepts are focused on are analyzed more thoroughly in this particular study. Written consent for children's participation in video-recorded documentations is given by the children's parents. Children's and teachers' names are anonymized in all public presentations.

The analysis concerns the opportunities children are given to develop their conceptions and learn to use numbers and is conducted in two steps: 1) the children's conceptions of number is characterized, 2) the analysis focuses what aspects that are emphasized when children express a certain conception including if and how this conception is challenged. The theoretical framework provides analytical tools to study such situations where the number concepts are challenged and what constitutes the development opportunity. We are in accordance with the theoretical framework looking for situations where children or adults "open up dimensions of variations" (Marton, 2015), meaning that a certain aspect of the learning object is emphasized and thereby made possible to explore and learn the meaning of. It is in particular *contrast* that we look for and how it enables different dimensions of variations to be opened up.

Results – Acts that challenge toddlers' number concepts

We know from studies with older children that some critical features of number are necessary to discern to use numbers successfully in arithmetic problem solving. In this study we see the same aspects' importance and can describe how the lack in discerning necessary aspects results in different conceptions of numbers. The sequence of counting words is quite common in different activities, but the sequence is mostly used for naming items or as any nursery rhyme. The number concept is then limited to an ordered string of words without relation to numerosity. It is more seldom found that the children use the sequence of counting words to find out how many objects there are in a set (in a cardinal sense). However, there are examples of children using number words to describe sets and quantities and thereby expressing a conception of number as quantitative relations. The following presentation will discuss how toddlers' conceptions of number are challenged in their interaction with peer and adults, with analytical focus on which aspects that are made possible to explore for the further development of number concept.

Describing objects or sets

Number words are by many toddlers used as a rhyme that is fun to recite. Some relate the reciting to groups of objects, one-to-one, and a few children use number words to describe sets of objects. The difference between these ways of using number words is in the conception of numbers as ordered names (emphasizing an ordinal aspect) or numbers as describing a set (a cardinal aspect brought to the fore). To challenge number concepts' meaning it is then necessary to open up these dimensions of variation to the child:

Harry (2:0): (looking in a picture book) Wow, horses.

Nancy (3:9): Yes. It's two horses.

Harry (2:0): (browsing through the book) There they are. There are also two.

The short but clear comment from the peer opens up for numbers' "manyness" and numerical relationships to be explored. Harry shifts his attention from "horses" to numbers, "two". However, numbers are not further challenged, such as by comparing two horses with more or less or generalizing to other sets of two.

Expressions of numbers as names of items are common in the literature and frequently observed in this study. This conception of number (or rather the number words) is characterized as a way to enumerate by giving each counted item a number name, attaching the name to the physical object. This means that number words do not refer to quantity - they are rather an ordered list of names applied to some items. In the short excerpt below the child Lou is counting stars in a book, over and over again, pointing her finger at one star at a time while saying the number words:

Lou (2:7): One, two, three, four, five, six, seven, eight, nine.

Arthur (3:1): That one is not nine!

Lou points at the stars while counting out loud. The second time she points at different stars than before, making Arthur react to the break in the pattern of named stars. Arthur acts as the counting rather is about "naming" the stars with number words, not differentiating any cardinal meaning. The order of naming them is then closely related to the physical objects, emphasizing ordinality.

Extending the setting

Mathematics is useful due to the transferability of notions and the abstract relationships that mathematics concepts describe. It is in this sense natural to think that generalization is a key to learning the meaning of abstract numbers. This is not uncommon in teacher-child interaction:

Elsa and Ann (adult) are sitting at the breakfast table.

Elsa (2:6): There's two (pointing at two plates on top of the other).

Ann (adult): There's another two (pointing at two other plates) and there's two pieces of bread, and there's also two (pointing at two spoons on a plate).

The teacher makes efforts to pick up the child's initiatives and generalizes the idea of sets of two. However, the observation does not reveal whether the child discerns the cardinal meaning of "two" applied to other settings as well or if the directed attention to other kind of objects takes away attention from the number's meaning of "twoness".

Comparing more-less

When numbers are used to describe sets, a possibility to explore number words' cardinal meaning opens up if sets of different quantities are made possible to compare. Subitizing enables children to compare and estimate small number of items, but this builds on the above mentioned condition, that focus is already directed towards numerosity.

Lou (2:7): (brings three different cat toys to a table) Here's two kitty cats.

Arthur (3:1): No, it's not like that, it's this many (showing three raised fingers on one hand). One, two, three (raising one finger for each said counting word).

Cardinality is undoubted critical for operating with numbers. It seems that comparison of different quantities, contrasting numbers as sets, makes this possible. Merely counting will promote ordinality as the primary aspect in focus, shorter or further on the sequence of counting words, but does not necessarily emphasize the "manyness" of the numbers.

Attention to the cardinal *and* ordinal meaning of number words

Observations show that many children, and teachers, express a rather strong focus on the aspect of ordinality, while the cardinal meaning of numbers is an aspect that is rarely opened up for exploration. The following excerpt is thereby an exception but important in our investigation of learning opportunities in preschool mathematics:

Alan (3:5): (has been asked to check how many children there are in the cloakroom) There is *one* and *one* and *one* and *one* and *one* and *one* and *one* and *one* and *one* and *one* and *one* and *one* there.

Gloria (adult): That's quite a lot. Or was there only one?

Even though the child is not using number words to express himself, he expresses number as an addition of ones. The teacher picks up the word he uses and opens up both the aspect of ordinality by referring to the number word "only one" and in the same occurrence also the aspect of cardinality when confirming that the boy had described a large number of objects.

The idea of directing attention to ordinality, as in the following excerpt, is however not always critical for a task and may thereby not be taken into consideration by the child:

Alice (2:4): (sits with a jig-saw puzzle with pieces shaped as fish and a number of dots on the board and corresponding number of dots on the fish) Just one dot and then just two (tries to fit a piece in the board).

Mary (adult): No, you have to look at the dots, here's three, then you have to see how many dots there are there [on the board]. Where should it go?

Alice (2:4): There! (pointing at the board where there are three dots)

Mary (adult): (pointing at each dot on the fish) There's one, two, three, where?

Alice (2:4): (pointing again at the board and the three dots)

Mary (adult): There's one, two, three.

Alice (2:4): It will go there (putting the rest of the pieces with 1–3 dots on their right places on the board). Here's only one dot (pointing at the last empty place on the board).

Alice's attention is directed towards the features of the fish that will not fit on the board. This conflict directs her attention to find clues or strategies to ease the task of finding right pieces. She discovers the dots and describes their difference in number "just one dot and then just two". The teacher supports her discovery but offers a counting strategy to make sure she finds sets of dots that are equally large. However, the attention to different aspects does not seem to meet, nor are they simultaneously considered as aspects of numbers. Alice did not count the previous two dots, she also seems to perceive equal number of dots if they are three, since she does not make any attempts to count in the same way as the teacher instructs her to do. It is thereby uncertain if the aspect of ordinality is necessary to emphasize since the child masters the task by focusing on the "manyness" of the sets. The number of dots is within the subitizing range, which has to be considered as a reason for not needing another strategy to compare the number of dots on the fish and on the board.

Discussion

This pilot project brings to the fore that children's conceptions of number seem to be characterized by partly numbers as names and sequence, parallel to using numbers to describe a set of objects. In some rare occasions these conceptions are emphasized simultaneously, enabling the fusion of ordinal and cardinal meaning, which is presumably an important condition for number concept development. An advanced use of number words includes both of these aspects and enables a child to solve arithmetic problems. Wynn (1992) raised this issue when presenting findings of 2–3-year olds' sense of numerosity in contrast to their emerging understanding of the counting system. The current study confirms that toddlers experience number in different ways, but provides insight to the activities and interactions where ordinality, cardinality and the fusion of them are put to the fore and thereby enable children to develop number concepts. According to Variation theory (Marton, 2015) such analyses are essential for understanding what constitutes the learning object (numbers in this case) and consequently how to support concept development by offering necessary aspect to be discerned by the child. Forthcoming studies will thereby shed more light on how the conception of numbers as names and rhyme can be challenged and related to numbers' "manyness". Critical is what role the different aspects play in number concept development; is cardinality superior to ordinality (c.f. Benoit et al., 2004), and perhaps more important, how are the aspects related in toddlers' number concept development?

It is of high pedagogical value to consider what activities toddlers encounter in preschool and what opportunities to explore ordinality and cardinality there are – the examples presented here show that it occurs in both planned and spontaneous situations. Further studies will also direct attention to teachers' acts to engage toddlers in such stimulating activities and how it can be made possible for children to extend their conceptions of numbers, as this pilot study indicates is possible.

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Perception, cognition and measurement of verbal and non-verbal aspects of the cardinal concept in bodily-spatial interaction

Morten Bjørnebye[1], Thorsteinn Sigurjonsson[2] and Tor Solbakken[3]

[1]Inland Norway University of Applied Sciences, Norway; Morten.Bjornebye@inn.no

[2]Inland Norway University of Applied Sciences, Norway; Thorsteinn.Sigurjonsson@inn.no

[3]Inland Norway University of Applied Sciences, Norway; Tor.Solbakken@inn.no

Research on how neuroscience and knowledge of human's endowment can inform educational practice is a young field. With reference to contemporary research and models that hold the idea that humans possess non-verbal inborn cognitive systems for estimating sizes and processing quantities, empirical data from a multi-case embodied design study on early learning of numbers is discussed. Based on observational data and on a discrepancy in the results of traditional and modified Give-N-tasks for determining the cardinal knower level, it is contended that designs which embrace innate spatially organized representations of numerosity through body-based metaphorical mappings, could guide the development of tools for measuring the cardinal concept.

Keywords: Cardinality, cardinal concept, measurement, embodied design, core knowledge systems.

Introduction

Recent research and discoveries in the field of cognitive development suggest that the individual's concept of cardinality in terms of fluency with exact enumeration constitutes a particular predictive factor for the development of mathematical skills (Aunio & Niemivirta, 2010). Furthermore, mastery of cardinality as a culturally-achieved word principle is regarded as a fundamental skill for later development of arithmetical abilities (Dowker, 2005). However, children find it challenging to understand the meaning of cardinality (Fluck et al., 2005). Some cognitive psychologists who develop models of numerical cognition posit a close relationship between the spatial and the numerical domains, and see the existence of spatially organized representations as the core of number meaning (e.g. Dehaene & Brannon, 2011). However, there is a dearth of research focusing on the underlying cognitive structures that support mapping between non-symbolic and symbolic representations of numbers (Mundy & Gilmore, 2009), and in particular inquiries into children's mathematical performances in naturally occurring testing scenes (Reikeras et al., 2012).

The aim of the study is to explore how the cardinal concept is measured, perceived and conceptualized in an educational design incorporating early learning of numbers. In the following, we introduce the conceptual framework of the study which is theories of core knowledge system.

The core systems of number representation and conceptual mapping

In the literature, there is a consensus that cognition is based on at least four domain-specific core knowledge systems for representing objects, actions, numbers and space, each mechanism being deeply rooted in human evolution with shared ontogenetically and phylogenetically abilities (Spelke & Kinzler, 2007). Two of these core systems are dedicated for representing numbers, i.e. the *Approximate Number System* (ANS) and the *Object Tracking System* (OTS), accounting for humans

basic numerical intuitions, and serve as the foundation for acquiring symbolic and cultural aspects of the number concept (Feigenson et al., 2004). Whereas the ANS supports rapid analog estimated representation of large magnitudes, the OTS is a cognitive system for tracking up to four objects in parallel (Burr et al., 2010). Subitizing, which is an immediate perceptual insight of the cardinal value of a small set of objects "without having to engage in conscious counting" (Wynn 1995: 36), is thought to emerge from the OTS-system. However, the exact nature and origin of subitizing is still in dispute (Piazza, Fumarola et al. 2011). Moreover, these core systems of object representations center on the spatio-temporal principles of cohesion and continuity, positing that objects maintain their connectedness and boundaries across movement in space and time (Spelke et al., 2007). Hence, from a core knowledge standpoint, this suggests that infants have an innate ability to grasp the ontological aspects of wholeness, numerosity and invariance of the cardinal concept.

A *conceptual mapping* concerns neural connections in the brain including core knowledge systems, and is described as a systematic set of relations between constituting elements where a target domain is understood in terms of a source domain (Lakoff & Johnson, 1999). Based on this model, the level of meaning in the transfer is determined by what the two domains have in common. This set of shared features or similarities is termed the "ground". For example, cardinal word knowledge within the subitizing range is thought to reflect a mapping between number terms or metaphorical expressions (e.g. "cat-four") in the semantic domain and the OTS (i.e. the cardinal value 4 is the ground). In a similar manner, approximate number word knowledge is supposed to reflect a projection between the ANS and cardinal labels such as "all", "many", "a part" or "about 20" (Gunderson et al., 2015). Thus, linguistic expressions that reflect estimated values or exact enumeration share the feature of pointing back to the collection of items as a part or as a whole, but these types of number representations are posited to be processed in different neural structures. Next, in order to illuminate how the core knowledge system is put into play in the design, a brief epistemological and ontological clarification of the cardinal concept is presented.

The concept of cardinality

The cardinality refers to the number of objects in a set, also denoted as the set size, magnitude or the quantity of a set. Formally, there are two main approaches to determining the cardinality of a set. The first method, which is associated with the cardinal word principle (Gelman & Gallistel, 1978), uses enumeration, that is the transitive meaning of counting to align the cardinal value to the set. The last number-word in a counting sequence points back to the group of tagged items, and hence ontologically reflects an aspect of the set as a whole. The second method compares sets directly using one-to-one correspondence. For example, a gestalt using two hands, two knees and two feet to tag an array of six leaves, is an embodied way of confirming that the quantity of the set of grounding limbs and the collection of leaves is the same. The non-verbal transitive relation emphasizes that the pair of knees is a part of the entire collection of six body-parts across movement in space and time. Thus, the aptitude to treat sets as wholes and parts of wholes reflects spatial and temporal features of the cardinal concept, and rests on the ability to grasp the idea that the value of the set is an invariant property across the form and shape of the boundary and the configuration of the items.

Methodology

Selection and intervention

Four 4-year olds were enrolled in a six-week program outdoors comprising ten one-hour sessions using direct methods to determine the cardinality. The cases were selected based on results on a Give-N-task for assessing understanding of cardinality (Schaeffer et al., 1974), as outlined below. Cardinal-four knower, Fia (3:9), was selected as a prototypical case, and cardinal principle-knower, Kate (4:2), and the two cardinal-one-knowers Chris (3:11) and Ted (4:1) to ensure maximum variation in the cases competences in representing numbers.

Situated in the vicinity of the kindergarten and contextualized by different N-dotted matrixes on the asphalt, the participants were guided to articulate their body-based mappings using corresponding number-metaphors (see Figure 1). For example, on a four-dotted matrix, they could embody and articulate the metaphorical expression “frog-four” performing a gestalt of four body parts tagging the dots.



Figure 1: Embodiment of spatial-structures using body parts

Data collection

The empirical data was assembled as video-recordings of intervention episodes and assessment tasks.



Figure 2: The modified small-scale (left) and large-scale (middle and right) Give-N-tasks

Pre- and post- Give-N-task. The children were asked to select a certain number of blocks from a set of fifteen blocks (e.g. “Can you give me three blocks?”). When the child had responded, the experimenter asked “Is that *N*-blocks?” If the child confirmed, a new task was given. Otherwise, the initial question was repeated. No suggestion was made to use a counting procedure to check incorrect responses. The knower level (c.f. Lee & Sarnecka, 2010) was determined by the highest number of correct responses given by the child in two out of three times (i.e. the criteria set by

Wynn, 1992), given that all preceding numbers had met the same criteria. Knower-levels above four are labeled cardinal-principle-knowers, and reflects the group that knows how counting works.

The "Create A-Metaphor-N-task" (only post-test, see Figure 2, left). The same procedure as the Give-N-task described above was applied but the question comprised the use of "number-metaphors". For example: "Can you find a cat-four?" and the experimenter asked "Is that a cat-four?"

The "Embody-Metaphor-N-task" (only post-test, see Figure 2, middle). A circle (d = 2.0 m) on the ground with 16 arbitrary distributed dots (d = 0.1 m). The experimenter asked can you jump or find a "Metaphor-N" in the matrix; for example, "Can you find a kangaroo-two and then a dog-four?" (Coded as 2+4, see Table 1). The question was repeated only if the child did not respond. No confirmatory question was posed after the child had completed the articulated embodiment. The initial position outside the circle was determined by the movement trajectory of the previous task.

Qualitative data analysis

A qualitative multi-case study approach using pattern matching and cross-case analysis as analytic techniques (Yin, 2009) was adapted to our purposes. The first step of the analysis identified unique patterns in the data material for each case, and meaningful units conveying information of non-verbal and verbal representations of numbers across testing scenes and activities were transcribed and coded in the qualitative analysis tool NVivo. A cross-case analysis was conducted in order to identify diversities, gaps and shared patterns across the cardinal-knower-level. Based on this, general patterns and discrepancies emerged and were synthesized.

Findings and discussion

Five hours of video produced 700 references in NVivo capturing situations where the four cases were engaged in bodily-spatial mappings of numerosity. Elaborated, the data shows 1182 occurrences of the seven most frequently used metaphors of articulated body-based mappings of numbers. For example, the use of the "cat-four" metaphor was reported 316 times. Moreover, the assessment shows that the cases skills in verbal production tasks span from cardinal-one-knowers to cardinal-principle knowers. With the exception of Ted who progressed to a cardinal-two knower-level, the results from the pre- and post-Give-N-tasks suggest that the intervention had no "effect". However, during intervention and on the two modified assessment-methods, the three subset-knowers showed domain-specific cardinal knowledge in terms of an ability via metaphors to map non-verbal and verbal cardinal knowledge above their assessed level in the ordinary Give-N-test. Subsequently in order to ensure an in-depth-analysis comprising the lower bound of the performances in the design, the examination will mainly focus on the behavior of Chris, this being the case with the lowest measured competence. Table 1 comprises a summary of his Give-N-results.

Give-N-Task Asked: "Can you give me N blocks?"			Create A-Metaphor-N-Task Asked: "Can you make a Metaphor-N?"	Embody-Metaphor-N-Task Asked: "Can you embody a Metaphor-N?"	
Task	**Pre**	**Post**	**Only post**	**Task**	**Only post**
One	1,2*	1,1,2	1	**Rooster-One**	1
Two	3,2,5	2,4,15,2	3,2,2	**Kangaroo-two**	2,2,2,2,2,2,2
Three	2,6,6,9,9	5,15	3,3,3	**Monkey-three**	3,3,3
Four	5,4,3	6	4,4,4	**Dog-four**	4,4
				3+1 (e.g. "Monkey-three and Rooster-one"	3+1
				2+3	2+3, 2+3
				2+4	2+4, 2+4
				4+2	2+4, 2+4
				3+3	3+3, 3+3
				4+4	4+4

* A follow up test established Chris as a cardinal-one-knower

Table 1: Results for Chris on traditional and modified Give-N-tasks

The traditional Give-N-task and the "Create a Metaphor-N-task"

According to the notion of conceptual mapping, the dissimilarities between the two things being compared in the Give-N-tasks might create an epistemic "tension" in terms of instability between two cognitive domains. Thus, and applied to the core knowledge systems, the question of discrepancy in the results narrows down to an explanation as to why the tension was resolved in a conventional manner in the projections involving ANS and OTS in the modified Give-N-tasks. Overall, the results of tasks above the three subset-knowers level of competence as measured by traditional Give-N-tasks suggest that the semantic mapping from a number word (i.e. the source) onto an unstructured collection of items (i.e. the target), did not share the cardinal value as a common feature (i.e. the ground). For example, the behavior for cardinal-one-knower, Chris, shows that the semantic expression "Give me *three* blocks" is frequently associated with verbal expressions such as "*many*" or "*all*", and in a concrete manner mapped as 2, 6, 6, 9, 9, 5 and 15 blocks respectively. Hence, the transfer gives no meaning as the mappings are reflected in arbitrary, unstructured and inconsistent distributions of blocks. Moreover, this suggests that the semantic label "three", with the exception from the first response, is mapped via ANS. However, on the two correct responses of post-testing of the two-knower-level (2,4,15,2), Chris accompanies his behavior with articulation of the metaphor "kangaroo-two" showing via the OTS an emergent ability to lean on figurative support in the mapping of the number-word "two". In contrast, on the "Create a Metaphor-N-Task", the tension of the mappings initiated by the contextualized metaphorical questions was resolved in a conventional manner showing that he is on a cardinal-four-knower level in a domain specific way. For example, on "find a monkey-three"-tasks, Chris consistently produced spatial structured assemblies of three blocks (see Figure 2, left). Hence, this suggests that the mapping from the number word "three" is processed via the OTS to produce structured configurations of sets. Furthermore, this chain of reasoning rests on the assumption that no counting procedure was employed. This premise lends support from previous research that suggests that subset-knowers seldom use any counting procedure on Give-N-tasks (Le Corre et al., 2006). Hence,

the verbal twist in the small scale "Give-N" question, from "Give-me-two-blocks" to "Can you find a kangaroo-two", suggests that the discrepancy in "knower-level" is on a linguistic level and that the knowledge assessed in the modified task is domain-specific and contextualized.

The "Embody-Metaphor-N-task" and behavior that reflects aspects of the cardinal concept

Below, we focus on the two core knowledge structures for representing numbers to examine the responses that cardinal-one knower, Chris, gave to the kangaroo-two and the cat-four task in the "Embody-Metaphor-N-task". Allowing the use of the commutative property of addition, the results in Table 1 show that Chris responded correctly to all four tasks, and the behavior was observed as articulated bodily-spatial representations "Kangaroo-two, Cat-four". Since Chris is standing outside the marked boundary of the matrix, we suggest that he cognitively represents the whole assembly of 16 dots using ANS. Based on the verbal instruction "kangaroo-two", Chris had to visually identify a configuration of dots that matched the metaphorical expression (see Figure 2, right). Although it is a difficult task to account for when the two systems for representing numerical information are activated, verbal- and non-verbal empirical data in combination with experimental findings of the signatures of the OTS and ANS, could provide supporting evidence to the various hypothesis (Barner et al., 2008). For example, one hypothesis posits that the first "container" was visually identified via a metaphor-word-mapping onto OTS, and the rest of the dots were cognitively processed as an estimated quantity through ANS. This suggests that the whole assembly of dots was decomposed in two subsets, one part being exact enumeration via OTS, the other being a constellation of the whole, i.e. an estimated number representation via ANS. Another explanation is that Chris subsequently processed the two dots via the OTS as a part of a whole estimated set. Either way, these lines of reasoning finds support in the claim that ANS interacts with OTS (Piazza, 2010), and moreover the suggestion that ANS constitutes a support in basic numerical processes of small numbers (Feigenson et al., 2013). Moreover, the findings of Burr et al. (2010) show that subitizing (OTS) rather than estimating needs attentional resources. Thus, a third hypothesis, which also lends support to the claim that ANS is defined for large and small sets (Cantlon et al., 2006), suggests that the two-dotted configuration was initially identified as an estimated value via the ANS, and as a result of an increased use of attentional resources, was subsequently processed as an exact numerical magnitude via the OTS.

Overall, the four cases demonstrated flexibility in their non-verbal and linguistic-metaphorical representations of numbers, and the diversity of embodied strategies was shared across cardinal knower-level and counting skills. However, the quality of the cases use of cultural skills for exact enumeration differs. While Kate as a cardinal-principle knower provided exact enumeration of embodied additive structures such as 4+4+4 and 1+2+3+4, the subset-knowers could represent the same arithmetic structure via articulated bodily-spatial mappings, for example three times "Cat-four" or "Cat-four, dog-four, bear-four" and "Rooster-one, kangaroo-two, monkey-three, cat-four". Hence, this gap in competence might inform of a potential learning trajectory.

Concluding remarks

The results reveal a discrepancy in sub-set knowers' behavior across representational tasks of numbers. In particular, the data show that subset-knowers possess an ability to map non-verbal and figurative expressions of cardinality in modified small-and large-scale Give-N-tasks above their

knower-level as assessed in traditional Give-N-tasks. We suggest that the most promising explanation concerns the suggestion that the formation of the cardinal concept rests on the core knowledge structures for representing numbers synchronized via direct methods to determine the cardinality. Based on this, the full bodily-spatial metaphorical mappings of the matrixes provided coherence and meaning to the cases ideas of invariance, wholeness and numerosity. Hence, the results correspond to a growing body of evidence suggesting the existence of a fundamental link between the non-verbal and verbally-created system for processing numbers (Feigenson et al., 2013). Furthermore, the consistency of the results in the modified Give-N-tasks suggests that the participants managed to transfer their metaphorical number knowledge across testing scenes. Thus, the design follows contemporary views on cognition that see mathematical ideas and reasoning as "embodied" and "imaginative" (Niebert et al., 2012), suggesting the use of metaphors and imageries as powerful "thinking tools" (English, 2013). Hence, the examination emphasizes that the cardinal value is a linguistic surfacing manifestation of a deeply grounded spatial structured concept, and that the use of metaphors might merge these two domains in a complementary manner. In conclusion, the examination shows that authentic assessment during intervention and the adapted Give-N-tasks provides a fine-grained picture of the multiple dimensions that proficiency of the idea of cardinality reflects.

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Kindergarten children's argumentation in reflection symmetry: The role of semiotic means

Svanhild Breive

University of Agder, Norway; svanhild.breive@uia.no

In this paper I investigate the characteristics of children's argumentation when they work with reflection symmetry. Using Toulmin's (2003) model for substantial argumentation, I illuminate structural aspects of the ongoing argumentation. In addition, I analyse the children's argumentation with respect to their use of semiotic means. Results show that children are able to argue for a claim in a quite complex manner. The study also illustrates the extensive use of semiotic means in children's argumentation. In every element in the argumentative structure, children use gestures and other semiotic means to mediate their ideas. It is actually impossible to make sense of the ongoing argumentation without considering the use of semiotic means.

Keywords: Argumentation, kindergarten, gestures, semiotic means.

Introduction

How children communicate their mathematical ideas is an important aspect in the attempt to understand children's reasoning in mathematics. In kindergarten children experience mathematical concepts through play and interaction with others. In their communication they justify and explain their mathematical ideas and in return they need to consider other's ideas and arguments. Thus argumentation can be seen as important for fostering children's mathematical learning.

This study is situated within a research and development project called the Agder project[i] (AP). One of the aims in the project is to investigate how researcher designed mathematical activities, developed in the project, stimulate mathematical competences. In this case study I observed one kindergarten teacher (KT) and a group of six 5-year-old children engaged in mathematical activities about reflection symmetry. The aim of this paper is to examine the characteristics of children's argumentation when they work with reflection symmetry. Furthermore, I examine what role 'semiotic means' (e.g. objects, linguistic devices and signs) play in the ongoing argumentation.

Following Toulmin (2003) and Krummheuer (1995), I regard argumentation as the practical business of choosing statements that serve the purpose of making an initial assertion reasonable and accountable for others. Argumentation is the production of an argument. An argument is then "the final sequence of statements accepted by all participants" (Krummheuer, 1995, p. 247). In addition, several arguments can serve as units in an expanded argumentation which again constitutes a new and extended argument. Toulmin also recognises that single statements can contain argumentative features. Just by making a statement you put yourself in a position of potentially being questioned.

Argumentation, acknowledged as an important means for enabling young children's mathematical reasoning, can be promoted through a dialogic approach to teaching (Mercer & Sams, 2006; Yackel & Cobb, 1996). Despite the increased focus on the role of mathematical argumentation for enabling young children's mathematical reasoning, little research has focused explicit on the role and characteristics of mathematical argumentation at kindergarten level. Pontecorvo & Sterponi (2002) found that children's reasoning in preschool activity unfolded "through complex argumentative

patterns" (p. 133). They emphasised that teachers should pay attention to the different ways children argue in order to facilitate children's "possibilities to practice, enrich and refine argumentative resources they have already acquired" (p. 139). Tsamir, Tirosh, and Levenson (2009) investigated different types of justification given by children between five and six years old, working with number and geometry tasks. Their study shows that young children are able to justify their statements by using appropriate mathematical ideas. Some children, in contrast, used their 'visual reasoning' as a way to justify their statements. When the researcher asked how they could know which bunch of bottle caps had more they answered "because we see", and they felt no further need to justify their answer or did not know how to do it. Dovigo (2016) investigated how argumentation promoted collaboration and problem solving in preschool (age 3-5). By comparing different ways of how argumentation took place in teacher-talk and peer-talk they found that peer-talk contributed very positively for promoting collaboration and problem solving. But at the same time they emphasised that if the teachers were able to guide the debate in a careful and exploratory way the teacher guidance could be a positive contribution to the development of the argumentation.

Theoretical framework

My theoretical stance is rooted in a sociocultural paradigm where interaction is regarded as the very engine of learning and development, (Vygotsky, 1978). As a consequence of adopting this theoretical stance, I regard argumentation as a cultural and historical activity. Children are not naturally born with the ability to argue. Argumentation is a communicative pattern which they learn through interaction with more knowledgeable others.

Interaction, specific for human beings, is characterised by the use of tools and especially by the use of language (Vygotsky, 1978). In recent years there has been a growing interest to study the interplay between gestures, language and thought both in mathematics education and in other domains. McNeill (2005) developed a theory where he regarded gestures as an integral part of language, not merely as a support for language. He regarded gestures as having an active and inseparable role in language and thought.

Not only gestures have been recognised as important for human reasoning. Radford, Edwards, and Arzarello (2009) talk about the importance of the multimodal nature of cognition; how different sensorial modalities – tactile, perceptual, kinesthetic become integral parts of our cognitive learning processes. Radford's (2002; 2003) theory of knowledge objectification emphasises how gestures, bodily actions, artifacts, (mathematical) signs and speech in cooperation affect mathematical reasoning. A special category of semiotic means of objectification that Radford (2002) considers is deixis. Deictic terms are words that have the function "to point at something in the visual field of the speakers" (p. 17), and cannot be fully understood without additional contextual information (e.g. "here", "there" "that", "this" etc.). All semiotic means play a significant role in mathematical mediation and reasoning. "Each semiotic means of objectification puts forward a particular dimension of meaning (signification); the coordination of all these dimensions results in a complex composite meaning that is central in the process of objectification" (Roth & Radford, 2011, p. 78).

The concept of argumentation used in mathematics and mathematics education is often related to the production of proofs. It is nevertheless important not merely to connect the concept of argumentation to formal logic as found in mathematical proofs. Toulmin (2003) distinguishes

between analytic argumentation, which is used in production of mathematical proofs, and substantial argumentation which is informal argumentation used in everyday practices. Substantial argumentation does not necessarily have a strict logical structure. Substantial argumentation gradually supports a statement by presenting relationships, explanations, background information, etc. (Krummheuer, 1995). Toulmin (2003) strongly emphasises that substantial argumentation should not be regarded weaker as or less important than analytic argumentation.

Toulmin (2003) developed a model for analysing structural and functional aspects of substantial argumentation with the aim to illuminate how statements are organised for the purpose of constituting an argument, and how a conclusion is established through the production of an argument. In Toulmin's model the core of an argument is based on three elements: claim (C), data (D) and warrant (W). The claim is an initial statement, for example an assertion or an opinion about something. To support the claim, the arguer needs to produce data. Data are facts or statements on which the claim can be grounded. A warrant is a justification of the data with regard to the claim. The warrant holds the argument together. It points to the relation between the data and the claim.

In addition, Toulmin's (2003) model contains three other elements, backing (B), qualifier (Q) and rebuttal (R). A backing is a statement that supports the warrant. It is like a special case of data that is provided as evidence for the warrant. The purpose of a backing is to answer "why in general this warrant should be accepted as having authority" (Toulmin, 2003, p. 95). A qualifier says something about the extent to which the data confirm the claim. Words like 'probably', 'presumably' etc. are often used as qualifiers. Rebuttals refer to exceptions or conditions under which the claim is true, often used subsequent to a qualifier, exemplified as "The claim is true except/unless/only if …".

Method for data collection and data analysis

In this case study I observed one KT in the focus group of AP and his group of six 5-year-old children engaged in mathematical activities about reflection symmetry. The activities had been developed in the AP, and as part of an in-service program for the focus group the KTs were asked to implement a number of activities with their children. I visited the kindergarten on two occasions with a one week interval. It was the KT himself that decided to work with reflection symmetry activities on both occasions. The method for data collection was observations and the sessions were video recorded and field notes were made.

I regard argumentation as a sequence of statements (both verbal and non-verbal) that serve the purpose of supporting an initial claim. Thus one criterion for selecting episodes from the transcript was that they should contain verbal communication and have more than one utterance from the children. Another criterion was that the episodes I selected should contain mathematics, and they should be linked to the lesson aim (reflection symmetry). In total I found 11 episodes from the transcript using these criteria. Ten of these episodes had more or less an argumentative structure. Six of the episodes had more than two turns and more than two argumentative utterances from the children. These episodes were analysed in depth according to Toulmin's (2003) model to identify the argumentative structures. In addition, I analysed each of the six episodes from a multimodal perspective. In fact, I *had* to look at multimodal aspects in order to be able to differentiate between the elements in the argumentative structure. I did not focus on any specific semiotic means and their significance for children's reasoning. Rather I focused on what kind of semiotic means children

used with respect to the different elements in the argumentation, and what role they played in constituting the argument.

Results

In this section I will present the analysis of one of the six episodes to illustrate the structure of children's argumentation, and what role semiotic means play in the ongoing argumentation.

In advance of this episode the children have been asked, by the KT, to find things in the room which they think are symmetric or as the KT says "equal on both sides". Each child is then asked to explain why they think the toy they have chosen is equal on both sides. In this particular episode one of the boys (John), who has chosen a trolley, is being asked to explain why he thinks the trolley is equal on both sides (or more precise; he is being asked if he think the trolley *is* equal on both sides).

KT: Maybe we should start with John, since he has a very large thing. John, is this equal on two sides?

John: Mmm (2)[ii] There (2) ((He lifts his trolley up from the table, and holds it in a straight forward position. Then he says "there" and nods his head)).

John: and there… ((He rotates the trolley 90 degrees, showing the side of the trolley and then nods his head while saying "there")).

KT: Aha!

John: and (2) there. ((He rotates the trolley 180 degrees, showing the other side of the trolley and nods his head again while saying "there")).

Elias: And there and there. ((Elias has already paid attention to the situation)).

KT: Can you see if this is equal Elias?

Elias: Look…

Elias: There, there (.) there, there (.) there, there and there, there. (.) And there and there, and (1) everywhere. ((He is pointing with his index finger to show where he thinks the trolley is equal. When he says "everywhere" he is letting his whole hand swipe over the trolley)).

KT: ((The KT lifts up the trolley and tries to show the symmetry line and explaining how the trolley is equal on both sides of that line)).

Elias: Everything is equal on both sides, even the wheels.

The structure of children's argumentation

Before this episode each child was asked by the KT to choose a thing that they thought was equal on both (two) sides. In this episode John has picked a trolley and just by doing so he has implicitly made the claim that the trolley is equal on both (two) sides. When the KT asks John; "John, is this equal on two sides?" John's claim is being challenged. The KT actually asks a yes-no question, but the question is still a quest for explanation or justification.

To argue for his claim, John lifts the trolley up in the air, like he wants to make it visible to the others. The first "there" and the first nod is also a part of this visualization which together constitute

the *data* he presents. It is important to notice that at this moment he holds the trolley in a straight forward position from his own point of view. He is not referring to any particular equal points on the trolley. The trolley itself is being presented as the data that supports the claim.

Then he is trying to present a *warrant* for his data by rotating the trolley 90 degrees while saying "there" and then back again 180 degrees while saying "there" again. Each time he is saying "there" he is nodding his head. John is presenting the warrant as two particular sides that are equal, and exemplifies the equality. The warrant (the example) relates the data (the presentation of the whole trolley) to the claim (the trolley is equal on two sides). Considering the time John is using while presenting the warrant and the way he utters the second and third "there", John does not seem very confident in his presentation of the warrant. Nevertheless, when John was walking around in the kindergarten trying to find a thing that had two equal sides, he considered the trolley for some seconds before he took it back to the table. This indicates that his choice was not completely random. John seems certain that the trolley is equal, but he is not quite certain how to justify it.

After John has presented the second "there", the KT utters "aha" (with a rising intonation at the end). By this utterance the KT gently appreciates John's contribution, even if the two sides that John presented thus far were not equal. While presenting his data and his warrant, John waits several seconds, and it seems that the KT thinks that John has finished his explanation after the second "there". From the children's point of view, the KTs "aha" gives Johns contribution authority and can be regarded as a backing of John's warrant-attempt. But from the KTs point of view, the "aha" was not meant as a backing, only as a gentle appreciation of his contribution.

Elias then contributes to the argumentation. By the utterance "and there and there" and a pointing gesture he is presenting another warrant for John's data. Elias is talking faster and more concisely than John. Because he is using his index finger rather than nodding he is also more precise in his communication and is able to point on specific points on the trolley, like the joints and the handles. The way Elias communicates indicates that he is more confident and has more knowledge about reflection symmetry. Elias is actually presenting several warrants for the data. Every time he says "there and there" and points at different corresponding points, he gives a new warrant. By presenting particular corresponding points, each warrant is exemplifying exactly where the trolley is equal. By repeating several almost identical warrants (presenting several examples) it seems that he is trying to communicate that *every* point on one side has a corresponding point on the other side.

After presenting several warrants Elias says "everywhere" while he is swiping his whole hand over the trolley. I interpret this as a generalisation of his previous statements (warrants), and thus a backing for the warrants because it answers "why in general this warrant should be accepted as having authority" (Toulmin, 2003, p. 95). The warrants are not independent examples of equality rather examples of a more general property of reflection symmetry.

When the KT shifts his attention to Elias in the middle of this episode, Elias answers by saying "look". The intonation of the utterance indicates that he is only introducing his coming explanation. I interpret his utterance as synonymous to "let me explain".

At the end of this episode, Elias says "everything is equal on both sides, even the wheels". The use of the word "even" in this sentence is very interesting. The word "even" I interpret as a qualifier for the claim. It says something about to what degree the data confirm the claim. Usually words like

"probably" or "presumably" are used as qualifiers, but in this case Elias is indicating that he is *very* certain that everything is equal on both sides, by saying "even the wheels". It seems that the probability for everything being equal increases since 'the critical points', the wheels, are equal. Why Elias regards the wheels as important points is hard to tell.

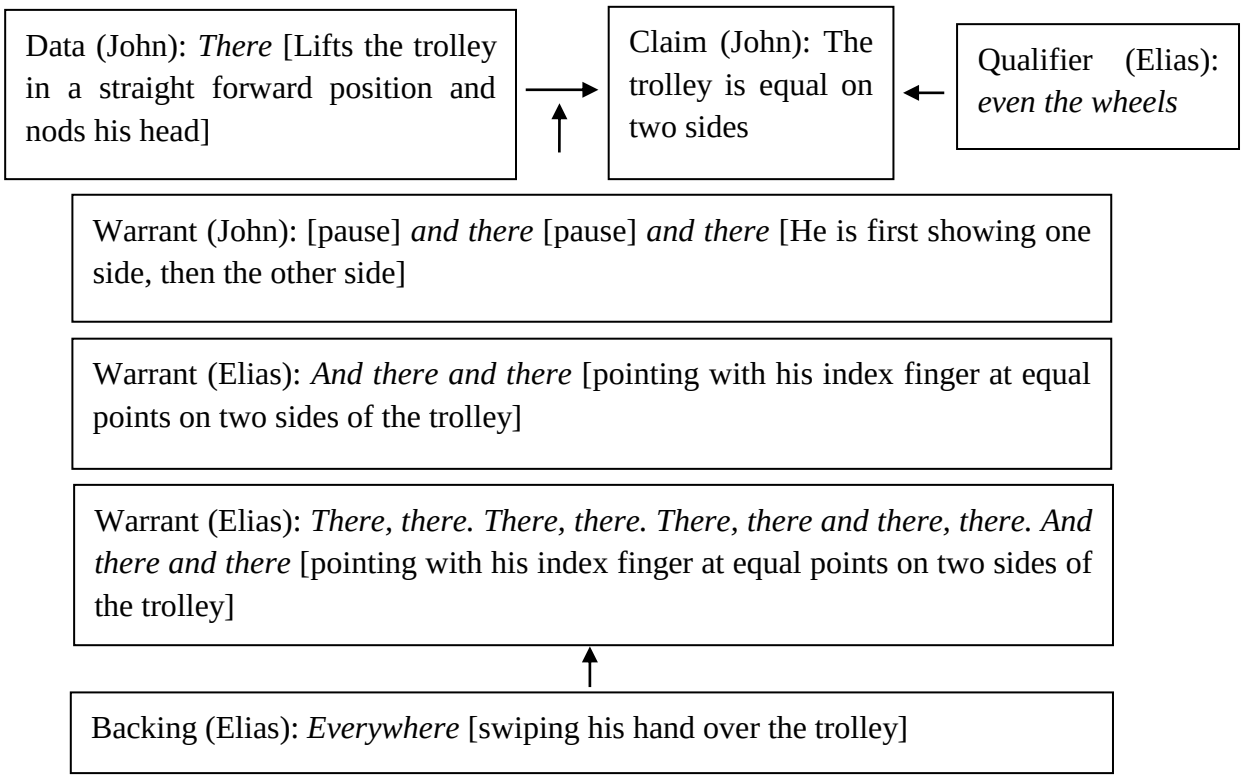

Figure 1: The structure of children's argumentation

This example illustrates the complexity of children's argumentation. They are able to present more than only the core of an argument. In this episode I found that some children are able to present both data, warrant, backing and even qualifier for a claim. In another episode that is not provided in this paper (because of the limited space) Elias was also able to present a rebuttal. He was able to modify his claim by giving examples of exceptions.

Discussion

This study shows that young children are able to argue for a claim in a quite complex manner. Using Toulmin's (2003) model to illuminate structural aspects of the children's argumentation, the results show that some children are able to use several of the elements in the model in their argumentation.

This study also illustrates the extensive use of semiotic means in children's argumentation. In every element in the argumentative structure, children use gestures and other semiotic means to mediate their ideas. This illuminates the significant role that gestures and other semiotic means play in children's communication and especially in their argumentation. (cf. McNeill, 2005; Radford, 2002; 2003; Roth and Radford, 2011). Deixis, in particular, are extensively used in the argumentation above. Both the data that John presents and the warrants that John and Elias present are based on the deixis "there" and the related pointing and nodding gestures. Even if John and Elias use different signs for mediating their ideas, both the nodding and the pointing gestures serve the same purpose,

namely to give contextual information to the deixis "there". It is actually not possible to get the whole meaning of the words "there and there" without including the pointing and nodding gestures.

The deixis and the related pointing and nodding gestures are not the only significant semiotic means in this argumentation. To be able to distinguish between the data and the warrant that John provides I had to interpret his related actions. When he presents the data he holds the trolley in a straight forward position, he is not referring to any particular equal points, only presenting the trolley as a whole, as if he wants to show the equality. The way he presents his claim corresponds with one of the findings in Tsamir, Tirosh and Levensons (2009), that some children based their justification on 'visual reasoning'. The trolley itself is being presented as the fact that supports the claim. In the warrant he is presenting two corresponding sides, as if he wants to give an example of the equality. Without interpreting these actions, it is impossible to distinguish between the data and the warrant, and thus fully understand the structural aspects of the ongoing argumentation.

The repetitive presentation of Elias' warrants and the swiping hand that generalises the repetitive warrants are other important semiotic means in the argumentation. By repeating "there and there" with corresponding pointing gestures Elias indicates that *every* point has a corresponding point on the other side of the symmetry line. When Elias says "everywhere", he swipes his hand over the trolley. This swiping gesture plays a significant role in the generalisation process of the points.

The results from this study point to significant features of children's argumentation and give important insights into *how* children argue. I think teachers could benefit from paying attention to the different ways children argue and being aware of the structural aspects in children's argumentation in order to provide opportunities for improving children's mathematical communication and reasoning (cf. Dovigo, 2016; Pontecorvo and Sterponi, 2002). But to be able to do so, the KTs also need to pay attention to how children make use of semiotic means in their argumentation. The Toulmin model revealed structural aspects of children's argumentation, but these structural aspects would not have emerged without considering the use of semiotic means. In line with Roth and Radford (2011) I would argue that all the different semiotic means play a significant role in the constitution of meaning.

In the example above we saw that Elias was able to use several elements in the model and demonstrated more confidence in his argumentation than John. A possible explanation could be that Elias is further in his appropriation of the properties of reflection symmetry than John. Maybe there is a correspondence between how far children have appropriated a certain subject and their ability to use several elements in the Toulmin model. This is thus a suggestion for further research on this interesting topic.

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[ii] (2) indicates approximately 2 second pause, (1) indicates approximately 1second pause and (.) indicates a pause less than one second

Making numbers: Issues in using manipulatives with young children

Sue Gifford[1], Rose Griffiths[2] and Jenni Back

[1]University of Roehampton, United Kingdom s.gifford@roehampton.ac.uk

[2]University of Leicester, United Kingdom rnag1@leicester.ac.uk; jenni@backfamily.me.uk

The Making Numbers research Project, funded by the Nuffield Foundation, has developed guidance for teachers of 3 to 9 year olds on the use of manipulatives in the teaching of arithmetic. The project, which consisted of a literature review, survey of teacher use and small-scale teaching investigations, identified key principles and issues for the use of manipulatives. This paper gives a brief overview of project findings, which are reported elsewhere, and discusses issues relating to the early years from the literature review.

Keywords: Early childhood mathematics, finger counting, patterns, representations, manipulatives.

Background to the project

Making Numbers is a Nuffield Foundation funded project to develop research-informed guidance for teachers of three to nine year olds on the use of manipulatives to teach arithmetic. The project has run for two years and includes a literature review, a survey of current practice and the development of exemplars of good practice through observation and small-scale teaching investigations. The resulting guidance has been published as a fully illustrated book for teachers with accompanying animations for use with children (Griffiths, Back, & Gifford, 2016). The findings of the project are reported in full elsewhere (Griffiths, Back, & Gifford, in press): this paper gives a brief overview, highlights some issues concerning the use of manipulatives with young children and exemplifies key pedagogical principles.

Our definition of manipulatives is "objects that can be handled and moved and are used to develop learners' understanding of a mathematical situation" (Gifford, Back, & Griffiths, 2015, p.1). This includes the use of everyday and structured materials, characterized by a pedagogical intention, which concurs with recent studies (Swan & Marshall, 2010; Carbonneau, Marley, & Selig, 2013). The project focus was on number sense, prioritizing for this age range counting, cardinality, comparison and composition of numbers. Hence the project title, 'Making numbers', emphasises flexibly decomposing and recomposing numbers, which was identified by Boaler (2009) as key to mathematical achievement.

From the survey and interviews with teacher groups, we found that manipulatives were mainly used in the early years and with older low achievers (Gifford et al., 2015). The most common were counters, interlocking cubes and Numicon 10-frame based number plates (Wing, 2001), followed by place value apparatus. Teachers' choice was influenced by commercial availability and past government policy, but for some it was serendipitous. Most expressed a lack of confidence about how to use manipulatives to teach different aspects of number.

The methodological approach of the literature review was to consider studies and theories from a range of perspectives, in order to gain insights into factors affecting children's learning. Sources included the history of pedagogy, cognitive and social constructivist theories of learning and quantitative and qualitative empirical studies. We found that experimental studies were contradictory

and inconclusive, identifying only crude factors, such as length of the study or the amount of instructional guidance, as shown by Carbonneau et al.'s (2013) meta-study.

The conclusion was that the effective use of manipulatives depended on some key pedagogical principles (Gifford et al., 2015). These included:

- the careful matching of both manipulatives and activities to the mathematical focus
- the identification and assessment of children's prerequisite understanding
- activities involving comparison, equivalence, analysis and generalisation
- discussion, requiring children to use manipulatives to justify reasoning
- linking manipulatives to abstract symbols
- creating an inclusive mathematics learning community.

Here we identify some issues from the literature review about teachers' use of manipulatives with young children. These concern fingers as manipulatives, discrete and continuous models for number and the learning potential of pattern activities. The pedagogical principles above are exemplified in a small case study from a teaching investigation.

Fingers as manipulatives

The benefits of drawing from a range of perspectives are evident in the consideration of fingers as manipulatives, which are of particular relevance to teachers of young children. Their significance has been underlined by neurological research: brain areas representing fingers and numbers are closely related, according to Wood and Fischer (2008). Young children use mental finger representations for numbers more than adults, suggesting that finger use is significant for the development of number understanding. Gracia-Bafalluy and Noel (2008) found that young children's finger awareness was predictive of their mathematical competence and that training in distinguishing fingers resulted in improvements in subitising, counting and comparing numbers. However, the way that fingers are used for counting varies in different cultures, including counting three to a finger, or counting hands as fives. Bender and Beller (2012) argued that the resulting number concepts also vary according to different languages, some of which support 10s structures more transparently. Jordan (2003) also found that children from low-income families tended not to use fingers to solve problems. This suggests that home practices differentially support children's number understanding and need to be taken into account by teachers.

How should children be taught to use fingers? Sarama and Clements (2009) advised teachers not to discourage children from using fingers until they were confident with mental strategies, in order to prevent reliance on finger counting. Marton and Neuman (1990), using a phenomenological approach, found that older higher attainers, who used recall and derived facts, showed numbers of fingers 'all-at-once', whereas low attainers continued to rely on using fingers to count on and back. 'Finger numbers' encouraged children to analyse numbers, developing subitised images and part-whole number understanding. This model therefore represents key number concepts more effectively. Sinclair and Pimm (2015) reported that using the app 'Touchcounts' rapidly resulted in three year olds showing 'all-at-once' finger numbers. This seems an important skill which young children might also learn in other ways, for instance, when singing number rhymes.

Discrete and continuous models of number

One current issue relating to young children is about the relative merits of discrete and continuous models of number, for instance using either counters or colour rods which represent numbers as lengths. Usually in England numbers are introduced by counting separate items. However, when children are later introduced to number lines, numbers are represented by intervals on a continuous line and children often count the numerals or marks, rather than the intervals between them. Sarama & Clements (2009, p. 119) recommended caution in using the number line "as a representation for beginning arithmetic", raising the issue of when and how to introduce it. However, neuroscientific evidence suggests that people intuitively see some kind of mental number line (Wood & Fischer, 2015). This supports renewed interest in teaching young children about number based on measuring lengths, as proposed by Bass (2015) following the approach of Davydov (1975). Bass pointed out that number "arises from measuring one quantity by another, taken to be the 'unit'" (2015, p. 100). He argued that introducing numbers as chosen units for measuring quantities provides a more coherent model of numbers which can also include fractions. It supports multiplicative and proportional reasoning and early understanding of algebraic principles such as inverse and commutativity. This argument also supports the use of colour rods.

Fuson (2009) argued that number paths or tracks are more comprehensible for young children, as they present adjacent squares which are more obviously countable than intervals on a line. Laski & Siegler (2014) found that children who regularly played number track games, reading the numbers aloud as they counted moves, improved their awareness of number magnitude and arithmetical achievement. It has subsequently been argued that children's engagement with ordinal number has been overlooked as a means of developing understanding. Sarama & Clements (2009, p. 93) suggested that children were linking the distance model with counting moves and numbers spoken:

> connections between the numerical magnitudes and all the visual-spatial, kinaesthetic, auditory and temporal cues in the games (i.e., all the magnitudes increase together: numerals, distance moved, number of moves, number of counting words etc.) may provide a rich mental model for a mental number line.

It therefore seems that we may underestimate children's capabilities to combine the various models to construct complex networks of numerical understanding. According to Clements and Sarama (2009), four year olds may be in the process of developing a mental number line connecting different 'quantification schemes', including discrete number ideas based on subitising and counting with continuous ideas about duration and length.

Some manipulatives have attempted to integrate discrete and continuous models of number. For instance, bead strings present countable beads, usually coloured in groups of five or 10, along a line: Beishuizen (2010) reported that teaching children to find a number on a bead string and then on a number line helped them to develop mental calculation. Some older apparatus, such as Montessori's numeric rods and the Stern counting board, present rods arranged as 'staircases' of discrete cardinal numbers, with numerals alongside. These afford a clear image of the value of numerals in sequence increasing by one, but are not common in schools, having been replaced by rods made of interlocking cubes. An approach we have developed is to use centimetre cubes alongside a ruler: this has the advantage of providing countable objects, while demonstrating that the individual items counted are

in the intervals between the numerals. This not only shows the numbers increasing by one and but also that each successive number includes all previous ones, the idea of hierarchical inclusion or 'nested numbers' (Clements & Sarama, 2009, p. 20). Combining both models also provides opportunities for children to discuss what is the same and what is different, as advocated by Harries, Barmby and Suggate (2008). This is an area where teachers need to be aware of the complexities involved in the differences between continuous and discrete models and further investigation is needed of how children reconcile these.

The potential of pattern

There is a range of possible factors affecting young children's understanding of manipulatives used to represent number relations. From Piagetian theory young children are seen as needing sensory experiences in order to learn, and from Vygotskian theory as being able to use symbols, such as fingers, from an early age. Recent theories suggest differences in young children's spontaneous and intuitive appreciation of number and pattern. Hannula and Lehtinen (2005) found that some very young children displayed a tendency to spontaneously focus on numerical features of a situation (SFON), while others did not, and this predicted later achievement. Mulligan and Mitchelmore (2009) found that children varied considerably in their awareness of mathematical pattern and structure (AMPS) and that this was also linked to mathematical achievement. They identified different stages of AMPS: some children focused on non-mathematical surface features of patterns, some children noticed one or two mathematical elements in a pattern and others could reproduce and continue patterns, by identifying the components and relationships. Mulligan and Mitchelmore also found that pattern awareness could be taught to young children, suggesting that this facility is learned from experience: Papic, Mulligan and Mitchelmore (2011) reported a successful intervention with four year olds, which improved awareness of pattern and mathematical structures. This presents a potentially powerful approach, both in building on young children's strengths with visuo-spatial memory and in developing pre-algebraic understanding.

Staircase patterns

One of our teaching investigations, into the learning of numbers to 20 with a class of 30 six year old children, exemplified the importance of pattern awareness as well as some of the key principles identified in the literature review. In a series of small group teaching sessions, we aimed to teach children about the numbers from 10 to 20, sometimes referred to as the 'teen numbers' (Gifford & Thouless, 2016). We used a 'staircase' pattern of 'teen numbers' made of rods of interlocking cubes arranged in number order. The rods for one to 10 were of different colours, with the rod for ten consisting of a stick of green cubes; the rods for 11 to 19 then had 10 green cubes with the rods for numbers 1 to 9 attached, repeating the same colour sequence. Numerals were presented alongside the rods 1 to 9. We invited the children to find a single rod, such as 18. We found that children used a variety of strategies to do this: most picked a rod at random and counted the cubes from one. Magda counted in ones along the 20 rod and then along the 18 rod and explained "I knew one less would be 19 and one less would be 18". Only one child identified rods by counting on from ten. Some children made errors in counting, either in the word sequence, or in matching words to cubes when pointing.

This activity demonstrated the pedagogical principle of carefully selecting manipulatives to match mathematical concepts and showed how this affected the accessibility of those concepts. We wanted

the children to recognize the sticks as representing the teen numbers as composed of ten and another number, but it was not easy to see there were ten green cubes and recognize the 'ten', unlike with Numicon plates which children had previously identified as 'ten'. However, the staircase arrangement enabled some children to recognize the 'one more than' pattern, which would not have been so evident with Numicon. When trying to find the 'thirteen' rod, six year Lucasz, who was not yet fluent in English, said, 'because after 2, 3. Because it's like ten, 1, 2, 3.' In doing this he made a sweeping gesture across the 11, 12 and 13 rods, then across to rods 1, 2 and 3. Later, Magda, whose home language was Czech and who was more fluent in English, said, 'It's a bit like counting up stairs. Like counting 1, 2, 3 but 11, 12, 13.' The children thereby implicitly identified two types of pattern, a repeating units pattern and the stair pattern of equal intervals of one, which is the most basic arithmetic sequence. This was evidence of early algebraic thinking, in that they noticed mathematical features, identified the relationship between elements and observed regularities (Blanton et al., 2015, cited by Kieran, Pang, Schifter, & Ng, 2016).

This activity thereby also demonstrated the principle of prompting analysis, by simply asking children to say how they identified a number. In our teaching investigations we have been impressed by the way patterns can fascinate children. On one occasion, when children were rushing by our table outside a classroom on their way to lunch, two six year old boys halted in their tracks in order to gaze at the staircase pattern and one child sat down to join us. We found that patterns like these engaged a range of children, who had differing expertise with numbers, in solving problems and noticing relationships. While some children may have had high levels of AMPS (Mulligan and Mitchelmore, 2009), the striking pattern made with manipulatives helped all children to focus on number relationships. Presenting number patterns in this way contrasts with other approaches which advocate that early years mathematics pedagogy should be based on realistic "context situations" (van den Heuvel-Panhuizen, 2008, p. 20).

The activity also prompted discussion, stimulating both Lucasz and Magda to express themselves creatively using language and gesture, as shown above, making connections and elaborating ideas. Lucasz may have been an example of a child using gesture to express emergent mathematics learning, as described by Garber, Alibali and Goldin-Meadow (1998) or he may have already been familiar with this pattern and been able to articulate it in Polish: we do not know. However, this showed that a lack of fluency did not prevent a child in the early stages of learning a language from trying to express a mathematical relationship, and also the importance of gesture in supporting mathematical discussion between children who are not using their home languages. The manipulatives pattern thereby supported the linking of different modes of representation, not only verbal and visual, but also kinaesthetic and, through the use of numeral cards, abstract symbols.

This activity also exemplified the need to identify and assess prerequisite understanding. Some children could not reliably count objects to 20, despite having been assessed as doing this the previous year. Most of the children counted from one instead of counting on from 10, even when it was pointed out to them that there were always 10 green cubes. This may have been because they lacked skills of counting on or were not yet able to unitise ten as an item. As Cobb (1995) pointed out, if children do not have a concept of 'a ten' they will not be able to 'see' it even when demonstrated. Similarly, some children could add 10 and 3 and instantly say "13", but seemed not to understand the inverse relationship sufficiently to apply this to decomposing 13 into 10 and 3 and did not use this knowledge

to identify 13 as 10 and 3. Therefore some children may have lacked the prerequisite understanding and skills to access what the manipulatives were intended to represent.

The final principle shown by this example, is that of an inclusive learning community. The children were not in 'ability' groups, as is common in English early years classrooms, in which case Lucasz and Magda would not have been in the same group. We do not know if Magda would have articulated the pattern in this way, if she had not first listened to Lucasz, who was able to hear his idea expressed differently. Latoya, in her second session, identified the 15 rod by counting on from 10, saying, '10, 11, 12, 13, 14, 15'; this was a skill she had not used previously but had observed another child using in the previous session. Both the lack of grouping and the open activity, allowing for a range of solution strategies, facilitated children in learning from others.

In our study we also observed successful practice, particularly with colour rods, which similarly showed children engaged by patterns and stimulated to discuss mathematical relationships. These experiences imply that developing children's focus on patterns is a promising avenue for mathematical pedagogy in the early years.

Implications

The issues discussed here suggest some potentially profitable avenues for future research with mathematical manipulatives and young children and some implications for practice. Firstly, we might build on home practices in finger counting and develop young children's use of 'all at once' finger numbers. We also might investigate how children reconcile differences between discrete and continuous models of number, through comparing and discussing representations. These issues highlight the need to consider carefully exactly how manipulatives might foster learning of particular aspects of mathematics. There are promising avenues for early years mathematics pedagogy in using manipulatives to build on children's interest in patterns and to develop children's own expressions of mathematical relationships. However, an important prerequisite for all of these is a focus on children's understanding of mathematical relationships rather than on performing calculations, as suggested by Ma (2015), which also has implications for early years mathematics curricula.

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A kindergarten teacher's revealed knowledge in orchestration of mathematical activities

Per Sigurd Hundeland, Ingvald Erfjord and Martin Carlsen

University of Agder, Norway; per.s.hundeland@uia.no, ingvald.erfjord@uia.no, martin.carlsen@uia.no

The aim of this study is to employ the Knowledge Quartet, proposed by Rowland, Huckstep, and Thwaites (2005), in order to characterise the kindergarten teacher's competence when orchestrating researcher designed mathematical activities for 5-year-olds. We are drawing on design research as a methodology in which design principles such as playful learning and inquiry are implemented in the activities. Our analyses show that knowledge-in-action and knowledge-in-interaction are revealed in the orchestration. Hence, the Knowledge Quartet is applicable in the kindergarten context – however, with some modifications due to kindergarten particularities.

Keywords: Design research, kindergarten, Knowledge Quartet, mathematics, orchestration.

Introduction

For about a decade, mathematics has been part of the Norwegian curriculum for kindergarten, a curriculum situated within a social pedagogy tradition. In a kindergarten setting in Norway, a kindergarten teacher (KT) is supposed to empower the children's mathematical explorations. However, the curriculum is not explicit as regards how KTs are to facilitate the children's mathematical explorations. In this study we characterise one KT's orchestration of mathematical activities for 5-year-olds in terms of knowledge employed by the KT. The following research questions are formulated for the study:

> In what ways may the Knowledge Quartet be employed in order to characterise the kindergarten teacher's competence?
>
> What knowledge-in-action and knowledge-in-interaction does a kindergarten teacher reveal in her orchestration of a mathematical activity on geometrical shapes?

The activities were designed by researchers in mathematics education (the authors of this paper among others), as part of an ongoing research and development project called "The Agder project"[i], in order for the participating children to engage with mathematical concepts and ideas.

In this study we use the metaphor of orchestration. By this metaphor we mean that it is the KT who is in charge of and leads the mathematical activity. She has to plan, think forward, act in the moment, follow up children's questions and comments, adapt questions for each of the children, etc. – an important and by no means easy role to fulfil. Our admiration of this complexity is thus considerable when we set out to design the mathematical activities and study how the KTs orchestrate the activities.

Theoretical lens: The Knowledge Quartet

In order to analyse in depth how the Knowledge Quartet may be employed in order to characterise the kindergarten teacher's competence and the revealed knowledge-in-action and knowledge-in-interaction on behalf of the KT, we draw heavily on the Knowledge Quartet coined by Rowland,

Huckstep and Thwaites (2005). The Knowledge Quartet was developed by Rowland et al. (2005) as a theoretical construct drawing on the profound and substantial work of Shulman (1986). Rowland and colleagues drew on videotapes from mathematics classroom lessons where pre-service teachers utilised their mathematical and pedagogical knowledge in their teaching. This quartet is used as a theoretical lens through which we analyse video data of one KT´s orchestration of one of the developed activities. We limit ourselves to one KT taking into account that "the quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom" (Rowland, Huckstep, & Thwaites, 2003, p. 97).

The Knowledge Quartet differs from the framework of Ball, Thames and Phelps (2008), mathematical knowledge for teaching (MKT), in that the former focuses on situations in the mathematics classroom through which the teacher's mathematics-related knowledge may be observed. Ball et al.'s (2008) framework describes different kinds of mathematics teachers' knowledge. Mosvold, Bjuland, Fauskanger, and Jacobsen (2011) used the framework of Ball et al. (2008) to analyse mathematics teaching at the kindergarten level. Mosvold et al. (2011) found that the MKT framework needs to be adjusted to the kindergarten context in order to be appropriately used. This is because (1) the work situation of orchestrating mathematical activities for a Norwegian kindergarten teacher is very different from the teaching situation of a U.S. mathematics school teacher; and (2) the tasks of teaching in the kindergarten setting are significantly different from that of the school setting. It is pedagogical activities which is the basis for learning activities in Norwegian kindergartens, not mathematical activities as such. These differences have been emphasised by Erfjord, Hundeland, and Carlsen (2012), through the lens of the didactic triangle.

In adopting a grounded approach to the analyses of data, Rowland et al. (2005) identified what they labelled the Knowledge Quartet, four dimensions along which mathematics teachers' "mathematics-related knowledge" (p. 255) may be analysed. These four dimensions are termed *foundation*, *transformation*, *connection* and *contingency*. Foundation encompasses the knowledge background of the KT; transformation and connection encompass how and to what extent knowledge is revealed in action as the KT implements and orchestrates the activity, hence dimensions described as knowledge-in-action. Contingency encompasses the KT's knowledge as this unfolds in interaction with the children, hence knowledge-in-interaction.

Foundation is a dimension of *propositional knowledge* used, adapted to our case, to address the KT's mathematical knowledge, the KT's knowledge of mathematics education, and the KT's view upon the purpose of mathematics education and how children learn mathematics. Analytically, we use this dimension to characterise both the mathematics and the didactical insights revealed by the KT in her orchestration of the activity.

Transformation is a dimension of *knowledge-in-action* which addresses the KT's choices of representations, demonstrations, and use of examples in her orchestration of the activity. The dimension focuses on the KT's ability to transform the mathematics "in ways designed to enable students to learn it" (Rowland et al., 2005, p. 265). Analytically, we use this dimension to describe the KT's orchestration of the activity on characterising two-dimensional geometrical shapes.

Connection is a dimension of *knowledge-in-action* as well, addressing to what extent the KT draws connections between various mathematics concepts and connections between various mathematics

procedures, alternative meanings for these concepts and different ways of carrying out procedures. Analytically, we use this dimension to characterise how the KT makes connections between the geometrical shapes and their features, as well as various ways of deciding what shape is what.

Contingency is a dimension of *knowledge-in-interaction*, a dimension that addresses how the KT interacts with the children through appropriately responding to children's contributions, to what extent she takes advantage of learning opportunities that emerge, and to what extent she makes the activity her own and deviates from the goals and foci of the activity. Analytically, we use this dimension to characterise how the KT responds to the children's ideas, the questions and comments she uses, whether she draws the children's attention towards particular mathematical ideas, and whether she makes the geometry activity her own and are unbounded by its original foci.

Design principles and context

In our design of the mathematical activities we drew on two main principles, *playful learning* and *inquiry approach* to the teaching and learning of mathematics. The design principle of playful learning emphasises that for children play and learning is one and the same thing. Playful learning encompasses both free play (child-initiated and child-directed play) and guided play (adult-initiated and child-directed play), where guided play is the principle used here. In guided play the KT orchestrates and literally guides the play in an adequate direction in order to reach pre-formulated aims for the children's play and to nurture the children's interest, curiosity, engagement, and (mathematical) sense-making (Weisberg, Kittredge, Hirsh-Pasek, Golinkoff, & Klahr, 2015). The design principle of adopting an inquiry approach to the teaching and learning of mathematics stems from Jaworski's (2005) inquiry as "a way of being in practice" (p. 103). That is, the KT and the children collaborate in order to achieve meaningful answers to prompts and questions, the children's curiosity is taken advantage of, and the children are guided into mathematical inquiries through the use of questions, being curious and excited about mathematical issues.

The KT that takes part in our study is a participant in the Agder Project, and by that also a participant in a professional development program. The professional development program within this project was based on four workshops of two days duration, focusing on Number, Measurement, Geometry and Statistics, combinatory and probability. In the third workshop, the KTs participated in lectures and group discussions on Geometry. Our data collection took place during the period when the KTs tried out two geometry activities. The observed KT is in her forties, educated for three years (180 ECTS) at university level, and is a well experienced kindergarten teacher.

Context

We as researchers are interested in scrutinising the KTs' processes of orchestrating our designed mathematical activities. Moreover, the KTs have been participants in a professional development program where we have contributed with lectures and feedback on previous orchestrations of mathematical activities. We study the case of Wilma and videotaped her orchestration of a geometry activity. Wilma had received a written instruction note for the geometry activity a couple of weeks in advance of our data collection.

Analysis and results

In order to employ the Knowledge Quartet (KQ) on empirical data, one has to go into the code level from which the four dimensions are extracted (Rowland, 2016, personal communication). Our analytical process started with a first phase of collectively reflecting on our collected data in total, consisting of two days with data collection in each of four different kindergartens. In doing that, we had the KQ in mind, seeking to use KQ terminology and codes to analyse our data. The second phase consisted of us collectively looking at video excerpts from the different kindergartens, resulting in our choice of the case of Wilma. The reason for choosing one of the sessions of Wilma was that this session, from our collective looking at the video, turned out to be the most promising in addressing our research questions. The KT observably revealed her knowledge in action and interaction, the children contributed with oral statements and questions, and interaction amongst the children was taking place. The third analytical phase consisted of transcribing the video from Wilma's session. The fourth phase consisted of us conducting collective in-depth analyses of the natural talk-in-interaction adopting the codes developed in the Knowledge Quartet.

The activities were presented in written form by us with the three headings *Equipment*, *Intention,* and *Implementation.* The particular activity we consider here, is the first part of an activity focusing on two-dimensional geometrical shapes. Under *Equipment* we wrote: Geometrical shapes: triangles, squares, rectangles, circles, trapezium, rhombus, while for *Intention* we wrote*:* The children are supposed to get experience in recognizing properties of different two-dimensional shapes. Furthermore, the children are supposed to practice mathematical argumentation with respect to features of the various shapes. As regards *Implementation,* we wrote: Let the children investigate the shapes and their characteristics. Let the children discover the shapes' differences.

Excerpt 1: Foundation

The following excerpts are taken from the initial phase of Wilma's orchestration of the geometry activity. Wilma shows and shakes a box containing two-dimensional paper shapes. One of the children responds that "It's shapes". Upon this, Wilma continues their conversation. The analytical contributory codes are: "awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures" (Rowland et al., 2005, p. 265).

Wilma:	Yes, that's correct. And with mathematical terminology we call them geometrical shapes. Are you able to pronounce that?
Sam:	I think it is cookies (Smiles as he says it).
Wilma:	John, are you able to pronounce that? Geometrical shapes?
John:	Geometrical shapes. (Several children simultaneously say "Geometrical shapes")
Wilma:	Yes, that is what they are called with mathematical terminology. Inside this box there are several of such shapes (She opens the box and shows it to all the children so that they may look inside the box).
John:	It looks like a puzzle.
Wilma:	Yes, it looks like a puzzle. That's true.

Sam: Yes. Are we going to puzzle with them?

Wilma: At least we are going to work with them, yes we are.

Ken: Can you pour them out?

Wilma: I was thinking of pouring them out. Then I want you to take a look at them. Currently, there are quite a few shapes and some of them are almost identical. Now you may take a look at them. (She pours the shapes out on the table; the children take some shapes each and say "that is small" and "a triangle".)

In this excerpt we argue that Wilma's orchestration is characterised by her foundation, both with respect to mathematics insights and didactical insights. The mathematical insights are revealed through Wilma's emphasis on the mathematical terminology through the twice expressed term "geometrical shapes". Furthermore, her mathematical insights are revealed through her choice of shapes that are congruent, shapes that are similar, and the variety of shapes included (various triangles, various quadrilaterals, circles of various sizes, ellipses, hexagons and octagons). The written material made by us suggested "triangles, squares, rectangles, circles, trapezium, rhombus" as shapes to work on in the activity while Wilma introduced many more shapes and terminology. Thus, we argue that the foundation here is Wilma's and not just her use of external provided foundation from us in the written form.

The didactical insights of Wilma are revealed through her way of establishing interest and curiosity among the children by shaking the box. The children's interest and curiosity about the shapes are nurtured further by her showing of the various shapes in the box. Wilma confirms that the shapes look like pieces of a puzzle and by that she makes a link between these shapes and the apparently well-known activity of puzzling. By establishing that link, Wilma also communicates a playful way of engaging with the shapes. Finally, in this excerpt, Wilma's didactical insights are revealed when she asks the children to inquire into the various shapes. By orchestrating the activity as playful, Wilma signals that she appreciates inquiry as a tool in order to learn mathematics. Additionally, the children's interest, curiosity, engagement, and mathematical sense-making are nurtured.

Excerpt 2: Transformation and Connection

Approximately ten minutes later, the following dialogue occurred, exemplifying the dimensions of transformation and connection. Relative to transformation, the analytical codes are: "choice of representation; teacher demonstration; choice of examples" (Rowland et al., 2005, p. 265). Relative to connection, the analytical codes are: "making connections between procedures; making connections between concepts; anticipation of complexity" (Rowland et al., 2005, p. 265). Some of the children had picked up quadrilaterals that they found interesting since they did not know their names and they were different from rectangles and squares. Then the KT said:

Wilma: Do you know what? These two quadrilaterals actually have other names with mathematical terminology. They have four edges (She counts "one, two, three, four" aloud while simultaneously pointing at the edges).

Susie: But what are they called then?

Wilma: That one is called a rhombus (she points at the rhombus while speaking).

Susie: Rhombus.

Wilma: Rhombus. And that one, do you notice that two and two edges are equal (she points at the parallelogram she shows). That edge and that edge (slides her finger along the two opposite, parallel edges), are equal, and that edge and that edge are equal (slides her finger along the two other, opposite, parallel edges). Its name is actually a parallelogram.

Sam: A paragram?

Wilma: Yes, a parallel o gram.

In this excerpt we see that Wilma carefully introduces two new shapes for the children. She gets everyone's attention by holding up and showing one shape at a time. She emphasises that both shapes are conceptually associated with quadrilaterals, by overtly counting the edges. At the same time she makes it obvious that the two shapes are particular kinds of quadrilaterals. This is pinpointed by saying that they have "other names", implicitly distinguishing these shapes from the familiar quadrilaterals square and rectangle.

Wilma's orchestration in this excerpt exemplifies transformation in that she uses one example of each of the two new shapes. She points at each of them in accordance with what she is saying, and she demonstrates how to classify geometrical shapes by counting their edges. Her orchestration also exemplifies contingency due to the deviations from the agenda set by the researchers in the designed activity.

The dimension of connection is also exemplified in this excerpt as Wilma draws the children's attention towards the particular features of one of the shapes. Wilma carefully focuses at the two pairs of edges that are parallel, one pair at a time, by sliding her index finger along the edges. She neither uses the mathematical concept of parallel nor equal length at this occasion. She only uses the feature "equal". The feature of parallelism is thus only implicitly focused. However, by sliding her index finger along the two edges, her gesture signals that they are of equal length.

In the end of the dialogue we also notice that Wilma is particularly focused at offering opportunities for the children to learn the name of the new shape. On two occasions, Wilma uses the name "parallelogram" for the shape. Both times she puts emphasis on the expression "parallelogram". We see that Sam tries to pronounce the name, but only partly succeeds. Wilma then slows down the pace in her pronunciation of the word, in order for Sam, and the other children, to pay attention to the new word. Hence, we observe that Wilma is eager to name new objects mathematically correct. Naming is an important element in these children's mathematical learning process. However, both in this example and in other examples the naming of shapes comes at a late stage, after the children have presented their sorting of shapes and the KT has orchestrated a discussion of the properties of the shapes. We consider this as evidence of her taking an inquiry approach in her orchestration.

Excerpt 3: Contingency

To exemplify this dimension of the KQ in Wilma's orchestration, we include examples of single moves which illustrate how she responded to the children's ideas, the questions and comments she used to make the children pay attention to various mathematical ideas. Even though she occasionally addressed her questions to one particular child, the other children still paid attention.

The analytical codes used were: “responding to children’s ideas; use of opportunities; deviation from agenda” (Rowland et al., 2005, p. 266).

Wilma drew the children’s attention towards the mathematical concept of sorting. Sam said: “Can we sort them?” (19). Then, some moves, but only a few seconds later, Wilma said: “Sam, what does it mean to sort?” (27). Wilma also drew the children’s attention towards mathematical features of the various shapes. One example was found when Sam said: “Yes, but these are small (points at the short edges of the rectangle). These two are equally long” (144). A few seconds later, Wilma responded to this utterance and asked all of the five children: “Does anybody know what the shape is called when two edges are quite long and two edges are shorter?” (147). With this question, she seeks to establish interest and curiosity, and thus to nurture playfulness in her orchestration.

Wilma likewise focused their communication around the similarities between the shapes. Susie talks about the fact that two shapes, two congruent triangles, may be joined in order to make a rectangle. She says: “Jack’s shapes have such…, but Ken’s do not have such when he puts them together” (196). Upon this statement of observation, Wilma responds immediately: “What happens when you put them together?” (197). Furthermore, Wilma addressed the children’s mathematical reasoning through questions like: “How did you figure out that one (points at one of the hexagons)?” (72); and “Do you want to tell the other children?” (227). These examples show that Wilma adopts inquiry as a way of being (cf. Jaworski, 2005) and responds to the children’s ideas and uses the opportunities that unveil in their interaction.

Discussion

In this study we have considered the following questions: In what ways may the Knowledge Quartet be employed in order to characterise the kindergarten teacher’s competence? and What knowledge-in-action and knowledge-in-interaction does a kindergarten teacher reveal in her orchestration of a mathematical activity on geometrical shapes? From what is possible to include in this relatively short paper, we are able to discern instances of all four knowledge dimensions through which her orchestration of the mathematical activity is informed. As such, the Knowledge Quartet (Rowland et al., 2005) has proven to be analytically useful when seeking to wrap up how a KT is able to orchestrate researcher-designed mathematical activities. At the same time, we observe that the KQ unfolds slightly different in the kindergarten context than in the school context, the context from which the KQ was originally elaborated and developed. In the Norwegian kindergarten context, it is highly rare, even inappropriate to orchestrate mathematical activities through long introductions and demonstrations. Concerning the issue of time, we see that the periods where the KT has the word are quite short, often only 5-10 seconds. Furthermore, in the kindergarten context it is unusual to give children extensive time to inquire into the mathematics without KT interference. This is however a usual case in the school context. KQ contributory codes, such as use of textbook, reliance on procedures, teacher demonstration, and making connections between procedures, are partially inappropriate and inapplicable in the kindergarten context. Thus, our modification of the KQ materialised as not taking these codes into consideration.

It is challenging to separate the dimensions from each other in the dialogues. In the dialogues the dimensions are intertwined, where one move may simultaneously exemplify several dimensions. The excerpts above are thus not mutually exclusive when it comes to the four dimensions of the

KQ. For the purpose of this paper, however, the excerpts are deliberately chosen to illustrate how the KT reveals her knowledge-in-action and knowledge-in-interaction with respect to this particular mathematical activity.

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Young children's aesthetic development in the context of mathematical explanation

Esther Levenson[1] Manya Raman-Sundström[2]

[1]Tel Aviv University, levenso@post.tau.ac.il

[2]Umeå University, manya.sundstrom@umu.se

Mathematicians routinely report that beauty is both a reward and a motivation for the work they do. However, how and to what extent children can appreciate mathematical beauty is an open question. This exploratory study looks at young children (ages 6-12, with a focus on the younger years) as they evaluate different explanations of claims about even numbers and triangular numbers. While our results are fairly speculative, we provide case studies which illustrate possible kinds of aesthetic reactions, and some of the factors which might impact on those reactions.

Keywords: Aesthetics, explanation, even numbers, triangular numbers, affect.

Introduction

There is little doubt that mathematicians have rich, aesthetic lives (Sinclair, 2004). While recent research has attempted to characterize what exactly this aesthetic life consists of (Raman-Sundström, & Öhman, 2016) or what is meant by aesthetics in mathematics in the first place (Rota, 1997), evidence suggests that aesthetic reactions are common in, and perhaps even central to, the working lives of mathematicians. According to some, aesthetic reactions include pleasure, tension, surprise, and a sense of being compelled (or repelled) (Marmur & Koichu, 2016). Interestingly, neuroscientists suggest that the same region of the brain is involved for judging mathematical equations and works of art (Zeki et al., 2014).

Recently, there has been interest in studying aesthetics in mathematics education. While some of those studies have investigated school children's aesthetic reactions while working on mathematical problems (e.g., Sinclair, 2006), few have focused on young children. Yet, research has shown that young children are capable of sophisticated proof-like reasoning (Maher & Martino, 1996), justification, and argumentation (Tatsis, Kafoussi, & Skoumpourdi, 2008). Thus, it seems reasonable to ask whether young children could be capable of aesthetic experiences, and if so, at what age. Are young children able to experience the satisfaction of finding a good explanation? Do they find pleasure in coming to understand an explanation?

This paper presents an explorative study of the possible aesthetic experiences of young children. Mathematics, like other aesthetic subjects, provides experiences that have the potential to draw people in. It also provides a reward or a sense of satisfaction (Sinclair, 2004). What makes the experience "complete" is the presence of *both* a startup phase and a reward phase. We study the presence or absence of aesthetic reactions of young children (age 6-7) by comparing their behavior to an older cohort (ages 9-10). We find that the younger children, while lacking some of the insight of the older children, have what we might call "aesthetic dispositions" that allow them to enjoy and to be curious about fairly complex mathematical tasks.

Theoretical background

One of the central questions of aesthetics is whether beauty is objective or subjective. According to Marmur & Koichu (2016), those that consider mathematical beauty as objective (such as Dreyfus and Eisenberg, 1986) list characteristics such as clarity, simplicity, brevity, and conciseness when judging theorems and proofs. In other words, beauty is an intrinsic property of the mathematical object. Those that consider beauty to be subjective, claim that mathematical beauty is in the eye of the beholder and that experience, age, knowledge, and culture contribute to aesthetical views. Marmur and Koichu (2016) integrate both views, concluding that when discussing school mathematics, we may hypothesize that a mathematical problem might elicit an aesthetic experience among students because of its simplicity or surprising result. Ultimately, however, students may or may not have an aesthetic experience depending on, among other things, the pedagogical setup of the problem.

In their study of university students, Marmur & Koichu (2009) found that surprise was integral to experiencing mathematical beauty. They found that students who referred to a solution as beautiful had first struggled with the problem, had put significant effort into finding the solution, and ultimately were surprised at the simple and unexpected solution. Struggle was also a factor in Brinkmann's (2009) study of middle and upper school students' appreciation of mathematical beauty. A problem was considered to be beautiful if it had a certain degree of complexity, yet felt solvable. Eberle (2014) investigated students' (ages 8-10) aesthetic attractions when evaluating geometric tessellations. Students referred to several characteristics of the geometric objects which contributed to their appreciation, such as real world connections, color, complexity, and uniqueness. Eberle (2014), as well as Sinclair (2001) also noted the generative role of aesthetics when students were involved in inquiry-based tasks. In both studies, aesthetics led students to engage and play with the mathematics, guiding them when deciding which direction to pursue.

The above studies related to aesthetics with regard to problems, solutions, and geometric objects. In our study, we focus on mathematical explanations and young children's appreciation of those explanations. Previously, Levenson (2010) found that fifth-grade students have preferences regarding different types of explanations. Students' preferences were based on clarity, brevity, relatedness, and because the explanation was perceived as fun. Although some of the reasons students mentioned for their preferences are reminiscent of aesthetic evaluations given by older students, and even mathematicians, the focus of that study was not specifically on aesthetic appreciation or satisfaction from an explanation. In this study, we draw on a theory of explanation developed by Gopnik (2000), which helps explain what makes certain explanations satisfying. Gopnik suggests that an explanation consists of two parts, the why? (the 'hmmm….' phase) and the because! (the aha! or wow! phase depending on how surprising the result is for the individual). An exploration can have a why? without a because! and vice versa. Both phases are needed for an explanation to be found satisfying. Moreover, Gopnik suggests that both the seeking and the satisfaction from finding a good explanation are part of our human nature.

The aim of this study is to begin an exploration of young children's appreciation of mathematical beauty. Specifically, we ask: Do young students' aesthetic reactions (or non-reactions) to mathematical inquiry and explanation differ from that of older students? Are younger and older

students capable of an experience that contains both the hmmm… and aha! phase of a mathematical explanation?

Methodology

Data was collected using semi-structured interviews with two cohorts of children, one aged 6-7 years old, and one aged 9-10 years old. The older children worked with explanations about triangular numbers, namely that the number of dots in the nth triangular number is $n(n+1)/2$. This cohort included four fifth grade girls, all from the same class, who sat together in a group with the interviewer in one of the girl's houses. The discussion began with introducing the girls to triangular numbers, discussing the number of dots in the first five triangles, and then asking them to come up with the number of dots in the 100th triangle. After giving them time to work on the problem, the girls were shown two solutions and asked to evaluate each solution.

The younger cohort of three children worked individually with the interviewer with explanations of the claim: the sum of two even numbers is always even. Two interviews were conducted in the house of the child and one was conducted in the house of the child's grandmother. Each interview began in the same way, asking the child to say if he or she could give examples of even numbers and to say why those numbers were even. After confirming that the children were familiar with even numbers they were given the following question: what would happen if you add two even numbers, would the answer be even or odd? Children were given time to think and reply. Although the interviewer had several explanations on hand for the children to evaluate, as will be shown in the next section, only one child was asked to evaluate explanations.

One of the difficulties of studying aesthetics in children, or even with mathematicians, is how to detect an aesthetic experience. While there may be bodily clues, such as changes in eye-dilation or neural correlates (e.g., Zeki et al., 2014), a natural place to start is by simply listening to what people say (see Wickman, 2006) and watching for engagement (or disengagement) during the experience. In this study, we take this approach as a first approximation, using key words (taken from the background studies) such as "Wow!" and "Funny!" as markers for a general aesthetic experience.

Findings

Below we present four episodes, one with older girls and three with younger children. What is striking about the young children is that there is no sense of surprise. The children seem to lack the 'hmm….' needed to build the wow! or to even warrant an explanation.

Fifth-grade girls, age 11

Trying to figure out how many dots will be in the 100th triangle proves challenging to the girls. When they realize that they would have to sum all of the numbers from 1 to 100, the interviewer gives them some time to work this out and then explains to the girls the Gaussian method for summing an arithmetic sequence. She lines up the numbers from 1 to 100 in one row and on top of that row, lined up the numbers from 100 to 1 (see Figure 1), explaining that this shows how many dots are in each row of the 100th triangle.

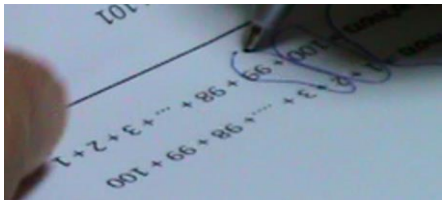

Figure 1: Summing the numbers from 1 to 100

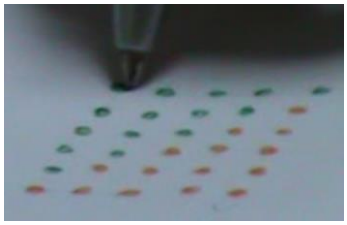

Figure 2: Combining two triangles

To explain why one multiplies 100 by 101 and then divide by 2, the following discussion ensues:

Esther: Two triangles of 100, right? Now look what we have here. (Esther circles the 100 and 1 and the 99 and 2).

Girls: Ahh!

Esther: 1 and 100, 2 and 99, 3 and 98.

Trina: Wow! It's great. It's the same thing.

The girls are surprised that the sums all add to the same number, 101. Their remarks of "Ahh" and "Wow" indicate their pleasure in this simple conclusion. After showing the girls this explanation, the interviewer shows them a second method, that of drawing two congruent triangles, inverting one and placing it next to the first, thus creating a rectangle. The number of dots in the triangle is then equal to the area of the rectangle divided by two (see Figure 2). After establishing that the girls understand both explanations, the girls are asked to compare the two methods.

Esther: Which explanation of the method gives you more satisfaction?

(All of the girls point to the second explanation with the dots.)

Amanda: You simply see that you do this times this, and then divide by 2 because you have 2 triangles.

Hailey: Because instead of computing it all, this is easier and simpler and in front of your eyes.

Amanda: Also, it draws attention more. It's more fun, but not just more fun, it's like, it goes more into your head.

In the second segment, the girls claim to like the second explanation better because it is simpler and because they can "see it." This hints at their appreciation for the aesthetic value of efficiency, which might have been enhanced by their struggle to find the solution.

Zev, age seven

Zev is seven years old, attends first grade in Israel, where learning about even and odd numbers is part of the curriculum in school. Zev is able to list several even numbers as well as several odd numbers. When asked why eight is an even number he says, "because four is even and … because four is even and it's… and also… the second four is even." Note that he does not stress that 8 could be written as the sum of two equal whole numbers, but rather that both of the addends, in this case fours, were both even numbers. When asked why 10 is an even number, the following discussion ensues:

Esther: OK. Is ten an even number?

Zev: Yes.

Esther: How do you know that ten is even?

Zev: Because odd plus odd is even.

Here Zev was probably thinking that 10 results from 5 + 5, both odd numbers. Realizing that Zev seems already familiar with summing odd numbers, the interviewer asks about the sum of two even numbers:

Esther: And what about an even number plus an even number?

Zev: Even.

Esther: Always?

Zev: Yes.

Esther: Can you tell me how you know that even plus even is always even?

Zev: Four plus four, eight. Eight pus eight, sixteen.

Esther: How do you know that sixteen is even?

Zev: Because ten is even, and six is even.

For Zev, it seems that his working definition of an even number is of a number that can be written as the sum of two other even numbers. Although this is a recursive definition, it does not seem to bother him and the outcome of this conception is that Zev does not even recognize the question of what might be the sum of two even numbers. Zev has no sense of hmm…. When Zev is shown another explanation for why the sum of two even numbers is always even (that every even number can be written as the sum of twos and thus the sum of two even numbers can also be written as the sum of twos), Zev says that the explanation is boring because he already knew that.

Anna, age six

Anna is six years old and attends kindergarten in Israel. Although even and odd numbers are not part of the kindergarten curriculum, Anna was able to list several even and odd numbers and claimed to have learned about them from her teacher. When asked why eight is an even number she responded, “Because each one has a partner.” When she was asked to explain what an even number is, she said, “That the two of them have a partner. That each of them has a partner.” What Anna is alluding to in her own language is that an even number may be written as the sum of twos. After talking about even numbers for a few minutes, we discuss the sum of two even numbers:

Esther: What would happen if I added an even number with another even number?

Anna: It would be even.

Esther: How do you know?

Anna: Because both of them are not separate, it never separates from the second.

Esther: What do you say! Are you sure that this is always this way?

Anna: Yes. It will never be that, uhmm, a partner doesn’t run away from the pair.

Anna has a very specific conceptualization of even numbers, that an even number represents an inseparable pair. She draws on this conceptualization to explain why the sum of two even numbers

must always be even. Engaging further with the problem, Anna takes out a bunch of wooden blocks from a basket (without counting) and proceeds to pair up blocks. When the interviewer asks if she can say if she took out an even or odd number of blocks she readily says that she took out an even number of blocks because on the table, every block is paired off. To summarize, Anna is drawn in to the problem, has an explanation, but she does not struggle with the claim.

Leila, age seven

Leila is from Sweden, has just turned 7, and attends kindergarten. Although learning about even and odd numbers is not part of the curriculum, she is able to list several even numbers (perhaps hearing about them from her parents or friends). Although prompted to think about the possible sum of any two even numbers, Leila shows no interest and instead, with the use of colorful cubes, begins to explore a self-generated conjecture involving pairs of twos. She puts together six pairs of cubes to represent an even number and then wonders if seven pairs will still be even.

Leila:	I have a question. You can't have an odd number of twos.
Manya:	How are you thinking?
Leila:	Like, if I have these (pointing to the six pairs of cubes), you can never have an odd number. Look here. I have 1, 2, 3, 4, 5, 6. If I have this many (she takes another pair), now I have enough…. odd numbers of twos… because there's seven… does it get an odd number or an even? I think it is actually funny. If you have an odd number of twos, it even gets an even number. That I think is really funny.
Manya:	I see. Why is that?
Leila:	I don't know, because it feels like it's actually pretty funny.
Manya:	Do you think it's funnier to have an even number made of an odd number of twos than an even?
Leila:	What do you mean – an even number of odd number …? Yeah. It gets an even number if you have an odd number of twos. Because I figured it out of these.

This discussion suggests that autonomy might play a role in having an aesthetic experience. Leila generates her own question (What is the sum of two odd numbers?) and the resolution of the question (that an odd number of twos can be even). She thinks that this conclusion is funny. While she might have had a similar reaction to a given statement, the affect seems to be closely correlated to ownership of ideas. In the last two lines, the researcher asks her to compare two statements. Leila replies "what do you mean"? The statement is funny to her because she "figured it out of these."

Comparing the older cohort to the younger cohort

The following table summarizes the observations from the data presented above and the children's aesthetic experiences (AE). A full circle represents that an AE took place, a dotted circle reflects a partial experience, and an empty circle that no AE was detected. In all of these cases, we can see that the path to having an aesthetic experience consists of several distinct phases. One needs to be engaged; there should be some build-up, some crucial moment, and then some release. Only the 10 year olds seem to have a complete experience.

Children	Age	Task	AE	Behavior
Trina, Hailey, Amanda	10	Triangular numbers		Have an aesthetic experience, marked by both surprise and satisfaction. They appreciate a solution which is simple and "goes into their head".
Zev	7	Even numbers		Has a conception of even numbers, but is uninterested in explaining why the sum of evens is even. He "already knows that".
Anna	6	Even numbers		Has a conception of even numbers which goes along with the claim about the sum of even numbers. Is engaged and involved, but is lacking a "hmmm…."
Leila	7	Even numbers		Takes control. Explores her own hypothesis. Tests if an odd number of twos can be even. Finds the result "funny".

Table 1. Types of aesthetic experiences for the triangular number and even number tasks

Build-up takes place when they explore the question, and their interest increases as the interviewer shows the explanation involving pairs of numbers. After having time to digest this information, the interviewer shows another explanation which further increases their interest. Each time a new explanation is understood, the girls say words like "wow!" and "aha!". The fact that they have this kind of reaction, we claim, is because they had time to explore and to start generating their own explanations. We also cannot rule out that there might be some developmental issues, such as the children being old enough to abstract and/or take in the explanations given.

In contrast to the fifth-graders, the younger students had limited or no aesthetic experience. Zev is strikingly uninterested in any explanation at all. We suspect that his disinterest came from the fact that he had been told in school that even + even is even, so there was no tension left to resolve. Leila has some interest in explanation, but not for the question given to her. Rather, she generates her own question about whether an odd number of pairs can result in an even number. She finds this result funny, indicating some level of surprise, which she quickly believed despite her initial expectation. Anna is drawn in to the explanation activity, but did not seem to have a full aesthetic experience. Unlike the fifth graders who could "see" why the explanations held, Anna simply states her conception of even numbers in terms of pairs or partners, and claims that any sum of pairs will still be even. She does not give an actual reason, which might be because she did not experience the hmmm… phase of explanation. She is not bothered by any alternative, so no relief or satisfaction is expressed.

Conclusion

One of the challenges of this study was to find tasks that might elicit an aesthetic reaction. We attempted to find tasks that would be suitably challenging, yet accessible to each of the age groups. In the end, the older children worked on a new task, presented not only in general manner, but with an iconic illustration, while the younger children worked on familiar (at least for two children) general characteristics of numbers. Thus, it might be that the different conditions affected the aesthetic experiences. Taking these limitations into consideration, there is still the possibility of developmental differences in aesthetic experiences. In the naïve view, children have and rely on concepts, but are not yet puzzled. Because of this lack of puzzlement (an essential ingredient

according to several researchers (e.g., Gopnik, 2000; Marmur & Koichu, 2009)), there is no tension in their mathematical exploration, nothing to be resolved, and so no aesthetic experience is possible. In contrast, in the mature view, children are engaged and puzzled. They are more open to explanations because they themselves have struggled with the questions. This kind of behavior is possible among quite young children (Leila, at age 7, has a very small amplitude aesthetic experience when she generated her own conjecture), but might be more likely to occur the more autonomy is given to the students, the more challenging the task, and the more supported the students are to not to give up when they think they already have the answer. As an exploratory study, this paper has begun a discussion regarding young children's possible aesthetic experiences when working on mathematics. Additional research is needed to continue this discussion.

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Discovering regularities in a geometrical objects environment

Bozena Maj-Tatsis and Ewa Swoboda

University of Rzeszow, Poland; bmaj@ur.edu.pl; eswoboda@ur.edu.pl

We present the analysis of a research conducted with first-grade pupils with a focus on their ability to notice and use regularities in three-dimensional geometrical objects by using playing blocks. The results show that children had difficulties in reconstructing the figure and retaining the regularities in the invisible part of it.

Keywords: Early years mathematics, patterns, regularities, geometrical objects.

Introduction

Discovering regularities is considered one of the most important processes involved in mathematics. Actually, developing an awareness of patterns is a significant step towards generalisation. Children are expected to be able to recognise, describe, duplicate and extend patterns, even at a young age. These assumptions are expressed in the Polish curriculum (Podstawa Programowa, 2011), in which the use of patterns and regularities appear in the first grades of primary schools; for example, the pupils are expected to be able to draw a particular pattern. However, much often the attention of the teacher is focused on the manual abilities of the students and not on mathematical thinking. In the next classes the topics do not play a significant role, according to the Polish curriculum.

In our study we designed a series of lessons for the first grade pupils using playing blocks. By observing constructions presented at pictures the children were asked to construct the same figures. In order to do so, they had to recognize a geometric pattern and/or a repeating pattern (Zazkis & Liljedahl, 2002), and then use their observations to their work. Our analysis was focused on children's ability in noticing regularities and in continuing them in their constructions. In our paper we describe a particular task in which the children had to reconstruct a three-dimensional figure and build an invisible part, which could be done by using the same pattern. We try to investigate to what degree the children subordinated their actions connected to building the invisible part of the construction to regularities of the visible part of the construction. Our theoretical framework is presented in the next section, followed by our methodology, the results and analysis, and finally our conclusions.

Theoretical framework

Discovering regularities in the primary school

In mathematics teaching exploring patterns is closely related to the development of mathematical thinking and reasoning: "Patterns are the heart and soul of mathematics" (Zazkis & Liljedhal, 2001, p. 379). This is underlined in many studies (e.g. Clements & Sarama, 2009; English, 2004; McGarvey, 2012; Threlfall, 2005). It is thus very important for every child from a young age to have contact with patterns; it is considered a way "to make connections to the world around them" (McGarvey, 2012, p. 310). As Frobisher and Threlfall (2005) state, in their first years of schooling students should develop abilities to describe, complete and create patterns. Tasks that involve patterns encourage students to verbalize their ideas, thus improve their communicational ability (Garrick, Threlfall & Orton, 2005). Moreover, searching for a regularity is an extremely effective method during solving

mathematical problems. English and Warren (1998) suggest a patterning approach as a way to introduce the concept of variable. Mulligan and Mitchelmore (2013) introduce the construct "Awareness of Mathematical Pattern and Structure" in order to study the development of structural understanding; moreover, they claim that this development is related to the big mathematical ideas of generality and equal grouping.

Although patterns and regularities are considered vital, in some cases they do not stand on their own in the curricula and teachers see such activities as merely an enrichment to more "traditional" activities. On the contrary, we believe that "algebra, and indeed all of mathematics is about generalizing patterns". (Lee, 1996, p. 103) Therefore, patterns should be observed and studied in many different manifestations in mathematics.

Three-dimensional geometry in the early mathematics classroom

Although small children have many experiences with handling three-dimensional objects within their play, early geometry curricula usually place three-dimensional geometry to higher educational levels (Swoboda & Vighi, 2016). In Polish primary schools, space geometry is almost absent in the first three grades. According to the National Curriculum (Podstawa Programowa, 2011), the only relevant topics are recognising and naming basic geometrical two-dimensional figures. Generally, the tasks proposed in the textbooks have a reproductive character and are directed into building the concept of geometrical object. Much often the tasks impose scientific terminology. There is lack of tasks which could be a cognitive challenge connected to geometric problem solving. The tasks in which the pupil has to use some geometrical properties of objects or some observed relations are also missing in the first years of school mathematics. Moreover, there are very few tasks related to three-dimensional geometry. Such tasks have a significant influence on development of thinking and imagination of a child who is surrounded by three-dimensional objects.

Context of the study and methodology

The starting point for our research was a general question concerning the degree that 6-7-year-old children can grasp regularities formed in a three-dimensional environment. The research was conducted by a preservice teacher (Falger, 2016) – a master student supervised by the second author of the paper – among 16 first grade pupils (6-7-year-old). It consisted of a series of four lessons of around 45 minutes and it was realized in the period of February – April 2016 every two weeks. The preservice teacher was known to the pupils. The research tools were series of tasks, which were based on creating a construction by wooden playing blocks of dimensions 3 cm × 3 cm × 3 cm in three colours: red, blue, and yellow. The classes comprised a sequence of activities which started by building "towers" and "snakes" (considered as one-dimensional figures) then by "walls" of different shapes and regularities (considered as two-dimensional figures) and continued by three-dimensional constructions which were also based on some regularities. During the classes the children were asked to build some constructions presented on a picture; sometimes this was accompanied by building the figure by the teacher. All the lessons were video recorded; additionally, photos were taken. All the phases of pupils' work were reconstructed and analysed by using the video; the process of analysis was supported by the photos and the notes of the preservice teacher.

For the purpose of this paper we will focus on the last lesson which consisted of two different activities. The first task was to build a construction which was presented to the children on a picture

(Figure 1, shape on the left) and then the pupils were asked to continue the figure. The second task was to build the construction presented on a picture (Figure 1, shape on the right) and built by the preservice teacher by using the blocks. For a short time the pupils could observe it in order to familiarize with it (still a part remained invisible to the pupils) and after that the building was destroyed. The children were expected to rebuild it. At the end of the activity the teacher asked them individually: "Which blocks did you use where it was not visible? Why?". In this paper we will analyse the second activity.



Figure 1: The first and the second construction during the fourth lesson

Both constructions were classified as visual geometric patterns (Zazkis & Liljedahl, 2002) in which we can observe two types of regularities: the colour repetition ("repeating pattern", Zazkis and Liljedahl, 2002) and the heights of the "towers" which increase by one block ("geometric growth pattern that can be quantified", McGarvey, 2012). The first figure can be considered a two-dimensional one and the second figure three-dimensional. In the second construction there is an "invisible" part which has to be built by the children.

Our analysis was mainly based on the videotaped process of building the constructions of every child; additionally, we collected data from a discussion with the pupil at the end of the task. All aspects of pupils' works emerged during the process of analysis, according to a grounded theory approach (Strauss & Corbin, 1998). We focused on answering the following research question: To what degree children will subordinate their actions for building the invisible part of the construction to regularities existing in its visible part? In particular, we have formulated the following research questions:

a) Were the pupils able to reproduce regularities in the visible part of the construction?

b) Did the invisible part of the construction retain any regularities during the students' work? Which student activities led to the retain of the regularities?

Results and analysis

In this section we will present the results of the second activity during the fourth lesson. Firstly, we present a detailed analysis of the works of three pupils as characteristic cases. All pupils' names are pseudonyms.

Ania's work (Figure 2)

The girl started her work from constructing a "tower" which contained four red blocks. Then she created a blue "tower" of three blocks and located it next to the initial one. The next step was one red two-blocks "tower" which she put next to the blue one. The wing was finalised by one blue block. After that, she started building another "tower" from four blue blocks. Next to it she located a red two-blocks "tower" and one blue block. She had left one blue block which she put at the top of the construction, half at the red and half at the blue "tower".



Figure 2: Ania's work

Ania used a different regularity of colours than in the picture; her "towers" were the same colour and occurred by turns. She consequently kept that rule. She built her "towers" decreasingly, the first part of her work retained the regularity of heights but in the second "wing" she made a mistake. Her work started from the invisible part and that somehow determined the rest of the construction. The only visible element of the highest "tower" was red, which could cause the decision of the colour choice for the whole "tower". Ania built the whole "tower" in one colour and that influenced the fact that the rest of "towers" were also one-colour. The process of creating one "wing" with her own ad hoc rule could pull her back from analysis of the regularity at the picture. During building of the second part of the construction she repeated her own rule and she changed only the first colour into a blue one. She also forgot that the four-blocks "tower" appeared only once in the construction. By putting the last block she probably tried to keep the regularity of the heights of the "towers" and to balance the two contradictions of the right and left "wings".

Kuba's work (Figure 3)

He firstly built the "right wing" and then the "left wing" of the construction, according to the template. Then he connected the two separated "wings" with the edges of the cubes. He completed the construction by a "tower" of four blocks: blue, red, blue, and red. He continued the regularity created by the second "left wing".



Figure 3: The phases of Kuba's work

Kuba worked very precisely and systematically; all his movements were intentional. He created the construction correctly. He recognised and used the regularity of colours and heights. The last part of the figure – the invisible "tower" – was an effect of continuation of the regularities previously created and connected to the construction of the "left wing".

Julia's work (Figure 4)

The girl started the construction from the invisible "tower" in the order: blue, red, blue and red block. Then she continued the "left wing" by constructing the "towers" of three blocks: red, blue and red, then two blocks: blue and red. She finalised the "left wing" by a red block. In that way she created a regular wall, concerning the colours and heights. Then she constructed the "right wing" by a "tower" of three blocks, then two-blocks "tower" and the last blue block.

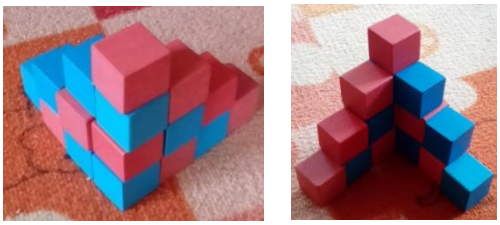

Figure 4: Julia's work

Julia was the only one who fully succeeded in the task by starting her construction from the invisible part. That part was a result of her imagining the continuation of one of the walls (the left wing). She applied the observed pattern without testing it on the visible parts of the figure. Thus, she started from realising the imagined part and then she reproduced the pattern in the opposite order. Her building of the "wing" was done by lower "towers" of alternate colours, thus we can conclude that the base of her first decision was determined by firstly ascertainment of the height (four blocks) and calculation back: red – blue – red – blue. Even the contradiction between the left and the right "wings" did not affect her self-confidence.

The analysis of the process of building the second construction (Figure 1) of all 16 pupils, led us to the following aspects. We have to note that one work may contain more than one aspects:

1. Regularity of colour – a pupil puts red and blue block in a staggered manner;
2. Grouping colour – a pupil is grouping the colours by building short series of same colour blocks;
3. Regularity of shape – the figure's shape is reconstructed correctly;
4. Partial regularity of shape – a pupil reconstructs only one part of the figure (one "wing"); s/he has difficulties in building the second "wing".
5. Student's own shape – a pupil builds a construction which is not related to the template.
6. The invisible part built first – a pupil starts the work from the invisible "tower";
7. The invisible part built as last – a pupil starts from the visible parts and at the end of work s/he competes the invisible "tower";
8. Colour regularity of the invisible part – a pupil uses the same rule of the colours which was observed in one of the "wings" of the construction;
9. Lack of colour regularity of the invisible part – a pupil does not use the rule of the colours.
10. Proper height of the invisible part – a pupil builds a "tower" by using 4 blocks;
11. Improper height of the invisible part – a pupil uses less or more than 4 blocks.

Table 1 presents the pupils' ways of work in relation to the aforementioned aspects. We may notice that 15 works contain more than four aspects; each aspect present in a particular work is marked by x. Only Bartek's work does not contain any aspect: this boy did not build anything during that activity; he was just repositioning the blocks. He justified this in the following way: "I don't want to build anything. I have already built something before (referring to the previous task)".

No.	Pupil's name	Characteristic aspects of the pupils' works										
		1	2	3	4	5	6	7	8	9	10	11
1.	**Kuba**	x		x				x	x		x	
2.	Zosia	x		x				x	x		x	
3.	Dawid	x		x				x	x		x	
4.	**Julia**	x		x			x		x		x	
5.	Maciek	x		x				x		x	x	
6.	Franek	x			x			x	x			x
7.	Borys	x	x		x		x		x			x
8.	Paula	x			x		x		x		x	
9.	**Ania**		x			x	x			x		x
10.	Wiki	x	x			x		x		x	x	
11.	Staś		x			x				x	x	
12.	Leon		x			x	x			x	x	
13.	Kamila		x			x	x			x		x
14.	Basia		x			x	x			x		x
15.	Jola		x	x			x			x	x	
16.	Bartek	-	-	-	-	-	-	-	-	-	-	-

Table 1: Characteristic aspects of the pupils' works

Only four children managed to build the correct construction by using the observed patterns: Kuba, Zosia, Julia and Dawid. They all used the regularity of colour (aspect no.1) and preserved the regularity of the heights of the "towers" (aspects no.3 and 10). They created the "invisible tower" which retained the colour regularity of one "wing" (aspect no.8). In Kuba, Zosia and Dawid's constructions the highest "tower" was a consequence of their work on the "wings" and was created as the final element. Julia started her construction from the "invisible" element. The remaining eleven children (without Bartek) tried hardly to complete the task. They used the experience from previous lessons by putting the red and blue blocks in a staggered manner (aspect no.1) or grouping them according to the colours (one "tower" red, then blue or bases of the "towers" are blue and the next level of the "towers" are red, etc. – aspect no.2). Five of the mentioned children used the regularity of colour (aspect no.1) and three of them transferred their observation into the invisible part, where they applied that regularity (aspect no.8). Two pupils out of eleven were able to reconstruct the shape by using the shape regularity together with space imagination (aspect no.3) but failed to keep the colour regularity (aspects no.1 and 9). Five children did not notice that the invisible "tower" is the highest one and contains 4 blocks (aspect no.11; usually the "tower" was built from three blocks, in one case it was five blocks). During reconstructing the processes of the pupils' works it can be noticed that the buildings were changed when the children reached the invisible "tower" and then they usually lost their regularity. It seems that the invisible part of the figure dominated the pupils' trials of constructions and most of their difficulties were caused by the fact that it was hard to imagine what is behind of the figure. Four examples of the constructions are shown in Figure 5.



Figure 5: The works of Staś, Leon, Kamila and Jola (from left to right)

Although almost all children did not have any problem in building the first construction (Figure 1) in which they used the observed visual geometric patterns: the repetition of the colours and increasing the heights of the "towers", building the second construction seemed to be very difficult. While in the first task all children were able to use their experience from the previous lessons, in the second task the experience was not enough for some of them. We may claim that working with geometrical patterns in a three-dimensional environment is much more difficult that than in a two-dimensional one and that patterning skills obtained in a two-dimensional environment are not easily transferred.

Conclusions

In early years mathematics it is expected that children will easily recognise patterns, particularly they should impose regularities on visual images and describe the rules that help to extend or predict those regularities (McGarvey, 2012). Many researchers express the opinion that recognising patterns is one of the most important skills which are necessary for algebraic thinking (e.g. Lee, 1996). On the one hand, research has shown that "perceiving a pattern is not difficult. Students successfully recognize patterns by imposing structural regularities onto visual and symbolic phenomenon" (McGarvey, 2012, p. 313). On the other hand, our study has shown that many children failed to construct the presented pattern. Only four out of sixteen pupils performed the task correctly and, additionally, retained the regularities in the invisible part. Although the teacher did not mention that the hidden "tower" should follow the rule of the whole figure, they felt the need to continue it. The task turned out to be difficult; however, many children presented their trials in keeping the regularities they observed. They demonstrated their good intuitions and imagination. In most works we could find a rule which was dominant and influenced the final performance.

Our data shows that even if the children had difficulties in reconstructing the figure, they were motivated and their work was intentional. The combination of two regularities and the three-dimensional geometrical object was challenging and such activities brought new experiences to the pupils. The tasks using three-dimensional geometric patterns unveiled that the children in early years need such stimulations. Such activities engage pupils in discovering regularities but also in experiencing and "touching" geometrical objects.

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Combining historical, foundational, and developmental insights to build children's first steps in mathematics

Ana Millán Gasca[1], Elena Gil Clemente[2], Ilaria Colella[3]

[1]Roma Tre University, Rome, Italy; anamaria.millangasca@uniroma3.it

[2]University of Zaragoza, Spain; elenagil@unizar.es

[3]Roma Tre University, Rome, Italy; ilaria.colella@virgilio.it

How can we design mathematical instructional activities that reveal children's early mathematical competence in an analogous way as school activities that exploit children's mother language competence? We put forward a list of mathematical conceptions young children may have been taught previously and some subsequent actions developed to observe them and guide their first steps in mathematics in an instructional context. The list has been developed on the basis of insights from modern axiomatic presentation of arithmetic and geometry contrasted with historical results and epistemology of mathematics. We discuss the application of the list in a singular context, a group of eight 3 to 8 year-old Spanish children with Trisomy 21, to show the suitability of this tool for revealing early mathematical competence.

Keywords: Experiential learning, early years education, special needs, mathematical skills.

The central issue: Enhancing young children's competence in mathematics

The starting point of our research was a situation of stagnation in two different and singular educational contexts: the encounter with mathematics of Italian primary first graders and the special needs of children with Trisomy 21(or Down syndrome).

Actual current praxis in Italian first grade classrooms[1] and the available research and teaching materials regarding mathematics for children with Trisomy 21 show a remarkable similarity in two aspects. Firstly, from the point of view of contents, there is a focus on numeracy and especially on teaching and learning of numerals and written arithmetic and a clear exclusion of geometry. Secondly, from a wider cultural point of view, we witness a marked lack of confidence in the relationship between mathematics and children's feelings and mind. (Millán Gasca & Gil Clemente, 2016). Both aspects are interrelated. As mathematics is socially viewed as a key component of our modernity, school is forced to cope with the difficulties of its teaching and learning. Consequently, teachers must concentrate on the traditional hardcore, that is, numeracy and practical tasks,

[1] A quite homogeneous didactical praxis was identified thanks to the relationship of the Roma Tre University Department of Education with schools (private, state, urban and rural, and with students from different social backgrounds) in the Lazio area in 2006-2014. Although compulsory mathematics teaching or initiation for 3 to 5 years old children is not laid down, in some Italian preschools, including the wide number of state Montessori schools, mathematics is a part of the curriculum and goals. A sharp contrast is experienced between the good results obtained in these latter schools, where pupils usually show a deep interest and curiosity towards number and geometry, and a general situation of difficulty, anxiety and fear of primary school first graders starting in the first months of first grade where schoolwork is focused on exercises about the writing of the numerals from 1 to 9 and moving on to addition and number symbols with two digits from 11 to 20 after 3-4 months.

narrowing the goals of the learning to obtain instructional success. Centrality of arithmetic and abandonment of formative goals are especially noticeable when teaching children with Trisomy 21[2].

Our working hypothesis was that early mathematical competency, in the same way as linguistic competency, could be enhanced and analyzed in terms of naive conceptions. For this aim, we developed a list of items (including concepts and observable actions) to be used in the design of focused mathematical instructional activities, able to bring out young children's mathematical competency (avoiding initiating children in mathematics through written arithmetic), in much the same way as school activities exploit children's mother language competency (avoiding initiating children in linguistic expression through grammar). After testing the suitability of the list with a group of twelve 4 year-old children schooled in Lazio (Italy) (Colella, 2014), we faced the challenge of using it to enhance the mathematical competence of a group of children with Trisomy 21 in Spain.

Naive mathematical conceptions

The fact that *there is a lot of mathematical life* "*before school*" or *"before being taught"* has been pointed out by authors such as Martin Hughes (1986), Margaret Donaldson (1978) and Liliana Tolchinsky (2003). Usage-based theory of toddlers´ language acquisition (Tomasello, 2003) offers a description of the key situations leading to the first holophrases in a joint adult-child attentional frame. It helps us to understand the precocity of children and their interest and enthusiasm regarding numbers as well as geometry.

The empirical examples considered by Hughes among children in Edinburgh's Department of Psychology nursery, as well as by Karen Fuson (1988) regarding her two daughters, recorded in the mother's diary she started at 1 year and 8 months, were observations of what are considered as *naive arithmetical* conceptions, that is, conceptions that have been observed in children independently of schooling and instructional design. Fuson and Hughes go beyond Piaget and his collaborators' work on the roots of children's understanding of arithmetical ideas because they avoided concentrating solely on the search for a developmental path and on the isolation of spontaneous cognitive development. Instead, they tried to come closer to children's thoughts and feelings through close interviews, observations and task experiments, considering that elementary teaching *enters the scene* in the wider context of children's human experience and development.

We have extended this available research in arithmetic to encompass geometry, following René Thom's views (Israel, Millán Gasca, 2012; Millán Gasca 2016), as shown in later pages.

Insights from historical and mathematical perspectives on primitive objects and relationships

In order to identify geometrical items we drew inspiration from Federigo Enriques (1924-27). He focused on the instructional meaning of the identification of the "primordial", primitive, undefined concepts of the modern axiomatic description of arithmetic and geometry considered in their historical context (Israel & Millán Gasca, 2012).

[2] Elisabetta Monari (2002) pointed out that the "old tree of mathematics" (the traditional view of school mathematics as introduction to written arithmetic) should be replaced by a "new tree", where different cognitive potential could appear.

It can be noted that the pivotal role of the acquisition of sequence of number-words in arithmetic is strikingly coherent with the modern Peano's (1899) axiomatic description of natural numbers. Information on undefined primitive objects (*number* and *one*) and their relationship (*successor*) is contained in axioms, such as, "*one is not the successor of any number*" or "*if two numbers have the same successor they are the same number*" that evoke counting. In addition, the first recursive definitions of addition and multiplication and the definition of "greater than" start from this information. Fuson's description of early arithmetical conceptions includes all of these mathematically critical ideas regarding an ordinal view of natural numbers defined in a child's way, together with the cardinal views and measures uses to which children are exposed nowadays (these views were also central in the ancient origins of number and of extensions of the concept of natural number). This attention to mathematically well-identified different number situations in children's experience is crucial to draw indications for instructional activities.

Following the same path, we paid special attention to the role of Hilbertian (1902) undefined concepts (point, straight line, plane), relationships (congruence, lie in, lie between) and first definitions of objects and relations (angle, segment, circle, triangle, polygon, greater than...) deduced from the axioms, in the building of naïve geometrical conceptions in young children´s minds. For example, in relation to the concept of line (paradigm of the continuum) and straight line, possible naïve conceptions regard line as a path; line as a stroke on a sheet of paper with a pencil; lining up, or walking along the minimum distance between two positions. In relation with the concept of point we can observe whether they know to stand at a fixed point, or if they are able to draw them. Talking about change of direction can be a way to introduce the concept of angle.

Inspired by these primitive concepts we considered performance regarding "solving simple geometric problems", such as drawing a circle freehand; cutting a circle; comparing circles (cutting and overlapping); drawing a straight line connecting two points; joining several numbered points freehand with a straight line; drawing a non-straight line; answering questions related to which object is longer, bigger or thicker...

Naïve arithmetical and geometrical conceptions list

The list we have developed, based upon primitive arithmetical and geometrical concepts and the previous developmental insights, includes some possible naïve mathematical conceptions that children can have. Following Fuson's point of view (1988), we looked for a web of conceptions, including connections between geometry and numbers and some obvious relationships, and not for a systematic building of a theory. Naive conceptions also include the ideas on symbols belonging to the oral dimension of language (not centered at all on the decimal positional numeration system.).

As for mother language, naive mathematical conceptions include competence together with errors and misunderstanding in a dynamic setting, where exposure to new experiences or situations helps the child to correct by him/herself previous ideas or accept and include corrections from peers or adults.

From the list, we have also developed a guide for observation in action. This guide consists of several activities intended either to observe this possible mathematical competency (activities simulating a non-instructional "informal" context, such as domestic or playground experiences) or as "opportunities to learn" (becoming proper instructional, teaching activities). These activities

should be embedded in children's overall living experience and should have a human sense for them (Donaldson, 1978). Furthermore, they should at the same time, help to actually generate learning, that is, to guide first steps in mathematics. Of course the border between observation and generation of learning is not sharp. For instance, during the time expended in exploring a question, many children may learn something or reinforced their knowledge (taking into account besides that not every child has the same previous informal opportunities to build naïve conceptions).

Relating to arithmetic, activities such as *bringing enough pencils for everybody, telling the cook how many people to prepare lunch* may bring out naïve conceptions, skills children already possess such as *knowing some number words, knowing some part of the number sequence, counting things.* These conceptions are connected with the primitive arithmetical concepts, *idea of number one*, or *successive number*. However, children can also bring into play these conceptions to actually be able to use their knowledge to bring enough or the exact number of pencils or to try to give an answer to the cook or even to answer correctly.

In relation with geometry other activities like *walking down a road, holding a thread between two children, folding a sheet of paper neatly* are suitable for bringing to light naïve conceptions of *path* and *line,* which are closely related to the primitive concept of *straight line.* In the same way, children can use these conceptions to learn, as instance, how to distinguish a straight line from a curved one.

First steps in mathematics for children with Trisomy 21.

When faced with the mathematical instruction of young children with Trisomy 21, we had to take into account the adverse general context of confusion about goals and contents mentioned at the beginning. In this context, children with Trisomy 21 appear to be in a clear disadvantage due to their well-known difficulties with arithmetic, lack of effective proposals for teaching and misunderstanding of the role of the discipline in their personal development.

There is also a problem in assessing the actual mathematical knowledge of children with Trisomy 21 (Faraguer, 2014) attributed to their scarce skills in oral and written language and their avoiding behaviour when put in stress situations (Wishart, 1993). This has lead to evaluations based upon interviews with parents or professionals (Faraguer, 2014) and consisting of solving decontextualized tasks (Zimpel, 2016). Such evaluations use to show a poor performance in mathematics by people with Trisomy 21.

From the success obtained using the list of naïve conceptions with a group of 4 year-old Italian children with no previous exposure to mathematics (Colella, 2014), this list appeared to be a suitable tool to make a proper assessment of the previous mathematical ideas of the children with Trisomy 21. We could also use this assessment as a basis for the building of an accurate teaching programme, that focus on formative values of mathematics without giving up to placing high expectations on the children.

Methodology

The experience[3] consisted of a twenty-hour workshop over ten months with a group of eight children between 3 and 8 years old (three aged 3, two aged 5, two aged 6, and one aged 8) without previous selection[4]. The workshop was conducted by a team of four volunteer special education teachers and devoted the first three months to an exhaustive exploration of their naïve arithmetic and geometrical conceptions.

It was a *study case*[5] framed in what it is known as *research for practice* (Faraguer, 2014). Throughout the sessions we made an experiential observation, which allowed us to write a narration of the living experience (Van Manen, 2013) and prepare a final description of the naïve conceptions of each child in relation to the items we have observed.

Development of the workshop

Firstly, we have to adapt the original list to make it suitable to the group of children. Table 1 shows the conceptions definitively explored.

Arithmetical conceptions	Geometrical conceptions
Numbers (any ideas) Counting (transitive and intransitive) Cardinality Subitizing Zero Spontaneous symbolic representation of quantity Resolution of simple arithmetical problems	Idea of point Line and idea of continuous Idea of straight and of non-straight Ideas of angle Ideas of round and circle Ideas of triangles and quadrilaterals Ideas of sphere and other regular solid figures Resolution of simple geometrical problems
Measure of time Distance Use of cardinal numbers to measure a distance (steps)	

Table 1: Some arithmetical and geometrical naïve conceptions

Secondly, throughout the three two-hour sessions devoted to the exploration of their naïve mathematical conceptions, we faced the challenge of designing activities also adapted to features of children with Trisomy 21 (for example most of them did not speak, so we could not use dialogue to build mathematical knowledge). We practiced oral sequence when counting balls to decorate a Christmas tree or when counting time playing hide-and-seek. We worked with the concept of

[3] Carried out in the context of the PhD thesis of the second writer devoted to the exploration of geometry with children with Trisomy 21, following a careful consideration of geometry in Édouard Séguin's approach. (Gil Clemente, 2016)

[4] Families who decided to participate in the research, were members of a local association in Zaragoza (Spain) and have more confident outlook than those of older children with a disappointing experience of primary school.

[5] It is a common methodology with children with Trisomy 21, because as Monari (2002) pointed out, these studies open the path to more general ones that usually confirm results obtained in singular cases.

straight line folding a letter to the Wise Men. We walked along paths to discover new worlds or join points to discover secret drawings. We also compared the length of swords before fighting as a way to compare magnitudes. We understood geometrical concepts through mimesis when training for having an adventure and we discovered surprising similarities among different familiar objects (balls, fruits, caps, towers, boxes, tins or tubes…).

We must highlight the importance of applying these activities in a happy play context in which we could witness their individual and group processes of learning without interference or pressure.

Results

In spite of the limitations inherent with an experience carried out in a formal context and not in their real life[6] we obtained some useful conclusions to guide our later research.

Most of the children, especially the youngest, had very limited initial arithmetical conceptions. Only three of them were able to count to nine and the rest hardly know the numbers "one, two, three". They could only subitize one or two objects, except the eldest child who reached six objects. They made a lot of mistakes reciting oral sequences (they counted objects more than once or forgot objects when counting) and counting objects or drawings (although they counted objects better than drawings). Only the three eldest children had well established some conception of cardinality and these children were able to solve some very simple arithmetic problems (such as "give me n" or answering to the question "n plus m" with low numbers and only by counting). Surprisingly, hardly any had difficulties in understanding zero in several ways, consistently with the research made by Zimpel (2016): most recognized the cipher, some knew that it was the number before one in the number sequence and some said the word "zero".

However, their initial geometrical conceptions were much better. Through the use of their bodies, movement and mimesis they showed their understanding of point as a fixed position (standing on it without moving), of line as a path (making an effort to go along it without bending) and of straight line as the minimum distance between two points (they all walked straightly when asked to go from one teacher to another). The eldest ones were also able to distinguish none-straight lines and named them as "curves". They all had an idea of a circle as a round (they knew how to sit in a circle or how to turn on themselves). However, they had scarce ideas of polygons (they showed more difficulties in recognizing triangles than in recognizing quadrilaterals). Surprisingly, they had a special ability to discover the similarities among every day solids. Acquisition of skills related with drawing differed substantially from one child to another due to the delay in motor development common in Trisomy 21.

Their greatest difficulties in geometrical conceptions had to do with every aspect related to measure (counting steps, for instance, was almost impossible for all of them, even for the eldest one) due to the strong relation between measure and numbers. They also showed a poor performance in understanding the relationship "to be between two objects or two persons", basic for the acquisition of the concept of segment.

[6] We observed only one of the children, the eldest, in his everyday life. From this observation we made a diary that was very useful for extending our research (Gil Clemente, 2016).

The most remarkable conclusion was the enthusiasm and good disposition showed by all the children when facing mathematical tasks: only one child did not engage in the activities proposed; the other children enjoyed the activities and concentrated on them; many families told us their children were looking forward to coming back and doing "mathematics". This widely confirmed our initial thesis about the natural relationship between mathematics and childhood, even for those with disabilities.

The results of our observation show a path to the possibility of seizing the power of geometry for developing some abstract thinking processes in children with Trisomy 21. This is consistent with the role attributed to geometry by Séguin (1846,1866) for awakening ideas in disabled children's minds and with recent research regarding the strength of abstraction in Trisomy 21 (Zimpel, 2016).

Final remarks

The experiences carried out with the two groups of children in Italy and in Spain, indicate that this approach to the encounter with mathematics actually stimulates knowledge building on a solid basis by avoiding the non-involvement of children in school mathematics and is therefore a promising path for future research. It runs in contrast with normal standardized numerical school exercises, by proposing items connected to the development of a relationship of *intimacy* with abstract mathematical objects such as points, segments or numbers (Thom, 1971) which should lay the basis for further introduction to symbolic thought.

Introducing geometry in children's education as a result of the confidence in the relationship between mathematics and childhood[7] helps children to develop this abstract thinking. We have confirmed this idea with the development and application of subsequent teaching sequences based mainly on geometrical concepts after the exploration of the naïve conceptions described (Colella, 2014; Gil Clemente, 2016).

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How do preschool teachers characterize their own mathematics teaching in terms of design and content?

[1]Hanna Palmér and [2]Camilla Björklund

[1]Linnaeus University, Sweden; hanna.palmer@lnu.se

[2]University of Gothenburg, Sweden; camilla.bjorklund@ped.gu.se

Preschool mathematics may look very different in different contexts. These differences concern both what mathematics children are offered to learn and how the learning of that mathematics is orchestrated. In this paper we present an ongoing study on how Swedish preschool teachers characterize their own mathematics teaching in terms of design and content. The target preschool teachers are those working with the youngest children aged one to three. We present two examples of how these preschool teachers describe and characterize their mathematics teaching in terms of design and content and we discuss possible contributions to research and practice.

Keywords: Content for learning, design for education, mathematics, preschool teachers.

Introduction

Preschool mathematics does not only prepare for future schooling but – and maybe even more important – provides "young children with rich and engaging intellectual stimulation" (Ginsburg, 2009, p. 405). From a time when young children were considered to be almost incapable of learning mathematics the question today is seldom whether or not mathematics belongs in preschool but rather how to organise mathematics teaching (Björklund, 2014; Cross, Woods & Schweingruber, 2009; Perry & Docket, 2008). However, the recognition of importance and the increased attention do not automatically imply consensus regarding how preschool mathematics should be designed or what constitutes an appropriate content. Cultural issues may explain some of these differences but there are also differences within countries and seemingly similar educational contexts (Palmér & Björklund, 2016).

There are several studies where researchers observe, analyze and often evaluate mathematics teaching and teachers in preschool (see Sarama, Clements, Wolfe & Spitler, 2016; Tirosh, Tsamir & Levenson, 2015). More seldom are the preschool teachers asked about how they themselves analyze and/or evaluate the observed teaching. In this paper we present an ongoing study on how Swedish preschool teachers working with the youngest preschool children, those aged one to three, characterize their own mathematics teaching in terms of design and content. To what extent do their characterization coincide with researchers and other preschool teachers? Maybe preschool teachers use different words to describe "the same kind of" mathematics teaching or reverse, maybe they use the same words to describe "different kinds" of mathematics teaching. The aim is to find out how these preschool teachers themselves conceptualize their mathematics teaching practice. In this paper we will present the frame for analysis we intend to use, to investigate one of our research questions, that is:

- How do these preschool teachers characterize their teaching in terms of design and content?

Since the study is ongoing the focus of the paper is on the rationale of the study and methodological layout with only brief discussions of two examples. First in the paper we will give some background of preschool mathematics and Swedish preschool. After this we present the study followed by two examples of the empirical material we are to work with. Finally, we discuss what we believe this study will contribute with to research and practice.

Preschool mathematics

Preschool mathematics is an issue of current debate and may look very different in different contexts. These differences are found both in *what* mathematics children are offered to learn and *how* learning of that mathematics is orchestrated (Cross, Woods & Schweingruber, 2009; Perry & Dockett, 2008). While some emphasise basic number facts and applying computational procedures (Westwood, 2011) others emphasise advanced mathematical activities focusing on a broad spectrum of content (Claessens & Engel, 2013; Seo & Ginsburg, 2004). Regarding *what* mathematics children learn in preschool numbers and quantity are often emphasised but also concepts of space, shape, pattern, and order are central in early mathematics learning (Sarama & Clements, 2009). However, there are studies showing that the depth and quality of how the content is made an object of learning varies, where for example spatial relations, shapes and patterns are rarely problematized (Björklund & Barendregt, 2016).

One way to characterize *how* mathematics is taught is to distinguish between naturalistic, informal and adult guided learning experiences (Charlesworth & Leali, 2011). Naturalistic learning experiences are initiated and controlled by the child. A naturalistic learning experience can turn into an informal learning experience if a teacher starts to interact with the child in a way that knowledge may be reinforced, applied or expanded. Adult guided learning experiences are those being pre-planned by the teacher involving some direct instruction. Björklund's (2014) study of meaning making of mathematical concepts highlights the complexity of designing preschool mathematics education. She found three ways in which teachers planned and acted to facilitate conceptual growth among 4- and 5-year-olds. One way of approaching mathematical concepts was to give the children individual traditional tasks to solve ("I give you *x* number of items, can you divide them into half?"). Another way of approaching the same concept was to "hide" the mathematical content in problem solving tasks, such as games and every-day tasks. The former approach, which was clearly goal-oriented and adult guided turned into a task of "doing" where the children primarily waited for their turn but not directing attention to the mathematical content rather than the joy of being given a task to solve. The latter, which also was carefully planned by the teacher to stimulate certain concept development, failed in establishing intersubjectivity because of the children's different attention to play the game and finish the task, rather than stop and reflect on the mathematical content within the tasks. Even though the children happily engaged in the activities, the mathematics was not in focus of attention. A third way of approaching mathematical content was framing a concept in narratives where the teacher could orchestrate the direction of a story and in that manner direct the children's attention towards an intended object of learning. It turned out that this approach appealed to the children and engaged them in problem solving where concept development was made possible. This third approach was also characterized as more perceptive to the children's suggestions and creative solutions. Thus, designing teaching for preschool mathematics is a delicate work, where abstract and "invisible" mathematical principles are to be

made explicit. Preschool didactics is to make the invisible visible to the child (Pramling & Pramling Samuelsson, 2011). Björklund's (2014) study is one example of this, since the focused attention has to be made common for both teacher and child, whereas the design of the activity may constrain or enable learning.

Swedish preschool

Swedish preschool, in which the present study is conducted, is situated within a social pedagogy tradition (Bennett & Tayler, 2006) where care, socialisation and learning constitute a coherent whole and is part of the formal education system. Preschool is offered to children between the ages of one and six, and similar to other Nordic countries (Reikerås, Løge & Knivsberg, 2012), the youngest children attending preschool are increasing in number. In Sweden, 94% of all 4–5-year-olds are enrolled in preschool or similar pedagogical practice and 88% of all 2-year-olds attend preschool or an equivalent practice (National Agency for Education, 2016).

The preschool curriculum includes several mathematics-related goals, for example that preschool should strive to ensure that each child "develop their understanding of space, shapes, location and direction, and the basic properties of sets, quantity, order and number concepts, also for measurement, time and change". Another example is to ensure that each child "develop their ability to use mathematics to investigate, reflect over and test different solutions to problems raised by themselves and others" (National Agency for Education 2011, p. 10). These are however not goals for children to attain but instead provides direction for content and activities.

Based on the curriculum, each preschool chooses the approaches most appropriate for its own setting. Preschool teachers and child-minders are the two main types of pedagogues working in Swedish preschools. Child-minder is an upper secondary school education while to become a preschool teacher; one must complete a three and a half year university programme in preschool teacher education. Preschool teachers educated after 2001 have studied mathematics teaching in their degree, but the preschool teacher profession is mostly characterized as "educational generalists", without specialization in any particular subject.

Theoretical framing

Since we want to investigate how preschool teachers characterize their own teaching we needed to develop a framework that included the dimensions of *what* and *how*. To capture both these dimensions we have used Bernstein's (1999) notions *vertical and horizontal discourses* together with Claesson, Engel and Curran's notions (2014) *basic and advanced content.*

Bernstein (1999) uses the notions vertical and horizontal discourses to distinguish between different kinds of knowledge. A discourse characterized by coherence of content, hierarchically interconnected procedures, specialized language, systematically organized activities focused on general knowledge is a vertical discourse. A discourse characterized by location within communities, high relevance in the situation, every-day language, segmentally organized and maximized encounters with persons and habits is a horizontal discourse. In this study the notions of vertical and horizontal discourses is used to describe the dimension of *how*.

Claesson et al. (2014) define mathematics content as basic or advanced depending on whether the majority of children in the group focused on have mastered the content or not. Thus, basic

mathematics imply mathematics content that the majority of the children already know but that still is new for others while advanced mathematics is new content for the majority of the children. In this study the notions of basic and advanced is used to describe the dimension of *what*. However, basic or advanced will not be based on groups of children mastering some content or not, but on the preschool teachers' view of the content in each situation being characterized. Together these four notions can be used to characterize different contexts of mathematics in preschool as in Figure 1.

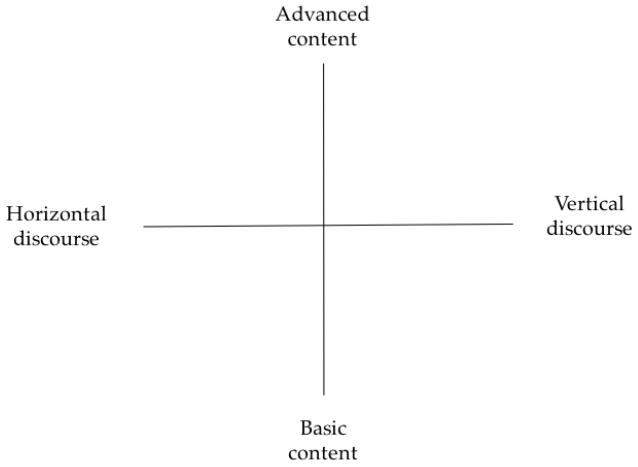

Figure 1: Connecting horizontal and vertical discourse with basic and advanced mathematics.

The two extremes basic and advanced content are to be understood as differences when it comes to which mathematics being focused on while the two extremes horizontal and vertical discourses are to be understood as differences when it comes to design. The axis basic and advanced content illustrates if the content is considered as basic or advanced, in other words if the children engaging in an activity will be familiar with and master the content or will it be a challenge. On the left side (horizontal discourse) it is sufficient that this content is part of every-day activities and routines with no need to make it explicit for the children. On the right side (vertical discourse) mathematics is the starting point with no need for applications. Thus, every-day is the starting point in the horizontal discourse and mathematics is the starting point in the vertical discourse. Along the line there is a gradual shift and somewhere in the middle there is a shift concerning everyday life or mathematics being the starting point for the design of preschool mathematics.

The study

The authors of this paper have been part of a national network for several years that focus on toddler mathematics in preschool settings. A consortium of preschool teacher educators from different Nordic universities initiated the network with a special interest in the youngest children's mathematics learning and didactical challenges in early childhood education. There are approximately 30 active members in the network. On the network's spring-meeting 2016, the current study was presented and the members were invited to participate in generating data for analysis. Thus, the selection of participants is information-oriented and deviant (Flyvberg, 2002) which imply that we have chosen teachers that we know are interested in teaching also the youngest

children in preschool mathematics. This selection is based on the research focus not being *if* these teachers teach mathematics but instead how they characterize the mathematics they teach.

At the network meeting the study was presented verbally and afterwards the information was also e-mailed to the participants. Until the autumn-meeting 2016 the participants who wanted to (participation is of course voluntary) were supposed to document "eight situations where toddlers encounter mathematics" on a pre-prepared form. First they were asked to "describe the situation". They got some extra help by the questions: Who was present? What mathematical content? What happened? Next they were asked to describe how the situation started. Was the situation spontaneous or planned? If the situation was planned, on what grounds? Then they were asked to describe their own as well as the children's actions in the situation. What did they do and say? What did the child/children do and say? To find out how these preschool teachers themselves characterize the teaching situations they describe, they were asked to place the situation in a picture like Figure 1 above. If they wanted to they could motivate their placement. Having the preschool teacher to characterize the situations based on *what* and *how* makes it possible to examine what they associate with expressions as everyday mathematics, advanced content for toddlers etc which in turn may develop the professional language of preschool mathematics. Finally they were asked to estimate how common a situation like the described one, is for this/these child/ren.

Two examples

The current study is ongoing and we have only a small sample so far and tentative results. Therefore, we will here present two examples of documentations submitted from two of the network members to illustrate the framework and how it can be used as an analytical tool.

Example 1

The first described situation is about a child aged two years and ten months. She and one preschool teacher are sitting together. This situation was planned by the preschool teacher based on the child's interest in sorting activities. The mathematics content is named as "sorting".

The preschool teacher gives the girl a box with small plastic bears in different sizes and colors and asks the girl if she can sort them. The girl answers, "yes I can" and starts to pick in the box. She picks up one bear and at the same time naming its color. She says "blue, yellow, red and green. Do we have more colors? Yes we have more blue bears".

The preschool teacher describes her own actions as "confirming what she [the girl] was saying" as well as "keeping the other children who wanted to take the bears away". She writes that she asked the child if she could count the bears. The child then answered, "yes I can but now I don't want to because I want to wear a dress instead".

This situation is described as occurring two or three times a week and is by the preschool teacher categorized as in Figure 2 below.

Example 2

The second example is a described situation with a child aged exactly two years. The mathematics content is named "training volume". The situation arose spontaneously outdoors. The girl is standing together with three other children in a puddle. She takes 2-3 shovels with water and pours

it into a bucket. Then she pours the water out again. This procedure is repeated over and over again for about 15 minutes. After about half the time another child aged two years and ten months starts to pour water into the same bucket. The only thing the first girl says during the 15 minutes is “pour in”, this as a call to the second child. The situation is described as occurring two or three times a week and is by the preschool teacher characterized as in Figure 2 below.

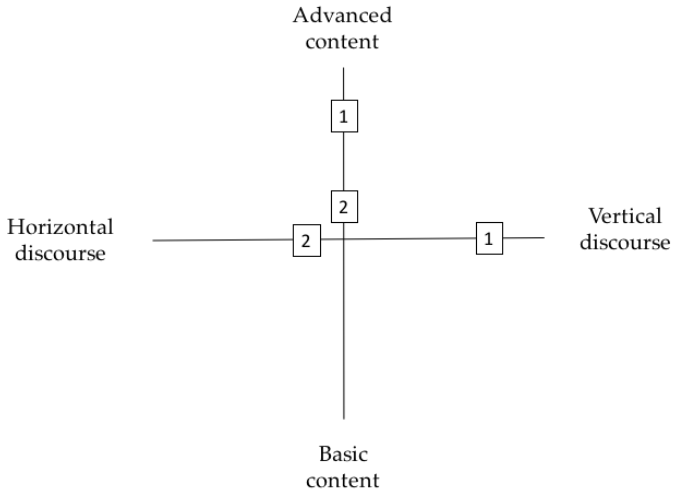

Figure 2: Example 1 and 2 as placed in the figure by the preschool teachers.

As mentioned, our selection of participants is information-oriented and deviant (Flyvberg, 2002) why our results will not reflect toddler mathematics in all Swedish preschools. However, the empirical material will provide some insight into the context in which the youngest children in Swedish preschool meet mathematics as well as which situations these preschool teachers think of as mathematical situations. In relation to the first example in this paper one could question the preschool teacher naming the mathematical content as “sorting” instead of describing sorting as an activity with the aim to make visible mathematical concepts as shape and size. Furthermore, one could consider if the content is to be deemed as advanced in relation to the explicit child in the situation. In relation to the second example one could question what the child is engaged in. Is she exploring volume or pouring water more as a scientific activity? Other questions that can be raised are if the invisible is made visible to the children in the situations as well as if focused attention becomes common for both teacher and child? Questions like this are about the situations constraining or enabling the learning of mathematics.

Expected contribution to research and practice

The preschool teachers focused on in this study are working with the youngest preschool children, those aged one to three. Based on the national network on toddler mathematics we know that these preschool teachers are interested in teaching mathematics. What we want to investigate is how they themselves characterize their mathematics teaching in terms of design and content. Since the study is ongoing we cannot present other than tentative results since only few examples of empirical material are collected so far. In this final section we will discuss what we believe this study can contribute with to research and practice.

The question of whose perspective that leads the interpretation becomes focal when starting to look into this kind of empirical data. “Volume” may be considered a quite advanced mathematical concept, since it demands attention to three dimensions and the spatial relationship between length, height and width of an object, for example. The preschool teacher may on the other hand consider the act of pouring water as a very simple exploring activity without further consideration of the complexity that the activity may entail. However, the child’s object of learning might very well be of natural scientific nature or a motor skill exploration whereas the mathematical content is left for the observer to interpret, without any conclusions of the mathematical learning value made possible.

Another reflection regards how the preschool teachers interpret vertical and horizontal discourses as well as basic and advanced mathematics for these preschool children. What similarities and differences can be found? As mentioned, one possibility is that preschool teachers use different words to describe “the same kind of” mathematics teaching or reverse, maybe they use the same words to describe “different kinds” of mathematics teaching. Making such similarities and differences visible may develop the professional language of preschool teaching in mathematics.

Since the study is conducted within the frames of a national network on toddler mathematics we believe it is important to contribute to this practice. One way of doing this is to use the empirical material to investigate to what extent the members characterize the same situation similarly. One way to do this is to ask some of the preschool teachers to present one of their situations and then let all the others do a categorization. When they place the situations into Figure 1 they define what they consider to be vertical and horizontal discourses as well as basic and advanced mathematics for these preschool children. Thus, collective but not joint categorizations can be the starting point for discussions about what we mean by spontaneous versus planned mathematics, vertical versus horizontal discourses as well as advanced versus basic mathematics.

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"Two, three and two more equals seven" – Preschoolers' perception and use of structures in sets

Priska Schöner and Christiane Benz

University of Education Karlsruhe, Germany; schoener@ph-karlsruhe.de, benz@ph-karlsruhe.de

There are numerous studies that confirm the importance of different skills in determining cardinality of sets for coherent mathematical learning. Some of these skills can be supported in kindergarten in a playful way. In this article we will first discuss whether it's possible to distinguish the two processes of perceiving quantities and determining the cardinality of sets and/or when they may coincide with each other. It will also be investigated how the perception of structures in sets develop in children at preschool age and how this is used to determine cardinality in various ways.

Keywords: Perceiving (sub-) structures in sets, determining cardinality of sets, (structural) subitizing, preschool education, early mathematics education.

Introduction

"Two, three and two more equals seven". This statement of a five year old girl may sound simple but profound mathematical developments underlie this statement. From the age of five to six years, the numerical knowledge increases strongly. This period seems to be a fruitful phase for the development of mathematical abilities (Weinhold Zulauf, Schweiter, & von Aster, 2003, p. 229). One of these developing abilities is the ability to perceive structures in sets and the usage of these in determining the cardinality of the set. The example above shows that the girl perceives a structure in a presentation of seven dots with the sub-structure two-three-two. This perception leads to a specific determination of the cardinality. The background knowledge about these issues is important as it enables professionals to ask adequate questions in order to support children individually or to offer suitable playing and learning areas. In particular, the different possibilities of the (individual) perceiving of structures play an important role, which will be described in detail in the following sections.

Perceiving (sub-) structures in sets

Perception does not just happen incidentally but on the contrary it is a very active process. By focusing the attention and one's own interest on the characteristics of a certain object, information from the surrounding area is actively chosen (Goldstein, 1997, p. 108). In the interviews described below, the attention is drawn to the cardinality of presented objects. Although not all ongoing cognitive processes when seeing and perceiving objects can be illustrated here, it shall be briefly referred to the gestalt psychology, in which the question is central as to why some things are perceived as a unity and others are not (Goldstein, 1997, p. 170). In order to answer the question, several gestalt principles (gestalt laws) were formulated. The six main important laws are the law of good gestalt (pithiness), the law of similarity, the law of closure, the law of proximity, the law of common fate and the law of past experience (ibid. pp. 170–176). The law of proximity indicates that things which are close together appear as belonging together (ibid. p. 173). The arrangement of objects as representations of sets may due to their distance, indicate a possible grouping of the objects. The emerging structured perception based on the grouping can then promote a determination of the set of objects in "a single glance". In psychological and mathematical education research this "perception in a single glance" and the simultaneous determination of the set is described as subitizing. Here, it is assumed that a set

with up to four objects can be determined in a glance, without counting the objects (e.g. Mandler & Shebo, 1982). It has not been clarified whether a quick counting process lies behind subitizing (Gelman & Gallistel, 1986) or whether it's about a non-counting process (cf. Dornheim 2008). Clements and Sarama (2009) distinguish between perceptual und conceptual subitizing. Perceptual subitizing is when you just "see" how many objects there are. So it is possible to identify the cardinal number instantaneously. Conceptual subitizing is used when "seeing the parts and putting together the whole" (ibid. p. 9). With conceptual subitizing it is possible to subitize more than four objects. Clements and Sarama (2009) use the term conceptual subitizing whenever a recognized structure is used for determining the cardinality, as in the case 'counting on', for example (cf. Schöner & Benz, in press). For the following analyzes it is important to use a term which describes the following situation only (for a set with five elements or more): the perception of structures in a set and the immediate naming of the number. In this case, the term "structural subitizing" is used (ibid.). So the term structural subitizing means that the determination of the cardinal number of objects coincides with the process in which the set is perceived in structures. The terms "perceptual" and "conceptual" subitizing are not used in this article. Even children at preschool age "can perceive structures in representations of quantities and use this for the determination of numbers" (Benz, 2013, p. 11). The girl who is quoted at the beginning of the introduction "Two, three and two more equals seven" determines the cardinality of the set by means of structural subitizing when she recognizes the structure and immediately knows that the answer is seven. In this case, she can use the structure for determining the cardinality of the set. Söbbeke (2005) states that children use the "visual structuring ability" as a method in perceiving and using structures in arrays of objects. This also means perceiving sets not as every single item but as (different) groups of items. This ability to change from focusing every single item to perceiving and identifying structures in sets is important for the numerical development (Hunting, 2003), as well as for the part-whole understanding (Krajewski, 2008; Benz, Peter-Koop, & Grüßing, 2015). In this paper, the emphasis is on the visual level, because each time the cardinality of a presented set is considered. This is the basis for the later more abstract processes with numbers, in which composing and decomposing numbers play an important role. Additionally, to the question whether and how the structured perception of sets develop in five to six year old children, it is illustrated if and how further possibilities to determine the cardinality of sets can be observed due to the perception.

Research questions

The following two research questions are investigated in this paper:

(1) How does structured perceiving of sets develop in children at the age five to six?

(2) How do children at preschool age use structures for determining the cardinality of sets?

Design

In this paper, a case study will show how structured perceiving of sets develops in children aged five to six years. In the case study it is also examined how children of this age use structures to determine the cardinality of sets. The case study is part of a larger study. In order to understand the framework of the case study, the structure of the entire study is described first. The whole study is an efficacy study with a treatment group and a control group with more than 100 children aged between five and six years (cf. Table 1). It's a panel design so the same children were interviewed three times (t1, t2,

t3) to evaluate whether and how they perceive and use structures for determining the quantity of a collection of objects. The first interview (t1) was before the intervention. Then the intervention happened. The second interview (t2) took place shortly after the intervention. The third interview (t3) was conducted as a follow up interview.

Between t1 and t2 the treatment group got a box with different materials and games, which were suitable to discover and facilitate the structured perceiving of sets in a playful way. The kindergarten teachers were instructed and used these materials one to three times a week together with the children. At time t2, the kindergarten educators were interviewed with the help of a structured interview in which six questions were formulated. Amongst other questions, it was asked which developments were observed in the children's perception and usage of structures. In many studies concerning the determination of cardinality, a set is presented only for a short instance in order to investigate whether the children perceive the set simultaneously or not (e.g. B. Clarke, D. Clarke, Grüßing, & Peter-Koop, 2008). As it can't be eliminated that the sets may possible be determined by a quick counting process, despite a short time of presentation, a pre-study with 27 children was conducted. Pictures with dots were presented to the children and also for only a short instance before they were asked to mention the cardinality of the dots. Statements made by the children such as for example "I have counted them again, as they could not be seen anymore" or "I have looked at the dots on a picture in my head" indicate that visual conception may play a role here. (Schönhammer, 2009, p. 178). Because a short duration of presentation does not allow a reliable conclusion whether the children use subitizing or whether they have counted the objects (quickly), the study at hand works without a time restriction. Therefore, there was no time limit for the children to look at the picture in determining the cardinality of the set of dots. To take a closer look into the processes of perceiving sets and determining cardinality, the investigation method eye-tracking was used to record the eye movement of the children. In this paper we will focus only on one part of the interview, the part with arrays of dots. Pictures with different numbers of dots were presented on a monitor. Before the interview, the children were instructed to say how many dots they can see, as soon as they know the answer. When the children said a number the interviewer asked how they came to the result. Interviews of nine children from the treatment group were previously evaluated and in the following section a thereof selected case study is presented.

First results and interpretations

This section analyses how perceiving structures in sets develop in children in preschool age and whether and how they use these structures for determining cardinality of sets. The following results and interpretations will be illustrated by means of a single case study. Liam is a child from the sample described above which took part in the study. His age at the three investigations is illustrated in Table 1.

Time of investigation		Age of each time of investigation
t1	29th September 2015	5 years, 2 months (5;2 years)
t2	16th February 2016	5 years, 6 months (5;6 years)
t3	5th July 2016	5 years, 11 months (5;11 years)

Table 1: Liam's age at each investigation

The following Table shows a section of the interview. It is based on one item, which was chosen as an example for a longitudinal analysis on all three time points of investigation.

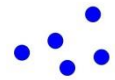
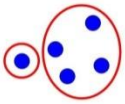
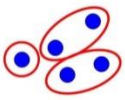
t1		Liam: One, two, three, four, five. *(He counts the dots aloud.)*
	Interpretation:	Perception: Set as individual elements Determining cardinality: Counting all
t2		Liam: Four and one are one, two, three, four, five. *(When counting he points the finger at each dot.)* I: Where is the one? Liam: *(He points to the leftmost dot.)* I: Thank you.
	Interpretation:	Perception: Set in structures Determining cardinality: Counting all
t3		Liam: Five. I: How did you find out that there are five? Liam: Here are two und again two and here is one. That results in five.
	Interpretation:	Perception: Set in structures Determining cardinality: Structural subitizing

Table 2: Case study – Liam

At the first interview t1 (cf. Table 2), Liam perceives the presented set of the five blue dots as single elements and uses to determine the cardinality of the set, the counting strategy of "counting all". He uses this determination strategy for all items, independent from the kind of objects shown to him. He always counts aloud and often uses additionally, his fingers as a counting aid by pointing to each single object. Even in sets of two, three or four, he continuously uses this strategy.

At the second interview t2 (cf. Table 2), after the implementation for four months, his perception of sets changed. Liam is now capable to perceive sets in structures. In order to determine the cardinality, he still uses his familiar strategy of "counting all". It is noticeable that he first explains the structures no matter how the objects are arranged. Then in order to determine the cardinality of the set he starts to count the objects one after another each time. By means of the eye-tracking data it can be confirmed that he really perceives structures and does not look on every single item separately. With several items he only mentions the two partial sets and only when asked again by the interviewer, how many these make up together, he answers by counting each time all the objects separately. At sets of two,

three or four, he now uses nearly every time subitizing in order to determine the cardinality. This hypothesis can be confirmed by means of the eye-tracking data and thus the possibility of an eventual quick counting process can be excluded (Schöner & Benz, in press).

At the follow-up-interview at time t3, he increasingly succeeds to use his structured perception of sets in determining the cardinality, as described in the example above (cf. Table 2). He uses structural subitizing in order to determine the cardinality of the set by perceiving the partial sets, knowing then directly how many objects there are. At some items he goes back to his familiar determination strategy of "counting all".

In the example of Liam, a clear development especially concerning the structured perception of sets is visible. A very interesting aspect is the fact that the identification of the cardinality of a set is not *one* process, but seems to consist of *two* processes. There is on the one hand the process of perceiving a set, which in turn can be distinguished in three kinds of perceptions and the process of determining cardinality which can also be distinguished into three sub-groups. The following Figure (cf. Figure 1) illustrates these two processes and their possible relationship. The model is the result of a first evaluation and is developed by an inductive approach (cf. Benz, 2013; Benz et al., 2015, p. 134).

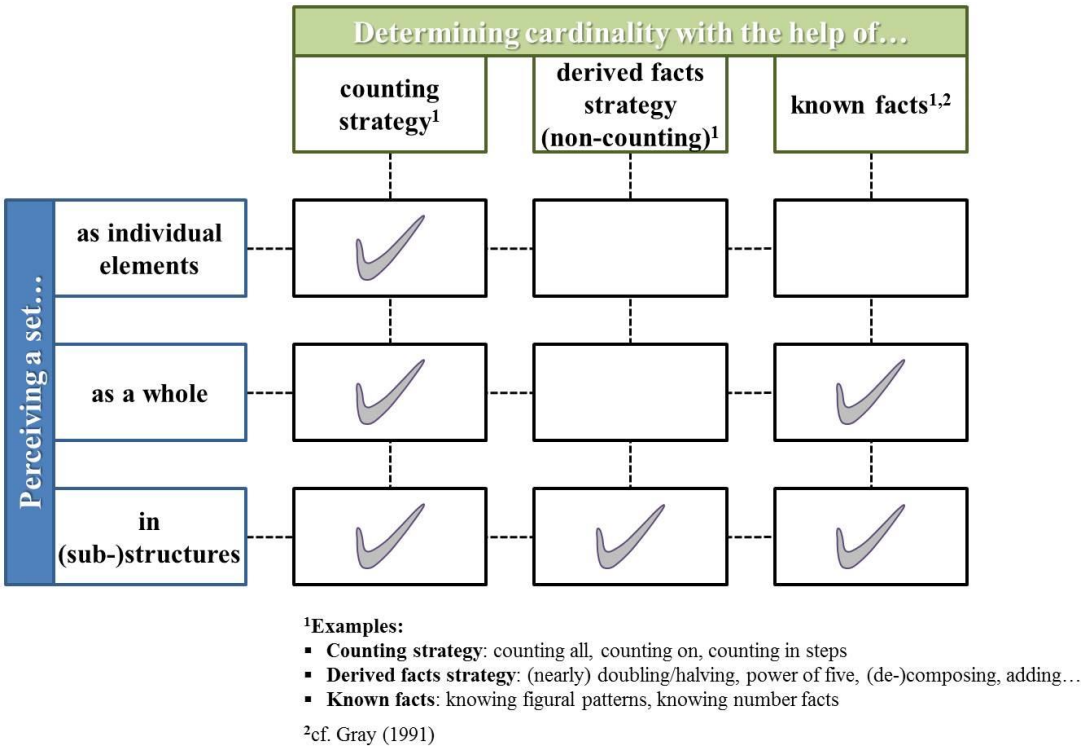


Figure 1: Two processes: Perception of sets and determining cardinality (cf. Schöner & Benz, in press)

The two processes of perceiving the structure of sets and determining the cardinality can run one after the other or coincide with each other. There are different possibilities of perceiving a set of objects (cf. Figure 1, blue boxes). Each of these three cases offers (partially different) strategies in order to determine the cardinality of the presented objects (cf. Figure 1). These processes that have been described can run successively. This is shown in Table 2 in the example of Liam at the second interview t2. He recognizes and names the structures that he perceives but he is not able to make a statement about the cardinality of the dots. This phenomenon can also be observed with many other children who were interviewed in the study at hand. In the example of Liam, the two processes

coincide at time point t3. He perceives the set in structures and then knows immediately that there are five dots altogether (structural subitizing). The perception of a set in structures offers not only the possibility of subitizing by knowing figural patterns (cf. Glasersfeld, 1982; 1987, p. 261) or by the usage of counting strategies for the determination of the cardinality of sets, but is also a necessary prerequisite in order to use non-counting derived facts strategies.

After the implementation of the tasks concerning the structured perceiving of sets at time point t2, a clear development can be observed in Liam's realization of structures. The detailed evaluations are not yet finished, but it is already conceivable that this development can also be observed in the interviews of many other children. The tendency of the follow-up study t3 is that this newly learned knowledge about the structuring of sets does increasingly stabilize itself and becomes more independent. In no case of the already evaluated interviews it is visible that the ability to perceive structures in sets is lost again. On the contrary, this newly acquired knowledge seems to be integrated as a familiar strategy in dealing with the cardinality of sets.

The kindergarten teachers from the participating kindergarten told that they could observe how children did use and explain structures in playing situations between t2 and t3, after the four months of implementation. "It [the structuring of sets] did become really independent". It was also significant that there were discussions among the children about it and arguments like: "One can put the five like this [four and one] or like that [two and three] or like that dice pattern. And it can well be seen like this [structure] and it cannot be seen very well like that, because it is mixed up." The preschoolers passed their newly discovered knowledge on to the younger children and explained to them their structured representation of a set. A mother told of a situation at home where her son arranged objects also in structures [four and five], explaining "look mum, this adds up in nine". The kindergarten teachers gave no purposeful suggestions to the materials at this time point, but still this higher attention of the teachers as well as of the children became independent and turned into an independent discovering and exploring.

Summary and conclusion

The following section is an attempt to answer the research questions and draw conclusions. In Figure 1 on the left side it is illustrated which possibilities might occur when perceiving sets. How to perceive sets, can thus be completely different. Initially, children seem to perceive a set predominantly as an arrangement of single objects. To answer the first research question it is helpful to look again to the case study with Liam. At time t1, he perceives several sets as single elements. After four months, he perceives the set by means of its structure. This kind of perception is still present at the last investigation t3. Here, the visual structuring ability becomes visible, as Elke Söbbeke describes (Söbbeke, 2005). This could also be observed in other children, who took part in this study. To perceive a set as a whole or in structures seems to be a natural step of development. The second research question investigates how children at preschool age use structures for determining the cardinality of sets. We examine again the example of Liam. At the first interview he uses his familiar counting strategy "counting all", in order to determine the presented set. Also, at the second point of investigation, he uses this counting strategy in order to determine the cardinality although he is now able to perceive the structures. It is obvious here that the process of perceiving sets must not coincide with the process of determining the cardinality of sets, but that these two processes may happen independently of each other. Within the following months, Liam learns to use the perceived structures

in order to determine the cardinality of sets. At the third point of investigation, it is obvious that through his structured perception of sets, he is now able to use non-counting strategies to determine cardinalities. In the described item, he uses the strategy "structural subitizing." To perceive a set in structures is therefore an important precondition for the usage of non-counting strategies and for replacing counting strategies through calculating strategies in primary school (Gaidoschik, 2010).

It is possible to support the structured perception of sets in a playful way already in kindergarten. Designing mathematical playing and learning environments which are mathematically substantial and which will enable the children to act in a discovering and exploring way through adequate games and materials, is a precondition for supporting the perception of structures in sets. In addition to providing such materials, kindergarten teachers should act inspiringly and supportively within this learning environment, in order to support this development in children. On the one hand they must be competent concerning mathematical contents but on the other hand they must be able to connect situational observations and perceptions with pedagogical-didactical activities. The knowledge about the processes of perceiving sets and determining the cardinality of sets, as it is illustrated in this paper, may serve as a basis for a differentiated, constructive and individual support in a playful learning environment.

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A framework for designing inquiry-based activities (FIBA) for early childhood mathematics

Chrysanthi Skoumpourdi

University of the Aegean, Department of Preschool Education Sciences and Educational Design, Rhodes, Greece, kara@aegean.gr

Nowadays it is widely agreed that students should be involved in critical and creative thinking about mathematics concepts and ideas in order to organize and consolidate their thinking, as well as to reflect and construct their own learning. Inquiry-based mathematics education, as a teaching and learning method among many others, seems to be a fundamental way to initiate and support students' effective involvement to the teaching/learning process. In this method, in which the role of students, teachers and activities is changing, the teachers must be also able to design and implement mathematics activities that are open to an inquiry-based process. In this paper, a framework for designing inquiry-based activities (FIBA) for early childhood mathematics is proposed, accompanied with an example, aiming to assist teachers in their instructional design.

Keywords: Inquiry-based activities, early childhood mathematics, framework.

Introduction – Theoretical background

In the last decades, it is argued that instructional design should support a critical (Skovsmose, 1994) and a creative mathematics education (Leikin, 2009). The teaching that was based on the simple reproduction of given conceptual contents and the transmission of knowledge has yielded space to the teaching based on student's own activities, that contributes to a meaningful learning and stimulation of their own identity as an emerged process (Van Oers, 2010).

In this new context, in which mathematics education is based on inquiry[1], teaching includes exploration, practical action, developmental thinking, connection of mathematical ideas and contents, problem solving and posing, collaborative learning, as well as autonomy to the development of ideas and methods (Artigue & Baptist, 2012; Ulm, 2012). Clarke and Clarke (2002) describing the effective mathematics teaching, set as a starting point the use of unusual problems without suggestions for their solution and the use of materials and other resources relevant to the content of the problems but also to the students' interests and needs. Moreover, they propose to address the classroom as a 'community of learning', that develops mathematical discourse and encourages the expression of students' ideas and strategies, focuses on the big ideas of mathematics, uses informal assessment methods to support instructional decisions, as well as facilitates students to act and think. In a "community of inquiry" teachers learn from students and students learn from teachers solving and posing mathematical problems to each other (Goodchild, Fuglestad & Jaworski, 2013). In this perspective, qualitative teaching, takes place by encouraging the creative and critical participation of students (Radford, Schubring & Seeger, 2011) and adoption of

[1] The IBME (Inquiry Based Mathematics Education) perspectives are based on approaches that have been developed in the field of mathematics education, such as the problem-solving tradition, among others, which tend to shape it in a particular way. But, in the problem-solving tradition the focus was on 'teaching problem solving' by developing problem solving skills and associated metacognitive competences, whereas in the IBME the focus is on 'teaching *via* problem solving'.

exploratory and expressive ways of learning. As Sierpinska argues, “it makes sense to assume that learning – mathematics or mathematics teaching – is either inquiry-based learning or it is not learning at all” (2016, p. 55).

In the inquiry-based mathematics education, the roles of students, teachers and activities, as factors that jointly shape the instructional design, are changing. Students are no longer considered as the passive recipients of knowledge but as co-constructors and co-researchers of their knowledge assuming greater responsibility for their actions (Lau, Singh & Hwa, 2009). They are at the center of the learning process; they actively participate in it and are strongly engaged with mathematics problems. They are actively involved in explorations and discussions, and they pose questions, that were usually posed by teachers, in order to express an opinion, to share and explain their ideas, to make assumptions and generalizations, to present their solutions to a problem, to justify their creations and to support or prevent the ideas of their peers. When students are forced to describe their strategies in detail and justify them, they promote their understanding. Students’ creative participation in the teaching and learning process of mathematics is related with producing new ideas, giving meaning to symbols, materials and tools, facts and procedures, understanding math problems, making plans and devising ways to solve them, as well as finding ways to assess the logic of their solution (Haylock, 1987).

The role of the teachers, since they are not the key persons of the teaching/learning process, is also changing and instead of being the sole sources of mathematical knowledge, who only focused on practicing the mathematical operations and procedures, they become supporters of children’ mathematics constructions (Lau et al., 2009). They work cooperatively with students to create the appropriate conditions and organize the process and the ongoing opportunities for them, to explore, to make connections, to build mental representations and to develop mathematics concepts based on their prior or/and informal knowledge through the interactions with others. They cultivate to students the need to communicate their actions, their creations, their solutions with the use of materials or other auxiliary means, to discuss their mathematical ideas, to share their thoughts, to cooperate (Shriki, 2010). They help students to look for multiple solution methods and multiple outcomes for a situation (Tsamir, Tirosh & Tabach, 2010), to identify relationships and common characteristics in different cases, to reflect and justify their thoughts and actions. Teachers transfer the problem-solving responsibility to students without, at least initially, intervening and suggesting ways of thinking. They encourage children to think and present their solutions to the rest of the class. The discourse, which includes the presentation of students' results, solutions, strategies and methods, highlights students' own considerations and encourages the exchange of their ideas (Ryan & Williams, 2007). Teachers pose questions related to students' ideas in such a way, as to explain their thinking and imagining, as well as giving meaning to their actions and developing their own understanding. It is important that the posed questions are linked with the task and its solving. The questions are not supposed to fulfill teachers' desire to teach mathematics, but have to encourage the emergence of multiple strategies, to make clear the relationships between these strategies, as well as to give opportunities to students to integrate the pieces of their knowledge.

Activities’ type, in the new cooperative, critical and creative classroom context, is changing. Activities arise from tasks that are problem-based with sufficient openness for inquiry-based learning (Ulm, 2012). The context of these tasks can be authentic or not, known or unknown to

students and can be associated with everyday activities, stories, games, workshops etc. Complex and creative situations allow various solutions and reasoning. They enable the use of students' informal knowledge and empirical reality. They are 'realistic' experiences, supported by a variety of materials and other means (Varol & Farran, 2006). Materials that are related to the task are used and aim to help students to investigate the situation, to find answers to the questions posed, as well as to pose their own questions. The main aim of the activities is not just to be carried out by the students, because it is the lesson of mathematics and they should do so, but the challenge to deal with the situation presented to them. They challenge students to connect mathematics with their daily life (Van den Heuvel-Panhuizen, 2005) and encourage them to describe the different ways in which they perceive things. Activities that are realized in many ways require students to conjecture, to interpret, as well as to justify their thoughts (NCTM, 2007). They encourage reflection and communication for students, to construct mathematical meaning using skills and knowledge they already hold (Varol & Farram, 2006). Moreover, they have to be interesting, effective and developmentally appropriate inducing the active participation of students (Tzur, 2007) to mathematics discourse. The solution of the problem is not readily apparent. To solve the problem, students have to use something more than a routine solution or an algorithmic process. The solution of the problem may be proposed by a student or a group of students. It may occur during the discourse or be suggested by the teacher. The mathematical discussion is completed when the whole class agrees to a mutually acceptable solution/s for the task, which can consist of many answers. In each case, children "earn" if the discussion, that took place, has set the basis for them to understand that the proposed solution is reasonable for this reflection. Then, when possible, the solution is modified and generalized. The generalization begins when reasoning gets independent, when it does not refer to the specific context in which it was created, and continues when the common characteristics of the different ideas are combined. Tasks can be either structured or unstructured but are open to an inquiry-based implementation.

The design of creative and critical activities that can be realized within an inquiry-based process that encourages students to decide for themselves when and how to use a method, a process or a strategy, can lead to meaningful learning. Such activities allow the creation and communication of students' reasoning cultivating skills such as prediction, fast perception of information, systematic reasoning, critical and creative thinking, problem solving and posing, assessment, comparison and correlation, as well as generalization, skills that are necessary for the future citizens (Sarama & Clements, 2009). This type of activities is not usual in the mathematics classrooms because its design and implementation is difficult (Sierpinska, 2016). It is easier for teachers to teach facts, procedures and problem solving processes through structured situations even if these situations are often not understood by the students and do not influence their thinking. Children, often, because they have to deal with routine and structured mathematical tasks, that require the implementation of specific strategies, do not sense when and how to use a mathematical procedure to solve a problem. Research results pinpoint that it is essential for the mathematics education to include also inquiry-based activities (Artigue & Baptist, 2012). Thus, a framework for designing inquiry-based activities for early childhood mathematics (FIBA) is proposed, accompanied with an example, to assist teachers in their instructional design.

Framework for designing and implementing inquiry-based activities (FIBA)

Taking into consideration the above main points of inquiry-based mathematics education, a framework for designing and implementing activities is proposed. The framework consists of seven stages (figure 1):

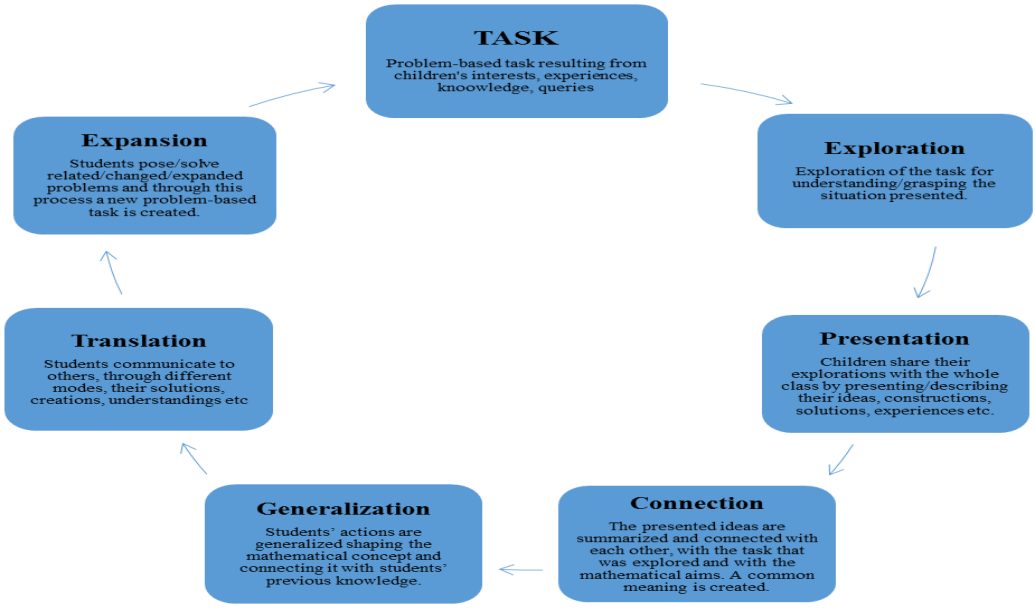

Figure 1: The seven stages of the FIBA

1. Task (key person: teacher): A problem-based task is invented and presented, by the teacher, through a context, resulting from children's interests, experiences, knowledge, queries (emergent from a previous free talk and sharing experiences with the students in the class which is based on the curriculum and is in accordance with a mathematical purpose). The task, which may have one or more solutions, is designed in such a way that it problematizes and incites children, in order to engage them in a problem solving and posing process. The problem to be solved could be a non-standard, unfamiliar, a bit complex and novel situation in order not to be solved just by applying existing knowledge and already-known strategies, but through exploration.
2. Exploration (key person: students): Children (individually or in groups) use their own (informal) problem-solving strategies to explore the problem introduced by the scenario, to choose/use materials and other auxiliary means, to make conjectures, to pose questions to each other and to the teacher for understanding/grasping the situation and to suggest solutions, in order to 'solve' the problem. In that stage, students have the opportunity to reflect and think about the problem on their own, before sharing their thoughts with their peers. They are also free to discuss their ideas about the problem with their peers before presenting them to the whole class.
3. Presentation (key person: students): Children share their explorations with the whole class by presenting/describing their ideas, constructions, solutions, experiences etc. Teacher, in that stage, is an observer and organizer of each team's presentation, orchestrating students' contributions, posing questions to help children describe, explain and open out their explorations. He/She also encourages students to pose questions to their peers, from other groups, in order to ensure that they understand all the presentations.

4. Connection (key person: teacher): Teacher, in cooperation with students, summarizes the results, poses questions and encourages students to ask questions that connect the presented ideas with each other, with the task that was explored and with the mathematical aims, in order to construct the common meaning that the classroom would share. Teacher's questions have to encourage mathematical thinking and reasoning and can be of several types (Carlsen, Erfjord & Hundeland, 2010). At that stage, it will become apparent if cooperative grouping strategies are effective in promoting classroom discourse.
5. Generalization (key person: teacher): Teacher is generalizing (and mathematizing when and if possible) students' actions, shaping the mathematical concept, connecting it with students' previous knowledge and giving feedback to them.
6. Translation (key person: students): At that stage, students are asked to communicate to others (students from another class, family etc.), their solutions, creations, understandings etc. through different modes—verbalizing, gesturally and schematizing (Skoumpourdi, 2016a).
7. Expansion: Students are asked to pose/solve related/changed/expanded problems.

In all the above stages, teachers are also responsible to take students' questions and comments into consideration, turn them into learning opportunities incorporating them to their instructional design, creating a new problem-based task.

The Pattern King: An example of an inquiry-based activity

Given that, the importance of patterning is increasingly highlighted in recent years and that, to further improve the performance of children on patterning, teaching interventions are necessary from the early years of schooling (Skoumpourdi, 2016b), in this chapter, an example of a pattern task[2] is presented, through the seven stages of the FIBA, supporting an inquiry-based teaching/learning process of mathematics.

Task: "The Pattern King forgot the password of his secret room and thus he cannot enter. Fortunately, he has photographs of the passwords he used to use and he asks your help to construct them". The students, (individually or) in groups, pick a drawing with chain schematization/password[3].

Exploration: Each (student or) group explores its schematization. They pose questions to the teacher to understand the situation. They observe how it is made and suggest ways for creating it. After the discussion, they choose the associated materials (connected shapes[4]) in order to construct the pattern/password and they create a/the chain.

Presentation: Each (student or) group presents the structure of its creation to the whole class,

[2] This is just an example of an inquiry based pattern activity. Taking into consideration the pattern's four functional characteristics (Skoumpourdi, 2016b), as well as students' pattern abilities and their type of performance (Skoumpourdi, 2013) a wide range of critical and creative pattern tasks can be designed which can engage children in challenging patterning experiences. A different combination of these characteristics leads to different pattern tasks with a varying degree of difficulty.

[3] Examples of chain schematizations/passwords  

[4] Connected shapes 

bringing side to side the construction with the schematic representation. The teacher is posing questions to ensure effective and understandable presentations. Also, students are encouraged to pose questions to their peers. The questions can be: “Can you describe us what did you construct? Which shapes did you use? Which shapes are similar? How many shapes did you use? Why did you put the triangle in this place? Which shapes are repeated? Do you observe any pattern?” etc.

Connection: Teacher, in cooperation with students, orchestrates a mathematical discourse trying to connect presentations/creations with each other, with the Pattern King’s problem, as well as with the pattern concept in order for a shared meaning to emerge. Questions can be: “What do you observe to this construct? In what, these constructs differ? To what are these constructs similar?” etc.

Generalization: Teacher is shaping the pattern concept, connecting it with students’ previous knowledge to generalize it. Questions are more general and are related to the pattern construct. For example: “Are these patterns the same”? “What is repeated in this pattern?” etc.

Translation: “Can you draw your creation to show it to the students of the other class?” In this stage students ‘translate’ their constructs and understandings for patterns and shapes to the paper.

Expansion: “A magician invited the King to play a game. He was sure that he would win him, to get his palace. The magician has two dice, one with numbers and one with shapes. Each of them—the king and the magician—would throw the dice and would take as many shapes as the dice indicate. The one that would create the longest chain with patterns will be the winner. Come to play the game to help the King make a very long pattern chain.” In that stage students expand their understanding by creating their own patterns.

At the end of the inquiry process, students would have used the connected shapes to copy the patterns or to construct their own ones. They would have translated their constructions to schematic representations, using paper and pencil, as well as verbalized them describing the pattern construct. They would also have discussed about the shapes’ names (circle, triangle and rectangle) and their orientations, as well as about length comparison.

Conclusions

Teachers, in their educational practice, are asked to invent mathematical tasks for their students, to design activities relied on them and implement them, as well as to reflect on the outcomes of the implementation. Yet it is not easy for teachers to design inquiry-based activities and implement them in classroom. They prefer to stay within the comfort of the usual school tasks which are routine and structured and they design teacher-centered activities that reproduce the given conceptual contents and are solved by memorizing facts and processes.

But if we take into consideration the argumentations of the last decades for the inquiry-based mathematics education, the design and implementation of inquiry-based mathematics activities, which are beyond the usual school tasks, will help students to develop their curiosity and creativity, their ability for critical exploration, reflection and reasoning and their autonomy as learners leading them to mathematical understanding (Artigue & Baptist, 2012).

Using the proposed framework one can design and implement activities, for early childhood, supporting an inquiry-based teaching/learning process of mathematics. Through the seven stages of the framework which include: 1. the formulation of the task, 2. the exploration of the problem posed

by the task, 3. the presentation of the explorations, 4. the connection of the presentations with mathematical aims, 5. the generalization of the connections and mathematical concepts and ideas, 6. the translation of the understandings on other modes and 7. the expansion of the initial task to a modified one, students' and teachers' roles alternate. During their cooperation they are both the key persons, the co-constructors and co-researchers of the teaching/learning process.

We do not support that all of mathematics activities should be designed and implemented in an inquiry-based process but we do believe that such activities should be adopted to develop different skills in students and to actively involve them in their own learning.

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The role of gestures in supporting mathematical communication for students with language delays

Helen Thouless

Independent Consultant, United Kingdom; hthouless@yahoo.com

Despite the fact that discourse is an important facet of mathematical learning, most research on students with language delays learning mathematics has focused on their procedural fluency, with limited focus on their communication of mathematical reasoning. This study focused on two first grade students with language delays as they engaged in choral counting, an instructional activity designed to encourage mathematical discourse. Qualitative analysis of the techniques they used to express their mathematical ideas found that the students' use of gestures in relation to an artefact supported their mathematical communication.

Keywords: Mathematics instruction, language impairments, nonverbal communication, discourse analysis, pattern identification.

Introduction

Discourse mediates mathematical learning (Forman, 2003) by providing a conduit for students to participate in mathematical practices, particularly when discourse is defined as comprising all forms of communication—including language, gestures, symbols, and artefacts (Lerman, 2001). The importance of discourse to mathematics education is shown by its prominence in educational policy documents (see e.g. NCTM, 2000), which state that mathematics instruction focused on discourse should enable students to express their mathematical ideas, analyse the mathematical thinking of others, and clarify and consolidate their own understanding of mathematics.

Gersten et al. (2009) found that the process of encouraging students with learning disabilities to verbalize their thoughts is effective, and yet it is uncommon to see teachers encouraging mathematical verbalizations from students with disabilities. This is because the dominant instructional paradigm for teaching students with disabilities is teacher-led algorithmic instruction (Jackson & Neel, 2006), which is characterized by the teacher demonstrating a step-by-step procedure for completing a specific type of problem, and the students then using these same procedures to solve similar types of problems. This type of instruction leaves little space for independent student verbalizations. In this paper I explore what students with language delays learn about communicating mathematical ideas by engaging in an instructional activity—choral counting—that encourages students to engage in mathematical discourse.

In this study I use the term language delays to mean that the students had persistent difficulties with expressive and/or receptive language that interfered with their academic competence and had been present since early childhood, but was unrelated to low cognitive ability, hearing loss, autism, or other known causes.

Conceptual framework

This study is influenced by a sociocultural framework in which learning is defined as the transformation of participation in a cultural practice (Rogoff, 2003). The cultural practice examined in this study is discourse about mathematical ideas. Amongst the community of mathematicians, discourse between mathematicians about axioms and conjectures is an important cultural practice that allows them to refine and improve knowledge; this is different than the one-directional discourse that commonly occurs in school mathematics classes, with the teacher imparting knowledge to the students (Lampert, 1990). This study examines the transformation of practice as the students move from the type of discourse typical in school mathematics towards disciplinary discourse as they learn several practices that mathematicians engage in when discussing mathematics: making assertions and presenting evidence (Lampert, 1990).

Several researchers have used a sociocultural framework to understand how students with English as an Additional Language (EAL) participate in mathematical discourse (Turner, Dominguez, Maldonado, & Empson, 2013). It is often assumed that students with EAL will struggle to participate in mathematical discussions, but Turner, Dominguez, Maldonado, and Empson (2013) found that these students increased their participation when the teacher invited their participation, validated their participation by responding positively to their contributions and accepted a variety of resources as valid forms of communication including gestures, objects, artefacts, and the students' home language.

These findings about how students with EAL can be encouraged to participate more in mathematical discussions, may help us support students with language delays to participate more in mathematical discussions. At present students with language delays are assumed to fare better in environments that limit peer interactions (Griffin, League, Griffin, & Bae, 2013), however, since "content learning is inseparably bound up with language learning and vice versa" (Barwell, 2005, p. 207), students with language delays may actually need more opportunities to participate in mathematics discussions than typically developing students. They may need more practice communicating mathematically, just as they need more practice communicating in other modes. This means that mathematics lessons should be designed to support students' language goals as well as their mathematical content goals. These language goals will be more readily addressed with mathematical discussions than by direct instruction.

Significance of research

The research question explored in this paper is: What do primary students with language delays learn about communicating mathematically as they interact during choral counting? This study contributes to the field of mathematics education by helping researchers and practitioners understand more about the intersections of language performance and mathematics learning by examining a group of students who are rarely asked to communicate their mathematical ideas as they learn mathematical content.

Methods

Self-study

This study is an example of self-study action research, as I was both the Special Educational Needs (SEN) teacher for the participants and the researcher in this study. I used the position of the teacher to investigate an issue, try a new method, and examine it systematically (Ball, 2000).

This type of research has several advantages and disadvantages in regards to validity. My established relationship with the students meant that I knew the history of shared understandings within the class and understood the children's use of language (Ball, 2000), and could use this knowledge to understand to what the children were referring, thus increasing the validity of the results. On the other hand, as their teacher I had a vested interest in seeing the students learn, which is a threat to the validity of the results. As an attempt to offset this threat to validity I triangulated the data with several other sources of data. The reliability of this study would have been increased if I was able to include a report of inter-rater reliability for the results.

This study was motivated by my own experiences teaching mathematics as an SEN teacher. I had tried to teach mathematics through direct instruction for several years and was dissatisfied with the limited progress that my students were making in mathematics. Therefore I decided to try a new instructional activity that emphasized mathematical discourse—choral counting—and to examine this new activity systematically to discover whether it is a fruitful way to work with young students with language delays.

Participants

The participants in this study were two first grade (6-7-year-old) boys who received small group SEN services in the areas of mathematics, literacy, and communication in an urban area of the U.S.A.

Martin[1] and Ali were both members of my primary special education mathematics class (PSEM). They had both qualified for SEN services under the category of Developmental Delays. Although the category given to them by the school district was Developmental Delays, which suggests global delays, the term Language Delays more accurately reflects their difficulties. These students showed delays in their language development, but no delays in their self-help or motor skills, and only minor delays in their social skills. Therefore, I use the label Language Delays to refer to these students' disabilities.

Martin and Ali were selected because at the beginning of the study they both had Individual Education Plan (IEP) goals related to counting, were in first grade, had language delays, and remained in my mathematics class throughout the duration of the study. There were four other students in their mathematics group, but these students did not receive SEN services in my class through the entire duration of the study.

[1] All names are pseudonyms.

Procedures

I chose to study the instructional activity of choral counting (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010) because it is an activity that incorporates both appropriate mathematical content for students in first grade and an opportunity for the students to engage in mathematical discussions. There are two sections to the choral counting activity: 1) rote counting, and 2) pattern identification and expansion. It is during the second section of the activity that students engage in mathematical discourse by expressing their own mathematical ideas.

In choral counting the teacher has to first choose an appropriate counting sequence for the students. For these students the counting sequences were by ones, twos, fives, tens, or backwards by ones. When counting by ones, the count started from a number in the low double digits because the students were very familiar with counting by ones from one. These counting sequences were selected because they were identified as the essential counting sequences for first grade students in the Washington k-12 Mathematics Standards (Office of Superintendent of Public Instruction, 2008), which were the relevant state standards in the time and place where this study was situated. Once the teacher has introduced the counting sequence to the students, the class counts together while the teacher strategically records the count on the board so that certain patterns emerged.

After writing three or more rows or columns the teacher stops the count and asks the students to identify patterns in the numbers. Once a student has stated a pattern they can be asked to extend, compare, or justify their pattern, and other students can be asked to build on what the first student has said.

The students in the PSEM class engaged in choral counting approximately weekly from November until March. They then continued to participate in choral counts once or twice a month from April until June. This resulted in eleven choral counting lessons over the year, each of which took approximately 20 minutes to enact. I additionally recorded my lesson plans, my reflections of each lesson, and asked the students to complete independent counts. Although I do not report my analysis of these additional sources of data in this paper, they did triangulate the data from the videos.

Data analysis

The data collected and analysed for this article consisted of two episodes from the videotaped choral counting lessons. The data were collected during a single school year, and were analysed systematically by drawing upon both sociocultural theory (Rogoff, 2003) and interaction analysis methodologies (Schegloff, 1997). In the initial analysis of the data, I used open coding to produce concepts, which were revised with further analysis. The resulting claims and assertions are based in the data and are therefore empirically grounded.

Results

In this paper I show how the students incorporated gestures towards the artefact into their interactions, and why this increased their communication of mathematical ideas.

In the first couple of choral counting lessons all of the students responded to the quest for patterns purely verbally. When asked what patterns he saw, Martin responded, “A square right there…Ooh! A triangle…Circle…Rectangle.” These verbal responses did not communicate enough information

to help his classmates or teacher understand why he was responding with shape names, and the conversation about shapes petered out.

On his next turn Martin initiated a new form of communication about the numbers. Instead of remaining in his seat, he came up to the board and gestured towards the numbers he was referring to (see Figure 1):

Martin: I see zeroes

Teacher: Where. Tell me where. [Martin got up.] Tell me

Martin: Right here. [Martin went to the board and pointed to the zeroes in the final column. As he counted he pointed to each zero.] Ten. Twenty. Thirty. Forty. Fifty.

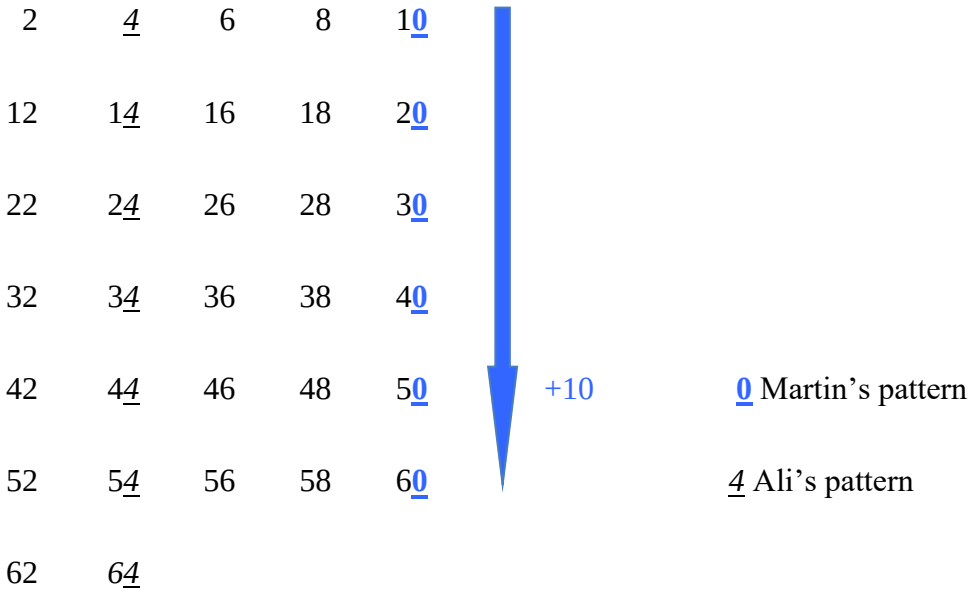

Figure 1: Count by 2s, December

In this interaction Martin defied the teacher's expectations that he would respond to the question purely verbally. Instead he adopted some of the teacher's communication style from the previous interaction with Ali by using the artefact to gesture towards the relevant numbers, but also innovated to convey more information. While the teacher had used a gesture that simply indicated the digits under discussion, Martin used a cohesive gesture that united two separate but related aspects of his idea (McNeill, 1992). Martin used pointing to indicate the repetition of the written zeroes in the ones place, while verbally reading the count by tens; in this episode Martin used a cohesive gesture to express his emerging understanding of the links between the symbolic and verbal representations of number. This corresponds to Garber, Alibali, and Goldin-Meadow's (1998) finding that children often use gesture to express emergent learning.

Martin's initial comment about "zeroes" was clarified and expanded by his use of gesture. This expansion allowed the teacher to respond to his statement and prompt him to further expand on his idea.

Teacher: What are they counting on by?

Martin: Tens. [Confident voice.]

Teacher: Yeah. Tens. [Writes +10 beside the zeroes with an arrow pointing down.] That's a great answer.

The positive response that Martin achieved through this interaction encouraged other students to take up his innovation and by episode 4 it had become a norm for the students to come to the board when they were trying to communicate which numbers they were discussing. For example, in episode 4 Ali communicated that he saw a similarity between the numbers in the first row (see Figure 2) by both verbalizing the numbers he was discussing and pointing to the relevant numbers. The words that he said, "Tens…got a number one zero, one one zero," could have been easily misconstrued but because he came up to the board and pointed at the relevant numbers, his listeners understood exactly which tens he was talking about.

10	1**10**	2**10**	3**10**	**410**
20	120	220	320	
30	130	230	330	
40	140	240	34*0*	
50	150	250		
60	160	260		
70	170	270		
80	180	280		
90	190	290		
100	200	300		**10** Ali's pattern

Figure 2: Count by 10s, January

The use of gestures allowed Ali to clearly communicate his idea and the mutual understanding engendered by this exchange of ideas allowed his teacher to extend the conversation (Goldin-Meadow, 2003).

Teacher: What number will be here? [Teacher points to the right of 310.]

Ali: Four

Teacher: Four hundred [Writes 4.]

Ali: Ten. [Teacher writes 10.]

In this exchange Ali went beyond the initial statement of the pattern to extend his pattern to the next column while incorporating an unstated arithmetic sequence in the hundreds.

Discussion

This action research project improved the participating students' educational outcomes, challenged assumptions and provides a basis for a call to social action, which are all important goals for action research (Kincheloe, 2003; Somekh & Zeichner, 2009).

The instructional activity of choral counting improved the students' educational outcomes (Kincheloe, 2003) by transforming their participation in the cultural practice of mathematics discourse by challenging the assumption that students with language-delays will not be active participants in discussions around mathematics because they find the language too difficult (Fazio, 1999). Although both boys found it difficult to express their ideas verbally, they were actively engaged in the mathematical discussions and used gestures to enhance their communication. Thal, Tobias, and Morrison (1991) found that students with specific language impairments are often worse at gesturing than their peers, but that those whose gestures develop normally will later catch up with their peers in verbal speech. Therefore it is important to encourage the use of gestures among students with language delays and this study showed that allowing students with language delays to gesture and physically interact with the numbers can support their participation in mathematical discussions. My call to action is to encourage other teachers to involve their students with language delays in mathematical discussions and to encourage them to use gestures and artefacts to express meaning.

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Preschool teachers' variations when implementing a patterning task

Dina Tirosh, Pessia Tsamir, Ruthi Barkai, and Esther Levenson

Tel Aviv University, Israel

dina@post.tau.ac.il, pessia@post.tau.ac.il, ruthibar@post.tau.ac.il, levenso@post.tau.ac.il

It is often recommended to engage young children with patterning activities. As part of a professional development program, nine preschool teachers were introduced to repeating patterns and were given the materials and instructions with which to implement an extension task with children. This study presents the various ways teachers implemented this task and investigates the impact of the various implementations on children's success in extending the repeating patterns.

Keywords: Repeating patterns, tasks, preschool teachers

Introduction and background

For several years, we have been providing professional development for preschool teachers guided by the Cognitive Affective Mathematics Teacher Education (CAMTE) framework (e.g. Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2014). Our aims are to promote teachers' knowledge and self-efficacy for teaching mathematics to young children. An essential element of pedagogical-content knowledge is knowledge of tasks (Sullivan, Clarke, & Clarke, 2009). In Israel, where this study took place, there is a mandatory mathematics preschool curriculum, but few curricular materials are available. Thus, introducing preschool teachers to appropriate mathematical tasks is essential. However, studies have shown that even when providing teachers with a task, and with explicit instructions for carrying out that task, teachers may implement the task in different ways (Bieda, 2010). In turn, different implementations may affect the cognitive load of a task or a student's conceptualization of key mathematical ideas (Stein, Grover, & Henningsen, 1996). This paper investigates the various ways preschool teachers implement a given repeating pattern task.

Repeating patterns are patterns with a cyclical repetition of an identifiable 'unit of repeat'. For example, a pattern of the form ABBABBABB… has a (minimal) unit of repeat of length three. The importance of engaging young children with pattern activities is supported by mathematicians, mathematics education researchers, and curriculum developers (Sarama & Clements, 2009). Pattern exploration and recognition may support children as they learn a variety of mathematical skills developed at this age. For example, recognizing repeating patterns may help children develop skip counting, such as 5, 10, 15, 20, 25, 30 ... where the ones digit forms the pattern 5, 0, 5, 0, … Recognition and analysis of patterns can also provide a foundation for the development of algebraic thinking and provide children with the opportunity to observe and verbalize generalizations as well as to record them symbolically (Threlfall, 1999).

Several studies have investigated ways in which young children engage with repeating patterns. For example, Seo and Ginsburg (2004) found that young children build block towers with an ABAB pattern. Fox (2005) observed young children painting stripes in ABAB patterns as well as one child who painted four sets of an ABC pattern and then said, "Look at my pattern" (p. 317). Waters (2004) observed a young girl who created a necklace out of game materials and described her necklace as "diamond, funny shape, diamond, funny shape" (p. 326). Papic, Mulligan, &

Mitchelmore (2011) found that some preschool children may be able to draw an ABABAB pattern from memory by recalling the pattern as single alternating colors of red, blue, red, blue, basically recalling that after red came blue and after blue came red. However, when shown a more complicated pattern such as ABBC, they could not replicate the pattern.

This study focuses specifically on the task of extending a repeating pattern. Pattern extension tasks mostly include showing the child a pattern and requesting the child to continue it. Papic, et al., (2011) reported that many children succeed at these tasks without necessarily recognizing the unit of repeat. Instead, they use the "matching one item at a time" strategy, also known as the "alternation strategy" which is especially successful with simple ABAB patterns. Rittle-Johnson, et al., (2013) found that some children reverted to producing an ABAB pattern while others could not produce more than one unit of repeat correctly when extending an ABB pattern. Similarly, Swoboda (2010) found that for some four-year old children, continuing a pattern means duplicating the unit of repeat once, and no more. In other words, both the complexity of the unit of repeat, as well as the number of times the unit is to be repeated, seemed to contribute to the difficulty of the task.

Another factor which may impact on the difficulty of extending repeating patterns is whether (or not) the given pattern ends with a complete unit of repeat. In one study, children were shown given repeated patterns and asked to consider extending the patterns by choosing between different possible continuations, some appropriate and some not appropriate. Children had greater success extending patterns which ended with a complete unit, than extending patterns which ended in a partial unit (Tsamir, Tirosh, Barkai, Levenson, & Tabach, 2015). Furthermore, several of the appropriate continuations would have extended the pattern in such a way as to end the pattern with an incomplete unit of repeat. Fewer children chose those possibilities as appropriate, even though they were correct extensions.

From the above studies, we see that there are several variables that may be taken into consideration when engaging with pattern extension activities: the structure of the pattern, the length of the unit of repeat, the number of times a unit appears in a pattern, and if the presented pattern ends in a complete unit of repeat or not. However, these variable all have to do with the repeating pattern. Might there be other variables that need to be considered when requesting children to extend a repeating pattern? Our first question is: Given an extension task and a set of repeating patterns, what are the various ways preschool teachers implement the task? Considering variation theory of learning and that learners may experience objects in various ways (Ling & Marton, 2012), our second question is: What can we learn about children's patterning abilities from the different implementations?

Methodology

This study took place within the context of a professional development course for preschool teachers, focusing on patterning for young children. Twenty-three preschool teachers participated in the program. All had a first degree in education and between 1 and 38 years of teaching experience in preschools. The entire program was planned for 21 hours. The teachers met seven times over a period of about four months in the local professional development center in their area. Approximately five of the seven sessions were devoted to repeating patterns with the other two focusing on number concepts. All lessons and tasks were planned by the four authors of this paper.

During the program, teachers were introduced to different patterning tasks as a tool for promoting their mathematical and pedagogical knowledge for teaching patterns in preschool. For the final project of the program, teachers were instructed to choose two of the tasks that were presented and analyzed during the course, and implement and video those chosen activities with one child. Those videos were then analyzed and discussed together in terms of children's solutions. In this paper we investigate teachers' implementations of one task (see Figure 1). Nine teachers (T1-T9) implemented this task, each with one child from their preschool (C1-C9). It is important to note that the task, along with the explicit instructions, was presented to the teachers by the teacher educator, who demonstrated how the task should be implemented. Furthermore, this task was not meant to be an instructive task, but instead an evaluation task in the sense that it was meant to assess children's ability to extend various repeating patterns.

Present the child with one pattern at a time. For each pattern prepare two or three separate containers, each container containing cutouts of triangles, squares, or circles. For example, when presenting the first pattern, place before the child two containers, one with blue squares and one with red triangles. For each pattern ask: What comes next? This question is repeated three times so that in the end, the child will have added three elements to the pattern.

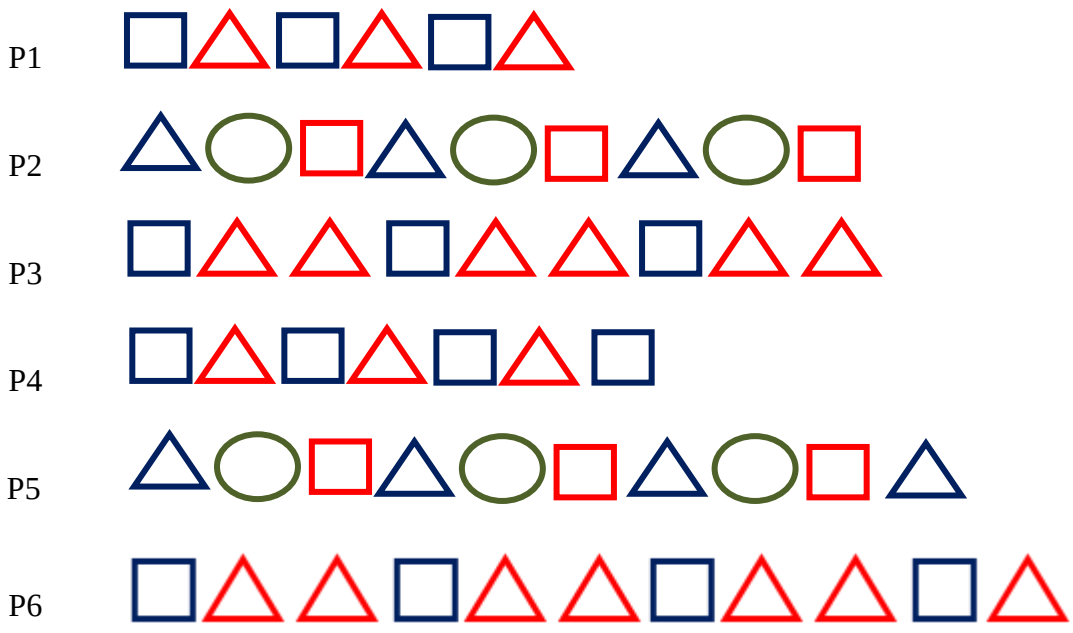

Figure 1: What comes next?

Note that there are basically three different structures, from the simpler AB, to the more complex ABC, and the even more complex ABB (e.g., Rittle-Johnson, el al., 2013). In addition, the first three patterns end in a complete unit of repeat and the last three do not. In other words, the sequencing of patterns goes from the simple to the more complex.

Various ways of implementing the task

We first note that none of the teachers changed the given patterns. Some variations came about from not implementing the tasks according to the instructions. For example, although teachers were told

to prepare five separate containers for each possible cutout, according to shape and color, and only present to the child those containers which contained cutouts for that pattern, only one teacher actually followed this instruction. Four teachers did separate the elements into five containers according to shape and color, but then kept all of the containers on the table, no matter which pattern was being extended. Four other teachers separated the elements into only three containers according to shape (e.g., putting blue and red squares in one container), and then placed all three containers on the table, no matter which pattern was being extended. Another explicit instruction which was not followed was the sequencing of the patterns. Two teachers did not present the patterns in the order given above. One teacher showed the patterns in the following order: P4, P6, P3, P4, P1, and P2. The second teacher used the following order: P4, P3, P2, P1, P6, and P5.

Some variations in implementation seemed to come about because no explicit instructions were given as to what to say to the child before beginning the task. That is, teachers were instructed to ask for each pattern "What comes next?", but were not told what to say when sitting down with the child and introducing the task. Six teachers stated at the beginning of the task, as they placed the pattern down on the table, "Here is a pattern." Five teachers asked the children to say out loud each element from the beginning of the pattern. It might be that the teachers thought that saying out loud the elements would allow children to hear the repetition of the unit of repeat and thus enable the children to pick out the correct next element. For example, at first, T5 did not ask C5 to read out loud each element of the pattern. She told her that there was a pattern and then asked her to pick out the element that should come next. After waiting a bit and seeing that the child sat still and did nothing, she then requested the child to say out loud each element of the pattern from the beginning. After that, C5 continued with the task and picked out the next element (correctly). T5 then continued with this instruction for each additional pattern (and answered each one correctly). Other teachers did not wait to see what would happen, but from the beginning requested that the child say out loud each element. T7 began her interview with C7 by saying, "Let's read the pattern together, let's read." At that point, C7 did not read the pattern but stretched out her arm to take a blue square (the correct element for extending the pattern) from one of the containers and place it at the end of the given pattern. T7 stopped her, despite that C7 chose the correct way of extending the pattern, and said, "No, sweetie. Wait a minute. First, let's read it." Three of the five teachers who requested children to read out loud the patterns, and an additional two teachers, asked the child they interviewed to say what was in the containers.

Variations in task implementation also occurred while the child was actively engaged with the task. Some of those variations were queries into why the child chose one or another element. For example, when engaged in the second pattern (P2), one child mistakenly took a blue square, but then immediately switched it with a correct red square. The teacher then inquired, "Why didn't you put down the blue square and why did you put down the red one?" This type of intervention did not interfere with the child's performance, but was instead a way for the teacher to listen to the child's way of thinking. In this case, the child answered, "because here (pointing to the pattern), the square is red." In other instances, the teacher's intervention came about even before the child took action. For example, when placing on the table P4 (the first pattern that did not end with a complete unit of repeat) T4 said to the child, "Now pay attention." After placing on the table the last pattern, T4 said, "Now look closely at the pattern, and also look carefully where it ends." This type of intervention has the potential to alternate a child's performance. In this case, despite all these warnings, C4

incorrectly extended all of the last three patterns (those that did not end with a complete unit). Some teachers intervened when the child chose an incorrect way of extending the pattern. For example, when T6 asked C6 to extend P3 (the pattern with an ABB structure), C6 incorrectly added ABA. The teacher then pointed to each element in the unit of repeat and said, "Look closely. Square, triangle, and …" C6 then responded correctly, "triangle." Interestingly, C6 had previously extended P2 in an incorrect manner. Although placing the correct shapes to extend the pattern, the child chose incorrect colors. In that case, T6 did not intervene, and instead said, "Good." Perhaps the teacher was satisfied that at least the child had chosen the correct shapes. However, when it came to placing incorrect shapes for P3, the teacher (T6) intervened.

Children's performances

Results of children's performances on the task, for each pattern, are shown in Table 1 according to structure and if the pattern ended in a complete unit (Comp.) or an incomplete (Inc.) unit. An extension of the pattern was only considered correct if the child successfully extended the pattern by three elements. As can be seen, children performed better extending a pattern that ended in a complete unit of repeat than a pattern which did not end in a complete unit.

Structure	P1 (AB Comp.)	P4 (AB Inc.)	P2 (ABC Comp.)	P5 (ABC Inc.)	P3 (ABB Comp.)	P6 (ABB Inc.)
Frequency	8 (89)	6 (67)	6 (67)	6 (67)	2 (22)	4 (44)

Table 1: Frequency (%) of successfully extending each pattern (N=9)

We now address the question of whether different implementations affected the children's success in extending the pattern. Comparing results of children who read out loud the elements of the pattern before extending the pattern, and those who did not, the relative frequency of success was slightly greater for those children who did not first read out loud the pattern (see Table 2).

	Reads out loud the pattern (N=5)	Does not read out loud the pattern (N=4)
P1 (AB Comp.)	4(80)	4(100)
P2 (ABC Comp.)	3(60)	3(75)
P3 (ABB Comp.)	3(60)	3(75)
P4 (AB Incomp.)	3(60)	3(75)
P5 (ABC Incomp.)	1(20)	1(25)
P6 (ABB Incomp.)	1(20)	3(75)

Table 2: Frequency (%) of success per variations in reading out loud the pattern

Regarding the placing of elements in containers, results (see Table 3) indicated that in general, the way the elements were presented made little difference to the children's ability to extend the pattern. Taking a closer look, for the first three patterns that ended in a complete unit of repeat, there was a higher success rate when the elements were separated by shape and color. However, when the patterns did not end with a complete unit of repeat, there was either no difference or there was a higher success rate when the elements were separated only by shape.

	3 containers separated only by shape (N=4)	5 containers separated by color and shape (N=4)	2 or 3 containers, only necessary elements (N=1)
P1	4(100)	4(100)	-
P2	2(50)	3(75)	1(100)
P3	2(50)	3(75)	1(100)
P4	4(100)	2(50)	-
P5	2(50)	-	-
P6	2(50)	2(50)	-

Table 3: Frequency (%) of success per variations in containers.

Regarding other differences in implementations, few affects were noticed. For example, among the six children who were told explicitly before beginning the task that there was a pattern which needed to be extended, three children extended correctly only two of the six patterns; the other three correctly extended three, five, and six of the patterns. Among those who were not explicitly told that there was a pattern (three children), a similar variance in success rates was found. The same variance in success was noted regarding children who were requested to say which elements were in each of the containers.

When analyzing the errors made by children, we found that the most prevalent mistake when attempting to extend a pattern that did not end with a complete unit of repeat was to continue the pattern as if it had ended in a complete unit, i.e., adding the first three elements from the beginning of the pattern. For example, C2 continued P4 by adding a square, triangle, and then a square. Another type of mistake which occurred for patterns both that ended and did not end with a complete unit of repeat, was to continue the pattern with ABAB despite there being a different structure to the given pattern. This occurred for C6 who added BA to an ABC structure, and also an ABB structure. Likewise, C9 continued P6 by adding BABABABA. Another type of mistake was taking the correct shape, but with the wrong color, as was demonstrated above by C2. This last type of mistake was directly related to the way the task was implemented. Obviously, if only the correct colors of shapes would have been on the table, this type of mistake could not occur.

Summary and discussion

As part of the professional development program, teachers were supplied with the materials for implementing the repeating pattern task. They were given laminated strips of paper with the patterns printed on them in color. They were given the matching pictures of colored squares, triangles, and circles to be cut out and placed in containers. They were even told what to ask each child. Yet, many variations occurred when implementing the task. Some of the variations occurred in the setup of the task, specifically with placing the elements in containers. Some of the variations occurred in the midst of implementing the task. When reflecting with the teachers on their implementations, it became apparent that these variations occurred spontaneously, without planning for them ahead of time. And yet, several of the teachers had the same ideas, such as having the children read out loud the pattern. Knowledge of tasks includes knowing the affordances and constraints of that task (Watson & Mason, 2006). It could be that the teachers saw this task as affording the opportunity to

review with the children names of two-dimensional figures. According to Zaslavsky (2008), teaching tools include not only materials, but other kinds of resources, such as language and time. It could be that teachers were incorporating the tool of language into this given task, having children say the names of the shapes in the pattern. It could also be, similar to other studies (Stein, Grover, & Henningsen, 1996) that teachers were attempting to lighten the cognitive load of the students by telling them that there was a pattern and having them say out loud the names of the shapes which repeated themselves. In any case, as teacher educators it is important to be aware that teachers may implement a given task in various ways. In fact, in our program, the teachers brought the videos of their implementations back to the program, and as a group, we viewed them together. This, in turn, enhanced the teachers' knowledge of tasks, including their knowledge of the way children engage with repeating patterns tasks.

Despite the variations in implementations, most of the outcomes were consistent with previous research. For example, children had greater success when extending patterns which ended in a complete unit of repeat than those which did not (Tsamir et al., 2015). Children in this study made similar errors as children in other studies, such as extending an ABB pattern with ABA (Rittle-Johnson, et al., (2013). Can we conclude then that variations seen in this study had no impact? Certainly, a study with nine children is not enough to make such a conclusion, but it does leave us with an interesting question. How come the variations seen in this study (e.g., telling the children that there was a pattern, reading out loud the elements of a pattern, placing the elements in various containers) did not seem to impact on children's performance? The answer to this question perhaps lies in acknowledging the essence of repeating patterns, which is the unit of repeat and its structure. None of the variations in implementations focused the child on the unit of repeat. What seemed to impact on results was the complexity of the structure as well as if the pattern ended in a complete unit of repeat. By reviewing these results with preschool teachers, noticing the variations as well as the little affect they had on children's performances, we strengthen teachers' appreciation for the structure of a pattern, and promote their knowledge for teaching repeating patterns in preschool.

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Teaching intervention for developing generalization in early childhood: The case of measurement

Marianna Tzekaki and Evangelia Papadopoulou

Aristotle University of Thessaloniki, Greece; tzekaki@auth.gr and evanpapado@sch.gr

Abstraction and generalization, lying at the heart of mathematical activity, attract the interest of many researchers who mostly examine generalizing processes in patterns and algebra. Given that earlier approaches questioned the possibility of developing generalizing capabilities in early ages, in our research we attempt to examine if an appropriate teaching intervention could change this initial assumption. For the needs of this work, 23 preschoolers participated in a seven months teaching intervention with relevant tasks in various topics: figures, patterns, measures and numbers. In this paper only for length measurement is presented. The children were pre and post examined in tests designed to examine their knowledge and initial generalizing abilities in this topic. The results indicated that their long-term involvement helped these young children to improve their abilities to (a) reflect on their own activity, (b) express their ideas and (c) reach to some concluding remarks, first related to their own personal experiences and later to more general thoughts.

Keywords: Generalization, abstraction, measurement, early childhood, teaching intervention.

Introduction

A 6-year-old child saying, "*When two distances begin and end together, this does not mean they are equal, we need to measure them....*" shows a preschooler able to express in a more detached from his/her experience and general way a relation concerning length comparison. This reveals a generalizing ability that, lying in the heart of mathematical activity, attracts the interest of research in mathematics education.

Current studies related to generalization and generalizing processes concern mainly patterns and approaches in algebra for older students (Hitt & Gonzales-Marin 2016; Warren, Trigueros, & Ursini 2016; Zazkis, Liljedahl, & Chernoff, 2008; Lannin, 2005), while relevant investigations of younger children's generalizing abilities also are connected to patterns and structures (Mulligan & Mitchelmore, 2013).

In our study we attempted to explore generalizing skills in early ages. More specifically, our research question focused on whether an appropriate teaching intervention could support young students' disposition related to noticing properties, relations and structure and developing a first level of more general conclusions. Our research supports the efforts for a meaningful and effective early mathematics education, by accepting generalizing abilities as an important component of it.

Abstraction and generalization

Humanity starts dealing with ideas related to abstraction and generalization quite early, pursuing answers related to the development of concepts in human's mind. Similarly, psychologists systematically investigated abstraction and generalization, as integral parts of humans' conceptual formation, supporting, thus, many explorations in mathematics education. For mathematics, *abstraction* is considered as a process during which students directly reorganize former structured

mathematics in a new mathematical structure (Hershkowitz, Schwarz, & Dreyfus, 2001). This process (as part of the generalization process) has been studied by many researchers. We could underline the advances of Dubinsky (1991) and APOS, Sfard (1992) and the theory of reification, Hershkowith, Schwarz and Dreyfus (2001) and epistemic actions, as well as a variety of combinations of these approaches (Tzur & Simon, 2004).

Generalization could be identified as the level at which students, starting from specific situations, proceed to more general ideas and conclusions identifying patterns, structures, relationships, rules etc. (Kaput, 1999). Similarly the idea of generalization, as an integral part of mathematical development, holds an important place in mathematics education. Starting from Harel and Tall (1989), Tall and Vinner (1981), Fischbein (1993), Bills and Rowland (1999), Radford (2001), Becker and Rivera (2003) and Sriraman (2004), many researchers have proposed models regarding theoretical approaches for this conceptual elaboration. These models refer to concept formation (as important stage of generalization), reflection and communication (as important reason for generalization), different kinds of generalization (extension, reconstruction, disjunction/ results and process of generalization), elements of generalization (grasping and expressing) and generalization as objectification (factual, contextual, symbolic).

As mentioned earlier, most of this research is related to generalizing processes in patterns and algebraic elements, and certainly examines these processes in older students. Their findings can be organized in the following strands: generalizing abilities, generalizing procedures and strategies, teaching intervention aiming at improving generalizing skills. Related to *generalizing abilities,* Yeap and Kaur (2008) studied structure and pattern recognition with the use of heuristic tools and technology as well as the development of metacognition and critical thought. Related to *teaching intervention*, Zazkis, Liljedahl and Chernoff (2008) investigated the use of appropriate examples and tasks.

The elements or actions and elaborations that consist the integral parts of generalization cannot be defined so easily. However, elements such as reflection of action and expression of more generalized properties, rules, relationships and structures deriving from specific tasks, problems and practical experience could be identified as being essential in this conceptual elaboration and are placed in the center of students' mathematical development. Therefore, mathematical education needs to cultivate these skills from early ages and progressively improve them in the course of mathematics education.

Starting from this basic question, our research aimed at examining (a) the possibility of developing generalizing capabilities in early childhood, and (b) teaching interventions that could improve these skills in young children. Our wider research concerns teaching and findings related to four topics of mathematics curriculum: geometric figures, patterns, geometric measurement and numbers. In previous communication (Tzekaki & Papadopoulou, 2016) we presented results concerning geometric figures and numbers, while in this paper we are limited to data related to geometric length measurement. We believe that the most interesting part of this presentation is not exactly the specific topic but the development of generalization capabilities in 5 years old children.

Measurement and generalization

Measurement is an important every day activity, that mathematically connects geometry and numbers, but its approach is not so evident in early years or even later (Bragg & Outhred, 2004).

Length measurement can be identified as a procedure that divides a continuous magnitude of an object to a number of specific parts (units) and connects it to the number of iteration of these parts. Thus, the conceptualization of length measurement could be analyzed in *several steps* that demand understanding and generalization (Battista, 2003):

1. Identification of length as a special (unchangeable) attribute of an object.
2. *Transfer* of this attribute to another object (intermediate).
3. *Comparison* of lengths by means of intermediate (arbitrary or conventional).
4. *Covering* with different equal units (arbitrary and conventional) and quantify this covering.
5. Connecting unit iteration (and thus measurement) to a number (Sarama & Clements, 2009).

Our research related to length measurements examines whether young children succeed to understand and express different generalized ideas for three of these steps: transfer, comparison and covering. More specifically, *generalizing levels of each of these measurement steps* was expressed and examined, both as *procedure and relation*, as following:

- *Generalization related to the transfer of a length to an intermediate* (2) means that children could identify the equality of the lengths of objects and intermediates.

- *Generalization related to the comparison of lengths (or distances) by means of intermediate* (3) means that children could identify the starting and ending points of objects and intermediates and the equality of their lengths.

- *Generalization related* to *covering with different units* for the needs of a measurement means that children could understand the role of equal units, the different quantification when the units are different and the role of numbers in the procedure of measurement.

Children's explanations both in tests and in classroom interchanges during the teaching intervention were classified according to this analysis.

Methodology

For the needs of this research related to length measurement, 23 preschoolers participated in a three weeks teaching intervention with relevant measurement tasks. There were also pro- and post examined in tests specially designed to examine their knowledge about measurement and their generalization level, before and after the intervention.

Pre and Post tests

The tests items in length measurement included four (4) items in accordance to the aforementioned analysis:

1. For the transfer of a length, a real object (for example a frame) hung on a specific height on the wall had to be transposed to another place.

2. For the *comparison of lengths*, *cards* of different width and height had to be ordered (their dimension were changing inversely, the thinner was the taller, and the thicker was the shorter).

3. For covering with different units and comparison there were two items: the children had to compare *unequal distances* in zigzags with common starting and ending points drawn on worksheets and

covered *with equal units,* as well as *equal distances* in zigzags with common starting and ending points, but covered *with different* (non equal) *units*.

Two of the items concerned lengths, while the other two distances. Also, two of them were real situations with material while the other two were representations of comparison situations. The children were individually interviewed on these tasks and were questioned about their reflections and conclusions. Their answers were organized in the following stages.

1st Stage: The child does not express any procedure or relation related to measurement.

2nd Stage: The child starts with a holistic measurement approach of procedure or relations that can explain his/her doing, e.g. transferring an object a child says "*I thought to put it in the same height*.." showing an imaginary line with his hand.

3rd Stage: The child starts conceptualizing length measurement and can explain the procedure presenting some relations, e.g. comparing distances in zigzags by means of batons, a child says "*for this side* (showing the starting point) *they are the same, but from the other side* (showing the end) *this one* (showing) *looks longer*…"

4th Stage: The child starts conceptualizing the measurement as iteration of units and can explain almost the whole procedure and relations, e.g. comparing distances in zigzags by means of matches, a child explains "*this one is longer because it has 1,2,3,…,6 matches and the other has 1,2,3,4 matches…*"

5th Stage: The children explain all the relations that are needed for the measurement, e.g. comparing also equal distances in zigzags by means of matches, a child explains "*they start together and end together and have the same number of matches".*

Teaching intervention

After the pre-test the children were involved in "generalizing experiences" concerning length measurement. They first worked with relevant measurement tasks in groups and then presented their results in the whole class. The teaching intervention consisted of five (5) lessons with eight (8) tasks. The tasks concerned all five steps of length measurement (length/distance - width – height, presented earlier) with

- Length transfer, *direct and indirect comparisons* and estimations.

- Indirect comparisons by means of *arbitrary of conventional units* (unit covering and number assignment).

- Indirect comparison by means of *conventional units with iteration* and number assignment.

- Length and height measurement by means of meter.

Tasks with *length transfer* proposed the use of batons or strings to measure, for example, the height of the position of a frame that has to be hung on a different wall. *Indirect comparisons* suggested covering with meters or other convectional units to measure and compare, for example, itineraries in stories or dimensions of furniture. Similarly, other *comparisons* presupposed the iteration of units for these measurements or comparisons.

The most important part of this intervention was the closure of activities, when children were systematically encouraged (a) to identify common characteristics, relationships and properties in the different encountered situations, (b) to express more general ideas and (c) to formulate conclusions or other overall rules about length measurement.

The discussion and the generalizing questions aimed at the understanding of measurement procedure and its principles and, more specifically, of the equal partitioning of the continuous attribute of objects, the unit iteration and the equality relations (with starting and ending points).

Results

The results coming from the comparison of the pre and post tests indicated that, after a series of tasks and systematic discussions, young children were able to improve their abilities to reflect on their own activity, to express their ideas and reach to some kind of conclusions, initially 'locally' related to their own personal experiences or the specific task (e.g. way of doing it) and later to a more general level regarding mathematical ideas (e.g. whether are equal or nor) or even formulating a rule or proposition (e.g. to measure two distances we have to…).

Findings related to measurement: Pre and post tests

The *pre-test findings* indicated that children initially didn't recognize the procedure of measurement, but approached comparisons globally and estimated lengths and distances based on their own experiences. For example, comparing distances covered with batons they didn't count the number of batons but based their responses on personal judgments of closeness. Their justifications were idiosyncratic, practical or kinesthetic, e.g. "*because I think so*…", or "*This line is bigger because in this place there is only one animal, so the thunderbolt killed more…".* They ordered objects based on their figure, while they covered them with units that were overlapping or had gaps. They understood that there was an equality or inequality of dimensions, but explained it based on morphological elements, e.g. "*it is more pointy…"*.

The analysis of children's responses *after the intervention* indicated a significant improvement regarding the identification of the procedure of measurement, as they did successful measurements and explained their actions based on measurement principles. They identified the iteration of units (with no overlapping and gaps) explain e.g. "*we don't put the sticks as we like, but one after the other, watching that they are not overlapping…"*. They were able to order objects based on metrical characteristics (length and height) and explained the measurement they have done. The children's improvement is illustrated in the following table (Tab. 1).

Issues of Tests	*Success pre, %*	*Success post, %*
Transfer of heights	0	87,5
Comparison of weights and heights	33,33	66,66
Comparison of unequal distances (in crooked lines) by means of batons	29,16	95,83
Comparison of equal distances (in straight lines) by means of matches	0	66,66

Table 1: Results before and after the teaching intervention

Students' conclusions related to measurement

Examining the children's answers to the interviews and the records from their work in the classroom, we can support that the preschoolers are able to conceptualize length measurement and its principals and moreover describe it, as procedure and relations at a more general level. The following examples present the ideas developed by the young children and the level of generalization achieved after involving in generalizing exchanges. At the end, the teacher gathered the children's conclusion about measurement in drawings and utterances.

The children's utterances expressing their concluding remarks about *the transfer of heights*:

- *We must measure one distance before doing the other*
- *The two distances must start form the same point*

The children's utterances expressing their concluding remarks about *covering with units*:

- *It shouldn't be one over the other and crooked*
- *We don't leave gaps, we put start to start and back to back*
- *We put them in a line*

The children's utterances expressing their concluding remarks about the *measurement of distances*:

- *When two distances are straight and start together and end together, then they are equal*
- *When two distances are zigzag and start and end together, it doesn't mean that they are equal, we have to measure to see which one is longer and which one is shorter*
- *When two distances are zigzag and don't start and end together, it doesn't mean that they are not equal, we have to measure them.*

Discussion

In general, this study shows that the development of generalizing capabilities in early childhood is possible on the basis of appropriate teaching approaches that encourage reflection, activity justification and concluding communication. The development of these abilities depends generally on a modification of young children's focus from personal to 'impersonal' ideas, and, thus, from 'local' to more general as a way of functioning in the mathematical class. This development was observed and recorded in all topics: figures, patterns, measurement and numbers (Tzekaki & Papadopoulou, 2016).

The development of children's ideas and generalizing remarks reaffirmed the importance of teaching approaches that orient the class to generalizing experiences (Sriraman, 2004). More specifically in the identification of the measurement procedure and relations by the children, their utterances showed that they succeeded to overcame the specific content of the tasks and proceeded to the conceptualization of measurement principles and invariants (Bills & Rowland, 1999). The children passed gradually from local generalizations related to every one of their actions and tasks to more general related to all their activity and finally to general principles of the measurement that recorded in the specific for their age way. The choice and the sequence of tasks, the group work in the classroom and the exchange between groups aiming at arriving systematically to more general remarks are the main factors that led to this result.

The findings gathered from this topic, as well as from all other topics, justify why it is possible but also of imperative to exercise children from early age, with appropriate activities and elaborations with respect to their age and their way of thinking, to generalizing skills as significant part of their mathematical development (Tzekaki, 2014).

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Discussing school mathematical narratives in early childhood future teacher education

Yuly Vanegas[1], Joaquín Giménez[2] and Marjorie Samuel[3]

[1]Universitat Autónoma de Barcelona, Spain; yulymarsela.vanegas@uab.cat

[2]Universitat de Barcelona, Spain; quimgimenez@ub.edu

[3]Universidad Católica del Maule, Chile; msamuel@ucm.cl

In this work we recognise how a group of future teacher of Early Childhood Education, analyse narratives about rich school experiences. We presented a professional task, in which we wanted to see how future teachers recognize initially the potential of the mentioned experiences to promote mathematical processes. We recognise that future teachers give a limited value to the problem-solving process and have difficulties in recognizing the processes of reasoning and proof. We found it is not because of a mathematical previous weak formation, but rather it points to the need of analysing school practices and narratives as good examples of action.

Keywords: Early childhood, future teachers, mathematical reasoning, problem solving.

Introduction

NCTM (2000, 2013), as other new national curricula, suggested that teachers could design better rich school activities, if they can identify the power to develop mathematical processes in their classrooms. In Spain, authors like Alsina (2014), pointed out that to achieve a quality of mathematics education for early years, it is important to implement curricula focusing on mathematical processes in a systematic way. It is important for the teaching and learning of mathematics to use mathematical models in relevant everyday contexts.

Preservice teacher's mathematical knowledge plays an important role when teaching mathematics. It is clear that for early years, almost no one remembers about his own experience. Therefore, their knowledge for teaching at early years is limited, and based upon personal theories and preconceptions (Jaworski & Gellert, 2003). "Little improvement is possible without direct attention to the practice of teaching ... [h]ow well teachers know mathematics is central" (Ball, Hill & Bass, 2005:14). In this paper, we assume that narratives are a wonderful way to allow students to personalize mathematics (Kurz& Bartholomew, 2013) and develop mathematical knowledge. In such a framework, the use of narratives is a powerful tool for teacher professional development and an useful research methodology for those interested in the study of teachers, teaching and teacher development (Ponte, 2001).

Llinares, Fernández, & Sánchez-Matamoros (2016) pointed out that for teacher education purposes, it is important to promote that future teachers (FT) grow on mathematical understanding by noticing mathematical aspects when future teachers analyse school experiences. It is also important for FT to know about designing rich practices. What about Early Childhood Education? In some countries as Spain, there is a global curriculum for Kindergarten, without any explicit mathematical goals for children's mathematical knowledge. Therefore, FT must have a preparation to understand the emergence of mathematical objects and processes from good early school practices. In the case of Spain, it is difficult to involve FT in implementing designed tasks, as it is for continuous training. There is a danger when considering the practice of teachers, and therefore presumably the experience of learners, of focusing exclusively on pedagogic practices, without reflective processes (Shulman, 1986). It seems to be the main reason for the need of mathematics reflective activities for prospective kindergarten teachers. In this presentation, we want to analyse the initial position of future teachers

when they analyse narratives about school experiences, in order to find how they relate the richness of school actions and the emergence of mathematical objects and processes.

Theoretical framework

As Ponte (2001) wrote, we use teacher narratives as a way to represent a school experience for oneself or for others. A narrative involves three basic elements: i) a situation involving some conflict or difficulty, ii) one or more agents that act on that situation with their own intentions, and iii) a temporal sequence of related events in which the conflict is resolved in a certain way. It involves people, settings, and events that take place in a given time. The acceptance of a story is oriented by convention and by "narrative necessity" (Bruner, 1991). In pre-service teacher education, narratives such as these provide good starting points to discuss issues faced by a teacher in making curriculum decisions and conducting classroom instruction. Some authors, like Chapman (2008) and Ponte (2001), mainly use narratives to reflect on future mathematics teachers' thinking and actions in relation to mathematics and mathematics teaching and learning, aiming to broaden their understanding of new curriculum orientations.

In this paper, we use narratives to study teacher's knowledge when analysing innovative teaching practices (Ponte, Oliveira, Cunha, and Segurado, 1998). In the current study, we focus on identifying, planning and enriching mathematics practices, made by others. We consider such activity as a fruitful starting point for inquiring into how FT should anticipate the enactment that will occur. Some future teachers of early childhood want to know and copy nice school experiences. Instead of this, this research leads to understand that richness of a school activity relates to the possibilities for emergence of mathematics objects and processes. Our aim is to recognise the initial reflection of future teachers about the emergence of mathematical objects and processes, as a noticing professional competency (Jacobs, Lamb & Philips, 2010). In this presentation, we focus on the analysis of problem solving processes, and analysis of the reasoning and proof processes. We assume that the practices in reading, analysing, and discussing narratives generate a number of insights that provoked to modify future teachers' planning for instruction. Many studies have reported that narrative data helps to validate the learning results as a basis for understanding of human actions (Polkinghorne, 2007), and to understand the role of intentionally drawn school practices (Font, Godino & Gallardo, 2013).

Methodological issues

We use a qualitative, naturalistic research perspective (Creswell, 1998) focusing on capturing and interpreting the participants' thinking about narratives as a case study. To achieve our aim, we designed a professional task. The professional task is structured in two parts. In the first part, FT are asked to read two articles: "The map of a treasure" (De Castro & Escorial, 2006) and "Where is Paula?" (Feixes, 2008).These articles describe two school experiences about geometrical itinerary aspects and spatial references, made with children of 5-6 years old. We select these two narratives because they describe rich (Woodham & Pennat, 2014) and high quality (NAEYC & NCTM, 2002) of school experiences for children from 3 to 6 years of age. Both school experiences use a continuous dialogue in the classroom. They also explain didactical orientations to planning and managing activities, connecting geometrical objects and processes. During the first experience, the idea is to promote a problem solving approach to "know about" the space of the school, as a provocation to find the place where a treasure is. Children spontaneously use paper and pencil itineraries, written codifications, and gestures to discuss in groups how to go from one point to another among other decisions. During the second experience, the teacher proposed to talk with Paula (a child that left the school to come back to her country, Uruguay). The five years-old children immediately ask where Paula is. A nice discussion about at what time do we call her, helps the children to discuss how the difference of time relates to the difference places in Earth, and different hemispheres. Our interest in

choosing these two narratives is to look at two sides of the problem of situating points in the real world space: the local or short distance problem (narrative 1) and global world problem (narrative 2).

In the second part of the professional task, four questions are posed to guide the analysis of the two narratives read in the first part. These questions are: 1) Talk about what raise your attention after first reading. 2) If you should recommend to your teacher friends these experiences, what do you explain for them? 3) What is the role of the teacher in both experiences? 4) Why do you think there are rich practices that develop the emergence of children's mathematical thinking?

The participants in our study are 33 FT of Early Childhood Education at the Barcelona University. To describe how they explain the emergence of mathematical objects and processes in the two narratives, we collect all the individual responses to the questions proposed in the second part of professional task. To analyse the data, we used a tool raised by Coronata & Alsina (2014). This tool includes five categories that correspond to the five mathematical processes proposed by the NCTM, (2000). For each of these categories, 6-7 indicators are provided for evaluation. In our study we only consider the indicators of problem solving and indicators of reasoning and proof processes.

The analysis takes three moments: a) The research team answer as experts giving a set of the processes observed; b) the future teachers' answers are analysed by using the methodological tool cited above; c) the research team explain some hypothesis about why the results appear. We assume that some text is related to one or more indicators, if there is a sentence evocating such principle by means of discursive argumentation (Gee, 2014). General sentences are not considered. For instance, the sentence "The teacher develop the capacity of creating arguments to explain children's curiosities in reference to mathematical concepts as distance, space and time" is assumed as relating to an indicator of reasoning and proof. But a sentence like "It is considered the interest of children" is not assigned to the indicator, because it is a fuzzy simple comment without any explanation given to what is the text or mathematical idea in the narrative.

Results

Many mathematical objects are easily identified by almost all the FT, but it is not enough explained how these objects emerge from the examples given in the narratives. In general comments, many FT talk about the differences among space by using time, and the idea of having different periods in a year. They talk about the meridian as a reference for timing. They assume the need of codes in order to represent itineraries among other geometrical objects.

Many FT also consider that problem solving is a common framework in both school experiences, but they explained some of the mathematical processes superficially (See Table 1). We find that future teachers identify issues related to problem solving more easily than other processes, and have difficulties in explaining aspects related to the processes of "reasoning and proof".

In Table 1, we associate examples of the responses made by FT to each indicator and we include an expert comment when FT identify problem solving process in the narrative 1, as an example of the use of indicators for problem solving. Some FT tell us that children's participation in narrative 2 stimulate imagination and creativity. Nevertheless, we only observed two out of 33 FT on where is it possible to see such promotion of creativity as an inquiry problem solving process.

The main indicator found is related to the assumption that contextualisation plays a role in problem solving activity. Nevertheless almost a half of the students write sentences in which FT talk about problem solving without any explicit indicator

In some cases, as FT14 and FT 24, they do not express any sentence about contextualisation and interest. We also see that 27% of the FT talking about mediators and interest are the same as those

who talk about contextualisation. It is possible that the lack of processes recognised relate to the mathematics background of the FT.

Indicators	% FT n=33	Examples of FT's responses	Responses from experts
Questions generate inquiry and exploration	6%	*"Helps to develop math thinking... Formulate questions, hypothesis, to find answers, explanations..."* (FT 3) (FT 6)	"Where is Paula" is a challenging starting question. Teacher promotes exploration when asking for information at home about time zones
Propose open problems	--	--	The teacher use children's open (non easy) proposals to analyse cultural influences about spatial relations
Contextualise in familiar contexts	52%	"*learn from the surrounding environment"*(FT 2)	The teacher contextualise to travelling problems to see what is invariant and changing in different positions
Promote discussion and participative debate	3%	*"contrast and reveal with different representations as figures, pictures or the use of dialogue the different knowledge*"(FT25)	The teacher promotes discussion about the need of comparing points and itineraries in the space, the use of references
Maintain the interest and curiosity	13%	"*…From the beginning, through their (children) questions we observed that is of its interest"*(FT2) "*children achieve different learning from curiosities"* (FT7) similar (FT9)	The school teacher focus on a lived experience to base a set of continuous problems related to the use of images to solve the problems
Use different type of mediators when solving	30%	*"The maps used by children are different from those done by adults"* (FT 9) *"…It is a dream…to know about now it is early morning…"*(FT 24)	Children construct and read maps in big spaces with the Earth globe as a powerful semiotic mediator. It emerges the idea of meridian line as a reference. The experience itself has an emotional background.
Reinforce the process using different support	6%	*"…using trial and error, children structure math knowledge…"*(FT14)	The teacher promotes the use of the starting situation to promote the use of spatial relations and references

Table 1: Responses associated to the indicators of problem solving process (narrative 1)

We can see that the FT do not relate some verbal aspects as discussing as a part of the problem solving process for narrative 2 (See Table 2). They realise that contextualisation is the main aspect behind problem solving activity for having good answers. It is expected that the FT talk about the role of mediators in this initial moment of analysis, but the justification they give is very limited focusing on having a "meaningful task" without any relation to a specific mathematical knowledge. They talk about "learn from the surrounding environment" without explaining that the need of a reference line (Greenwich Meridian) appears when we have numbers to indicate points in the space (initial idea of geographical coordinates). This could indicate a rather weak mathematical background of the FT, and the need for having a professional reflection about what mathematical knowledge emerge from a mathematically interesting activity like this. Precisely, this is the role of training process and professional activity never done before. The results show that narratives help to focus the reader's attention for recognising more mathematical connections (distance/speed; codification/itineraries; real world/representation) than expected.

Indicators	% FT n=33	Examples of FT's responses	Responses from experts
Questions generate inquiry and exploration	6%	"*build hypothesis, elaborate representations"* (FT3)	The teacher promotes different strategies as situating, identifying, recognising, building hypothesis
Propose open problems	--	--	There are open strategies, but no open problems
Contextualise in familiar contexts	36%	*"To know what is a map, which is its use*" (FT4)"…*The problem promote meaningful knowledge"*(FT3)	The context of "find treasure" help to identify the role of registers when solving itinerary problems.
Promote discussion and participative debate	9%	"…*and the colleagues were able to decode the information"* (FT9)	The dialogue gives challenges for coding/decoding processes
Maintain the interest and curiosity	3%	"*Children take and search at home more different maps … helping the comprehension and motivation*" (FT4)	The use of a school as a milieu, and the aim of arriving to a treasure, ensure interest and give opportunities for maintaining interest as a long job
Use different type of mediators when solving	15%	"*The maps done by children were functional, …they served for the purpose of representing a space indicating the place of a treasure"* (FT9)	The maps, are used to identify itineraries, distances, directionality

Table 2: Responses associated to the indicators of problem solving process (narrative 2)

It is difficult to find explicit children's arguments and reasoning in the narrative of "Where is Paula?" In fact, the teachers tell us many sentences (as "we always ask why") about the use of argumentation and reasoning, without explaining all the details. Nevertheless, in the children's pictures we can observe that they talk and argue when they observe the Earth globe, or when they talk about "Uruguay is far away". We only find general statements about reasoning and proof as we can see in Table 3 and Table 4.

Indicators	% FT n=33	Examples of FT's responses	Responses from experts
Helping to develop student's thinking	12 %	"*Children structure their mathematical knowledge*" (FT 13).	The Teacher promotes reflections and arguments about the invariance of day/month/year but different time and season, by seeing to the Earth globe.
Inviting to explain conjectures	6 %	*"The need to have good questioning"*(FT4) "*revealing initial ideas and preconceptions*" (FT 16)	The practice promotes to use arguments relating conjecturing about the need for having time references
Promoting to control conjectures	3%	"*to establish hypothesis to understand zones having the same time…to understand why it happens in the world… and which are the lines that make the difference, when situating places in the map*" (FT 16)	The teacher promotes some deductive reasoning. To argue, Bernat uses if…then as a deductive reasoning. Describes the relation between numbers (+1) and going to the right. Also (-1) means going to the left.
Questioning to evaluate arguments	3%	*"...develops the ability to create arguments explaining the concerns of children which refer, in this case, to space and time (FT 1) "being aware of time and its difference*" (FT 17).	The teacher promotes adjusting variables and control validity when talking about "that fit with those from other sources as it is the case of having information about time zones"
Promote reasoning by giving feedback	--	*"...the children in both experiences offer the possibility to establish arguments and generating hypothesis"* (FT 9).	Teacher promotes inference levels of spatial reasoning: What it is possible to see; what I see without specific attention to particular students
Promoting divergent thinking when arguing	3%	*"Permits the children to observe, to explore, and to determine what is more important or less…"* (FT 16)	Promoting children arguments about why do we say "Uruguay is far away" The need of relating two points in the space. The need of having a global view to understand it, by arguing that the flight is long (according time)
Promoting discoveries, analysis and arguments	6%	*"About the project…it pays my attention the amount of information that children can draw (extract ideas and conclusions) from the maps….arriving to conclusions that children assume as the best and right*" (FT2)	The teacher promotes the need of connecting children's surprises to math or science knowledge (children see Paula as summer dressed)

Table 3: Responses associated to the indicators of reasoning and proof process (narrative 1)

Clements & Sarama (2009), tell us about the need for promoting reasoning since early years, however, FT in the research do not identify the amount of possible quotations in the narratives relating reasoning and proof. We observe that, in general, many of the future teachers do not offer specific mathematical examples from the narratives to illustrate the reasoning and proof processes.

It is surprising that inquiry attitude not seems to be considered as part of a problem solving process, perhaps because FT have the belief that the most important for a problem solving is to have a right solution. In fact, many of the future teachers' comments do not pay attention to the role of the teacher giving opportunities for continuous problem posing moments, promoting hypothesis and conjectures. The FT explain that the teacher in both narratives promote the use of arguments, but none of the FT mentions the importance of feedback.

Indicators	% FT n= 33	Examples of FT's responses	Responses given from experts
Helping to develop student's thinking	15 %	*"In this practice… the teacher ask questions to influence children's reasoning, to improve their thinking*" (FT 19)	The teacher tell us explicitly the use of the students' natural environment to promote spatial thinking
Inviting to explain conjectures	3 %	*"The teacher tries to improve autonomy to reflect, to produce hypothesis about the ways of coding"(FT28)*	The teacher promotes the emergence of Students' conjectures about maps and big distances o represent travelling by using descriptions.
Promoting to control conjectures	6 %	*"The teacher permits that children explore information given by the maps, and select which information is relevant and which one is not"(FT 19)*	The teacher tells about reflections and argumentation, but it is not explicit how the teacher controls the conjectures, because the focus is the codification process.
Questioning to evaluate arguments	6 %	*"Sharing and contrasting* (FT 17)*"Discussing about his or her discoveries*"(FT 23)	They discuss to have a common result, but different representations and evaluate the representation used
Promote reasoning by giving feedback	--	--	The teacher find that David, use a comparison (small globe vs big Earth) to reason that from Madrid to Lanzarote you must use a flight.
Promoting divergent thinking when arguing	--	--	The teacher promote that Children adjust their arguments about which objects must be in the map, where to place and how to represent them.
Promoting, analysis and arguments	6 %	*"..revealing initial ideas and preconceptions*" (FT 16)	The teacher promote the emergence of Students' ideas (Luke : Itinerary as a set of steps; David what we can learn from Earth globe

Table 4: Responses associated to indicators of reasoning and proof process (narrative 2)

After this professional task, we devote some time for collective reflection not detailed in this paper. Some new processes appear as: "The teacher drives arguments, and promotes different possible contents and meanings" (FT 19) or "by dialoguing, the teacher gives immediate answers to students, reinforcing children's knowledge about coordinates" (FT 16), or "Children almost prove their conjectures, in a way that surprises us" (FT 9).

Conclusion

Future teachers are able to identify many mathematical objects and some processes implicit in the narratives analyzed. We see less process than it was expected. The main one is problem solving. The analysis promoted by the two professional tasks has allowed us not only to characterize some aspects of professional noticing of the future teachers, but also to establish a basis for recognizing the role of mathematical discourse (Adler & Ronda, 2015). In fact, after the implementation of the professional task, we find a more structured discourse of future teachers, more connections, didactic arguments, recognition of a greater number of processes and more justifications. With this research, we enlarge the conjecture done by Llinares et al. (2016) that noticing also promote deeper subject-matter understanding of pre-school's future teachers. Therefore, we consider that the implementation of this type of tasks is relevant in the training of future teachers of early childhood education.

Finally, it stands out that discussing and reflecting on school narratives such as those presented here has allowed future teachers to contrast school practices, different from those they have traditionally observed giving mathematics knowledge improvement.

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Early years mathematics learning – Comparing traditional and inclusive classroom settings

Andrea Peter-Koop and Miriam M. Lüken

Bielefeld University, Faculty of Mathematics, Bielefeld, Germany

andrea.peter-koop@uni-bielefeld.de; miriam.lueken@uni-bielefeld.de

Keywords: Early years mathematics, inclusive education, assessment.

Introduction

Following recommendations by the UNESCO, an increasing number of primary and secondary schools in Germany take on children with special needs, who were traditionally taught in special education schools, and include them in their classrooms. One organizational approach is to place the children with special needs in one class and provide a teaching team consisting of a primary teacher and a special education teacher for this class during the majority of the lessons. The study reported in this poster seeks to investigate the mathematics learning of four parallel classes during their four years of primary school in Germany. While three of these classes are taught in the traditional way with one teacher responsible for (mathematics) instruction, the fourth class is taught by a teaching team.

Theoretical background and research interest

Studies on the learning of children with special needs in inclusive compared to exclusive settings (i.e. in homogeneous groups predominantly in special education schools) in Germany, Austria and Switzerland suggest that children with special needs with respect to their school learning frequently underachieve when taught in special education schools (Wocken, 2005). Children with special needs who attend regular schools with inclusive classrooms, in contrast, demonstrate significantly higher achievements (Dessemontet et al., 2011). Already in 1998, Feyerer, who had reviewed findings from studies on inclusive learning, concluded that children without special needs taught in inclusive classes in regular schools showed at least the same, and sometimes better achievements than their peers in traditional settings. Klemm and Preuss-Lausitz (2012) in their meta analysis of more recent studies in inclusive classroom settings found that children with special needs in inclusive classes demonstrate higher cognitive gains than their peers in special education schools, while both, children with and without special needs, who are taught in inclusive settings demonstrate substantial gains with respect to their social skills. However, none of the studies conducted in this context explicitly looked at achievement in mathematics.

Methodology

The longitudinal study (2015–2019) reported in this poster compares the mathematics learning and achievement as well as the social and emotional school experience of four parallel mathematics classes (n = 100) of one primary school over their four years of primary school education. Three of the classes are taught in the traditional way with one primary teacher being responsible for mathematics instruction, while one of the four classes is taught by a teaching team consisting of primary teacher and a special education teacher. Included in this class are four children with special needs. Children's mathematics learning is measured using the task-based Early Numeracy Interview (ENI) and associated Growth Point Framework (Clarke et al., 2002) that was first developed in

Australia and then translated and adapted to the German curriculum (Peter-Koop et al., 2007). Over the four years of primary school the ENI, which involves individual interviews, is conducted five times, i.e. at the beginning of school and at the end of each school year. In addition, a standardized test on primary children's social and emotional school experiences, FEESS (Rauer & Schuck, 2004), is conducted at the end of each year level with the whole class. Lesson observations during the second and fourth year are also planned in order to look at the characteristics of collaboration in the inclusive classroom between the primary teacher and the special educator (in mathematics lessons) as well as at the differentiation of mathematical content.

First results

The analysis of the ENI data shows the highest gain for the inclusive class taught by a teaching team. This class showed the weakest mathematics knowledge and understanding in terms of the Growth Points and it will be interesting to see how the students continue to develop in comparison to the other three classes in that year level. At the beginning of their second primary school year, the inclusive class also shows significantly lower values in parts of their social and emotional school experiences (measured by FEESS). The significant differences relate to the sub-categories "peer acceptance", "climate in the classroom" and the "children's self-concept".

We are proposing this work to TWG13 as to date little is known about young children's early school experiences during their first year of school, that not only looks at their mathematics learning in an individualized and detailed way, but also examines the social and emotional involvement first graders experience during their first school year. Another interesting aspect will be to see how these experiences may change over the years and how they relate to their mathematics learning.

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Making maps in kindergarten

Geir Olaf Pettersen and Monica Volden

UiT The Arctic University of Norway geir.olaf.pettersen@uit.no, monica.volden@uit.no

Keywords: Kindergarten, maps, perspective.

Maps have been used in Norwegian kindergartens in several ways, from topological to more topographical maps. The aim of this study is to develop knowledge about how exploring the terrain around their kindergarten can help children develop spatial orientation. The children in the kindergarten made different maps with pictures from a digital camera. We also used a big picture (size A1) taken with a drone from above the kindergarten. The assumption was that making the map and examining the picture would help the children translate the world around them into a scaled-down, two-dimensional representation. Clements and Sarama's (2014) framework for development of spatial orientation, navigation, and maps will be used to analyse the activity, which is based on the students' learning trajectories for spatial thinking. The children have to be familiar with the area and special landmarks, then they need to experience the one-to-one connections between the real world and objects or icons on the map. Here, it can be helpful to use drawings, such as a table with legs, so that children can recognise them from their own perspective. Clements and Sarama argue that it is important to work with mathematical questions such as 'Which way?', 'How far?', or 'What is it?' They also argue that students need experience to become competent users of maps (Clements and Sarama, 2014). Bjerva, Græsli, and Sigurdjónsson's (2011) model for map reading skill, *kartstigen*, will be used, but an aerial picture instead of a two-dimensional map will replace Level C. Young children's ability to use aerial pictures has been investigated in past literature (Plester, Richards, Blades, and Spencer, 2002).

This leads in to the following research questions: *What kind of orientation experience do children get from activities with maps in kindergarten? How do children understand the representation of the area around the kindergarten from maps they have made themselves and from aerial pictures?*

Five children, age 5, were used in the present study. The study was designed and carried out by researchers and kindergarten teachers, and consists of three separate parts. First, the kindergarten teachers took the children on a walk from the kindergarten to a small lake nearby, with which the children were familiar. During the walk, the children were invited to find special landmarks, and the teachers took pictures of them. The following week, the children, teachers, and researchers made maps from the pictures. These maps were topological and showed the path from the kindergarten to the lake, with respect to the order of the pictures. After finishing the maps, the children tried to find their way to the lake with the help of the maps. During this walk, the kindergarten teachers and researchers encouraged the children to stop at each landmark on the maps. The aim was to investigate whether the children were able to make a connection between the real world and the maps. Finally, the children were shown a big aerial picture of the same area to investigate how they managed to navigate using a picture with a bird's-eye (vertical projection) perspective.

Data were recorded with video cameras and transcribed. The study design is an explorative case study design (Cohen, Morrison, & Manion, 2007).

Early findings

The children did not have much difficulty making a connection between the real world and the landmarks on the maps they made. However, when they were presented with an aerial photo of the same area, they were not immediately able to find the kindergarten. Monica is one of the researchers:

Monica:	What's on the picture?
Children:	Kindergarten
Monica:	Where is it? (Children pointing in opposite directions)
Monica:	Why do you think that?
Ann:	Because it is grey, and it got that kind of thing.
Erik:	Yes, it got that colour. Grey
Ann:	It is the kindergarten.
Monica:	What kind of things are outside the kindergarten? On the playground?
Ann:	The playground. What? There is no playground. Then it's not there.



The kindergarten, from the children's point of view, has the shape of a big, grey rectangle. This was the first thing they looked for. When Monica asked whether they could see the playground, they realised it could not be the kindergarten. After examining the picture for a while, they were given their self-made maps of the area. They were now able to make a connection between the landmarks on the map and the picture, even though they were taken from a different perspective.

Preliminary findings indicate that the maps were helpful for the children's experiences, in connecting landmarks on the map with the real world, but the children experienced more difficulties finding the kindergarten on an aerial picture.

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Learning of numerosity through a designed activity with dice and towers

Reinert A. Rinvold

Hedmark University of Applied Sciences, Norway; reinert.rinvold@hihm.no

Keywords: Numerosity, counting, continuous quantity, objectification, design.

Research topic and theoretical approach

The aim of this study is to develop both activities for the learning of numerosity in three year old children and theory describing the process of learning. The theoretical framework is Luis Radford's theory of knowledge objectification. His theory gives a central role to semiotic systems used within culturally and historically bound practices and social interaction in mathematical activity and learning, but also assumes an intimate and dialectical relationship between mathematical thinking and the material world (LaCroix, 2012). According to Radford (2008, p. 222), "… mathematical objects are fixed patterns of reflexive human activity incrusted in the ever-changing world of social practice mediated by artifacts.". Knowledge objectification is a matter of actively and imaginatively endowing with meaning the conceptual objects that the child finds in his or her culture (Radford, 2008, p. 223). Counting is a central mathematical activity that children normally develop. It is a procedure which children do not initially connect with properties of sets. Counting is interwoven with other quantity related cultural activities, such as for instance the How-many task, set construction and measurement. Set construction means building a set when its numerosity, for instance, is specified by a number word, as in the Give-a-Number task (Wynn, 1992). Davydov (1975) has advocated that perceptual comparison of quantity is developmentally prior to numerosity. When lengths are divided into equal units, the combination of perceptional length comparison and assignment of number words can be seen as a rudimentary example of measurement. The How-many task requires that the child assigns a number word to a presented set. The special role of the last counting word in a count is a pattern in the interface between counting and the How-many task. Children often emphasize the last counting word or insert "and" before it. Such linguistic devices are examples of semiotic means of objectification.

> These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call semiotic means of objectification. (Radford, 2003, p. 41)

Typically, children for some time think that the counting procedure is the answer to the How-many question. To initiate processes of deeper objectification of numerosity, it is proposed that engagement in other uses of number words than counting is vital.

An activity with towers and dice

In the poster presented at the congress, one designed activity with the intention of integrating counting and other quantity related cultural activities was outlined. A hypothesis behind the designed activity is that perceptual length comparison may be a semiotic means of objectification for connections between counting, the How-many task and set construction. The core of the activity

is to roll a dice and build a tower with the number of bricks the dice shows. The children sit on the floor and walk or crawl to fetch the dice when it is thrown. The plastic bricks are easy to pick up and fit exactly into the holes of the dice. An alternative way of building a tower is to put one brick into each hole in the top face of the dice and then put the bricks onto the stick. Putting the bricks into the holes can also be used to evaluate whether a tower is correctly built. A variation of the activity is that a puppet, the Easter bunny, makes a mistake when building the tower indicated by the dice. The puppet is hidden behind a screen so that the children do not see what it does. The children then are given the opportunity to respond and possibly correct the mistake.



Figure 1: The towers and dice activity

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Kindergarten class – inquiry and magnets

Ana Filipa Silva[1], Conceição Costa[2] and António Domingos[3]

[1] Associação Pró-Infância Santo António de Lisboa, Portugal; educadorafilipa@gmail.com

[2] Escola Superior de Educação de Coimbra, Portugal; ccosta@esec.pt

[3] Universidade Nova de Lisboa, Faculdade de Ciências e Tecnologia, Portugal; amdd@fct.unl.pt

Keywords: Inquiry, kindergarten, interdisciplinary, socio-culturalism, early mathematics education.

Background

We will present results from children aged five engaging in an activity and learning about magnets and inquiry in a kindergarten session. We want to understand how they learn and grow and in what ways the kindergarten teacher orchestrates the activities in order to engage the children in magnetism through inquiry. The study is part of a broader project in the kindergarten, aiming mainly to understand how and what skills and learnings are developed through understanding and inquiry. How do children develop inquiry skills (making observations, asking questions, making predictions, designing investigations, analyzing data, and supporting claims with evidence)? How do they conceptualize mathematics (classification, representing mathematical ideas in drawings, types of reasoning, number sense, comparison of sizes, problem posing)? The results show the importance of the questions (Carlsen et.al, 2010) and an appropriated orchestration for creating an inquiry interdisciplinary learning environment.

The Portuguese curriculum for kindergarten education (age 1-6 years old) requires an integrated and global approach to subject areas. It seems that few kindergarten teachers integrate physics into their teaching, and according to Tu (2006) only half of kindergartens are equipped with an area for experimental activities and few of those kindergarten teachers promote science and mathematics tasks. If such an approach were performed early, it would allow scientific reasoning and it would foster a better understanding of science with effect upon the children´s achievement (Eshach & Fried, 2005). In this study, we adopt: an interdisciplinary approach between science and math; the perspective on early math of Clements and Sarama (2009); and a sociocultural perspective on learning and development. We view learning as a social and a situated process of appropriation where individuals make concepts, tools, and actions their own through collaboration and communication with others (Rogoff, 1998). Also, as Hedges and Cullen (2012), we consider that in the early childhood context, participation is more active than mere presence, which in itself may not engender learning without attention relationships, content, change, context and cultures. This sociocultural perspective is useful for our emphasis on the orchestration of participation in social magnet activities.

The theoretical stance of our study is in accordance with Inquiry-Based Science Education (IBSE), (Worth, Duque and Saltiel, 2009). It is an approach to teaching and learning science and math that comes from: an understanding of student learning; the nature of science inquiry which may be represented as a set of four stages (explore, investigate, draw conclusions and communicate); a focus on content and heavy dependence on the local context and the interest of students and teachers. The inquiry–based approach has important principles such as: direct experience is key to conceptual understanding; students should be taught skills (making observations, asking questions,

making predictions, designing investigations, analyzing data, and supporting claims with evidence); reasoning, talking with others and writing both for oneself and for others.

Method and results

The study was carried out in one private kindergarten in Lisbon and adopted a qualitative research methodology under the interpretative paradigm, emphasizing meanings and processes. The researcher, first author, took the dual role teacher-researcher, conducting the study with her own 25 children (aged five years) in her own environment. In this study, we collected empirical data through the use of video camera and audio as well as field notes from one session out of seven sessions implementing the IBSE approach. Interaction and communication were captured as children engaged in magnet activities and the teacher orchestrated the group of children who were participating. Mathematical features used by children were elicited through the class interactions. We decided to analyse deeply the participation of the children in that interdisciplinary environment session above mentioned in looking for supporting the remaining research analysis.

The results imply that children's learning is influenced by their active participation and by the kinds of questions and learning opportunities that emerge, and how kindergarten teacher respond to children's ideas and orchestrates the activities. The collected data from that session indicate such an interdisciplinary environment involved playful learning and inquiry. The children developed content and competencies in science such as vocabulary (*to attract* and *magnet*) and experimentation. The children also developed: imagery, quantitative and critical reasoning; number sense (object counting, counting mental images, comparing numbers and early addition); as well as representing mathematical ideas in drawings. The children had opportunities to pose and to solve problems. Children connected ideas in play with the whole class in the game "fishing" and they learnt through collaboration and communication with one another.

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Slow down, you move too fast!

Jorryt van Bommel[1] and Hanna Palmér[2]

[1]Karlstad University, Sweden; jorryt.vanbommel@kau.se

[2]Linnaeus University, Sweden; hanna.palmer@lnu.se

Keywords: Abstraction, combinatorics, representations, pre-school.

Short description of the research topic and theoretical framework

The focus of this poster is on the links between representations and solutions when 6-year-olds work on combinatorial tasks. The task can be explained as follows: In how many ways can three bears sit on a sofa?

Studies of young children's representations often focus on informal and formal representations. Abstraction is often perceived as a goal and teachers often value when young children use abstract representations. Heddens (1986) introduced semi-concrete and semi-abstract as connections between concrete (objects) and abstract (signs) representations. He referred to representations of real situations, for example, pictures of real items, as semi-concrete, whereas he referred to symbolic representations of concrete items, where the symbols or pictures do not look like the objects they represent, as semi-abstract. Connected to the task presented above, concrete implies real bears and a sofa, semi-concrete implies images of (resembling) bears, semi-abstract implies a symbolic, not resembling, representation of the bears and abstract implies the use of formal signs. As for the concrete level, the 'real' bears can be replaced by three play-bears, or by engaging in a role-play, enacting the situation. In both semi-concrete and semi-abstract representations, the colours of the bears and their symbols are often kept identical.

When conducting the combinatorial task in 13 pre-school classes in Sweden (125 children, age 6), children showed different levels of abstraction in their solutions of the problem. In a previous paper we problematized the apparent relation between the level of abstraction and the number of duplicate solutions. Surprisingly we found that the children who used semi-concrete representations in their documentations were more systematic in the process of finding solutions with less duplication than the children who used semi-abstract representations. We argued that a more abstract level seemed to reduce the problem from bears on a sofa to putting three coloured dots on a paper. It seemed that children moved too fast to the next level of abstraction. Internalisation of the problem had not occurred yet which led to a reduction of the problem as described above. Simultaneously, we also concluded that documentation within the semi-concrete level was very time-consuming for children, as it takes a lot of time for the children to draw bears (Palmér & van Bommel, 2017).

Methodology

How could we slow down these children without slowing down the process of documentation? An application was especially designed to let children work on the semi-concrete level, and simultaneously, the time-consuming issue of drawing the bears was taken into account. The aim of the application was to provide an opportunity to develop an understanding of combinatorics and its

systematisation by letting children work within the semi-concrete level. The design-principles for the application concerned a possibility to adjust the level of the problem, resulting in a choice of the number of bears and the size of sofa. It also resulted in the possibility to save solutions as well as to offer an insight to all outcomes. By doing so, we wanted to explore the semi-concrete level more explicitly and focus on the learning opportunities this level of abstraction can offer the children.

During the autumn of 2016, 6-year-olds (about 60) from different preschool-classes have tested the application. Our reflections concern the learning opportunities created by using the digital form of documentation (in addition to the paper and pencil forms). After using the application, we let the children work on similar combinatorial tasks but only by using paper and pencil.

Preliminary results

Our preliminary results show that the children who have worked with the application do not document as many duplicates when using semi-abstract representations as the children in our previous study (Palmér & van Bommel, in press). Additionally, we see an increase in the systematic way they organise and search for solutions. These first results of the study also indicate that the children indeed develop significant skills when working within the semi-concrete level, for instance they seem to have obtained a good understanding of what the concept different combinations entails. A special note has to be made towards the possibility to adjust the level of the task in the application. For example, some children explored the situations of two bears on a three-seat sofa and four-seat sofas with different number of bears. Such explorations led to discussions of similarities between two and three bears on a three-seat sofa and mathematical aspects of the tasks were discussed at another level compared to paper and pencil lessons.

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Poster can be downloaded at:
https://www.kau.se/files/2017-02/cerme_vanBommelPalmér_Slowdownyoumovetoofast.pdf

TWG14: University mathematics education

Introduction to the papers of TWG14: University mathematics education

Alejandro S. González-Martín[1], Irene Biza[2], Jason Cooper[3], Imène Ghedamsi[4], Thomas Hausberger[5], Alon Pinto[6], Fabrice Vandebrouck[7] and Olov Viirman[8]

[1]Département de Didactique, Université de Montréal, Canada; a.gonzalez-martin@umontreal.ca

[2]School of Education and Lifelong learning, University of East Anglia, UK; i.biza@uea.ac.uk

[3]Department of Science Teaching, Weizmann Institute of Science, Israel; jasonc2107@gmail.com

[4]University of Tunis, Tunisia; ighedamsi@yahoo.fr

[5]Institut Montpelliérain Alexander Grothendieck, CNRS, Université de Montpellier, France; thomas.hausberger@umontpellier.fr

[6]Department of Science Teaching, Weizmann Institute of Science, Israel; allalon23@gmail.com

[7]Laboratoire de Didactique André Revuz, Université Paris Diderot, France; fvandebrouck@irem.univ-paris-diderot.fr

[8]Department of Mathematical Sciences, University of Agder, Norway; olov.viirman@uia.no

Keywords: University mathematics education, service courses, transition to and from university, mathematics in non-mathematics disciplines, university teachers' practices and knowledge

Introduction

TWG14 (*University Mathematics Education*, hereafter UME) was launched in CERME7 (Nardi, González-Martín, Gueudet, Iannone, & Winsløw, 2011) in recognition of the growing area of research in university mathematics education research. This area, although sharing in many cases approaches, methods and research topics with other areas in mathematics education research, has its own distinctiveness: the institutional characteristics at university (or postsecondary education in general) are usually quite different from those in compulsory education and do not always follow national curricular guidelines; the training of teachers (when existing) is also different from the training followed by primary and secondary teachers; there are many cases of classes with a large number of students, and teaching approaches are usually different – the amount of personal work expected from the students is also much higher; students' personal experience and aims are different than in compulsory education; very often mathematical notions and reasoning are dealt with at a higher level of complexity and abstraction; etc.. The fast growth of this research area, both outside and inside CERME, is evident in the breadth of research publications in this field and was recognized by the ERME community, in inviting the three-year leader of this TWG to present a summary of UME research as a CERME10 plenary lecture (Nardi, 2017).

Within CERME, the number of papers submitted to TWG14 has been increasing steadily since its inception. This year, we received a record number of 64 papers, with 47 getting accepted for presentation (of which, 41 are published in the proceedings, together with 17 short contributions). This indicates a substantial increase in comparison to the 31 full and 14 short contributions in TWG14 in CERME9 proceedings (Nardi, Biza, González-Martín, Gueudet, Iannone, Viirman, & Winsløw, 2015). Additionally, the substantial number of papers led to the split of TWG14 into two *isomorphic* groups (TWG14A: 23 accepted papers and TWG14B: accepted 24 papers) which ran in parallel and common sessions during the conference. This introductory paper summarises the works presented in both groups, as well as our common discussions.

Outside of CERME, the number of handbook chapters focusing on UME or dedicating some sections to it (from the pioneering Artigue, Batanero, & Kent, 2007, to the most recent Coupland, Dunn, Galligan, Oates, & Trenholm, 2016; Larsen, Marrongelle, Bressoud, & Graham, in press; Rasmussen & Wawro, in press) acknowledges the recognition of this research area for its specificities, as does the launching of the *International Journal of Research in Undergraduate Mathematics Education*. Moreover, the activities of TWG14 have also led to two major contributions: the creation of the *International Network for Didactic Research in University Mathematics* (INDRUM) with a bi-year ERME topic conference (see INDRUM2016 and INDRUM2018); and, the publication of a *Research in Mathematics Education* special issue summarising some of the works presented during CERME7 and CERME8, and discussing the use of *institutional, sociocultural and discursive approaches to research in university mathematics education* (Nardi, Biza, González-Martín, Gueudet, & Winsløw, 2014).

In CERME10 we intended to cement and expand further this work, as well as welcoming contributions from across the board of research approaches and topics: the teaching and learning of advanced university mathematics topics (including proof); transition issues "at the entrance" to university mathematics, or beyond; the training of university mathematics teachers; challenges for, and novel approaches to, teaching mathematics at university level (including the teaching of students in non-mathematics degrees); the role of ICT tools and other resources (e.g. textbooks, books and other materials) in the teaching and learning of university mathematics; assessing the learning and teaching of mathematics at university level; collaborative research between university mathematics teachers and researchers in mathematics education; and, theoretical and methodological approaches to research into the teaching and learning of university mathematics.

In the large number of papers received, we identified some continuities, but also some ruptures, with previous iterations of TWG14. For instance, there is still a large number of papers following sociocultural, discursive, and institutional approaches, although a considerable number of papers using cognitive approaches was also present. Moreover, among the papers focusing on a specific mathematical notion, calculus and analysis are still predominant, although we also received papers discussing other topics (such as group theory, linear algebra, or logic). There is also a small but growing number of papers addressing the teaching and learning of more advanced topics, such as algebraic topology, ring theory, and quantum mechanics. The number of papers addressing the use of mathematics as a service course (i.e., mathematical course offered to non mathematics specialists) is still growing, and papers focusing on engineering were again predominant, although we also received papers addressing the use of mathematics by biologists, economists, and physicists. The number of papers addressing teachers' practice, training, and knowledge has also grown considerably, showing the increasing interest that this area of research is gaining. Conversely, papers proposing experimental interventions (in particular, using technology) are still rare in the group. Finally, we note that a larger number of quantitative studies were presented at CERME10 in comparison to previous CERMEs. Among the accepted papers, we could identify six main themes (although we are aware that some papers fit in more than one theme): students' learning of specific topics; students' experience and affective issues; interventions; didactical transposition and use of resources; mathematics for non-mathematicians; and, teachers' practices and knowledge. In what follows, we follow the structure of these themes to summarise the main results presented during the conference. Due to the large number of papers and to space limitation, we have not included the content of the short contributions in the summary.

Themes and paper contributions

Students' learning of specific topics

Nine papers can be seen as contributing to this theme. Aaten, Deprez, Roorda and Goedhart show the difficulties with applying Lithner's framework in order to analyze 'hybrid' types of students'

reasoning when solving integration tasks in undergraduate calculus; the authors argue that this framework may need the addition of reasoning types that make use of some kinds of recall. Biza employs Sfard's commognitive framework to investigate first-year students' meaning-making of the tangent line, finding that they engage with analytical, geometrical and algebraic discourses in their substantiations about tangents, sometimes engaging with more than one discourse in the same response, and sometimes separating them across different responses. Chellougui uses Copi's system of natural deduction as a frame to investigate students' difficulties in producing a valid proof in mathematical contexts that involve multiply-quantified statements (e.g. the definition of an order relation in elementary set theory). Hanke and Schäfer use eight categories of mental images of continuity to show that students' mental images of real-valued continuous functions can be expressed by different forms of communication, which, in turn, depend on whether mental images are used or are explicitly verbalized. Juter investigates how students understand continuity and differentiability (and their links) during and after a calculus course, with a focus on students' choices of representations, both espoused and enacted; her study identifies that only students who preferred formal theoretical representations were able to produce formal proofs, as well as a strong coherence between students' espoused and enacted preferences of representations. Mai, Feudel and Biehler study first-year university students' personal concept definition of a vector; they identify several misconceptions and note that a vast majority of students state geometrical concept definitions that are not fully adequate and may cause conflicts in their learning of linear algebra. Ovodenko and Tsamir describe students' grasp of the notion of inflection point, and offer a detailed classification of reasons students offer to identify a point as an inflection point, and a point as a non-inflection point. In the field of abstract algebra, Ioannou studies discursive shifts in year-2 mathematics students' learning of group theory, drawing attention to some commognitive conflicts between the new discourse and other mathematical discourses, including advanced mathematics (e.g. set theory) and high school mathematics. Thoma and Nardi, also taking a commognitive approach, study first-year students' learning of the notion of variable, drawing attention to commognitive conflicts between the notion of variable, and even the notion of number, between school and university mathematics. Both these papers show that university instructors are aware of common student errors, yet they may not be aware of the conflicts that underlie these errors.

Students' experience and affective issues

Eight papers can be seen as contributing to this theme. Using data from questionnaires and exploratory factor analysis (EFA), Anastasakis produces a typology of seven different types of resources that engineering undergraduates use; he refers to the role of tools within Activity Theory and uses also Wartofsky's categorization of artefacts to propose an interpretation of these resources as seven different modes of action in students when studying mathematics, concluding that the way we usually classify resources does not necessarily reflect the way these resources are used by students. In a study of first-year engineering students' note-taking, Andrà uses a narrative approach where students' notes are seen as re-tellings of a story told by the teacher; she focuses on how the students condense the mathematical content in their notes, and what conditions might prompt students to act as 'scribblers' or 'thinkers'. Bampili, Zachariades and Sakonidis conduct an in-depth analysis of one student's process of transition from secondary to tertiary mathematics studies; they consider this transition from a rite of passage perspective, finding connections between the social, academic and mathematical dimensions of the transition, for instance, an interaction between emotions and the student's reconstruction of her mathematical thinking. Griese and Kallweit report on a quantitative analysis of the relationship between patterns of learning behaviour and examination outcome in first-year engineering courses. Kaspersen, Pepin and Sikko describe a quantitative analytical tool for evaluating students' mathematical identities and investigate the relationships between mathematical identities and grades in university mathematical courses. Kürten reports on a preliminary course for engineering students, and shows how this course can be

designed to influence students' self-efficacy. Liebendörfer, Hochmuth, Biehler, Schaper, Kuklinski, Khellaf, Colberg, Schürmann and Rothe propose a taxonomy of the goals of 44 mathematical learning support services offered by universities in Germany; the taxonomy suggests a range of educational goals (e.g. learning the language of mathematics or strategies for studying) and system related goals (e.g. reduce dropout rates or increase passing rates). Marmur and Koichu investigate the relationships between key affective events and the mathematical discourse in two university mathematics lessons where two similar problems were considered.

Interventions

Five papers can be seen as contributing to this theme. Fredriksen, Hadjerrouit, Monaghan and Rensaa study the introduction of a flipped classroom approach in an engineering course at a Norwegian university, focusing on the emerging tensions when students are introduced to a novel approach with videos and quizzes; using the Cultural Historical Activity Theory (CHAT) approach, they identify some tensions attributed to the changes of rules and expectations, as well as the lack of shared understanding in the community of students about the mathematical topic, their preparation for and participation in the sessions. Hogstad and Isabwe describe the use of a digital tool that combines mathematics and kinematics aiming to help students to better grasp integrals; using the theory of instrumental genesis, they investigate the activity of two groups of students with the tool, and identify the pragmatic and epistemic values of students' techniques for solving some given tasks. Kondratieva and Winsløw develop activities dedicated to helping students relate familiar practical tasks from calculus with theoretical ideas of more advanced courses in analysis; their approach is based on a theoretical model of the calculus-analysis transition, using the notion of praxeology from the anthropological theory of the didactic (ATD), and the associated strategies from Klein, to deal with students' challenges in this transition. Lecorre uses the scientific debate methodology developed by Legrand to design and implement, at the transition between high-school and university, specific tasks on double-quantified statements (the Q^2-game) that may raise the need for conventions of interpretations before they are introduced through mathematical formalism. Schmitz and Schäfer investigate the potential of designing a course in linear algebra and in analysis using the Abstraction in Context framework to increase students' motivation and ability to engage in concept construction; their results indicate that the new courses seem to help students in the transition from school to university mathematics.

Didactical transposition and use of resources

Two papers can be seen as contributing to this theme. Ghedamsi investigates the mathematical organization of complex numbers in the official textbook in Tunisia at upper secondary level; mainly building on Sfard's three stages of cognitive development and on Duval's theory of semiotic representations, she identifies three didactical variables that can be used to efficiently influence students' activities and learning process of complex numbers when they enter tertiary levels. Jovignot, Hausberger and Durand-Guerrier analyze the implicit complexity of a proof presented in a textbook, which involves the concept of ideal in ring theory; using ATD's construct of praxeology, the mathematical organization related to abstract algebra is modeled into structuralist praxeologies, highlighting the intertwined relationships between algebraic, set-theoretic and logical praxeologies and, as a consequence, the inadequacy of such proof for students' self-learning.

Mathematics for non-mathematicians

Seven papers can be seen as contributing to this theme. Feudel analyses the use of derivatives in economics to introduce cost functions and marginal cost; his data indicate that many of the students participating in his study, after their calculus course, just identified the derivative as an amount of change, without showing a clear understanding of the differences and connections between the derivative in mathematics and in economics contexts. González-Martín and Hernandes Gomes analyse the use of integrals in the Strength of Materials for Civil Engineering course to introduce

the notion of bending moment in the study of beams; using tools from ATD, their analyses show that, even if bending moments are introduced as an integral, the proposed tasks do not mobilise elements related to integrals from a calculus course. Kortemeyer and Biehler investigate the mathematical skills and knowledge required in undergraduate engineering using quantitative and qualitative analytical tools developed particularly for this study. Quéré uses tools from ATD to study French engineers' mathematical needs in the workplace; using data from 237 French engineers, he identifies mathematical notions they use, but also the need of 'mathematical abilities' that allow them to use mathematics not only as a tool. Rensaa uses grounded theory techniques to investigate engineering students' own descriptions of what they mean by 'learning linear algebra'; she identifies an apparent contradiction: to describe what they have learned, students emphasize conceptual more than procedural approaches, but in order to know that they have learned something they refer to solving specific tasks in the discipline. Viirman and Nardi describe a series of activities designed for Norwegian students of biology on biology-related mathematical modeling, and follow the learners' path from ritualized participation in mathematical routines towards more explorative participation; they suggest that highly scaffolded tasks, that explicitly state which routines students should invoke, may inadvertently contribute to students' ritualized participation in mathematical discourse. Wawro, Watson and Christensen analyze one student's meta-representational competence as he engages in solving a quantum mechanics problem involving concepts from linear algebra; they correlate this type of competence with abilities to solve tasks that require thinking in, using, and relating different notation systems from physics to mathematics.

Teachers' practices and knowledge

Ten papers can be seen as contributing to this theme. Branchetti analyses the resources, orientation and goals in the intended practices of a high school mathematics teacher with a PhD in mathematics, in relation to the topic of real numbers; the analysis indicates that orientations concerning the epistemology of real numbers, the goals of mathematics education in the high school and students' conceptions and difficulties lead the teacher to choose a very intuitive approach, missing the opportunity to benefit from his knowledge and expertise as a research mathematician. Cooper and Zaslavsky analyse a case of a mathematician/mathematics educator co-teaching partnership in an undergraduate course on Mathematical Proof and Proving; they find that the mathematician's main concern was with the written proof and its "correctness", whereas the mathematics educator showed a sensitivity to the person behind the proof, and to pedagogical aspects of proof and proving, suggesting that this type of co-teaching might be a way of achieving relevance for teaching in mathematics courses. Farah combines an ATD perspective with a sociocultural approach to identify institutional features that influence and transform the working habits of students in the context of French preparatory classes for business schools; she finds a great stability among the teachers' practices she investigates, these practices being strongly linked to the specific institution in which they occur. Fernández-León, Toscano-Barragán and Gavilán-Izquierdo use the horizontal and vertical mathematisation to study the conjecturing and proving approaches of a research mathematician working in a Spanish university; the analysis suggests that these practices (both in a horizontal and a vertical way) interact with each other when mathematicians create new knowledge. Florensa, Bosch, Gascón and Ruiz-Munzon report on a professional development course for mathematics lecturers in engineering; using tools from ATD and the construct of study and research path, they carried out modelling activities with a group of lecturers, allowing the introduction of some ATD notions to empower lecturers to question and put under vigilance the dominant epistemology at university. Jaworski, Potari and Petropoulou draw on their previous research to theorise and characterise university mathematics teaching within an Activity Theory perspective, developing an example concerning a lecture course in calculus with first year undergraduates; the Teaching Triad at the micro level of goals and actions, together with Activity Theory at the macro level, are used together to capture the complexity of the teaching situation,

addressing for instance the ways the lecturer engages his students and provides for their needs. Meehan, O'Shea and Breen examine 'brief but vivid' accounts of their lectures they wrote during a first-year undergraduate calculus course, and investigate the kinds of decision points they faced and how these decisions were triggered. Pinto observes and analyses the decision making and the shift of choices of an experienced mathematics teacher (Alan Schoenfeld) while he teaches a mathematical problem solving course; the Teaching for Robust Understanding of Mathematics framework (TRUmath) is used to unpack the conflicts that may underlie teachers' dilemmas and to explain their decisions. Püschl investigates the discussion patterns of teaching assistants in Germany with specific focus on how they work on tasks in small group tutorials, suggesting a typology of five discussion patterns of tasks: heuristic, pragmatic, student-oriented, problem-oriented, and minimalistic. Stewart, Thompson and Brady examine one mathematician's thought processes as he taught a course on algebraic topology; adopting the perspective of Tall's three worlds, they investigate how the teacher moves between the formal, symbolic and embodied worlds, and how he uses written handouts to ease students' movement between the worlds, particularly from the embodied to the formal, which he sees as the most challenging for students.

Current developments in TWG14

As we mentioned earlier, one of the themes that has grown considerably in this CERME10 concerns teachers' practices and knowledge. The presented papers offer a variety of theoretical and methodological approaches to study teachers' practices and decision making in the preparation of their teaching as well as during their actual teaching. As evident in the papers discussed in the conference, the investigation of teaching in its complexity seems to demand the use of more than one theoretical perspective. Moreover, some of the papers have proposed ways of collaboration between researchers and teachers towards better research insight as well as further development of teaching through a research-based reflection of teachers on their practice. These works have facilitated the discussion on teacher education and professional development at university level; an area of significant teaching interest that seeks further research.

Regarding students' learning of specific topics, CERME10 contributions continue to deepen our understanding of aspects of students' learning. This year there was more interest in how learning of specific topics can be seen also in relation to students' studying practices that go beyond these topics and specific courses. For example, students' learning can be seen in relation to how they use resources, take notes or experience transition issues.

Furthermore, this year we discussed five papers proposing interventions and reporting on the evaluation of the implementation of these interventions. Also, there were studies of tensions between innovative approaches and students' experiences, especially when these approaches contradict students' expectations. As in previous CERMEs, the number of papers proposing interventions is not high (Winsløw, Gueudet, Hochmuth, & Nardi, in press), and the account of experimental uses of digital technologies is still low. Although a range of studies proposes innovative approaches, more research is needed in this area, particularly studies that go beyond specific contexts and groups of students.

Finally, another growing area in TWG14 concerns mathematics for non-mathematicians. The papers presented this year show different degrees of collaboration with experts from other disciplines, as well as the importance of understanding the needs of these disciplines and their use of mathematical notions. From research about the learning of specific topics that happens to be conducted, for instance, with engineering students, the field has moved to study specific uses of mathematics by professionals (this year from biology, economics, engineering, and physics). We believe that this is an important shift of focus, and we expect to see in the future more papers studying the use of mathematics (and the professional needs) of several categories of professionals, as well as how mathematics can be taught by targeting these professional needs.

Reflection and ways forward

In this concluding section we reflect on the research in UME so far and suggest ways forward in terms of two directions: general questions about teaching and learning at university level; and, the role of mathematics as a service subject.

Regarding the first direction, general questions about teaching and learning at university level, we have noticed that research in UME usually reports on studies conducted in a specific educational context. As a research community, we would like to see more research conducted in joint efforts by colleagues from different countries. Strengthening communication between mathematics educators and mathematicians is also necessary towards collaborative research projects that engage mathematicians and suggest innovative approaches for future practice. There are more occasions recently where researchers in mathematics education are invited by mathematics teachers to share experiences and views and to contribute to curricular development decisions; collaborations of this type are very welcomed by our community,

Furthermore, although there is a considerable number of studies connecting students with mathematics, as well as teachers with mathematics, we would like to see more studies connecting teachers and students (teaching with learning). Aiming towards this connection may lead to new theoretical and methodological developments. Finally, we also discussed that there is a growing body of research about mathematicians teaching non-mathematics students, but there is still little research on how non-mathematicians teach mathematics as a service subject.

Regarding the second direction, the role of mathematics as a service subject, we would like to see more research on the different challenges and priorities that may occur in service courses. To name some examples, in service courses teachers may encounter large and heterogeneous groups; the content is not necessarily in the teacher's research area; there can be consequences for teachers' promotions (between giving a course of his/her specialty or a general course), which may have an impact on their motivation and practices; etc.. There is also little research on the epistemological analyses of what it means to teach mathematics to other disciplines; what makes the use of mathematics necessary in other disciplines; and, why mathematics is used as it is used in other fields. These investigations may lead to the identification of possible ruptures – and conflicts for the students – with how the content is presented in the mathematics courses. Another way forward can come from the use of discursive approaches, which would allow studying the discursive difference between communities. In most of these cases, we see the value of the collaboration of UME researchers with experts of other disciplines towards a research agenda that can address these questions.

Finally, in a general way, we would like to see more research that goes beyond single case studies, as well as research projects that expand small-scale studies to a bigger scale. We also notice that most of the papers in our group address mainly one of the themes we listed earlier, but we see the benefit of research that connects these themes by addressing the complexity of the teaching and learning of mathematics at university level. In all these scenarios, it is possible that mixed-methods studies will become more necessary. Regarding contributions to practice, the accumulated body of research results in UME should contribute to the development of research-based teacher training programmes for university teachers. Furthermore, there is a growth in the amount of mathematics learning support, and institutions are developing mechanisms to better guide and support students' learning of mathematics; we need to develop research about these new mechanisms offered to students, as well as about their impact and connections to what students learn in lectures.

This brief account of the presentations and discussions held in TWG14 during CERME10 aims to summarise our activities during the conference, as well as to invite the reader to explore the papers (long and short contributions) included in these proceedings. Our exchanges will continue in

different fora and we hope to meet again the participants to pursue our discussions and reflections, and to foster collaboration. Until we meet again in CERME11, the next meeting will take place in April 2018; we invite all participants (as well as newcomers) to join us in INDRUM2018.

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Undergraduates' reasoning while solving integration tasks: Discussion of a research framework

Aaltje Berendina Aaten[1,2], Johan Deprez[2], Gerrit Roorda[1] and Martin Goedhart[1]

[1] University of Groningen, the Netherlands; a.b.aaten@rug.nl; g.roorda@rug.nl; m.j.goedhart@rug.nl; [2] KU Leuven, Belgium; johan.deprez@kuleuven.be

In this paper we investigate the extent to which the research framework on reasoning developed by Lithner (2008) is adequate for characterizing undergraduate students' mathematical reasoning. We conducted a small number of individual task-based think-aloud interviews in which students solved integration tasks. Several examples illustrate how we characterized reasoning types by using the framework. However, we found that some reasoning types were not covered by the framework. We propose to extend the framework by introducing a reasoning type that is mathematically founded but not creative, and as a consequence, may be intertwined with imitative reasoning.

Keywords: Mathematical reasoning, undergraduate students, calculus.

Introduction

Students' mathematical reasoning while solving mathematical tasks is not always as well-founded as it appears, as has already been highlighted by Vinner (1997) in his article on pseudo-conceptual and pseudo-analytical thought processes in mathematics learning. Moreover, undergraduates' reasoning in the domain of calculus is found to be susceptible to ill-founded reasoning (Lithner, 2003). Because of its relevance for mathematics teaching, students' mathematical reasoning demands further investigation. Different frameworks on mathematical reasoning have been reported in literature. Some frameworks focus on argumentation used while solving proving problems (e.g., Blanton & Stylianou, 2014; Stylianides, 2008). Zandieh and Rasmussen (2010) constructed a framework on mathematical reasoning that distinguishes informal and formal reasoning, which seems most suited for investigating students' understanding of abstract concepts. Carlson and Bloom (2005) combined the problem solving phases Orienting, Planning, Executing, and Checking with the use of problem solving attributes Resources, Heuristics, Affect, and Monitoring. Their framework appears capable of identifying students' problem solving activities and the influence of cognitive, affective and/or metacognitive factors. However, this framework does not incorporate the foundation of mathematical reasoning. Lithner (2008) constructed a framework which does incorporate foundations of students' strategic decisions in solving mathematical tasks. This framework is often referred to when characterizing reasoning as either imitative or creative, which are the two main categories in the framework (e.g., Jäder, Sidenvall, & Sumpter, 2016; Jonsson, Norqvist, Liljekvist, & Lithner, 2014). However, the framework offers greater detail, by also defining Memorized Reasoning, and Familiar, Delimiting, and several types of Guided Algorithmic Reasoning (Lithner, 2008). Since this detailed characterization of mathematical reasoning based upon its foundation appears useful for identifying students' reasoning, we selected this framework for our research. Based upon our experiences in applying this research framework, we discuss the framework's possibilities and limitations.

Theoretical framework

Lithner (2008) defines reasoning as:

> The line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it (Lithner, 2008, p. 257).

Based on observations of students who solve mathematical tasks, Lithner describes various ways in which students choose a mathematical strategy to solve a task, pointing out students' 'predictive argumentation' (reasoning for choosing a strategy) and 'verificative argumentation' (reflection upon implementation of strategy), where strategy "ranges from local procedures to general approaches" and choice "is seen in a wide sense (choose, recall, construct, discover, guess, etc.)" (Lithner, 2008, p. 257). The resulting framework is visualized in figure 1.

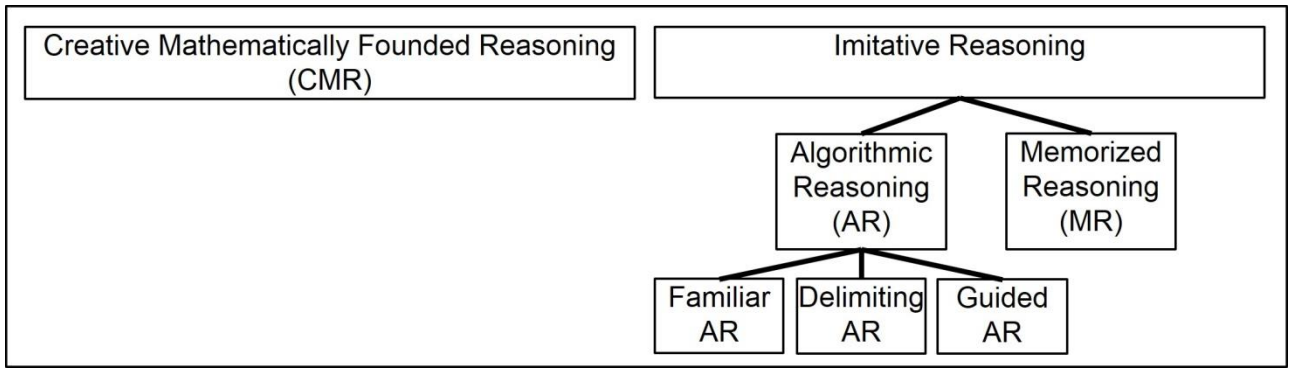

Figure 1: Visualization of reasoning framework as described by Lithner (2008)

The framework distinguishes two main categories, Creative Mathematically founded Reasoning (CMR) and Imitative Reasoning. CMR[1] refers to reasoning that is based on intrinsic mathematical properties, that is novel to the student (the reasoner) and for which the student has arguments (Lithner, 2008). Imitative reasoning is described as reasoning in which an algorithm or answer is recalled in some way. Imitative reasoning is divided into Memorized Reasoning and Algorithmic Reasoning. Memorized Reasoning implies that the student recalls a complete answer, for example a definition or a proof that is learnt by heart. Algorithmic Reasoning occurs when a student recalls an algorithm. Lithner's framework altered over time (see Lithner, 2003, 2004, 2008); in this study we applied the framework as described in Lithner (2008).

The definitions by Lithner (2008) for each of the reasoning types are listed in Table 1. In the definition of Delimiting Algorithmic Reasoning (see Table 1), the term 'set' of algorithms requires some explanation. Lithner (2008) clarifies that if no guidance is available and if the task is unfamiliar to the student, then the student must choose an algorithm from the 'set' of algorithms the student knows, based upon some kind of connection to the task.

[1] In earlier versions of the framework (e.g., Lithner, 2004), creative mathematically founded reasoning (which was then named Plausible Reasoning) was subdivided in a global and a local subtype, but this distinction has not remained.

Reasoning type	**Criteria**
Creative Mathematically founded Reasoning	Three criteria: "Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created." "Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible." "Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning" (Lithner, 2008, p. 266)
Imitative Reasoning	No definition is given. Imitative Reasoning is subdivided into Memorized Reasoning and Algorithmic Reasoning.
Memorized Reasoning	"The strategy choice is founded on recalling a complete answer. The strategy implementation consists only of writing it down." (Lithner, 2008, p. 258)
Algorithmic Reasoning	"The strategy choice is to recall a solution algorithm. The predictive argumentation may be of different kinds (see below for examples), but there is no need to create a new solution." "The remaining reasoning parts of the strategy implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached." (Lithner, 2008, p. 259)
Familiar Algorithmic Reasoning	"The reason for the strategy choice is that the task is seen as being of a familiar type that can be solved by a corresponding known algorithm." "The algorithm is implemented." (Lithner, 2008, p. 262)
Delimiting Algorithmic Reasoning	"An algorithm is chosen from a set that is delimited by the reasoner through the algorithms' surface relations to the task. The outcome is not predicted." "The verificative argumentation is based on surface considerations that are related only to the reasoner's expectations of the requested answer or solution. If the implementation does not lead to a (to the reasoner) reasonable conclusion it is simply terminated without evaluation and another algorithm may be chosen from the delimited set." (Lithner, 2008, p. 263)
Guided Algorithmic Reasoning	Text-guided Algorithmic Reasoning: "The strategy choice concerns identifying surface similarities between the task and an example, definition, theorem, rule, or some other situation in a text source." "The algorithm is implemented without verificative argumentation." (Lithner, 2008, p. 263) Person-guided Algorithmic Reasoning: "All strategy choices that are problematic for the reasoner are made by a guide, who provides no predictive argumentation." "The strategy implementation follows the guidance and executes the remaining routine transformations without verificative argumentation." (Lithner, 2008, p. 264)

Table 1: Definitions of reasoning types, derived from Lithner (2008)

It is important to note that the foundation of Creative Mathematically founded Reasoning is explicitly stated, while this is not the case for Imitative Reasoning: CMR is by definition founded in intrinsic mathematical properties, while the foundation of Imitative Reasoning is not clearly stated. The definitions of various sub-categories of Imitative Reasoning contain criteria like 'surface relations', 'surface considerations', 'surface similarities', 'no predictive argumentation', and 'without verificative argumentation'. This terminology appears to stem from earlier work: Lithner (2004) distinguished mathematically founded reasoning and superficial reasoning, where the latter was based upon surface properties and not upon mathematically relevant properties. Although in Lithner (2008), 'imitative reasoning' is not defined as founded in superficial or surface properties, many of the subtypes are (see Table 1). Moreover, all examples and explanations given by Lithner (2008) do refer to situations in which the reasoning is founded in so-called superficial properties and not in intrinsic mathematical properties, which is a criterion for CMR.

Certain studies have already used the framework to characterize students' reasoning. Boesen, Lithner, and Palm (2010) employed the categories of Memorized Reasoning, Algorithmic Reasoning (without subcategories) and extended the category of Creative Mathematically founded Reasoning by defining the two subtypes Local CMR and Global CMR (in accordance with Lithner (2004)). Sumpter (2013) applied the framework to label episodes of students' reasoning in a study on the role of beliefs in mathematical reasoning, and showed three examples which were all labeled as Familiar Algorithmic Reasoning. Jäder et al. (2016) used the framework to discern whether students' reasoning was imitative or creative. We remark that these studies have not used all sub-categories of the framework as described by Lithner (2008). In the case of Sumpter (2013), only one type of reasoning was discussed. Although these studies did make use of the framework, none of them did explicitly reflect upon its applicability. Since we consider the framework a worthwhile addition to literature on mathematical reasoning, we investigate its applicability for characterizing students' reasoning. The research question we thus aim to answer is: to what extent is the framework by Lithner (2008) adequate to characterize undergraduate students' mathematical reasoning?

Methodology

The data in this study originates from interviews with three first year mathematics bachelor students, one male and two female, of varying mathematics proficiency levels, determined by previous exam scores. These students are a sub-sample of a group of 12 students participating in a longitudinal study that investigates the development of mathematical reasoning. The students are majors in mathematics at the University of Groningen (the Netherlands) or the KU Leuven (Belgium); universities which offer courses in a wide range of domains at undergraduate and graduate level. The individual task-based think-aloud interviews lasted for approximately 1.5 hours each and are administered by the first author at the end of the students' first undergraduate year. Students were permitted to use a list with basic calculus formulas, which did not include elaborate integration formulas. The students were asked to explicate their thinking while solving tasks and, after each task, to answer the questions: "How did you come to think of using this strategy?", "How certain were you that this strategy would help you solve the problem, and why?", and "Have you seen this type of task before?". The interviews are video and audio recorded.

We used tasks to create a situation in which the students must choose a suitable strategy from a wide range of possible strategies. We considered integration tasks suitable for this purpose since the

students had learnt various mathematical strategies for solving integrals, such as partial integration, substitution, partial fractions, or Euclidean division, in the courses they had taken so far. These considerations led to selection of various tasks, amongst which $\int \sqrt{9-x^2}dx$ and $\int \sqrt{x^2-9}dx$. Both integrals can be simplified through inverse trigonometric substitution, e.g. $x = 3\sin(t)$ or $x = 3\cos(t)$ to solve the first integral, and $x = 3/\sin(t)$ or $x = 3\cosh(t)$ to solve the second integral. The students had taken courses in integral calculus in which they solved similar tasks, amongst many other types of tasks. The explicit discussion of these types of integrals had already taken place earlier in the academic year. Based upon teaching experience we expected that these tasks at the time of the interviews would be non-trivial to many students.

While integration tasks may be regarded as tasks that solely require application of procedures, these tasks can arouse various types of mathematical reasoning in students. Considering and selecting suitable procedures is a process in which Creative Mathematically founded Reasoning as well as Imitative Reasoning can become visible. Familiar Algorithmic Reasoning can be used if the student recognizes the problem type and recalls the corresponding algorithm; Delimiting Algorithmic Reasoning if the student does not recognize the task but recalls various algorithms such as partial integration or substitution of some kind; Creative Mathematically founded Reasoning can be employed if the student is not able to recall a solution strategy but instead constructs a solution or reconstructs a forgotten reasoning sequence, such as drawing a rectangular triangle and deducing a suitable substitution. We did not expect Memorized Reasoning, since the solutions to the tasks are extensive. Guided Algorithmic Reasoning also appeared improbable, since example solutions were unavailable and the interviewer would not offer any hints. The available list with formulas however could serve as inspiration.

Transcripts of the task solutions are split into episodes. An episode begins at the first consideration of a strategy (or a set of strategies) and ends when the strategy is abandoned and a new strategy is about to be considered. Using the framework to characterize parts of a solution is similar to the method of Lithner (2008) and Sumpter (2013). The first author tried to characterize each of the episodes through the definitions given by Lithner (2008). If this was unsuccessful, the difficulties were described. The findings from this analysis were discussed with the other authors until agreement was obtained.

Results

Below we describe several reasoning episodes from our data, which illustrate how we characterized reasoning using the framework and which problems we encountered.

Familiar Algorithmic Reasoning?

Example 1: $\int \sqrt{x^2-9}dx$; student A chose to rewrite the integrand by splitting it into two terms and next integrating them separately: $\sqrt{x^2-9} = \frac{x^2-9}{\sqrt{x^2-9}} = \frac{x^2}{\sqrt{x^2-9}} + 9\frac{-1}{\sqrt{x^2-9}}$. The student had applied this strategy earlier in the interview, when (unsuccessfully) solving $\int \sqrt{9-x^2}dx$. The student explained: “The task was similar to the former task, so I figured I could try using the same approach.” We observe that the task is recognized as a familiar type, which made the student decide to apply the same strategy as before. We characterize this reasoning as Familiar Algorithmic Reasoning.

Example 2: $\int \frac{x}{\sqrt{x^2-9}}dx$; this sub-task arose while student A worked on $\int \sqrt{x^2-9}dx$. This sub-task is of a standard form because the derivative of x^2-9 is in the numerator, making $t=x^2-9$ an appropriate substitution. Student A effectively chose this strategy and explained afterwards: "Because I knew that the derivative of this (points at x^2-9 in the denominator), was something like this (points at the numerator) […] So I applied substitution to this (points at x^2-9), because this (points at the numerator) would then be eliminated.". We observe that the student noticed the relevant mathematical characteristics of this task and knew what algorithm would solve tasks of this type. Either the task type was familiar or the student constructed the approach. The ease with which the student came to this conclusion and the fact that this type of task has been practiced extensively make us expect the student to be familiar with the task type: we characterize the reasoning as Familiar Algorithmic Reasoning. However, the student clearly founds the reasoning in intrinsic mathematical properties, which is very different from the reasoning that occurred during Example 1, where the student appears to base the strategy choice solely on familiarity with the task type.

We conclude that Examples 1 and 2 provide two rather distinct reasoning types while both satisfying the criteria for Familiar Algorithmic Reasoning.

Memorized Reasoning or Creative Mathematically founded Reasoning?

Example 3: $\int \frac{1}{\cos\theta}d\theta$; this sub-task arose while student C worked on $\int \sqrt{x^2-9}dx$. The student searched the primitive of $\frac{1}{\cos\theta}$ with respect to θ, which is $\ln|\sec\theta+\tan\theta|$. "It was something like $\ln$ to the power…, $\ln$ of, wait. $\int \frac{1}{\cos\theta}d\theta$. it does not have to be so complicated. There must be something that I overlook. […] eh. Ah, no, wait wait, hey. sec θ times … Secant tangent? There was something about that. $\frac{\sec\theta}{\ln|\sec\theta\tan\theta|}$. I rely on my memory now, because I have solved those integrals. I know it's an integral with secant, with ln. (student calculates the derivative of $\ln|\sec\theta\tan\theta|$, infers it is not correct) What was it like? […] wait, I think I know. $\ln|\sec\theta+\tan\theta|$ is (student calculates derivative) $\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}$. Then you can cancel this ('$\sec\theta+\tan\theta$' in numerator and denominator) and then you obtain indeed… I knew it was something with $\ln$."

We observe that the student solves the task by making use of answer recall, but also reasons on the intrinsic mathematical properties of the task to be successful. In the framework, the only reasoning type that makes use of recall of an answer is Memorized Reasoning. However, the second criterion of Memorized Reasoning is not fulfilled. The strategy implementation was not just writing down the answer, since the answer was constructed and verified building upon the intrinsic mathematical properties of the components involved in the reasoning. On the other hand, the category of Creative Mathematically founded Reasoning does not reflect the important role of memory in this solution. This example shows hybrid reasoning with elements from Creative Mathematically founded Reasoning and from Memorized Reasoning.

Intrinsic mathematical properties or surface properties?

Example 4: $\int \sqrt{9-x^2}dx$; student B rewrote $y=\sqrt{9-x^2}$ to $x^2+y^2=9$ and remarked it is a circle: "That gives a nice circle. Then you have got the radius, a circle with radius 9, radius 3, I mean. It's not transformed, so you get this. Circular coordinates. Let's take a look at circular coordinates […] Then you get $x=r\cos\theta$ and $y=r\sin\theta$. So $r^2\cos^2\theta+r^2\sin^2\theta=r^2$. […] This is of course… This is just $r^2=r^2$ because $\cos^2\theta+\sin^2\theta=\ \ldots\ r^2$." The student stops using this strategy.

We observe that the student considered the circular coordinates (polar coordinates) since the integrand made the student think of a circle. The strategy appears to be selected based upon intrinsic mathematical properties of the task. However, the student employed the circle coordinates in an ineffective way, which shows that the student did not know why the property of the task, that it concerns a circle, implies the use of circle coordinates. The strategy of using circle coordinates are selected only because the task concerned a circle, therefore the foundation for strategy selection should be regarded as based on the task's surface properties. This example raises doubts concerning whether it is always possible to distinguish a surface property from an intrinsic mathematical property.

Conclusions and discussion

The framework by Lithner (2008) provides means to highlight foundations that underlie students' reasoning when solving a mathematical task. However, we faced several difficulties when employing the framework as an analysis instrument. Examples 1 and 2 concern rather distinct reasoning episodes, while both satisfy the definition of Familiar Algorithmic Reasoning. Whether or not the student provides mathematically founded reasons is a relevant characteristic but not included in the definitions. In Examples 2 and 3, the predictive argumentation of a strategy was imitative (based on recall of any kind), but verificative reasoning was founded in intrinsic mathematical properties of the task. These examples reveal that the framework does not cover such 'hybrid' types of reasoning. Example 4 confronted us with the more fundamental issue how to decide whether reasoning is based on 'surface properties' or on 'intrinsic mathematical properties'.

A way to improve the applicability of the framework is to include reasoning types that are mathematically founded as well as make use of some kind of imitative reasoning. This is not the same as local CMR (Lithner, 2004), which is reasoning that is partly Creative Mathematically founded Reasoning while the remainder is Imitative Reasoning. We propose that reasoning can be mathematically founded without being creative, and in addition, that mathematically founded reasoning can be intertwined with imitative reasoning. Whether a property is an intrinsic mathematical property or a surface property appears to depend on the student's understanding, e.g. of why a certain task property leads to a certain strategy selection. Distinguishing between the use of surface properties and intrinsic mathematical properties therefore requires a more complete picture of the students' reasoning as a whole. These suggestions are based upon difficulties faced when applying the framework on a small number of reasoning episodes within the domain of integration. To obtain a framework adequate to characterize any type of mathematical reasoning not only requires thorough investigation of specific examples, but also requires investigation of the structure of the framework such that the framework will be decisive for each reasoning episode to be characterized.

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A categorisation of the resources used by undergraduates when studying mathematics

Marinos Anastasakis, Carol Robinson and Stephen Lerman

Loughborough University, Mathematics Education Centre, Loughborough, United Kingdom;
M.Anastasakis@lboro.ac.uk, C.L.Robinson@lboro.ac.uk, S.Lerman@lboro.ac.uk

This paper presents results from a survey exploring the kind of resources that engineering undergraduates (N=201) use when studying for their mathematics modules. By using Factor analysis, we were able to produce a typology of these resources. The resulted typology was further analysed by combining Leontiev's version of Activity Theory and Wartofsky's hierarchy of artefacts. This helped us to draw links between the tools that students use and infer about their learning actions.

Keywords: Blended learning, educational resources, student centered learning, activity theory, Wartofsky.

Introduction

Historical records suggest that using tools has always been inseparable from expressing and doing mathematics (Roberts, Leung, & Lins, 2012) with numerous examples demonstrating the capacity of tools to influence the development of mathematics itself as a scientific discipline (Laborde & Sträßer, 2009)[1]. Since the end of the nineteenth century many different types of tools have been used for the teaching and learning of mathematics (Kidwell, Ackerberg-Hastings, & Roberts, 2008) however for many the term "technology" corresponds to electronic or digital in nature tools (e.g. calculators) something that manifests a kind of historical amnesia (Roberts et al., 2012). Students nowadays have access to a plethora of digital/online resources that they could use alongside more "traditional" ones (e.g. textbooks, lecturers or their own notes) and thus blend their learning. As a matter of fact, Masie (2006) asserts that this has been always the case as students were always combining resources in order to support their learning. The notion of blended learning (BL) has been introduced almost 30 years ago however, the term has not been clearly defined yet (Bos & Brand-Gruwel, 2016; Graham, 2013; Torrisi-Steele & Drew, 2013). The three main definitions of BL used in education are (Sharma, 2010): BL as a *combination of face-to-face and on-line teaching*; BL as a *combination of technologies* and; BL as a *combination of pedagogical methodologies*. In this paper, we are mainly referring to BL as a combination of technologies or better as a combination of tools since each era has its own technologies not only digital ones. Despite what its name implies, BL has received criticisms and some authors argue that it should be rather called *blended teaching* because current views adopt a teacher-centred and not a student oriented perspective by focusing on the resources that instructors choose for their students (De George Walker & Keeffe, 2010; Oliver & Trigwell, 2005). If students indeed blend their learning by mixing different resources not just digital or the ones provided to them by their institution, *what kind of* resources *do they blend* and *how can these* resources *be classified*? In terms of the resources that students use, we have found a lack of empirical studies exploring the kind of resources that undergraduates *themselves choose and use*, with previous studies focusing

[1] Although many authors consider the terms "tools", "artefacts", "instruments" or "resources" to be different, for the purposes of this paper we treat them as having the same meaning.

mostly on digital/online or institutionally-led resources and thus neglecting the "blended" nature of their learning (for a short review see Anastasakis, Robinson, & Lerman, 2016). In effect, our sense of the field aligns with comments from authors suggesting that the student perspective has not been taken into account in the BL literature (e.g. Ituma, 2011; López-Pérez, Pérez-López, & Rodríguez-Ariza, 2011). Having previously identified the kind of resources that a sample of engineering undergraduates uses when studying mathematics (Anastasakis et al. 2016), our aim in this paper was to propose an empirically based typology of these tools.

Theoretical framework

Long before the dominance of the world wide web and computers, many researchers had emphasised the significance of tools in our everyday activities; a well-developed theoretical account of human praxis which emphasises the role of physical tools is second generation Activity Theory (AT) (Leontiev, 1981). From an AT perspective, our relationship with the "objective" world is mediated by tools: the central role of tool mediation within this framework is due to the fact that tools shape the ways we interact with reality and they reflect past people's experiences and practices (Kaptelinin & Nardi, 2006). In AT, activities have a hierarchical structure and they consist of three layers (*Ibid*.): at the top is the activity itself directed towards the object of the activity (our overarching goal); in the middle lie actions (what we do) directed at goals (what we want to achieve) and; finally, at the lower layer we find operations (non-conscious, routine processes) which are directed to conditions (non-subjective factors that affect our actions). In sum, AT asserts that tools are the means by which subjects are trying to achieve their goals and in this sense tools are bounded with a subject's practice. Despite the central position that tools hold in AT, little is written from this perspective in regards to how they can be categorised; among the little accounts found in the literature, is Wartofsky's typology of artefacts. Wartofsky (1973) considered tools as the genes of our cultural evolution and proposed that they can be classified into *primary*, *secondary* and *tertiary* artefacts. As primary are considered the tools themselves (Engeström, 1990); secondary artefacts represent "modes of action using primary artefacts" (Cole, 1996, p. 121) and they "synthesise the ways and procedures of using instruments and materials" (Miettinen, 1999, p. 189). Finally, tertiary artefacts are those which emphasise creativity (Kaptelinin & Nardi, 2006) and "transcend[s] the more immediate necessities of productive praxis" (Wartofsky, 1973). As an example of what constitutes a primary and a secondary artefact, Bussi and Mariotti (2008) refer to the abacus: the abacus itself is a primary artefact and the ways of using abaci for counting, keeping records or making computations represents a secondary artefact. Engestrøm notes that Wartofsky's typology is closely related to Leontiev's levels of activity (Petersen, Madsen, & Kjær, 2002): primary artefacts correspond to the level of operations/conditions, secondary to the level of actions/goals while tertiary to the level of the activity (Engeström, 1990; 2015). Our focus in this paper are primary and secondary artefacts.

Method

This study is part of a doctoral project that aims to identify the kind of tools that undergraduates use when studying mathematics, how these tools are used and the reasons for using them. During the autumn term, a paper-based questionnaire was administered to four different groups of second year engineering students in Loughborough university and in total 201 completed it. Loughborough has one of the largest cohorts of engineering students (over 3000 undergraduates) in the UK and a well established provision of Mathematics Support (http://www.lboro.ac.uk/departments/mlsc). It has also

led on significant projects producing high quality printed material (e.g. the HELM project: http://helm.lboro.ac.uk). The questionnaire consists of three main parts and its design was guided by Activity Theory (AT) (Leontiev, 1981). Here we report only on the part related to the resources (tools) that undergraduates use. In this, students were explicitly asked to identify how often they use a list of 14 resources on a 6-point semantic scale (1/Never, 2, 3, 4, 5, 6/Always) with two additional open ended items for other resources not listed in the questionnaire. The list was based on our literature review, five in depth interviews with undergraduates conducted in 2015 and the resources that Loughborough University offers to students e.g. the Learn website (university's VLE). The list of resources was carefully generated and encompasses a great variety of tools available to students; in this way it reflects -to a certain degree- students' reality as learners when it comes to the resources they use. Students were also asked to identify which five of these 14 resources they use the most (top-5) and rank them in a descending order (not reported here).

Analysis and results

Summary statistics

Results for the tools that students use are presented in Figure 1. These results have been already presented elsewhere (Anastasakis et al., 2016) but we include them here for clarity. By using each tool's mean, we categorised them into three main groups: tools with a mean greater than or equal to 4.5 were characterised as *high-use*, those with a mean between 3 and 4.5 were assigned into the *mid-use* group while resources with a mean between 1.5 and 3 were put into the *low-use* group.

Factor Analysis

Exploratory Factor Analysis (EFA) is a statistical method aiming at grouping variables which have something in common (i.e. they correlate with each other). This enables researchers to identify latent constructs in the data that cannot be measured otherwise directly. Each group (or cluster) of variables is then called a factor and the variables consisting each factor are thought to be measuring the same underlying/latent construct i.e. each factor represents one underlying construct. An initial EFA was performed on all the 14 variables for tools. We used an oblique rotation because the underlying constructs sought in our data were expected to be related (all variables are related to tool-use after all). Our sample's adequacy was measured by the Kaiser-Meyer-Olkin measure of sampling adequacy and found to be above the minimum value of .5 (KMO=.711). Bartlett's test of Sphericity also showed that the correlations between variables are not 0 i.e. the correlation matrix is not an identity matrix: this is true when the significance value for this test (p) is less than .05 and in our case, it was p<.001.

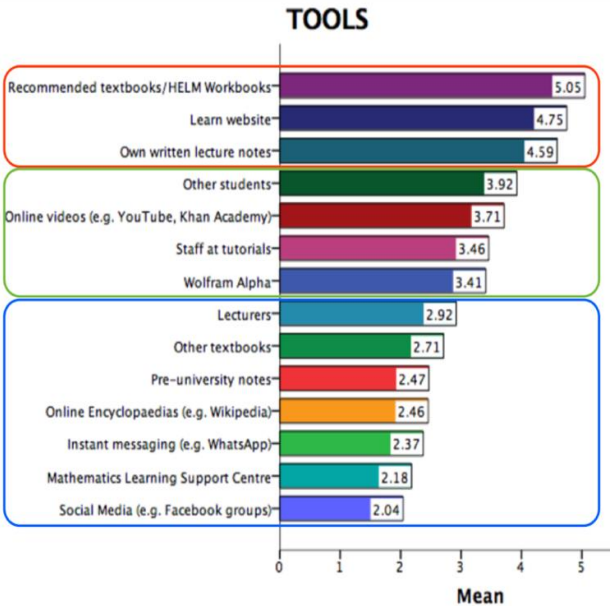

Figure 1: Tools and their grouping based on their mean (high-use: red, mid-use: green, low-use: blue)

Pattern Matrix[a]

	Factor		
	1	2	3
Mathematics Learning Support Centre	.733		
Other textbooks	.643		
Lecturers	.638		
Pre-university notes	.479		-.31
Staff at tutorials	.367		
Instant messaging (e.g. WhatsApp)		.733	
Other students		.710	
Social Media (e.g. Facebook groups)		.495	
Online Encyclopaedias (e.g. Wikipedia)			.523
Online videos (e.g. YouTube, Khan Academy)			.390

Extraction Method: Principal Axis Factoring.
Rotation Method: Oblimin with Kaiser Normalization.[a]

a. Rotation converged in 6 iterations.

Table 1: Pattern matrix of the final EFA (please note that values below .3 are omitted)

Structure Matrix

	Factor		
	1	2	3
Mathematics Learning Support Centre	.698		.313
Lecturers	.644		
Other textbooks	.643		
Pre-university notes	.521	.334	
Staff at tutorials	.374		
Instant messaging (e.g. WhatsApp)		.740	
Other students		.724	
Social Media (e.g. Facebook groups)		.497	
Online Encyclopaedias (e.g. Wikipedia)			.537
Online videos (e.g. YouTube, Khan Academy)	.327	.302	.435

Extraction Method: Principal Axis Factoring.
Rotation Method: Oblimin with Kaiser Normalization.

Table 2: Structure matrix of the final EFA (please note that values below .3 are omitted)

The scree plot was used as a criterion for determining how many factors should be kept. Two lines that best summarise the scree plot were drawn with the intersection of these lines (called point of inflection) indicating how many factors are present in our data, excluding the factor on the point of inflection (in our case 3 factors). We additionally examined both the pattern and structure matrices and decided to exclude the variables "own written lecture notes", "HELM workbooks", "Learn website" and "Wolfram Alpha" from our subsequent analysis. This was done because these variables were either having factor loadings below the cut-off value of .364 that we used based on our sample's size (see Stevens, 2002, p. 374) or because only one variable was present on a single factor (the goal of EFA is to group similar to something variables). Based on this analysis, we run a second EFA on the 10 remaining items with an oblique rotation (KMO = .711, $p<.001$) by requesting a 3 factor solution. Both the pattern and structure matrices were interpreted (Tables 1 and 2). Factor loadings (numbers at structure and pattern matrices) can be thought as the correlations between each variable

and the factors and they represent how well a variable "fits" into a factor. The final obtained factors included the following variables:

- **Factor 1** (5 variables): "Mathematics Learning Support Centre", "other textbooks", "lecturers", "pre-university notes" and "staff at tutorials"
- **Factor 2** (3 variables): "other students", "instant messaging" and "social media"
- **Factor 3** (2 variables): "online videos" and "online encyclopaedias".

At this point, we decided to treat the 4 variables not loading on any Factor ("HELM workbooks", "own written lecture notes", "Learn website", "Wolfram Alpha") as latent constructs too. This decision was made for two reasons: first, these variables did not load on any Factor (i.e. not relating with other variables); and second, the nature of each resource is different and unique when compared with the other resources, thus they can be thought "measuring" something on their own. By adopting a descriptive approach (Rummel, 1970) we named the 7 identified types of resources as follows:

1. **The "official" mathematical textbook**: "HELM workbooks"
2. **Students' lecture notes**: "own written lecture notes"
3. **University's VLE**: "Learn website"
4. **The calculator**: "Wolfram Alpha"
5. **Teaching staff**: Factor 1
6. **Peers and communication tools**: Factor 2
7. **External online tools**: Factor 3

From the above types of resources, "teaching staff" (Factor 1) contains 5 variables which they seem not fitting together; is it reasonable to interpret together different in nature variables such as "pre-university notes", "other textbooks" and "lecturers" for example? In our opinion, it makes good sense since they correspond to students' direct interactions with university's teaching staff ("Mathematics Learning Support Centre", "lecturers", "staff at tutorials") and ways that students interact indirectly with teaching staff; this includes the use of resources probably suggested by teaching stuff ("other textbooks") or resources which are the product of prior interactions with a person holding a teaching position e.g. A-levels tutor ("pre-university notes").

Discussion

Our aim for this paper was to produce a typology of the resources that engineering students in our sample reported using. By performing an Exploratory Factor Analysis on our data, we were able to identify 7 different types of resources that undergraduates in our sample reported using (Table 3, left column). From a Wartofskian point of view, all the tools used by undergraduates when examined separately are primary. On the other hand, secondary artefacts represent "*modes of action* using primary artefacts" (Cole, 1996) and they "synthesise the ways and *procedures of using instruments* and materials" (Miettinen, 1999, p.189, our emphasis); this means that our proposed typology corresponds to the different *secondary artefacts* that undergraduates use. When examined from an AT perspective, the typology of tools corresponds to *thematically related actions* that students undertake when studying mathematics. This is because actions in AT are the "...specific interactions that people have with artefacts and other people..." (González, Nardi, & Mark, 2009) i.e. as actions we account the processes of *using* a tool (types 1, 2, 3, 4 and 7) and/or *interacting* with other subjects (types 5 and 6). Our Wartofskian and AT-based interpretations are also consistent from a statistical

point of view: students were asked how frequently they use a resource thus, the nature of each variable is related to using a tool or interacting with a person i.e. an action from an AT perspective (*Ibid.*) or the ways of using primary artefacts. Thus, from both a Wartofskian and AT perspective, the 7 different types of resources that undergraduates use, correspond to the following secondary artefacts/actions (Table 3, right column):

1. Studying the mathematical textbook
2. Taking notes during a lecture
3. Accessing institutionally provided material (online)
4. Performing (complex) calculations
5. Interacting with teaching staff
6. Interacting with peers (in-person or virtually)
7. Searching for external/alternative material online

Typology of Tools	Secondary Tools (Wartofsky) - Actions (AT)
(1) The "official" mathematical textbook	Studying the mathematical textbook
(2) Students' lecture notes	Taking notes during a lecture
(3) University's VLE	Accessing institutionally provided material (online)
(4) The calculator	Performing (complex) calculations
(5) Teaching staff	Interacting with teaching staff
(6) Peers and communication tools	Interacting with peers
(7) External online tools	Searching for additional/alternative resources of information

Table 3: Proposed typology of tools (left) and their representations as secondary artefacts and actions

Conclusion

In this paper, we analysed survey data from a cohort of second year engineering students (N=201) about the kind of tools they use when studying for their mathematics modules. In contrast with common approaches found in the literature, we did not focus only on digital/online or institutionally provided resources but rather we incorporated a variety of resources that students have at their disposal. An Exploratory Factor Analysis of our data allowed us to identify 7 different types of tools that students in our sample reported using: the "official" mathematical textbook ("HELM workbooks"); their own written notes; university's VLE ("Learn website"); a sophisticated calculator ("Wolfram Alpha"); university staff (Factor 1); peers and social apps (Factor 2); and non-institutional online tools (Factor 3). By adopting Wartofsky's hierarchy of tools these dimensions represent secondary artefacts i.e. 7 different ways that students use primary artefacts or students' modes of action when studying mathematics: studying the mathematical textbook, taking notes during a lecture, accessing institutionally provided material, performing calculations, interacting with teaching staff, interacting with peers and searching online for additional/alternative sources of information. This interpretation is consistent with AT because these dimensions represent students' actions when studying mathematics. One important implication of our analysis is that although some resources are

different (e.g. people, digital) they may be used by students in a *similar* way: this was the case of Factor 1 which contained different in nature resources. This result contradicts our common assumptions when categorising resources (e.g. people, digital, online etc.) and adds an empirical basis for the argument that the way we usually classify resources does not necessarily reflect the ways these resources are used. Because of the nature of our data, we could only infer about the nature of the 7 types of tools that undergraduates use by only examining the resources included in each type. However, our preliminary analysis of 14 interviews suggests that our interpretation aligns with these resources' actual use: for example, students who interviewed reported using Facebook for communicating with peers when having an issue with mathematics (either by using Messenger or by posting a question on Facebook groups created by undergraduates). Our intention for the future is to complement the survey data with data gathered with other methods (interviews and diaries). Finally, we are of the opinion that the results of our analysis (students' learning actions), highlight the temporal nature of all primary tools used in learning and suggest that our future research foci in mathematics education should be on the ways that these tools are used rather than the tools themselves.

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An analysis of freshmen engineering students' notes during a preparatory mathematics course

Chiara Andrà

Politecnico di Milano, Italy; chiara.andra@gmail.com

I analyse the notes of a group of first year engineering students who attended a course in pre-calculus mathematics. Being interested in verbalisation skills at the beginning of university, I adopt a narrative lens to analyse the notes: I see a lecture as a story being told by the teacher and the students' notes as re-tellings of the teacher's story. I focus on the way the students condense the mathematical content in their written notes in two distinct teaching formats: a traditional frontal lesson and the concluding phase of a classroom discussion after a small group activity.

Keywords: Secondary-to-tertiary education transition, students' notes, narrative approach.

Introduction

This paper is part of a wider research project on freshmen engineering students at the Polytechnic of Milan, aimed at understanding their difficulties during the first year of their studies. Recent research has found that mathematics at tertiary level is difficult for engineering students (see e.g. Gomez-Chacon, Griese, Rosken-Winter & Gonzales-Guillen, 2015). Gueudet (2008) provides a detailed overview of mathematics education studies concerning the transition from school to university and she identifies the theme that is the core of my research, namely: the students' organisation of knowledge that is new to them at the beginning of university.

Boesen, Lithner & Palm (2010) argue that the kind of task assigned to the students affects their learning: tasks with low levels of cognitive demand lead to rote-learning by students and, consequently, their inability to solve problems that are unfamiliar to them (for instance, the ones that require conceptual understanding). Breen, O'Shea and Pfeiffer (2013) define an 'unfamiliar task' as a task "for which students have no algorithm, well-reharsed procedure or previously demonstrated process to follow" (p. 2318), and provide evidence that this kind of tasks raises an awareness about the need for more than procedural understanding of mathematics, thus easing the students' transitions to university math practices. Also, the teaching strategies employed in the class can influence the development of one type of knowledge more than another: teacher-centred methods would favour the development of procedural knowledge and student-centred methods would favour conceptual knowledge (see e.g. Garner & Garner, 2001; Allen, Kwon & Rasmussen, 2005). Inspired by these studies, I investigate how the students organise the knowledge in their notes during a preparatory math course.

To take notes does not involve only mathematical ability. It involves also verbalisation skills. O'Neill, Pearce and Pick (2004) found that there is a correlation between performance in generating narratives and mathematical ability as early as in primary school years. With Nardi (2011), I recognise the centrality of the students' ability to use ordinary language to construct and convey mathematical meaning and I investigate undergraduate engineering students' notes in a preliminary math course at their first year at university. With Nardi (2011), I value the students' attempts to mediate the mathematical meanings through words, symbols and diagrams and I maintain that at the basis of the students' difficulties in dealing with a discursive shift from secondary to university mathematics there are: (a) undervalued verbalization and (b) premature compression. According to the former, "the

students undervalue, and often avoid entirely, expressing their mathematical thoughts verbally" (Nardi, 2011, p.2056); according to the latter, "students' mathematical writing is typically prematurely compressed, namely ridden with gaps, leaps and omissions" (ibid.).

Theoretical framework

Andrà (2013) examines the relationships between a teacher's lecture and the students' notes by viewing the lecture as a kind of story that the teacher tells and the students' notes as retellings of the teacher's story. A mathematical lesson seen as a story can be analysed in terms of its components: its characters, setting, action, plot, and moral (see also Bal, 2009). Mathematical objects, in fact, can be considered the mathematical characters of a story (Dietiker, 2012). They can play a central or a peripheral role, have multiple names, and have properties that can be introduced and developed gradually. The setting is the space where characters are placed. Sometimes the setting is not obvious, as it refers to underlying assumptions and/or axioms. The setting may also involve different registers, such as algebra or the Cartesian coordinate system. The action is that which the actor performs. In mathematical stories, the result of an action can be a change in an object or in a setting, or both. According to Dietiker (2012), we see that, unlike in literary stories, mathematical ones can change actions into objects (through reification). Andrà (2013) observes that the students miss important (teacher's) mathematical actions in their notes and more in general Morgan (1998) notices a relative absence of active verbs in mathematical writing. The moral can be seen as the intended message of the lesson. The plot is the sequence of actions and it involves the shaping of the story, which is linked to its aesthetic effects: for instance, the rhythm and the frequency of the story (Bal, 2009) may affect the students' focusing on the areas of emphasis of the story and foster his anticipative acts. Some moves might displace attention away from what the teacher wants to communicate: for example, repeating the name of a character often may lead the students to think that the actual name is important, or more important than its properties. Other moves may induce the students to believe that the setting is unimportant. These moves can be interpreted in terms of Rotman's (1988) schema, which distinguishes between invitations for the reader to be a "thinker" and ones that prompt the reader to be a mere "scribbler". This distinction provokes a further distinction, namely: to think about note-taking as mere consumption of mathematical meaning or to think about it as active production of meaning. It is possible to interpret Rotman's schema with respect to the students' notes in this way: a scribbler is a student who reports mainly the mathematical characters of the story and misses the actions, so that the notes result to be compressed and ridden with gaps, leaps and omissions (see also Nardi, 2011). A scribbler is also a student who avoids putting her thoughts in her notes, and limits herself to copying and/or reporting what the lecturer is saying/writing. The plot is the same plot as that of the story told by the lecturer. A thinker, instead, re-organises the content of the lesson, she (re)structures the plot so that it becomes accessible to herself even after the lesson ends. A thinker pays attention to the details and also records the mathematical actions so that her notes are not overly compressed and under-verbalised. In Andrà's (2013) understanding of Rotman's schema, furthermore, some moves invite the students to be thinkers while others invite them to be scribblers. In view of the findings of Boesen, Lithner & Palm (2010) we consider two scenarios: in the first one, the lecturer proposes a group activity on a conceptual and unfamiliar task and the students' notes are taken during the classroom discussion that follow the group activity; in the second scenario, the lecturer assigns a procedural task and corrects it on the blackboard. The interest is to see how the students' notes change (if so) in the two different scenarios.

Methodology

The Polytechnic of Milan, like many universities all around the world, organizes some courses before the beginning of the first semester, which have the purpose to recapitulate the basic knowledge that is necessary for the students to successfully attend the courses at the first academic year. One of these preparatory courses is on pre-calculus mathematics. Since three years, the course is organised according to a flipped classroom pedagogy: the students (are supposed to) watch a series of videos in a MOOC and at university the lecturers of the preparatory course involve them in groupwork activities aimed at deepening their understanding of the basic math concepts and expose the students to frontal lessons with routine exercises. The mixed method of teaching serves the purpose of both exposing the students to a new, "conceptual" teaching and to make them feel comfortable with teaching practices that are more typical of secondary school. During the first lesson of the preparatory course, among the exercises given about polynomials, one had a conceptual nature, since it said "The polynomial p(x) is divisible by the polynomial q(x) if…". This is a kind of task that is unfamiliar for Italian students, since it asks for reflection about the definitions and the students do not have a well-established procedure to resort to. It was first dealt with in small groups, then discussed at classroom level. During this last phase, the teacher wrote the steps of the solution at the blackboard and the students took notes. Another task had a procedural nature and was not unfamiliar for the students: two polynomials were given, p(x) and q(x), and the students were asked to divide p(x) by q(x). It was solved at the blackboard by the teacher. I compare and contrast the students' notes in these two different situations: one familiar and procedural (i.e., linked to actions), one unfamiliar and conceptual (i.e., linked to characters). In this study, the lecturer under consideration is also the researcher and the author of this paper. At the end of the lesson, the students were informed about the study and invited to provide their notes for research purposes. I am aware of the potential issues concerning this method of data collection, and I followed ethical guidelines in order not to expose the students to risk. Ten students offered their notes; of those I selected four to be analysed in this paper since they are contrasting. The students are identified with four fictitious names: Angela, Filippo, Roberto and Vincenzo. Their notes are analysed through a narrative lens, identifying: the mathematical characters; their setting; the mathematical actions, understood in terms of operations made on/by the mathematical characters; the plot, or the organisation of the content on the sheet of paper; the moral. The four students are inferred to be scribblers or thinkers by looking at these elements. The research questions read as follows: (a) how do students organise their notes? (b) which elements of the teacher's "story" are recorded, and which ones are discarded? (c) in which cases do the students act as scribblers and in which ones as thinkers?

Data analysis

Figures 1-8 report the four students' notes regarding the two tasks. Since they are in Italian, a translation is provided in the caption of each figure.

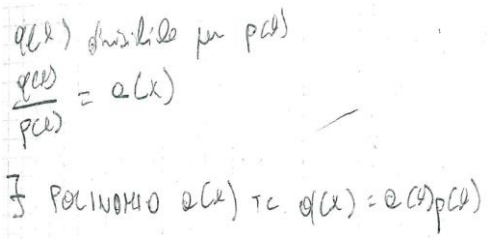

Figure 1: Vincenzo's notes about the "conceptual task". On the first row, he writes "q(x) divisible by p(x)". On the second row, he writes a formula and in the last row: "polynomial a(x) s.t. q(x) = a(x)p(x)".

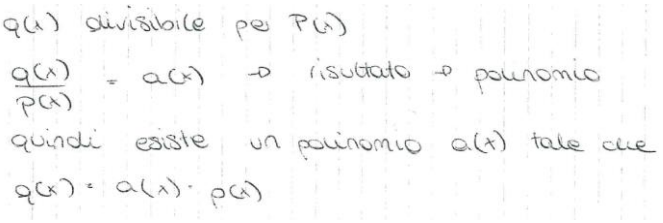

Figure 2: Angela's notes about the "conceptual task". On the first row, she writes "q(x) divisible by p(x)". On the second row she writes the ratio, then she adds an arrow and writes "result", then another arrow and "polynomial". On the third row, she writes "hence there exists a polynomial a(x) such that q(x) = a(x)p(x)".

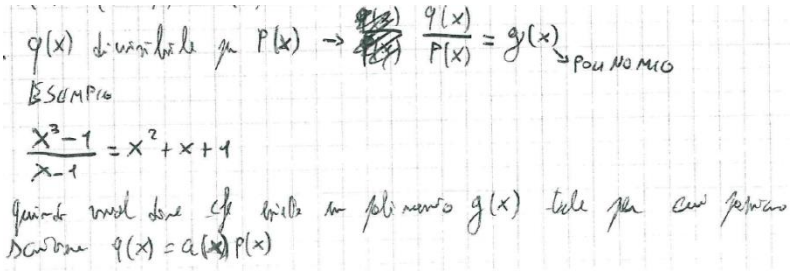

Figure 3: Filippo's notes about the "conceptual task". On the first row, he writes "q(x) divisible by p(x)", then he draws an arrow and writes a ratio, from which he draws another arrow and writes "polynomial". On the second row, he writes "example" and in the third row he writes a formula. In the last row: "hence it means that there exist a polynomial g(x) such that I can write q(x) = a(x)p(x)".

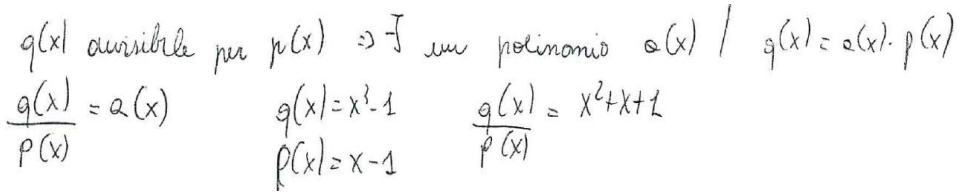

Figure 4: Roberto's notes about the "conceptual task". On the first row, he writes "q(x) divisible by p(x)", then he draws an arrow and writes "a polynomial a(x) | q(x) = a(x)p(x)".

Vincenzo's notes (Figure 1) report exactly what the lecturer wrote on the blackboard. From these notes, we can see that the story has: (A) two mathematical characters, p(x) and q(x) and there's a relationship between them, being the latter divisible by the former; (B) the mathematical action is a division; (C) the moral is that there exists a polynomial a(x) such that q(x) = a(x)p(x). Vincenzo's plot is linear: each row is put below the other one, with no connections. The plot of Angela's story (Figure 2) is also linear: (A) is followed by (B) that is followed by (C). But she also adds an arrow on the right side of (B) and she writes "result", then another arrow and then "polynomial". These

details have been told by the lecturer orally. For Angela, it is worth noticing that the result of the mathematical action is a polynomial (which is a(x), a new character), namely a character that still has the same properties of the other two. While the setting was implicit in Vincenzo's notes, it emerges in Angela's ones: the setting is the set of polynomials. Instead of the mathematical symbol for "there exists", she writes in words and she also adds "hence" at the beginning of (C): the moral is made more explicit and the verbalisation is less condensed compared to Vincenzo. Filippo (Figure 3) organises the plot in a non-linear way: he writes (A) and on the same row he writes (B), to which he draws an arrow and writes "polynomial". Like Angela, Filippo also remarks about this detail. Vincenzo and Angela do not write down the example, while Filippo does. Hence, a story in the story is told: it's the story of the two characters that become two particular polynomials. At the end of the story, Filippo writes (C) in a fashion that is similar to Angela's.

Filippo's story is slightly less linear than the stories re-told by Angela and Vincenzo, but the student that writes a different plot is Roberto (Figure 4): he writes (A), an arrow, then (C) on the first row, namely he puts at the first line the characters and the moral, then on the second row he writes the action, which is (B), and the story in the story, namely the example. We can infer that Roberto is a thinker, since he re-organizes the knowledge, while Vincenzo is a scribbler, since he reports the story in a linear way. Angela and Filippo also act as scribblers: in a sense, we can say that they are more accurate than Vincenzo, since they report more details, but do not re-organise the content of the lesson as Roberto does. Roberto, in fact, does not only remark what is worth noticing, he establishes a hierarchy in the mathematical content: characters and moral on the same, first row, and the action plus the example on the same, second row.

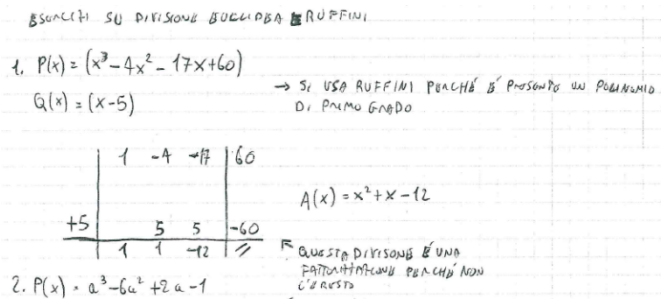

Figure 5: Filippo's notes about the "procedural task". On the first row, he writes "Exercises on euclidean division and Ruffini's division". To the right of the first arrow he writes "Ruffini is used because a first order polynomial is present". To the right of the arrow pointing to -60, he writes "this division is a factorisation, because there's no reminder".

If we look at the "procedural task" in Filippo's notes (Figure 5), we notice that he employs a more linear structure compared to the "conceptual" task. He writes: (0) the title of the story ("Exercises on the Euclidean division and Ruffini"), then (1) he presents the characters P(x) and Q(x), then (2) the series of actions in the Ruffini's grid. The new character, (3) A(x), the result of the actions, is present to the right of the grid. At the right side of the paper he adds comments that are connected to the "story" by means of arrows: such comments better characterise, and justify, the actions that are made. Like in the conceptual case, we see them as details that are worth to be noticed by Filippo. Like in the conceptual case, we can say that Filippo is an accurate scribbler. Vincenzo's notes (Figure 6) also have a linear structure with no connections between (1) and (2), or between (2) and (3). Vincenzo also adds the comment "we use Ruffini when we divide by an order-1 polynomial", but this comment about the actions is put below the characters with no arrow.

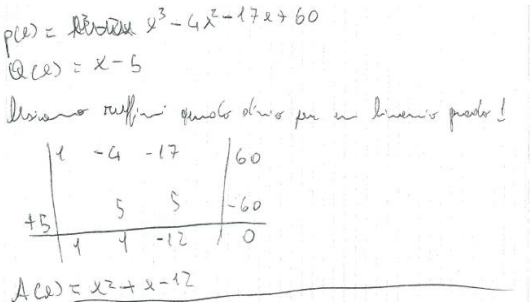

Figure 6: Vincenzo's notes about the "procedural task". The first sentence reads "We use Ruffini when we divide by an order-1 polynomial".

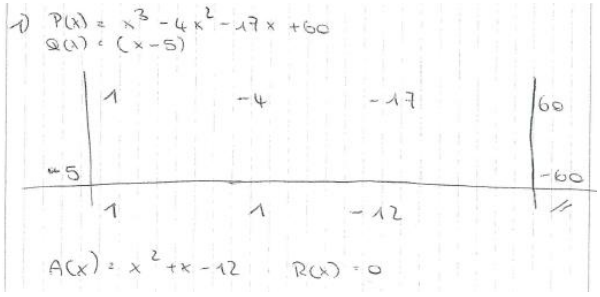

Figure 7: Angela's notes about the "procedural task".

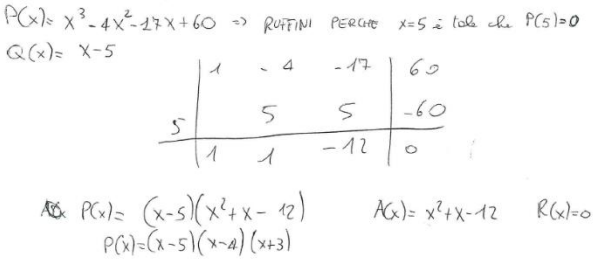

Figure 8: Roberto's notes about the "procedural task".

Angela's notes (Figure 7) reflect exactly what the lecturer wrote on the blackboard. The notes have no words, just symbols: we can say that there's only one register present, the symbolic one. She records the characters, i.e. (1), the actions, i.e. (2), and the new character that results from the action, i.e. (3). Differently from Filippo, whose notes have a rather linear structure, Angela's ones are even more linear and essential, as if she wants to record just the essential facts. Like Vincenzo, Angela is an (inaccurate) scribbler.

Roberto (Figure 8) records (1), then (2), then (5)-(3)-(4) on the same line. To the right of (1) he remarks "Ruffini, because x=5 is such that P(5)=0", hence noticing a detail that is different from the ones recorded by Filippo and Vincenzo and also less general than those: Ruffini's algorithm can be used for any value of x when q(x) is an order-1 polynomial, not only for those polynomials where the value of x is a zero. Since the lecturer has said something different (see Filippo's or Vincenzo's notes), we can infer that Roberto added a detail that generated from his own knowledge about polynomials. As for the conceptual task, Roberto reorganises the space of the sheet and we can infer that he acted as a thinker.

Discussion

I discuss the data analysis in terms of what the distinction between scribblers and thinkers can add to: (1) our knowledge of the students' verbalisation skills (responding to the question which elements of the teacher's "story" are recorded, and which ones are discarded?); (2) our understanding of how the students organise their knowledge (how do students organise their notes?); (3) whether the teaching

strategies have an influence on conceptual and procedural understanding of mathematics (when do the students act as scribblers and when as thinkers?). I would like to underline that I am not valuing "thinker" over "scribbler", on the contrary I am interested in seeing which elements of the lesson provoke either modality in students. Comparing Angela's and Vincenzo's notes, I tend to say that both them act as scribblers in both tasks. I can further infer that Angela is a scribbler because the course is recapitulating mathematical concepts that are familiar for her: she probably does not need to put so many details in her notes. Angela's notes of the procedural task report only what has been written on the blackboard. Looking at her notes for the conceptual task, furthermore, I commented that she had time to record details that are worth to be noticed and she didn't give us the impression that she was rushing to keep the pace of the lecturer. For Vincenzo, it is a completely different story: he records only what has been written on the blackboard during the conceptual task and during the procedural one he added "We use Ruffini when we divide by an order-1 polynomial" to his notes with respect to what has been written on the blackboard. I can see that Vincenzo is struggling to remark all that is relevant, since the lesson is difficult for him. A conclusion that can be drawn from these observations is that a student acts as a scribbler in two cases: either if the mathematical content is too easy for her, or if it is too hard.

Nardi (2011) pointed out that the students under-verbalise and hyper-condense the mathematical discourses. As regards the conceptual task, I can see that Vincenzo and Roberto condense the mathematical content more that Filippo and Angela, but Roberto does it in a completely different way compared to Vincenzo: Roberto compresseseorganises the content, to have the character and the moral on the same row, and the actions plus the example on the second row, while Vincenzo linearly puts the elements of the story one after the other. Andrà (2010) analysed the teaching styles of university lecturers and concluded that in a blackboard modality (namely, when the lecturer is mostly writing on the blackboard) the students have to adjust the pace of their note-taking to the pace of the lecturer's writing. By comparing Vincenzo's and Roberto's notes, indeed, I can imagine the former making an effort in dealing with a pace that is too fast for him, to the point that he does not have time to record the details that Angela and Filippo remarked, while Roberto seems to stop and think (fast) where he wants to put what is told by the teacher. Angela and Filippo can be seen as accurate scribblers, and it seems that they tend not to hyper-condense the math content. As well, Angela and Filippo tend not to under-verbalise when they take notes on the conceptual task. Why are some students more accurate scribblers than others? Andrà (2010) interpreted this difference in terms of each student's ability to keep the pace of the lesson at the blackboard, but I would also add that it depends on the student's views: for some students, it seems necessary to record all the possible details, while for others it seems a question of being brief. Looking closely at Angela's notes, and comparing her notes on the conceptual task and on the procedural one, I can see a difference: in the first case, she adds details and comments that she discards in the second case. Vincenzo does not add comments in neither case, and Filippo accurately adds details in both cases, hence Angela seems to be the student on which the procedural vs conceptual nature of the task provokes different modalities of taking notes, and actually the conceptual nature of the task invites her to remark more details. This seems to have an impact on her verbalisation skills and on conceptual reflection.

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The transition from high school to university mathematics: A multidimensional process

Amalia-Christina Bampili[1], Theodossios Zachariades[1] and Charalampos Sakonidis[2]

[1] National and Kapodistrian University of Athens, Greece; amabab@gmail.com, tzaharia@math.uoa.gr [2] Democritus University of Thrace, Greece; xsakonid@eled.duth.gr

The transition from secondary to tertiary mathematics encompasses a complex interaction of social, academic and mathematical context changes, including a vast array of emotions, beliefs and issues. The present paper reports a study of the difficulties faced by a first year undergraduate student in a Mathematics Department during her transition from secondary to tertiary education through the lenses offered by a rite of passage framework. Data were gathered over the student's first two semesters of attendance predominately through interviews. The results indicate difficulties she faced regarding the mathematical content and a powerful interaction between emotions and the reconstruction of her mathematical thinking.

Keywords: Transition, university mathematics, rite of passage, reconstruction.

Introduction

The secondary-tertiary transition is itself an exciting and often confusing experience for students. After tough examinations, the successful students have yet to adjust to new learning environments, new modes of study, and above all, higher expectations.

The problems encountered in the transition from high school to university mathematics are common in every educational system worldwide. Several researchers identify a "gap" between school and university mathematics content (Luk, 2005; Kajander & Lovric, 2005; Winsløw, 2013), while others identify important changes that affect students during the secondary-tertiary transition. These include the new academic and social environment as well as the shift required to a different mathematical way of thinking and studying (Cherif & Wideen, 1992; Tall, 1992).

The aim of this paper is to study the ways in which a first-year student in the Department of Mathematics at the University of Athens dealt with transition issues through the lens offered by a rite of passage framework, focusing on the ways that changes in the student's social life and the academic environment shaped the reconstruction of the mathematical thinking required.

Literature review

The transition from high school to university mathematics could be seen as an interaction of many transitions: social, academic, mathematical content transitions as well as others (Alcock & Simpson, 2002). University as an institution and university mathematics are encountered as a new world, with a new language and new rules that make the novice student feel like a foreigner (Gueudet, 2008).

With respect to the social dimension, Hernandez-Martinez et al. (2011) considered the social aspects of transition as the most important when entering university. Students argue that the beginning of university life can be a quite scary and nerve-racking phase for many but also an "exciting" personal opportunity to develop in a "better environment". Some recall being quite shy in the beginning but becoming more confident over the first year. Going to college is about "working harder" but also about expanding social life. The change from a structured, parent-disciplined life to a self-disciplined

university life is difficult. First-semester students claim that the change of the education environment, new expectations and unlimited freedom are the biggest problems (Cherif & Wideen, 1992; Clark & Lovric, 2008).

Concerning the academic dimension, students in transition undergo changes requiring an adjustment of learning strategies, time management skills and a shift to more independent studying. They experience changes in teaching and learning styles. They often encounter a higher level of competitiveness among their unknown colleagues (Clark & Lovric, 2008). The new environment demands a different type of critical thinking, something for which students are not necessarily prepared (Cherif & Wideen, 1992).

As far as the mathematical content is concerned, first-year university students often face

> the move to more advanced mathematical thinking [which] involves a difficult transition, from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions (Tall, 1992, p. 1)

Furthermore they are confronted with a significant change from a computational to a proof-based learning and teaching approach. Some concepts learned at high school need to be reconstructed at the tertiary level thus increasing the transition's difficulties. In tertiary mathematics courses students are exposed to the introduction of abstract concepts and formal reasoning; they witness an increased emphasis on the precision and rigor of the mathematical language, and this is very new for them (shock of the new) (Clark & Lovric, 2009). The relevant literature seems to agree that more relational and conceptual understanding as well as more flexibility in solving mathematical problems compared to high school mathematics is expected (Breen, O'Shea, & Pfeiffer, 2013). In other words, a shift from "instrumental understanding" to more "relational understanding" is required.

Theoretical considerations

We employed the rite of passage approach (Clark & Lovric, 2008) to explore the ways in which the subject of the study dealt with transition issues. We considered the transition from high school to university mathematics as a rite of passage, a concept explored in anthropology and in other disciplines (e.g. in cultural studies). French anthropologist Arnold van Gennep (1960) (in Clark & Lovric, 2008) described and analyzed certain events that, in one way or another, create a "crisis" in an individual's life. He observed that these "life crises" (e.g., birth, betrothal, marriage, or death) possess a similar general structure, and based on this, developed a three-stage theory of what he called rites of passage. In the *separation* stage, the person experiencing a crisis gets "removed" from the rest of the community (family, social group, etc.). The process of achieving necessary changes constitutes the *liminal* stage. In the *incorporation* stage, the person learns about the community that she/he will belong to at the end of the rite. With the support of members belonging to the communities involved, she/he is supposed to find her/his place in the new community. Applied to mathematics, the model suggests that one could analyze problems and issues in transition by studying their dynamics within three stages: (a) separation (from high school) which takes place while students are still in high school, and includes anticipation of forthcoming university life; (b) liminal (from high school to university) that includes the end of high school, the time between high school and university, and the start of first year at a university; (c) incorporation (into university) concerning roughly the first year

at a university (Clark & Lovric, 2008). Although Clark & Lovric (2008) suggest applying the rite of passage model with regard to the mathematical content only, we utilize a methodology for revealing the dynamics and the connections within all three dimensions (social, academic and mathematical content).

The study

Situated within the literature reviewed above, the study reported here is part of an ongoing research project aimed to examine the interface between social, academic and mathematical content aspects of the transition from high school to university mathematics. In particular, the research questions pursued in the study were as follows:

1. What was the dynamics exposed in each of the three stages of the rite of passage along the three dimensions (social, academic and mathematical content)?
2. How do academic and social dimensions interact to shape the passage from the liminal to the incorporation phase regarding mathematical content?

Greek students who want to enter University go through hard preparation to pass the exams in their last high school year. During their final high school year, most of the students undergo a strictly structured life program, including many hours of daily study almost always under the guidance of school teachers and private teachers in paid courses after school. They are introduced to Calculus, coming across proofs, the emphasis of teaching being, however, more on computational than conceptual learning/understanding. As first-semester university mathematics students, they are introduced again to Calculus but this time in formal terms, more as Mathematical Analysis. This constitutes a qualitatively big jump for their thinking. Furthermore, there is hardly any support around provided either by the academic staff, in the form of learning advisors, or by higher-years students and/or the Students' Association.

In October 2015 we started surveying incoming first-year students (October 2015-June 2016), collecting information. Twelve students volunteered to be interviewed individually to help us look thoroughly at the issues described above. Four semi-structured interviews (in the beginning of the first semester, before the semester exams, in the middle of the second semester and before the second semester exams) were carried out, each lasting between 25 and 45 minutes; these were audio-recorded and fully transcribed. Students were asked about their conceptions of university mathematics, how their experience of mathematics at school differed from that at the university, how their study habits or ways of working had changed, how they felt being a member of a new academic environment and how they dealt with the changes in their social-personal life.

One of these students, Nefeli (a pseudonym), is the focus of this work. We chose Nefeli because her responses during data collection strongly indicated that she was undergoing a rite of passage regarding mathematics: although she was doing well in mathematics (her grades were good at school and also in the university entry exams, 16/20 on average), in the beginning of her first university year she felt that perhaps it had not been a good decision to study mathematics. She was negatively affected because of the overwhelming changes imposed in her lifestyle and the new academic environment that strongly influenced her studies. She even considered quitting. Only after the first semester exams did she started adapting to the new environment, and at the end of the first year she almost felt well adjusted.

Results

Nefeli's representative comments and thoughts related to transition and expressed in the contexts of the four interviews were organized along three dimensions, social, academic and mathematical content, within each of the three phases of a rite of passage, as presented in Tables 1, 2 and 3. In the following, some central issues emerging along each of these dimensions and across the three phases are discussed.

The social aspects of the transition were seen by Nefeli as among the most important (but also worrying) issues. She highlighted mainly two of them: (a) the home-university distance and (b) her relationships with classmates and friends (Table 1).

Social dimension	**Separation phase**	**Liminal phase**	**Incorporation phase**
(a) the home-university distance	$S_{1:}$"School was near my home". (1st interview)	$S_{2:}$"I am negatively affected because of the long home-university distance". (1st interview) $S_{3:}$ "I manage time better, but I'm still undergoing a total change in my former well organized life". (2nd interview)	$S_{4:}$ "I have the opportunity to manage my time as I want, although not so effectively all the time". (3rd interview)
(b) her relationships with classmates and friends.	$S_{5:}$ "I try also to spend some time with my friends from school and neighborhood which is not easy…they hardly understand that I have to study hard". (2nd interview)	$S_{6:}$ "Some interesting people I have met here helped me to adjust myself to the new environment". (1st interview) $S_{7:}$ "With my classmates I have the feeling that we discuss mostly issues about our studies but in a competitive way". (2nd interview)	$S_{8:}$ "I met some higher-year students who helped me a lot to adjust to the new environment". (3rd interview)

Table 1: Social aspects through the three transition phases

Nefeli experienced big changes in the new academic environment (academic dimension). A vast array of answers is identified in her interview responses: from great expectations for a creative teacher-

student relationship and academic staff support to her statement that some professors do not care at all if students understand their lectures. Two critical features are (a) teacher-student relation and (b) lack of support (Table 2).

Academic dimension	**Separation phase**	**Liminal phase**	**Incorporation phase**
(a) teacher-student relationships	$A_{1:}$ "I had great expectations for a creative teacher-student relationship". (1st interview)	$A_{2:}$ "I couldn't understand what was written on the blackboard". (1st interview) $A_{3:}$ "I believe that professors and students are not close enough.... Professors take it for granted that students understand mathematics. They have many academic expectations from them. I am afraid to ask the professor, if I don't understand something, because he may think that I am stupid". (2nd interview)	$A_{4:}$ "I have to say that some professors guided us well enough…I felt better asking questions and the truth is that I did not receive a negative treatment from the professors". (3rd interview) $A_{5:}$ "I was positively influenced by the guidance of some teachers who inspired me to listen to them". (4th interview)
(b) lack of support	$A_{6:}$ "I have expectations for academic staff support, like in high school". (1st interview)	$A_{7:}$ "I am negatively influenced by the absence of help from the Student Association and the absence of a Student Learning Advisor". (4th interview)	$A_{8:}$ "…my adjustment was getting better after a long time with great mental and spiritual effort...". (4th interview)

Table 2: Academic aspects through the three transition phases

Regarding studying mathematics (mathematical content dimension), Nefeli lost her self-confidence at the beginning. As time went by, she confronted studying mathematics as a challenge: to turn her

disappointment and stress to something powerful and effective. She highlighted two main issues (a) the psychological impact of the "unknown subject" and (b) the new way of studying (Table 3).

Mathematical content dimension	**Separation phase**	**Liminal phase**	**Incorporation phase**
(a) the psychological impact of the "unknown subject"	M_1: "I thought that I was good in mathematics because of my good school grades and because I passed the university entrance exams also achieving good grades". (1st interview)	M_2: "When I started studying university mathematics, I was desperate. I was wondering if I had taken the right decision". (1st interview) M_3: "I am still thinking that maybe it wasn't a good decision to study mathematics. If I could say only one thing that I still struggle with, this is the difficulty of the subject. … I felt I turned my love to mathematics to something sick….". (2nd interview)	M_4: "I am in a position now to say that the more I study mathematics the more I love mathematics and I am happy with my choice". (3rd interview) M_5: "I feel more confident. The exams were less demanding than I expected. I passed the exams with good grades". (4rd interview)
(b) the new way of studying	M_6: "I experienced a big change. In high school, we did not pay much attention to the conceptual understanding. Teachers told us what to study and how". (1st interview)	M_7: "I try to change the way of studying. I try very hard on my own to understand. I try to deepen more in definitions and theorems". (1st interview)	M_8: "…I realized that to do well on the first semester exams, I had to use my "simple" knowledge inductively to solve a problem, rather than knowing many things". (3rd interview)

Table 3: Mathematical content aspects through the three transition phases

The results show the dynamics and the connections identified within all dimensions through the three transition phases. As we follow Nefeli's steps, we can see that in the separation phase she had to deal with her expectations concerning her social and academic life (S_1, S_5, A_1, A_6) and "move away" from her former way of living and studying (M_1, M_6), which is characteristic of this phase. Some of these

changes affected her almost until the end of the first year (for example the lack of studying support). She struggled a lot to achieve necessary changes (a process assigned to the liminal phase), something that also affected her self-confidence as a math student (A_2, A_3, M_2, M_3, M_7). Her great mental and emotional effort as well as the support of some higher year students and the influence of some inspiring professors (S_8, A_8, A_4, A_5) helped her to take the next step. After the first semester exams and more clearly near the end of the first year, it looks like she had also managed to find the necessary way of studying (M_8). Overall it seems that she was close to finding her place within her new community, which is a feature of the incorporation phase (S_4, M_4, M_5). Her success in Calculus I and II exams (8/10 and 10/10) can be seen as a positive outcome of her efforts.

Discussion and conclusions

Regarding our first research question, we found that the rite of passage framework brings out the dynamics of all dimensions. We followed Nefeli passing from one well-defined, established and accepted position in life to another, which is equally well-defined, established and accepted (Clark & Lovric, 2008). Nefeli saw university as an institution and university mathematics as a new world, with a new language and new laws that made her feel like a foreigner (Gueudet, 2008). She experienced a big change in her social and academic life which affected her studies as noted by Hernandez-Martinez et al. (2011). She struggled with the shift from "instrumental understanding" to more rational and conceptual understanding (Breen et al, 2013). As Tall (1992) suggested, in order to achieve the transition, students should adopt a new way of thinking, a prerequisite also acknowledged by Nefeli. Organizing her thoughts and comments within the phases of a rite of passage, we could identify some initial steps of the necessary shift to a new "mathematical self-identity" needed.

Regarding our second research question, the results of our analysis reveals dynamics and connections between all three dimensions (social, academic and mathematical content). To pass from the liminal to the incorporation phase concerning the mathematical content, Nefeli had to feel better in the new academic environment and also try to deal effectively with her social life. For example, she appears to shift from the position that she felt undergoing a total change (in her former well organized -by others- life) during the liminal phase, to finding some positive aspects in her new self-disciplined university life ("I have the opportunity to manage my time as I want" and "I realized I had to use my "simple" knowledge inductively to solve a problem). This is in accordance with Tall's (1991) position that:

> Advanced mathematics, by its very nature, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively. It involves a struggle…and a direct confrontation with inevitable conflicts, which require resolution and reconstruction (p. 252)

We consider that our study constitutes a good starting point for exploring specific transition issues more extensively. A deeper investigation of the interaction between different aspects of transition from high school to university mathematics is needed. The analysis of other students' interviews indicates that the rite of passage lens allows for critical social and academic aspects shaping the passage to the new 'mathematical world' to be identified. Overall we view studying university mathematics as a multidimensional process requiring the reconstruction of mathematical thinking.

Fulfilling this reconstruction demands a repositioning of the student considering the new social and academic community. To this end, the institution should systematically offer students' support, and in a well-organized manner, since the lack of which, as the results indicated, might affect students' self-confidence and successful adjustment to the new environment.

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"Points", "slopes" and "derivatives": Substantiations of narratives about tangent line in university mathematics students' discourses

Irene Biza

University of East Anglia, Norwich, UK; i.biza@uea.ac.uk

This paper reports from a study on first year university mathematics students' meaning making of tangent line, especially in their transition between mathematical contexts: algebraic, geometrical and analytical. The analysis draws on the commognitive approach (Sfard, 2008) in order to identify characteristics of responses to a questionnaire in which 182 students were asked to explain in simple words the tangent line, describe its properties, provide its definition and identify if a drawn line is a tangent of a given curve. Findings suggest that students engage with analytical, geometrical and algebraic discourses in their substantiations about tangents, sometimes by engaging with more than one discourse in the same response or/and across different responses in the same script.

Keywords: Tangent line, mathematical discourse, derivative, narratives, routines.

Introduction

Research reports students' difficulties with their meaning making of the tangent line to a function graph. These difficulties have been attributed to students' encounter with the tangent line in different mathematical contexts (e.g. Euclidean geometry, analytic geometry or analysis), disconnection between algebraic or analytical approaches (e.g. rate of change, slope, derivative or tangent line formula) and graphical approaches (e.g. limiting position of secants, visualisation of tangents) and differences between a global perspective (relation of the line and curve as a whole) and the local perspective (relation of the line to the curve at a specific point). Most challenging cases are: when the tangent has more than one common point with the graph (e.g. $f(x)=\sin x$ at $\pi/4$) or coincides with the graph or a part of it (e.g. when the curve is a straight line); tangency at inflection points (e.g. $f(x)=x^3$ at 0); and, points in which the limit of the difference quotient from the left and the right are different real numbers (e.g. $f(x)=|x|$ at 0) or infinity (e.g. $f(x)=\sqrt{|x|}$ at 0) (Biza, Christou & Zachariades, 2008; Castela, 1995; Park, 2015; Vinner, 1991).

In this paper, I draw on my previous research on students' perspectives about tangent line (Biza et al., 2008; Biza & Zachariades, 2010) by analysing not only what lines students recognise as tangent or not, but also by considering how they justify their choices. The conjecture I examine here is that students use a range of arguments from geometry, algebra and analysis to justify their choices that go beyond the correctness or not of these choices. With this analysis, my objective is to gain insight into how students make meaning of mathematical objects – in the case of this paper tangents – through their communication about them. To this aim, I analyse first year university mathematics students' responses to a questionnaire about tangent line by drawing on the commognitive approach (Sfard, 2008). In what follows, I introduce the main tenets of the commognitive approach that I employed in the analysis, and the methodology of the study. Then, I present preliminary findings from the analysis and I discuss them also in relation to potential implications for teaching.

Theoretical underpinnings of the study

According to the commognitive approach (Sfard, 2008) communication about mathematics in written or verbal responses is not a *window* to thinking but an inseparable part of this thinking that makes sense only in the context in which this communication takes place. A mathematical discourse is defined by four characteristics: *word use*, *visual mediators*, *narratives* and *routines*. *Word use* includes the use of mathematical terms (e.g., in the context of this study, 'tangent', 'derivative' or 'direction coefficient') as well as everyday words with a specific meaning within mathematics (such as 'touch', 'region' or 'point'). *Visual mediators* include mediators of mathematical meaning (e.g., function graphs, diagrams, geometrical figures or symbols) as well as physical objects. *Narratives* include texts, written or spoken, which describe objects and processes as well as relationships among those (e.g., definitions, theorems or proofs), and are subject to endorsement, modification or rejection according to rules defined by the community (e.g., 'a tangent line is a line that has one common point with a curve' is an endorsed narrative for tangents in Euclidean geometry but not in analysis). *Routines* include regularly employed and well-defined practices that are used in distinct, characteristic ways by the community (such as defining, conjecturing, proving, estimating, generalising and abstracting). Sfard elaborates three kinds of routines: *deeds*, *explorations* and *rituals* where explorations are categorised into *substantiations*, *recall* or *constructions* (ibid, pp. 223–245). Recently, there has been increasing interest in discursive approaches and, especially, in university mathematics teaching and learning research, discursive approaches are gaining more momentum (Nardi, Ryve, Stadler, & Viirman, 2014) in the investigation of university teachers' discursive practices (e.g. Park 2015; Viirman, 2015) and student learning (e.g. Güçler, 2016).

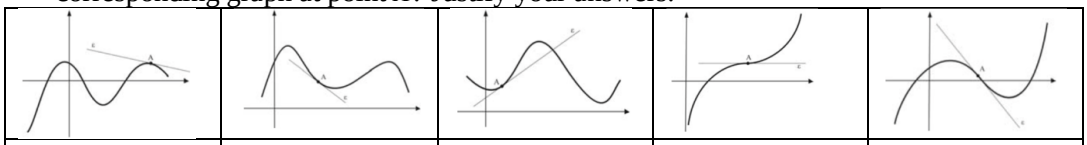
In this study, students' responses to a questionnaire are seen as acts of communication and thus part of their meaning making about tangent lines. Mathematical routines such as investigating if a line is a tangent (see question Q3 in Figure 1) can be explorations that include recall of previously endorsed narratives, substantiations of narratives about *why* a line is (or is not) tangent or constructions of new objects such as formulae and graphs. However, there are differences in the mathematical discourses about tangency in analysis, geometry and algebra. For example, in Euclidean geometry, whether a line is tangent or not depends on the number of common points and the relative position between the line and the curve (geometrical routines) because a tangent line to a circle has one common point and keeps the circle to one side (geometrical narratives). In analysis, tangency is checked locally (analytical routine) and is defined by the derivative at a point (analytical endorsed narratives) which is the slope of the line (algebraic narrative). In algebra, the tangent line will be justified through calculations (algebraic routine) of the slope and defined through its equation or the vector that gives its direction (algebraic narratives). Identifying how students' responses to a questionnaire engage with these discourses is the focus of this paper.

Methodology

Data reported in this paper were collected from a questionnaire administered to 182 first year university students (97 female) from mathematics departments in Greek universities. All participants had been taught about the tangent line in Euclidean and analytic geometry, and in elementary analysis courses in Years 10, 11 and 12, but not yet at university as the questionnaire was administered at the beginning of their first year. The questionnaire included tasks (see a sample of questions from the questionnaire in Figure 1) in which the students were asked to explain in their

own words the tangent line (Q1); to describe properties of it (Q2); to identify if a drawn line is a tangent line of a given curve (Q3); to construct the tangent line, if it exists, of a given curve through a specific point on the curve or outside the graph (Q4 and Q5); to provide definitions (Q6), to write the formula, and to apply the formula in specific cases (Q7 and Q8). In questions Q3, Q4 and Q5 only the graph was provided and no formula of the corresponding curve was given; students were asked to identify or construct the tangents based on the graphs and justify their choices. The proposed curves were chosen to reflect students' common difficulties with tangent lines identified by previous research (Biza et al., 2008; Castela, 1995; Vinner, 1991). For example, the corresponding line: had more than one common point with the curve (e.g., in Figure 1, Q3.b and Q3.c in comparison to Q3.a challenge the geometrical routine of checking the number of common points and the relative position between the line and the curve) or passed through an inflection point (e.g., in Figure 1, Q3.d and Q3.e challenge the geometrical routine of the relative position between the line and the curve) – for more about the questionnaire design see Biza et al. (2008).

Q1: Explain, in simple words, what you are thinking when you hear the term "tangent line".
Q2: Write as many properties as you can think of about the relationship between a curve and its tangent line at a point A.
Q3: Which of the lines that are drawn in the following figures are tangent lines of the corresponding graph at point A? Justify your answers.



Q3.a	Q3.b	Q3.c	Q3.d	Q3.e

Q6: What is the definition of the tangent line of a function graph at its point A?

Figure 1: Questionnaire sample

In earlier analysis (Biza & Zachariades, 2010), student choices in questions Q3, Q4, Q5, Q7 and Q8 were characterised according to their correctness and analysed quantitatively. This analysis suggested a classification of students regarding their perspectives on tangent line and its relation with the corresponding curve into three groups with *analytical local* perspectives (closer to the tangent line in the context of analysis – 25.8%); *geometrical global* perspectives (more relevant to the tangent line in the context of geometry – 17.6%); *intermediate* perspectives between the analytical local and the geometrical global perspectives (56.6%). Although this classification indicated a spectrum of students' perspectives about tangency, it does not grasp the subtlety of these perspectives as they were evident in students' choices and justifications of these choices. To this aim, student responses to questions Q1, Q2 and Q6 and their justifications in questions Q3, Q4 and Q5 were analysed qualitatively. Part of this analysis focuses on the mathematical discourses students engaged in in their responses (analytical, geometrical and algebraic) with specific emphasis on the words used, routines and narratives, substantiation of these narratives, how this discourse is related to their choices (correct or not) in the questionnaire and the consistency of student responses across the questionnaire. This paper discusses preliminary findings from the 182 student responses to the items: Q1, Q2, Q3a-e and Q6 presented in Figure 1.

Student justifications on why the sketched line is a tangent or not

Justifications students offered in order to accept or reject a tangent line when it does not have any other common point and keeps the graph at the same semi-plane (Q3.a) or when it has other common points sketched (Q3.c) or not (Q3.b) are summarised in Table 1.

Justification	Script Example
Rejection of the line as a tangent	
Common points between line and curve, global view	"No [it is not a tangent], the line has 2 points in common with the function graph"
Relation of the line and the curve, global view	"No it is not a tangent, although it touches[1] the function graph at the point *A*, it cuts [the graph] at another point"
Relative position of the line and the curve, global view	"[The line] splits the curve in two semi-planes"
Only local acceptance of the tangency	"Not [a tangent] in general […] in a small interval (δ>0 (x-δ, x+δ) it is [tangent]"
Derivative / differentiability	"Although the function is differentiable at *A* and thus it has a tangent, the extension of the [line] ε that goes through *A* has another common point with the function and as a result it is not a tangent"
Acceptance of the line as a tangent	
Common points between line and curve, local view	"[It is a tangent, b]ecause if we consider a small region (κ, γ) around the point *A* where [the line] ε is tangent we can see that [the line] ε does not touch any other point"
Relation of the line and the curve, local view	"Yes [it is tangent] because it touches exactly at [the point] *A* and it does not cut it [the graph]"
Relative position of the line and the curve, local view	"The part of the function graph which is close to the point *A* is located at the same side of the line ε"
Common points between line and curve, global view	"It [the line] has one common point with the curve"
Relative position of the line and the curve, global view	"$f(x)$>(ε)"
Slope of the line	"Yes, the line ε is tangent at *A*, the slope equals to the direction coefficient of the line"[2]
Derivative / differentiability	"It is [tangent] because it has slope [equals to] the derivative of the function at this point"
Opposite rays	"The rays ε_1, ε_2 which are tangents at *A* are opposite"
Other	"There is only one tangent at the point *A*" or "There is a limit which is the same from the left and the right side or "ε: it is tangent, the point *A* is defined and belongs to the domain of the graph"

Table 1: Student choice justifications to questions Q3.a, Q3.b and Q3.c

[1] Data have been translated from Greek to English. In Greek, the noun *tangent [line]* (*εφαπτομένη [ευθεία]*) and verbs such as *being tangent, abut, touch* (*εφάπτεται*) have the same origin. In Greek, the excerpt: "No, it is not a tangent because although it touches the graph …" sounds contradictory ("Όχι, δεν είναι εφαπτομένη γιατί αν και εφάπτεται στη γρ. παράσταση …"), one explanation is that the noun "tangent" draws on the mathematical discourse, whereas the verb "touches" draws on the everyday discourse.

[2] In the Greek curriculum, the "direction coefficient" is the coefficient *m* in $y=mx+b$, that indicates the slope of a line.

Justifications students offered in order to accept or reject a tangent line when the tangency point is an inflection point (Q3.d and Q3.e) are summarised in Table 2.

Justification	Script Example
Rejection of the line as a tangent	
Common points between line and curve, global view	"No [it is not a tangent], because the curve and the line cut each other in several points"
Relation of the line and the curve, global view	"It is not [a tangent] because [the line] penetrates the curve"
Relative position of the line and the curve, global or local view	"It [the line] intersects the function graph by going to its both sides"
Inflection point / concavity change	"It [the line] is not tangent because *A* is inflection point"
Change of function formula	"Because the formula of the graph changes"
Derivative / differentiability	"The graph does not have tangent at [the point] *A* because the graph is an image of a function *f* and $A(x_0,f(x_0))$, $f''(x)=\kappa$ for $x<x_0$ and $f''(x)=\lambda$ for $x>x_0$ $\kappa\neq\lambda$ close to x_0"
Solution of the corresponding system of simultaneous equations (line and curve)	"[The line] ε is not tangent because the system line – curve has one solution and not a double solution"
Other	"more than 1 lines can be sketched through point *A* with at least one common point with the graph" or "if we consider the figure as two figures with *A* as the unique common point the line ε is a tangent of the two figures. If we consider the figure as a whole the [line] is not a tangent of this figure»
Acceptance of the line as a tangent	
Common points between line and curve, local view	"The ε has only point in common with C_f in the region $(x_A-\kappa, x_A+\kappa)$, $\kappa>0$ and very small"
Common points between line and curve, global view	"The line ε is tangent because it has one common point with the curve and the concavity of the graph changes at this point" (these participants rejected the line when it had more than one points in common)
Opposite rays	"It is [tangent] but for the right part of the function after *A*, tangent is the right part of the tangent and respectively for the left [part]"
Slope of the line	"Yes, the line ε is tangent at *A*, the slope equals to the direction coefficient of the line"
Derivative / differentiability	"derivative equals to the slope of the tangent"
Inflection point / concavity change	"It is [tangent] and the [point] *A* is inflection point"
Other	"The ε is tangent at the point *A* especially internal"

Table 2: Student choice justifications to questions Q3.d and Q3.e

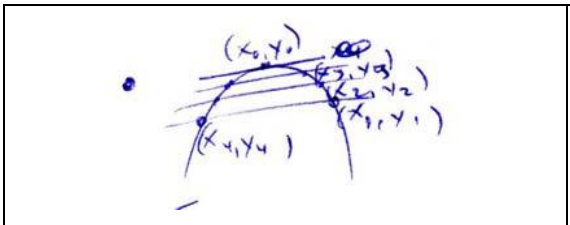
In both set of questions students engage with analytical, geometrical or algebraic discourses. They use narratives such as derivative, differentiability, intervals, regions close to the tangency point, inflection point or concavity (analytical discourse); common points, relative position of curves, same-plane or ray (geometrical discourse); and, slope or system of simultaneous equations (algebraic discourse). Routines include checking for common points or for the relative position between line and curve (geometrical discourse) or for derivatives (analytical discourse) or slopes (algebraic discourse). Routines are applied locally around the point *A* (analytical discourse) or globally for the whole figure (geometrical discourse). Indicatively, of the justifications offered in Q3.c, 83.3% were geometrical (either global or local); 12.1% analytical; and, 4.6% a mixture of analytical and geometrical/algebraic. Whereas, in Q3.d, 70% were geometrical (either global or local); 2% algebraic; 22% analytical; and, 6% a mixture of geometrical and analytic/algebraic.

Similarly, the word use includes verbal descriptions as well as terms and symbols from geometry, analysis and algebra. The relation of the line and the curve especially at the point *A* are described in a range of ways, not necessarily with consistent (in terms of the different discourses) meaning. For example, in questions Q3.d and Q3.e where point *A* is an inflection point the line *intersects* (*τέμνει*); *pierces* (*τρυπάει*); *cuts* (*κόβει*); *crosses* (*διαπερνά*) or *bisects* (*διχοτομεί*). The same word can be used with a range of meanings in different scripts or across the same script. For example, "intersection" may mean the common point regardless of the position of the curve in relation to the line, whereas in other cases (in the same or different scripts) it means the split of the curve into parts. The analysis aimed to identify evidence not only regarding the common points but also regarding the overall relation of the line and the curve and their relative position. These subtle differences are not always evident in student responses.

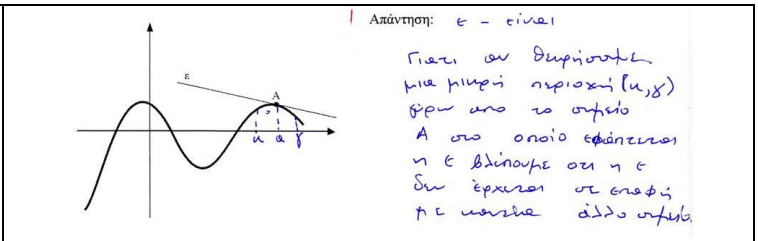
Furthermore, the analysis indicated the use of endorsed narratives from more than one discourse in the same response or/and across responses within the same script. For example, student S[149] who performed well in all questionnaire items, writes in Q2:

> $f'(x_A)=\lambda$ the direction coefficient. $f'(x_A)=\lim_{x\to x_A}\frac{f(x)-f(x_A)}{x-x_A}$ [analytical narrative]. At this point it [the line] has one "double" [his emphasis] common point with C_f [algebraic narrative]. It can have other common points with C_f, $x\neq x_A$ [geometrical narrative]

and he sketches the graph in Figure 2a. In Q3.b and Q3.c he accepts the line because "it satisfies all the conditions" and in Q3.d he writes: "The [line] ε is [tangent] because f is differentiable at x_A and ε has one (double) common point with C_f in the region $(x_A-\kappa, x_A+\kappa)$, $\kappa>0$ and very small" [a mixture of analytical and algebraic endorsed narratives applied locally].



2a: S[149]'s response to Q2



2b: S[123]'s response to Q3.a

Figure 2: Student responses

Student S[123], on the other hand, who had difficulties with accepting tangency at an inflection point writes in question Q1: "The tangent line is a line that touches a graph at a point *A* and in a small area around it [the point] it does not intersect the graph" [geometrical narratives applied locally]. Then, in question Q2 he writes: "The slope of the tangent at the point $A=(x_0, f(x_0))$ is equal to $f'(x_0)$" [analytical narrative]. In Q3.a and Q3.b he accepts the line and justifies the choice: "Because if we consider a small interval (κ, γ) around the point *A* where [the line] ε is tangent we can see that [the line] ε does not touch any other point" (Figure 2b) [geometrical narratives applied locally]. In Q3.d he responds "ε – it is not [tangent] because it bisects f" [geometrical narrative].

Another student (S[261]), who also had problems with tangency at inflection points, writes in Q1: "Gradient (λ), the tangency point, the formula of the line, $f(x)=\lambda=(y_2-y_1)/(x_2-x_1)$, $M(x,y)$, $y_2-y_1=f(x)(x_2-x_1)$" [algebraic narratives]. Then in Q2 she responds: "The [point] *A* is a tangency point

and belongs to the figure. It satisfies the equation of the tangent as well as of the figure. It is $\lambda=\tan\omega$ (the angle between the figure and x'x)" [algebraic narratives]. In Q3.a and Q3.b she accepts the line and justifies the choice: "(ε) is tangent at *A* [the line is] at the same side of the graph" [geometrical narratives applied globally]. In Q3.c she writes "(ε) is tangent at *A* only. There [the line is] at the same side of the graph" [geometrical narratives applied locally]. I have highlighted "there" in her response as an indication of the focus at the area around point *A*. In question Q3.d she responds: "(ε) not tangent. It intersects with the graph going through both its sides" [geometrical narratives applied locally]. Finally, in Q6 she responds by using mainly geometrical narratives with symbolisation from analysis:

> The tangent line of a function graph at a tangency point $A=(x_0, f(x_0)$, that belongs to the function and the tangent, is a line that intersects the function without going from its one side to the other but remains at the same side of the function with only one common point the [point] *A*. [her emphasis]

Discussion

This paper reports on my first attempt to draw on the commognitive approach to analyse 182 first year university mathematics students' justifications about tangent line. My initial conjecture was that students engage with different discourses (geometrical, algebraic or analytical) even for the substantiation of similar choices regarding the tangent line. Although the findings presented in this paper cover only a small slice of the data, in terms of questionnaire items, I would say that there is evidence supporting my conjecture. Students engage with analytical, geometrical or algebraic discourses in terms of the endorsed narratives and routines I identified in their responses. Also, the word use includes verbal descriptions as well as terms and symbols from geometry, analysis and algebra. Additionally, the same justification may engage with more than one discourse or/and different discourses across the script. Also, it seems that in several responses there are arguments that use analytical endorsed narratives (derivative etc.) applied through geometrical routines (check the tangency globally). I note here that my analysis considers students' responses in relation to the discourses of the mathematical community in different mathematical areas and not in relation to their correctness. Use of analytical narratives, for example, do not necessarily 'secure' the correctness of the response and, the other way around, a correct choice does not necessarily draw on a coherent and consistent justification. Furthermore, the type of the task may also affect the type of discourse students engage in. For example, a graphical question (e.g. sketch the tangent line) may trigger geometrical discourses whereas an algebraic question (e.g. calculate the formula) may trigger analytical or algebraic ones. The reported findings are indicative and further investigation is in process also in relation to inconsistencies within a response with potential commognitive conflicts between geometric, algebraic and analytical discourses of tangent lines as well as to the extent student responses are mediated by the task formation.

This work contributes to our insight into what students bring with them when they join post-secondary mathematics courses and I credit to the commognitive approach the deepening of this insight in the case of students' meaning making of tangency. I envisage teaching implications of the outcomes of this analysis in calculus or analysis introductory courses. For example, the observed mismatch between lecturers' and students' discourses (Park, 2015) would be dealt by explicitly addressing commognitive conflicts with the use of appropriately selected examples – see (Biza,

2011) about the role of examples in student meaning making of tangency.

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Teaching didactics to lecturers: A challenging field

Ignasi Florensa[1], Marianna Bosch[2], Josep Gascón[3] and Noemí Ruiz-Munzon[4]

[1]Escola Universitaria Salesiana de Sarrià – Univ. Autònoma de Barcelona, Spain; iflorensa@euss.es

[2]IQS School of Management, Univ. Ramon Llull, Spain; marianna.bosch@iqs.url.edu

[3]Departament de Matemàtiques, Univ. Autònoma de Barcelona, Spain; gascon@mat.uab.cat

[4]TecnoCampus Mataró-Maresme, Univ. Pompeu Fabra, Spain; nruiz@tecnocampus.cat

Lecturers' professional activity is, at least, twofold: research and teaching. However, their professional development is generally mostly based on research achievements and little effort is made to empower lecturers overcome the difficulties experienced during their teaching activities. We postulate that didactics of mathematics can be a powerful tool to help lecturers question and reorganize the knowledge to be taught, and to make them aware of the conditions enabling and the constraints hindering new modalities of teaching mathematics, more based on its use as a modelling tool to approach open questions. We present in this paper a first edition of a professional development course for lecturers designed for and experienced in an Engineering School in Barcelona. The results obtained are then used for a subsequent course redesign to be conducted with lecturers of a similar university school.

Keywords: Lecturer education, mathematical modelling, Anthropological Theory of the Didactic

Introduction

Traditionally, lecturers' development courses have not been considered relevant by research in teacher education. This is a normal phenomenon considering universities' criteria when hiring lecturers and evaluating those already lecturing: mainly research activities and merits are considered. In contrast, lecturers' didactic or pedagogical education is usually ignored or, at most, considered as a positive complement. The absence of regular lecturers' teaching training is a worldwide phenomenon with few – and not always successful – exceptions. In the United Kingdom, the Higher Education Academy (HEA), the UK Professional Standards Framework (UKPSF) and its accreditation process made a first attempt to incorporate lecturer training as a requirement to teach in UK universities (Department for Education and Skills, 2003). Nevertheless, this program that was thought to be central in lecturers' professional development has finished as a volunteer training and accreditation scheme for both individuals and institutions involved in teaching at higher education (The Higher Education Academy, 2011).

We consider that, as long as their activity has a clear twofold character based on research and teaching, in addition to the traditional training in research (Master's Degree and PhD program), lecturers also need an explicit pedagogical and didactic education. In fact, universities are among the sole existing teaching institutions where teachers are not required an explicit training course on teaching and learning processes. We consider that this crucial difference should not be accepted as a given: the conditions of existence of a university teacher education course have to be studied, especially with the possibility to base it on contents emerging from research in didactics.

In order to have a first set of empirical data to evaluate the conditions of existence of such a course for lecturers at university level we designed a course for 14 lecturers of an Engineering School in

Barcelona (www.euss.es). Lecturers participating in the course teach Analysis (3), Strength of Materials (4), Physics (2), Electronical Technology (2) and Informatics (2). We took as starting point the frame of "study and research paths for teacher education" (SRP-TE) based on recent investigations in the Anthropological Theory of the Didactic (ATD) for pre-service and in-service secondary teachers. The lecturers' course was experienced in February 2016. We present the design principles and results of this first edition, as well as the subsequent re-design for new editions, to overcome the experienced difficulties and take advantage of its potential strengths.

University teacher education: A field to be explored in ATD

Courses for pre- and in-service lecturer professional development are an unexplored field in research. There exists very little literature regarding this subject and the few experiences reported involve only general pedagogical contents not taking into account the very nature of the knowledge involved in the teaching and learning processes. It is important to highlight that no paper on this field was presented at the last CERME9 (neither at TWG 14, *University Mathematics Education*; or at TWGs 18, 19 and 20, *Teachers' Knowledge, Practices and Education*), or at groups regarding teacher training or university teaching at the last ICME 13, except for a preliminary version of this paper (Florensa, Bosch, & Gascón, 2016b). The structure of ICME13 Topic Study Groups about teacher education is especially revealing at this respect: there were four groups on teacher education, two (in and pre-service) centered on the elementary level and two on the secondary level, but none on the tertiary level. At the recent conferences on Mathematics Education in North America, only Ellis presented research on teacher assistants training (Ellis, 2014a, 2014b).

Regarding the presence of papers in journals about lecturers' education we have found very little production: only two papers (Guasch, Alvarez, & Espasa, 2010; Postareff, Lindblom-Ylänne, & Nevgi, 2008) and the *Handbook on Teaching and Learning in Higher Education* (Fry, Ketteridge & Marshall, 1999). We have developed a research from the initial year of publication to the end of 2015 in these journals: *Educational Studies in Mathematics, Higher Education, Journal of Mathematics Teacher Education, Mathematical Thinking and Learning, Journal of Teacher Education, Recherches en Didactique des Mathématiques, REDIMAT, RELIME*.

As said before, we consider that research in didactics can be used as the basis for courses on lecturer education regarding teaching and learning processes. Our starting hypothesis is that results emerging form secondary teacher education can be used at this level. Our results will be used to partially confirm this assumption. The *Solid Findings in Mathematics Education on Teacher Knowledge* (Education Committee of the EMS, 2012) state explicitly that "content knowledge" (CK) is necessary but not sufficient for teaching. The report of the Education Committee highlights as crucial notions to be developed in teacher education the "pedagogical content knowledge" (PCK) (Shulman, 1987) and the different dimensions of the "mathematical knowledge for teaching" (MKT) (Ball, Thames, & Phelps, 2008). Both approaches clearly go further than the traditional conception of teaching as transmission of knowledge and consequently ask for changes in teacher education concerning the way mathematical knowledge should be approached.

We use the Anthropological Theory of the Didactic (ATD) as a main framework for the design, experience and analysis of the course. The last investigations on teacher education in ATD show that the use of notions such as PCK and MKT do not ensure researchers/educators to include a

questioning of the nature, selection and organization of the contents to be taught (Ruiz-Olarría, 2015). Under the ATD approach, the role of teacher education is not limited to enrich teachers' pedagogical performance, but also to provide them with tools to contest the so-called *dominant epistemology* and emancipate from it when designing study processes (Gascón, 2014).

This questioning and reorganization of the knowledge to be taught is not spontaneous for teachers (nor for lecturers) because they tend to assume the institutional dominant epistemology as their own. The way proposed by ATD research to locate it at the core of teacher educational processes has very much evolved in this last decade. It started with a first experience in secondary teacher education based on the "questions of the week" (Cirade, 2006) and nowadays takes the form of an inquiry-based device called "study and research path for teacher education" (SRP-TE), which starts from a problematic question appearing in the field of the teacher profession and leads to the search, development and analysis of alternative teaching proposals (Barquero, Bosch, & Romo, 2015, 2016). The main idea of the SRP-TE is to generate a practical and theoretical questioning of the school activities linked to the teacher professional initial question. It is structured in five modules:

- M0: Formulation and first exploration of the generating question Q_0 of the SRP-TE, for instance one of the kind: "How to teach (a specific content)?" which is to be partially answered at the end of the process.
- M1: Living a "study and research path" (SRP) as a student. The main goal is to make teachers encounter an unfamiliar inquiry-based activity related to Q_0 that could exist in a normal classroom of the considered educational level.
- M2: Adaptation of the lived SRP to be experienced in a real school situation. During this adaptation, many of the institutional restrictions teachers should face are expected to show up. They can thus be afterwards analyzed from an epistemological, didactic and *ecological* perspective (what can "live" and under what conditions in a given educational setting).
- M3: Experimentation, management and carrying out of in vivo and a posteriori analyses of the adapted teaching proposal.
- M4: Joint elaboration of a critical analysis of traditional teaching practices and the possibilities (and limitations) of introducing new proposals, as well as generation of a partial answer to Q_0.

During the development of SRPs-TE for secondary school teachers, an epistemological tool has been adapted and developed to facilitate the analysis of the SRP and the questioning of school contents: what we call "question-answer maps". Following other authors, we consider these maps, which are used as a key tool in ATD research, as a powerful instrument for teacher education:

> We hypothesize that such a representation is sufficiently close to teachers' concerns, and also captures such essential parts of a didactic design, that one could use it as a tool for collaboration and communication with and among teachers, regarding concrete teaching designs (Winsløw, Matheron, & Mercier, 2013, p. 281)

Some preliminary and promising experiences exist in using these maps in teacher training courses to describe the dynamic and collective aspects of mathematical activity (Barquero, Bosch, & Romo, 2016; Florensa, Bosch, & Gascón, 2016a; Jessen, 2014). The work with the maps seem to be useful for teachers in order to describe knowledge in inquiry activities and to act as a counterpoint of the official curricular organization of contents.

Research questions

The work presented in this paper is considered as an exploratory design (Singh, 2007) to obtain and analyze a first set of data from the first implemented course and to redesign it to be applied in another institution. The specific research questions that will be studied are:

RQ1. The role played by question-answer maps in teacher education: Do they help lecturers describe, analyze and design inquiry and modelling processes and the involved knowledge?

RQ2. Does the course empower lecturers to identify the dynamic and collective nature of the lived SRP in contrast to the static, individual and compartmentalized dominant conception of knowledge?

Course description

The engineering school where the course was implemented keeps a four-hour time slope with no teaching for all lecturers all Wednesdays: they use this time for professional development, meetings, pedagogical courses or activities. In fact, it is a Salesian university with a special concern about teaching and learning processes, as well as students' personal evolution. The course was structured in six two-hour sessions during three weeks, and the central question to be partially answered was: "Could modelling be the main motivation of my subject? Which conditions enable and which constraints hinder this modelling activity?"

Because of the time restriction, the five-module structure of the SRP-TE had to be adapted. The six sessions appeared to us (designers and course leaders) as a short course. However, they finally seemed to be enough for the work planned. Of course the *true work* is to be carried out afterwards, when lecturers decide to introduce some new proposals in their subject based in the work initiated at the course. During this application phase teachers implementing SRPs asked for help to the researchers-educators, thus extending the real duration. We planned the course as follows:

- 1st session: Explicitly state the professional question Q_0 and shortly present the ATD framework including the notions of praxeology, Herbartian schema and media-milieu dialectics, topogenesis, mesogenesis and chronogenesis (Barquero & Bosch, 2013). They seemed to be well understood and some of them were mobilized during the 4th session.
- 2nd and 3rd sessions: A SRP was proposed to be carried out in groups of up to three lecturers. "Taking into account the incidence index of the last 9 months of a dengue outbreak: could you forecast the incidence index for the next 3 months (already known)?" (Figure 1)
- 4th session: Lecturers generated a question-answer map of the lived SRP including aspects such as media-milieu dialectics. One of the generated maps can be seen in Figure 2.
- 5th session: Lecturers are invited to create new small groups with the colleagues teaching the same subject. They are asked to design a SRP by choosing a generating question in their field trying to overcome some observed didactic facts such as the absence of raison d'être, the disconnections of topics or the poverty of the experimental work, among others.
- 6th session: Sharing some possible teaching proposals and conclusions of the course.

Year	Incidence index
0	31
1	179
2	438
3	454
4	587
5	1176
6	1543
7	1859
8	2373
9	696
10	0,1
11	0,05
12	0

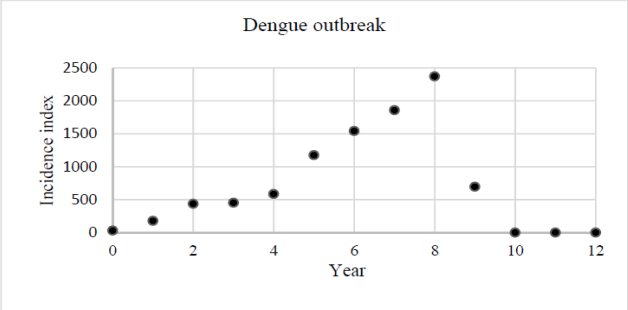

Figure 1: Data used for the lived SRP

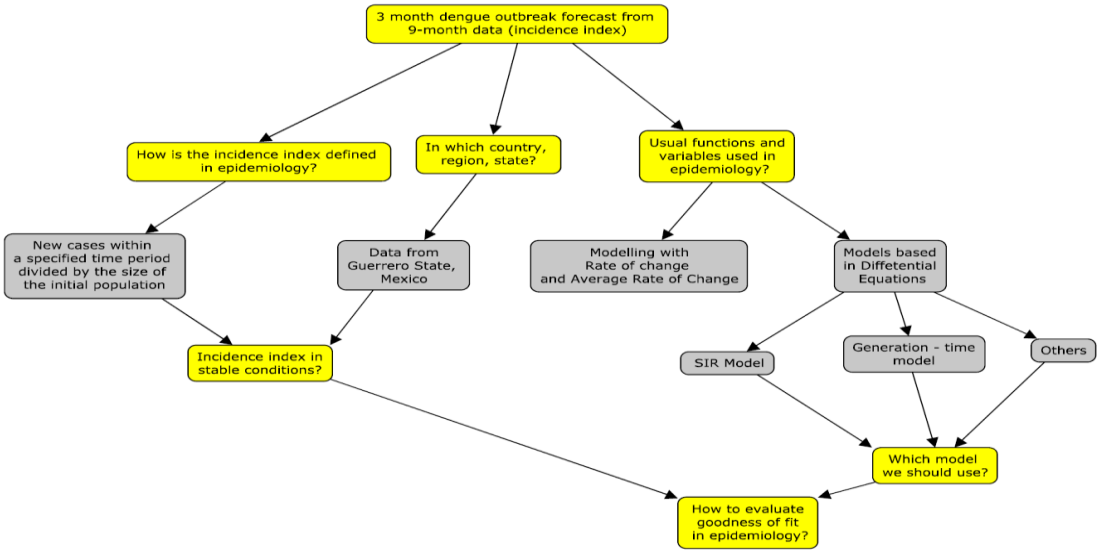

Figure 2: Question – answer map of one of the groups

In the introduction to the 5th session, lecturers were invited to identify didactic facts that they would like to overcome through the new didactic proposal. The goal was not to implement the inquiry by itself, but to identify how the *dominant epistemology* in the institution is related to these problematic phenomena and roughly propose new possible epistemological and didactic organizations to face them. The question-answer maps were the tool provided to lecturers to carry out this work. During the implementation of the course, some of the contents that we initially considered as difficult had an easier reception than expected (especially the notion of media-milieu dialectics) and, on the contrary, some basic notions were difficult to share with the participants, for instance the description of contents in terms of questions instead of topics.

In order to obtain data to evaluate the course, all the questions-answer maps of all groups, both from the analysis of the modelling lived activity and from the a priori design of the SRP, were collected. We have also obtained data from a final survey filled in by all lecturers attending the course. The survey was structured in three main blocks. The first block addressed general aspects of the course such as duration, balance between individual and team work, time structure, etc. The second block asked about content-related aspects of the course like the work developed with question-answer

maps and with the media-milieu dialectics. Finally, the survey asked the lecturers about the possible consequences of the course on their teaching activities: changes in the conception of knowledge, dynamics and collective aspects of activities, and availability of new designing and evaluating tools.

Results and discussion

The question-answer maps regarding the dengue outbreak SRP shew up how the inquiry was capable to connect fields usually disconnected in the traditional curricular organization of contents. For example, the map of Figure 2 reveals that functions, differential equations, regression, average rate of change and epidemiological notions are deeply interrelated. An interesting fact emerged when analyzing different maps from different working groups: depending on their lecturing field, they approached the problem quite differently. For instance, Mathematics lecturers' work was centered on finding a mathematical model fitting the data, whereas Chemistry lecturers' work evolved around the epidemiological data, the notion of "incidence index" and searching scholar literature regarding other similar outbreaks. The different teaching fields of lecturers permitted to share different visions of the knowledge at stake in the proposed SRP. The use of the maps was a key factor to describe this connection of fields usually lacking in school institutions.

The second part of the survey about the content of the course reveals that the work developed by lecturers with the question-answer maps and the media-milieu dialectics was difficult for them (more than 70% of the teachers found it hard or very hard) but at the same time they identified this work as "easily applicable to design and manage new teaching and learning processes" (more than 70% of the lecturers found contents and tools of the course easy to use and to implement). Regarding the consequences of the course on the lecturers' teaching practices, the survey showed that it helped (more than 90% totally agreed) to change their previous conception of knowledge towards a dynamical-collective conception in terms of modelling activities.

The third source of evidence are the maps generated by the lecturers as a priori analysis for an SRP to be experienced in their subjects. In total, six maps where generated by lecturers, all of them with a generating question and making explicit the didactic facts intended to be overcome. Two of these a priori SRP designs where experienced during the spring semester, starting just after the lecturers' course. These two emerging SRP have been experienced and managed only by lecturers that followed the course and did not have any other didactic experience or training. This fact is especially interesting because with the analysis of these experiences a first set of data can be collected regarding the conditions of existence of SRPs at the university level led by lecturers with almost no direct connection with research in didactics. This first experience in lecturer education seem to preliminary validate Winsløw et al. (2013) hypothesis about the use of question-answer maps in teacher education and confirm Barquero et al. (2016) results. Lecturers have worked with the maps and have used them to both model a lived study process and a priori analyze their own designed SRP. Moreover, the maps have been used to compare the knowledge mobilized during a specific SRP and the school knowledge. The Q-A duplets appearing in the map were used as the elements to contrast with curricular requirements.

The course also appears as a good tool to empower lecturers to question and put under vigilance the *dominant epistemology* at the university. It produced a discussion (and thus enabled a reflection) on what knowledge has to be taught at the university and how the modelling activity with its dynamics

and collective aspects could be considered. Regarding the conditions of existence of a lecturer course based on the ATD, it seems that the described conditions make it viable and that some lecturers have taken it as an opportunity to redesign their teaching and learning activities. However, an important aspect to take into account is the fact that one of the leaders of the course is also a lecturer in the considered Engineer School, what certainly affected the good predisposition of the attendees due to his personal leadership in the institution. This particular condition has to be considered in new editions of the course and the question of its reproducibility remains open.

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High school teachers' choices concerning the teaching of real numbers: A case study

Laura Branchetti

Politecnico di Milano, Italy; branchetti.laura@gmail.com

The goal of this paper is to present a case study in which a high school teacher with PhD in Mathematics was asked to answer a questionnaire concerning the teaching and learning of real numbers and then he was interviewed in order to investigate the interplay between his resources, goals and orientations in the decision-making process. As a main result, I show how some orientations concerning the epistemology of real numbers, the goals of mathematics education in the high school and the students' conceptions and difficulties lead him to choose a very intuitive approach to the teaching of real numbers and to leave aside all his expertise as a mathematician.

Keywords: Real numbers, teaching, high school, teachers, tertiary education.

Epistemological issues concerning real numbers and continuum

The relation between the continuum and the real numbers is often considered as something intuitive and to be taken for granted (Lakoff & Nunez, 2000), but as it is well known to the experts in history and epistemology of mathematics, this is one of the most complex issues to face dealing with the foundations of mathematics. This topic deals indeed with very relevant challenges. I will just highlight some aspects that are relevant to characterize the orientations of a teacher concerning the epistemology of real numbers and continuum. In *Continuity and irrational numbers,* Dedekind (1872, transl. 1901) stated: "In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. [...] For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis. The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus [...] depend upon theorems which are never established in a purely arithmetic manner" (p. 1-2). Dedekind came to the construction of R as the field of the rational cuts, stressing that the new numbers - irrationals - were creations necessary to identify the points of a line and the numbers and making explicit that the assumption of the property of continuity of the line is nothing else than an axiom. In this paper, I just report Dedekind's approach, since it's particularly relevant to analyse the case study I present here; for a complete dissertation see Bell (2014).

Challenges with teaching and learning of real numbers and continuum

The topic of teaching and learning real numbers and continuum in the high school and the university have been investigated in several countries and nowadays a lot of results are available; in particular, the researches concern the difficulties of high school and university students and prospective teachers. In this paper, I report just some examples, but the literature is very rich (for a complete review see Voskoglou & Kosyvas, 2012). First of all, cognitive issues "resonate" with the very relevant epistemological issues: space-temporal intuitions and metaphors (Bolzano, 1817), and the formal

approach, based on static and rigorous should be clarified by teachers. Indeed, according to Lakoff and Nunez (2000), the identification between objects such as lines, points numbers and sequences, that is very usual at school and in the University, hide the intrinsically metaphorical nature of the relation between natural continuity and "formal continuity". The main students' difficulties reported in several studies concern: irrational numbers; infinity; points of a line; density and continuity; and, number line. Students from high school to university are often not able to define correctly the concepts of rational and irrational numbers, like if rational numbers in general remains isolated from the wider class of real numbers (ibid., 2012). In particular, students' ideas concerning the relation between 0.999… and 1 (Tall & Schwarzenberger, 1978) and, in general, concerning the meaning and the use of decimal representations (Margolinas, 1988) are usually very ingenuous and seem to be product rather of a spontaneous generalization from finite to infinite numbers than of a structured learning path. Tall (1980) observed a recurrent phenomenon, that was defined *dependence*: there are more points in a longer segment than in a shorter one, based on the generalization to infinite cases of what has been learnt of the biunivocal correspondence of finite cases. Tall (1980) also observed among the students an intuitive model of the line in which points are as much as they need to fill a segment with physical points, that are "non-overlapping marks" (p. 3) and the quantity of numbers is proportional to the length of the segment. This intuitive model, without a suitable elaboration, could influence the learning of properties like density and completeness if the model of real numbers considered is a line. Also, Bagni (2000) alerted from the false illusions concerning the introduction of properties of real numbers in the graphical domain. Bergé (2008) highlighted that in the transition from calculus to analysis in the university, it's necessary first to change the conception of the number line. Fischbein, Jehiam and Cohen (1995) carried out research based on the assumption that the irrational numbers could be counterintuitive because of their high complexity but, contrary to what they had hypothesized, they found out that many high school students and prospective teachers overcame the barrier quite easily in suitable contexts, paying attention to the potential difficulties and presenting them to the students.

Research problem and framework

In this work, I face the problem from the point of view of teaching in the high school, focusing the attention on high school mathematics teachers' intended reported choices concerning the teaching of real numbers in the high school; in particular the focus is on what considerations and necessities guide their decisional process. Schoenfeld's model (2010) provides a tool to distinguish between three main factors that may influence the teachers' choices: *resources* (mathematical and pedagogical knowledge); *goals* (educational, instructional and social aims); *orientations* (beliefs concerning knowledge, concerning teaching-learning processes). I used this model to design a written questionnaire answered by the teachers in the first part of this research. Orientations may be very general (what mathematics is, what learning is) or more specific and may concern epistemological (what is the role of real numbers in the history of mathematics, what are real numbers necessary for), cognitive (what is difficult for the students, what is a good choice to help students) or ecological aspects of the didactical activities (what teachers must do within an institution, what are the aims of the teaching in high school). Some of these orientations act as criteria to make choices. Since real numbers and continuum both in the history and in the teaching-learning processes oscillates between the two poles of intuition and formalization, I owe special attention to the orientations concerning rigor and intuition.

Methodology

This *case study* is part of a research carried out for a PhD dissertation concerning the teaching and learning of real numbers in the high school, in which I involved 89 Italian high school teachers with very different backgrounds. I will discuss in a *case study* the role of teachers' resources, orientations and goals in a teacher's decision making intended process, according to Schoenfeld's model (2010). In Italy, according to the national curricula, an introduction to calculus and some theorems of analysis are proposed to students in the end of high school. According to Bergé (2008) what characterize more the transition from calculus to analysis is the transformation of intuitive models of the line into a formal construction of R. Usually, in high school, teachers are asked to introduce: limits, continuous functions and related theorems (Bolzano, Weierstrass), derivatives, Rolle and Lagrange theorems, examples of computations of limits and derivatives, Riemann integral, examples of integration based on Torricelli-Barrow theorem and examples of differential equations. Italian textbooks are usually based on formal definitions and intuitive examples that are not suitably connected each other. Usually intuitive approaches to the definition and use of real numbers are proposed in the first 4 years of high school (roots, points, decimal numbers). Then, in the last year, definition and procedures of calculus and theorems are introduced through formal expressions and using concepts such as limits, convergence towards a point and open intervals. The research question in this study is: *how does the interplay between resources, goals and orientations about teaching and learning of real numbers and continuum in the high school affect a high school teacher's intended choices?* I designed a study with a questionnaire and follow up interviews. The teacher first was asked to answer an online questionnaire structured in order to investigate his knowledge, goals and orientations. The first questions concerned the teacher's background and training. The teacher was asked to answer questions about the main properties of the set R, the construction of R starting from Q, the definition of limit points. In the next section of the questionnaire the teacher was asked what they thought to introduce R was necessary for and to comment on teaching materials or parts of lessons concerning real numbers and the line, from different points of view (construction of $\sqrt{2}$, correspondence between points of a line and numbers, algebraic and graphic approach to inequalities). Then the teacher was interviewed in order to make him declare his choices concerning teaching and learning of real numbers in depth and to follow the thread of his thoughts, in order to make the orientations emerge in relation to the epistemological, cognitive and institutional issues that were emerging time after time. The interview was a semi-guided one: a general question concerning the way the teacher introduces usually real numbers in the high school and the motivations of the choices; a particular question concerning the relation between numbers and points of a line; a particular question concerning the way the teacher presents the enlargement from *Q* to *R i*n relation with the line; a general question concerning the relevance in the last year of high school (in particular for calculus and analysis) of the previous knowledge concerning real numbers and the way connect the two.

I identified the following *a priori categories*:

1. *Resources*
    a. Mathematical knowledge
        i. The teacher knows the main properties of R (complete, ordered, Archimedean field)

ii. The teacher knows at least one construction of R (Dedekind, Cantor, Hilbert, …)

iii. The teacher knows that a limit point of a set A can be defined in every dense set

iv. R as a complete ordered field is necessary for Analysis

2. *Goals*

a. Institutional

i. To introduce intuitively some real numbers, after providing some examples and proofs of irrationality of some numbers introduced in geometrical constructions

ii. To formalize real numbers in order to give foundations to the theorems of Analysis

b. Personal

i. To construct a set in which it is possible to define the most used continuous functions (exponential, logarithmic, …)

ii. To construct a set in which it's possible to solve equations (but the ones with complex solutions) and inequalities

iii. To construct a set in which it's possible to formulate some theorems of Analysis and to define limits, integrals and derivatives

3. *Orientations*

a. Epistemological

i. R is complete in the sense of the continuity of the line

ii. A theoretical construction of R is not necessary to develop calculus and formulate theorems of Analysis, since it was constructed after the theorems

iii. R is a set of points of a line

iv. The representations of real numbers (points, decimal numbers, …) are all equivalent (framed in the same theory)

v. A postulate is necessary in the constructions or axiomatization of R

b. Cognitive

i. R is intuitive for students; students have preconceptions of real numbers

ii. The construction of R is too abstract for students

iii. Students prefer simple and concrete things, even if they don't understand everything

iv. It's important be make the lessons intuitive for students

v. It's important to be rigorous and consistent during the lessons

c. Ecological

i. It's important to respect institutional constraints

I analyzed data of the questionnaire using *a priori* categories concerning the different dimensions of

the teacher's profile - goals, dimensions and knowledge - and then I looked for further emerging, unexpected phenomena to frame in this research background and to compare them with further literature review. Using a qualitative analysis, I labeled the relevant features concerning the three dimension and the declared choices. Then, I looked at the interview searching for sentences that could confirm the teacher's belonging to the categories I used in the questionnaire and to look for new relevant elements emerged in the interview. Finally, I looked for a relations between the different categories in order to interpret the teacher's choices in terms of the interplay between resources, goals and orientations.

Data analysis

I report first the background and the teacher's answers in the first part of the questionnaire, to show that the teacher showed advanced mathematical knowledge. I label the sentences with the codes presented before.

Background: Master, PhD in Mathematics and National qualification for Mathematics and Physics high school teachers*;* 5 years of experience as a teacher

He studied real numbers: at the University in a course of Analysis and at school

Properties: two operations make real numbers a field (with characteristic 0); total order, compatible with the operations; complete [R-a-i]

Construction: Two equivalent constructions: 1) the method of Dedekind's cuts (separating elements of two sets whose union is Q and that have no maximum or minimum in Q, e.g. $\{x \in Q : x^2 < 2\}$ and $\{x \in Q : x^2 > 2\}$; 2) quotient of set of the Cauchy's sequences with convergent ones [R-a-ii]

Limit point: It's possible to define it also in Q; the set $\{x \in Q : x < 0\}$ has 0 as a limit point [R-a-iii]

Then, I report the most relevant results of the data analysis carried out in the second stage:

1. the properties of real numbers are necessary to introduce only differential and integral Calculus, sequences and series. [G-b-iii]
2. a video in which the graphic and algebraic solution of linear inequalities are presented as two different solutions should be changed because the solution is the set of numbers that satisfy the equation and only the representation may be graphic or algebraic [O-a-iv]
3. a tutorial in which a concrete problem involving measures of courtyard containment is used to present the "reality of irrational numbers" helps the student to create good images of real numbers, even if something is not convincing [O-b-iii]
4. a video in which the correspondence between R and points of a line is showed using a point moving on a line, with the extreme indicated by a decimal number with one decimal digit, can't help to grasp the correspondence between real numbers and points of a line, because only an origin is fixed and not a unit and it's difficult to justify negative numbers [O-b-v]

In the interview, some further relevant aspects emerged concerning the teacher's knowledge and cognitive and epistemological orientations and goals. Since new categories emerged, I label the synopsis with a priori but also with a further category (NEW_C_i).

To present the following synopsis, I use a chronological criterion:

1. students have preconceptions of the relations between numbers and point of a line. He presents real numbers intuitively, as points of a line, and to base on this intuition all the definitions, also

very formal (e.g. limits, Cauchy-Weierstrass continuity, decimal numbers, ...) [O-b-i]
2. this is enough to "do what we have to do", a "pseudo-mathematics" [NEW_G_1]
3. real numbers are imagined as the real line, with an abuse of language, that makes a few damages at this level but may have many advantages [G-b-i&ii]
4. it's important to be coherent with mathematics [O-b-v] without saying it to the students [O-b-iii]
5. to introduce analysis "seriously" R is needed [R-a-iv]
6. formal definitions are not useful but were only useful to clear "the conscience of Dedekind" and that "Euler did so many good things without formalizing R" [O-a-ii]
7. history confuses students [NEW_O_1]
8. students can't understand very much of real numbers in the high school [O-b-ii]
9. asked to declare his choices concerning the introduction of continuous functions he referred to formal definitions (Cauchy-Weierstrass approach) that are traditional in Italy [G-a-ii]
10. there is a "parallelism between geometrical and algebraic postulates" [O-a-iii]
11. R and the line are the same object [O-a-iv]
12. we live between two truths, the 'pure mathematical' and the 'operational' one [NEW_O_2].
13. none really use R and the line is really strange; maybe teachers are disappointed because no one really knows what numbers are, thinking at infinite convergent sequences and the definition of new numbers (irrationals) that are limits of convergent rational sequences [NEW_O_3]
14. he uses representations like decimal numbers, the line, the roots only in order to make operations with them [O-a-iv] but never deepen their meaning and mutual relations [O-b-iii]
15. it's simpler for the students and for the teachers to "sneak off the theoretical crevices" [O-b-iii]

Discussion and conclusions

The teacher is a PhD in Mathematics (Analysis) and attended teachers' training courses. The knowledge he showed about the topic is advanced. The pedagogical knowledge has never been taken in account by the teacher to support argumentations, while his orientations, reflections and experiences are used to motivate his statements during the interview. Also, he never quoted explicitly the institutional constraints. He declared to choose usually to avoid completely the formal introductions of real numbers and the historical issues and to simplify as much as possible. Even if he's aware also of some epistemological issues, his choices are very traditional and are suitable calculus but not for analysis (Bergé, 2008). Asked to declare his choices concerning the introduction of continuous functions he referred to formal definitions (Cauchy-Weierstrass approach) that are traditional in Italy. He declared to switch suddenly from intuitions of continuity and a set of numbers with different representations to a formal implicit meaning of R, used in the hypothesis of theorems without a contextualization and without stressing the epistemological implications of such a step. The teacher conflicted with the true relevance of formal constructions of real numbers: sometimes he said it's necessary, sometimes it seemed just a fancy of some mathematicians. He's convinced that some representations of real numbers can't be interpreted in high schools – even if he never considers not to use them – so he prefers the students to use them without being aware of their complexity. Moreover, he's convinced that not only the students have limitations dealing with real numbers but there is an epistemological issue: there are two truths, the 'pure mathematical' and the 'operational' one. Furthermore, he showed "epistemological doubts" concerning the deep meaning and the existence of irrational numbers. To sum up what could seem to be only didactical and cognitive

motivations (he wants the students to understand; simplifying and omitting is always better for students) hide – or at least are accompanied by – deep epistemological unsolved doubts and noisy ambiguities highlighted by the teacher, declared several times during the interview, both spontaneously and answering the interviewer's questions, as confirmed by relevant sentences like "The line is… is perceptual. No.. it's not perceptual, is stem from... you don't see. But … what is the line?"; "In practice is it useful for anything? It was useful for the purpose of a clear conscience for Dedekind but it's not useful at all"; "I take this point, limit of a function. A bit an approaching to the border of the abyss, keep the feet ... approach something that doesn't exist ... infinite rational paths doesn't imply to be rational. It's something that maybe we don't understand very well too ...". The teacher's orientations concerning the cognitive aspects of teaching and learning real numbers (intuition and preconceptions) that seemed in the beginning the most relevant motivations towards an intuitive oversimplification, considered helpful for students, are thus deeply intertwined with his epistemological orientations. Firstly, his orientations concerning the uselessness of formal definitions do not motivate him to look for suitable teaching strategies and, on the contrary, act as factor that reinforces his naive orientations towards what is better to foster in students' learning processes. Secondly, his epistemological doubts, hidden under perfect formal definitions, encourage him to keep the "Pandora's vase" closed and to avoid to face his own uncertainties, thinking that for the students it's absolutely better not to know them in order to keep on trusting him and let him going on presenting the "pseudo-mathematics" that is enough in the high school. His orientations and his decision to use a very traditional, internally disconnected and full of "theoretical crevices" approach to the teaching of real numbers have been proved to be unsuitable by a lot of researchers both from a general point of view and for the specific problem of real numbers. I can state that, in this investigation, it emerges that an advanced mathematical knowledge, even very significant, doesn't imply the use of this knowledge in teachers' choices: doubts and personal orientation can lead the teacher to use a trivial and sterile approach for all the complex issues that characterize real numbers and continuum from an epistemological point of view, taking the risk to create at least the same problems to the students that teachers with a weaker background in mathematics would create. The teachers, without suitable reflections and teacher training courses, could also reinforce their motivation towards such a choice mixing in their mind personal orientations and expected students' cognitive features, justifying and hiding with the last ones the epistemological uncertainty. The main implications of the study are the following:

1) mathematicians, even with a PhD in Analysis, in their transition to a teaching profession may miss the opportunity to benefit from their knowledge by not being completely aware of the epistemological issues of the importance of formalizations into teaching;

2) mathematicians who become high school teachers, in order to become able to design good teaching and learning activities for their students concerning real numbers, should be trained not only from the disciplinary point of view, but also from the epistemological and the didactical one.

These observations are particularly relevant in the country in which I carried out my investigation, from an institutional point of view, where often the significance of the epistemological and the didactical background of teachers is a central in the debate between policy makers (and some teachers) and the community of researchers in mathematics education.

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Proof and formalism: The role of letters' logical status

Faïza Chellougui

Faculty of Sciences of Bizerte, University of Carthage, Tunisia; chellouguifaiza@yahoo.fr

In mathematical activity, especially in proof, it is a fairly common practice to change letters' logical status without giving any indication of these changes. Nevertheless, in some cases this practice is likely to hidden invalid steps in the proving process. In this paper, I show on an example that Copi's system for natural deduction provides a methodological tool that allows us both to anticipate where such invalid steps could appear and to analyse students' proof productions.

Keywords: Formalism, proof, letters' logical status, quantifiers, natural deduction

Introduction

Difficulties met by university students in the use of logical formalism are well documented in the literature (e.g. Selden & Selden 1995, Dubinsky & Yiparaki 2000, Chellougui 2009). An important issue on the development of formalism in university mathematics education is to reduce ambiguities conveyed by natural language, in order to foster the understanding of mathematical statements and to improve the development of proving skills. However, Durand-Guerrier and Arsac (2005) have underlined a rather common practise in mathematics textbooks of changing letters' logical status in a proof without giving any indication of these changes. This practice introduces additional ambiguities in the proving procedure. The aim in this paper is to address the conjecture that university students who are not able to cope with these ambiguities face difficulties with proofs.

In this paper I use the system for natural deduction (i.e. a formalisation of mathematical reasoning) suggested by Copi (1954) in predicate calculus (i.e. an extension of propositional calculus that deals with the internal structure of propositions, with symbols for properties, relations, quantifiers and individuals) as a tool for checking the validity of proof. A main interest of this system is to make explicit the difference between bound variables (i.e variables in the scope of a quantifier) and generic elements (i.e any individual element of the domain of quantification at stake)[1]. In the first part, I present the Copi system which has been used for both *a priori* and *a posteriori* analysis in my research. The second part focuses on the letters' status in a proof from the logic and didactic point of view. To this purpose, I present results demonstrating student's difficulties with letters' status at the beginning of their university studies. In the third part, I present results from first year university student's responses in a proof where the use of letters challenges the validity of the proof (Chellougui, 2009).

Natural deduction system

I first introduce the natural deduction system proposed by Copi (1954) and show that this system provides tools to detect invalid steps in a proof by remaining as close as possible of the usual modes of reasoning of mathematicians (Durand-Guerrier & Arsac, 2005). In my work, I use this system in

[1] "[…] the proof of a universal statement, apart from the case of proof by induction, is always done by the method of the generic element: in order to prove a statement of the kind "for all x ∈ E, P(x)", you prove P(a) for a generic element a ∈ E, then after verifying that only properties of a that are common to all elements in E were used, you conclude that P(x) is true for every x in E." (Durand-Guerrier and Arsac, 2005, p.153)

the frame of predicate calculus, where four rules for introduction and elimination of quantifiers are introduced. In Figure 1, I summarize a presentation offered by Durand-Guerrier (2005) of these rules completed by some specific restrictions needed for preserving validity.

The first rule of inference concerns elimination of the universal quantifier; it is called Universal Instantiation (UI) and states that:

"(...) any substitution instance of a propositional function can validly be inferred from the universal quantification. We can express this rule symbolically

$$\frac{(x)\Phi x}{\Phi v}$$ (where v is any individual symbol)[1]" (Copi, 1982, p.50)

This rule expresses that *what is true for all is true for any*.

The second rule is the dual of the first one and concerns the introduction of the universal quantifier; it is called Universal Generalisation (UG) and states that:

"(...) the universal quantification of a proposition can validly be inferred from a substitution instance with respect with the symbol y. Our second expression for this quantification rule is

$$\frac{\Phi v}{(x)\Phi x}$$ (where v denotes any arbitrarily selected individual)" (ibid., p.51)

This rule necessitates a restriction; you must be sure that no assumption other than the property expressed by Φ has been done. Obviously, it is build "by analogy with a fairly standard mathematical practice"(ibid., p.50)

The third rule concerns the introduction of the existential quantifier; it is called Existential Generalisation (EG) and states that:

"(...) the existential quantification of a propositional function can be validly inferred from any substitution instance of that propositional function; (...) Its symbolic formulation is:

$$\frac{\Phi v}{\exists x \Phi x}$$ (where v is any individual symbol)" (ibid., p.52)

The fourth rule is the more delicate to use. It concerns the elimination of the existential quantifier; it is called Existential Instantiation (EI) and states that:

"(...) from the existential quantification of a propositional function we may validly infer the truth of its substitution instance with respect to an individual constant which has no prior occurrence in that context. The new rule may be written as

$$\frac{\exists x \Phi x}{\Phi v}$$ (where v is an individual constant having no prior occurrence in the context)" (Ibid. p. 52).

Figure 1: Copi's rules (Durand-Guerrier, 2005, pp. 413-414)

This system can be used to control locally the validity of mathematical proofs. It can act as an intermediate between the usual practice and the completely formalized system. The rules of introduction and elimination of quantifiers from Copi address the semantic dimension, because there is an introduction of letters referring to generic elements. In ordinary textbooks, most often, both operations of elimination and introduction of quantifiers are either absent or partial (Durand-

Guerrier & Arsac, 2005; Chellougui, 2009). By making explicit the rules of introduction and elimination of quantifiers, Copi's natural deduction system allows the identification of implicit steps, especially in cases of proof that use multi quantified statements where there is, at least once, each of both universal and existential quantifiers. In my research, I use Copi's natural deduction as a tool for *a priori* analysis in order to anticipate possible invalid steps in the proving process of a given statement and as a tool for *a posteriori* analysis of the proofs offered by undergraduate students. In a following section, I present an example of such analysis.

The logical structure of mathematical statements: Changing letters' status

Durand-Guerrier and Arsac (2005) have investigated the letters' logical status in mathematics teaching. In their work, they refer to the predicate calculus in order to analyse quantified statements with a focus on the variables' dependence. They analyse a specific mistake which appears in proofs where one applies twice or more a statement of the kind "*for all X, there exists Y such that R(X,Y)*", abbreviated to AE statements, and a student may ignore that in that case, *a priori*, "*Y depends on X*". The misuse of AE statements in calculus have been demonstrated in an invalid proof of Cauchy's mean value theorem (Figure 2, Durand-Guerrier and Arsac, pp. 151-152):

Theorem 1 *(mean-value theorem). Let us consider two real numbers a et b such that* $a<b$ *and a real function f defined on the closed interval* $[a;b]$. *If f is continuous on* $[a;b]$ *and differentiable on the open interval* $]a;b[$, *then there is a point c in the open interval such that* $f(b)-f(a)=(b-a)f'(c)$, *where* $f'(c)$ *is the first derivative of the function f at c.*

Theorem 2 *(Cauchy's mean-value theorem). Let us consider two real numbers a and b such as* $a<b$ *and two real functions f and g defined on the closed interval* $[a;b]$. *If f and g are continuous on* $[a;b]$ *and differentiable on* $]a;b[$, *if* $g(b)\neq g(a)$ *and if the first derivative* g' *of g is never equal to zero on* $]a;b[$, *then there is a real number c in* $]a;b[$ *such that* $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$.

Figure 2: Two theorems: *mean-value theorem* and *Cauchy's mean-value theorem*.

We can read:

> "A proof rather often provided by first year science students consists of the following deduction of Theorem 2 from Theorem 1:
> Function *f* satisfies the conditions for applying Theorem 1; hence there is a number *c* in $]a;b[$, such that $f'(c)(b-a)=f(b)-f(a)$. Also *g* satisfies the conditions for applying Theorem 1; *hence* there is a number *c* in $]a;b[$, such that $g'(c)(b-a)=g(b)-g(a)$. As g' is never equal to zero on $]a;b[$, then $g'(c)\neq 0$; hence $g(b)-g(a)\neq 0$. The result follows from the quotient of the above two equalities.
> This proof is invalid; one can prove it by considering two functions for which it is not possible to choose the same number *c*." (Durand-Guerrier et Arsac, 2005, p.152).

According to the authors, the error from a logical point of view is the following: since c is a bound variable following an existential quantifier, it cannot denote a particular real number. However, the existential elimination that must be applied here allows to consider a real number *r* such that:

$$f'(r)(b-a)=f(b)-f(a).$$

When thus applying the same rule to *g*, it is necessary to consider a real number *s*, that may or not be equal to *r*, such as $g'(s)(b-a)=g(b)-g(a)$. It is important to notice that this logical analysis depends only on the logical structure of Theorem 1, and not on the mathematical meaning of the letters *f*, *c*, *r*, etc. The same reasoning allows us then to derive the quotients' equality:

$$\frac{f'(r)}{g'(s)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

but this does not provide a proof of Cauchy's mean value theorem (Durand-Guerrier & Arsac, 2005).

This example illustrates the difficulties linked with the logical status of letter in proof and proving. In the next section, I illustrate one example in Algebra how the use of Copi's natural deduction allow us to anticipate student's difficulties and to analyse their proofs.

An example in elementary set theory

In the context of my PhD (Chellougui, 2004) conducted in Tunisia, I distributed a questionnaire to ninety-six mathematics students arriving at university in November 2001 (details on the analysis and main results can be found in Chellougui, 2009). In this paper, I focus on a specific example that I analysed in detail in order to highlight the methodological relevance of Copi's natural deduction for *a priori* and *a posteriori* analysis of proofs. In the Tunisian university, the first elements of elementary set theory including equivalence relation, order relation and binary relation, are taught at the beginning of the first academic year. The example I discuss here regards the proof that a given binary relation $\Re$ is an order relation (Figure 3). My main objective was to identify precisely students' difficulties in the use of multi quantified statements in proof and proving.

We consider the set $\mathbb{N}^{*2}$ endowed with the relation $\Re$ defined by: $\forall(p,q)\in \mathbb{N}^*\times \mathbb{N}^*(p\Re q\Leftrightarrow\exists n\in \mathbb{N}^*; p^n=q)$. Show that $\Re$ is an order relation.

Figure 3: The exercise submitted to students

I hypothesised that the students would be able to recall each of the three properties that an order relation checked: reflexivity, antisymmetry and transitivity, because they have met this type of questions in the course and in the series of exercises, although the formalisation of an order relation was new to them. The three definitions of the properties above were given to the students in the general case of a binary relation $\Re$ as follows:

Reflexivity: $\forall p\ p\Re p$. Formulation containing one universal quantifier.

Antisymmetry: $\forall p\forall q(p\Re q\wedge q\Re p \Rightarrow p=q)$. Formulation containing two universal quantifiers.

Transitivity: $\forall p\forall q\forall s(p\Re q\wedge q\Re s \Rightarrow p\Re s)$. Formulation containing three universal quantifiers.

In this paper, I focus on the proof of *antisymmetry* for a binary relation whose definition involves an existential quantifier, leading to a rather complex logical structure as will be shown in the *a priori* analysis. I first present some elements of *a priori* analysis; then I present results from students' responses.

[2] $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$

Mathematical and logical analyses of proof of antisymmetry

The definition of the binary relation $\mathfrak{R}$ involves two universal quantifiers in the beginning of the formula and an existential quantifier in the second part of the equivalence.

In order to anticipate the difficulties that the students could meet in proving that the given binary relation owns the property of antisymmetry and to make explicit the steps needed for a complete proof, I provide (Figure 4) a mathematical and logical analysis, using the Copi's system for natural deduction (see Figure 1) with a specific focus on introduction and elimination of quantifiers

(1)	$\forall p \forall q\ (p \mathfrak{R} q \Leftrightarrow \exists n \in \mathbb{N}^*\ p^n=q)$	Premise[3]
(2)	$a \mathfrak{R} b \Leftrightarrow \exists n \in \mathbb{N}^*\ a^n=b$	UI on (1)
(3)	$b \mathfrak{R} a \Leftrightarrow \exists n \in \mathbb{N}^*\ b^n=a$	UI on (1)
(4)	**[** $a \mathfrak{R} b \wedge b \mathfrak{R} a$	Auxiliary premise
(5)	$a \mathfrak{R} b$	Simplification on (4)
(6)	$\exists n \in \mathbb{N}^*\ a^n=b$	Modus Ponens on (2) and (5)
(7)	$b \mathfrak{R} a$	Simplification on (4)
(8)	$\exists n \in \mathbb{N}^*\ b^n=a$	Modus Ponens on (3) and (7)
(9)	$a^m=b$	EI on (6)
(10)	$b^k=a$	EI on (8)
(11)	$a^{mk}=b^k$	Property of the power
(12)	$a^{mk}=a$	Transitivity of the equality on (10) and (11)
(13)	$a=1$ or $m=k=1$	Mathematical properties
(14)	$a=b$ **]**	Consequence of (13)
(15)	$(a \mathfrak{R} b \wedge b \mathfrak{R} a) \Rightarrow a=b$	Introduction of $\Rightarrow$ on (4) and (14)
(16)	$\forall p \forall q\ (p \mathfrak{R} q \wedge q \mathfrak{R} p \Rightarrow p=q)$	UG on (15)

Figure 4: Formalisation of the proof in the frame of Copi's natural deduction

This formalized demonstration starts with a universal premise followed by four successive universal instantiations; twice in (2) with two different letters *a* and *b*, and twice in (3) with the two same letters *a* and *b*; so, one works then with two generic elements. In (4) an auxiliary premise is introduced to express the antecedent of the property of the antisymmetry on the generic elements (this is a standard way to prove a conditional statement in the frame of Copi's natural deduction: proving B under hypothesise A provides a proof of A$\Rightarrow$ B). The two existential statements (6) and (8) are followed by two existential instantiations with two different letters: *m* in (9) and *k* in (10).

[3] A proposition upon which an argument is based or from which a conclusion is drawn.

The mathematical argument is developed in (11) to (14) using mathematical properties without the quantifiers. The passage from (13) to (14) does not appear in this demonstration. It can be expressed in the following way:

(13): $a=1$ or $mk=1$; First case: $a=1$ then $a=b=1$; Second case: if $mk=1$, with $m\in\mathbb{N}^*$ and $k\in\mathbb{N}^*$, then $m=k=1$; finally, in both cases, $a=b$ (14)

Framing the proof in Copi's system allows us to anticipate potential flows likely to appear in students' responses from (6) and (8) to (9) and (10), respectively, in case the restriction rule for two successive existential introductions is not applied.

I provide an *a priori* classification of the answers that I will use for *a posteriori* analysis.

Category 1: answers for which two different letters are quantified existentially.

Category 2: answers for which the same letter is quantified existentially.

Category 3: answers for which the same letter is used without existential quantifier,

Category 4: answers where two different letters are used without existential quantifier.

Classification of students' answers

Among the ninety-six students that answered the questionnaire 80 students have produced a proof of the antisymmetry property for the given binary relation.

1) There are 17 copies in category 1 (about 21%) with representative examples that illustrate this category (Figure 5).

Student 1: $\forall(p,q)\in\mathbb{N}^*\times\mathbb{N}^*;\begin{cases}p\mathfrak{R}q\\ q\mathfrak{R}p\end{cases}\Leftrightarrow\begin{cases}\exists n\in\mathbb{N}^*,\ p^n=q\\ \exists m\in\mathbb{N}^*,\ q^m=p\end{cases}\Leftrightarrow p^{nm}=p\Leftrightarrow\begin{cases}p=1\\ \text{or}\\ nm=1\Rightarrow n=m=1\end{cases}$

Then $\begin{cases}\text{if } p=1\Rightarrow p=q=1\\ \text{if } nm=1\Rightarrow p=q\end{cases}$

Student 2: $\forall(x,y)\in(\mathbb{N}^*)^2\ \begin{cases}x\mathfrak{R}y\Leftrightarrow\exists n\in\mathbb{N}^*\ /\ x^n=y\\ \text{and}\\ y\mathfrak{R}x\Leftrightarrow\exists n'\in\mathbb{N}^*\ /\ y^{n'}=x\end{cases}\Leftrightarrow x^n=y^{nn'}=y\Leftrightarrow x=y$

Student 3: $\forall p\in\mathbb{N}^*,\ \forall q\in\mathbb{N}^*;\begin{cases}p\mathfrak{R}q\Leftrightarrow\exists n\in\mathbb{N}^*,\ n=m,\ p^n=q\\ q\mathfrak{R}p\Leftrightarrow\exists m\in\mathbb{N}^*,\ m=n,\ q^m=p\end{cases}\Rightarrow n=m\Leftrightarrow p=q$

Figure 5: Responses of Student 1, Student 2 and Student 3

In the solution of student 1, the mathematical argument for step (11) to (14) is fully developed. This is not the case of student 2: the mathematical argument is absent, there is no mathematical justification for the equality of both natural numbers x and y.

Student 3 used two different letters in the existential statements, however student 3 declared them to be equal. In this case we can make the hypothesis that the student attributed the same value to m and n in order to satisfy the validity of the statement. Another hypothesis could be that student 3 thinks that the equality is necessary.

2) There are 27 copies in category 2, that I have subdivided in two cases:

(a) The existential quantifier is present twice (21 copies), see Figure 6.

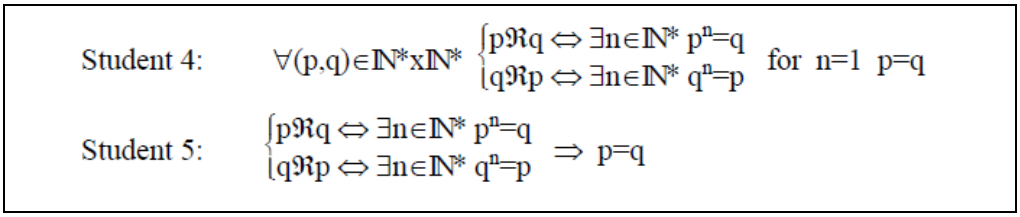

Figure 6: Responses of Student 4 and Student 5

In the production of student 4, there is a presence of both quantifiers, the student tries to construct the object *n* existentially introduced in such a way as to have the conclusion. So, to verify the property of antisymmetry and have the equality *p=q*, the student takes, for the natural number n, the value *1* which is a solution of the equation $p^n=q$ of unknown *n*. Also for the student 5, where there is absence of universal quantification, the construction of the object *n* is implicit and the conclusion is immediate for the student.

(b) The existential quantifier is present only once (6 copies), see Figure 7.

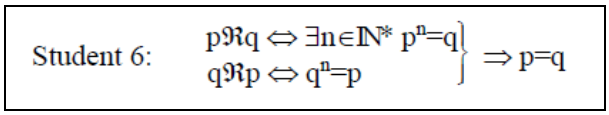

Figure 7: Response of Student 6

I notice with the student 6, that the existential introduction of *n* for the relation $p\Re q$ is also considered for $q\Re p$. Let us note here that, on one hand, both variables *p* and *q* are not introduced and that, on the other hand, the elimination of the number n is not declared. This illustrates the fact that there are implicit arguments in the use of variables and in the steps required to prove the antisymmetry property.

3) There are 36 copies classified in category 3. I have considered in this category copies where the letter is used for both statement and is not in the scope of a quantifier. I have consider it as an implicit existential introduction without taking in account the restriction rule. The example in Figure 8 is typical of answers in this category.

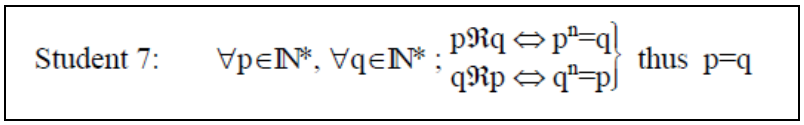

Figure 8: Response of Student 7

In the production of student 7, the variable *n* is the same in the two equivalences, and there is no mathematical argument supporting the conclusion. It is possible that the student wrote directly the conclusion *p=q* to fulfill the antisymmetry property; another possibility is that he considered that it was obvious that both equations $p^n=q$ and $q^n=p$ allow to conclude that *n* is *1* and to deduct the equality.

4) There is no copy in category 4, provided that we consider that the existential introduction may remain implicit.

These results confirm my hypothesis that the complexity of the logical structure on the side of quantifiers is likely to create an obstacle for the students to provide a correct mathematical argument. In particular, it is noticeable that the only students that provide sound mathematical

arguments are those in category 1, i.e. those who take in account the restriction rule for existential introduction.

These results also highlight the fact that in case of two successive applications of an existential definition that *a priori* requires the use of different letters, many students use the same letter. In other words, they do not respect the restrictions on the names of objects associated with the rule of existential instantiation. We could suppose that the symmetry in $p\mathcal{R}q$ and $q\mathcal{R}p$ triggers the choice of the same letter; however in line with other results (e.g. Durand-Guerrier et Arsac 2005) we would say that this students' practice could be found in various other contexts.

Conclusion

In this paper, I aimed to illustrate the relevance of our methodology relying on Copi's natural deduction that allowed detailed *a priori* analysis of proofs and *a posteriori* analysis of students' productions. From the *a priori* analysis of the proof of the antisymmetry property of the binary relation at stake, I identified possible invalid steps. The *a posteriori* analysis of the proofs provided by students has shown that such invalid steps appeared in many answers and that in some cases, this invalid steps prevent the students from identifying the mathematical property required for providing a valid proof.

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A mathematics educator and a mathematician co-teaching mathematics – Affordances for teacher education

Jason Cooper[1] and Orit Zaslavsky[2]

[1]Weizmann Institute of Science, Israel; jasonc2107@gmail.com

[2]New York University, USA; oritath@gmail.com

Prospective secondary school teachers are required to take undergraduate courses in mathematics, which may be of limited relevance for their teaching. In this study, we investigate affordances of co-teaching for achieving such relevance. This is a qualitative study of an undergraduate course on Mathematical Proof and Proving, co-taught by a professor of mathematics education and a professor of mathematics. Analyzing an episode critiquing three different proofs, we show that the mathematician was concerned mainly with the written proof and its "correctness", whereas the mathematics educator showed a sensitivity to the person behind the proof, and to pedagogical aspects of proof and proving. We propose that such a course may help students reconcile conflicts between how mathematics is taught and practiced in university and in high school, and suggest such co-teaching as a model for achieving relevance for teaching in mathematics courses.

Keywords: Mathematical proof and proving, teacher education, undergraduate mathematics.

Background and setting

The study reported herein was carried out in the context of an undergraduate course on mathematical proof and proving (MPP) taught at a major university in the USA[1]. Typically, such courses are the responsibility of mathematics departments, yet this course was conceived and designed through collaboration between the university's department of teaching and learning and its mathematics department, and was co-taught by a professor of mathematics education – the second author of this paper, herein MEI (Math Educator Instructor) – and a professor of mathematics, herein MI (Mathematician Instructor). The goal of this collaboration was to capitalize on the two departments' complementary fields of expertise – mathematics education and mathematics. This setting provides a unique opportunity to investigate differences and interactions between these two different types of mathematical expertise. Some preliminary findings were reported by Sabouri, Thoms, & Zaslavsky (2013) and by Zaslavsky & Cooper (2017), where some aspects of the co-teaching were discussed.

The course was open to a wide range of students, but the majority of participants were enrolled in teacher preparation programs, for which it was a required course. Accordingly, we situate our study in the context of the mathematical preparation of pre-service secondary-school teachers. These students have two different needs for MPP. In the short term, they must be proficient in university routines of proving in order to succeed in their mathematics courses. The notion of formal proof is new for many university students, and the transition to the kind of mathematical proving that is required in undergraduate courses is known to be difficult (Harel & Sowder, 2007). In the longer term, the majority of these students will become secondary school mathematics teachers. As such, they will be expected to teach MPP in their classrooms and to assess students' proficiency in and understanding of mathematical proving. Thus they are faced with the challenge of two transitions,

[1] Data collection was supported by the National Science Foundation under Grant No. 1044809.

first from high school to university MPP, and then back to high school. One way of addressing this challenge is to offer two separate courses, taught in mathematics departments and in schools of education, leaving it up to students to reconcile differences between MPP in the two contexts. Our study suggests an alternative; perhaps a course that is co-taught by professors of mathematics and of mathematics education will help reconcile the different, sometimes conflicting notions of MPP at high school and undergraduate levels. This is the overarching question that guides our study.

Literature review and research questions

Secondary school teachers are usually required to take some university level mathematics (ULM) courses in their pre-service training; in some contexts, they are even required to hold an undergraduate degree in mathematics or in a related field. Yet many ULM courses, taught in mathematics departments, deal with advanced mathematical content whose relevance for teaching is not immediately obvious. A number of researchers have investigated affordances of ULM courses for teaching, using a methodology of teachers' self-reporting (e.g., Adler, et al., 2014; Even, 2011; Zazkis & Leikin, 2010). Such methodologies have been found to be of limited value; teachers tended to report on general affordances of learning mathematics from mathematicians, however, Zazkis & Leikin (2010) found that teachers were generally unable to specify in what ways they made use of ULM in their teaching. In our study we utilize a different methodology for revealing affordances of the MPP course for teaching, in analysing the teaching in this unusual setting.

According to Harel & Sowder (2009), mathematicians teaching undergraduate courses are not fully aware of student difficulties in learning MPP. The conception of co-teaching in our context was similar in some aspects to the way it is utilized in special education (Friend et al., 2010), letting the mathematician and the mathematics educator share the responsibility for the content, while leaving it up to the educator to attend to students' "special needs". Thus, we hypothesize that the mathematician will take responsibility for epistemic aspects of the course, while the educator will take responsibility not only for addressing the students' mathematical difficulties, but also for the course's relevance for teaching. This hypothesis is consistent with Cooper's findings (2016) in his study of a professional development course for primary school teachers taught by research mathematicians, where the mathematicians took responsibility for the mathematical content, while the participating teachers themselves took responsibility for achieving relevance for teaching. Accordingly, our research questions are:

1. What are the instructors' views on MPP? In what ways are their views similar or different?
2. What are the affordances of different views on MPP for the students as future teachers?

Theoretical framework and epistemic analysis

We join researchers such as Nardi et al. (2014) in taking a Commognitive approach (Sfard, 2008). Mathematical "knowledge" is conceived as a particular community's established modes of communication, called discourses. These are constituted in commonly used keywords (e.g., "proof", "given"), in narratives that are endorsed or rejected by the community (e.g., proofs), in visual mediation that is considered useful (e.g., two-column format of a proof), and in repetitive routines (e.g., proving). The very different mathematical discourses of two communities - research mathematicians and researchers in mathematics education – differ in their use of common keywords, in the types of routines they engage in, and in the rules and norms that determine which narratives

will be endorsed and which will be rejected, and these differences are grounded in the communities' activities – mathematical research on the one hand and mathematics education on the other.

Proof is a *genre* in mathematical discourse, a type of narrative that is expected to adhere to a community's conventions. It usually includes text, and may employ a variety of means to visually mediate its mathematical ideas. A proof may be endorsed or rejected based on the metarules of a community's mathematical discourse. Proving is a routine in mathematical discourse, with the goal of producing a proof. Ideally, the prover should take responsibility for producing a "valid" (i.e. endorsable) proof, which may subsequently be endorsed or rejected by a teacher or by peers (classmates or fellow researchers). Routines of proving, and the proofs that are produced, are governed by very different metarules in school and in university.

Method and data

The study reported herein is part of a larger study, for which extensive data were collected, including video recording and field notes of each of the 13 lessons taught in 2013, audio recording and field notes of weekly meetings held following each lesson, email exchanges among members of the teaching team, and students' written homework with TAs' comments and grading. The rich data offers insight into the instructors' intentions, however, in the current analysis we are concerned with affordances of the co-teaching as it played out, and thus focus our attention on the video recordings of the lessons, and, in particular, on a single episode from lesson 8, selected for being rich in interactions between the MI and MEI. We assumed that the two instructors referring to each other's ideas, possibly disagreeing with each other, would reveal their tacit norms and ideas about MPP, and highlight differences between them.

Analysis

In the following episode, MI and MEI discuss a homework assignment, where an exemplary proof had been provided for the claim $|x \cdot y| = |x| \cdot |y|$. Three other responses (also ostensibly proofs) were included in the assignment, and students were asked to critique them. These "proofs", labeled 2.1, 2.2, 2.3, were copied onto the board, and were discussed.

Format: we present one or more utterances followed by a short annotation raising points that we later elaborate in the discussion. For brevity, utterances not relevant for our analysis were omitted.

Sample proof 2.1

MI One thing I told you in the very first lecture is to do what? Write the "given" and write RTP [remains to prove] … This is a mistake that is commonly made because people confuse what is given and what you need to prove.

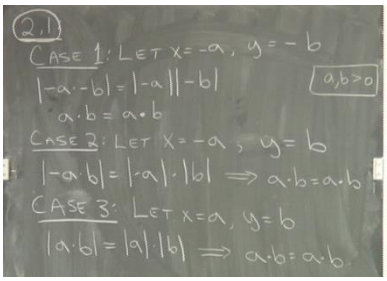

Figure 1: Proof 2.1

MI considered proof 2.1 unacceptable, since it begins with what needs to be proven, and manipulates it to obtain the incontrovertible $a \cdot b = a \cdot b$. He believes that following a simple rule of writing the given and writing RTP at the outset can help students avoid this common mistake.

MEI [Where] we provided proof [in the HW], we actually wrote the given and what you have to find, and if the people who wrote this proof [2.1] started by saying: "given", then there would be no confusion of what should the last line be. The last line being verifying this.

MEI elaborates MI's point, but soon after gives a different perspective on this rule of thumb:

MEI It's more than writing what's given and what you have to prove. It's also accounting in each line … what is the status of what you wrote. Is it given, is it a known fact that you bring from some other place, which is fine. You have to annotate and write where it comes from, how you got to there... If we can infer the following [line], we have to say it... We need these words to make sense of what the status of each line is. Because you'll suddenly ask what is this? How do I know it? It's a mean of communication, but it's also a means of sort of control of what you're doing… …We need to know [where it came from], not just us to follow you, but mainly for you to produce a correct proof. If you skip and don't account for each line, you're more likely to make a mistake.

MEI stresses the importance of accounting for everything that is written. The most obvious reason, stressed by MI earlier in the lesson, is that these are the norms of the genre; this is the way you communicate with others within the mathematical community. However, MEI's words suggest additional considerations. First, in her words "we need to know… not just to follow you" she appears to be conscious of the pedagogical setting, where instructors need to follow the student's thinking in order to assess their work. Additionally, she sees the written proof not only as a means of communicating with others, but also for communicating with oneself (i.e., thinking). Her word "control" alludes to metacognitive aspects of proving, and MEI seems to be suggesting that following norms of writing a proof may contribute to the process of producing a valid proof.

Sample proof 2.2

MI The proof started by saying... this [$(xy)^2 = x^2 \cdot y^2$] is correct, right? Does [it] imply [$\sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2}$]?

MEI The thinking of this person was that they're taking what we need to prove and squaring it and getting there

For MI, it is the proof that is "saying" something, whereas MEI draws attention to the thinking of the person behind the proof.

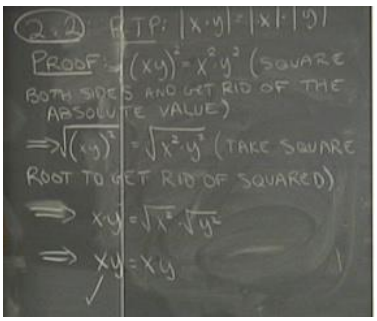

Figure 2: Proof 2.2

MI I'm asking you simply does anyone in this room disagree up to this step?

The implication of MI's question ("does anyone disagree") is that the proof may be correct (at least "up to this step") regardless of the thinking of the person who wrote the proof (e.g., squaring the RTP instead of beginning with the given).

MEI This would be ok without this [striking through lines in the parentheses "square both sides and get rid of the absolute value"]. Because what it says here, that you took this as given in a way, and squared it and got this.

In spite of MEI's previous attention to the prover, she is now showing how the written proof can be fixed by striking out the parts that were a consequence of the prover's misguided thinking.

MI Sorry, I did not read that. I was just looking at the equations. Absolutely right, this is making the same mistake [starting with RTP instead of with Given].

MEI I do want to say very nicely that at least the logic of the thinking was clear here, because whoever did it provided the explanation, and this is easier to follow, and important.

MI, in assessing the correctness of the proof, had not read the text in parentheses. This further supports the suggestion that he is concerned with the proof as a mathematical product (i.e., the equations), and not with the prover's thinking, which is represented in the explanations. MEI, in contrast, values not only the mathematical correctness of the proof, but also the clarity of thinking that is revealed. Here again she is showing concern for pedagogical aspects of MPP.

MI Whoever wrote the proof had a very good idea at the beginning, and choked at this point [$xy = xy$]

Here MI refers for the first time to the person who wrote the proof. However, his claim that this person "had a very good idea" is not justified. This claim was based on the fact that the first line on the board - $\sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2}$ – can serve as the beginning of a valid proof. In this he is ignoring the proving process and the thoughts of the prover, as reflected in the words that MEI crossed out.

Sample proof 2.3

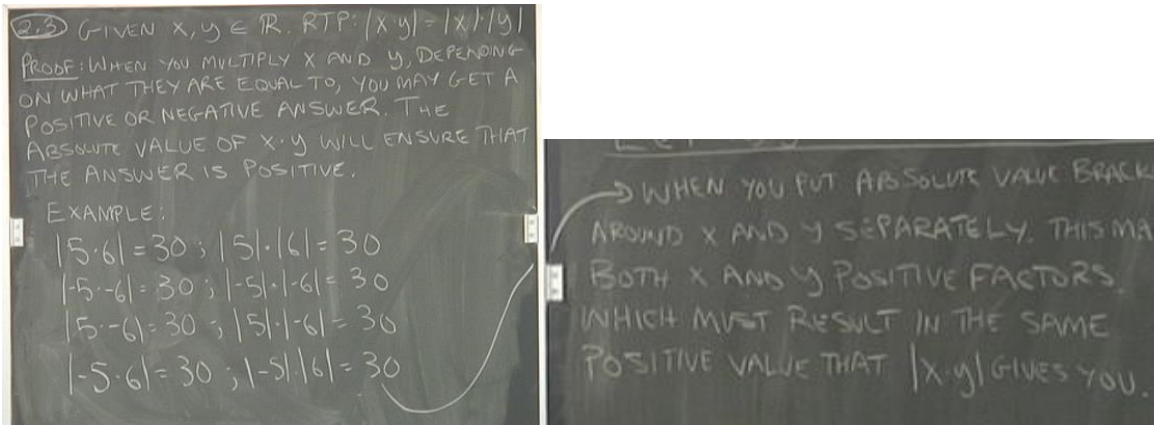

Figure 3: Proof 2.3

MI How many of you think this is a proof? How many of you don't think this is a proof?

MEI What do you think J's opinion is? Does he think it's acceptable or not?

In her question, MEI is allowing for the possibility that the students' opinion will be different from MI's, but she is suggesting that they should be coming around to seeing things Jim's way.

MI It is a proof. It's a badly written proof, but it is a proof. Whoever wrote this made a lousy job of writing [it]. That's the only mistake that person made... because they don't know how to write a proof. How to present it. So the mistake here is not in the content but in the presentation. The person ended up writing two paragraphs for two lines.

These words highlight three aspects of a proof in MI's discourse. There is the end product, what he calls the "presentation", which in this case he considers "lousy" (Jim later rewrites the proof in two condensed lines of mathematical expressions). There is the "content" of the proof, the mathematical

ideas that underlie the presentation and are revealed in it, which in this case are valid. Finally, there is the thinking of the prover, which MEI is keenly aware of, but Jim appears to be ignoring.

Discussion

In this section, we discuss findings from the analyzed episode, along with some additional findings from other episodes whose analysis we omitted for brevity. Regarding our first research question, we found, as hypothesized, that MEI and MI stressed different aspects of MPP. We present similarities and differences between their discourse, as it pertains to proof and proving.

The human element in a mathematical proof

MI and MEI both held the view that a text purporting to be a mathematical proof must adhere to specific norms of communication. For Jim, the question of validity was central: does the proof begin with the given, end with what needs to be proven, and is each line in the proof mathematically justified. This is a consequence of the communicational role of proof in his discourse – to convince members of the community that a claim is valid. MEI, too, was concerned with the mathematical validity of arguments, yet she was also sensitive to pedagogical aspects of proving, and considered the prover's communication with a teacher, who is not only assessing the validity of a mathematical text, but also the nature of the mathematical thinking that produced it. There were also differences in the instructors' attitude to the prover's communication with herself. MI saw two distinct phases in proving, a draft phase (which he called a "scratch") where the prover does her thinking and is not accountable for what is written, and the final product which will be scrutinized by others. Thus, for MEI a student's proof should reveal the prover's underlying thinking, whereas for MI the final product should conceal thinking. MEI held a more integrated view regarding the phases of producing a proof, where the "accounting" in the final product could serve a metacognitive role in the process of proving, by "controlling" the flow and minimizing mistakes. These differences in the instructors' attitudes to proof and proving are evident in their use of language. MI spoke of what the proof "is saying", whereas for MEI it is a human agent who is "saying" something. Jim frequently asked if the students "agree up to this step", where he is referring to a step in the written proof. In sample proof 2.2 MI did not pay attention to the student's thinking, as reflected in the text in parentheses. It was MEI who suggested striking out these lines, but though this would "fix" the proof, she realized that it would not fix the thinking of the person who produced it. MI, on the other hand, when speaking of the process of producing proof 2.2, attributed a "good idea" to the prover, based on the fact that a valid proof could have begun with the first line. He felt that after this promising start the person had "choked", and this just a few seconds after he conceded that the prover had in fact made "the same mistake" as the prover in sample 2.1 - beginning with what needs to be proven.

In spite of MI's attention in sample proofs 2.1 and 2.2 to proof as a text, in proof 2.3 he suggested that this text may be a "representation" (i.e., a visual mediation) for something else. Though the proof was badly written, he felt that there was no mistake in the "content" of the proof. The nature of this underlying content and its relationship with its representation as a text is not clear. MI does not appear to be alluding to a human agent's thinking, but this point requires further analysis of his discourse, drawing on additional data.

The role of proof in mathematical discourse

Later in the lesson MEI pointed out how the pedagogical context of the course may give a distorted view of the role of proving in mathematics: "Mathematicians do proof in order to establish theories… But what happens in this course, because the focus is on how to really construct proofs, sometimes we're doing it about facts that may be trivial to you… we may be giving you a wrong message". This may explain the importance MEI attributed to a comment from MI in lesson 6, regarding a proof of the claim: if $n > 10$, then $n^5 - 6n^4 + 27 \geq 0$. At MEI's prompting, MI showed that a careful analysis of the proof reveals that the expression is not only greater than 0, it is in fact greater than 40,000! Why should this be important? Jim explained that "you might need [this "stronger" fact] later on in the proof". MEI re-voiced this idea, but we suspect that for her it had a second role, in addressing "the wrong message" about the nature of proving. The task, given by some anonymous agent, was to prove "greater than 0". In showing more than was required, MI and MEI were modeling proving as an investigation, where the prover has some agency in deciding what to prove.

Affordances of the co-taught course for future teachers: Explicit and implicit goals

We hypothesized that a course on MPP co-taught by a mathematician and a researcher in mathematics education would have special affordances for futures teachers, in presenting and reconciling different aspects of MPP that are crucial for teaching. Our analysis has demonstrated some affordances.

The explicit goal of the course was transitional – to help students learn the mathematics department's norms of MPP, and to develop proficiency in university level routines of proving. This goal was addressed by both instructors, with Jim taking a leading role. MEI accepted and encouraged Jim's role as the mathematical authority, in calling on him to take over when crucial mathematical issues were at stake. It was MI who modeled the kind of proving that will be expected in advanced university courses. Furthermore, if students internalize MI's discourse, they may eventually bring a commitment to mathematical precision and rigor to their own classrooms. MEI did not disagree with Jim. However, in stressing other aspects of MPP, we feel that she was modeling not only how to produce an acceptable proof, but also how to teach MPP. In her attention to the thinking behind students' proofs she was modeling how these future teachers should be concerned not only with what their students are writing, but also with the mathematical thinking that drives their work. She was further concerned with some meta-level issues, such as the investigative nature of mathematical activity, and the development of metacognitive skills of self-monitoring the process of proving.

Thus, students were exposed to two different perspectives on MPP, both of which they will need to internalize as teachers. However, MI and MEI by and large taught in the *one teach, one observe* approach (Friend et al., 2010), and their points of view remained disjoint. In fact, being aware of their differences and wishing to present a unified front, they often tried to resolve them at the planning stage of lessons (Sabouri, Thoms, & Zaslavsky, 2013). Had they taught in the *teaming* approach, described as "representing opposite views in a debate, illustrating two ways to solve a problem, and so on" (Friend et al., p. 12), there may have been opportunities to openly discuss differences, encouraging students to reconcile different aspects of MPP relevant for their future teaching.

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The organization of study in French business school preparatory classes

Lynn Farah

American University of Paris, France; lgf00@mail.aub.edu

This paper presents parts of a doctoral research pertaining to the study of mathematics in French business school preparatory classes. In what follows, we identify the main features of the institutional devices designed and implemented by two mathematics teachers in their respective classrooms in order to influence and transform the working habits of their students. To do so, we rely on qualitative analysis of data collected mainly through interviews and questionnaires. Our conceptual framework borrows constructs from the Anthropological Theory of Didactics as well as several works in the sociology of education field.

Keywords: Mathematics learning, preparatory classes (CPGE), organization of study, teaching devices, teachers' practices.

Context

Student failure in mathematics during the first university years is a widespread problem in many countries including France, but it does not seem to affect in the same way students of all French higher education institutions. In fact, there are in France alternative institutions such as the *Classes Préparatoires aux Grandes Écoles* (CPGE in what follows) students achieve much better results in mathematics than those enrolled in regular French universities, as is reported in official statistics provided by the ministry of national and higher education and research[1]. The CPGE prepare students over two academic years after obtaining the French baccalaureate to enter the *Grandes Écoles*, which are mainly business schools or engineering schools, by passing the *concours*, national competitive written and oral exams specific to each type of school which students take by the end of the second preparatory year. In the French educational systems, the two preparatory years at the CPGE are equivalent to the first two years of undergraduate study at university and do not lead to obtaining a degree. The CPGE have three streams, scientific (S), business and economics (EC) and literary (L), which each have different tracks.

Our study focuses on the CPGE in the continuation of the work of Castela (2011). These institutions differ widely from regular French universities in elements commonly considered as the main causes of student failure (Farah, 2015b, chap.II, section 4). They are known for their selectivity in recruiting students who have obtained exceedingly above-average results throughout high school and in the French baccalaureate, as well as their supportive culture, which fosters student collaboration and provides them with close follow-up, in a relatively rigid high-school-like system within stable moderate-sized classrooms. In fact, these institutions resemble more the European and North American universities than the French universities in terms of teaching methods and student-teacher relationships. Therefore, it is important to point out that although our study is conducted in a

[1] Source: http://www.enseignementsup-recherche.gouv.fr/cid75181/reussite-et-echec-en-premier-cycle.html

specific environment, its results help us to understand more general issues that concern other institutions in France and in other countries**.**

With the change of institution, from high school to CPGE, students face a significant rift with respect to the work they have to complete in order to succeed in mathematics. In fact, in the CPGE, students are expected to develop without teacher supervision and in addition to the tasks that are prescribed to them, significant autonomous personal work in mathematics that is not necessarily in the continuity of what ensured their prior success in high school. Moreover, assessment in the CPGE is entirely conditioned by the nature of the examinations of the *concours*. In mathematics, it is cumulative, covering the content of both preparatory years, which is never the case neither at university nor even in high school. In this paper, we are interested in the ways these institutions shore up their students and help them develop a new working mode in mathematics, geared towards the CPGE requirements, during the first year of scientific track business school preparatory classes (ECS). Therefore, we focus on the relationships that exist between institutional and students' personal organization of study in mathematics.

Conceptual framework

We are mainly interested in the institutional dimension and its impact on students' learning in mathematics. We claim that through their ways of functioning, the CPGE institutions help their subjects (the students) construct a new working mode in mathematics adapted to the CPGE requirements. We refer to the foundations of the Anthropological Theory of Didactics (ATD) to examine the weight and action of these institutions (Chevallard, 2003). We endorse Chevallard's (2003) description of an institution as a social system that allows and imposes on its subjects, that is people who occupy different positions within the institution, ways of doing and thinking. Subjects are hence submitted to collective constraints and expectations that regulate their actions and thus subjugate them (in French, *assujettir*). For our study, we consider first, at a global level, the CPGE institution within which individuals occupy the positions of student, teacher, and administrative staff… At a local level, we focus first on the teaching of mathematics in the broader sub-institutions, the EC stream then the ECS track. Next, we consider the teaching of mathematics in the school institution. Lastly, we examine the institution of the mathematics classroom of each teacher, with two main positions: teacher and student.

Regardless of the level of institution in question, it is important to bring forward the idea of organizational stability emphasized by Darmon (2013) and Rauscher (2010). Darmon identifies institutional devices that are shared among CPGEs, which put students to work while supervising them. In accordance with the ATD hypotheses, Rauscher advances that subjects of the CPGE institutions occupying the teacher position (per discipline, hence in mathematics in particular) predominantly share common experiences and background traits. They thus form a distinctive social group, as a result of several interacting mechanisms, and take decisions as a team, or a tribe *(tribu)* as Chevallard (2003, p.89) would refer to it. The hypothesized continuity and stability within the CPGE, as to the norms of the teacher profession and the study organization created by each teacher, enables us to foresee the influence of the global CPGE institution on the students' work.

The work of Darmon (2013) in sociology allows us to clarify certain important aspects of the functioning and role of the CPGE institution, through which it exerts its subjugation actions and the

resourcing of students' personal work. Darmon defines a specific type of institution based on the socializing functions of the CPGE and examines it as an institution where a specific type of person is manufactured. According to her, these *"enveloping institutions"* (p.10) shape and transform the students through preparatory institutional socialization processes. Therefore, she analyzes the different daily functioning devices that make it possible for the institution to exert its effects on the students *("surveillance, sanction, examination and pressuring techniques"* p.16). It appears that this subjugation process in undertaken by taking into consideration the individuals involved; this is far from common in higher education practices in France and sounds highly paradoxical. In fact, Darmon puts forward the fact that the CPGE strives to soften the preparatory violence. She describes the institution as being powerful but not totalitarian, violent but concerned about the well-being of its members, it operates by individualizing to the extreme rather than homogenizing, thus reinforcing its take over the individuals which are its members (2013, p.28). Her findings converge with those of Daverne and Dutercq (2013) who put forth the regretted yet accepted pressure to which CPGE students are subjected as well as the personalized adaptation of teaching.

Furthermore, we sought to develop the institutional dimension of our research from the point of view of the teachers. We hence considered two levels: the first once pertains to the way teachers are subjected to the CPGE institution; the second one is related to the mathematics classroom of each teacher, the local institution s/he creates thanks to stable devices which we seek to identify. We believe that the subjugations to the CPGE generate an environment in which each teacher enjoys a given autonomy and can freely express his/her individuality within the boundaries of the common CPGE teacher culture highlighted by Rauscher. Using Darmon's work, we bring forward the CPGE institutional functioning analysis in order to explore how the socializing function is exerted.

Therefore, based on the different didactical and sociological elements of our conceptual framework, we address the following research question in this paper: which institutional devices lead to the transformation of the students' personal work mode in mathematics, at both levels of an institution, ranging from the global CPGE institution to the local teacher classroom institution?

Methodology

A first phase of our study, which is beyond the scope of this paper, was entirely centered on students' personal work in mathematics. Using questionnaires and interviews of first year students from two ECS track preparatory schools in Paris (see Farah, 2015a), we gathered data about the organization of mathematics courses, the teaching methods, the assessment tools and the resources provided by the teachers to put students at work and accompany them in the study of mathematics. Based on this, we sought to approach the practices of the teachers by examining the teaching devices they design and implement in their classrooms as well as their meta discourse (Robert and Robinet, 1993, p.1). We must clarify that the word "discourse" refers to verbal expression, i.e. the use of words to exchange thoughts and ideas. It is not a theoretical construct borrowed from a conceptual framework. As for the word "meta", we refer to Robert & Robinet's definition (1993) whereby a teacher's discourse contains meta elements, i.e. about mathematics and about the ways of doing and learning mathematics. The second phase of our research followed from this.

To answer the research question addressed in this paper, we relied on qualitative analysis of data collected from two mathematics teachers of the schools involved in our study. We started with data

obtained through semi-structured interviews conducted with each of the two teachers about the devices instituted in their classrooms. Then, we designed and had each teacher complete two questionnaires. The first one, inspired from Rauscher's thesis (2010), is about their career path and their choices with respect to teaching in the CPGE, which we believe determine their position and impact their subjugation within the CPGE institution. The second one, inspired from Darmon's book (2013), is about the assessment and pressuring devices the teachers implement in their respective classrooms to put the students to work, as well as their ways of softening preparatory violence in terms of the support and comfort they bring to the students.

We used Qualitative Content Analysis of the interview transcriptions and questionnaire answers to put together a description of institutional devices implemented by each teacher (local) and those common across the different institutional levels (global). The narratives were analyzed thoroughly, manually, line by line, in a search for keywords and vocabulary terms constituting the teachers' discourse about the ways students should study mathematics, while focusing on anything pertaining to institutionalization, regularity, and insistence on specific actions by the students or the teachers. Our search was structured around the following themes that determined the analysis rubrics of our content analysis: taking notes in class, managing work and revisions overall, studying between two mathematics sessions, using resources, preparing for an exam, the *colles*[2], collaboration between students, student difficulties. We then resorted to triangulation to confirm the information obtained from the teacher-designed instruments by comparing it with what we had gathered in the first phase of our study through the students. We must clarify that, besides the things that converge with the information gathered from the students, we had very few elements that would allow us to determine the propinquity between the teachers' statements and what actually takes place in their classrooms. In fact, one could be surprised that, in an analysis of teaching practices, there have been few filed observations. This limit is due to practical constraints in terms the duration of a doctoral thesis. The final output of our analysis is presented in the form of a description of the different institutional devices that organize and shape students' personal work in mathematics.

Main findings

The findings show that the teachers seek to put their students to work and mold their study methods in mathematics through numerous collective devices instituted in their classrooms. In addition, they closely follow-up on each student's work in mathematics through customized individual devices. Thanks to the latter, the teachers develop and apply diverse pressuring techniques in order to ensure the students' intellectual training and their successful passing of the *concours*. We provide below a description of the main devices, which are either dictated by the global organization of mathematics study in the CPGE institution and thus revealing how the teachers are subjugated to their institution, or specific to one of the more local sub-institutions (for more details, see Farah, in press).

[2] A *colle* is an assessment tool specific to preparatory classes. In mathematics, it classically takes the form of a one-hour bi-monthly oral examination, in groups of three students working individually but simultaneously, answering lesson questions and/or solving problems on the board, managed by a *colleur* who is present to supervise and grade the work.

The teacher's course and the follow-up beyond

The mathematics course organization and progression are the first aspects of guidance to students' work. For both teachers, when they explain a mathematics lesson, their first priority is to retain the students' attention while encouraging them to actively participate by regularly asking questions. The lesson is completed and illustrated through examples and exercises, which are solved in class or at home, then corrected in class. During regular classroom sessions, if needed, teachers wrap up the work that they have previously started during practical solving sessions (called *Travaux Dirigés* or TD). Theses special sessions give students room to work on exercises in small groups, thus fostering discussions with the teacher as well as classmates.

Both professors involved in our study use a handout as the baseline for the lesson explanation; they distribute it to students either systematically or occasionally. Depending on the teacher, the class or the chapter, this handout can be exhaustive or having blanks to complete, and teachers modify it regularly in order to tailor its contents to the level of the students and their capabilities (concentration, understanding, note taking ability) and the course pace is slowed down or increased accordingly. The main objective behind the use of such a device is to save note-taking time in class and ensure that students don't make mistakes in copying key elements. A typical handout contains mathematical definitions and notations, propositions and theorems with occasional short proofs, lesson examples and application exercises. During the lesson, the teachers spend most of the time completing the missing proofs then provide additional examples. They explain to students that the proofs are the basis of mathematics, and repeatedly underline the practical and generic aspects to be extracted. On the contrary, little importance is given to statements of theorems. This is an example of a specificity of mathematics teaching in this CPGE stream as opposed to the insistence on academic knowledge in universities. In addition, the teachers formulate several remarks that are not solely about theoretical mathematical knowledge. In fact, in addition to the mathematical content, the teachers make comments related to practical knowledge. These are part of their meta discourse which contains technological elements (Castela, 2011) used to bring forward the know-hows linked to the mathematical content, thus allowing them to accompany students in their study.

In addition to the time dedicated to lesson explanation, exercises solving and correction, both teachers ensure to always be available to assist the students in the learning of mathematics outside the classroom. They are willing to answer questions, provide explanations, recommend and even correct additional work despite believing that the workload they assign is already enough (regular exercise sheets and occasional extra exercise sheets with their correction for some chapters). They usually encourage students not to look for more resources (textbooks, online) and focus on what they provide due to time constraints. Moreover, the teachers hold weekly tutoring sessions to ensure that students are getting all the needed help within the institution. Through their extended availability and individualized follow-up, the teachers are ensuring that all their students are provided with the necessary assistance for their learning, while they control and organize their study. They are thus softening the preparatory violence through surveillance and examination. This is one important manifestation of the CPGE teacher culture that is absent in French universities.

The recurring discourse about the ways of studying

The teachers encourage their students to regularly study their mathematics lessons and solve the assigned exercises (for both the regular sessions and the TD) and they always explain to them how they should proceed to do so. The teachers emphasize the importance of reading a mathematics lesson actively and critically. The objective is learning the keys lesson elements while thinking about them and asking the right questions to first understand then memorize. According to both teachers, validation of the learning should be done by playing-back important lesson contents mentally, then preferably in writing.

In addition, they both stress the crucial role of decontextualizing in mathematics learning. To do so, they underline the significance of both the results brought through proof and the use of generic components of reasoning, in addition to the techniques used in standard exercises which students must be able to acquire and plough back in other situations. In fact, they preach strategic exercise solving whereby students are expected to identify standard problems and recognize methods, techniques and tricks that can be used to solve them, which can also be applied to other problem situations. Thus, according to both teachers, students should have a transfer-oriented approach to exercises rather than one that favors only practice or reproduction (Castela, 2011), the latter are dominant among successful university students but are deemed ineffective in the CPGE. The teachers also insist on the necessity of doubling efforts until mastery is attained when facing difficulties in solving an exercise.

We can refer to the notion of constructive help proposed by the teachers to guide students in studying the lesson, solving exercises and decontextualizing proof and exercises, when working on a daily basis between two mathematics sessions, as well as when preparing for an exam. In fact, we have identified several features of help common to both teachers in their discourse, about expected ways of studying mathematics and practical knowledge pertaining to the techniques which could help students gain know-hows relating to the awaited tasks. These illustrate the convergence of learning methods regularly repeated by two different teachers of the same stream and track.

The assessment tools

In order to ensure that the students are completing the assigned work (lesson and exercises) and to identify their weaknesses and difficulties in mathematics before the graded exams, teachers use personalized informal evaluation techniques during classroom sessions (both regular and TD). They often resort to oral interrogations about the lesson notions by randomly calling on students or choosing those who are inattentive or fall behind. Also, while the students are solving exercises in class, the teachers go around to check what they have done, they assess their understanding and help when needed. Then, the teachers encourage the students to engage in discussions about the exercises' solutions before correcting them or asking a student to do so. One of the two teachers gives special care to exercises preparation by the students prior to class. In order to push students to maintain regular work, he periodically calls students to the board and collects notebooks without prior warning whenever he notices that the work has not been fully done, without necessary grading any. These are all examples of surveillance and sanction techniques that allow the institution to monitor and redirect the work of the students' work.

The teachers have several types of more formal assessment devices, institutionalized at the global CPGE level, which allow them to evaluate the degree of investment and understanding of their students. Firstly, they use all sorts of written evaluations. Teachers mainly resort to short quizzes focused on the mathematics lesson content (definitions, theorems…) at the beginning of the school years to push the students to study, however they state that they cannot maintain them throughout the year due to time constraints. They also have monthly exams (called *Devoirs Surveillés* or DS), and bi or triannual mock *concours* which are summative and are conducted in conditions similar to the official *concours*. One of the teachers quizzes his students about the correction of previous DS exams thus allowing the students to detect and address their weaknesses. In addition, teachers assign and grade homework sets (called *Devoirs Maison* or DM) on a monthly basis and they usually invite students to work on those in small groups. All of the above are examination and pressuring techniques used across the CPGE institution, with specificities of each sub-institution.

Last but not least, the *colles* are the most important assessment tool that teachers use to evaluate their students in a highly customized manner. We summarize the main perks they list about this institutional device, which are for most specific to the case of mathematics *colles* in the ECS track, since their organization and functioning changes across disciplines, tracks and streams. The *colles* impose on the students a work and study regularity, which is certainly stressful and tiring for some, but the pressure is eventually seen as beneficial for the majority. Mathematic *colles* sessions are described as similar to private tutoring sessions where students can discover their weaknesses, ask questions, obtain additional explanations and a new point of view, and practice by solving additional exercises. Further to these mathematical learning related aspects, the *colles* are characterized by their interpersonal feature and the know-hows and social skills they teach (stress management, oral presentation, self confidence) which go beyond the scope of the classroom or even the school. Therefore, the *colles* are viewed as a summary of the best things the CPGE have to offer in terms of learning environment for their students (Daverne and Dutercq, 2013, p.182). They are to many teachers the secret to students' success in CPGE (ibidem, p. 182), despite the difficulties and constraints they are subjected to.

Discussion and conclusions

On one hand, we can conclude that the teachers who took part in our study are heavily involved in their students' learning. To accommodate the needs and level of a *"new population"* (ibidem, p.7) of CPGE students, more diversified in terms of academic and social backgrounds, teachers redefine their teaching modalities and pedagogical devices and adjust the level of their expectations. Daverne and Dutercq state that if some young students have good working habits when they enroll in the CPGE, none yet have the general culture nor the confidence needed to face the concours, which requires from teachers a high level of commitment towards them and a constant care for their moral (ibidem, p.8). Hence, the teachers participate in the didactical and pedagogical organization of their students' autonomous study thanks to the advice they provide and the devices they institute and regularly adapt according to their needs and capabilities. They are therefore clearly dedicated to their students' success. This is also reflected through the closeness in the student/ teacher relationships, which we do not tackle in this paper (for more information, see Farah, 2015b).

On the other hand, although the use of the varied pressuring techniques in mathematics differs among teachers and depending on the students' dispositions, the techniques themselves remain redundant across teachers and classes. This brings forward their generality and continuity within the EC stream of the CPGE institution, of which they become a specificity. As a matter of fact, we find in the teachers' discourse common features underlining the coherence in the practices of teacher tribes per class as well as the stability of practices within each preparatory school, within the ECS track, and even within the entire EC stream. Regardless of the level of the institution, the devices used are specific to the teaching and learning of mathematics, even though we do not examine them with respect to a specific mathematical content in this paper. We conclude that the coherence of practices noted between the two teachers involved in our study concurs with what the sociological studies of Rauscher (2010), Darmon (2013), and Daverne and Dutercq (2013) have identified.

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How mathematicians conjecture and prove: An approach from mathematics education

Aurora Fernández-León, Rocío Toscano-Barragán and José María Gavilán-Izquierdo

University of Seville, Department of Didactics of Mathematics, Seville, Spain

aurorafl@us.es, rtoscano@us.es, gavilan@us.es

This work makes a contribution to the research line that studies the mathematical practices of research mathematicians, when developing their research, with the aim of improving the teaching and learning of such practices in an educational context. To be precise, we focus on the mathematical practices of conjecturing and proving in order to identify their characteristics as a basis to formulate a model. To address this problem, we consider Rasmussen, Zandieh, King and Teppo's (2005) theoretical constructs (horizontal and vertical mathematising) and report results from a case study of a research mathematician.

Keywords: advancing mathematical activity, conjecturing, proving, research mathematicians.

Introduction

Freudenthal (1973), in relation to mathematics and its learning, asserts that

> there is no doubt that pupils should learn mathematizing […] There is no mathematics without mathematizing […] This is what follows from the interpretation of mathematics as an activity. (p. 134)

On the other hand, Dreyfus (1991) points out the relevance of mathematising by highlighting the processes through which mathematical knowledge is constructed (discovering, defining, proving, modelling, etc.). These processes are relevant research focuses in mathematics education by the importance of its teaching and learning. In this sense, assuming mathematics as an activity, Rasmussen, Zandieh, King and Teppo (2005) also note the significance of considering mathematical practices developed in a learning setting.

From the mathematics education perspective, mathematicians have been seen as study subjects. In fact, Tall (1991) already points out that "we are cognizant of the fact that it is essential to understand the nature of the thinking of mathematical experts to see the full spectrum of mathematical growth" (p. 3). In this regard, several researchers study mathematical practices of mathematicians with the aim of proposing models that describe such practices. For instance, Ouvrier-Buffet (2015) proposes a model that describes how mathematicians develop the mathematical practice of defining. Weber and collaborators focus on aspects of mathematician's work associated to the process of proving (Mejia-Ramos & Weber, 2014; Weber, Inglis & Mejia-Ramos, 2014; Weber & Mejia-Ramos, 2011). Furthermore, Weber and Mejia-Ramos (2011) devise "a model for how mathematicians read the proofs of others" (p. 341). These inquiries lead us to consider the study of the mathematical activities that take place when research mathematicians generate conjectures and proofs.

In our study, conjecturing and proving refer to those activities carried out to generate conjectures and proofs respectively. For us a conjecture is a statement that can be true or false, appears reasonable, "has not been convincingly justified and yet it is not known to be contradicted by any examples, nor is it known to have any consequences which are false" (Mason, Burton & Stacey, 1982, p. 58). On

the other hand, we consider Weber and Mejia-Ramos's (2011) definition of proof: "the socially sanctioned written product that results from mathematicians' attempt to justify why a conjecture is true" (p. 331).

For two reasons, we consider the practices of conjecturing and proving as two sides of the same coin. Firstly, for reasons related to mathematics, Peirce's contributions (1997) on the three type of reasoning: abduction, induction and deduction, applied to mathematics, justify the joint consideration of both processes (Fernández-León, Toscano & Gavilán-Izquierdo, 2016). Specifically, abduction refers to the provisional adoption of a hypothesis, deduction traces out the probable and necessary consequences of a hypothesis and induction is the verification of a hypothesis by experiments (Peirce, 1997). Secondly, from mathematics education, Alibert and Thomas (1991) point out that "[t]he formulation of conjectures and the development of proofs are two fundamental aspects of a professional mathematician's work" (p. 215).

This work contributes to the research line that studies the mathematical practices of research mathematicians, when developing their research, with the aim of improving the teaching and learning of such practices in an educational context. To be precise, we focus on the mathematical practices of conjecturing and proving in order to identify their characteristics as a basis to formulate a model.

Mathematisation: Theoretical perspectives

In the last decades, the specific meaning of the term "advanced mathematical thinking" has been a subject under discussion. Tall (1992) associates this expression with the formal use of definitions to describe concepts and the logical deductions of theorems based upon them. He points out that such grade of sophistication is the highest level of mathematical thinking, but not the activities to reach it. With this in mind, Rasmussen et al. (2005) propose an alternative characterisation of advanced mathematical thinking, called "advancing mathematical activity", that mainly focuses on general mathematical practices (defining, classifying, conjecturing, etc.) instead of on particular mathematical contents. They emphasise the progression and evolution of students' reasoning in relation to their previous activity when participating in a variety of different socially or culturally situated mathematical practices.

Rasmussen et al. (2005) also suggest that each mathematical practice can be described by using two different dimensions, the so-called horizontal mathematising and vertical mathematising. These two terms are firstly used by Treffers (1987) to describe what he calls "progressive mathematising". Treffers refers to horizontal mathematising as the activities of "transforming a problem field into a mathematical problem" (p. 247) and to vertical mathematising as those proper activities of the process of reorganisation within the mathematical system itself. Notice that the idea of mathematisation is originally formulated by Freudenthal (1973), who understands mathematics as a human activity. This author indicates that "there is not mathematics without mathematizing" (op. cit., p. 134). He defines mathematisation as the activity of organising matter from reality and within the mathematics discipline. Freudenthal (1991) assumes Treffers' dimensions, although expressing their meanings in the following way:

> horizontal mathematisation leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly; this is vertical mathematisation. (Freudenthal, 1991, pp. 41–42)

For him, these two forms of mathematisation are not separated worlds, have the same status in practice and can take place at all levels of mathematical activity.

In our research, we assume Rasmussen et al.'s (2005) approach about horizontal and vertical mathematising. These authors also slightly adapt and modify Treffer's constructs. For them, horizontal mathematising refers to those activities used to formulate a problem situation in such a way that it can be mathematically addressed subsequently. Thus, horizontal mathematising also includes problem situations that are properly mathematical and is mainly related to initial or informal ways of reasoning. On the other hand, vertical mathematising refers to those activities built on horizontal activities with the aim of creating new mathematical ideas or realities. They use these constructs to characterise the practices of symbolising, algorithmatising and defining. An important coincidence they find among these three mathematical practices is the relation between "creating" and "using". They argue that both actions occur when these practices are carried out, although with a different role in each dimension. In particular, in horizontal mathematisation, people create (definitions, algorithms, etc.) "to express, support, and communicate ideas that were more or less already familiar" (Rasmussen et al., 2005, p. 70) and products of this dimension are used within their mathematical problematic situation. On the other hand, in vertical mathematisation, new mathematical realities are created and using promotes "movement from the particular to the more general and in some cases the more formal" (op. cit., pp. 70–71). The authors state that vertical activities often give rise to other horizontal activities. Vertical mathematisation may be the setting for a new horizontal mathematisation, which subsequently can lead to vertical mathematisation, and so on, creating a chain of progressive mathematisations.

Our study aims to identify characteristics of the practices of conjecturing and proving of research mathematicians to describe and explain how they develop them. For this purpose, horizontal and vertical dimensions proposed by Rasmussen et al. (2005) are considered. Thus, the research question behind this study is: Can horizontal and vertical mathematising constructs describe and explain the mathematical practices of conjecturing and proving of research mathematicians?

Methodology

In this research, we assume a qualitative methodology. In particular, we have adopted a case study methodological approach. With the aim of answering the research question above, we consider an inductive analysis, that is, the different categories emerge from the data. In this work, we discuss one case of a research mathematician.

Participant and context

The participant (Anna, pseudonym) is a research mathematician, understanding as such those who have a Ph.D. in mathematics and have published research papers also in mathematics. Specifically, Anna is a teacher that researches in mathematical analysis (functional analysis) and has more than five years of experience in university teaching. The case study of Anna presented here is part of a larger research study which aims, among others, to refine and thus improve the analysis shown below. The results reported in this paper are based only on this participant's case.

The research instruments

The data for our study are obtained from different sources: interviews, working documents and research reports. Four semi-structured interviews are conducted. The first of them aims to obtain basic information from the researcher; the following interviews revolve around the mathematical research carried out by Anna. Specifically, we discuss her research results collected in different research reports (papers, posters and beamer presentations in conferences). We also talk about the personal working documents used in the development of her research. Some of these documents are related to situations that led to successful outcomes and some others to situations which were less successful.

Data analysis

The analysis process (interpretative analysis) considers the dimensions horizontal and vertical mathematising from Rasmussen et al.'s (2005) approach to organise the mathematical activities that take place when conjecturing and proving. Specifically, this process identifies relevant events in the data and assigns meanings according to the theoretical framework: first, they are classified as actions linked to the mathematical practices of conjecturing or proving. These actions are subsequently linked to the horizontal or vertical dimension of the corresponding practice, according to the characteristics of these two dimensions. For instance, when proving, the activity of *Detecting patterns in examples* (see description below) is considered horizontal since when detecting patterns in examples one systematically expresses or formulates (based on experimentation) how certain property (the problem situation) holds in such a way its generalisation can be addressed in the vertical category *Formalising findings with examples.* Once that double assignment is finished (conjecturing-proving and horizontal-vertical), taking the nature of the events in consideration, the categories emerge in such a way that each event is seen as an "example" of a category.

Results

In this section, categories of activities that emerged from the analysis of the data are presented. Consequently, the offered classification is not based either on the content of the research (geometry, analysis, etc.) or the mathematical method considered in each practice (proof by contradiction, etc.). Although space prevents the exemplification of each given category, we provide several excerpts from Anna's answers and documents to illustrate some of them. Specifically, we denote each example by "Example x.y", where "x", that may be 1 or 2, refers to the mathematical research situations in which the example arises and "y", that varies between 1 and 6, indicates the precise moment (in the temporary sequence) of the mathematical research situation in which the excerpt appears.

Although conjecturing and proving are closely related practices (Fernández-León et al., 2016), we describe them separately for expository reasons. Notice that the informal character of horizontal mathematising and the almost simultaneity of these two mathematical practices make complicated to differentiate if the features of the researcher's informal activities are linked to the construction of a proof or, on the contrary, of a conjecture.

How mathematicians conjecture

Three categories of horizontal nature (C.H.a, C.H.b, C.H.c) and two of vertical one (C.V.a, C.V.b) have been characterised during the analysis of Anna's case. These categories do not describe a linear

process, but they are interconnected many times. We begin by describing the activities that characterise the horizontal component of the practice of conjecturing.

C.H.a) *Detecting patterns*. Experimentation with mathematical objects (a triangle, a number, a Hilbert space…) and in relation to a certain characteristic or observable property. To be more precise, logical reasoning and informal activities with mathematical objects involved in detecting a certain pattern in a concrete mathematical context.

Example 1.1- Anna: When dealing with the research question about whether all complete CAT(0) spaces satisfy the (Q4) condition, we started to check what happened to spaces with constant curvature, since the other two extreme cases had already been checked by the authors of the paper (Hilbert spaces and R-trees). Firstly, we checked that the hyperbolic space, with curvature -1, satisfied property (Q4). That conclusion led us to a conjecture.

Example 2.1- Anna: We considered the analytic expression of the modulus of convexity of the sphere, a geodesic space that is not linear: $\delta(r,\varepsilon)=1-\frac{1}{r}\arccos\left(\frac{\cos r}{\cos\frac{\varepsilon}{2}}\right)$; and tried to prove its monotonicity with respect to "r" through the first derivative. We did many calculations but no conclusion could be established. Many experiments with the software Mathematica were also done to see if the modulus of convexity of the sphere was monotone with respect to the variable r. However, we couldn't derive any conclusion from hand calculations, which was the necessary for any future publication.

C.H.b) *Testing conjectures*. Verification or rejection of a certain conjecture by specific examples.

Example 1.3- Anna: We started to checked property (Q4) in more CAT(0) spaces, specifically, on gluing CAT(0) spaces. These experiments allowed us to reject the conjecture "every CAT(0) space has property (Q4)".

C.H.c) *Modifying statements*. Experimentation with the components of an already existent conditional proposition (proved or not, that is, a proved proposition or a conjecture) consisting in modifying its hypothesis or conclusion.

The activities that characterise the vertical component of the practice of conjecturing are given next.

C.V.a) *Formalising patterns*. Generalisation and formalisation of a certain pattern observed in horizontal mathematising activities. Specifically, a pattern observed in the horizontal dimension is used to formulate what is known as a conjecture.

Example 1.2- Anna: We conjectured that "every CAT(0) space has property (Q4)".

Example 2.2- Anna: After seeing many different plots with Mathematica of the cited function, we conjectured that "The modulus of convexity of the sphere is nonincreasing with respect to r".

C.V.b) *Formalising modifications of statements*. Formalisation of the modifications of the hypothesis or conclusion of an already existent conditional proposition (proved or not) which gives rise to a conjecture.

Example 1.4- Anna: We changed the conjecture on the (Q4) property by formulating a new one that was more probable in the light of the example checked before (gluing CAT(0) space): "any CAT(0) space with constant curvature satisfies the (Q4) condition".

How mathematicians prove

Two categories of horizontal nature (P.H.a, P.H.b) and four of vertical one (P.V.a, P.V.b, P.V.c, P.V.d) have emerged from the data. We start by describing the activities that characterise the horizontal component of the practice of proving.

P.H.a) *Detecting techniques or tools within proofs*. Careful study and examination of the characteristics and steps of other proofs related to the proof to be built. When dealing with the construction of a new proof, it is common to seek proof techniques, in other proofs, that may fit in well with the new proof. Notice that this description is consistent with the claim by Rasmussen et al. (2005) that many specific activities of this dimension are of organisational and clarifying type.

P.H.b) *Detecting patterns in examples.* Experimentation with specific examples that satisfy the hypotheses of a given conjecture with the aim of detecting patterns that could be extended to more general settings for the proof in process.

Example 1.5- Anna: After formulating the new conjecture, we considered another space of constant curvature, the sphere, to see which pattern was followed when checking the (Q4) condition in such space. We observed that it was very similar in both spaces.

In the sequel, the activities that characterise the vertical component of the practice of proving are given.

P.V.a) *Selecting and applying demonstration methods.* Selection and application of demonstration methods (direct, by contraposition, by contradiction, etc.).

P.V.b) *Using proof techniques.* Use of proof techniques or tools found in the horizontal dimension.

P.V.c) *Applying known results.* Application of known results to build chains of logical implications.

P.V.d) *Formalising findings with examples.* Extension and formalisation of the findings and calculations with examples in the horizontal dimension.

Example 1.6- Anna: After revising those examples, we extended the calculations in the examples to general cases and proved the conjecture. In fact, what was done was the following: we wrote the proof in the hyperbolic space and wrote what to do in the general case. This type of reasoning is very common in our papers if the extension is very easy and useless.

Results of this study may show the still strong influence of formalism on mathematical research.

Conclusions

In this research, different categories are identified (and organised through the constructs horizontal and vertical mathematising) to describe and explain the mathematical practices of conjecturing and proving. These categories are consistent with previous results of other studies, for instance, inductive reasoning described by Peirce (1997) is closely related to our category *Testing conjectures.* The "Examples x.y" shown in the previous section corroborate, for both practices, the interrelationship between horizontal and vertical mathematising activities. Notice that the identification of these different categories may contribute to elaborate a model that characterises both mathematical practices.

The categories identified may give information about instruction processes to improve students' understanding of the practices of conjecturing and proving and of their products (conjectures and proofs). Such understanding highlights and emphasises the duality conjecturing/conjecture and proving/proof, so that it is consistent with what mathematics (as a product of mathematicians) reveals. On the other hand, Harel and Sowder (1998) identify three different students' proof schemes: external conviction proof schemes, empirical proof schemes and analytical proof schemes. A tentative idea to develop in future works is to check whether the role we have identified induction to play in the horizontal dimension of the practice of proving, *Detecting patterns in examples*, may help us to facilitate the transition of students from empirical schemes to analytical schemes.

We characterise the results of this inquiry as exploratory. That is, the classification described above must be refined and validated, in future research, in other different contexts (with other mathematical contents, other mathematical cognitive levels, etc.).

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Students' interpretation of the derivative in an economic context

Frank Feudel

University of Paderborn, Institute of Mathematics, Germany; feudel@math.upb.de

The concept of derivative plays a major role in economics. One important competence for students of economics is to interpret values of the derivative in an economic context. In the study presented in this paper it was investigated how students of economics interpreted values of the derivative in an economic context before and after their Calculus course at university. Before the course only very few students were able to interpret these values adequately. After the course about half of the students were able to state an adequate economic interpretation of values of the derivative, mostly as amount of change of the function while increasing the output by one (marginal) unit, which is a common interpretation in economics. However, the data indicates that many of these students just identified the derivative with that amount of change without understanding the differences and the connection between these two different mathematical objects.

Keywords: Derivative, economics, concept image, economic interpretation, marginal cost.

Introduction and embedding of the research

In order to maximize their profit firms have to make many profound decisions, for example about possible investments. An important tool that aids economists to make optimal decisions is marginal analysis. Baumol and Blinder (2015) listed marginal analysis even as one of the most important ideas of economy. In marginal analysis the consequences of making relative small changes from the current situation are examined (Ruffin & Gregory, 1990). Typical problems are the effects on cost or revenue if the output is increased by a small amount (normally one unit), or the effects on the demand of a product if the price is increased by a small amount. Marginal analysis is closely connected with the mathematical concept of derivative, which serves as a tool to measure the effects of these small changes on cost or revenue. Therefore, students of economics should have a proper understanding of the mathematical concept of derivative and its use in marginal analysis, which is currently investigated in my PhD-Thesis (supervised by Prof. Dr. Rolf Biehler).

Unlike for students of engineering or physical science, whose understanding of the derivative has been examined more extensively (Bezuidenhout, 1998; Bingolbali & Monaghan, 2008; Çetin, 2009; Maull & Berry, 2000), only little research about the understanding students of economics have of this concept exists. Only two studies are known to the author dealing with the understanding of rate of change by students of economics (Mkhatswa & Doerr, 2015; Wilhelm & Confrey, 2003), and none of these involves the concept of derivative explicitly. To use the derivative in marginal analysis the students have to interpret calculated values of the derivative in an economic context. Interpreting values of the derivative in contexts is not easy for students, as shown by Bezuidenhout (1998) for students of engineering. He showed, for example, that many of these students could not interpret values of the derivative of a stopping distance function $S(v)$ (v is the velocity of a vehicle) adequately. They overgeneralized that the derivative is the acceleration or the velocity, or they had problems with the unit, which shows that they did not have a profound understanding of the derivative itself as rate of change. The economic interpretation of the derivative has the additional difficulty that it does not directly correspond to any of the usual mathematical representations of the

derivative, which is explained in detail in the next section. Therefore, it can be expected that students of economics have even more difficulties in understanding this interpretation. To address this conjecture, the study presented in this paper is guided by the following research question: "How do students of economics interpret the derivative in an economic context before and after their Calculus course?" The results will extend the knowledge about students' difficulties in the understanding of the derivative with a focus on students of economics, who have been rarely considered until now.

Theoretical background of the study

The economic interpretation of the derivative

In physical contexts, the derivative is often interpreted as rate of change (for example as speed), which directly corresponds to other representations of the derivative like the slope or the limit of the difference quotient. The common economic interpretation, however, represents a different mathematical object, which is now explained for cost functions. If $C:[0;\infty[\rightarrow[0;\infty[$ is a cost function and $C(x)$ represents the cost of a given output x, $C'(x)$ (called marginal cost) is often interpreted as the additional cost while increasing the output by one unit (Schierenbeck & Wöhle, 2003). This additional cost, exactly calculated by $C(x+1)-C(x)$, differs from the derivative $C'(x)$ in its numerical value and in the unit. Hence, this interpretation has to be connected to the students' other knowledge of the derivative and needs to be justified. A typical justification is via the approximation formula $C(x+h)-C(x)\approx C'(x)\cdot h$ for h close to 0 with the additional argument that $h=1$ is really small in economic contexts. A more detailed description of the connection between the derivative and its economic interpretation as additional cost can be found in Feudel (2016).

Some books of economics add the word "marginal" to the unit when describing marginal cost. The marginal cost is then the cost that arises if the output is increased by a marginal unit and is written as $C'=\frac{dC}{dx}$ (e.g. in Reiß (2007)). In economic literature the term "marginal unit" is often used as a synonym for a very small, but finite, unit. In Dyckhoff (2002), for example, the term is used to emphasize that one unit is small enough and that the derivative $C'(x)$ can be used as approximation for the difference quotient $\frac{\Delta C}{\Delta x}$ for $\Delta x = 1$. Reiß (2007) emphasizes furthermore that it depends on the context whether a unit can be considered as marginal or not (example of the book: if water utility is measured in cubic meters a marginal unit might be a cubic millimeter).

Theoretical tools for the study

An adequate economic interpretation should be part of the students' conceptual knowledge of the derivative, which can be described with the term *concept image* by Tall & Vinner (1981). The *concept image* describes the total cognitive structure that is associated with the concept. This includes are all mental pictures, associated properties and processes. Concerning the derivative the students' *concept image* should contain the different representations of it, the differentiation rules, connections to other mathematical concepts like monotonicity, and in the case of students of economics also an adequate economic interpretation of the concept. To understand this interpretation properly students should connect it to other knowledge of the derivative they already have from school. The process of making these connections, called synthesizing by Dreyfus (2002),

is a special challenge in the case of the common economic interpretation of the derivative as additional cost because this interpretation does not directly correspond to any of the other representations of the derivative, and, as described above, justifying it needs some argumentation.

Knowledge about the derivative covered in the students' Calculus course

In the course the study took place in (University of Paderborn, Germany), the derivative concept itself was covered in two lectures. In the first lecture the definition of the derivative as limit of the difference quotient, which was also visualized by secant lines "converging" to the tangent line, and the differentiation rules were presented. Afterwards the unit of the derivative in comparison with the unit of the original function was discussed in the case of a cost function *C*. In the second lecture the economic interpretation was presented and justified. Two possibilities were given in the course:

1. Approximation of the additional cost of the next unit

This interpretation was justified in the course via the above mentioned approximation formula $\Delta C \approx C'(x)\cdot\Delta x$ that was derived by deencapsulating the limit in the definition of the derivative and using the limit as an approximation of the difference quotient.

2. Additional cost of the next marginal unit

It was illustrated in the lecture with the help of the tangent line in the case of a convex function that the error between ΔC and $C'(x)\cdot\Delta x$ in the approximation $\Delta C \approx C'(x)\cdot\Delta x$ becomes smaller, the smaller Δx is. It was explained that the limiting process $\Delta x \to 0$ results in a fictional equation $dC = C'(x)dx$, in which dx was called a marginal unit.

Some lectures later, after the introduction of the concepts of monotonicity and convexity, the connection between the derivative and these two concepts was discussed. In a last step the derivative was used as a tool to solve optimization problems.

Besides the lectures the students had to solve problems referring to the content of the lectures. The problems were solved by the students in small groups. One week later the solutions were presented on the board in the lecture hall. Relevant for this study is that the problems also included a task to interpret the value $C'(5)$ of the cost function $C(x) = 8x^2 + 10x + 700$ in an economic context.

Methodology of the study

Data Collection

Students of economics at the University of Paderborn were administered a pre-test addressing their previous knowledge of the derivative concept in September 2015 in a voluntary bridging course before their math courses. The pretest contained the following task, to check if an adequate economic interpretation was part of the students' *concept image*:

> A company produces pens. The cost (in euro) for the production of a number of *x* pens can be described with a cost function with the following equation:
>
> $C(x) = \frac{1}{30000}x^3 - \frac{1}{100}x^2 + 2x,\ x \geq 0$. It can be determined that $C'(200) = 2$. Interpret this result in the above context.

After their Calculus course in February 2016 the students had to take an exam to finish the course successfully. In the exam the students also had to answer a similar task:

> Let $P:[0;\infty[\rightarrow \mathrm{R}$ be a profit function of a company, which manufactures a product in an unlimited and indivisible amount. The profit is measured in units of money, the output measured in units of quantity. It is known that the derivative function *P'* is called marginal profit. You get to know that $P'(73)=0.2\,\frac{\mathrm{GE}}{ME}$ (GE = units of money, ME= units of quantity).
>
> State an economic interpretation of this value.

As described in the previous section, the students were familiar with that type of task.

Data analysis

The answers to the two tasks were categorized by the author with quantitative content analysis. Besides two categories given by the interpretations "additional cost of the next unit" and "additional cost of the next marginal unit", which were derived from literature in economics (see "Theoretical background"), the categories were created inductively because it was not clear what answers the students might state. This led to a system of 10 categories for the task in the pretest (other answers, which were all wrong, were given by less than 3% of the students). The answers in the first three categories were adequate economic interpretations (detailed descriptions in Figure 1).

Category	Description	Prototypical Statement
1. Additional cost of the next unit	The students state that $C'(200)=2$ means that the additional cost of the next unit (if 200 units are produced) is 2 euro (or round about 2 euro).	"It means that if the production is increased from 200 units by one unit, the cost increases by 2 euro."
2. Additional cost of the next marginal unit	The students state that $C'(200)=2$ means the additional cost of the next marginal unit is 2 euro or round about 2 euro. It was ignored if the units of money are also called "marginal".	Not mentioned in the pretest. An example from the exam (interpretation of $P'(73)=0.2$ for a profit function *P*) is: "At a production of 73 units of quantity the production of one more marginal unit would increase the profit by 0.2 units of money."
3. Growth rate of cost	The students state that $C'(200)=2$ means that the cost increases by 2 euro per pen (at a production of 200 pens).	"At a production of 200 pens the cost increases by 2 euro per pen."

Figure 1: Adequate response categories for the task to interpret $C'(200)=2$ for a cost function *C* in an economic context

The answers in the categories 4-10 were not adequate economic interpretations (detailed description in Figure 2): the answers in categories 4-5 contained at least a correct idea, the answers in categories 6-8 were wrong economic interpretations, and the answers in categories 9-10 were no economic interpretations and therefore were wrong answers to the task as well.

Category	Description	Prototypical Statement
4. Gradient of cost at the point	The students state that $C'(200) = 2$ means that the cost increases by 2 euro at a production of 200 pens.	"At a production of 200 pens the cost increases by 2 euro."
5. Cost per unit	The students state that $C'(200) = 2$ means that the cost per pen is 2 euro.	"The cost per pen at a production of 200 pens is 2 euro."
6. Total cost	The students state that $C'(200) = 2$ means that the cost for 200 pens is 2 euro.	"At a production of 200 pens the cost is 2 euro."
7. Cost doubles or halves at the point	The students state that $C'(200) = 2$ means that the cost doubles or halves at a production of 200 pens.	"At this point the cost doubles for each pen further produced."
8. Other wrong interpretation	Another wrong interpretation is stated or the answer does not make sense.	"The 201st pen costs exactly the same like the 200th pen."
9. Slope at the point	The students state that $C'(200) = 2$ means that the slope of the cost function is 2 at $x = 200$. There is no connection to economics.	"Slope at the point 200 is 2."
10. No valid answer	The students state a (maybe) right answer, which does not have to do anything with the task, or state no answer or answer fragments.	1. "No idea." 2. "If $C'(x)$ positive, then the profit"

Figure 2: Not adequate response categories for the task to interpret $C'(200) = 2$ for a cost function C in an economic context

After the development of the category system the data was re-coded by a student to check inter-rater reliability. The reliability coefficient Cohen's Kappa was $\kappa = 0.82$, which is good.

The 10 categories mentioned above were also used to categorize the answers to the task in the exam to interpret $P'(73) = 0.2$ for a profit function P. However, three additional categories that contained more than 3% of the answers had to be added (Figure 3). In addition, the category "cost doubles or halves" was adapted to "Increase of profit by 20%".

Category	Description	Prototypical Statement
11. Marginal profit at the point	The students state that $P'(73) = 0.2$ means that the marginal profit at a production of 73 is 0.2 units of money per unit of quantity.	"At the point $x = 73$ the marginal profit is 0.2 units of money per unit of quantity."
12. Growth of marginal profit	The students state that $P'(73) = 0.2$ means that the marginal profit increases by 0.2 either at a production of 73 units or if the production is increased by one (marginal) unit.	"If one changes the production from 73 units by a marginal unit, the marginal profit increases by 0.2 marginal units."
13. Extreme point at the point	The students state that $P'(73) = 0.2$ means that the profit has an extreme value at 73 units of quantity or that 73 units is the optimal output.	1. "The company has a maximal profit at the output of $x_{opt} = 73$." 2. "x_{opt} is located at 73."

Figure 3: Additional response categories for the task to interpret $P'(73) = 0.2$ for a profit function P in an economic context

Results

The students' answers to the task to interpret $C'(200) = 2$ for a cost function C in an economic context before their Calculus course are shown in Figure 4.

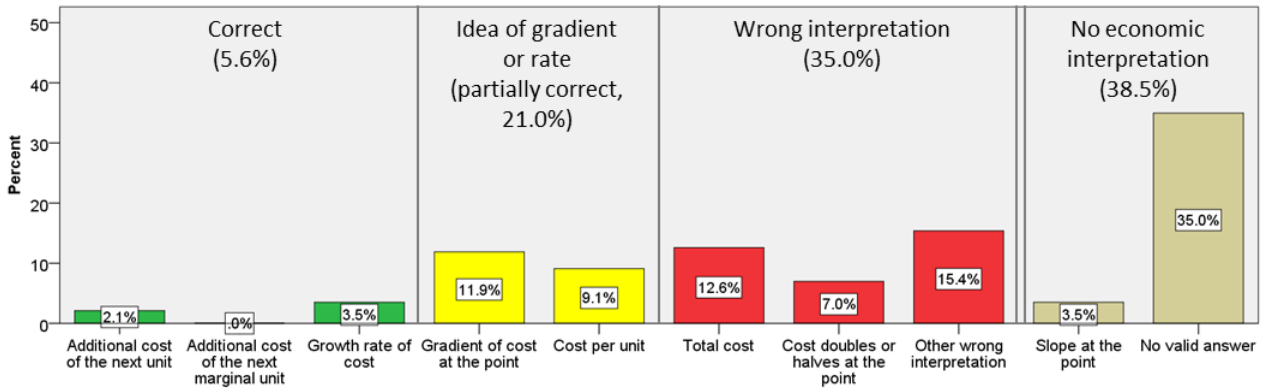

Figure 4: Students' answers to the task to interpret $C'(200) = 2$ for a cost function C in an economic context before their Calculus course (N = 143)

As can be seen in Figure 4, the economic interpretation of the derivative of a cost function C as the additional cost of the next unit was not known by the students from school, and is not as intuitive that the students stated it spontaneously. Instead, some students tried to use their knowledge about the derivative being the slope at a point (category: Gradient of cost at the point) or the rate of change (categories: Growth rate at of cost or Cost per unit). Others had misconceptions.

The students' answers to the task to interpret $P'(73) = 0.2$ for a profit function P in an economic context in the exam after their Calculus course are shown in Figure 5.

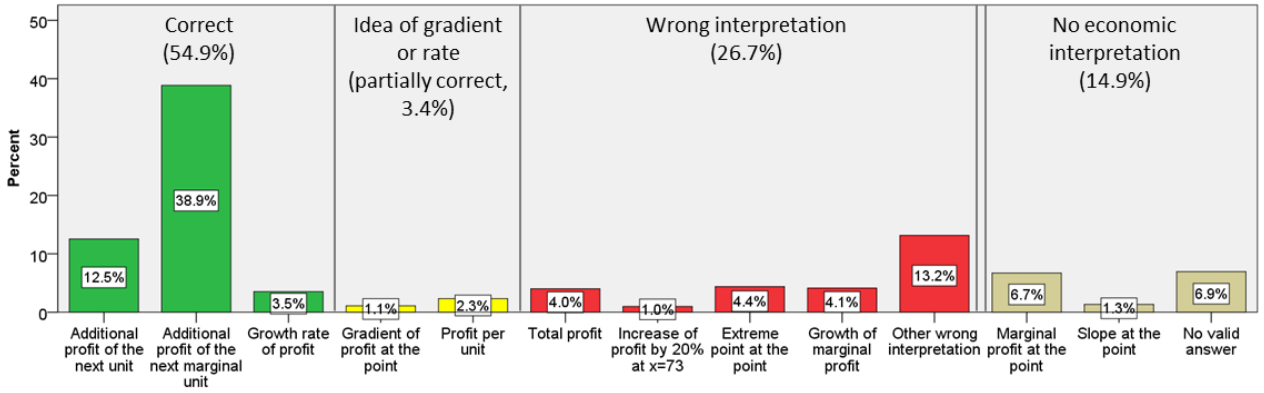

Figure 5: Students' answers to the task to interpret $P'(73) = 0.2$ for a profit function P in an economic context in the exam at the end of their Calculus Course (N = 821)

As can be seen in Figure 5, more than half of the students knew the common economic interpretation of the derivative of a profit function as additional profit of the next (marginal) unit after the course (but still various misconceptions occurred). Thus, an adequate economic interpretation of the derivative was part of their *concept image* for about half of the students. However, the data indicates that many of these students did not really integrate their economic interpretation of the derivative with the rest of their concept image, i.e. they did not fully understand the differences between the derivative $P'(x)$ and its economic interpretation as profit of the next unit (differences in the numerical values and in the units). Of the 422 students who interpreted the derivative as additional profit, 100 students (23.7%) mentioned "unit of money per unit of quantity" as corresponding unit, which is the right unit of the derivative but not of its interpretation as additional profit. Furthermore, of the 103 students who interpreted the derivative as the additional

profit of the next unit, only 20 students (19.4%) mentioned in their interpretation that the value of the derivative $P'(x)$ is just an approximation of the additional profit $P(x+1)-P(x)$, although the lecturer had emphasized that this has to be stated explicitly in the interpretation (and had shown graphically the error on the board). This indicates that many students probably did not understand the differences between the derivative $P'(x)$ and its economic interpretation. They just identified both objects, which leads to an incoherent *concept image* of the derivative (Tall & Vinner, 1981).

Limitations of the study

Due to organizational problems it was not possible to match the students in the pretest and the exam. So it is not possible to investigate the progress of individual students with the data. It is just possible to compare the amount of students giving certain answers before and after the course.

Discussion and conclusions for further research

The study shows that an adequate economic interpretation was not part of the students' *concept image* of the derivative (with few exceptions) when entering university. Hence, this interpretation should be introduced in the course with caution and connected to the rest of their *concept image* of the derivative concept. After the course an adequate economic interpretation of the derivative as the additional cost/profit of the next (marginal) unit was part of their *concept image* for about half of the students. However, the data indicates that many of these students did not understand the differences between the derivative as a mathematical concept and its economic interpretation. For example, one quarter of the students who interpreted the derivative as additional profit gave the wrong unit (a unit of rate). They did not distinguish between the derivative being a rate of change and its economic interpretation being an amount of change, which is also documented in literature (Mkhatshwa & Doerr, 2015). Furthermore, the data indicates that many students were not aware that the value of the derivative is just an approximation for the additional cost/profit of the next unit. Students having these problems probably did not connect the economic interpretation of the derivative to their *concept image* of the derivative as a pure mathematical concept properly.

To find out to what extent the students really integrated the economic interpretation of the derivative as additional cost/profit into their *concept image* as intended in their Calculus course (i.e. that they know the differences and the connection between these two different mathematical objects, and can justify the identification of them), further research is necessary. Therefore, in spring 2015, an interview study addressing this identification directly was conducted to find out to what extent the students really integrated the economic interpretation of the derivative into their *concept image* of the derivative concept (and not just memorized it) and what cognitive obstacles occur during the justification of the economic interpretation of the derivative in the way it was done in the course.

The above mentioned research, however, only takes a cognitive perspective into account. There are more factors influencing the way students interpret the derivative in an economic context like institutional practices of the two institutes involved in their education (institutes of mathematics and economics). Taking these institutional influences into account, which has great potential for further research, would, however, require other theoretical perspectives like ATD (Bosch & Gascón, 2014).

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Exploring tensions in a mathematical course for engineers utilizing a flipped classroom approach

Helge Fredriksen[1], Said Hadjerrouit[2], John Monaghan[3] and Ragnhild J.Rensaa[4]

[1]University of Agder & The Arctic University of Norway, Norway; Helge.Fredriksen@uia.no

[2]University of Agder, Norway; Said.Hadjerrouit@uia.no

[3]University of Agder, Norway; John.Monaghan@uia.no

[4]UiT – The Arctic University of Norway; Ragnhild.Rensaa@uit.no

Flipped Classroom approaches to teaching are becoming increasingly popular in higher education, but there is a lack of empirical research. We present here a study performed during an engineering course for 20 students at a Norwegian university, on student appropriation towards Flipped Classroom through interviews, questionnaire, video/quiz usage statistics and classroom filming. We approach this research through an activity theoretical framework, focusing on tensions experienced as the students try to tackle the demand of video preparation and active learning in class. In line with much of the recent research on the topic, we find that most students seem to appreciate more collaboration with peers and teacher. However, there is also evidence that the new form of teaching creates various tensions; a minor part of the cohort demonstrates conflicting beliefs about mathematics learning, resisting the active learning part of Flipped Classroom.

Keywords: Activity theory, flipped classroom, tensions.

Introduction

Flipped Classroom (FC) is most commonly known as a method that arranges the lecturing part of the teaching as homework through videos. This is considered the out-of-class part of the FC. When students come to class, the stage is set for learning in a student-centered manner, using various problem-solving activities (Bergmann & Sams, 2012). This is considered as the in-class part of FC. Both are vitally important for the FC learning model to work. The out-of-class video learning "primes" the students for the crucial in-class active phase (Seyfedine, Kadry & Hami, 2014), where hopefully the active "learning-by-doing" understanding and adaptation takes place. The idea is that through well-designed activity sets in class, the teacher has the opportunity to challenge the students at both a collaborative and conceptual level in this phase (Wan, 2015).

In this paper, we describe a study that was conducted in the spring term of 2016 at a Norwegian university, where students in their first year of engineering studies were exposed to several interventions of the FC way of teaching. The teaching setup in this university is well suited for flipped teaching. It is a small campus, with only 20-30 students per-year in a 3-year long bachelor study in computer engineering, allowing for a tight integration between the students and the teacher.

Many studies on the implementation of FC seem to indicate that motivation might increase among students in mathematics (Franqueira & Tunnicliffe, 2015; Kadry & El Hami, 2014; Roshan, 2015). However, there are also research studies that indicate the opposite. Wasserman, Quint, Norris, and Carr (2015) found that students in a flipped calculus III class were critical to the use of class time for group work. Strayer (2015) reports that students felt "lost" and disengaged with the material sooner than students in the traditional classroom did. Ramaglia (2015) did a comparative study between

flipped and non-flipped high school and middle school mathematics classes in her PhD thesis, but failed to find consistently increased peer-to-peer activity among students. Referring to these mixed results from other studies, it seems interesting to gain more insight into what kind of tensions, strains and possible resolutions of these can be observed in a FC realization. Based on this background, our research questions are formulated as follows:

Research Question: What are the tensions that emerge from students' attempts to appropriate change towards FC facilitated by videos and quizzes?

Theory

We believe that learning can best be understood when considered as a common enterprise among students and teacher, emerging as culturally negotiated in an environment based on constructive criticism. Turning to Cultural Historical Activity Theory (CHAT) (Engström, 1994), we have a rich theoretical basis to put FC in a broader perspective.

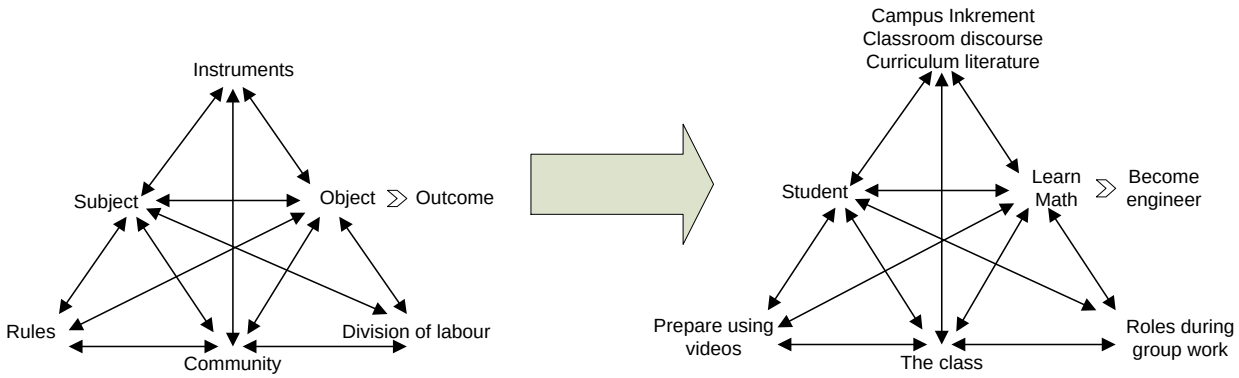

Figure 1: The CHAT triangle adopted to the Flipped Classroom from a student's perspective

The primary part of the activity system in this study is the student as a subject in her object-oriented activity to learn engineering mathematics. In attempting this, she uses various instruments. The most important ones are Campus Inkrement (a virtual learning environment used for distributing videos and quizzes), classroom discourse and curriculum literature. The dominant new rule governing FC compared to "traditional teaching", is the video preparation part, forming the out-of-class component of FC. The mathematics in the video should form a common ground of knowledge for the community consisting of students and teacher. For the division of labor, we consider how students attain various roles in their collaboration to solve tasks in-class.

In any activity system there exist tensions and contradictions. Engström (1994) summarizes activity theory in five principles, and among these, he mentions contradictions as one of the leading sources for change and development. Basically, contradictions can be defined as a misfit within elements of an activity system, between them, and between different activity systems. Engeström (1987) argues that four levels of contradictions are present in an activity system, and identified tensions in interactions within and between activity systems. The contradictions can be identified at four levels: primary, secondary, tertiary, and quaternary. If we apply this model in the context of FC, we can describe the contradictions as follows:

1. The primary contradictions occur within the elements or components of FC as an activity system, e.g. within the community of students and teacher.

2. Secondary contradictions arise between the elements of FC, or when two or more elements of FC conflict with one another, e.g. between the community and subject (for example between the class and the individual student), between the object and the community, or between the rules and the community, etc.
3. Tertiary contradictions arise when a new and advanced method or artefact is used to achieve an objective, e.g. when videos are introduced as a new artefact to teach mathematics.
4. Quaternary contradictions occur between FC (as activity system) and another activity system.

Methodology

We performed two separate periods of FC teaching during the second semester of study year 2015/2016. We performed data collection by issuing an anonymous questionnaire, doing three semi-structured interviews and two rounds of classroom filming. In addition, students' usage statistics of video and quizzes were collected through the Campus Inkrement software. As our theoretical stance is in the socio-cultural field, an interpretative research paradigm was chosen. The questionnaire and the interviews were performed after the students' first encounter of FC teaching, informing us on student impressions on pedagogical and technical impressions with the learning platform chosen for distributing video/quizzes, the in-class group work activities and the quality of interaction with the teacher and the other students. Episodes relevant for the enlightenment of the research questions of the paper, tensions and student appropriation towards FC, are highlighted in the results section.

Campus Inkrement (CI) as a mediating artefact

Preceding each in-class session, a corresponding out-of-class session of videos and quizzes was presented to the students in CI, which is a web-application fulfilling the role of the out-of-class component of FC. Built from the ground up to be consistent with the FC teaching design, the teacher/researcher also has the capability to highlight video watching statistics and quiz results for the individual student. From a student perspective, CI brings the opportunity to give feedback on how well the student understood the current topic on a scale 1-5. In addition, self-perceived effort can be reported on a similar scale. The student also has the opportunity to ask for further guidance from the teacher on specific topics. This opens up for an out-of-class possibility for students to prompt the teacher for assistance without revealing their uncertainty to peers in-class.

FC implementation

In this class, there were 20 students following the course. Before attending the spring term, these students had all background from a 10 ECTS (European Credits) calculus based Math-1 course with traditional lecture-based teaching. The course in the spring term that was subject for FC teaching was labelled Math-2, consisting of 10 ECTS containing series, Fourier and Laplace transform, recursion equations, proofs and optimization on functions with two variables.

After having informed the students thoroughly about the new form of teaching in the beginning of the term, we started out the term with one month of FC teaching in January. The topic for this first round of FC teaching was sequences and series, studying criteria for convergence, and in the end Taylor expansions and Maclaurin series. Although we did not influence the curricula, obligatory assignments and exam, we could plan and implement FC as we saw fit, including the teaching performed in-class. The teaching consisted of two or three 90-minutes sessions each week. To prepare

each in-class session, 3-4 videos each of 8 to 15 minutes in length were available for the students. In between the videos, quizzes directly related to video contents were given. The videos presented the mathematics in a chalk-and-talk fashion, screen-capturing teachers writing using a tablet, including some demonstrations made in geogebra. We produced 12 of the 36 videos, the rest were collected from online resources mainly from Khan Academy (https://www.khanacademy.org). The videos were procedural in content, in the sense that there was little time to go into proofs or elaborate on deeper concepts. This choice was intended to make the video homework manageable in length for the limited out-of-class time. In line with FC ideas, in-depth understanding should be elaborated in an in-class setting.

After this first attempt at FC teaching, we spent the middle of the course teaching traditionally with other teachers involved. The reason for this shift was the necessity for collecting feedback through interviews with a representative selection of students, in addition to an anonymous questionnaire. This to inform us on potentially needed adjustments in the second phase of FC. At the end of the term, we ran two more FC teaching weeks on the introduction of functions with several variables, linearization of these, partial derivatives and optimization. On most occasions, specially adopted task sheets were prepared for in-class active learning to provoke discussion and in-depth conceptual reflection about the mathematics, the purpose being to raise the abstraction level.

Results

These three sources of data, the questionnaire, the interviews and the filming, provide the possibility for us to triangulate findings. As this is a paper investigating tensions in the CHAT sense, we have been actively looking for excerpts where such qualities are prominent.

Questionnaire

At the beginning of March, we invited all students participating in the class to answer an anonymous questionnaire. Here we asked the participants to agree or not on fifteen statements, in a 5-point Likert scale fashion, about various features of our FC implementation. The purpose of this was primarily to inform us towards the next iteration of FC. Additionally, the questionnaire contained three open-ended questions, prompting the students to express their opinions about the method with their own words. n=15 out of N=20 students responded.

We have chosen to highlight three responses to the open-ended question: “What did you feel was most inconvenient with this method of teaching and learning mathematics?” While other questions highlighted the positive sides of FC, the three statements below are representative for most of the answers to this question, and are important for the analysis of tensions:

> “Personally, it works better for me when I spend time on my own with the tasks. Thus, the session in the classroom became wasted for me. I believe I should learn new things in the class, and then work on my own with the topic afterwards, and then turn to the videos for assistance.”
>
> “I got “pushed away” from the classroom using this method, since I do not like to work on tasks in groups. I feel that group work is difficult since many do not understand the topic 100%, which means that many just do not participate in the discussions.”
>
> “Group work was unsuitable, since mathematics is a more “individualistic” subject.”

Interviews

In addition to the questionnaire, we performed interviews with a representative set of students in a semi-structured fashion. Due to time constraints, we had to limit the sample to three persons. This group of students was chosen as a representative sample according to gender and age, but also due to observed willingness to make critical remarks about the teaching. The interview tried to dig a bit deeper into topics of engagements, impressions about videos, group work and interaction with the other students and the teacher and lasted for about 30-40 minutes. With respect to our consideration of tensions, we present interview excerpts from students with positive and negative views on FC.

The first interviewee was an engaged student in mathematics, with almost 100% attendance in class. He favored learning by videos over traditional learning, and liked the fact that the teacher was more available for questions than traditional lecture-based teaching. As the problematic part of all the group work, he pointed at troubles with fluency in using new mathematical vocabulary. However, he noted that by trying to communicate verbally the task with the others in the group, it became easier to understand how to solve it for himself.

The second interviewee had most of his career from offshore industry but turned to engineering studies for health reasons. He had been away from mathematics for a long time and sometimes struggled to keep up with the pace in the group work

Student: I did not like the specially adopted tasks we got for the class session, and the way we worked in the groups was very inefficient for me. Because many in the class are above me, I am stuck behind the rest during the work.

Interviewer: Ok, but you liked to prepare using the videos?

Student: Yes, I liked that very much.

Interviewer: But you think it would be easier for you to find the answer to the tasks if you were all by your own solving them?

Student: Not easier, but it would have been a better way for me to understand them, since I would be alone to think it over, instead of the others in the group just working fast through them.

Interviewer: So you weren't able to engage in the conversation and participate with your own thoughts?

Student: Not to the degree I wanted.

Both interviewees 1 and 3 expressed concerns about using specially adopted tasks for the in-class work. They worried about the tasks not having sufficient relevance for the final exam, and would rather spend time solving tasks from the textbook. I chose to not include excerpts from the third person being interviewed, since there were little indications of tensions in this interview.

Classroom filming

During the second FC intervention period, we filmed two in-class sessions. We filmed several of the groups, primarily motivated by how the out-of-class teaching affected in-class group work. Two or three students worked together solving problems related to the videos, but on a slightly higher level

than the examples used in the videos. One episode in particular caught our attention. One student in a pair (let us call her Silvia) attended class seemingly well prepared and brought notes with her that she had taken from the videos. The other student in this pair (let us call him Nick) seem to be quite unprepared. A study of CI usage statistics confirms this impression. He did not bring notes, and barely spoke during the beginning of the episode. Silvia on the other hand expressed interest in how the formula of linear approximation for functions in two variables came about. The formula referred to is the well-known linearization

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial y}{\partial y}\Delta y$$

The first author hinted that this formula was based on an extension of the single variable case that was derived in the video, but Silvia was clearly not satisfied with this, wanting to know more. In addition, she was the dominant speaker in the group: In the 22 minutes that the episode lasted, we counted 488 words spoken by Silvia, whereas Nick spoke 236. He did catch up in the last third of the episode though, speaking almost as much as Silvia does then. We observe from the videos that this occurs after he had listened carefully to her struggles with the problems and the conversation that she had with the teacher in connection with this.

It was evident from the CI user statistics that many students had not prepared in the last period of FC teaching. This was influencing their progression in-class, although many seemed to use other means of catching up with the topic. They used other resources such as the curricula book, discussions with more knowledgeable peers like prepared students and the teacher, and even to some extent looking at the videos in-class on their own laptop.

Discussion and summary

As previously discussed, activity theory can be used to depict tensions in the FC teaching. Studying the activity triangle in Figure 1, one of the most prominent changes in FC compared to traditional teaching are the rules. These undergo a radical change in FC, enforcing video preparation for in-class active learning.

There are two important observations we would like to highlight. Firstly, the second statement from the questionnaire excerpts hints towards a lack of understanding among several group members (the community) about the mathematical topic at hand. It seems that many members in the group had not grasped the mathematics in the videos, or they simply had not watched them, leading to a breach in the quality of student group collaboration.

The filmed episode of Silvia and Nick confirms this impression. Nick appeared to struggle to follow the arguments of Silvia, although there were evidence that he somehow changed from being an ‘eavesdropper’ of her struggling and collaboration with the teacher to becoming an active participant. However, Nick was not playing as the part of a collaborating peer, and thus failing to support Silvia in the discourse. We believe that this was due to his lack of preparation using the videos.

Classroom discussion is considered a vital instrument of learning in FC, and it constitutes a major tension if this is not taking place inside a group. We consider this as a secondary contradiction between rules and community. As previously explained, the major rule to consider in the student FC activity is the necessity to arrive at the in-class session being ‘primed’ by the out-of-class session. If

a major part of the group has failed to do this, the in-class discourse, considered a CHAT instrument, is hampered. Thus, this contradiction could also be seen as a secondary contradiction between rules and instrument.

Considering the data excerpts, we can also mention the tensions below, even though not substantiated through triangulation as the one already mentioned:

1. Tension in expectations/beliefs/rules: Students expect to be "taught" by the teacher, but FC rules and division of labour directs students towards learning through collaboration with peers (subject – division of labour tension), (secondary contradiction).
2. Students disagree with the new rule that tasks should be solved during class time. Preference towards solving them in solitude (object – rules tension, students feel this is not the best way to learn math), (secondary contradiction).
3. Students need to adopt to a new paradigm of work: Preparation through video lessons requires discipline, which results in tension, especially when a heavy workload is expected in courses taken in parallel (subject – rules tension), (tertiary contradiction).
4. Fluency in discourse. Problems expressing the mathematical problems verbally to other students. (subject – instrument (discourse) tension), (tertiary contradiction).
5. Students failing to keep up with the others during group-work (subject – community tension), (secondary contradiction).

As we discussed earlier on, in the filmed classroom episode with Silvia and Nick there is indication that students who have prepared by engaging in the out-of-class work seem to express themselves fluently in the mathematical problems, and in addition seem eager to learn more about the concepts behind the procedural mathematics shown in the videos. This provides empirical evidence (though only a single case) of the potential of FC to motivate students to strive towards a higher level of abstraction.

Validity and reliability issues

This paper must be viewed in the context of a report on a pilot study. More elaborate studies will be carried out during 2016/2017 and 2017/2018 with engineering students in the same institution. Thus, there is little rigid design according to how data have to be collected to obtain optimal analysis and results. Handpicked excerpts from the data material were chosen to highlight the findings. There is also the issue of the researcher being present as the teacher, a classical objectivity dilemma found in many small-scale educational research settings. However, as we are presenting the data using several methods, both quantitatively and qualitatively, we can to some degree state that we have made valid triangulation of the findings.

Conclusion

Our analysis of the data collected in this study, shows evidence that there exist several tensions in FC; some of these could be expected from the outset, while others are surprising. Data seem to point towards various aspects of the *active learning* being the most problematic part for many students towards a FC realization. Considering the activity system of the student, a secondary contradiction or tension materializes between the rules and the community, since many students was not adhering properly to the out-of-class part of FC. This is also seen to hamper the in-class discourse, considered to be an important instrument of learning.

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A micro-model of didactical variables to explore the mathematical organization of complex numbers at upper secondary level

Imène Ghedamsi

University of Tunis, Tunisia; ighedamsi@yahoo.fr

When complex numbers are introduced, there may be some mathematical choices that go beyond the learning processes requirements. Research highlights the necessity to take into account the epistemological aspects of complex numbers in order to cope with students' potential difficulties with these numbers. In this paper, we analyze the contents of the mathematical organization planned by the upper secondary institution to introduce these numbers by using a micro-model of didactical variables. Our results underline a lack of organization and put forward some learning criteria that could be deployed to design tasks for introducing efficiently complex numbers.

Keywords: Didactical variables, complex numbers, mathematical organization.

Introduction

One of the most useful approaches to introduce the complex numbers postulates the existence of the solution ($\sqrt{-1}$ or i) of the equation $x^2 + 1 = 0$, and enlarges by the same time the set of real numbers by including such solution in a way that the sum and product rules could be naturally generalized (Ghedamsi & Tanazefti, 2015). Yet, the real number and the complex number are utterly unlike, specifically because the former is associated to a concrete measuring process. Some pioneer research on complex numbers (Artigue & Deledicq, 1992; Rogalski, 2002; Rossel & Schneider, 2003) underline the complexity of learning these numbers by emphasizing the gap between the epistemological aspects of these numbers and the mathematical organization chosen to introduce them. Three fundamental epistemological aspects are highlighted by these researches and may give some details about this gap:

- In history, imaginary numbers are firstly used as a tool by the Italian algebra school to resolve cubic equations; they appeared as the square root of a negative number in the numerical expression given by the formula of the real solution ($\sqrt[n]{a+\sqrt{-b}}+\sqrt[n]{a-\sqrt{-b}}$) with no more mathematical meanings.
- Imaginary numbers were being used by mathematicians long before they were first properly defined as complex numbers. The progress of the imaginary from the statute of a simple tool towards a mathematical existence as an object was supported by their efficiency to solve geometrical and infinitesimal calculus problems. Mathematicians made free use of them by applying the permanence principle which consists on generalizing real number rules to these numbers.
- The manipulation of imaginary numbers in history evolved through the use of several mathematical and semiotic representations. The use of several procedures, particularly those related to the multiplication of vectors, led to a geometric representation of imaginary numbers in the Argand (or complex) plane by identifying them to both a vector and a point. This kind of representation is not the first one; mathematicians started thinking about trigonometric representations of imaginary numbers along with their emergence by using infinitesimal calculus rules. The key role of these semiotic and mathematical representations reinforced the utility of imaginary numbers and putted forward the necessity to firmly entrench them as a mathematical object.

Many researches emphasize the major role of the organization of mathematical activities in the teaching and learning of mathematics as it shaped what could be taught and how this could be done (textbooks, syllabus, etc.). For instance, the study of the mathematical organization of Calculus concepts in the transition between secondary school and university highlights several problems in the way used to introduce mathematical topics that could potentially affect students' process of learning (Bressoud, Ghedamsi, Martinez-Luaces & Törner, 2016). This paper seeks to analyze this phenomenon in the case of complex numbers taking into account the epistemological aspects mentioned above, especially through their connection to the development of students' work with these numbers. We build on a networking of frames in order to identify the didactical variables - defined as the parameters that influence students' work (Bloch & Ghedamsi, 2005), related to complex numbers. Then, we use these variables to give a global vision of the choices of the mathematical organization related to the introduction of these numbers.

Theoretical frames

Sfard (1991) argued that three phases shaped both the historical and the cognitive development of a mathematical concept. She particularly underlined subtle differences between historical and cognitive phases by means of the example of complex numbers. From the historical point of view, the three phases are defined as follow: "*1) the preconceptual stage, at which mathematicians were getting used to certain operations on the already known numbers […] manipulations were treated as they were; as processes and nothing else […]; 2) a long period of predominantly operational approach, during which a new kind of number begun to emerge out of the familiar processes […], at this stage, the just introduced name of the new number served as a cryptonym for certain operations rather than as a signifier of any "real" object […]; 3) the structural phase, when the number in question has eventually been recognized as a fully-fledged mathematical object […].*" (Sfard, 1991, p. 14). These three stages are firmly consent with the history of complex numbers. In adequacy with these phases, the cognitive development of students is categorized into three stages: 1) *interiorization* where students become aware of the processes that gives rise to the concept; 2) *condensation* where students start combining and generalizing processes; 3) *reification* where the concept achieves the status of mathematical object by unifying the various processes in a structure. In this sense, the mathematical organization of complex numbers should cope with activities that enable students to first build skills of computation with square roots through active involvement of processes including those related to number representations; and secondly to realize through a huge variety of cases the practical perspective of these numbers (Sfard, 1991).

The object level of this concept is closely linked to the algebraic structure (or mathematical category) of the set of complex numbers. Each category or structure refers to one mathematical representation. In the language of category theory, the establishment of an isomorphism between sets of the same algebraic structure allows a person confronted to a new set to detect similarities and connections to familiar objects or sets, and to organize efficiently the new set. For instance, the isomorphism of fields $\varphi : (\mathbb{C}, +, .) \longrightarrow (\mathbb{R}^2, +, .)$, $a + ib \longrightarrow (a, b)$ permits to link the abstract representation of the complex numbers to a more concrete one. Furthermore, the use of functors to transit from one category to another permits to translate a difficult problem from one mathematics area into an easier problem in another area; this is the case of the trisection problem which is just solved by using complex numbers. This knowledge thus permits to orientate and to organize mathematical thinking.

At the transition from the end of secondary school to university, two categories of mathematical structures are implicitly used in the teaching of complex numbers: the category of Euclidian space and the Field category. To achieve the object level, these categories (and more) should become "*the ultimate base for claims on the new object's existence*." (Sfard, 1991, p. 20). Of course, this could not be done at the transition between secondary school and university! To do so requires some foundational notions that are actually not taught at these levels. Nevertheless, the study of the mathematical organization of complex numbers by means of category theory provides information about the mathematical representations concerned by the institution and their design as well as those involved in a process of conversion between two mathematics areas. These information are associated to the mathematical culture of complex numbers, and they are useful for the teachers as they become aware of substantial details that influence the development of the learning process. Thus, the investigation of the upper secondary mathematical organization by using categories should not be outlawed simply because of the lack of mathematical notions at this level. However, it is fundamental to stress the distinction between the semiotic representations of an object as signifiers - that are organized into semiotic registers, and what is signified thus the mathematical representation. In the case of complex numbers, one semiotic representation may evoke more than one mathematical representation and vice versa. For instance, $z, \overline{z}, |z|, \arg(z)$, etc. constitute the elements (or the scripts) of what we called the intrinsic register; students' work with complex numbers in the category of Euclidean space could be done by using both the intrinsic scripts and the geometrical ones. This is the case of the representations of three collinear points: A, B and C are collinear points, if there exists a real number $\alpha \neq 0$ such that $z_B - z_A = \alpha(z_C - z_A)$ which is equivalent to $\arg(\frac{zC - zA}{zB - zA}) \equiv 0(\pi)$. Further study on the semiotic registers involved in the same category or in the transition between two categories is unavoidable. A switch between two semiotic registers is a conversion which refers to the same signified being in the same category. This conversion creates a new semiotic representation that does not involve the formation of a new object. In the case of the translation from one category to another, this switch engages a process of creation of a new mathematical representation of the same object. The process of conversions between semiotic representations as well as between mathematical representations performed the cognitive flexibility of the students and enabled the enlargement of the representations field of these numbers (Duval, 1995).

This theoretical synthesis highlights the impact of at least three didactical variables on the learning process of complex numbers. These variables lead to a micro-model that we use to investigate the institutional mathematical organization of complex numbers: DV1: The use of the permanence principle which concerns the generalization of real numbers rules (in the categories of field and Euclidian space) to the complex numbers; DV2: the use of the process-object duality; and DV3: the use of the semiotic and mathematical representations. We classify the mathematical organization in terms of practical blocks containing types of tasks and techniques to solve these tasks (Chevallard, 2006). This classification gives an overall patent description of the institutional requirements that permits to analyze the mathematical organization of complex numbers by involving the three variables of the stated micro-model. In this paper, we investigate this mathematical organization by means of two didactical variables that are DV2 and DV3.

Empirical context

Complex numbers constitute one of the most important topics introduced in algebraic courses at the end of the secondary school in Tunisia for 16/17 years old students. Courses follow the contents of the unique official textbook used by the teachers as their own syllabi; almost all mathematics teachers adhere strictly to this textbook. The mathematical organization concerned by this study is taken from this textbook.

The modeling of the whole mathematical organization of complex numbers into praxeologies underlines the existence of 14 types of tasks designed T1 until T14; each one can be solved by using more than one technique. These techniques are generally algorithmic and indicated in the terms of the questions. For instance, to solve the type of task *T7: Determine an argument of a complex number*; students have the possibility to simply employ the property of the argument of the product of two numbers, they can also use cosine and sinus properties. Each task is a block formed by the type of tasks and the involved technique. The frequencies of these tasks (each task can occur more than once) in the whole organization are as follows:

T1: Determine the cartesian representation of a complex number	20
T2: Solve an equation using complex numbers	5
T3: Determine the conjugate of a complex number	2
T4: Determine the affix of a point or a vector	7
T5: Determine the modulus of a complex number	12
T6: Spot points in the complex plane	16
T7: Determine an argument of a given complex number	5
T8: Determine the trigonometric representation of a complex number	11
T9: Determine the kind of a quadrilateral (rectangle, square, etc.)	2
T10: Determine a set of points of the complex plane	11
T11: Determine the kind of a triangle (isosceles, equilateral, etc.)	1
T12: Prove that three points are collinear	1
T13: Determine the position of two lines (parallel, secant, etc.)	1
T14: Determine the sinus and the cosine of an angle	1

Table 1: Types of tasks and frequencies

The tasks are organized according to the categorization of the course into three sections: 1) the first section concerns the introduction of complex numbers via the traditional approach mentioned in the beginning of this paper; T3, T2 and T1 are the tasks used in this section particularly to prove and to exemplify the proprieties of the product and the sum of complex numbers and of their conjugates, by using the cartesian representation; 2) in the second section, the affix and the image notions, and the modulus and its proprieties are introduced with no details about the necessity to draw on the complex plane; T6, T5 et T4 are the tasks involved in this section; 3) the third section introduces the argument and its proprieties, and the trigonometric representation of complex numbers; T7 and T8 are mostly studied here. The tasks T9 to T14 are not concerned by a specified section, they are considered as integrative tasks that permit to use a variety of techniques. Using this classification, we can now

analyze the mathematical organization of complex numbers with a particular focus on the micro-model of the selected didactical variables.

Some results

Pseudo operation level of complex numbers

In opposition to the cognitive development principles as highlighted by Sfard (1991), the mathematical organization focus from the beginning, on the object level of complex numbers in way that: 1) the *reification* phase is "imposed" to the students with no tasks that allow the unification of the processes into a structure; 2) the *preconceptual* level of these numbers is missing, and the *interiorization* phase is limited to the manipulation of several representations of the object which is already introduced via one of them, this manipulation is mainly guided by the questions; 3) the *condensation* phase, which is supposed to illustrate the operation level of these numbers, is mostly neglected. It is important to precise that the same type of tasks could be associated to more than one level of complex numbers: preconcept, operation, or object. Specifically, the frequencies of the tasks referring to these levels are as follows:

	Objet	**Preconcept**	**Operation**
T1: Determine the cartesian representation	20	0	0
T2: Solve an equation using complex numbers	5	0	0
T3: Determine the conjugate of a complex number	2	0	0
T4: Determine the affix of a point or a vector	3	4	0
T5: Determine the modulus of a complex number	11	1	0
T6: Spot points in the complex plane	5	11	0
T7: Determine an argument of a given complex number	5	0	0
T8: Determine the trigonometric representation	10	1	0
T9: Determine the kind of a quadrilateral (rectangle, etc.)	0	0	2
T10: Determine a set of points of the complex plane	1	8	2
T11: Determine the kind of a triangle (isosceles, etc.)	0	0	1
T12: Prove that three points are collinear	0	0	1
T13: Determine the position of two lines (parallel, etc.)	0	0	1
T14: Determine the sinus and the cosine of an angle	0	0	1

Table 2: Complex numbers levels and frequencies

About 2/3 of the whole tasks concerns the object level of complex numbers, and only deals with algorithmic computation in the field category or the Euclidian space category of $\mathbb{C}$. The only process, referring to the supposed *preconceptual* level of complex numbers, consists on the strict manipulation of complex numbers representations by "juggling" from one to another. These manipulations are isolated, indicated in the statements, and do not probably lead to any kind of *interiorization*, this is the case, for example, of the geometric interpretation of the cartesian representation and vice versa. Only two tasks from those related to the operation level could actually be considered as an effective training to combine and unify processes that show the practical perspective of complex numbers (tasks T13 and T14). Tasks T9 to T14 should clearly highlight the role of complex numbers to

overcome complex computations of the geometrical problems, but with a certain choice of data, the students can simply apply geometric rules with no reference to these numbers. But the mathematical organization that is investigated in this paper does not permit to underline the role of complex numbers in simplifying extremely complicated computations; the data associated to the tasks T9 to T12 are not thought to take advantage of the operation level of these numbers.

The amalgam of complex numbers representations

Four kind of semiotic representations structured into four registers intervene in the mathematical organization of complex numbers: 1) intrinsic register (In., $z, \overline{z}, |z|, \arg(z)$, etc.); 2) cartesian register (Ca., scripts using $a + ib$); 3) trigonometric register (Tr., scripts using $r(cos\theta + isin\theta)$); 4) graphic register (scripts using cartesian coordinates G.a.c., or polar coordinates G.a.p.). These semiotic representations are employed in several ways by mean of the two mathematical categories of $\mathbb{C}$ used in this mathematical organization: field category and Euclidian space category. The analysis of the tasks using complex numbers representations is structured into three levels depending on the conversion process: 1) no conversions between semiotic representations as well as between mathematical representations; 2) only conversions between semiotic representations; 3) conversions between both semiotic representations and mathematical representations. In the case of this mathematical organization, each conversion between mathematical representations is followed by a conversion between semiotic representations. Depending on the techniques used, the same type of tasks could be associated to any of the three levels mentioned above: object, preconcept and operation. For instance, the tasks related to *T1: Determine the cartesian representation of a complex number* can be solved in the same register (computing powers of *i* in the cartesian register), or by moving into another register (from the graphic register to the cartesian one). Some techniques used to solve tasks referring to T8, T3, T2, T1 and T10 do not require conversions, the frequencies of these tasks are shown in the table below:

Mathematical category	Semiotic register	Occurrence
Field	Cartesian	26
	Intrinsic	1
	Trigonometric	2
Euclidian space	Cartesian	1
	Intrinsic	5
	Graphic/cartesian coordinates	1

Table 3: Frequencies of tasks with no conversions

About 1/3 of the whole tasks (26 out of 95) of the mathematical organization concerns the work on the Field category using the Cartesian register. These tasks involve simple computations by the mean of the properties of the operations in the $\mathbb{C}$ field. Tasks from almost all the types are concerned by the conversions between semiotic representations:

Mathematical category	Semiotic conversion	Occurrence
Field	(Ca.→Tr.)	2
	(In. →Ca.)	1
Euclidian space	(Ca.→In.); (Ca. →Tr.)	1; 6
	(Ca. →G.a.c.); (G.a.c. →Ca.)	24; 5
	(Ca. →G.a.p.); (G.a.p. →Ca.)	4; 1
	(In. →Tr.); (G.a.p. →Tr.)	6; 1
	(In. →G.a.c.); (In. →G.a.p.)	4; 2

Table 4: Frequencies of tasks with semiotic conversions

The most important tasks that require semiotic conversions within the Euclidian space are those related to the determination of the graphic representation of a complex number by interpreting geometrically a given relation with complex numbers. This interpretation is based on a standard employ of the properties given in the textbook. Finally, only two tasks need a double conversion: T13 (Euclidian space./G.a.p.→Field/Tr.→Euclidian space./G.a.c.), and T14 (Euclidian space/Ca.→Fied/Tr.).

Conclusion

The analysis shows that the mathematical organization of complex numbers involves some values of the didactical variables that are theoretically identified. This result corroborates their validity to examine students' learning expectations in the case of complex numbers. However, the approach used to introduce these numbers, firstly as object and implicitly as an element of a field set, avoids the possibilities to engage efficiently in tasks that deal with the operation level of these numbers as well as with valuable conversions of semiotic representations in the category of Euclidian space. Moreover, this approach makes it difficult to organize contents related to conversions between mathematical representations so that students can use them by their own in the future. But the role of the institutional mathematical organization is manifest to overcome the potential changes that should occur in the way students are required to work with complex numbers at university level. On the other side, the micro-model of didactical variables related to complex numbers reveals a high level of cognitive flexibility that is required for learning complex numbers specifically at the beginning of the university level: differentiate between real numbers rules and complex numbers ones; use conversions between mathematical representations to solve geometrical problems; make autonomously semiotic conversions; involve several categories of the set of complex numbers to solve problems outside and inside mathematics, etc. These requirements are a source of students' main difficulties with complex numbers (De Vleeschouwer, Gueudet & Lebaud,, 2013 ; Ghedamsi & Tanazefti, 2015). Specifically, Barrera (2013) highlights students' difficulties to interpret the product of complex numbers by means of plane's transformations in the category of Euclidian space. The crucial question is then: how to design efficiently the introduction of complex numbers in ways that minimize transition issues to the university? This study leads us to conjecture how it is possible to tackle such question by means of the micro-model of didactical variables. More precisely, three criteria may be considered for designing efficiently the mathematical organization that aimed the introduction of complex numbers:

1) use several values of the didactical variables and specifically those related to the conversions between semiotic and mathematical representations; 2) highlight the distinctions between the different categories of the set of complex numbers; 3) focus firstly on the operational level of complex number and improve its use in the resolution of different kind of problems. Further studies on students' learning of complex numbers are needed to examine the efficiency of these criteria.

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How are Calculus notions used in engineering? An example with integrals and bending moments

Alejandro S. González-Martín[1] and Gisela Hernandes Gomes[2]

[1]Université de Montréal, Canada; a.gonzalez-martin@umontreal.ca;

[2]Université de Montréal, Canada; gisela.hernandes.gomes@umontreal.ca

Research has shown that mathematics courses in engineering programmes present students with a number of difficulties, some of which stem from a disconnection between mathematics course content and the professional activity of engineers. Using tools from the anthropological theory of the didactic (ATD), we examine how the drawing of bending-moment diagrams is introduced in a classic textbook used in engineering programs. Although the notion of integral is used to teach this topic, the techniques used rely mostly on geometrical considerations (and not on integral techniques or theorems), and the justifications provided are a mix of (incomplete) mathematical discourse and professional justifications, with implications for students' learning.

Keywords: Mathematics for engineers, ATD, praxeology, integral.

Introduction and background

Mathematics is an important subject in many scientific and technological fields, including engineering. However, the difficulties university students face in their mathematics courses can lead them to abandon their professional aspirations (Ellis, Kelton, & Rasmussen, 2014). Research in engineering and mathematics education has shown that these difficulties manifest themselves in at least two points in a student's learning pathway. First, researchers have stated that students find the progression from secondary to tertiary education to be very difficult, especially when it comes to mathematics (Rooch, Junker, Härterich, & Hackl, 2016), and that they possess unsatisfactory mathematical readiness for engineering courses (Bowen, Prior, Lloyd, Thomas, & Newman-Ford, 2007). Second, a disconnect between mathematics courses and professional courses in university engineering programmes has been identified. For instance, Loch and Lamborn (2016, p. 30) stated that "mathematics is often taught in a 'mathematical' way with a focus on mathematical concepts and understanding rather than applications. The applications are covered in later engineering studies." This disconnect may create a "gap in the students' ability to use mathematics in their engineering practices" (Christensen, 2008, p. 131). This gap can be aggravated by the fact traditional engineering courses are usually separated into two groups: basic science courses in the first two years (such as mathematics and physics), and technical courses specific to each area of engineering in later years. Regarding this, Winkelman (2009, p. 306) indicated that "the first 2 years are typically devoted to the basic sciences, which means that students may only encounter engineering faculty in the third year of study". Some effort has been made to bridge the gap between mathematical and engineering practices, for instance by linking basic mathematical methods to applications (Rooch et al., 2016) or by introducing courses on mathematical modelling and problem solving early on in engineering programmes (Wedelin, Adawi, Jahan, & Andersson, 2015). These initiatives seem to have positive effects on student learning.

Tertiary mathematics education research has identified a number of difficulties encountered by

Calculus students; however, there is a lack of research on how teachers of professional engineering courses consider and use the mathematical tools taught in prerequisite mathematics courses. In general, it is expected that students in second- or third-year professional courses have grasped the mathematical notions taught in their earlier courses. We are interested in studying how Calculus notions – which students are expected to master – are used in professional engineering courses; in particular, whether they are used in the same way as in Calculus courses. Specifically, our research analyses the presentation of Calculus notions in a classic engineering textbook. We anticipate that this analysis will help Calculus teachers in engineering programmes understand how the notions they teach are used in higher-year professional courses, which may lead to a reflection on the connections (or lack thereof) between the content of Calculus courses and that of professional courses. In this sense, we adhere to Castela's (2016) position on the issue of choosing appropriate mathematics for professional-oriented programmes: "mathematicians need to take some distance with their own culture […]. They have to reconsider the following questions: which mathematical praxeologies are useful for such engineering or professional domains? What needs would be satisfied? Which discourse makes the mathematical technique intelligible? This is actually an epistemological investigation that we consider as a prerequisite to the design of mathematics syllabi for professional training programs" (p. 426).

Theoretical framework

Because we are interested in how mathematical notions are used in Calculus and professional engineering courses, we believe that an institutional approach is appropriate for our research. In particular, Chevallard's (1999) anthropological theory of the didactic (ATD) provides useful tools for analysing mathematical activity, since it considers that human activities are institutionally situated, and, consequently, so is knowledge about these activities (Castela, 2016, p. 420).

A key element is the notion of *praxeology* (or *praxeological organization*), which is formed by a quadruplet [T / τ / θ / Θ] consisting of a type of task to perform T, a technique τ which allows the completion of the task, a discourse (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. In analysing tasks, we identify the *practical block* (or *know-how*) which is composed of types of tasks and techniques. The *knowledge block* describes, explains and justifies what is done, and is composed of the technology and the theory. These two blocks are important elements of the anthropological model of mathematical activity that can be used to describe mathematical knowledge.

Our research identifies specific *praxeologies* present in professional courses; we analyse how Calculus notions are applied in these courses and whether this application reflects how the notions are usually presented in Calculus courses. In this case, analysing the *practical block* of these *praxeologies* allows us to identify specific tasks that require the use of Calculus notions, whereas analysing the *knowledge block* allows us to identify the justifications given in using these notions, and compare them with the justifications usually given in Calculus courses. We consider the work of Castela (2016), who identified that "when a fragment of social knowledge, produced within a given institution I, moves to another one I_U in order to be used, the ATD's epistemological hypothesis states that such boundary crossing most likely results in some transformations of knowledge, called transpositive effects" (p. 420). Her model (p. 424) proposes that in the boundary-crossing process, some (or all) elements of the original *praxeology* may evolve, and it ascribes the same level of

importance to types of problems and techniques as to concepts and theories. However, unlike Castela, we do not analyse the same type of task in two institutions, but rather a single *praxeology* specific to engineering and the use of mathematical tools within it.

Methodology

As we stated in the introduction (agreeing with Castela, 2016), in order to analyse how mathematics are used to solve problems in a given professional field, we must first understand and define these problems. We believe this is best achieved in collaboration with professional practitioners. To determine how Calculus notions are applied in professional contexts in engineering courses, we contacted an engineering teacher who holds Bachelor and Master of Civil Engineering degrees. Over the past 28 years this teacher has taught a variety of professional engineering courses at Brazilian universities, in engineering programs that meet international standards. He has also enjoyed a career in structural systems and reinforced concrete since 1986, developing projects and serving as a consultant. We interviewed him in March 2016 to understand how he uses Calculus notions in his professional courses. The interview and post-interview exchanges covered his way of teaching, the books he uses and the course notes he produces, focusing on his way of presenting different notions. For the purposes of this paper, we have chosen to analyse the introduction of shear force and bending moment and, specifically, how integrals are used to introduce this topic. At his university, shear force and bending moment are introduced in the second year of the programme, in the Strength of Materials for Civil Engineering course (students take Calculus in their first year). Three classic international textbooks are listed in the course syllabus (all translated into Portuguese), the main reference being the book by Beer, Johnston, DeWolf and Mazurek (2012).

The teacher indicated he primarily follows the structure of the main reference book in teaching shear force and bending moment. Therefore, this paper focuses on the book's content; we are currently analysing the complementary material provided to students, as well as the content of the interview, which will be the source of future papers. In analysing the textbook, we identified how notions are introduced, the type of tasks associated with them, and the type of *praxeology* developed, paying particular attention to the *practical* and *knowledge blocks* and the role of mathematical tools and discourse within these blocks.

It is also important to note that in the prerequisite Calculus course at this instructor's university, certain properties and results are proved while others are simply stated. For instance, the connection between the sign of the derivative and the monotonicity of the function (θ_1) is present and used in some tasks (such as the drawing of functions), as well as the connections between differentiability and continuity (θ_2).

Shear and bending forces: a summary

The content introduced in this part of the course is related to the analysis and design of beams, an important aspect of civil and mechanical engineering. Generally, loads are perpendicular to the axis of a beam (*transverse loading*), which produces bending and shear in the beam. These

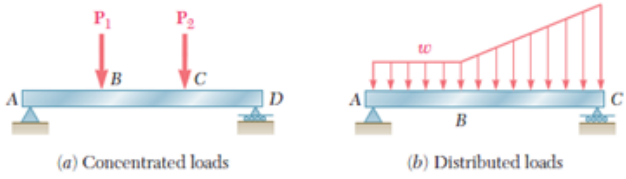

Figure 1: Types of loads (Beer et al., 2012, p. 346)

transverse loads can be concentrated (measured in newtons, pounds, or their multiples of kilonewtons and kips), distributed (measured in N/m, kN/m, lb/ft, or kips/ft), or both (Figure 1).

When a beam is subjected to transverse loads, any given section of the beam experiences two internal forces: a shear force (V) and a bending couple (M). The latter creates normal stresses in the cross section, whereas the shear force creates shearing stresses. Consequently, the criterion for strength in designing a beam is usually the maximum value of the normal stress in the beam.

Therefore, one of the most important factors to consider in designing a beam for a given loading condition is the location and magnitude of the largest bending moment. To determine this location, students are introduced to techniques for drawing bending-moment diagrams, defining M at various points along the beam and measuring the distance x from one end.

Data analysis and discussion

Although the main reference book develops its theoretical content in a well-structured way – which allowed us to grasp the notions presented – is it possible that students do not read it. Research examining how engineering students use their mathematics books seems to indicate that students pay little attention to theory, focusing instead on tasks (Randahl, 2012). We are not aware of research that looks at the way engineering students use their textbooks in professional courses.

The content addressing the drawing of bending-moment diagrams is presented in Chapter 5 (*Analysis and design of beams for bending*) of Beer et al. (2012). The chapter starts by introducing the different types of beam and loads, and the notions of load (w), V, and M. Section 5.1 introduces the relations between, and the directions of, the forces V and M in different sections of a beam, according to the type of load. In this section, calculations are made based on the idea that the sum of forces must equal zero, using formulae introduced earlier in the book. Sketches of bending-moment diagrams result in configurations such as the one shown in Figure 2. Obviously, someone with a background in Calculus could start to make a connection between the diagrams for V and M. However, this connection is not made in the textbook until section 5.2 (*Relationships between load, shear, and bending moment*).

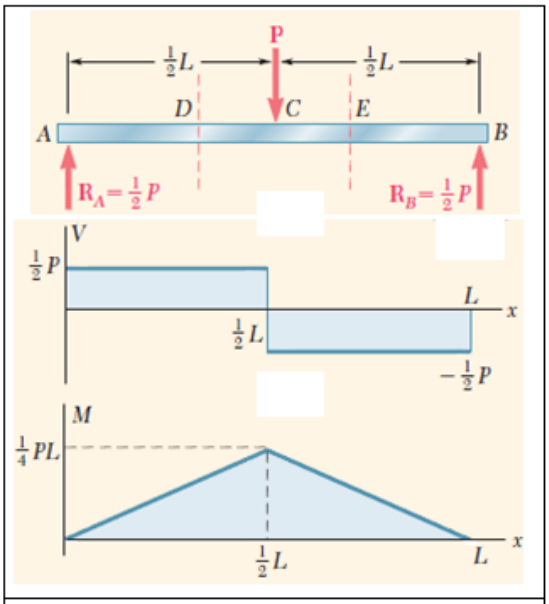

Figure 2: Example of bending-moment diagram for concentrated loads (Beer at al., 2012, p. 350)

The technique used in section 5.1 is quite rudimentary, but section 5.2 defines more explicitly (using derivatives and integrals – for this reason we focus on the content of this section) the relationships between w, V, and M to facilitate the drawing of bending-moment diagrams, which is the type of task (T_E) to solve. Section 5.2 presents a new *praxeology* (related to the one in section 5.1) that introduces the calculation of V and M at two adjacent points, x and Δx. Expanding on results from section 5.1, the authors arrive at $\Delta V = -w\,\Delta x$ and state: "Dividing both members of the equation by Δx and then letting Δx approach zero: $dV/dx = -w$. [This] indicates that, for a beam loaded as shown in [the given figure], the slope

dV/*dx* of the shear curve is negative" (p. 360). We have two remarks about this. First, the book avoids the writing of limits. Including limits could help make a connection with mathematical *praxeologies* present in the prerequisite Calculus courses (for instance, when defining derivatives and shifting from Δx to *dx*). Even if the technology used to arrive at the final expression is based on content previously taught in a Calculus course, it is not certain that every student will make the connection, since tasks addressing the passage from Δx to *dx* are not very numerous in Calculus courses. Second, the book links *dV*/*dx* with the notion of slope, but (surprisingly) relates the latter to a single case (illustrated with a figure), rather than explaining it as a general principle using the technology θ_1 introduced in the Calculus course. This could lead some students to think that this connection between the slope of *V* and *w* applies only to the given figure. Although the notions (and their properties) introduced through T_E are defined using tools from Calculus, they are not explicitly linked to technologies (such as θ_1) derived from Calculus. Finally, the expression is integrated between points *C* and *D* to obtain: "$V_D - V_C = -\int_{x_C}^{x_D} w dx$" and "$V_D - V_C =$ – (area under load curve between *C* and *D*)."

In general, although the textbook uses elements of Calculus, it avoids explicitly using the kind of notation and properties that have been institutionalised in Calculus courses (such as θ_1 and θ_2 mentioned above). For instance, the books states: "[$dV/dx = -w$] is not valid at a point where a concentrated load is applied; the shear curve is discontinuous at such point" (p. 361). Here, the author avoids a clear statement about continuity and differentiability (available in θ_2). As Castela (2016) pointed out in a different context, we believe that the authors are seeking to develop another kind of knowledge, strongly correlated with a professional context. Employing techniques similar to those used to find *V* (and again, avoiding the writing of limits and saying instead "and then letting Δx approach zero"), the expression $dM/dx = V$ is deduced and the authors state: "[this] indicates that the slope *dM*/*dx* of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value (i.e., no concentrated load is applied). [It] also shows that $V = 0$ at points where *M* is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending". Interestingly, once again, the book's authors avoid using explicitly a technology derived explicitly from Calculus (θ_1), making it less likely that students will make the connection. They finally deduce that: "$M_D - M_C = \int_{x_C}^{x_D} V dx$" and "$M_D - M_C =$ area under shear curve between *C* and *D*."

We can see that the book avoids explicitly using properties previously institutionalized in Calculus courses, which leads to a kind of praxeology in which Calculus tools are written but geometric techniques are favoured. We do not mean to say these techniques are wrong; however, they could result in a knowledge gap, as some students may not recognise the same object (*integral*) that they

encountered in their Calculus course. For instance, the first solved example (t_1) (Figure 3) presents a uniformly distributed load w. Using previous formulae, the reaction forces in the extremities are deduced (equal to $\frac{1}{2}wL$), which allows the deduction of $V_A = \frac{1}{2}wL$ and $V - V_A = -\int_0^x w\,dx = -wx$, leading to $V = V_A - wx = \frac{1}{2}wL - wx = w\left(\frac{1}{2}L - x\right)$. Note that the notation differs from that in the theoretical section, and the expression depends on the parameter w (introducing a technique τ_1 that differs from what was previously presented and that does not address the presence of w); however, the latter is not highlighted, and a graph is drawn (Figure 3c), taking for granted that students can interpret a graph depending on a parameter (ignoring students' known difficulties with parameters; e.g. Furinghetti & Paola, 1994). The maximum value of the bending moment is obtained by calculating the area under the positive triangular region $\left(M_{\max} = \frac{1}{2}\frac{L}{2}\frac{wL}{2} = \frac{wL^2}{8}\right)$, and the curve is hand-drawn (another technique that does not address that M has been introduced as the integral of V). The authors conclude with: "Note that the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree), and the bending-moment curve a cubic (third degree). The shear and bending-moment curves are always one and two degrees higher than the load curve, respectively. With this in mind, the shear and bending-moment diagrams can be drawn without actually determining the functions $V(x)$ and $M(x)$" (p. 362). A single case is used to introduce an important technological element (θ_E) that is helpful in solving T_E (drawn by hand), but this element is not justified in general, even though introducing V and M as integrals (showing that their coefficients can be deduced as primitives) would allow the use of a technology derived from the Calculus course for this justification. The book instead chooses to introduce a "rule" (θ_E) indicating that the student simply has to add one and two degrees, respectively, to draw $V(x)$ and $M(x)$. The next solved problem has students calculate (again using formulae from section 5.1) the values of forces in extremities of intervals as well as areas using geometry. Students are asked to draw by hand the bending-moment curve (Figure 4), even for cubic functions. This way, given the original diagram (Figure 4-top), students can deduce the value of V, which will be constant at certain intervals, and deduce its value at D and E specifically, while simply linking them with a straight line. Once a student has drawn the graph for V, it is possible to calculate the areas under each segment to deduce the values of M in B, C, and D, linking them by hand.

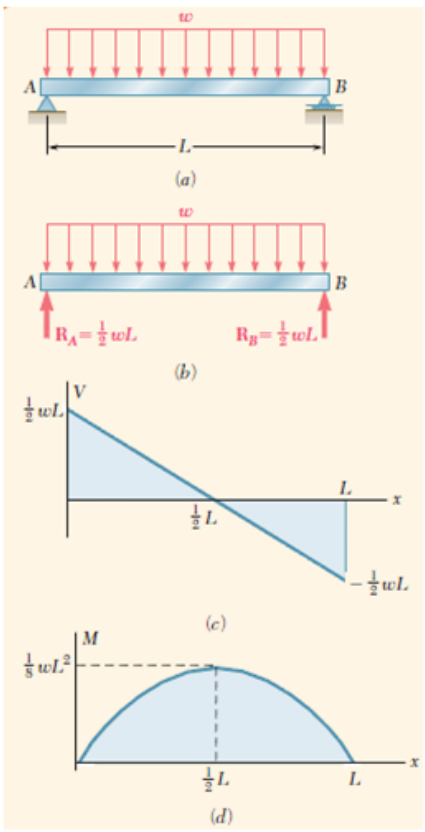

Figure 3: Solved example (Beer at al., 2012, p. 362)

In summary, the book introduces a *praxeology* to solve the problem of drawing bending-moment diagrams (T_E); however, although related notions are introduced using mathematical tools such as integrals, the technologies rely on implicit mathematical results without clearly identifying them, favouring a more professional perspective. The techniques presented are limited to calculating certain points on graphs and linking them using geometric properties, which hinders students' ability to make connections with the techniques and notions introduced in their Calculus course. Notions are presented as integrals but this fact is not explicit in the book's techniques nor in the technology; because it is possible to ignore the book's explanations when focusing on techniques, it is not certain that students will connect this content with content previously studied in Calculus courses. The book introduces a *praxeology* in which the *practical block* is clearly presented [T_E, τ_E], but where the *knowledge block* (mainly θ_E) mixes statements from mathematics and the engineering profession, leaving many facts implicit. Furthermore, this type of task does not justify all the content and techniques previously learned in Calculus courses regarding integrals.

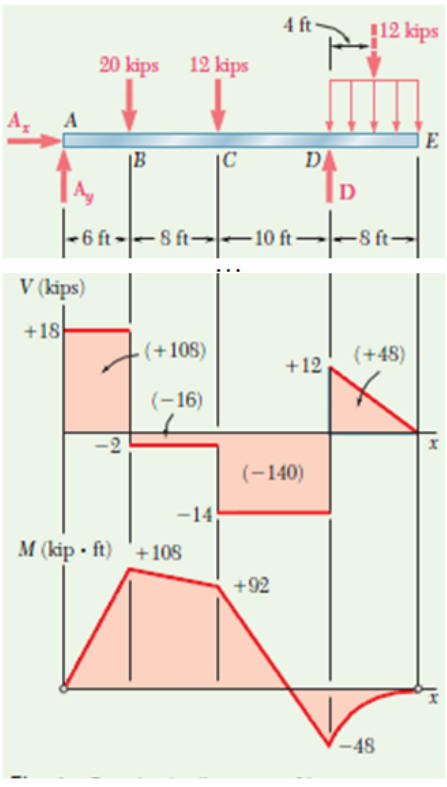

Figure 4: Solved example (Beer at al., 2012, p. 363)

Final remarks

In this paper we analysed the process of boundary crossing (Castela, 2016) of content related to integrals, and examined how this content is used as technique and technology in a *praxeology* proper to civil and mechanical engineering. The literature has identified disconnections between mathematics and professional engineering courses (Christensen, 2008; Loch & Lamborn, 2016) and our research has allowed us to pinpoint one of these disconnections. Furthermore, we believe the tools provided by ATD allow us to study *praxeologies* and identify the connectivities and disconnectivities between the content in mathematics courses and professional courses.

It may be argued that the study of integrals in engineering programmes is motivated by the simple fact that "engineers use integrals". However, we believe that the way integrals are taught in Calculus courses follows acknowledged *mathematics praxeologies* (those which are accepted and recognized by the institution of mathematics research; Castela, 2016, p. 421). These *mathematics praxeologies* ignore the use of integrals in professional courses. The crucial question, evoked in the introduction, of "what needs would be satisfied?" seems to be ignored by the *praxeologies* developed in Calculus courses, resulting in two different uses of *the same* object. We intend to analyse the entire content of the book related to sheer forces and bending moments, as well as the course notes, to provide a more detailed portrait of the use of integrals in this content. This work will be followed by further analysis of other engineering-related content, which will allow us to better understand the use of Calculus content by engineers and pinpoint possible gaps experienced by engineering students.

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Students' view of continuity: An empirical analysis of mental images and their usage

Erik Hanke[1] and Ingolf Schäfer[2]

[1]University of Bremen, Germany; erik.hanke@uni-bremen.de

[2]University of Bremen, Germany; ingolf.schaefer@uni-bremen.de

We investigate university students' mental images of continuity of real-valued functions by analyzing the answers of a questionnaire administered to Bachelor students at the University of Bremen. Our conception of mental images is based on concept images in the spirit of Tall and Vinner (1981) and the Grundvorstellungen (basic ideas) present in German subject-matter didactics (vom Hofe & Blum, 2016). For this purpose, we introduce the notion of "communicative simulacra." Furthermore, we catalog students' mental images of continuity that appear within this and preceding studies and demonstrate results on their acceptance in the study group. The used taxonomy and results are part of the first author's master's thesis (Hanke, 2016).

Keywords: Continuity, mental images, Grundvorstellungen, concept image, acceptance.

Introduction

More than 20 years ago Moore (1994) did an empirical analysis on the difficulties students face when they are required to give formal proofs. He identified among other factors that the students had little intuitive understanding of the concepts and their concept images were not adequate to perform certain proofs. Moreover, Selden and Selden (2013) argue that the ability to choose the right conceptual representation is a vital part in proving and generally in problem-solving activities.

In the context of analysis the concept of continuity is one of the most fundamental notions needed to do rigorous proofs. It is well known that students have difficulties with the notion (Tall & Vinner, 1981). This paper focuses on the mental images that future math teachers, pure math and applied math students have, which mental images they find acceptable and which they use to solve tasks. For the notion of continuity from the point of view of mathematics as well as mathematics education we refer to Tall (2013).

Theoretical background

In the German tradition of subject-matter didactics the notion of *Grundvorstellungen* (regularly translated as "basic ideas") has gained much attention: The idea of Grundvorstellungen was extracted by vom Hofe (1995) after an analysis of related ideas of didactics of arithmetics and college-preparatory didactics by pointing out the importance of creating internal representations of mathematical notions in the learners' minds (vom Hofe & Blum, 2016). According to Kleine, Jordan and Harvey (2005) Grundvorstellungen link mathematics and reality by pointing out that modeling is a central mathematical process which fulfills requirements of mathematical literacy (application, structure and problem orientation): The authors argue that this is only possible after having acquired internal representations of mathematical concepts, so called Grundvorstellungen, which connect learners' experiences and mathematical knowledge with real life. Primary Grundvorstellungen are directly related to concrete objects and actions in the environment of the learners whereas secondary Grundvorstellungen consist of imaginative actions with mental representations (vom Hofe & Blum,

2016). The latter are particularly relevant for the notion of function and special classes thereof such as continuous real-valued functions.

We prefer to regard the essence of Grundvorstellung, using a subject-matter-didactical analysis, as a predominantly normative (or even prescriptive) approach to find internal representations learners should acquire in order to be able to recognize and use a mathematical notion in inner-mathematical or applied fields. But the idea of Grundvorstellungen is complemented by the wish of mathematical didactics specialists to observe actual mental models or images, respectively, that learners really develop (vom Hofe, 1995; vom Hofe & Blum, 2016; Kleine, Jordan, & Harvey, 2005).

The notions of *concept image* and *concept definition* by Tall and Vinner (1981) have been foundational for the existing literature on university students' conceptions of elements of analysis such as differentiation, integration but also limits and continuity. The concept image comprises "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Besides, the concept definition is a form of words to specify the concept and to communicate it. It can be *personal* or *formal*, thus individually shaped or widely accepted by the mathematical community (Tall & Vinner, 1981). A formal concept definition rather reflects a normative viewpoint on what is actually forced to belong to the concept. We argue that concept definitions are part of the overall of concept images of a mathematical notion. Contrariwise, if learners are confronted with an existing concept definition they develop *concept definition images*, a part of their concept images that expressly comprises their associations with the definition. Additionally, it is understood that learners enter their acquisition process of a newly introduced concept with preexisting concept images (Tall & Vinner, 1981), and that teaching persons and environments can influence the acquisition of concept images (Bingolbali & Monaghan, 2008).

Since both the idea of Grundvorstellungen and concept images lack a distinctive description of what actually counts as internal representations, a conceptualization of *mental images* (Vorstellungen) was developed in (Hanke, 2016) which seems more appropriate to address the subtleties of precise and distinguished research questions in the scope of mental images. Mental images are substantiated as *individual* constructions and therefore *re*constructions of all kinds of mathematical notions. They are of *singular*, *regular* or *epistemological* nature, can be subdivided into *mental pictures* (Vorstellungsbilder) and *mental actions* (Vorstellungshandlungen) (Weber, 2007). Due to the premise of being able to be communicated, mental images can be shared as well as accepted, rejected or even imposed on somebody.

The most important idea for our study—and in general empirical research—is the fact that mental images cannot be observed. Thus, the only way to do empirical research about mental images is to study their *communicative simulacra*, the transformation of a potential inner world of a learner into observable entities such as spoken words, written solutions to exercises and so on (Hanke, 2016). In particular, communicative simulacra do not reflect normative assumptions on a notion as it is the case with Grundvorstellungen. In case of answers to a direct question on mental images (e.g. "What is *your* intuitive meaning of continuity?") we will speak of *exclamatory simulacra*. With this terminology we emphasize the fact that what is actually communicated by a learner depends on the occasion of communication and does not necessarily reflect the full entity of associations of the learner. We cannot even be sure that the learner is aware of the intentions of the researcher when

asked about mental images. Rather, we find blurrings of the actual mental images of learners that could potentially be sharpened by further qualitative analysis. In particular, exclamatory simulacra are shaped by the learner's understanding of the concept in question and are only a subset of communicative simulacra which, in turn, can be expressed by different forms of communication. Here we concentrate on descriptions of communicative simulacra.

Based on Moore's (1994) and Selden's and Selden's (2013) conclusions, we believe that the mere knowledge of definitions, the ability to reproduce them or the setup of mental images for a mathematical concept do not necessarily mean that the students are able to use the concept. Also, we believe that the more mental images students have the more they are able to apply at least some of these in inner-mathematical situations or in contexts. Moore's (1994) term *concept usage* is related to our idea of distinguishing between exclamatory simulacra of mental images and the usage of (probably different) mental images as required in the third section of our questionnaire. The second part of the questionnaire provides insight in the *acceptance of attitude* (Einstellungsakzeptanz) and *acceptance of usage* (Nutzungsakzeptanz) (Weber, 2007) (see next section).

The review of central papers (Bezuidenhout, 2001; Núñez & Lakoff, 1998; Schäfer, 2011; Takači, Pešić, & Tatar, 2006; Tall & Vinner, 1981) on concept images and related results on students' conceptions of continuity lead to the classification in Table 1 of eight possible mental images that are reported in the literature following Mayring's (2015) methodology of *qualitative content analysis*. We emphasize that these categories are representatives of communicative simulacra identified in the literature and we do not intend to judge about their formal or normative correctness.

Connections of continuity to the concept of integration is hardly ever noticed explicitly and therefore in case of appearance subsumed under miscellaneous. Likewise, the concept image of "pulling flat" the graph of a real-valued continuous function (Tall, 2009, p. 487) could not be found in any of the students' responses. It seems to be related to the rubber band metaphor often used in topology and usually is not part of standard German textbooks or lectures on analysis in one variable. Additionally, Schäfer's (2011) Grundvorstellungen for real-valued functions (controlled stability while wiggling at a point, possibility of approximation at a point and connectedness of the graph) are subsumed in the categories of Table 1.

#	**Category**	**Example**
I	Look of the graph of the function	"A graph of a continuous function must be connected"
II	Limits and approximation	"The left hand side and right hand side limit at each point must be equal"
III	Controlled wiggling	"If you wiggle a bit in x, the values will only wiggle a bit, too"
IV	Connection to differentiability	"Each continuous function is differentiable"
V	General properties of functions	"A continuous function is given by one term and not defined piecewise"
VI	Everyday language	"The function continues at each point and does not stop"
VII	Reference to a formal definition	"I have to check whether the definition of continuity applies at each point"
VIII	Miscellaneous	

Table 1: Categories for mental images of continuity

Setup of the study and methodology

Our research questions have been:

1.) What mental images do students express by exclamatory simulacra?

2.) What mental images do students accept and make use of in argumentation?

3.) Is there a difference between students who want to become teachers and those studying pure and applied mathematics with regard to mental images or concept usage?

We distributed a questionnaire to 54 Bachelor students (first-year pure and applied mathematics and second-year mathematics teacher students) in Bremen after the completion of a lecture with exercise classes on Analysis I (Hanke, 2016). The course covered approximately the content of Binmore (1982). The chosen methodology of the questionnaires is very similar to the one used often to investigate concept images (e.g. Tall & Vinner, 1981; Bezuidenhout, 2001; Nordlander & Nordlander, 2012). Our questionnaire, described in detail below, is an extended version of those described in Tall and Vinner (1981) and in particular Schäfer (2011). No questions concerning applications are given in order to identify students' conceptions of continuity solely related to the mathematics itself. New is the differentiation as described in the taxonomy of communicative simulacra and the comparative approach of acceptance of attitude and acceptance of usage, i.e. if students accept certain concept images and if they can apply those different images to some example functions.

To identify the different types of simulacra, the types of questions we provided different stimuli. In the first part of the questionnaire the students were asked to freely verbalize what the intuitive meaning of continuity from their point of view is. In the second part we probed the acceptance of attitude of the verbalizations of mental images presented to the participants in fictive statements on a 6-point Likert scale (totally decline (0), …, totally accept (5)) (Table 2). Furthermore, the third part of the questionnaire focused on acceptance of usage of mental images since we asked to give arguments for whether the following functions in Table 3 are continuous at the respective locations with multiple mental images. We have chosen the functions because they have discontinuities of different kind: the right and left hand side limit of $g(x)$ exist and do not coincide as x approaches 0, and the limits of $f(x)$, and $h(x)$ respectively, do not exist as x approaches 1, or 0 respectively, while the graph of f is disconnected in every neighborhood of 1, whereas the graph of h is connected in every neighborhood of 0.

#	Description	#	Description
1	Having minima and maxima is characteristic for continuity	5	Controlled wiggling
2	Limit definition of continuity	6	Connection to differentiability ("a function is not continuous at a point if it cannot be differentiated at that point")
3	Weierstraß ε-δ-definition / preimages of small open intervals contain small open intervals	7	Graph has no jumps
4	Graph has no holes	8	Graph does not swing too much back and forth

Table 2: Mental images of continuity probed in the second part of the questionnaire

1.) $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 0 \text{ for } x \in \mathbb{R}\setminus\mathbb{Q} \\ x \text{ otherwise} \end{cases}$ at $x = 0$	2.) $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 0 \text{ for } x \in \mathbb{R}\setminus\mathbb{Q} \\ x \text{ otherwise} \end{cases}$ at $x = 1$
3.) $g\colon \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 \text{ for } x < 0 \\ 0 \text{ otherwise} \end{cases}$ at $x = 0$	4.) $h\colon \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ at $x = 0$

Table 3: Different functions in the questionnaire

Summarizing, we are interested in the threefold of mental images through communicative simulacra: exclamatory simulacra and the usage and acceptance of mental images observable in communicated outcomes. Due to page restrictions, we limit ourselves on an overview and provide some statistics. For instance, the flexibility of usage and acceptance of individuals will be preserved for an upcoming paper.

Results and discussion

All answers to our questionnaires were categorized according to Table 1. Multiple responses of the students were not only possible but desired and multiple categorization of a single answer into the categories was also possible. The categorization of the answers to the first question "What is the intuitive meaning of continuity from your point of view?"—i.e. the exclamatory simulacra of mental images of continuity—led to the following: five students did not give an answer, 37 answers fell into only one category, eleven in two and the one remaining answer in three categories. Around 70% of the overall codes were found in "Look of the graph" (I) and all the other categories appeared in no more than 10% of the cases each.

The "Look of the graph" (I) is the dominant mental image among students when asked to give one. Nevertheless, some students are able to accept other mental images as well. Figure 1 illustrates this. While the items "Graph has no holes" (4) and "Graph has no jumps" (7) are accepted by the majority, items close to the limit (2) or Weierstraß definition (3) of continuity have 40% to 50% acceptance. The fact that the majority wrongly connects differentiability as a necessary condition for continuity may be a problem of the item in the questionnaire which included two negations. Based on the very high rejection rates for the items „Having minima and maxima is characteristic for continuity" (1) and „Graph does not swing too much back and forth" (8) in the second part of the questionnaire, it seems to be certain that these are not common misconceptions about continuous functions.

The functions *f*, *g* and *h* in Table 3 were all known to the students and had been part of the course in analysis and also of the exercises. In contrast to Tall and Vinner (1981) we did not give a picture of the graphs. The functions *f* and *g* seem familiar to the majority of students so about 50% are able to give a correct answer. The function *h* seems more complicated and most students do not answer the question at all and about half of the answers are false. This item is one where there is a real difference between those who study to become teacher and those who want to work as mathematicians. In the latter group the percentage of a correct answers is about twice as high (Hanke, 2016).

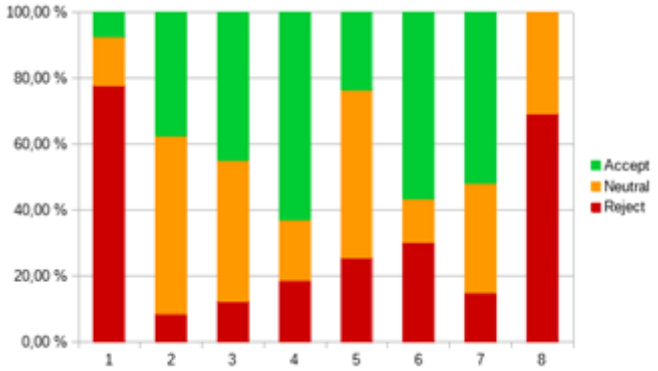

Figure 1: Acceptance of item on Likert scale from 0 to 5: Reject (0, 1), Neutral (2, 3) and Accept (4, 5) in percent of answers

To identify group differences between the different study groups (pure and applied math students vs. future teachers), we counted the occurrences of every category in Table 1 in the answers of the students for each of the questions. Concerning the overall usage of certain mental images measured with the coding of all answers to all functions of the third part of the questionnaire Fisher's test on the resulting contingency tables did not yield a significant result. We interpret that there are no observable differences in the acceptance of usage of the different study groups. Using the Kruskal-Wallis-Test, we could not find statistically significant differences except for the acceptance of the limit definition (2) in part two of the questionnaire ($p < 0{,}03$) where teachers students tended to express their acceptance with higher values on the Likert scale than the others.

Comparing the results with Schäfer (2011) we identified more detailed mental images via their exclamatory simulacra (categories 4 to 7). While the concept image of "look of the graph" (I) was dominant here as well, it was not so dominant in Schäfer's study (2011). We see a more diverse pattern in the argumentation for the three functions instead.

The most interesting part of the empirical results is that the same mental image is used by the students either to justify a wrong or a correct answer (based on their judgment whether the given function is continuous or not; cf. Figure 2): The look of the graph (I) was used most frequently for a correct but also for a wrong answer. For example, the graph of function *h* is connected (i.e. has no jumps) but the function is discontinuous at the origin. Among the answers to this function we found e.g. "Yes [*h* is continuous], since [it is] going through, without gaps or jumps," or "The function looks continuous, since it does not 'jump'." General properties of functions (V) were even used more often for a wrong than for a correct judgment of continuity. Again, for the function *h*, some students argued it is "discontinuous because of a pole," but also *h* is "continuous at $x=0$, since it lies in the domain of the function." For this function we could also find various different justifications for (dis-) continuity like "it gets area-like at the origin" or "it wiggles too much." This is also the function where more wrong than correct answers were justified with mental images.

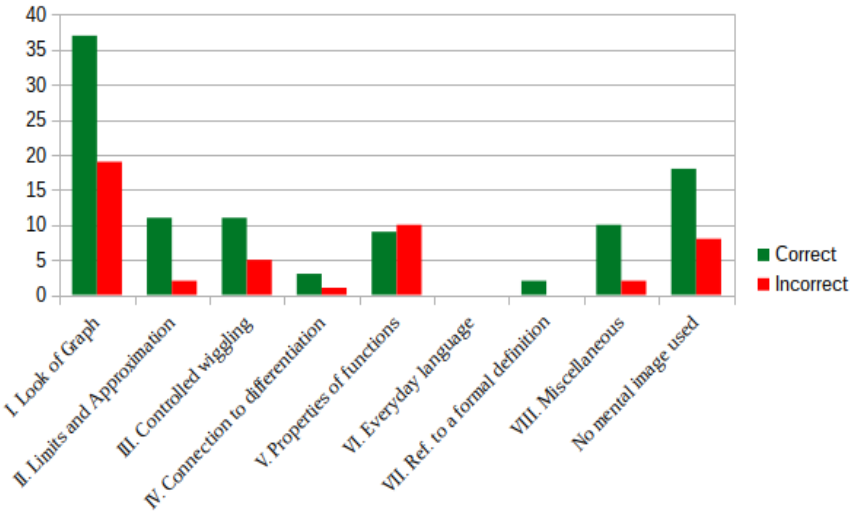

Figure 2: Numbers of correct and incorrect answers per category summed up for all four functions

Outlook

In this study we provided a taxonomy that guides empirical research related to mental images of mathematics students in several directions. We pointed out that communicative simulacra of mental images of real-valued continuous functions depend on the context in which mental images are used: Figures 1 and 2 show that the spectrum of mental images used or accepted by the students we investigated is broader than the spectrum of explicit exclamatory simulacra. We also found out that in the overall of justifications of (dis-) continuity a single mental image does not exclusively help or misguide. A first step into mental images of metric space-valued continuity is also given in (Hanke, 2016).

We believe that future research on teachers', doctoral students', tutors' or university lecturers' conceptions of continuity will provide insight into similarities and differences between social groups in the overall process of teaching and learning of a particular mathematical notion. This will be of particular importance for the teaching of real-valued continuity in today's university classrooms. Since continuity is disappearing from the curricula in secondary schools in Germany, it would be interesting to find out how the teaching of continuity in secondary schools implicitly or explicitly influences the mental images of beginning university students. Particularly, the question on the stability of mental images arises. Focusing on the analysis courses taught at secondary schools, an upcoming research area are teachers' judgments of the adequacy of teaching continuity in schools as a prerequisite for important facts on differentiability and integration such as the fundamental theorem of calculus.

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Using the theory of instrumental genesis to study students' work with a digital tool for applying integrals in a kinematic simulation

Ninni Marie Hogstad[1] and Ghislain Maurice Norbert Isabwe

University of Agder, Norway; [1]ninni.m.hogstad@uia.no

Digital tools are increasingly becoming part of mathematics in Higher Education, some of which are used pedagogically. An example is Sim2Bil, a digital tool that offers mathematical tasks about a simulation of cars. Students can solve the tasks using integrals. Applying the theory of instrumental genesis, in which techniques are analyzed in light of epistemic and pragmatic value, we studied the tool when used by groups of engineering students. We observed the students applying techniques such as instrumented and pen & pencil for solving the tasks. The techniques required the combination of understanding integrals kinematically (as distance travelled), graphically (as area under a graph) and as a calculation with symbols (finding the anti-derivative). In fact, the mathematical tasks provided opportunity for students to address at least one task, and we had students solving the most demanding ones.

Keywords: Digital tool, engineering students, instrumental genesis, epistemic value, pragmatic value.

Introduction

Solving mathematical tasks involves using different tools. These tools might be compasses and rulers, but also language, symbols, gestures, and digital tools. From a didactical point of view, it is important to study how such tools function, how students work with these tools and how the tools can be incorporated into educational practices. Within education, tools not only serve to do mathematics, such as carrying out calculations quickly, but they can also be pedagogical instruments for learning mathematics (Artigue, 2002). In our study, we focus on digital tools. Examples of digital tools for doing mathematics are Mathematica (www.wolfram.com/mathematica/) and MatLab (www.mathworks.com), which are numerical computing environments used in academia and industries. Examples of digital, pedagogical tools for mathematics are Geogebra (www.geogebra.org) and MIT Mathlets (http://mathlets.org).

Our research is positioned within the education for engineers. The integral is one of the mathematical concepts to be learned. It can be perceived in many different ways, for example as an object (as a function or as an area) or as a process (calculating the anti-derivative or taking the limit of a Riemann sum). Researchers have found that students have difficulties conceptualizing the integral (e.g. Jones, 2013; Swidan & Yerushalmy, 2014). Derouet (2016) has focused on the relation between integral, area and probability, in which the integrand is the probability density. We take an approach where we study the relation between integral, area and distance in which the integrand is the velocity. We are representing the integral as a mathematical model for making objects move under certain conditions.

There are several digital tools used for the learning of integrals. On the internet one can find applets, in which the integral is demonstrated with the Riemann Sum with an interactive slider for showing the limiting process. Berry and Nyman (2003) used a different tool, namely one that can record

displacement of an object and then graphically display both a displacement-time graph and a velocity-time graph. These kinematical graphs were meant to assist students in visually making a connection between the function and its anti-derivative. Yerushalmy and Swidan (2012) describe another tool that, given a function and its graph, generates graphically an accumulation point graph. Further on, Swidan and Yerushalmy (2014), presented the Calculus Integral Sketcher, which allows students to construct and drag a primitive graph when a function graph is given. In all cases, students' learning is supported by graphical means. We want to extend the research on pedagogical, digital tools for learning the concept of integrals by combining graphical and kinematical approaches with dynamic animations. Therefore, we considered a tool aiming to engage students to (1) combine different topics of their curriculum, such as calculus and kinematics, (2) calculate and interpret graphs etc, and (3) collaborate.

We studied Sim2Bil which can be used for groups within higher education where integrals and kinematics are part of the curriculum (students in the natural sciences, engineering, etc.). The tool requires users to work with velocity functions, of which the integral represents the distance travelled. The tool's name comes from a Norwegian word for car, 'bil', derived from automobile. The tool consists of mathematical tasks connected to dynamic animations. In addition to studying this tool, we want to study collaborative work between students, because inter-personal mediation is considered an important aspect of future education (Lowyck, 2014).

Figure 1 shows the interface of Sim2Bil. The top left part shows an animation of two cars driving in a straight line from a starting line to a finish line. This is the simulation area. The lower left part shows two separate graphs for the velocity-time function of both cars. The areas under the graphs represent the distance travelled by the cars. This is the graph area. The bottom right part is the space for the velocity functions of the cars. A user can set in parameters for polynomial velocity functions. This is called the formula area. Also, within this area there are buttons to click on. 'Formula1' and 'Formula2' give the generalized kinematical formulae for average and instantaneous velocity, and displacement, which include the integral symbol. Other buttons can hide/show the cars and graphs. When pressing the Start button in the bottom right corner, an animation starts showing the cars run, and at the same time the areas under the graphs are animated: the grey areas increase with time.

The tasks in the top right area were especially designed for the animations. In connection with the animations of cars driving by taking over each other and finishing at the same time and the graphs, the tasks were about making these cars run under certain conditions. Without the tasks, the animations of moving cars will not be very meaningful, and vice versa.

In the study presented in this paper, we investigate how Sim2Bil is used by groups of students. We use the theory of instrumental genesis to analyze our data. Our research question is: what forms does the instrumental genesis of students working with Sim2Bil take?

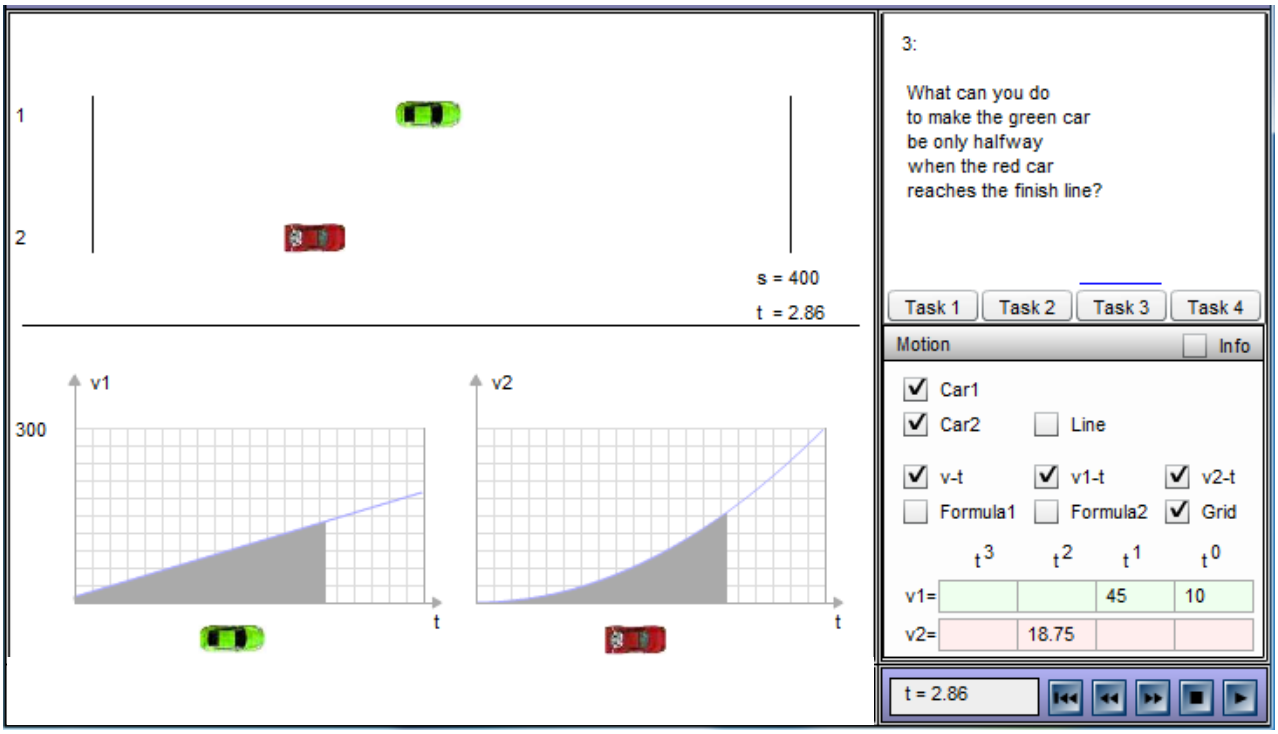

Figure 1: Interface of Sim2Bil.

The theory of instrumental genesis

The theory has assisted researchers to study students' activities in CAS environments (e.g. Drijvers, 2003; Guin & Trouche, 1999) and dynamic geometry environments (e.g. Alqahtani & Powell, 2016). First, we will explain the constructs instrumentalisation, instrumentation and technique which play an important role in the theory. Further, we will explain how techniques can have epistemic or pragmatic value.

Any artifact, a physical object, will remain a bare artifact for a person if the person does not know what to use it for. However, an artifact might turn into an instrument if there exists a meaningful relation between the person and the artifact (Rabardel, 2003). Drijvers and Trouche (2008) give a good example of how a hammer, an artifact, is turned into an instrument if a person has skills and experience of how to use it properly. The distinction between artifacts and instruments does not lay in a physical change of the objects, but a transformation of the way a person thinks about and practically uses the object.

This transformation process is a learning process and it is called instrumental genesis. The transformation process works in two directions as explained by Trouche (2004): towards the artifact and towards the user. The first direction involves the user learning to use the artifact. This embraces the activities in which an artifact becomes an instrument for a user, and about how the action influences the user's activity and knowledge. This direction is called *instrumentalisation*. The other direction concerns the user using the artifact meaningfully for tasks. This is called *instrumentation*. As Artigue (2002) points out, this process involves developing ways for solving tasks.

We will analyse students' work with Sim2Bil. Our intention is to understand how students are appropriating the tool and the transformation process of the tool becoming an instrument. Since

there exists a dialectic relationship between an artifact and a user (Gueudet & Trouche, 2009), we will look into both directions described above for the investigation.

Epistemic and pragmatic value of techniques

While doing mathematics, students can use several techniques, which may involve doing calculations and making drawings. A technique within the theory of instrumental genesis means "a manner of solving a task" (Artigue, 2002, p. 248). Artigue distinguishes between instrumented and paper & pencil techniques. Techniques yield a result and, therefore, they have a value. The value can be epistemic or pragmatic. Epistemic value involves that the technique has a meaning for the students related to the mathematical objects involved. We observe this, for example, when students find solutions to symbolic equations by creating a graphical representation with a graphing tool to use the zeros of the graph instead of solving the equations algebraically or numerically.

A technique might also have pragmatic value. Artigue (2002) explains that we can observe pragmatic value when a technique is applied by students who are focused on the productive outcome of that technique (e.g. to have a quick answer). Further, she states that epistemic value might be less recognizable than pragmatic value since the latter concerns the appearance of immediate results. We observe this, for example when students are randomly guessing an equation.

Methods

To investigate students' work with Sim2Bil we needed to observe students interacting with this digital tool. Hence, we created an environment in which students use the tool (tasks, animations), and other resources such as calculators, pens and paper.

As a default setting, there are two velocity functions given in the interface so that a student can press the Start button, see the cars run and the areas under the graphs grow. The two default velocity functions make the cars finish simultaneously. In this way, the forthcoming tasks were framed, which all have requirements on how the cars should finish. With a total of four tasks, Sim2Bil integrates tasks and animations.

The first task requires students to press the Start button and explain what the shaded areas represent. This task gives a visual introduction to the tool offering an association to distance as represented by area under a graph. After Task 1, there are Tasks 2–4, in which students are asked to find velocity functions that fulfill different requirements for the running cars. In Task 2, students are asked to find other functions so that the cars run with different velocities and arrive at the finish line at the same time. This will require students to translate the kinematics into mathematics and find functions v(t) so that $\int_0^4 v(t) = 400$. In Task 3, they are asked to make the green car be only half way when the red car reaches the finish line, thus $\int_0^4 v(t) = 200$. In Task 4, they are asked to make one car have half the velocity of the other when they arrive at the finish line simultaneously.

Our study was carried out at two different universities, at each of which we worked with one group of three students. All were within their first year of engineering studies. Group 1 comprised three boys, while Group 2 included one boy and two girls. All the six participants were not familiar with

Sim2Bil. In their lectures, the students of Group 1 might have seen a similar tool as Sim2Bil, including a button to press on screen for making an animation run. All volunteered to participate in this study outside regular lectures. The groups sat in a room with a table and one laptop with the tool in front of them. They were informed that we would study how they interact with the tool, and that they would not be assessed. Both groups were given an unlimited time for group work, and it turned out that Group 1 spent 45 minutes and Group 2 used one hour.

There are some differences in how the groups were treated, with regards to parts of the interface and how the tasks were given. In the interface, the top right area showed an unused menu for Group 1 and they received the tasks on paper. For Group 2, the Tasks 2–4 were given on screen. In the formula area, the students of Group 1 could write in parameters to make up to third degree polynomials (see Figure 1). Group 2 students could write in any expressions. Since Task 1 has a different nature than the others, as it does not ask for mathematical expressions, we gave this task orally to Group 2, at about three minutes into the session. In this way, we could see whether the Start button would be quickly found. The data collection consisted of video recordings of the students' group work. The first author was present with both groups and two cameras were used at different angles to capture students' writings, gestures, and screen activity. The video recordings were transcribed fully for Group 1 and largely for Group 2, and analyzed in light of instrumental genesis. In particular, techniques the students used to solve the tasks were identified by going through the videos and transcriptions, and analyzed in terms of epistemic and pragmatic value.

Results

By analyzing the students' group work regarding the process of *instrumentalisation*, we observed that Group 1 easily found the Start button at the beginning. They saw the cars driving to the finish line and arriving together, the growing graphs and the increasing shaded areas. Then they read Task 1. They related the areas under the graphs with the distance covered by the cars.

Group 2 was not asked to press the Start button at the beginning. They started by explaining to each other what they saw on the screen:

Dana: We are going to work with the relationship between velocity and time. (…) I'm thinking we are supposed to come up with functions like this (points at the formula area) related to the graphs (points at the graph area). Isn't it?

Jeff: Mhm… Okay, it looks like we have the formula for the velocity of the first car… and for the second car. So, uhm….

In the episode above the students related velocity functions to the graphs and velocity to the cars. After this, Jeff turned to the researcher and asked: "Excuse me, what are we supposed to find out here? We are supposed to…". The researcher asked them to press the Start button. Then, they pressed Start and saw the animated cars and graphs, and they related the areas under the graphs to distance covered by the moving cars. Thereafter, they started working on the remaining tasks.

Based on the observation of Group 2, we saw that guidance of finding the Start button was needed. At first sight, the screen offers much information, so this button can be overlooked. By pressing Start, they were introduced to the tool's functionalities and to the conditions of the tasks. The animation showed two cars running differently but finishing simultaneously. We interpret that once

the Start button was found, it enabled the appropriation of (1) the operations of the animations and (2) the conditions of the tasks.

In regards to the process of *instrumentation*, the students used several techniques. For example, they set parameters in the formula area and pressed the Start button and watched the cars run. Another instrumented technique was to insert parameters and notice how a graph of a function looked like, dismissing it when it went too low. These techniques were applied by both groups within each task. On some occasions, we observed some students set in parameters seemingly at random in the formula area. Then, the graphs were noticed or they saw the cars run. On other occasions, the instrumented techniques were done as a final check whether their paper and pencil solutions were correct (paper & pencil techniques are explained below). These instrumented techniques had pragmatic value, since it was a quick check to see whether inputs were correct. The students might have reasonably guessing parameters, and on some occasions, the students explained why the animations appeared the way they did connecting it with algebraic expressions of the functions. The techniques connect symbolic, graphical, and kinematical representations. Therefore, it can be argued that the techniques also had epistemic value.

We observed both groups use paper & pencil techniques for the tasks 2–4. They calculated the integral as anti-derivative, but knowing that they calculated the area under the graph and that it was the distance covered by the cars. In task 4, Group 1 also used a technique consisting of making rectangle and triangle drawings and making area calculations based on the fact that one area needs to be equal the other one. The techniques have a pragmatic value since the focus was on the productive potential (finding velocity functions) and it was a way of checking their answers. It can also be argued that these techniques, inherent symbols, had an epistemic value, since integration can be regarded as anti-derivation and an area can be regarded as distance travelled.

Both groups discussed how to solve each task. Also, one group occasionally used gestures to visualize the cars and graphs in a way they could not visualize through instrumented techniques (e.g. get a specific graph). Also, gestures were used to support their imagination in discussions on the requirements. When one group started on task 4, they used their hands to gesture the run of the cars, mimicking the cars take over each other and finish together. Especially, the use of gestures occurred when the students faced challenges in finding parameters to solve the tasks.

For creating polynomial velocity functions, the tool included a hint for the problem-solving process. We observed some students being “stuck” while doing calculations and clicking around on possible buttons for a clue. The formula buttons confirmed to them, that they needed an integral, but it did not help them in calculating the integral or in mathematizing the requirements of the tasks.

At the end, the one group related their answer (v1=100, v2=-25t+150) to different parts in the areas on screen:

Erik: Can we prove that the answer is correct? Yes, by the calculations again. (…) But we can also see it here that it stops on half of the other (point at the screen). So, it stops on 40. So, you can see that on the graph, actually.



Sam: Oh, yes and you can see that it is a relation between

the area under the graph…which is the integrated of the velocity.

This episode shows that the students were reasoning on how they can check whether their answer is correct. Erik mentioned the *paper & pencil* technique (calculating the triangle and rectangle, area), and the *instrumented* technique (notice the graphs on screen including the equal areas and the velocity of the red car is half the velocity of the green car). Sam repeated the relation between areas and distance.

Discussion and conclusion

Research has focused on different approaches for learning the concept of integral. In some studies, the integral was visualized as Riemann sum. In our contribution, we have taken another approach: we investigated how students work with Sim2Bil, which is a digital tool designed for university level engineering studies that include mathematics and kinematics. Two groups of students were offered tasks and animations to make cars fulfill different requirements. They needed to collaboratively understand the distance travelled as an area under a graph.

The theory of instrumental genesis allowed us to investigate students' activities in a technological environment as previous studies have shown. In particular, focusing on particular aspects of the theory such as instrumentation, instrumentalization and technique, we were able to analyse the ways students used Sim2Bil.

In the theory of instrumental genesis, a distinction is made between learning to use a tool, and using the tool for solving tasks. For learning to use the tool, the Start button played a dominant role. By pressing the Start button, students observe how the velocity functions make the graphs appear and the cars run. Thus, students were introduced to the functionalities of the tool and the conditions of the tasks. However, if students weren't told to press Start, it could take long before they discover the dynamic animation. For using a tool for solving tasks, also known as instrumentation, Artigue (2002) distinguishes between two types of techniques, *instrumented* and *paper & pencil* techniques. In tasks 2–4, our students needed pencil and paper for calculating the integral as anti-derivative, and making drawings of areas which they also calculated. Their instrumented techniques consisted of generating cars running and generating graphs. In both groups, the instrumented techniques were always the final activity in each of the tasks 2–4. With this technique, they could check whether the functions they had found met the conditions of the task.

Several techniques were used to solve the tasks. The instrumented techniques in the groups played a role as a check whether solutions are correct. Thus, they had pragmatic value. At the same time, the instrumented techniques had epistemic value because integration can be regarded as anti-derivation, and an area can be regarded as distance travelled. With task 1, our students learned the operations of the animations and the conditions of the tasks within only a few minutes. The Start button was included to get the cars running. This framed the forthcoming tasks. The challenge laid in finding functions for solving the remaining tasks. To overcome these challenges, the students used several techniques as explained above supported by gestures and discussions.

The study has some limitations. Sim2Bil is different from other tools in which students can construct mathematical objects (drag triangles, construct families of parabolas, etc). It is more of a question generator, the animation "explains" the task, and the students can use the animations or the graphs to check their answer. Additionally, the findings of this study may not be generalizable to larger groups of students since we have only observed two groups of students using the digital tool.

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Investigating the discursive shift in the learning of Group Theory: Analysis of some interdiscursive commognitive conflicts

Marios Ioannou

University of the West of England (Alexander College); mioannou@alexander.ac.cy

This study aims to investigate undergraduate mathematics students' learning experiences in a first course on Group Theory. I have used the Commognitive Theoretical Framework to examine incidences of interdiscursive commognitive conflict that emerge due to incommensurabilities with other areas of mathematics, such as Set Theory. Data is comprised of students' coursework, interviews and other secondary data. Analysis suggests that incidences of incomplete mathematical learning emerge when students need to cope with the notions of set, group, subgroup, and their elements in the same mathematical task. In addition, analysis suggests that students can often successfully produce a technically valid proof, without necessarily having full grasp of the involved concepts, indicating a ritualistic participation in Group Theory discourse.

Keywords: Group theory, theory of commognition, discursive shift, commognitive conflict.

Background

Unlike other areas of university Mathematics, such as Calculus and Analysis, the learning of Abstract Algebra, and Group Theory in particular, has been investigated to a significantly lesser extent. The first studies focusing on the learning of Group Theory emerged in the early nineties, adopting, mostly, an acquisitionist[1] perspective and within the Piagetian tradition of studying cognitive processes and errors (e.g. Dubinsky et al., 1994; Asiala et al., 1998). Other studies investigated issues such as difficulties students face with the level of abstraction of this particular mathematical subject (Hazzan, 1999), students' reaction to the semantic abbreviation and symbolisation (Nardi, 2000), the importance of visualisation (Ioannou and Nardi, 2010; Zazkis et al., 1996), and novice-students' difficulty with the process of proof (Weber, 2001).

These studies have highlighted the pedagogical challenges that students, as well as educators, face in the learning of this particular course, mostly due to its abstract nature, the often unclear, to the novice students, raison d'être of the fundamental concepts, as well as the consequent tension due to historical decontextualisation of these concepts (Nardi, 2000). Leron and Dubinsky (1995, p. 227) suggest that "[t]he teaching of abstract algebra is a disaster, and this remains true almost independently of the quality of the lectures."

Far more scarce are studies in this particular area that analyse learning from a participationist[2] perspective and in particular through the lenses of Commognitive Theoretical Framework (Nardi et al., 2014). Ioannou (2012), among other issues, investigated the intertwined nature of object-level

[1] Acquisitionists consider human development "as proceeding from personal acquisitions to participation in collective activities". (Sfard, 2008, p.78)

[2] According to participationism, "patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual." (Sfard, 2008, p. 78)

and metadiscursive level of mathematical learning in Group Theory, focusing on the *intradiscursive*[3] commognitive conflicts (see Ioannou (2016) as an example) but also on the *interdiscursive* commognitive conflicts, namely in relation to the incommensurability with other areas of mathematics. This study is a ramification of the second category. In particular, the aim of this study is to investigate incidences of commognitive conflict that emerge due to the incommensurability between various mathematical concepts in other mathematical fields towards the learning of Group Theory. For instance the notion of a set, as this has been learned in secondary mathematics education, or in the introductory course of Set Theory, and the newly introduced notion of group, focusing both on these notions as well as their elements.

Theoretical framework

As mentioned above, Commognitive Theoretical Framework (CTF) by Anna Sfard (2008) adopts a participationist perspective on learning and teaching. This fact sets CTF apart from Behaviourism and Cognitivism, in an ontological, epistemological and methodological level. Unlike the acquisitionist perspective, Commognition considers the object of developmental change to be the human activity and not the individual. Moreover, by using CTF, one should not aim to analyse the students' skills or the mental schemas of the various concepts but the discourse itself, as the principal object of attention. In fact this last characteristic of CTF is what distinguishes it from the other participationist approaches (Sfard, 2008).

Focusing on mathematical discourse in specific, unlike other scientific discourses, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, i.e. "a system that contains the objects of talk along with the talk itself, and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). CTF defines discursive characteristics of mathematics as the **word use** (the mathematical vocabulary, including the keywords that are used, not always exclusively, in schools and academia), **visual mediators** (the visible objects that are used as part of communication), **narratives** (any sequence of utterances that describe objects, relations and process, such as definitions, theorems and proofs), and **routines** (repetitive patterns characteristic of mathematical discourse) with their associated metarules, namely the *how* and the *when* of the routine.

A useful notion of CTF, especially for this particular study, is **commognitive conflict**, which is defined as a "situation that arises when communication occurs across incommensurable[4] discourses" (Sfard, 2008, p. 296). Commognitive conflict is considered "a gate to the new discourse rather than a barrier to communication, both the newcomer and the oldtimers must be genuinely committed to overcoming the hurdle" (Sfard, 2008, p. 282). Therefore, an aim of this study is to identify these

[3] These are conflicts that emerge within the particular mathematical discourse, e.g. a conflict that may occur for concepts such as subgroups and normal subgroups (both within the discourse of Group Theory).

[4] Incommensurable discourses are the discourses that differ in their use of words, visual mediators, routines or their rules of substantiation. In addition, they may allow the endorsement of seemingly contradictory narratives, due to the fact that they do not share criteria for deciding whether a given narrative should be endorsed or not. (Sfard, 2008)

situations in the undergraduate mathematics students' attempts to solve problems, which involve the newly introduced notion of group.

Other important notions within the CTF that are important for this study are the **rules of discourse**, namely the object-level and the metalevel rules. **Object-level rules** are defined as "narratives about the regularities in the behaviour of the objects of the discourse" (Sfard, 2008, p. 201). In other words these are rules that are directly related to the definition of the various objects, e.g. group, subgroup, coset, etc. **Metalevel rules** "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201). In other words metarules govern the process of proof of new (to novice students) mathematical results.

Consequently, Sfard (2008, p. 254) describes two distinct categories of learning, namely the object-level and the metalevel learning. Moreover, **object-level learning** "expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse". In addition, **metalevel learning**, which involves changes in the metalevel rules of the discourse "is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses". In the context of this study, object-level rules could be considered the rules governing the elements of the set X or the group G, whereas metalevel rules could refer to the proof that an algebraic structure is indeed a subgroup.

Methodology

This study is a ramification of a larger research project, which conducted a close examination of Year 2 mathematics students' learning experiences in their first encounter with Abstract Algebra. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year.

The Abstract Algebra (Group Theory and Ring Theory) module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data included the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, there have been analysed the staff and student interviews, and the coursework solutions. The interviews, which covered a wide spectrum of themes, were fully transcribed, and analysed with comments regarding the mood, voice tone,

emotions and attitudes, or incidents of laughter, long pauses etc., following the principles of Grounded Theory, and leading to the "Annotated Interview Transcriptions", where the researcher highlighted certain phrases or even parts of the dialogues that were related to a particular theme. Furthermore, coursework solutions were analysed in detail, after the data collection period, using the CTF, and mostly focusing on issues such as students' engagement with certain mathematical concepts, the use of mathematical vocabulary and symbolisation, and the application of discursive rules.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complaint, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., were addressed accordingly.

Data analysis

This study focuses on the application of object-level and metalevel rules that govern the mathematical concepts under study, namely, groups, subgroups, sets and their elements, investigating also the emerging commognitive conflicts. A priori analysis suggests that there are two likely commognitive conflicts: the first is related to sets (in school mathematics, sets come with a binary operation, whereas, in university mathematics, sets such as X in $Sym(X)$ do not); the second is related to functions (in school mathematics, functions operate on algebraic structures, whereas in Group Theory they play a double role, namely, operating on sets, and being themselves members of a set with a binary operation). Such commognitive conflicts have appeared in five of the thirteen students' solution of the following task: Suppose X is a non-empty set and $G \leq Sym\ (X)$. Let $a \in X$ and $H = \{g \in G: g(a) = a\}$. Prove that H is a subgroup of G.

Interestingly, students, despite their problematic application of object-level rules of the involved concepts, were often able to apply the involved metarules correctly, and produce a valid proof (e.g. for the claim that H is a subgroup of G). This fact possibly indicates that proving, as assessed in this course, may not always require an explorative participation in the proof process and complete grasp of the involved mathematical notions, but rather can rely on a ritualistic[5] performance of new routines. Moreover, successful application of metalevel rules does not necessarily imply that all the involved mathematical concepts have been fully objectified[6]. Due to limited space, below there will be demonstrated two examples of students' responses.

The first example of interdiscursive commognitive conflict, as this has been suggested in the a priori analysis above, appears below in the solution of Student A. The student has not grasped the fact that the operation refers to the group G and not to the set X. Apparently, he seems to have tried to apply it to $a \in X$ in an effort to prove inverses. He has not realised that X is a set and not a group, and therefore there is no defined binary operation on X.

[5] Rituals are defined as "sequences of discursive actions whose primary goal (closing conditions) is neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people" (Sfard, 2008, p. 241)

[6] Objectification is defined as the "process in which a noun begins to be used as if it signified an extradiscursive, self-sustained entity (object), independent of human agency" (Sfard, 2008, p. 300).

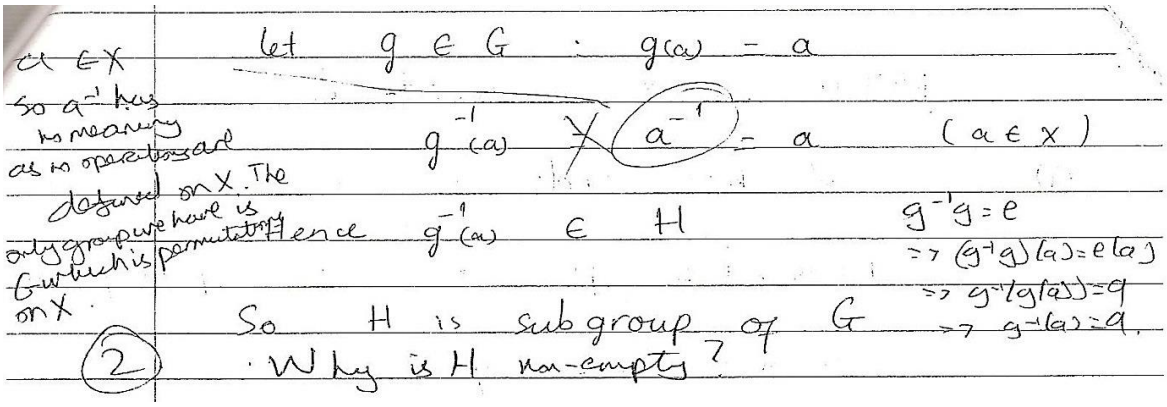

Figure 1: Example of commognitive conflict of Student A

In Figure 1, one can identify a commognitive conflict that emerged due to the discursive shift from the secondary school mathematical discourse, where all mathematical sets have algebraic structure, and in particular a binary operation with some properties. The notion of a set without an operation is new for these students. Furthermore, it proves to be particularly confusing to deal with this new kind of object in the context of a course on Group Theory, where some structures, namely groups, do have a binary operation, and others do not. In addition, another commognitive conflict is related to changes in acceptable notation. In particular, the notation gH is possibly confused with $g\ (X)$. This "abuse" of notation, where gH stands for "the set of all gh for h in H" may contribute to the confusion. An underlying commognitive conflict is due to the fact that in a university mathematics discourse, abuse of notation is often acceptable where it does not cause mathematical ambiguity. However, notation that is mathematically unambiguous may nevertheless be pedagogically confusing.

The second example, related to the second commognitive conflict of the a priori analysis, appeared in Student B's attempt to solve the aforementioned mathematical task, as seen in Figure 2. Although her solution demonstrates that she has a structural understanding of the required proof, yet she is still unable to practically do it. She applies accurately the routine for a set to be a subgroup, nevertheless there is an inaccurate application of object-level rules of the concepts of set and group and consequently several inaccuracies in her attempt. The first one is related to the expressions $g(a_1), g(a_2)$, … and a_1, a_2… where a is used to signal an element of H, and which is possibly a result of a deep confusion regarding the elements of the groups, G, H and $Sym\ (X)$ versus the elements of the set X, as well as the operation of the permutation on a set versus the composition of permutations. This inaccuracy may be considered as a result of a commognitive conflict regarding the notion of function. In the old mathematical discourse functions operate on algebraic objects, which are usually numbers. In Group Theory, functions (such as permutations) play a dual role – they operate on sets as in the old discourse, where often these sets have algebraic structure, but, in addition, they are also themselves objects of an algebraic structure (in this case group), where there is a binary operation (function composition). This dual role contributes to the incommensurability of the discourses.

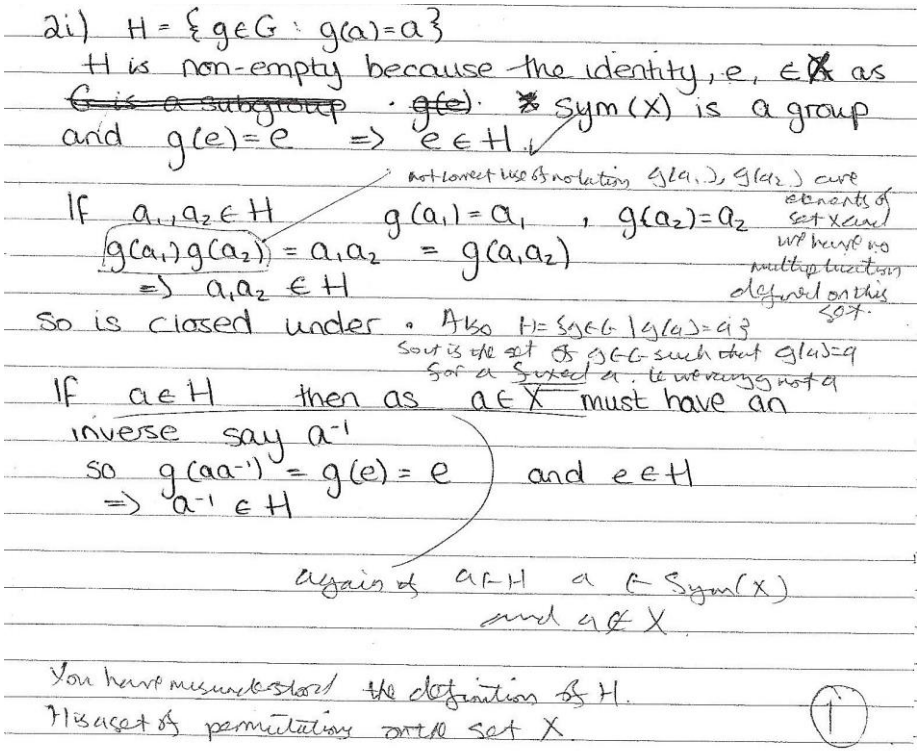

Figure 2: Example of commognitive conflict of Student B

Moreover, Student B uses the incorrect expressions $g(a_1), g(a_2)$, referring to elements of the set X, but under operation that is not applicable in X. She does not have a clear view of what is X and what is H, i.e. that X is a non-empty set and that H is a subgroup of G with a certain condition. At some point she also writes $a \in H$, which is not true since a is an element of X. Student B's incomplete object-level learning is also revealed in the following statement:

> I found it quite hard, because... I got a bit confused with this um… $Sym(X)$ and stuff, but – so I don't – I started it but then I weren't sure, whether I was doing it right, so I kind of have stopped, and I'm gonna go ask for help. To like – because I – I don't like, if I'm doing something and I'm not sure if it's right, I don't like to carry on because I don't want to do it all wrong.

The Lecturer, with the statement below, reinforces the claim that a number of students had an incomplete object-level learning regarding the elements of groups and sets.

> It is interesting, you know you have got a set in the group and somehow separating out in their minds the different roles of the elements of the setting up around and the group which is acting, is something that, you know, somehow they don't have a picture in their mind of – so they – you know writing a string of symbols round like $g_1g_2(a)$ its – the sort of – the distinction between the elements of the group and the elements of the set is something that is not necessarily clear.

Lecturer's opinion highlights the importance of examining students' object-level learning of the relevant mathematical concepts as well as their efficiency with the process of proof, something that can be investigated using CTF. It also reinforces the claim of unresolved commognitive conflicts as these occur in the discursive shift from the secondary education mathematics discourse to the formalism of Group Theory discourse. In particular, this study is in agreement with the Lecturer about

the fact that in the discourse of Group Theory there coexist structures with and without an operation, a new feature that is particularly confusing for novice students.

Interestingly, Lecturer's view is in agreement with the Seminar Leaders and Assistants', as this has been expressed in their final report on the 78 students' performance, as seen in Figure 3.

Q2(i) - At times a poorly done question. Many people were confused by the definition of H. The thing to note is that X and $Sym(X)$ (the permutations of the set X) are two separate entities and elements of one are not elements of the other. So here H is a subset of $G \leqslant Sym(X)$ and in particular H is the subset of those $g \in G$ which leave some fixed element a in X unmoved. So we are fixing a and looking for the permutations that do not move a, not a set of elements of X. Importantly we have no operations defined on the elements of X only on the elements of G, so $g_1(a) \cdot g_2(a)$ as some wrote is undefined since $g_1(a)$ and $g_2(a)$ are elements of X an arbitrary set.

Figure 3: Markers' comments on the 78 students' performance

Similarly, Markers' comments highlight students' difficulty to distinguish the various algebraic structures that coexist in the discourse of Group Theory. Students' confusion is due to the fact that that they often cannot distinguish the elements of the set and the elements of the group, but also they cannot always successfully attach the binary operation to the appropriate structure.

Conclusion

This study's aim was to identify incidents of incomplete mathematical learning in the context of Group Theory, focusing, when possible, on interdiscursive commognitive conflicts, related, in particular with the concepts of set, group and their elements. In agreement with other studies, Group Theory is a demanding subject, both from an object-level (Dubinsky et al. 1994; Nardi, 2000) as well as from a metalevel (Weber, 2001) perspective. The analysis above suggests that a frequent incidence of incomplete mathematical learning emerged during the discursive shift from a set (as students have learned in the secondary education mathematical discourse) to a group (new introduced concept in the discourse of Group Theory) and it also involved their elements. The first example of commognitive conflict emerged when Student A applied object-level rules relevant to the concept of group on set X, in which there is no defined binary operation. A second example of commognitive conflict, related to the first, was revealed through problematic use of notation, that displayed Student B's unclear view of what is X and what is H, i.e. that X is a non-empty set and that H is a subgroup of G. Finally, the analysis above suggests that students may have a structural understanding of the required proof, yet they are still unable to practically do it, indicating a ritualistic participation in the Group Theory discourse.

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Theorising university mathematics teaching: The teaching triad within an activity theory perspective

Barbara Jaworski[1], Despina Potari[2] and Georgia Petropoulou[2]

[1]Loughborough University, Loughborough, UK; b.jaworski@lboro.ac.uk

[2]University of Athens, Athens, Greece; dpotari@math.uoa.gr, enaprosena@yahoo.gr

We draw on our recent research to inspect again some of the theoretical perspectives we have been using to analyse data and to characterise teaching-learning in university settings. We focus particularly within a sociocultural perspective on Activity Theory (AT) and the construct 'the Teaching Triad (TT)', seeking to embed the TT within an AT perspective. To achieve this, we relate the Teaching Triad with aspects of the sociocultural setting both in and beyond direct interactions in face to face teaching. While this is mainly a theoretical paper, an example is taken from observations of teaching in university lectures in a Greek university to show how these theoretical perspectives have provided insights to the institutional and cultural complexities involved.

Keywords: University mathematics learning and teaching; teaching triad, activity theory, didactical triangle and tetrahedron.

Introduction to University Mathematics Teaching (UMT)

By University Mathematics Teaching (UMT) we refer to any or all the teaching of mathematics which takes place at university level. In our own corpus of work we are particularly interested in face to face teaching in lectures and tutorials in which teachers design their teaching for the benefit of students who attend their sessions. We are interested in uncovering relationships between teaching and learning within the full sociocultural context of university life. This includes the institutional setting as well as the cultures from which teachers and students make sense of the interactions in which they engage. In particular, we seek to know more about "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer, Smith & Horvath, 2010, p. 99). Our research addresses:

What is it that mathematics teachers do and think as they perform their teaching work in a university setting, and how does this relate to the mathematical meaning making of their students? (Jaworski, Mali & Petropoulou, 2016)

This question takes us into the didactical thinking of teachers who consider how best to enable students to think mathematically and develop understandings of mathematical topics; it includes teachers' pedagogic thinking in the ways in which they interact with students and use resources to promote students' engagement with mathematics; it includes also the ways in which teachers work within university affordances and constraints, the norms and expectations of university culture and their own educational histories, their views of mathematics and of what it means for students to learn mathematics and so on.

In our work to date we have used a number of theoretical perspectives to analyse data from teacher-student interactions in university mathematics teaching. Largely we have taken a broad sociocultural perspective in which we aim to address both micro and macro aspects of teaching. In

some of our work we have more specifically used Activity Theory to examine relationships and issues in teaching (e.g., Jaworski & Potari, 2009; Jaworski, Robinson, Matthews and Croft, 2012). Within some of this work we have used a theoretical construct, the Teaching Triad to address *micro* aspects of teaching while Activity Theory has addressed *macro* aspects, as we explain below.

In this paper our aim is to zoom in on connections and inter-relationships between these areas of theory as they apply in our research into teaching mathematics at university level. In order to contextualize these theoretical ideas, we include below an example from university lecturing. Since our focus is on the *theories* we are using in relation to the *activity of teaching*, we do not try to analyse the actual meaning-making of the students in our example.

Introduction to the Teaching Triad (TT)

The Teaching Triad (TT) is a theoretical construct developed from earlier research into the teaching of mathematics at secondary school level. It offers a way of characterizing mathematics teaching by acting as a tool for analyzing teaching data from classroom situations; it has also been used by teachers as a developmental tool (Jaworski, 1994; Potari & Jaworski, 2002). More recently it has been used to characterize mathematics teaching at university level and as an analytical tool at this level (Jaworski, 2002; Jaworski, Mali & Petropoulou, 2016).

Although NOT a triangle, the TT comprises *three* inter-related elements or domains of teaching: Management of Learning (ML); Sensitivity to Students (SS) and Mathematical Challenge (MC). These have been interpreted in terms of the interactions that take place within a classroom setting and, as such, focus on the *micro* aspects of teaching, without overt focus on the broader situational and cultural focuses, the *macro*. Briefly, *Management of Learning* describes the teacher's role in the constitution of the classroom learning environment by the teacher and students. It includes classroom groupings; planning of tasks and activity; use of textbooks and other resources, setting of norms and so on. *Sensitivity to Students* describes the teacher's knowledge of students and attention to their needs, affective, cognitive and social; the ways in which the teacher interacts with individuals and guides. group interactions. *Mathematical Challenge* describes the challenges offered to students to engender mathematical thinking and activity; this includes tasks set, questions posed and emphasis on metacognitive processing. These domains are closely interlinked and interdependent (Jaworski, 1994). Research has shown that a good balance between SS and MC is needed for effective teaching: a lot of SS, but little MC can lead to good teacher-student relations but low mathematical progress; a lot of MC but little SS can result in students feeling stressed or unable to succeed. When challenge and sensitivity are well balanced, the result is "harmony" – students are suitably challenged and stimulated while supported to achieve (Potari & Jaworski, 2002).

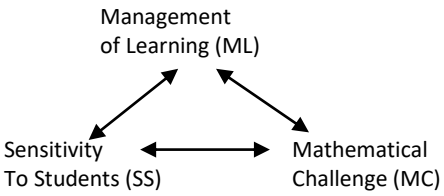

Figure 1. The Teaching Triad (Jaworski, 1994).

The TT is associated with another familiar construct, the Didactic Triangle (DT) which links Teacher (T), Students (S) and Mathematics (M) and draws attention to relationships

Teacher←→Student; Teacher←→Mathematics; Student←→Mathematics and links between these pairs (e.g., Rezat & Strässer, 2012). The TT expands the "Teacher" node of the DT, illuminating the links Teacher←→Student and Teacher←→Mathematics through the constructs SS and MC respectively while extending the DT to the wider classroom context through the construct ML. This wider context includes the resources a teacher uses in mediating between students and mathematics as expressed in the idea of a Didactic Tetrahedron in which there are 4 planes: the original DT linking TSM and the planes linking TSR, TRM and SRM (R=Resources/artifact; see Figure 2). .

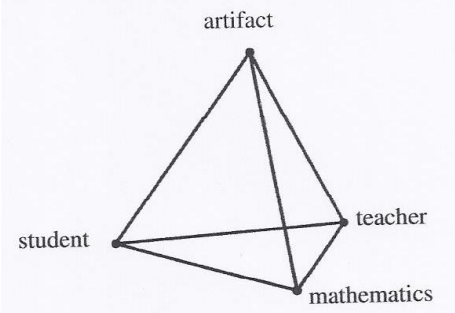

Figure 2 . The Didactical Tetrahedron (DTetra) (Rezat & Straesser, 2012)

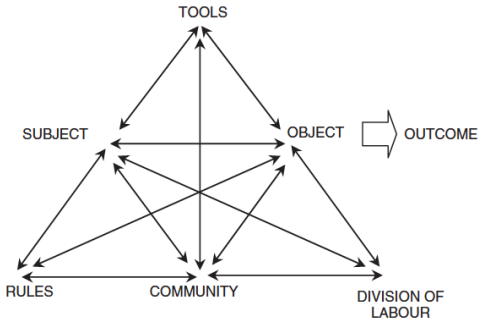

Figure 3. The Expanded Mediational Triangle (EMT) (Engestrom, 1999)

Embedding the TT into the sociocultural perspective

In this paper we re-examine the TT as a construct used within a sociocultural perspective and particularly its relationships to and within an Activity Theory analysis of teaching data. As a backdrop to AT we take Vygotskian perspectives involving particularly mediation, tool use, scientific concepts and the zone of proximal development (ZPD). Briefly, we see teaching as a process of mediation between teacher, students and mathematics (relationships are expressed simply in the DT and expanded in the TT). Teaching can be seen as mediating between student and mathematics: this is not a simplistic relationship but one with several dimensions which the TT serves to accentuate. The resources that a teacher brings to teaching (examples include mathematical symbolism, dynamic software, display media) are tools used in the teaching process; tools to facilitate learning (indicated by the extension of the DT to the DTetra). Scientific concepts are those distinguished by Vygotsky as involving theoretical learning in contrast with spontaneous concepts which arise from empirical learning (examples are mathematical concepts which need to be introduced by someone – they are not naturally occurring in everyday interactions). Daniels (2008, p. 314) cites Hedegaard (1998, p. 120) to suggest that "the teacher guides the learning activity both from the perspective of general concepts and from the perspective of engaging students in 'situated' problems that are meaningful in relation to their developmental stage and life situations". These words capture importantly the basic ideas of ML and SS in the TT of which we say more below. Daniels emphasizes the important relationship between the idea of scientific concepts and the ZPD as involving a teacher in bringing general theoretical knowledge to her interactions with students, while engaging students in concrete tasks from which scientific concepts can be abstracted. This suggests important relationships between a teacher's didactics and pedagogy – expressed simply, the former involving the transformation of mathematical concepts into tasks and activity for students and the latter involving the organization of the social setting to enable students' engagement with mathematics (together these form the basis of ML in the TT). Within the

ZPD, student engagement with a teacher's theoretical input can achieve better learning outcomes than would be achievable by a student's engagement with empirical tasks alone.

The concepts expressed extremely briefly above fit with the sociocultural perspective of A. N. Leont'ev, who makes the following point "in a society, humans do not simply find external conditions to which they must adapt their activity. Rather these social conditions bear with them the motives and goals of their activity, its means and modes." (A. N. Leont'ev, 1979, pp. 47-48). Here we focus particularly on Leont'ev's three layers of human action which constitute *Activity*. The outer, or top layer is labelled 'Activity' which according to Leont'ev (1979) is *always motivated*, although the motive might not be explicit. Within Activity, the second layer consists of the 'actions' of humans engaging in Activity. *Actions* are *goal-directed*, such that the goals are always explicit or conscious. In the third layer, actions include 'operations', which depend on the 'conditions' within which actions take place. In earlier research we have used Leont'ev's layers to explain issues and tensions which have emerged from analyses between teachers' and students' perspectives on mathematics teaching and learning (Jaworski & Potari, 2009; Jaworski, Robinson, Matthews and Croft, 2012).

If we think of the *Activity* of a university teacher teaching mathematics, within a university setting, subject to all the sociocultural forces within which the Activity takes place, we might think of the *motive* of this activity to be the mathematical learning of students participating within the complexities of this setting. *Actions* here are the *teaching actions* which take place as the teacher engages in the teaching process in relation to the mathematics which is the focus of teaching. Such actions are *goal* directed and relate to ways in which the teacher thinks about her teaching and acts in relation to her students. Thus, teacher intentions and theoretical perspectives form *goals*, and didactical and pedagogic processes form *actions* in this activity setting. The *operations* within this role, with which the teacher engages, are closely related to the practicalities of the role; for example, setting exams, creating VLE pages, assessing students' work. These operations must take place within the affordances and constraints of the university system which impose *conditions* on the operations.

Another model which is used very commonly to represent an Activity System, is the *Expanded Mediational Triangle* (EMT) from Engeström (e.g. 1998). This developed originally from Vygotsky's (simple) mediational triangle (the top part of the EMT) linking a subject with the object of her activity via the resources (tools, artefacts) employed in mediation. Engeström recognized the important mediational functions of other aspects of the sociocultural setting, such as 'rules', 'community' and 'division of labour' which expanded the roles of tools/artefacts, and which he added overtly in the EMT (see Figure 3). The 'rules' include university procedures and constraints, community includes both student and academic communities, and division of labour recognizes differences between student and teacher roles within the academic setting.

Example: Teaching in a lecture course in calculus with first year undergraduates

This example comes from the study of university lectures in first year calculus teaching in a four-year mathematics programme in the mathematics department of a Greek University (see Petropoulou, Jaworski, Potari & Zachariades, 2015). The lecturer is an established mathematician,

with extensive teaching experience, who is very popular among the students in this department. The teaching takes place in an amphitheatre with more than 200 students. The course is compulsory and its focus is theoretical with an emphasis on proofs; in this example, the mathematical focus is the convergence of series. The approach is new for the students who have previously experienced calculus in high school as a set of methods and computations. Students' expressed opinions and the very low success rate in the course examination suggest that this course is experienced as one of the most difficult during the four year programme. A large number of students take more than four years to complete their studies (the average time is 6.5 years) and some of the students have part time jobs in order to support their studies financially.

The lecturer is aware of these sociocultural issues and takes them into account in his teaching, as our analyses show. For example, he says, "I do know that students get lost in their first year and that most find mathematics too difficult…if you don't *pay attention as a teacher*, the average duration of their studies could easily become 7 or 8 years". (In analysis we see here SS in cognitive, affective and social dimensions as we explain below).

The lecturer's teaching appears rather traditional as he is seen mostly standing at the front of the room, writing at the board, "telling" or "explaining" the mathematics with rare interaction with students. Nevertheless, we see many elements of sensitivity, taking into account students' learning needs. By scrutinizing his teaching actions and goals, we see that his main teaching *goal* is to make the content relevant to students, with associated *actions* providing comprehensive explanations, highlighting subtle points that cause students' difficulties, linking informal and formal representations, making connections with students' prior school experiences, emphasizing the importance of the specific content in mathematics, in the course exams and in other courses.

At the beginning of this episode the lecturer reminds students that they know they can add a finite number of terms of a sequence i.e. they know that the sum of a finite number of terms always exists. He points out that a central question is whether the sum of an infinite number of terms exists.

He establishes the importance of this question by saying that this is exactly what we mean when we say that if it exists then the series converges. He also highlights that the "big difference here" is that the number of terms is infinite. By relating the convergence of the series to the existence of a sum, he attempts to help students to make sense of the meaning of convergence (which may still be difficult however). This can also offer a mathematical challenge that is possibly not appropriate for the students to respond to at this stage. It acts more as a situated problem for introducing the relevant theorems about the convergence of a series.

He sympathizes with the students, through a personal story about a teacher he had at school for whom the convergence was of great importance. We might say that this story supports their comfort zone, offering affective sensitivity. He then formalizes a basic proposition related to the necessary condition for a series to converge: "If a series $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $a_k \to 0$".

Here we see SS-cognitive in alternative expressions of the meaning of convergence, helping students to make sense of the concept of series convergence. We categorise the personal story as SS-affective/social, encouraging students in the lecture to have rapport with the lecturer and feel empathy with his approach to teaching them. These are pedagogic strategies which enable the

lecturer to proceed to a more formal didactic stage in his explanation in which he acknowledges a problem they might find in a text book on the topic. "The books write 'consider the sequence S_{n-1}'. But what is the sequence S_{n-1} if n=1? Is it S_0? S_0 is not defined! Ok?". His solution to this problem is to introduce a second sequence t_n: "Now, I define a second sequence t_n as follows – I am going to write down for you the terms of this sequence. First, I set something… let's say t_1, to be equal to 0. Then I set the 2nd term of t_n to be equal to S_1, the 3rd to S_2… Ok? … the 4th to S_3 etc. Namely I set t_1 to be 0 - you can set everything you want. So let t_n be S_{n-1}, if $n \geq 2$." He concludes this proof and then he offers a second proof based on the formal ε-δ definition. He compares the two ways by characterising the first way of proving 'the quick way' and the second ε-δ proof 'the slow way'. He provides all the details in both of these proofs highlighting the problem solving strategies that are usually used in proofs about series such as for example the use of partial sums.

These steps challenge students to engage with the mathematics of series in a more formal way. Perhaps this MC is scaffolded by the sensitivity observed in the earlier considerations. We see again the lecturer's drawing of students into his confidence in encouraging them to be critical of the text book, and in involving them in his reasoning for introducing the new sequence. We might see these careful steps on the part of the lecturer as his sensitivity in "paying attention as a teacher to his students' potential difficulties".

The lecturer subsequently takes the opportunity to remind students of the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$ the sequence of which tends to 0 but the series itself does not converge, and he uses this to justify that the inverse of the above proposition does not hold. He draws students' attention to the usefulness of this example in the forthcoming exams.

Further actions include providing resources and materials to students for their individual studying especially for those students who cannot attend the lectures, the structuring of the content, the teaching tools (board, supportive resources) and the traditional communication norms. In the analysis, Mathematical Challenge (MC) is often difficult to distinguish, appearing to be integrated into the SS. It is usually addressed through problem solving heuristics that are presented by him in explaining general mathematical strategies in specific cases of problems and theorems (e.g., the use of partial sums for proving the convergence of series) and by emphasizing metacognitive processes (e.g., comparing different solution strategies).

By referring to the EMT, we identify some links to the TT. SS is related to the lecturer's attention to the students' community (e.g., offering supportive resources for the students who do not attend the lectures, the delay for completing their studies). ML is related to the lecturer's attention to the university community (e.g., the tools that the lecturer uses and develops, the institutional rules such as examinations, large cohorts of students,) On the other hand MC is related to the community of mathematicians and to the mathematical practices that the lecturer brings into the classroom.

Discussion

In this example, the Activity is the sociocultural setting of teaching and learning. Seeing the teacher as *subject* (in the EMT) with the *object* of enabling students to make sense of the mathematical topic, mediators are the various artefacts/resources (such as the lecturer's board writing; his provision of on-line resources) as well as the cultures of students or teachers (student community.

academic community), differing roles of students and teachers (division of labour), and the expectations of university lectures/tutorials and the four-year programme (rules). In Leont'ev's terms the Activity is the whole, the lecturing, with the motive of enabling the students to learn basic concepts and theorems of calculus by taking into account their learning needs. We see actions and goals particularly in the activity of the lecturer: what he does to achieve the main goal of making the content relevant to the students, such as explaining mathematics at the board to ensure that students are provided with clear accounts of mathematical concepts with which they can work further, providing on-line resources to help students who must work to support their studies.

The Teaching Triad cuts across the Activity Theory frameworks to interrogate the activity of teaching. It captures the teacher's actions as related to mathematics and to the students (MC and SS). Through ML, we see the teacher's use of artefacts: tasks and resources, pedagogic strategies to include and engage students, orchestration of the environment to facilitate learning. MC can be seen in the ways the teacher presents or provides access to mathematics, linking with what the students know and with what they are expected to do in the course exams.

SS links the affective, cognitive and social elements of student engagement, rationalizing conventions and norms within the constraints and affordances of the institution. The triad presents a framework in which we see all the aspects of Activity through its three dimensions.

Elaborating further the elements of the TT discussed above, we see a close link between SS and ML in the teaching of this lecturer. SS has a strong social dimension apparent inside and outside the amphitheatre. For example, we see his concerns for providing clear explanations without interaction with the students as the institutional context and the affective constraints do not allow it. He says, "In an audience of 200 students, if you discuss with 2–3 of them, these probably will be the strongest students and the others will feel bad. … And finally nothing will remain on the board". He also takes into account students who cannot attend the lectures for various reasons (e.g., participating in social associations; socializing after the hard entry examinations; or having to get a job for financial reasons) by providing supportive online resources and materials. The lecturer teaches within the sociocultural setting described above. He engages with mathematics and with students: the fundamental relationships expressed in the DT. He uses a range of resources with which to engage students as expressed in the DTetra. The TT enables us to inspect these relationships in more depth, addressing the ways in which the lecturer engages the students and provides for their needs. We come to see that despite an approach that seems transmissive, he is nevertheless sensitive in social, affective and cognitive ways to what students need in order to make sense of the mathematics he offers. These needs relate strongly to elements of the sociocultural context in which the activity takes place including the number of students, the financial provision for their studies, and their struggles with mathematical formalism. We see lecturer's goals and actions, through which he demonstrates challenge and sensitivity, to relate fundamentally to his recognition of these contextual demands. While the AT frames (EMT and Leont'ev's layers) characterize teaching activity in its relation to context, the TT zooms in on the goals and actions to specify qualities of sensitivity and challenge and their management within the given context.

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Praxeological analysis: The case of ideals in Ring Theory

[1]Julie Jovignot, [2]Thomas Hausberger and [2]Viviane Durand-Guerrier

[1]Lycée Lamartine, Mâcon & Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier; juliejovignot@gmail.com

[2]Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier

The importance of studying structuralist praxeologies has been highlighted by Hausberger (2016). In this communication, we illustrate it on the case of ideals in Ring Theory. We provide a detailed study of a proof extracted from a textbook in Abstract Algebra showing that structuralist praxeologies involve interplay between intertwined algebraic, set-theoretic and logical praxeologies, revealing a hidden complexity.

Keywords: Mathematical structuralism, structuralist praxeologies, ideals in Ring Theory, logic.

Introduction

Hausberger (2016), with the introduction of the notion of *structuralist praxeology*, underlined the importance of *praxeological analysis* in the didactical study of phenomena related to the teaching and learning of Abstract Algebra at University level. His work is based on an epistemological investigation of algebraic structuralism that showed that mathematical practice in Abstract Algebra may be interpreted as an application of the *axiomatic method*, structures being used as tools by mathematicians in order to prove statements on objects. In the Anthropological Theory of the Didactics (Bosch &Gascon, 2014), a method is a *set of techniques*. In fact, ATD poses the general model that every human activity may be described by quadruples $[T,\tau,\theta,\Theta]$, called *praxeologies*, which correspond to the organisations it sets up: these combine a praxis (a type of tasks T and a set of techniques τ) with a logos that include two levels of description and justification of the praxis: the technology θ and the theory Θ. Hausberger (2016) made the assumption that clarifying the *structuralist techniques* may illuminate practices in Abstract Algebra, make their rationale more visible and ground them as a coherent whole. Hausberger (2016) described common tasks and techniques in the arithmetic of abstract rings and studied the structuralist praxeologies developed by students on a mathematical forum online. By contrast, the empirical data presented here is an extract of the solution of an exercise on Noetherian rings proposed by teachers in a textbook. The central mathematical notion at stake is the notion of *ideal*. By a detailed study of this example, we will develop the argument that structuralist praxeologies involve interplays between algebraic, set-theoretic and logical praxeologies, thus revealing a hidden complexity.

Structuralist praxeologies as intertwined algebraic, set-theoretic and logical ones

The notion of structuralist praxeology

Structuralist techniques are the by-products of the complete rewriting of classical algebra operated by Noether's school in the 1920s (Hausberger, 2013 & 2016). They are based on the now standard structuralist constructs: sub-structures, homomorphisms, isomorphism theorems, products or sums of structures, quotients, etc. Hausberger (2016) stressed that common tasks in Abstract Algebra may often be solved using elementary techniques. Whenever its logos block contains a theorem on structures, the praxeology may be called structuralist. Nevertheless, a gradation of its structuralist

dimension (loc. cit.) may be observed. In fact, structuralist praxeologies reflect the concrete-abstract and particular-general dialectics that are at stake in Abstract Algebra: tasks involving concrete and particular objects are completed by using abstract and general considerations on structures. Examples will be given in the sequel. The particularity of structuralist praxeologies that will be investigated in this article is that they often involve sub-praxeologies of algebraic, set-theoretic or logical type.

Algebraic and set-theoretic praxeologies

Noether qualified her own work of "set-theoretic foundation for algebra" (Hausberger, 2013), following Dedekind. On an epistemological point of view, it is characterised by the transition from thinking about operations on elements to thinking in terms of selected subsets and homomorphisms. The distinguished subsets are the kernels of homomorphisms, hence the normal subgroups in Group Theory and the ideals in Ring Theory. Noether uncovered the importance of the chain condition on ideals that led to the definition of Noetherian rings (see below). In other words, set-theoretic operations on ideals are connected to algebraic properties on elements. We will present below this connection by means of a "dictionary". It explains the intertwining of algebraic praxeologies (on the level of elements) and set-theoretic praxeologies (on the level of structures), but it leads also to the use of logical praxeologies, notably for the descent from the ideals toward the elements at stake.

Logical praxeologies

Many tasks in Abstract Algebra involve proof and proving, thus logical praxeologies. Durand-Guerrier (2008) has enlightened that the natural deduction developed by Copi (1954) provides a powerful tool to analyse and check mathematical proofs. In particular, it allows identifying those steps where mathematical arguments are silenced, supporting the claim that mathematics and logic are closely intertwined in proof. We will rely on Copi's natural deduction to describe logical praxeologies likely to appear in proof and proving: elimination and introduction of implication, universal quantifiers and existential quantifiers, restriction of the domain of quantification. The theory is the First order logic (Predicate calculus) and the technologies are logical theorems (i.e. statements true for every interpretation in any non-empty domain). In Copi's natural deduction, one deals with a generic non-empty universe, and some aspects need pragmatic control in order to ensure validity, as we will see below. The following table details common logical praxeologies that can be involved in a proof and hence in the study of structuralist praxeologies.

We provide triplets (type of tasks, technique, technology):

index	Type of tasks	Technique	Technology	Example of use
L1	Elimination of an implication	Asserting the antecedent – asserting the consequent	$[(P \Rightarrow Q) \wedge P] \Rightarrow Q$	Deduction based on a conditional theorem
L2	Introduction of implication	Recognizing that Q has been proved under the hypothesis P, and assert "$P \Rightarrow Q$"	$\neg(P \wedge \neg Q) \Leftrightarrow (P \Rightarrow Q)$	Conclusion of the proof of a conditional statement
L3	Elimination of a universal quantifier	Deleting the quantifier, introducing of a generic element of the universe, assigning this element to every occurrence of the variable in the open statement.	$[\forall x\ (F(x)] \Rightarrow F(y)$	Using a universal statement in a proof by generic element.
L4	Introduction of a universal	Given a true statement involving a generic element of a domain U,	No logical theorem. Need to control that the	Conclusion of proof by generic element.

	quantifier	assert the corresponding universal statement	element is actually a generic element of U (no other assumption on this element has been done)	
L5	Introduction of an existential statement	Given an element of the universe U satisfying an open sentence, assert that the corresponding existential statement is true.	$F(y) \Rightarrow \exists x\ F(x)$	Conclusion of the proof of an existential statement.
L6	Elimination of an existential statement	Given a true existential statement, introduce an element satisfying the corresponding open sentence.	No logical theorem. Need to control that the name of the element has not been used prior in the proof	Using an AE statement ("For all, Exists") in a proof.
L7	Restriction of the domain of quantification	Given a universal statement true in a domain A, assert it on a subdomain B of A.	$[(\forall x\ (A(x) \Rightarrow F(x)) \wedge (\forall x\ (B(x) \Rightarrow A(x))] \Rightarrow [(\forall x\ (B(x) \Rightarrow F(x))$	Fitting the statement with the antecedent of a conditional statement
L8	Transformation of a statement preserving its truth value	Substitute an equivalent statement to a given statement	In the case of implication: $[(\forall x\ (P(x) \Rightarrow R(x)) \wedge (\forall x\ (Q(x) \Leftrightarrow R(x))] \Rightarrow [(\forall x\ (P(x) \Rightarrow Q(x))$	Using the dictionary of properties elements/structures (*cf.* table 2)

Figure 1: List of a priori logical praxeologies according to Copi

The case of ideals in Ring Theory

The notion of ideal and its ecology

An *ideal I* of a ring $(A, +, \cdot)$ is, by definition, a subset of A with these properties: (i) I is a subgroup of the additive group $(A, +)$; (ii) if $a \in A$ and $x \in I$, then $a \cdot x \in I$. As part of her Master's degree dissertation, the first author conducted an epistemological and didactic study of the concept of ideal in order to explore the ecology, including the *habitats* and *niche* (Artaud, 1997) of this concept in French university education. This epistemological study started with the creation of ideal numbers by Kumer in 1847 and it enhanced the rise towards abstraction leading in the 1920s through the work of Noether to the concept we use today (Jovignot, to appear). As far as the ecology of the concept of ideal is concerned, the epistemological study allowed the identification of the following *a priori* main *habitats:* general Ring Theory (quotient rings and isomorphism theorems), arithmetic of abstract rings and elimination theory. Bearing on those results, Jovignot developed an analytical framework to identify *habitats* and *niches* of the notion of ideal in algebra textbooks addressed to undergraduates and Master's students. A first study of 3 textbooks has led to improve this grid, that was then applied to a sample of 7 French textbooks that were considered as representative of the ecology of the concept of ideal and of its use in the different post-secondary institutions in which this concept is taught in France. This study confirmed general Ring Theory and arithmetic of abstract rings as major *habitats* of the concept of ideal, but it also allowed the exhibition of *habitats* that had not been previously identified, such as the theory of modules and algebraic geometry, which suggests the importance of leading a complementary study in contemporary epistemology. Finally, elimination theory appeared, in our sample, only in the specialized computer algebra manual.

Ideals and structuralist praxeologies in the arithmetic of abstract rings

Arithmetic of abstract rings as a mathematical domain is characterized by a mathematical structure in "Russian dolls": it involves Euclidean, principal ideal domains (PID) and unique factorization domains (UFD), which generalize properties of the ring of integers, and mathematical theorems that state inclusions from the former class to the latter. Common tasks consist in proving that a given ring, for instance Gauss's ring of integers $\mathbf{Z}[i]$, belongs to a class or the other. More abstract tasks, such as the one that will be analyzed below, involve making new connections between such classes. The central notion is the notion of ideal. In fact, the class may be defined directly by a property on ideals (such as PIDs) or by properties on elements (such as UFDs) which may be related to properties of ideals by means of the following "dictionary" which was already mentioned above. This dictionary will be useful to understand used praxeologies in the task studied below.

index	Conditions of validity	Level of elements	Level of structures
D1		a divides b	(a) contains (b)
D2		a and b are associates	$(a) = (b)$
D3	$p \neq 0$	p is a prime element	(p) is a prime ideal
D4	*A* is a principal ideal domain	p is irreducible in *A*	(p) is a maximal ideal of *A*
D5	A is a unique factorisation domain	*d* is a *gcd* of *a* and *b*	$(d) = (a) + (b)$

Figure 2: dictionary of properties elements/structures

The task under study

In the next section, we will present the praxeological analysis of an exercise involving the concept of ideal which is extracted from a book addressed to Master's degree students preparing the French *Agrégation*[1]: Francinou, S. & Gianella, H. (1994). This book is widely used by university students in France. The authors sampled classical exercises in Algebra and provided proofs. In the chosen exercise, students are requested to establish a connection between Noetherian integral domains endowed with an extra property and PIDs. We clarify that this praxeological analysis is not a tool for teaching but could help us later for the design of experimentation with students.

The exercise is the following (our translation):

Let *A* be a Noetherian integral domain. We assume that every maximal ideal of *A* is principal.

1) Show that *A* is a unique factorization domain.
2) Show that every non-zero prime ideal is maximal, principal and of the form (p) where p is irreducible.
3) Show that *A* is a principal ideal domain. (loc. cit. p.57)

We will restrict our study to question 1. The authors introduce the following classical criteria, in which E designates the property of existence of a factorization and U the property of unicity:

A is a UFD if and only if:

a) every increasing chain $(a_1) < (a_2) < (a_3) <\ldots$ of principal ideals is stationary (equivalent to E)
b) every irreducible element is prime (equivalent to U)

[1] Competitive exam for prospective teachers for secondary and tertiary education

The proof provided by the authors is the following (our translation from French):

Since *A* is Noetherian, *A* satisfies (E). To establish that *A* is a unique factorization domain, it suffices to prove that if *p* is irreducible, the ideal (*p*) is prime. Let us consider a maximal ideal *M* containing (*p*). By hypothesis *M* is principal generated by *a*. Thus *a* divides *p*. Since *a* is not a unit (because $M \neq A$), *p* and *a* are associates and (*p*) = *M* is maximal. In particular (*p*) is prime.

Praxeological analysis of the task

Supplementing the proof

Reading the proof of the authors, it appears that a lot of steps remain implicit. In order to be able to study the full set of praxeologies involved in the proof, either explicit or implicit, we have supplemented it. We consider that the proof is complete when all the statements are obtained by natural deduction from previously established results or standard theorems in Abstract Algebra. We do not examine in detail in this paper the question of which of these supplements should be taught, but we will provide hypothesis that will be studied in further steps of this research. The steps of the proof presented in the textbook are numbered, our supplements appear in italic and are designated by letters whenever several steps are involved. The supplemented proof reads as follows:

1. Since A is noetherian, *A* satisfies (E).
a. Indeed, A is Noetherian so every increasing chain of ideals is stationary by definition.
b. In particular, every increasing chain of principal ideals is stationary.
c. So, thanks to the criteria, A satisfies(E).
2. To establish that *A* is a UFD it suffices to prove that if *p* is irreducible, the ideal (*p*) is prime.
a. Indeed, we need to show that every irreducible element is prime (criteria, b)
b. And "p is prime" is equivalent to "(*p*) *is prime"*
c. In fact, we will show that (p) is maximal. It is enough since every maximal ideal is prime in a ring.
3. *Let p be an irreducible element of A and* M a maximal ideal containing (*p*).
a. If there aren't any irreducible elements, we are done. In fact, irreducible elements exist since A is Noetherian, except if A is a field.
b. p is not an unit, so (p) is proper and M exists according to Krull's theorem.
4. By hypothesis *M* is principal. *Let a be a generator of M.*
5. Thus *a* divides *p*.
a. Indeed, (p) is included in M and M = (a), so (p) is included in (a).
b. And (p) is included in (a) if and only if a divides p.
6. Since *a* is not a unit (because $M \neq A$), *p* and *a* are associates - *indeed, a|p so there exists b in A such that p=ab; moreover, p is irreducible so, since a is not a unit, b must be a unit and p and a are associates* -
7. and (*p*) = *M* is maximal *since two principal ideals are equal if and only if their generators are associates.*
8. In particular (*p*) is prime.

Praxeological analysis

We present the praxeological analysis as a tabular; in the column labelled "steps", we are indicating in which steps of the proof the studied praxeology appears. Only tasks, techniques and technologies are mentioned; the theory in the sense of ATD is Ring Theory. We note S structuralist praxeologies and A algebraic ones.

steps	Type of task	Technique	Technology
1 – 8 (S1)	Show that a ring is UFD	Use of the criteria	Equivalence between the criteria and the definition of a UFD
1	L7		
2	L8		
2 b – 8 (S2)	Show that an element p is prime	Associate to p the ideal (p) and show that (p) is prime	The dictionary of properties elements/structures (D3)
2 c – 8 (S3)	Show that an ideal is prime	Try to show that the ideal is maximal	Every maximal ideal is prime
2 c	L1		
3 - 7 (S4)	Show that an ideal I is maximal	Take a maximal ideal M containing I and show that $M=I$	Krull's theorem
3 a	L6 (making explicit the two existential statements permitting the introduction of p and M)		
4 - 7 (S5)	Show that two principal ideals are equal	Show that two generators of those ideals are associates	The dictionary of properties elements/structures (D2)
5 (S6)	Show that a divides b	Show that (a) contains (b)	The dictionary of properties elements/structures (D1)
6 (A1)	Show that two elements a and b are associates	Show that a divides b; it is enough to conclude whenever b is irreducible and a is not a unit	Definition of units, irreducible elements and associates; a and b are associates if and only if $a \mid b$ and $b \mid a$

Figure 3: praxeological analysis of the task

Conclusions of our analysis

This praxeological study allows us to highlight significant characteristics of the praxeologies used by the authors that we summarize below.

The dictionary elements/structure is used along the proof; indeed the proof involves relationships between properties of elements of the ring (being an irreducible or a prime element) and properties of subsets (being a principal, maximal or prime ideal). Moreover, the algebraic notion of generator and the dictionary of properties elements/structures allow the replacement of common set-theoretic praxeologies (such as proving an equality of two sets by double inclusion) by more powerful algebraic praxeologies (involving A1). This cultural shift that is characteristic of structuralist algebra may be pointed out as a potential obstacle (previous praxeologies hindering the use of the new praxeologies to be acquired).

The structuralist praxeology S1 decomposes into several sub-praxeologies S2-S6, A1, L1, L7, L8. In the authors' proof, only structuralist steps of the proof are given; the steps involving algebraic and logical praxeologies are nearly systematically hidden. We may hypothesize that these authors see structuralist steps as the architecture of the proof and expect students to be able to reconstruct the missing elements by themselves. On the contrary, we will argue in favour of setting out the non-structuralist praxeologies and elaborate on their role in connection with structuralist praxeologies.

The nearly systematic omission of logical praxeologies raises the following issues and comments. First of all, the proof deals with generic objects, which is due to the level of generality of the statement of the exercise. It is already explicit in the statement itself, therefore both rules of elimination (L3) and introduction (L4) of a universal quantifier on the ring are not needed. In the sequel, the ideal *M* is introduced (step 3) without justification of its existence (Krull's theorem). The introduction of the generator of M is allusive and could be misinterpreted, letting think that this element has already been introduced. In both cases, the elimination of the existential quantifier (L6) remains implicit, letting thus implicit the statements themselves. The generic element *p* that plays a central role in the proof is not introduced, while it is a delicate step. Indeed, a classical way to prove a conditional statement by generic element is to introduce an element satisfying the antecedent, under the implicit assumption that such element exist; indeed, if not, there is nothing to prove (step 3.a). In addition, letting silent the restriction of the quantification domain (L7, step 1) hides the fact that this rule does not apply for existential statements, which might not be clear for some students. Finally, the substitution rule (L8) is a key for using the dictionary elements/structure by substituting a property of elements for a property of structure and vice-versa.

Giving such a proof requires the availability of the praxeologies cited above and a suitable understanding of their interrelations, or enough experience on the structuralist methodology in order to apply these praxeologies *en acte*. We hypothesise that the textbook's proof does not permit the appropriation of the structuralist praxeologies at stake. A didactical strategy to reach this goal may include, for instance, a "meta-discourse" on the crucial role of the dictionary elements/structure, together with making explicit the logical praxeologies whose role has been underlined above.

The particular construction of the proof (related to the decomposition of S1 into S2-S6-A1) can be understood by analysing the interplay between the blocks of the praxis and that of the logos of the different praxeologies engaged in the proof. However, the technological elements are seldom present in the proof written by the authors. For example, the properties of the dictionary are used but barely cited. Even if the students own in their praxeological equipment those technologies, the proof doesn't offer them the opportunity to identify those technologies in the context of the proof and thus build the associated structuralist praxeologies in order to be able to use them by themselves in another proof situation.

General conclusion and perspectives

Our praxeological analysis has highlighted the complexity of the chosen exercise. This complexity comes, in particular, from the decomposition of structuralist praxeologies into several structuralist sub-praxeologies and their interrelation with logical and algebraic praxeologies. These are fundamental in order to make the structuralist technologies practically operative. A sketchy proof which restricts to the structuralist steps, although it is seen as a clear and synthetic account by mathematicians, may therefore appear quite inadequate for self-learning by students who are not familiar with the structuralist methodology. In other words, our study contributes to break the "illusion of transparency" behind proofs that may be found in Abstract Algebra textbooks.

We aim to record and analyse the work of students who attempt to reconstruct the proof as we did, or to write a proof from scratch. In this forthcoming empirical study, our praxeological analysis will serve as an *a priori* analysis. It may also be used as a starting point in order to prepare clues for the

students and other types of didactical intervention, as well as to lead semi-structured interviews. Moreover, we intend to interview the authors of the book in order to get insights in their goals and motivations for the choices they made when writing down the proof. More generally, it is expected from these praxeological analyses, conducted on a larger scale, a deeper understanding of structuralist praxeologies with a view to setting up didactic engineerings dedicated to the teaching of structuralist concepts and in particular the ideal concept.

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University students' understandings of concept relations and preferred representations of continuity and differentiability

Kristina Juter

Kristianstad University, Sweden; kristina.juter@hkr.se

The aim of the study reported in this paper is to investigate how students understand continuity and differentiability during and after a calculus course. The students' choices of representations, both claimed and acted, were also studied. The study is part of a larger study of four student groups taking a calculus course. 207 students answered a questionnaire during the course and of them, 11 were interviewed after the course (the ones in this paper). Answers in questionnaires and interviews were categorised and compared. All students who preferred formal theoretical representations, and only those students, were able to produce formal proofs. The students' stated and acted preferences of representations were quite coherent, with only a few inconsistencies.

Keywords: Calculus, continuity, differentiability, understanding, representations.

Introduction

Learning means adaption of, building on and sometimes rejection of prior knowledge. Calculus at university level comprises numerous new things to learn for many students and the actual learning may take place a while after the teaching occasion or even the examination. Differentiability and continuity, the topics studied in this paper, are closely linked to limits that have been proven difficult to learn (e.g. Juter, 2005, 2012; Tall & Vinner, 1981). Nagle (2013) concludes, in her overview of research on transitions to formal limit conceptions, that there is a consensus in the results about the students' insufficiently developed concept images that do not allow them to formally understand limits. The transition requires students to go from a dynamic, discrete way of perceiving limits as processes to a static, continuous viewpoint where limits are regarded as formal objects. Nagle suggests an alternative introduction to calculus where more time is spent on informal conceptions to ease the transition to a formal definition. Raman (2002) found that students learning calculus do not seem to develop abilities to coordinate formal and informal aspects of mathematics unprompted, due to too little experience of such activities. It is therefore important to learn more about how students use formal and informal representations, deliberately or not, when they study mathematics. Exams also influence students' studying strategies. Bergqvist (2007) found that a vast majority of tasks from 16 university exams in introductory calculus from four different universities in Sweden only required imitative reasoning skills to pass. Mathematics learning is then endangered to become reduced to remembering routines rather than understanding concepts, processes and relationships, since students' strategies for learning are influenced by exams. The study in this paper further investigates formal and informal representations used by students to argue for relational properties of continuity and differentiability during a calculus course, and how they have developed after the course. The students' exams were divided in two, where the first part was a written routine problem solving exam and the second an oral exam where definitions and proofs were assessed. The students' preferred types of representations were also investigated and compared to their used types of representations and understandings to further explain how the students' use of formal and

informal representations compare to their learning processes. The research questions addressed in the paper are:

- How do students' relational understandings of continuity and differentiability during a calculus course compare to their understandings after the course?
- How do students' claimed preferences of representations match their actual use of representations?
- How do students' understandings and preferences of representations, spoken and acted, correlate?

Theoretical frame and some prior results

Students' understandings of mathematical concepts are reflected in their solutions, reasoning and other actions as traces (Juter, 2005) of their concept images, i.e. the total cognitive representation of a concept that an individual has in his or her mind (Tall & Vinner, 1981). Tall and Vinner define a person's concept definition for a concept as the words or symbols used to define the concept. Understanding a concept and being able to solve tasks involving the concept may be regarded as synonyms for some students, particularly if being able to solve tasks through imitative reasoning is enough to pass exams. The two ways of dealing with mathematics can however be distinguished according to their core features. Hiebert and Lefevre (1986) defined conceptual knowledge (p. 3) as a web of pieces of information well linked together with meaningful connections. Relations between concepts are abundant and significant. They defined procedural knowledge (p. 6) as knowledge requiring an input which the learner recognises and is able to perform a linear procedure on to obtain an outcome. No relational understanding is required for the process to be carried through. Strong and valid connections between concepts, i.e. conceptual knowledge, help learners to understand more as new information is embedded in, and supported by, their existing knowledge (Hiebert & Carpenter, 1992). Rich connections between concepts also reduce the burden of remembering pieces of knowledge and makes transfer within the concept image easier. Students are often unaware of the quality of links between concepts in their concept images, particularly if irrelevant or untrue links are mixed with true ones (Juter, 2011). A large number of links enables students to explain what they think determines a concept or a relationship between concepts. This can give a false sense of understanding if the links are incorrect, which in turn may lead to a situation where the student is unaware of any need for further work with the concept.

Connections between different representations of the same concept, as well as connections between different concepts, are important to create strong concept images. A function can for example be represented in different ways algebraically, by a graph, or in words. Santi (2011) addressed the issue of students understanding different representations of the same mathematical phenomenon or concept, e.g. tangents. He compared the limit process of a derivative in calculus with a cognitive perspective to a more embodied Euclidean approach of the tangent touching the curve in one point. Some students showed difficulties in seeing those representations as the same object. In a study of university students learning limits of functions (Juter, 2005), another example of incoherence in representations of a concept was apparent. Several students interpreted the formal theory as stating that limits are unattainable for functions, but when limits were used in problems they could see that sometimes functions could attain limit values (e.g. linear functions). When both these perceptions

were evoked simultaneously, the students became confused. Students meet different definitions and representations, depending on the context, e.g. intuitive descriptions, informal definitions and formal definitions (Jayakody & Zazkis, 2015). Jayakody and Zazkis presented two definitions of continuity based on limit definitions used at university courses. They concluded that students should investigate different definitions and their consequences to better understand the purpose of them. When investigating a function for continuity, the results may differ depending on definition choice, particularly if the definitions are learned intuitively rather than formally. An intuitive representation is here regarded as a perceived self-evident mental representation of a concept or phenomenon, as described by Dreyfus and Eisenberg (1982). An intuitive representation often lacks the benefits of formal strictness that can be useful in particular situations, e.g. determining if a function is continuous in a neighborhood of a given point. Developing conceptual knowledge may be difficult based mainly on intuitive perceptions. In the example with attainability of limits (Juter, 2005) some students misinterpreted the strict inequalities in the formal definition to mean that the function never can attain the limit value. The intuitive interpretation of that part of the definition overthrew the formal definition leaving the students with an incoherent concept image. Intuitive representations and other informal representations work as support for learning in many cases, but sometimes they are obstacles, particularly in a procedural learning approach where there are few opportunities to understand relations from deductive reasoning. In this study students formal and informal (including intuitive) representations of continuity and differentiability are studied and compared to the students' stated and acted preferences of representation forms.

The study, methods and sample

The 11 students focused on in this study were part of a larger study of 207 students enrolled in their first calculus course at university level. The course was not given in one particular program, so the students were from different disciplines, such as physics or mathematics. Their understandings of continuity and differentiability, and proving strategies of statements regarding the concepts, were examined (for prior results see Juter, 2012). The students were from four different groups taking the same course (different semesters). The duration of the course was 10 weeks and included basic calculus with limits, continuity, derivatives, integrals, differential equations and Taylor's formula. The students wrote an individual exam with focus on problem solving, mainly with calculations, at the end of the course and if they passed, they took an individual oral exam covering the theory of the course a couple of days later. The students answered a questionnaire when they had covered continuity and derivatives in the course. The 207 students in the study were all answering the questionnaire, which was more than 90% of the students attending the lectures. They filled it out after a lecture and had as much time as they wanted (they used up to about 30 minutes). The aim was to learn more about the students' understandings of the concepts and the relation between them, but also how they expressed their responses, e.g. formally or informally. The questions were for those reasons openly formulated. The first five questions were about what features continuous functions and differentiable functions have and what the concepts are used for. The questions relevant for the part reported here followed and they are:

1. Are all continuous functions differentiable? Justify your answer.
2. Are all differentiable functions continuous? Justify your answer.

The aim with these two questions was to see what types of representations the students would select to argue for their hypotheses. Before the data collection, they had seen examples and proofs that would enable them to answer both questions even though they were differently formulated than in the course. After the course, 11 of the students were individually interviewed. The students volunteered by indications in their questionnaires and were selected from their questionnaire answers to exemplify conceptual understanding, procedural understanding, formal use of theory and informal use of theory. The selected students are described after Figure 1. Each interview lasted about 30-45 minutes and was audio recorded. The questions were about the questions from the questionnaire and the students' answers to them, proving, examination forms and attitudes to mathematics. They were particularly asked if they agreed to their former statements in the questionnaire or not. The analysis of the interviews were tightly connected to the questionnaires and the students' development from them. Representation forms as well as mathematical content were analyzed and categorized.

Results and discussion

Figure 1 shows the students' answers to the two questions in the questionnaire (Q), if they agree or disagree (correctly or incorrectly) to those answers at the interview (I) after the course, and if the students managed to prove their statement in the second question (if so, in the questionnaire, Q, or the interview, I).

Stud.	*Continuous implies diff. Q*	*Diff. implies continuous Q*	*Agrees (I) correctly/ incorrectly*	*Disagrees (I) correctly/ incorrectly*	*Proves formally, Q or I*
Jonas	No, $\|x\|$	Yes, small change in x causes a small change in y	Correctly but he wants something added about intervals		Yes, I
Jack	No, $\|x\|$	Yes, no actual reason	Correctly explaining why $\|x\|$ is not differentiable		Yes, I
Jim	No, not $\|x\|$ and endpoints of $[a,b]$	Yes, differentiability is a stronger feature than continuity	Correctly		Yes, I
John	No, $\|x\|$	Yes, correct formal proof using the definitions of continuity and derivative	Correctly		Yes, Q
Felicia	No, $\|x\|$	Yes, same left and right limit, slope independent of chosen point in the neighbourhood of the point	Correctly agrees with first question and explains why $\|x\|$ is not differentiable	Correctly clarifies her answer to the second question. Thinks it was messily formulated	No
Fred	No, $\|x\|$	Yes, no reason	Correctly (some confusion)		No
Fay	Yes, no jumps in a neighbour-hood of an undefined point so same limits form left and right	No, a function may be differentiable on an interval, but not in the actual jump		Correctly but a bit vaguely justified in a formal attempt	No
Clara	No, only if defined for all points in an interval	No, no reason	Incorrectly		No

Carly	Yes, since they always have a slope	Answers that continuous implies differentiable again	Incorrectly on the first question, not really addressing the second		No
Celia	No, $\|x\|$	Yes, no actual reason	Agrees but adds error: In (0, 0) is $\|x\|$ not continuous		No
Carl	No, at peaks there are many different tangents. States that continuous implies diff. in another question	No, not a stair function	Correctly agrees on the first question	Correctly disagrees to his statement that continuous implies differentiable but unable to clarify the second question	No

Figure 1: Students' understandings of continuity and differentiability from questionnaires during the course (Q) and interviews after the course (I)

The students in Figure 1 are categorized in three groups, separated by different first letters in their fictitious names, depending on their responses to the two questions in the questionnaire and the interviews. In the first group (all names start with J), the four students correctly answered the questions in the questionnaires and interviews and came up with correct formal proofs. All four students used $|x|$ as a counter example to show a continuous non-differentiable function in the questionnaires. Three of the four students (all but John) did not prove their answers to the second question in the questionnaires, but they were all able to do so in the interview. Jim did at first not think he was able to prove his statement in the interview, but when he got started he was able to take it deductively step by step through knowledge about the concepts revealing a conceptual (Hiebert & Lefevre, 1986) approach to mathematics in this area. Jack had a similar task to prove at his oral exam and showed confidence in procedurally proving it in the interview, even though he was unable to prove it during the course in the questionnaire. In the second group, with three students, all names start with F. The students either answered correctly at the questionnaire and agreed with their answers in the interview (Fred and Felicia) or answered wrongly at the questionnaire and then disagreed in the interview (Fay). Felicia and Fred both used $|x|$ as a counter example the same way the students in the first group did. The students in the second group could show some confusion or small mistakes, but they answered correctly in a large sense after the course. The students did not produce any proof of the second question, but Fay made an attempt to do so when she was asked to try. She was however unable to see it through after she had written the definition for continuity where x tends to a and the definition for derivative where h tends to 0. It would probably work better for her if she had used a definition of derivative where x tends to a so she could combine the definitions easier; the lack of such flexibility could be due to her concept definitions. Comparisons of various definitions, as suggested by Jayakody and Zazkis (2015), could have helped her adjust her concept definitions to work together. She also thought that a limit is not an exact value, which can lead to problems understanding that a tangent in a point is unique if it exists, as Santi (2011) found. The third group comprises four students, all names starting with a C. These students were unable to correctly answer and/or justify their answers in the interviews. Carl and Carly both stated that continuous functions are differentiable in the questionnaire (Carl wrote it as an answer to another question where he was asked what features continuous functions have, so he gave two opposing answers in the questionnaire since he answered 'no' to question 1 above) and

Carly kept that opinion in the interview whereas Carl changed to a correct standpoint. He was however not able to correctly answer the second question. Carly thought that all continuous functions have a specific slope in all points and are hence differentiable. In the interview she thought that all differentiable functions are continuous in an open interval since the tangent does not fall over the edge at the endpoints. Carly had an intuitive (Dreyfus & Eisenberg, 1982), non-formal, way of explaining her thoughts as this example indicates. Celia got the answers correct but added erroneous explanations that did not seem founded in any conceptual knowledge, e.g. $|x|$ is not continuous at (0, 0).

There were various kinds of confusion in all groups, but in the first group it was only Jonas who lacked something about intervals in his own reasoning in the questionnaire, and this was sorted out in the interview. The other two groups showed more serious errors and confusion as described. The clarity in representation varied in the students' responses to questions in the study and the students used different types of representations to argue for their hypotheses. Figure 2 show the students' preferred representation styles as they described it and as they acted when answering the questionnaire (Q) and in the interviews (I). The category called "F theory" means students using formally expressed definitions and theorems. "Pictures" is a category for students using diagrams or other figures to explain. "Words" is a category for descriptions in words, including theoretical (but not formally theoretical), and informal (including intuitive) descriptions. The categories "F theory" and "Words" complement each other in the sense that formal representations are in the first category and informal representations in the second. The categories can be combined, e.g. "Theory" and "Words" in John's description where he formally stated a proof of the second question using formal definitions of derivative and continuity and explained the definition of continuity using words and not formal theoretical notation. John specifically stated that he did not use pictures ever, Clara stated that she read the formal theoretical parts, but did not use that way to express herself and Celia described her learning intentions to be shallow with no focus on formal theory. The categories for these three students are specified according to this in Figure 2. These are only narrow timespans to look at the students' mathematics representations so they may of course vary from what is reported here.

Students	*Says to prefer in interview*	*Preferences in action*
Jonas	F theory	F theory I, Words Q
Jack	F theory	F theory I, Words Q
Jim	F theory	F theory I, Words Q
John	F theory, Not pictures	F theory I, Q, Words I
Felicia	Pictures	Pictures I, Words Q
Fred	F theory	Pictures I, Q, Words Q
Fay	Pictures	Words I, Q, Pictures Q, F theory I
Clara	Reads F theory	Words I, Q
Carly	Pictures	Pictures I, Words Q
Celia	Not F theory	Words I, Q
Carl	Pictures	Pictures I, Q

Figure 2: Students' outspoken and acted preferences of representation forms in interview (I) and questionnaire (Q)

There is a rather good correlation between what representations the students said they preferred and what they used in this study. Fred's statements and actions were most apart as he said that he preferred formal theory, but showed no traces of it. Instead he used pictures and words as did Felicia and Fay in the same group. Only students in the second and third groups, i.e. students with names not starting with a J, stated that they preferred to use pictures in their reasoning. The tendency was also apparent in their actions. Carly, who was very visual in her explanations, preferred representations as pictures. Her mathematical development was not conceptually strong as her representations were vague and erroneous. Fay also preferred pictures, but she turned to formal representations when she was urged to try to conduct a proof (as afore described). Male students had a stronger focus on formal theory throughout, but this is a small sample so it may be a coincidence. All four students in the first group (names starting with J) said to prefer formal theoretical representations and correspondingly used formal theory. John even emphasized that he did not use pictures, which he also did not do in this study. No other student than these four both said to prefer formal representations and used formal representations in justifying claims. The four students were the only ones who could prove that differentiable functions are continuous (Figure 1). Three of them were unable to prove it in the questionnaire (or did not do it for other reasons) even though they had just covered the topic in the course, but managed to prove it in the interviews. One reason may be that many students learned the theory after the course for the oral exam since the theory was examined then. If so, they did not use much of the theory in problem solving or in making sense of mathematics during the course.

Celia stands out from the other students in Figure 2, as she showed traces of a concept image with quite weak connections. She was aware of the weaknesses since it was her strategy to learn mathematics shallowly and she kept on learning that way on purpose. Her stated approach to mathematics was procedural and she had no attempt to learn anything conceptually. This was also very clear in her responses in the interview and the questionnaire (see Figure 1). Celia had a representation of $|x|$ not being differentiable, but she did not know why. She had an intuitive sense of how it should be and kept that standpoint even though she had no means available in her concept image to justify or explain it. When she tried to explain she came to the wrong conclusion that $|x|$ is not continuous at (0, 0).

Conclusions

The changes in students' understandings of continuity and differentiability from the time right after they have learned the concepts (Q) to after the exams (I) were mainly correct adjustments. Some added errors occurred but the main type of changes were improvements of the concept images. It appears as if the students' conceptual understanding and use of theory had matured and small mistakes could be clarified deductively after the course. Students with more serious misunderstandings or insubstantial learning strategies during the course did however not show evidence of understanding the concepts better after the course (e.g. Clara and Carly). Most students' descriptions of what types of representations they used agreed with their actual usages in the data sample. A clear result is that all students in the first group claimed to prefer formal theoretical representations, all used them and all (and only they) were able to correctly prove the statement in the second question. The results of this study imply that further development of conceptual understanding after the learning situation may depend on students' preferred representation style.

Formal representations seem to be most useful for developing conceptual understanding of the concepts.

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Engineering mathematics between competence and calculation

Birgit Griese[1] and Michael Kallweit[2]

[1]Paderborn University, Germany, birgit.griese@math.upb.de

[2]Ruhr-Universität Bochum, Germany, michael.kallweit@rub.de

This study continues our previous research about the relationship between learning behavior and examination outcome in first-year engineering courses. So far, our findings have stressed the importance of making (continuous) effort and processing the weekly assignments (Griese, 2016; Griese & Kallweit, 2016), but learning behavior related to understanding was found to have little relevance. In this paper, we examine a consequent cohort of 458 students, and investigate the relationships between examination outcomes and deep learning strategies. This approach is better suited to assess competence rather than calculation routines, in accordance with the SEFI (Société Européenne pour la Formation des Ingénieurs) curriculum (Alpers, 2016). To reach this goal, variations of traditional exercises are planned to be gradually introduced in a mathematics lecture (and the appertaining assignments) for engineering first-years.

Keywords: Engineering, mathematics, curriculum, competence, assessment.

Introduction

There is a general awareness of "the struggle students endure in Service Mathematics courses" (Liston & O'Donoghue, 2009, p. 10), and of an orientation of future engineering education towards competencies (http://www.teaching-learning.eu; Alpers, 2011, 2016), following analogous developments in secondary education. This, however, is not intended to mean less skill in the handling of symbols, formulae, and operations, as the deficiencies in this area are often lamented. Rather, the notion is to keep "higher-level learning goals" (Alpers, 2011, p. 107) in mind: thinking, reasoning and modeling mathematically, posing and solving mathematical problems, as well as communicating in, with, and about mathematics (Alpers, 2011, p. 103f.), while not neglecting the traditional skills. This change must necessarily involve reforms in teaching and assessment (Entwistle & Entwistle, 1992), albeit gentle ones. Our research prepares the ground for a planned development study in this area.

Theoretical background and research approach

When introducing curricular innovations, it is advisable to investigate the initial situation as thoroughly as possible, so that changes can be researched adequately. This applies to the current state of teaching and assessment in engineering mathematics, as well as to predictors of success or failure. For mathematics majors, Rach (2014, p. 219) identified, among others, mathematical competence and school qualifications as predicting as much as 38% of academic success (in terms of passing a first-year module in calculus), whereas other researchers (Eley & Meyer, 2004) focus on aspects of the learning process (e.g. "systematic and principled use of examples", p. 449). The role of affective factors was investigated in depth by others (e.g. Andrà, Magnano, & Morselli, 2011; Liston & O'Donoghue, 2009), stressing the importance of enjoyment of mathematics, the individual's mathematical self-concept, beliefs, and motivation. Personality, or more specifically, interest in mathematics, is also an important factor (Alcock, Attridge, Kenny, & Inglis, 2014; Liebendörfer & Hochmuth, 2013). When aiming at competence-oriented learning of mathematics for first year

engineering students, their specificities and level of understanding is worth investigating (Entwistle & Entwistle, 1992; Khiat, 2010), and should be supported not only by an appropriate choice of tasks, but also by innovations in both teaching and assessment, such as "small group activity, a variety of forms of questioning, an assessed group project" (Jaworski & Matthews, 2011, p. 178) or journal writing (Glogger, Schwonke, Holzäpfel, Nückles, & Renkl, 2012).

We are interested in what learning behavior is promoted 4 for first-year engineering students, who are confronted less with proofs, but who have to deal with formal notations and who are expected to draw the connection between abstract theorems and calculation routines when aiming to succeed (see also Griese & Kallweit, 2016). Findings could also shed a light on how much conceptual understanding is required in service mathematics. We phrase our research objectives as follows:

RQ1: How do specific teaching practices relate to student learning behavior in first-year engineering mathematics courses?

RQ2: What clusters of students indicating specific learning behavior can be identified?

RQ3: What are the relationships between student learning behavior and academic success?

Methodology

Questionnaire

For our current survey we opted for items covering learning behavior under six aspects: weekly assignments (a1 to a8, 8 items), lectures (l1 to l5), tutorials (t1 to t4), deep learning (d1 to d8), surface learning (s1 to s4), and effort (e1, e2). The items were taken from Wild and Schiefele (1994), Himmelbauer (2009) as well as from Trautwein, Lüdtke, Schnyder, and Niggli (2006), via Rach (2014), and were slightly reworded to distinctly refer to mathematics. All items were rated on a 4-point Likert scales with extreme points (1) *not true* and (4) *true*. The survey was conducted three weeks before the end of the first semester. So, students had had ample experience (> 12 weeks) with academic work, had overcome the Christmas break, and the written examinations were looming. The mathematics lecture for students of civil, mechanical, and environmental engineering was addressed in the academic year 2015/2016, as well as the more advanced one for students of electric engineering and IT security, yielding a total of 458 data sets, complementing the 508 from our previous study (Griese & Kallweit, 2016).

Data analysis

In order to explore the structure of the questionnaire, we employed descriptive statistics, conducted explorative factor analysis (principal component analysis with orthogonal, i.e. varimax rotation) and calculated Cronbach's α for internal reliability.

Then, k-means cluster analysis was employed to identify different learner types who might show different patterns of academic success. Here, standardization of scale scores proved helpful for the characterization of the clusters. The average examination scores of the clusters were calculated.

Multiple linear regression was chosen to explore the influence of the different categories of learning behavior on academic success. For each participant, the items of one scale were combined by determining their means. These were used as predictors to calculate their influence on the outcome variable, *academic success*, represented by assessment points. Predictors were entered into or

removed from the linear model by means of the forward, backward and stepwise methods. Constants, coefficients b, their standards errors, standardized coefficients β, their significance values, R^2 and ΔR^2 were calculated. Missing data was eliminated pairwise in all analyses.

Results

Sample description

Out of the 458 students having answered our questions, 382 (83.41%) are enrolled in an engineering course (the rest gave no answer or were attending other courses). 74.61% of these are male, although the percentage varies over the different engineering courses (from only 49.35% males in civil engineering up to 90.23% males in machine engineering). The average age is 20.75 years (SD=3.30 years, median = 20 years), which means that the vast majority enrolled at Ruhr University almost directly after leaving school. About one quarter (25.57%) have a mother tongue different from German. About two thirds (67.42%) gained their general qualification for university entrance at a grammar school (German *Gymnasium*), and 70.05% attended an advanced course in mathematics when at school. 69.02% went to the preparation course offered by our university. Considering that 37.31% of the students state they got no more than average marks in mathematics at school, there may be a notable share of students facing problems with tertiary mathematics.

The sample of 262 data sets from students of machine, civil, and environmental engineering (who attended the same mathematics lecture) was chosen as it fit the sample from the previous year. 192 data sets could initially be matched via their individual codes to results from the written examination, and a further ten were matched by completing exactly one blank (out of the five defining a code). In order to avoid wrong matchings, this was only done in unambiguous cases. The resulting 202 data sets were then used for further explorations (meaning 60 questionnaires were eliminated for the purpose of research question three).

Exploration of items and factor structure

Some items showed prominent descriptive values. The items with the highest scores are l1 and t1 (M_{l1}=3.69, SD_{l1}=0.75, M_{t1}=3.58, SD_{t1}=0.85) which cover regular attendance of lectures respectively tutorials. Item l4 (see below) scored lowest, followed by t3 (M_{t3}=1.94, SD_{t3}=0.86, *I prepare for the math tutorials*).

The results of the explorations of the factor structure are presented here, complementing the outcomes from the year before, see Table 1. In summary, in 2014/2015, the theoretically implied six factors were identified with only one slight renaming of *effort* into *continuous effort*, but the internal reliability was compromised in three of the six scales (Cronbach's $\alpha < 0.6$): *surface learning*, *deep learning*, and *tutorials*. This was acceptable only because the factors relevant for academic success, *weekly assignments*, and *continuous effort*, had $\alpha > 0.7$. The explorations for the new data from 2015/2016 finally resulted in the same six scales as before (with a root mean square residual of 0.07). The similarities between the two years are notable, and as before, the total variance explained sums up to 48%.

Factor	Items 14/15	α in 14/15	Items 15/16	α in 15/16
Weekly assignments	a1, a2, a4, a5, a6, a7, a8	0.75	a1, a2, a5, a6, a7, a8	0.74
Continuous effort	e1, e2, d4, d7, d8, t3, l5	0.72	e1, e2, d4, d7, d8, t3, l5	0.63
Lectures	l1, l2, l3	0.72	l1, l2, l3	0.48
Surface learning	s1, s2, s3, s4	0.57	s1, s2, s3, s4	0.59
Deep learning	d1, d2, d3, d5, d6	0.56	d1, d2, d3, d5, d6, a4	0.61
Tutorials	t2, t4, a3	0.53	t1, t2, t4, a3	0.61

Table 1: Factors and their internal reliabilities, data from two years

Item a4 (*I only hand in the solutions of weekly assignments that I authored myself*) has changed its loading from *weekly assignments* to *deep learning*, and indeed it can be understood both ways, as a strategy for handling the weekly assignments, and as a deep learning strategy. The factor *continuous effort* shows only an acceptable internal reliability which does not improve when items are deleted or added. Item l4 (*During or after the mathematics lecture I ask questions if something is unclear to me*) again showed its inadequacy and was not entered into further calculations, so this item (with $M_{l4}=1.80$, which is the lowest value observed, and $SD_{l4}=0.91$) is not expedient. Item t1 (*I regularly attend math tutorials*), which was eliminated in 2014/2015 due to unilateral scores, now loads (in compliance with its conception) on *tutorials*, without worsening the internal reliability. The scale *lectures* has lost its cohesion due to the fact that it contains both the lowest and the highest scoring item (l4 and l1). The scoring may be connected to some changes in teaching style, thus addressing RQ1. Mostly, the internal reliabilities of the scales are within the range of acceptability or better (apart from *lectures* with $\alpha = 0.48$) and allow the use of five out of the six factors for further investigations.

Scale centers	A	E	L	S	D	T	# Students	Average assessment points
Cluster 1	0.55	0.54	0.21	-0.46	0.55	0.53	64	84.52
Cluster 2	-0.63	-0.62	-0.25	0.52	-0.63	-0.61	56	50.84

Table 2: Cluster analysis (k-means) for two clusters, standardized score values

Concerning RQ2, the data fitted best into two clusters, whose average standardized scale scores are presented in Table 2. The students in the first cluster show superior learning behavior under all the six aspects represented by the factors; they even employ less *surface* and more *deep learning* techniques (as pointed out by the pattern of algebraic signs), which consequently correlates to a higher number of achievement points in the written examination.

The scales (named with capital letters) show varying average scores, indicating the relevance students assign to them. *Lectures* (L), *tutorials* (T), and *weekly assignments* (A) score highest ($M_L=3.55$, $M_T=3.43$, $M_A=3.26$, $SD_L=0.57$, $SD_T=0.54$, $SD_A=0.52$), while *deep learning* (D), *continuous effort* (E), and *surface learning* (S) score medium ($M_D=2.87$, $M_E=2.82$, $M_S=2.31$, $SD_D=0.48$, $SD_E=0.43$, $SD_S=0.60$).

Predictor	b	SE for b	β	Sig.
(Constant)	-33.10	30.50		0.280
Weekly assignments	17.18	5.88	0.27**	0.004
Continuous effort	-3.43	7.02	-0.04	0.626
Lectures	5.27	5.25	0.08	0.318
Surface learning	-12.71	4.57	-0.24**	0.006
Deep learning	2.97	6.55	0.04	0.651
Tutorials	17.06	5.65	0.27**	0.003

R^2=.38, *** for p<0.001, ** for p<0.01, * for p<0.05

Table 3: Regression model with six predictors and outcome variable academic success

For answering RQ3, concerning the relationship between learning behavior and examination outcomes, linear modelling was employed. The correlations between the resulting six factors were limited to 0.45, allowing this method. The purpose of a linear model is to identify the factors (predictors) connected to an outcome variable (*academic success*, measured in achievement points in the written examination), as well as the direction (via the algebraic signs of b and β) and the strength of their influence (via the absolute value of the standardized β). In linear regression, the aim is to predict values of an outcome variable via a linear model of one or more predictor variables. Correlation between the predictor and the outcome variables is a condition for linear regression, but must not be interpreted as causality without further information. It can (but need not) mean causality in both directions (or even a common cause for both observations). Linear regression, however, has the advantage of distinguishing between predictors and outcome. It also provides estimates for the significance and the strength of the influence of each predictor on the outcome variable.

Predictor	b	SE for b	β	Sig.
(Constant)	-35.40	26.94		0.192
Weekly assignments	16.54	5.61	0.26**	0.004
Surface learning	-9.81	4.35	-0.18*	0.026
Deep learning	6.10	5.78	0.08	0.293
Tutorials	16.65	5.12	0.27**	0.001

R^2=.38, *** for p<0.001, ** for p<0.01, * for p<0.05

Table 4: Regression model with four predictors and outcome variable academic success

In our case, the direction of the influence (learning behavior on examination performance) is unsuspicious, even though we are aware of the fact that other variables (e.g. general intelligence, education before university) influence performance, too. As a first step, all six factors were entered into a linear model, resulting in the parameters presented in Table 3, showing the importance of *weekly assignments*, *tutorials*, and of avoiding of *surface learning* techniques.

The forward, backward, and stepwise methods for entering predictors into the model, respectively removing them, were employed, resulting in the four-predictor model shown in Table 4, which additionally comprises *deep learning* strategies (which, though not significant, increases the R^2 considerably from 25%), and explains 33% of the variance of *academic success*. The algebraic signs of the (significant) β values indicate the direction of the supposed impact of the predictors on the outcome variable: The more students engage in working on their *weekly assignments*, the more they actively partake in the *tutorials*, and the less they employ *surface learning* behavior, the more successful they are in the written examination.

Summary and discussion

The highest average scale scores were found for *lecture*s and *tutorials*, thus pointing out their central role in university teaching (in spite of new digital tools for distance learning). Again, the item on asking questions during or after lectures scores consistently lowest and loads unsystematically. Obviously, hardly any students dare to ask questions in the huge lecture hall comprising more than 800 seats. This item need not be used again in comparable courses. There is no scale with a mean score below 2.3 (all average scale scores are medium or high), which can be understood as an indication for the fact that our questionnaire covers only the learning behavior students report to engage in regularly; it may also be understood as a weakness of the questionnaire, as learning behavior *not* engaged in might also provide interesting revelations.

Some parameters of the new sample indicate a more competent cohort (e.g. the smaller share of students with a weaker educational background), although other findings show hardly any difference (e.g. gender, mother tongue ≠ German). The high scores for the items from the *lectures* scale are striking, it now has a distinctly higher average score (M_L=3.55 in 2015/2016; M_L=2.35 in 2014/2015) and has gained the top position, hinting that the students from this cohort attended the lectures more often and engaged in preparations or follow-up work more regularly. One reason for this may be a very different teaching approach in 2015/2016, which (among other features) involved the upload of script with gaps *before* lectures, as contrasted to uploads of complete scripts *after* lectures in 2014/2015. This distinct difference impacts on learning behavior and addresses RQ1.

Regarding RQ2, in the cluster analysis, two opposing groups of almost equal size emerge: one showing sensible, continuous, and diligent learning behavior (and consequently attaining more assessment points); the other is characterized by superficial and irregular learning or procrastination (and less points). It is remarkable, though, that their standardized scores for *lectures* are more similar than the scores from the other scales, which (apart from the fact that it reveals the weakness of this scale) allows the interpretation that the engagement in *lectures* is a less distinctive feature than other learning behavior. Considering how irregular the items from this scale score over the years, and the personality factors involved, this scale will probably stay problematic in future.

Concerning RQ3, the linear modeling stresses the relevance of working with the *weekly assignments* (as in the previous year) and attending *tutorials*. Again, *lectures* play no quantitatively relevant role, despite the high average score assigned to them now; but for reasons pointed out above this scale must be interpreted with care. The fact that *surface learning* techniques have a significant (and negative) impact in the final model is remarkable in view of the task types engineering students face in their first year. This may be hinting at an already changed assessment focus, but that would have

to be supported by a detailed and comparative analysis of the tasks from several years. *Deep learning* techniques were kept in the model as a complement and because they increase the explained total variance, although it can be argued that their contribution is weak and not significant. In contrast to other findings, *(continuous) effort* now does not contribute relevantly to explaining academic success, which is another indication of a change in assessment. On the whole, the more recent model paints a clearer picture of what is relevant or not in order to succeed in the examination than in the year before, when multiple choice tasks were involved.

Outlook on further research perspectives

The results form the basis for further research in which the tasks from the weekly assignments and the exercises in the written examination are examined more closely with the goal to gradually change them towards more competence-orientation, according to the suggestions by Alpers (2016). This would involve, for example, finding, describing, and correcting different types of mistakes in the calculation of an inverse matrix, instead of doing the calculation itself. The results gained from the explorations presented in this and a previous paper (Griese & Kallweit, 2016) will then supply the background against which the expected changes can be compared.

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The association between engineering students' self-reported mathematical identities and average grades in mathematics courses

Eivind Kaspersen[1], Birgit Pepin[2], and Svein Arne Sikko[1]

[1]Norwegian University of Science and Technology, Faculty of Teacher and Interpreter Education, Trondheim, Norway; eivind.kaspersen@ntnu.no, svein.a.sikko@ntnu.no

[2]Technische Universiteit Eindhoven, Eindhoven, The Netherlands; b.e.u.pepin@tue.nl

Arguments have been made that one purpose of learning mathematics successfully is for students to develop mathematical identities. Thus, since students are frequently evaluated with grades in university mathematics courses, a relevant question is how mathematical identities are associated with average grades. This study has measured engineering students' mathematical identities and compared these measures with grades in university mathematics courses, and a Welch's ANOVA conclude that the mean average grade amongst students with high mathematical identities is significant, and about one grade higher than students with low mathematical identities. Moreover, the variance is greater amongst students with low mathematical identities, which indicates a strong association between mathematical identity and average grade only when mathematical identities are high.

Keywords: Mathematical identity, Rasch, ANOVA.

Introduction

The transfer of mathematical knowledge from university to the world of work seems problematic. Specifically, evidence has been provided that "attainment" in university mathematics courses is poorly transferred. One example is an experiment that illustrated how 17 students and researchers all failed a mathematics examination they had previously passed, even the students who had recently passed the original exam with an "A" (Rystad, 1993). Moreover, selected studies illustrate how the mathematics is often hidden in "black-boxes" (e.g. Williams & Wake, 2007) in the world of work, and consequently, arguments have been made that the world of work seeks more general mathematical characteristics than what is typically assessed in standard exams (e.g. Hoyles, Wolf, Molyneux-Hodgson, & Kent, 2002). On a general note of education, Wenger (1998) argued that learning is about developing identities in communities of practice. In general, over the last decades, there has been an increased attention towards the construct of identity, and mathematical identity in particular (e.g. Axelsson, 2009; Black et al., 2010; Wenger, 1998). Thus, if the world of work seeks general characteristics of working mathematically, a relevant question is how mathematical attainment in university mathematics courses, as represented by average grades, is associated with mathematical identity. This paper addresses this question.

This study has examined the association between self-reported mathematical identities and average grades in university mathematics courses. From a Rasch calibrated instrument, previously validated in Kaspersen (2015), the students were categorised as having a "low," "medium," or "high" mathematical identity, and the paper will illustrate how the mean average grade of students with high mathematical identities was significant and about one grade higher than students with low mathematical identities. Moreover, the variance amongst students with low mathematical identities was higher than amongst students with high mathematical identities, although the difference was not

significant (p=0.06). The paper concludes that high mathematical identities are associated with high average grades in university mathematics courses. However, the same conclusion is not true amongst students with lower mathematical identities.

Theoretical framework

The construct of identity suffers from a lack of consensus on general philosophical issues (Cote & Levine, 2014). Specifically, identity is defined differently across different studies and paradigms, such as "a certain kind of person" (Gee, 2000, p. 99), "those narratives about individuals that are reifying, endorsable and significant" (Sfard & Prusak, 2005, p. 44), and "self-perceived mathematical knowledge, ability, motivation and anxiety" (Axelsson, 2009, p. 387).

This lack of consensus is typical in pre-paradigmatic fields (Kuhn, 1970). Unlike firm paradigmatic fields where well-established theories tend to guide the analyses, research in pre-paradigmatic areas has a more dialectical relationship between data and theory (Kuhn, 1977). This description is a fair representation of how the theoretical perception in this study was chosen. That is, no ready-made theory was chosen on pure faith. Rather, a definition of identity was established that was consistent with measurement (i.e., consistent to conclude some persons to have stronger mathematical identities than others), yet, influences by fragments of multiple existing theories. The following theoretical perspective and a wider discussion on practical significance has been provided in more detail in Kaspersen, Pepin, and Sikko (2017).

On another note, we do not regard theories as mirrors of some true reality. Thus, we do not believe that some theories *are* true, and that others *are* false. When we propose the following theoretical perspective, therefore, we are not refusing other perspectives, for instance, a narrative view on identity. Rather, we claim that *if* we choose the following perspective, then the practical consequence is that mathematical identity can be measured.

The perspective of mathematical identity relies on two assumptions. First, we assume that identity (originated from the Latin *idem*) is about sameness and distinction. As such, the position in this study juxtaposes perspectives that consider persons to have their unique identity. That is, persons are indeed unique. However, they can be defined as identical with respect to a set of characteristics, just like mathematical objects can be identified by certain characteristics while remaining unique on others. Moreover, since there exists an infinite number of characteristics, identities have a varying degree of complexity. That is, mathematical identity can be binary, linear, or multidimensional, and we argue that there is no ontological limit to the number of dimensions. Consequently, there exists no set of criteria that dictates when researchers have arrived at the final dimension. Hence, the choice of complexity can be nothing but pragmatic, and in this study, we have chosen a one-dimensional perspective on mathematical identity, whereby persons are distinguished on a continuum from having a low to having a high mathematical identity within the engineering education context.

Furthermore, if we accept that persons participate and contribute in multiple activities, a consequence is that each person has multiple identities, a position that is shared by many authors, for example Black and colleagues (2010) who, inspired by Leont'ev (1981), presented the idea of "leading identity." Since there is no limit to how many ways persons can be distinguished, we argue that there exists no limit to the number of identities, although the number of identities that individuals are consciously aware of is likely to be finite. Moreover, in this study, we take no definite position on the

relationship between identities. Thus, when we later will conclude that selected persons have (more or less) the same mathematical identity, we do not make claims about how these are related to the multiplicity of identities–for instance, whether they are central/leading or peripheral identities.

Second, we assume that identity is relational by nature. That is, persons can be concluded to be identical relative to a set of characteristics, only if the structure of these characteristics is person-independent. Thus, in quantitative studies, we reject the assumption that persons with the same score on some test or questionnaire are identical unless statistical evidence is provided that the items stay invariant across relevant subgroups. Hence, there likely exist contexts that are so different that comparisons of identities across these contexts do not make sense. Consequently, we argue that the methods that are applied to capture identities should also capture the level of invariance.

In conclusion, we define mathematical identity to be *where persons position themselves relative to the social structure of being mathematical within the activity in which they participate and contribute*. From a one-dimensional perspective, "the social structure of being mathematical" is a person-independent set of characteristics and their internal structure (i.e., their relative distance) that distinguishes persons on a continuum from having a "low" to having a "high" mathematical identity. "Where persons position themselves" is persons' positions relative to the social structure.

Method

To test the relationship between engineering students' self-reported mathematical identities and average grade in mathematics courses, a convenience sample consisting of Norwegian engineering students (N=361) was selected. 47 students attended an "Introductory course in mathematics," 71 students attended a "Calculus 2" course, 113 attended a "Calculus 3" course, 11 a "Cryptography" course, and 119 were students from a variety of courses in their normalised final year of education. The participants responded to a Rasch-calibrated instrument (Rasch, 1960), previously validated in Kaspersen (2015), that measures persons on a continuum from having a low to having a high mathematical identity relative to 20 uni-dimensional characteristics. The items in the instrument were collected from three sources: the literature, other related instruments, and from persons contributing in mathematical activities (e.g., students and lecturers). The validation of the instrument will not be discussed in depth, as details can be found in Kaspersen (2015). The person reliability, analogous to Cronbach's alpha, was 0.87. Moreover, from principal component analysis of residuals, the instrument was found to be sufficiently uni-dimensional for the purpose of measurement with a 1.99 unexplained variance (in Eigenvalue units) in a second contrast. Furthermore, the mean of the squared standardised residuals (outfit mnsq) and the information-weighted version (infit mnsq) (see e.g., Bond & Fox, 2003, p. 238 for a detailed description) indicated a sufficient data-model fit, with Item 6 and Item 15 as the most underfitting items (Table 1).

Rasch measurement requires additivity, uni-dimensionality, and invariance, and the probability of an observation is a function of the difference between a person's measure and a characteristic's measure (e.g. Wright & Stone, 1979). Thus, most response strings follow a Guttman-like structure with most deviations around the measure of the person. Consequently, persons with approximately the same measures, except those with large misfit, have, not only the same measures but also approximately the same combination of self-reported characteristics (and thus concluded to be identical with respect to these characteristics).

After the validation of the instrument, the respondents were categorised as having either low (measures lower than -1), medium (measures between -1 and 1), or high (measures above +1) mathematical identities (all measures are in logit units). The distance from the "low"/"medium" to the "medium"/"high" thresholds was about the same distance as one response category. Consequently, persons with "high" mathematical identities were expected to respond at least one category higher on each characteristic than persons with "low" mathematical identities. Subsequently, a one-way ANOVA was conducted to compare the association between mathematical identity and the self-reported average grade in mathematics courses at the University (from grade F=1 to grade A=15). However, since the Levene's (1960) test barely accepted the null hypothesis of homogeneity of variances (p=0.06), and the sample sizes across categories were unequal, the Welch's ANOVA was chosen since it is more robust to unequal sample size and variance.

Moreover, the assumption of normality was violated, and the grades were ordinal as opposed to interval measures. Since Welch's ANOVA assumes normal and interval measures, 10,000 simulations were made in R (R Core Team, 2015) to assess how these violations affected the robustness of the analysis. To ease this part of the analysis, we considered a transformed data set which had no difference in the mean across groups but was otherwise identical to ours–the assumptions of Welch's ANOVA were violated equally in the empirical study and the simulated studies. This transformation eased the interpretation since we could compare the results with the statistical ideal situation (perfectly normal interval data, equal sample size and variance). If our data set was as good as the ideal situation, we would expect the Welch's ANOVA to show a significant difference in about 5% of the simulations.

Specifically, from the empirical data frame, M, a new data frame, M', was made whereby each grade in the medium and high groups was shifted so that the mean of all three categories in M' were equal (i.e., keeping the sample sizes and distributions, but aligning the means). From M', 10,000 data frames, $M_1 - M_{10,000}$, were randomly sampled whereby the sample sizes in the three groups were equal to the original M. Subsequently, Welch's ANOVA was conducted on each simulated data frame. Since the result showed that 5.2% of the p-values in the simulations were less than .050, it was concluded to ignore violations of Welch's ANOVA's assumptions since they had only a trivial negative effect on the robustness.

Result

Mathematical identities

Due to the Guttman-like response strings, a rough interpretation of Table 1 is that most students with low mathematical identities (measures lower than -1) agreed with characteristics much lower than -1, and disagreed with those much higher than -1. That is, students with low mathematical identities often keep trying when they get stuck, but they rarely study proofs until they make sense (to them), they rarely like to discuss mathematics, they rarely derive formulas, etc. Likewise, students with medium mathematical identities (measures between -1 and 1) frequently keep trying, connect new and existing knowledge, and can explain why their solutions are correct, but rarely take the initiative to learn more than expected, rarely take the time to find better methods, etc. Students with high mathematical identities (measures above +1) agree with most characteristics in the instrument. A more thorough discussion is discussed in Kaspersen, Pepin, and Sikko (2017).

Item statistics: Measure order

Measure	INFIT MNSQ	OTFIT MNSQ	Item
1.91	.81	.83	1. Takes time to find better methods
1.58	1.08	.99	2. Takes the initiative to learn more
1.24	.91	.86	3. Thinks of times when methods don't work
.55	1.22	1.20	4. Struggles with putting problems aside
.51	1.05	1.07	5. Derives formulas
.45	1.36	1.37	6. (x) Likes to be told exactly what to do
.41	.96	.95	7. New ideas lead to trains of thoughts
.32	1.05	1.05	8. Likes to discuss math
.20	1.07	1.07	9. Makes his/her own problems
.05	.99	.99	10. Studies proofs until they make sense
.04	.86	.88	11. Moves back and forth between strategies
–.10	.87	.86	12. Tries to understand formulas/algorithms
–.20	.72	.74	13. Considers different possible solutions
–.26	1.03	1.05	14. Pauses and reflects
–.38	1.32	1.31	15. Finding out why methods do not work
–.47	.86	.86	16. Wants to learn more things
–.77	1.20	1.20	17. Visualises problems
–1.19	.71	.76	18. Can explain why solutions are correct
–1.83	.83	.88	19. Connects new and existing knowledge
–2.05	1.02	1.06	20. Keeps trying

Note. Item 6 was negatively coded

Items in their entirety in https://www.researchgate.net/publication/309740755_math_identity_questionnaire

Table 1: Characteristics of mathematical identities amongst Norwegian Engineering students

Moreover, it is evident from Table 1 how the identities in this study were situated amongst the engineering student context. That is, persons with measures, say, around 0.5 in other contexts would be identical to engineering students with the same measures, only if the same set of characteristics were proven to be invariant (i.e., calibrated to have the same structure) in both contexts.

The relationship between self-reported mathematical identities and average grade

Figure 1 illustrates the relationship between self-reported mathematical identity and average grade in university mathematics courses. The Welch's ANOVA showed that the association between mathematical identity and self-reported average grade was significant, $F(2, 110.79)=31.966$, $p=0.000$. Moreover, the mean of the self-reported average grade amongst students with high mathematical identities was about one grade higher than those with low mathematical identities. The Games-Howell test showed that the difference was significant between all groups with low-medium as the least significant ($p=0.001$).

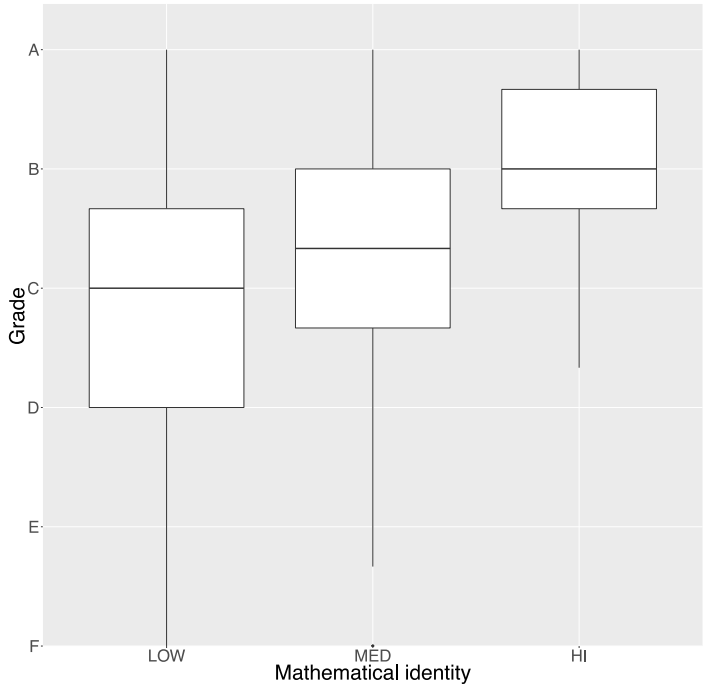

Figure 1: The relationship between self-reported mathematical identity and average grade in university mathematics courses

The unequal variance is also illustrated in Figure 1. Specifically, the variances decreased with the increase of mathematical identity. That is, high mathematical identities are associated with high self-reported average grade. However, there seems to be no limit to how low mathematical identities students can have and still get high grades.

Conclusion and discussion

In this paper, we have argued that the average grade in university mathematics courses amongst students with high mathematical identities is about one grade higher than amongst students with low mathematical identities, and the difference is significant. Moreover, we have shown that the variance of self-reported average grades amongst students with low mathematical identities is higher than amongst students with high mathematical identities. That is, students with high identities get, for the most, high grades. However, the grades of students with lower identities are more uncertain.

We have in this study examined the association, and not the causal relationship, between self-reported mathematical identities and average grades, and therefore we argue that the significance of the result is that it points the direction for future research. Specifically, we suggest future research to address the following:

First, replicates of this study should seek more precise measures. That is, the precisions of the mathematical identity measures can be improved by including more response categories (as long as they are sufficiently validated) and more items, particularly near the "gaps" (e.g., between 0.5 and 1.2 logits). Moreover, the precision of the average grade would most likely be improved if self-reported average grades were substituted with actual average grades.

Second, future research should seek a more causal relationship between identities and grades. Specifically, this study does not conclude that an increase in mathematical identity infers an increase in attainment.

Third, future research could study the significance of mathematical identity versus the significance of attainment. For instance, students can be categorised as having "low identities and low grades," "low identities and high grades," or "high identities and high grades," and subsequently studied with respect to other variables, for example, in the transition from university to the world of work.

Fourth, we argue that future research can transfer the design of this study to other samples and forms of testing students' attainment. For example, relationships between mathematical identity and measures on international standardised tests, such as PISA and TIMSS, can be tested. Accordingly, we argue that future research can nuance the debate on the significance of these tests. If some districts/countries are "teaching to the test," then one might hypothesise that a relatively great proportion of students in these districts/countries are in the "top left corner"–that is, students with low mathematical identities, yet, high measures of attainment.

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A praxeological approach to Klein's plan B: Cross-cutting from calculus to Fourier analysis

Margo Kondratieva[1] and Carl Winsløw[2]

[1]Memorial U. of Newfoundland, Canada; mkondra@mun.ca

[2]University of Copenhagen, Denmark; winslow@ind.ku.dk

In our previous work on Calculus–Analysis transition we independently explored the reasons of students' difficulties with studying analysis and observed that the problem is related to the discontinuity of students' experiences leading to their inability to interpret the (formal and more rigorous) ideas learned in analysis courses in terms of (practical) knowledge acquired in calculus courses, and vice versa. In this paper we continue and combine our work with two new contributions: a theoretical formulation of Klein's idea of a "Plan B" for teaching mathematics, applied to the transition in question; and a concrete student activity attempting to give flesh to this "plan" for the special case of introductory Fourier Analysis.

Keywords: Calculus, Fourier analysis, transition, praxeology

Introduction

Calculus and Analysis appear as related, but distinct subdisciplines in many contemporary university programmes. Calculus courses specialise in mathematical themes indicated by course titles such as "Integral Calculus", "Functions of several variables" or "Ordinary differential equations". Analysis courses, on the other hand, treat theoretical perspectives on these same mathematical themes, gradually moving from course titles such as "Real Analysis", "Fourier Analysis" towards more abstract areas such as functional and harmonic analysis. In short, calculus courses can be roughly characterized as teaching students certain calculation practices related to real and vector valued functions, with little theoretical precision or justification – while analysis courses tend to present "formal theory with little practice". This division is a didactical construct which is related to historical and institutional conditions (see Klisinska, 2009, for an in-depth analysis of the case of the "fundamental theorem of calculus").

The main reason for the division seems to be that the two types of courses cater to different student populations. While calculus courses are studied by a large cohort of students in natural and social sciences, much fewer students get to study analysis (mainly students of pure mathematics, theoretical physics and mathematical statistics). For these and other reasons, it may be difficult to change the course structure.

The transition from Calculus to Analysis presents mathematics students with several challenges (for examples, see Winsløw & Grønbæk, 2014). Here is a typical student formulation of some of these (interview with a student of the first author, summer 2016):

> In calculus courses we learn methods, but usually the why questions are not explained or proved. (...) However, analysis courses felt as separate. They were more theoretical than applied. I never grasped them as well as Calculus. It was often unclear, what it was leading to. I wish we had a better sense of connection between the theory we covered in pure math courses and the methods shown in applied math courses.

We have explored this perceived lack of "connection" in earlier papers (Kondratieva, 2011, 2015; Winsløw, 2007, 2016). In the present paper, we use the notion of praxeology (Chevallard, 2006) to represent the general "connection" problem in more precise terms, and - as a theoretical case study - to present a new proposal for "connecting" Calculus and one of the basic theorems in Fourier Analysis. Our research results are thus basically theoretical.

Theoretical framework

Chevallard (2006) defines a *praxeology* as a pair (P,L) consisting of a *praxis block P* and a *logos block L*. A praxeology is a minimal element of human knowledge, *P* representing the practical part - the "know how" - and *L* the intellectual part, the "thinking and explaining" – often called "know why". The two are interdependent:

> …no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast. Praxis thus entails logos, which in turn backs up praxis. For praxis needs support – just because, in the long run, no human doing goes unquestioned. (Chevallard, 2006, p. 23).

As we focus here on mathematical praxeologies taught and learnt at university, it is obvious that *praxis* (e.g. computing the Fourier series of a given function) is intimately connected to various forms of *logos* - from *ad hoc* explanations of standard techniques to theories involving general definitions, theorems and proofs. To compare and contrast the praxeologies developed in calculus and analysis courses, we consider that they represent various affinities with the praxeologies of present-day mathematicians, which we shall represent suggestively using Greek letters (Π,Λ). We can thus, as a first naïve model, propose that praxeologies taught and learnt in calculus courses are of the form (Π_i,L_i): the *praxis blocks*, including computational techniques, are identical to those used (for tasks of the same type) by professional mathematicians, while the logos blocks L_i are limited to informal explanations of smaller collection of practice blocks (like the various techniques for determining whether a series is convergent or not). On the other hand, analysis courses then focus on the scientific form of logos blocks. The taught and learnt praxeologies in such courses are therefore of the form (P_i,Λ_i) where each Λ_i constitutes a logos block consistent with the scientific model, while the praxis blocks P_i are didactic "afterthoughts" constructed to consolidate the acquisition of Λ_i. As mentioned in the introduction – such teaching practices often fail to motivate students for Λ_i and to provide them with a coherent, autonomous relationship with (Π_i,Λ_i). Our research focuses on how this issue can be addressed.

Taken together, calculus and analysis courses in principle provide students with praxeologies (Π_i,Λ_i) which, taken individually, are close to the scientific model. For instance, convergence tests used in Calculus praxis on series are now supplied with a theory involving precise definitions and proofs of the "criteria" for convergence. However, because the number and technical complexity of these praxeologies is quite high and the Π_i were taught in other courses, typically years before, some effort and support may still be needed for students to "assemble" individual praxeologies (Π_i,Λ_i). We can say that working along these lines corresponds to establishing complete but separate praxeologies within different small areas of mathematics, which is what Klein called "Plan A" for teaching: "Plan A is based upon more particularistic conception of science which divides the

total field into a series of mutually separated parts and attempts to develop each part by itself." (Klein 1908/1932, p. 77, see also Winsløw, 2016) Within this approach two praxeologies are related only through strict logical dependency at the theoretical level and only within strictly confined areas (which, in terms of what students actually acquire, may be surprisingly small).

However as explained by Klein, the scientific practice (historically as well as currently) involves more than isolated or strictly dependent praxeologies. Klein (1908/1932, p. 78) recommended that also elements of "Plan B" be included in mathematics teaching both in schools and at university:

> The supporter of Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon another. He prefers, therefore, the methods which open for him an understanding of several fields under a uniform point of view.

In terms of the praxeological model above, we may thus summarize the two "plans" or strategies for developing and connecting students' previous knowledge as follows:

Plan A. *assemble elementary praxeologies* (Π_i, Λ_i) from calculus and analysis elements, by establishing firm relations of type $\Pi_i \leftrightarrow \Lambda_i$. In fact, this is sometimes a possible function of the "fingertip" exercises, which constitute P_i in many courses and textbooks on analysis.

Plan B. *develop cross-cutting relationships among praxeologies* which could be of one of the following types (or combinations among them):

B1. Relating praxis blocks ($\Pi_i \leftrightarrow \Pi_j$) or logos blocks ($\Lambda_i \leftrightarrow \Lambda_j$)

B2. Relating otherwise unrelated praxis and logos blocks ($\Pi_i \leftrightarrow \Lambda_j$)

It may be more easy and common to develop relations of type B1, even if they certainly appear more often in "mathematician" praxeologies than in typical course teaching. We now present and analyse an example of student activity aiming at developing relations of the last type (B2): namely, that students connect a collection of praxis blocks Π_i (concerning trigonometry, integration and convergence) to a logos block (L_0) from Fourier Analysis.

A logos block from Fourier Analysis

For a 2π-periodic, piecewise continuous function $f: \mathbb{R} \to \mathbb{C}$, the Fourier series of f is defined as

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{, where } a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \text{ and } b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

In general the two infinite series may not converge at a point x. In 1829, Dirichlet gave one of the first sufficient conditions for pointwise convergence of a Fourier series. A version of this result which is usually formulated for *piecewise* continuous functions, is stated below in a special case to avoid technicalities. We refer to it as Dirichlet's theorem, although we don't use his original claim.

Theorem If $f: \mathbb{R} \to \mathbb{C}$ is a continuous 2π-periodic function with piecewise continuous derivative, the Fourier series of f is pointwise convergent to $f(x)$ at every $x \in \mathbb{R}$.

Here we outline the main steps of the proof that appears in a typical formal course on Fourier Analysis (see e.g. Folland, 1992, pp. 30–36 for the wealth of computational details omitted here):

1. First, it is shown that under weaker assumptions, such as *f* being square integrable and 2π-periodic, the coefficients a_n and b_n tend to zero as *n* tends to infinity. (In fact, one demonstrates this by showing that the series $\sum a_n^2$ and $\sum b_n^2$ are both convergent.)

2. Next, by direct computation, we rewrite the *N*th partial sum given by $s_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos nx + \sum_{n=1}^{N} b_n \sin nx$ as: $s_N(x) = \int_{-\pi}^{\pi} f(x+y)K_N(y)\,dy$, where K_N is the *Dirichlet kernel* defined by $K_N(x) = \frac{1}{\pi}(\frac{1}{2} + \sum_{n=1}^{N} \cos nx)$. Clearly $K_N(0) = (N+1/2)/\pi$ and $\int_{-\pi}^{\pi} K_N(x)\,dx = 1$. By clever use of the addition formulae, $K_N(x) = \dfrac{\sin(Nx + x/2)}{2\pi \sin(x/2)}$ for $x \neq 0$.

3. Finally, let $\Delta_N(x) = s_N(x) - f(x)$. Using 2., a straightforward set of computations yields

$$(*) \quad \Delta_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)\sin(y/2)\cos Ny\,dy + \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)\cos(y/2)\sin Ny\,dy,$$

   where g_x is defied by $g_x(y) = \dfrac{f(x+y) - f(x)}{2\sin(y/2)}$ for $y \neq 0$, and $g_x(0) = f'(x)$. In fact, $g_x(y)$ is a continuous for $y \neq 0$ and 2π-periodic function. At $y = 0$ $g_x(y)$ may have a jump discontinuity if $f'(x+0) \neq f'(x-0)$. The functions $g_x(y)\sin(y/2)$ and $g_x(y)\cos(y/2)$, notwithstanding the possible discontinuity of the latter at $y = 0$, are bounded, and thus, square integrable. Formula (*) shows that $\Delta_N(x)$ is simply the sum of *N*th Fourier coefficients of these two functions. Applying now 1., we conclude that the infinite Fourier series $s_\infty(x)$ converges to $f(x)$ because its partial sum $s_N(x)$ can be written as $s_N(x) = f(x) + \Delta_N(x)$, where $\Delta_N(x)$ vanishes as $N \to \infty$.

The key point of the proof is (*): to rewrite $\Delta_N(x)$ as a sum of two Fourier coefficients, together with the fact that the coefficients tend to zero as $N \to \infty$. According to the distinction we made above, the general result (and certainly its proof) does not belong to the realm of Calculus. When students are presented with the theory in a somewhat more general form, - they may not realize that the proof almost entirely draws on the notion of series convergence and on techniques known from Calculus. To make them discover that is the aim of the design that we present in the next section, focusing on the following special case:

Example. Applying the above Theorem for $f(x) = x^2$, extended periodically from $[-\pi, \pi]$ to $\mathbb{R}$, we get that the Fourier series converges to 0 at $x = 0$. Computing the Fourier coefficients, this gives

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ so that } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

The latter - striking - result can be derived by other means, as a variant of the famous Basel problem (see e.g. Kondratieva, 2016). One such approach is at the root of the design presented below.

Outline and *a priori* analysis of Student Activity

In continuation of earlier work of the first author (Kondratieva, 2016), we took the Example above as a point of departure for constructing a sequence of exercise-like activities that would lead

students through two approaches to computing the infinite sum considered in the Example: *part 1* consisting of a series of "calculus-like assignments" which, without saying so, go through the proof of Dirichlet's theorem in the special case where $f(x)=x^2$; *part 2* in which the students work directly with the result, as in the Example; and a *final reflection* in which the students are supposed to realize that the proof (known from a Fourier Analysis logos block L_0) amounts to nothing more than a generalization of the sequence of calculus techniques drawn upon in part 1. We notice here that the numbering suggests that the praxis and logos blocks thus connected through the activity are not, *prima facie*, connected - and, thus, the connection established is really of type B.

Part 1 begins with presenting the problem of determining the value of $S=\sum(-1)^{n+1}/n^2$. The praxis blocks acquired in calculus courses do not provide ready-made techniques to solve this problem; instead, students are invited to do so through "several preliminary problems":

1. Compute the integral $\int_{-\pi}^{\pi} x^2\cos mx\,dx$ for any natural number m (Π_1: integration rules).
2. Show that $\frac{1}{2}+\sum_{n=1}^{m}\cos nx=\frac{\sin(m+1/2)x}{2\sin(x/2)}$ (Π_2: trigonometric formulae).
3. Show that $u(x)=\begin{cases}x^2/\sin x, & x\neq 0\\ 0 & x=0\end{cases}$ defines a continuous function on $[0,\pi/2]$

   (Π_3: techniques to compute limits, including the special result $\lim_{x\to 0}x^{-1}\sin x=1$).
4. Find u' and show this function is bounded on $[0,\pi/2]$

   (Π_3, and Π_4: differentiation from first principles).
5. Show $\int_{-\pi}^{\pi}x^2\frac{\sin(m+1/2)x}{2\sin(x/2)}dx=8\int_0^{\pi/2}y^2\frac{\sin(2m+1)y}{\sin y}dy=8\int_0^{\pi/2}\frac{\cos(2m+1)y}{2m+1}u'(y)dy$ (Π_1).
6. Show that the integral in 5 converges to 0 as $m\to\infty$ (Π_1 and Π_3).
7. Finally, combine the results above to find *S* (Π_1).

The only slightly advanced praxis (technique) involved in the above appears in 6., where Π_1 is supposed to include something like $\left|\int f\right|\le\int|f|$ – or, alternatively, Π_3 should include a rule which permits to conclude that $\lim_{m\to\infty}\int_I f_m=0$ under appropriate conditions on (f_m).

Part 2 of the activity invites the students to compute the Fourier series of $f(x)=x^2$ and engage in some concrete computations related to its convergence which are in fact very similar to 1.-6. above.

The final reflection is supposed to make them discover the close parallel between the two parts.

In case they do recall elements of the proof of Dirichlet's theorem, they will recognize in Problem 2. the simplification technique of the Dirichlet's kernel in step 2., and in Problems 3.-6. – a way to get the convergence result of step 3. Meanwhile, step 1 appears more indirectly in the concrete case, where both the Fourier coefficients of *f* and the auxiliary coefficients, appearing in (*), can be computed or estimated directly. Indeed, many textbooks present Step 1 as a corollary of a more

general theorem on orthogonal sets in Hilbert space. This, together with the technicalities related to the possible non-continuity of *f*, contributes to the impression that the proof is way beyond simple techniques from Calculus. Nevertheless, comparing the proof with the proposed activity, the students could realize that in the special case $f(x) = x^2$the proof relies entirely on well-known praxis blocks (Π_i, i = 1,2,3,4). Certainly, this could establish a strong relation $L_0 \leftrightarrow (\Pi_i, i = 1–4)$ which might in fact be prepared by students' working *Part 1* above *prior* to encountering L_0.

Some experimental observations

To pilot and refine the above design before testing it with a larger group of students in a course on Fourier Analysis, we have done a preliminary study with five students who have completed at least 3 years of undergraduate mathematics program. These students were involved in summer research projects in mathematics at the Memorial University of Newfoundland. This involvement is an indicator of the students' high motivation and achievements in studying mathematics. The students volunteered to solve the problems from the activity (with no firm restrictions in time or access to any materials) and participate in a follow-up semi-structured interview. The students were asked whether they found the problems (a) familiar, (b) interesting, (c) easy/accessible; and whether they saw any connections between praxis and logos of parts 1 and 2. All students regarded problems 1-6 as familiar from their calculus courses, and they found them easy. In words of one student, "I loved that stuff when I was in my calculus courses, so I found these problems pleasant… And they are not difficult, too." While problems 1-6 were familiar to the students, they clearly indicated that no projects of nature similar to problem 7 were present in their study: "I think it is a cool layout. Nothing of this format was in my calculus courses, – when you need to use previous results to solve larger or more interesting problem." Students regarded the task of series evaluation as challenging but also most enjoyable: "The problems 1-6 were like baby steps… And they met together nicely in problem 7". So, at least these students were successful and appreciative of tasks in part 1. As for the accessibility of part 1 for an average student in a calculus course, we had overall a confirmative response: "I think it is accessible for a student who has done Integral Calculus.... if they are not confined to a very short period of time, then yes." Another student confirmed, "it could be a good exam sequence, more fun than just doing problems." However, a different perspective was also articulated: "…many students take this [Integral Calculus] course because it is a prerequisite for their programs, so maybe they would not be interested as much."

Among the five students only one had studied Fourier series in his courses, while others had heard the term but had very little familiarity with the subject. However, they all recognized the similarity in the technical praxis of parts 1 and 2, for example, that calculation of the Fourier series in part 2 resembles evaluation of integrals in problem 1 from part 1. Bridging the theory and connecting the idea of convergence of an individual series in part 1 and pointwise convergence of Fourier series was more challenging. This is where the role of an instructor might be critical: to help students to relate new theoretical constructs and ideas to familiar praxis. We realize that students' background makes a difference, however even learners previously unfamiliar with Fourier series seem to benefit from this activity. Students' responses based not on reproduction of known facts, but rather on reasoning related to their practical experiences, is an indication of establishing new mathematical relations. The following is an excerpt from an interview with students of the first author:

M.K.: Is it always possible to replace a function with its Fourier series in calculations?

Student 1: In my (applied) courses we were told that no (a function is not always equal to its Fourier series), but this was never proved. Now it kind of makes more sense.

M.K.: Do you think that familiarity with part 1 would help to exemplify general theory related to Fourier series and their convergence?

Student 1: Yes, definitely. I think it is more logical to go this way about discussing conditions of pointwise convergence of Fourier series. However, the experiences need to be close together in time, so that the second part occurs before students have forgotten the first portion.

The space available does not allow us to give the details of students' accomplishments and their impact on our design. We simply note that the sample students were by and large able to complete them and see the inner connections. Also, the students considered that building on the familiar computational tasks (1-6) on the one hand, and on new theoretical constructs (Example) on the other hand, organized around given problem (evaluation of the series S) was stimulating: "Suppose someone has a theoretical solution and I have a computational solution and they look completely different, but they give the same answer to the same problem so they have to be the same somehow… then I want to go back and find out why they are the same. I found it very interesting."

Conclusions

While calculus courses include praxis blocks Π_i compatible with those of professional mathematicians, their theoretical components are more informal and focused on algebraic computation. Moreover, these praxis blocks are often isolated from each other, as they occur within separate sections of textbooks and courses, and students typically don't get opportunities to apply them in combinations. When students meet Dirichlet's theorem, they are given a general and relatively complicated proof (in Analysis). In such courses, "simple applications" (such as the Example above) may be introduced as examples or exercises, to build an artificial practice block P_0 corresponding to the much richer logos block L_0. The fact that the general proof (L_0) is essentially linked to familiar praxis blocks from Calculus will then not appear. We propose that by replacing P_0 by a sequence of computational auxiliary tasks (1-7), similar to the steps 2 and 3 of the proof (L_0), two goals can be achieved. First, students will see how different praxis blocks $(\Pi_1,...,\Pi_4)$ from Calculus work together and combine to support P_0 by themselves. Secondly, this special case might help to prepare for the various general theorems on Fourier series convergence (L_0 and beyond) by relating it to the concrete and familiar elements $\mathsf{P}_1,...,\mathsf{P}_4$. This hypothesis will be investigated empirically. More generally, we hypothesize that situations which enable students to establish "cross cutting relations" $\Pi_i \leftrightarrow \Lambda_j$ are precise and possibly partial interpretations of Klein's Plan B. At the same time, constructing integrated praxis blocks such as $(\Pi_1,...,\Pi_4)$ above constitutes an essential complement to "Plan A" type courses. These constructions could emerge from detecting explicit links between different solutions of *interconnecting* problems (Kondratieva, 2011), as shown above. It will clearly necessitate a careful analysis of (central) theory blocks of more advanced courses, and resources found in reasonably well-established praxis blocks of previous

courses. So, while the general hypothesis may look fairly simply, realizing it in concrete cases - even theoretically - represents a non-trivial didactical research programme.

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The interface between mathematics and engineering – problem solving processes for an exercise on oscillating circuits using ordinary differential equations

Jörg Kortemeyer and Rolf Biehler

University of Paderborn, Germany; kortemey@math.upb.de, biehler@math.upb.de

In our project, we investigate the mathematical skills that are required in first-year courses of technical subjects of engineering bachelor courses, i.e., we do not look at the courses on pure math. We analyze four exercises from an exam in electrical engineering, which is compulsory for first-years. To solve such exercises, students have to combine their knowledge in electrical engineering with their skills in mathematics. We introduce a theoretical approach consisting of three elements: a normative solution called "student-expert-solution, "low-inferent analyses" for qualitative studies with students, and categorizations of written solutions. We describe the newly developed tools and details on one of the exercises and present the results from the analysis of the exercise in reference to the three concepts mentioned above. This provides insight into the interface between mathematics and engineering courses in the first year at university.

Keywords: Engineering mathematics, competence, differential equations.

Introduction

Engineering students at German universities are taught mathematical subjects as well as engineering subjects, which require some understanding of mathematical topics, at the same time. This leads to several challenges for the students: To begin with, the lectures on Mathematics for Engineering Students (MfES) and the Fundamentals of Engineering (FoE) are very often asynchronous. On the one hand, there is a deductive structure in the lectures on mathematics, which leads to a certain order of presentation of the different topics to assure understanding. On the other hand, there is also a standard way of presentation of the different engineering topics in the FoE-courses and because of that, mathematical topics are often needed earlier in the FoE-courses than they are presented in the MfES-courses. Moreover, there are different mathematical practices in MfES and FoE, for example in the use of vectors or differentials (see e. g. Alpers, 2015). There is a mismatch between the mathematics in MfES-courses, the mathematics at school level, and the "contextual mathematics" required in engineering tasks (see e. g. Redish, 2005).

At the beginning of the research, our central question was: how do engineering students solve tasks in basic engineering courses given this situation with two interconnected fields of competences - mathematics and engineering. We are interested in the modelling and assessing of explicit as well as implicit competences required and developed by students in this field. We investigate how students actually solve exercises in FoE-courses and which difficulties occur. Our focus is on four typical exercises of a FoE-exam after the first year at university. In this paper, we present a case study of these issues in the context of a single exercise on ordinary differential equations in the electrical engineering field of oscillating circuits. Our focus here is on the following research questions:

1. What are the expectations from students' solutions to an exercise on ordinary differential equations in a first-year electrical engineering course? 2. What are the characteristics of students' problem-solving processes (e. g. strategies, difficulties) in electrical engineering courses?

For the analysis of students' work we required a normative solution of the exercise, which was developed with engineering experts and which considers both fields of competences. This normative solution is based on relevant theoretical concepts that are presented in the next section.

Theoretical background

In this section, we present the theoretical tools that were used to develop the newly constructed methodology for our investigations. As a first step, the approaches deal with modelling processes using mathematical methods and mathematical problem solving. Both theoretical approaches are relevant, as they combine inner- and outer-mathematical solution parts and describe their connections. Next, we also consider actual solving processes that help us to supplement normative solutions with the steps students use when solving an exercise.

The first approach is the modelling cycle by Blum and Leiss (2007), which is used to describe idealized modelling processes of real world problems that can be solved using mathematics. In a broad outline, it divides the modelling processes into two distinct parts, the so-called "rest of the world" and "mathematics". Our second perspective is mathematical problem solving by Polya (1949), who intended to give advice to students on how to solve mathematical problems as well as applied problems referring to mathematics. He divides the solving processes into four phases: understanding the problem, devising a plan, carrying out the plan, and looking back.

For the analysis of actual solution processes of students, we use theoretical approaches developed by Redish and his working group, i.e., by Redish and Tuminaro (2007) and Redish and Bing (2008), in addition to the normative solution. Their approaches discuss the role of mathematical resources and knowledge in solving processes by pairs of physics students. Redish and Tuminaro (2007) distinguish three framings in qualitative solving processes: quantitative sense-making, qualitative sense-making, and rote equation chasing (without understanding the underlying physical situation). Bing (2008) looked at mathematical justification strategies and found four distinct types of justifications: calculation (a correctly done algorithm gives a correct result), physical mapping (the physical behavior is described correctly by mathematical results), invoking authority (the result is consistent with the lecture) and math consistency (the same mathematical approach is used in a similar situation). The theoretical background is presented in more detail in Biehler, Kortemeyer, and Schaper (2015).

The newly developed methodology and its aims

In order to do research in this interface of two interconnected competence fields, new theoretical approaches had to be developed. This section presents the three main approaches that were developed on the basis of the theoretical approaches mentioned above. As shown in Figure 1, the central theoretical tool is the SES, which builds on expert interviews and the theoretical frameworks of the modelling cycle and mathematical problem solving. The SES is our tool to answer the first research question, i.e., it gives idealized solution processes which we can expect from first-years. It is used to analyze and structure the video-graphed solving processes, which were transcribed using

LIAs, and the categorizations of written solutions. Details on the SES, the LIAs and the categorizations are presented in Figure 1:

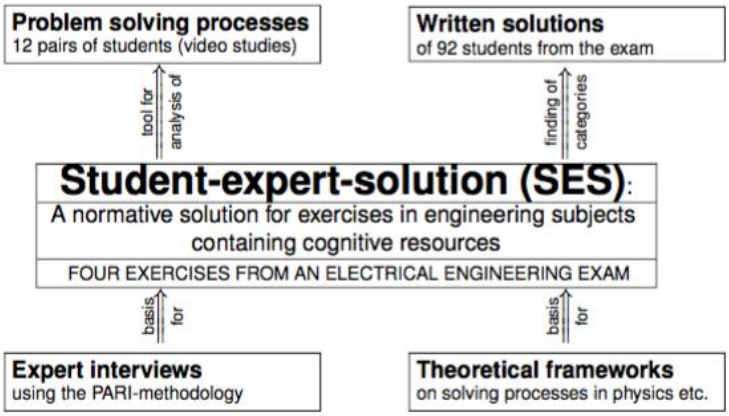

Figure 1: Diagram on the connection of the different elements of our analyses

Initially we asked the task designer and the electrical engineering experts to solve the exercises from the perspective of students who understood the contents of the courses in the first year of studies well. Afterwards we interviewed them concerning their solution processes. The aim of the expert interviews was to identify the explicit and implicit competence expectations of instructors in electrical engineering courses. We conducted the interviews using the Precursor-Action-Result-Interpretation (PARI) method by Hall, Gott, and Pokorny (1995) which is a task-based interview technique. This solution was then subdivided using the language of the modelling cycle and mathematical problem solving, which in combination structured the normative solution of the exercise. The solving processes could be divided into three main phases: mathematization, math-engineering working, and validation and each main phase was subdivided by Polya's four phases. The expert interviews and the structure shown in Figure 1 were the basis of the student-expert-solution (SES), which was used to finally sharpen the theoretical description, and as a basis for the further analysis of students' work. SESs are represented by two columns: the first column provides a normative solution to the exercise in detail and is structured as mentioned above. The second column contains a division of the problem-solving process into phases, as well as remarks given by the experts on expected mistakes, alternative solutions, and learning goals for the different phases.

One of our main interests is to describe real problem-solving processes of students for the four exercises using both qualitative and quantitative methods. Those analyses are based on the SES. We conducted video studies of problem solving processes of three to four pairs of students per exercise, who were asked to solve the exercise and to think aloud during the solution processes. The videos were transcribed with additional remarks on the activities (especially gestures) performed. We analyzed the transcripts using our concept of the low-inference analyses (LIAs) with the aim of finding differences to ideal solutions (the SESs) and to identify students' difficulties. The LIAs consist of four parts: First, there is the connection of the phases in the SES and the phases in the problem-solving processes of the students. Second, the differences between the idealized solution parts in the SES and the actual solution paths of the students are described. The third part consists of commenting and interpreting of the differences, which forms the basis to conceptualize and describe problem-solving strategies, which are expected to be more general than just the process in the actual

exercise alone. Finally, we connect the strategies we found with the strategies described by Redish and Tuminaro (2007) in general, and Bing (2008) in particular, in order to find typical strategies and challenges at the interface between math and engineering.

In addition to these qualitative studies, we also scanned 92 anonymized "solutions" of students from their written exams. In order to analyze the solutions, the phases in the SES were subdivided into the particular activities that students have to accomplish in order to solve an exercise. For example, the phase of the math-engineering work was subdivided into the forming and the evaluation of the formula. Each student's work was categorized using a partial credit system, i.e., they got two points if the activity was done correctly, or they got one point if it contains right parts (e. g. the solution would be correct if one multiplied it with a power of 10), or they got no points if the solution is completely wrong. The categorization "1" was subdivided into 1a, 1b, 1c etc. to distinguish different forms of mistakes. This provided quantitative results on the frequency of mistakes and – by combination of activities in contingency tables – the connection of successes in different activities. The results are used to confirm, refine, and enhance the results in the first two levels.

The SES for the analyzed exercise on ordinary differential equations

This section presents the first part of one of the exercises of the exam to exemplify our method and present exemplary results. It answers the first research question, which asks what we can expect from students in their first year at university. The exercise deals with oscillating circuits and transients. We present the problem setting as well as a short overview of the solution. This solution is enhanced by the remarks of the experts, which were elicited in the third phase of the PARI-interview. For the first time, this paper presents our total approach for an exercise using methods from MfES. In this exercise, the oscillating circuit contains a resistor R, an inductor L, a capacitor C and an ideal voltage source U_0. In summary, the students have to read the sketch – taking into account conditions on the switches S_1 and S_2 - to be able to form an ordinary differential equation (ODE) and then to solve it. The exercise starts with the circuit diagram shown in Figure 2. It consists of eight subtasks. In this paper, we concentrate on the first five subtasks, which deal with the left part of Figure 2.

Initially, both switches are open, and the inductor and the capacitor are totally discharged. At the moment $t=0$ the switch S_1 is closed, while S_2 remains open. In subtask 1 and 2 the students are to give the values of $u_C(t)$, the voltage at the capacitor, $i_C(t)$, the electric current in the capacitor, and $i_L(t)$, the voltage at the inductor, before and after opening S_1, i.e., before and after $t=0$. Solution: All three values are 0 before S_1 is closed, because the components of the circuit are initially assumed to be discharged. After closing switch S_1, $u_C(t)$ and $i_L(t)$ are still 0, as a voltage across a capacitor or an electric current through an inductor does not change discontinuously; a fact students learned in the lecture. $i_C(t)=U_0/R$ directly after the switching of S_1 due to Ohm's law and then declines due to the charging of the capacitor.

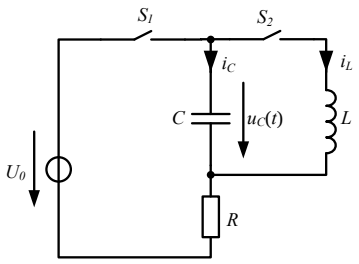

Figure 2: Sketch of an oscillating circuit containing the mentioned components

In subtask 3 the students are to form an ordinary differential equation for $u_C(t)$. Solution: We have to apply Kirchhoff's voltage law on the left part, giving $U_0=u_C(t)+u_R(t)$, and use the two component equations of the capacitor $Cu_C'(t)=i_C(t)$ and the resistor $u_R(t)=i_C(t)R$. The combination of those equations gives an ordinary differential equation (ODE) of first order, which is $u_C(t)+ RCu_C'(t)=U_0$.

The ODE is to be solved in subtask 4. Solution: The solution can be done using either the separation of variables combined with a variation of constants, or alternatively, the solution can be found by superposition of the solution of the homogenized ODE, one particular solution of the inhomogeneous ODE and the using of the initial value $u_C(0)=0$. The solution is $u_C(t)=U_0(1-e^{t/(RC)})$.

In subtask 5 the students are to sketch the voltage curve of $u_C(t)$. Solution: The graph of $u_C(t)$ starting at $u_C(t=0)=0$ approaches an asymptote at $u_C(t)=U_0$, because $e^{t/(RC)}$ converges to 0 for $t \rightarrow \infty$.

Developing the SES for this exercise

At first sights, the solving process can be divided into three phases: mathematization (the given circuit diagram, subtask 1 to 3), math-engineering working (subtask 4), and validation, which is partly done in subtask 5, at least, if the students know the physical behavior of the setting.

As stated in Biehler et al. (2015), there are differences to the modelling cycle in mathematization processes in exercises in basic courses of electrical engineering. Students do not construct a real model from a real situation– as suggested in the modelling cycle – but they need to have strategies to reconstruct the underlying real model - which was implicitly taught in the course (but usually not called model). This includes understanding conventionalized circuit diagrams. In contrast to the exercise on magnetic circuits that was presented in Biehler et al. (2015), the equivalent circuit diagram does not have to be produced by the students; they can use the given diagram directly for their mathematization. In both cases, students use implicit idealizations and are not necessarily aware that they are idealizations. The students have to "read" the diagram and recall and use its physical background in the first two subtasks. The didactic motive of the first subtasks is – as stated by the task designer – to remind students of applying Ohm's law. Then in subtask 3, the mathematization consists of two independent competences: either recognizing certain components and translating them into their equations, or alternatively the translation of the experiment set-up into mesh and node equations using graph-theoretical arguments in an application of Kirchhoff's laws. The result are three equations: $U_0=u_C(t)+u_R(t)$, $Cu_C'(t)=i_C(t)$, and $u_R(t)=i_C(t)R$.

In the next step, there are similarities to the modelling cycle, except that physical quantities are used instead of numbers. The left part can be mathematized by the three equations mentioned and using them, an entering of the "world of mathematics of physical quantities" is possible. Students have to

do equation management (see Biehler et al., 2015) to combine the equations in order to get a formula, which also contains one unknown quantity (given by a function in this case), while all the quantities are given in the exercise or have already been calculated. The equation management includes equations with functions as objects and leads to an inhomogeneous ODE of order one. A further characteristic of the equation management is that, unlike in the solving of systems of linear equations, there are no methods to find out whether there are enough or too many equations to get a solvable ODE. Asked for typical mistakes the experts said that the students have some problems in applying mathematical methods to solve the ODE. He also said, that for some students, the application of Kirchhoff's laws is hard, as they do not obtain all the required equations.

Students have learned two different algorithms to solve such ODEs. In the MfES-courses, they solve the homogenous ODEs by separation of variables and – using the solution of the homogenous ODE – they subsequently solve the inhomogeneous differential equation. In the FoE-course, they retain a solution by using superposition of the homogenous and the inhomogeneous solution. In the interview, the expert said that most students are able to set up the differential equation, the following solving of the ODE, however, is quite difficult for many students, especially finding the inhomogeneous solution. As the students work with functions instead of numbers or quantities, there is no difference in the use of the solving algorithm for ODEs, which was presented in the MfES-course. So, in this case, the solving process can be divided analogously to the modelling cycle, i.e., there is a "real" world (given by a conventionalized sketch), its translation using three equations and the solving in the mathematical world with quantities.

The solution of the inhomogeneous ODE, $u_C(t)=U_0(1-e^{t/(RC)})$, describes the behavior of the voltage in the capacitor in such a setting. Students know the qualitative behavior of this function from lab courses, which are obligatory in the first year at university. The didactic motive of the task designer was to make students see the connection between their solution of the ODE and the physical mechanisms they know from the lab courses, and use this as a validation strategy. Possible variations of exercises on this topic, which were suggested by the experts, can be either done by using further components (as in the right part of the sketch, which leads to a second order ODE) or by changing the setting of the circuit from a series connection to a parallel connection.

Analyzing the actual solution processes of the students

Selected results of the analyses of the videos in the LIAs. Three pairs of students worked on this exercise in our video studies. Each pair directly found the component equations using the concepts and the language of graph theory for applying Kirchhoff's laws was a bigger problem for two pairs: They were not sure whether one mesh equation would be enough to mathematize the setting, or if they also needed to have node equations, as there was a node between the two parts of the oscillating circuit. However, no pair started the equation management with an incorrect equation and they were also successful in combining them. In reference to solving the ODE, all three pairs used the superposition-method, i.e., they used the method presented in the FoE-course.

In subtask 5 the three pairs acted in different ways, which they described while thinking aloud. One pair found the solution of the ODE by inserting $t=0$ and realizing that the function converges to U_0. Another pair remembered the behavior they had seen in the lab courses, i.e., they knew that the graph should start at $u_C(0)=0$ (also known from subtask 1 and 2) and would converge to the value of

the ideal voltage source, so they applied their physical knowledge to get a mathematical representation of the result, i.e., they used the "mapping meaning to mathematics"-game (see Redish & Tuminaro, 2007). The third pair used both arguments, i.e., they drew the solution of the ODE and validated it with the physical behavior, saying it confirms the result of the ODE.

Some results of the analyses of the written exams. There is a connection between finding the component equations and the applying Kirchhoff's law: 84 of 92 students either did both types of equations right or both wrong. Here, 77 students were able to find the correct ODE; 56 of them solved the homogenous ODE correctly, i.e., for about 73% of the students solving the rightly formed ODE was no problem. Table 1 shows that all students who were able to solve the homogenous ODE could also solve the inhomogeneous ODE. Eight students only solved the inhomogeneous ODE correctly by finding one particular solution using physical arguments, i.e., they were able to solve at least one part of the task without applying any mathematical methods to solve ODEs, by looking instead at certain values of $u_C(t)$ that were known from the problem setting.

	Inhomogeneous solution: wrong resp. partly right	Inhomogeneous solution: right	Total
Homogenous solution: wrong	28	8	36
Homogenous solution: right	0	56	56
Total	28	64	92

Table 1: Connection between homogenous and inhomogeneous solutions

Summary and discussion of results

The solution processes of first-years (Research Question 1). The SES shows that this exercise has more similarities to the modelling cycle than the one presented in Biehler et al. (2015). Although the exercise uses quantities rather than numbers, it contains three distinct parts, which are analogous to the modelling cycle: mathematization, math-engineering and validation. The mathematization part consists of finding equations for the involved components and the experiment set-up by applying Kirchhoff's laws. In math-engineering these equations are combined in a purely mathematical way, they are solved using inner-mathematical algorithms. The validation part is attended to by a retranslation into the so-called real world by looking at the physical behavior.

The analysis of students' work (Research Question 2). In the mathematization part, most students were able to find both kinds of equations, and in the video-studies the biggest hurdle was, whether they had the right number of equations to get a solvable ODE. The component equations were cited from the FoE-lecture, i.e., the students invoked authority (see Bing, 2008). The students in our studies could either find both the component equation and the equations by application of Kirchhoff's laws or none of them. In contrast to the remarks of the experts, the same holds for the solution processes in the math-engineering part, i.e., more than 90% of the students who solved the homogeneous part correctly also solved the inhomogeneous part. Moreover, some students only solved the inhomogeneous ODE using physical arguments. The question remains is whether students realize that they can also apply another method from the MfES. In the validation part, students used different strategies, involving both mathematical as well as physical arguments, i.e.,

some students did all steps of the modelling cycle, while others argued using inner-mathematical arguments. They showed different justification strategies, analogous to justifications like calculation and physical mapping, as defined in Bing, 2008.

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Self-efficacy of engineering students in the introductory phase of studies

Ronja Kürten

University of Münster, Germany; ronja.kuerten@uni-muenster.de

In this paper we investigate the self-efficacy (SE) of engineering students in the introductory phase of studies. We focus on changes occurring in the first weeks, reasons for those changes, and the effects of a mathematics preliminary course on SE. Based on results of interviews with first year students in 2014, the preliminary course was adapted in the following year. In 2015 we interviewed first year students and collected questionnaires on SE. In the analysis we focus on mathematics, social, and general SE.

Keywords: Self-efficacy, transitional programs, undergraduate students, higher education.

Introduction

The start of studies at tertiary education institutions marks a new stage of life for young people. Students often move to a new location, meet new people, learn new rules, and are required to become accustomed to new educational settings. These changes can influence the students' self-efficacy (SE) beliefs (Schunk & Meece, 2006), which can be defined as "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (Bandura, 1997, p. 3). In academic contexts these beliefs correlate with perseverance, persistence, and achievement (Pajares, 1996; Schunk & Pajares, 2001), and therefore mediate study-success (Fellenberg & Hannover, 2006). Depending on the given task, different SEs are of importance: Engineering students for example face mathematical, social, and course-specific tasks. For this reason mathematics, social, study-oriented, and course-related SEs may influence the study progress. In this paper we focus on mathematics and social SE, as it seems reasonable to assume that they might be influenced by a mathematics preliminary course.

Van Dinther, Dochy and Segers (2011) reviewed 39 studies investigating factors that affect SE at higher education level, which were conducted between 1993 and 2010. None of these studies considered effects of preliminary courses that are common measures at the introductory phase of studies. Fischer (2014) investigates the perceived change in SE occurring during a mathematics preliminary course in his dissertation and finds that students perceive only minor changes in SE caused by the preliminary course. This study, however, offers no reasons for existing or non-existing changes. Another educational transition, namely the transition from primary to secondary school, is well investigated concerning SE beliefs. "Young adults often experience declines in their competence and efficacy beliefs as they make the transition from elementary to middle school […] [that] may result from changes in the school environment" (Schunk & Meece, 2006, p. 80).

The results of interviews conducted with first year students at the University of Applied Sciences Münster in 2014 suggested that several study-related SEs might change during the first weeks at university. The students reported rising uncertainty concerning the new style of teaching and learning at university and the finding and joining of a study group. Furthermore they feared to fail in their mathematics studies whilst being confident about success in other subjects. When asked for mastery experiences in mathematics during the first weeks they didn't report any preliminary course

experiences, and even improvement in the mathematics-tests didn't raise their confidence, as improvement was the least they had expected in a test that mostly differed only in numbers from the first one. As a consequence of these results, the preliminary course was adapted in 2015 to promote participation and cooperation of students.

In this paper the development of SE (mathematics and social) of engineering students at the start of studies is investigated. The study focusses on changes occurring during the first weeks at university, their reasons, and the role of the redesigned mathematics preliminary course regarding those changes.

Theoretical framework

SE is "competence-based, prospective, and action-related" (Luszczynska, Scholz & Schwarzer, 2005, p. 440), as shown in items such as "*I am confident to solve the systems of equations with* $x + y = -7$ *and* $x \cdot y = 30$" (Zimmermann, Bescherer & Spannagel, 2011, p. 2137) or "*I am confident that I can start a conversation with someone I don't know very well*" (Hermann, 2005, p. 107). Efficacy beliefs are multi-dimensional, varying in level, generality, and strength (Bandura, 1997). Generality of SE can range from task-specific SEs as used in the original definition of Bandura (1997) to domain-specific SEs such as mathematics SE or social SE to general SE. "Mathematics self-efficacy expectations indicate the belief of a person in his/her own competence to solve mathematical problems and tasks successfully" (Zimmermann et al., 2011, p. 2136). Engineering students are confronted with a considerable amount of mathematics in their studies, although they usually don't choose their course of studies for that reason. Social SE can be defined as "an individual's confidence in her/his ability to engage in the social interactional tasks necessary to initiate and maintain interpersonal relationships" (Smith & Betz, 2000, p. 286). Students beginning their studies at university face these tasks daily. "General self-efficacy […] reflects a generalization across various domains of functioning in which people judge how efficacious they are" (Luszczynska et al., 2005, p. 440). It can be defined as "the belief in one's competence to cope with a broad range of stressful or challenging demands" (Luszczynska et al., 2005, p. 439). We use this rather stable SE to control our results for testing effects.

SE beliefs influence goal-setting, motivation, or perseverance of people (Bandura, 1997). Empirical studies have shown that in academic contexts, low study-related SEs influence proneness to drop out of studies, whereas high SEs are supposed to promote study success (Fellenberg & Hannover, 2006). Pajares (1996) discovered that SE influences performance independently from and as strong as ability, and Schunk and Pajares (2001) showed that mathematics SE is a better predictor of achievement than self-concept, anxiety or prior experiences in mathematics. In general, SEs that slightly exceed actual skills are the most functional, as they "lead people to undertake realistically challenging tasks and provide motivation for progressive self-development of their capabilities" (Bandura, 1986, p. 394). SE can be developed through four main sources: enactive mastery experiences, vicarious experiences, verbal persuasion, and physiological and affective states. Enactive mastery experiences (failures or successes) are the most influential source, especially when they are attributed to personal effort. When no absolute measures of adequacy exist, social comparisons can function as source of SE. These vicarious experiences, i. e. observed experiences of models, are of greater influence when the observer perceives higher resemblance to the model. Verbal persuasion can be a third source of SE if significant others express faith or doubts in one's capabilities. This source is less influential than the first two. The last source mentioned by Bandura is that of

physiological and affective states. Stress reactions or feelings of joy in the face of certain tasks may influence one's SE if no other information is available. The way these sources influence one's SE is dependent on the cognitive processing, and therefore SE cannot simply be interpreted as the sum of prior mastery experiences (Bandura, 1997). Of these sources, enactive mastery experiences appear to be the strongest at higher education level (van Dinther et al., 2011).

Research questions and methods

Based on the preliminary findings and open questions described above, this study aims to find answers concerning the following research questions:

1. Do mathematic and social SEs change during the first months at university and if so, how do they change?
2. How does the mathematics preliminary course affect mathematic and social SEs?

At University of Applied Sciences Münster each year in September before the start of the semester, a mathematics preliminary course takes place. It consists of twelve modules: The first one – "How to study" – addresses general differences between school and university and other study-related aspects, such as time-management or preparation for exams. The second module – "practicing mathematics" – gives an introduction into reading and writing mathematics, set-theory, propositional logic and proofs. The remaining ten modules focus on contents of school mathematics from lower and upper secondary level (Kürten & Greefrath, 2016).

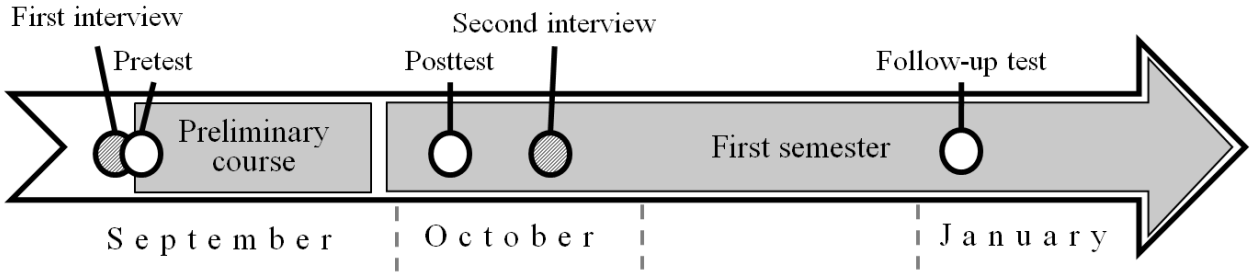

Figure 1: Time bar of the preliminary course and the accompanying surveys

For the quantitative part of the study, three data collection points were chosen: At the day before and at the first day of the preliminary course (pretest), in the first two weeks of the semester (posttest) and in the first two weeks of January (follow-up test, see Figure 1). Participation in the first test was mandatory for all students taking part in the preliminary course and voluntary for the rest. Participation in the posttest and follow-up test was voluntary because of organizational reasons. Each mathematics test consisted of 19 items associated with the contents of the mathematics modules of the preliminary course, with items that vary from test to test only in the used numbers or the given context. After completing the mathematics test, the participants submitted a questionnaire on statistical data, such as the type of school qualification and the elapsed time since the end of school, and a SE questionnaire. This questionnaire, in 2015, was composed of the German versions of three category systems: general SE (Hinz, Schumacher, Albani, Schmid & Brähler, 2006), mathematics SE (Zimmermann et al., 2011), and social SE (Hermann, 2005). The general efficacy part uses a four level Likert scale, while the other two operate with a five level Likert scale. The results of the general SE test were used to complement the specific view of the other scales and provide an opportunity to

detect testing-effects. The other scales were chosen due to the results of prior analysis (2014), which suggested changes in mathematics and social SEs. This specific mathematics SE scale was chosen, as it presents tasks that can be solved using school mathematics, the only mathematics most of the students taking part in our survey knew when filling in the pretest. The reasoning behind the choice of social SE scale lies in its specific design for students at tertiary education. It was translated into German, and the translation was validated by retranslating the items back into English and comparing the two English versions. These scales were chosen because they were already tested for validity, objectivity, and reliability with satisfying results (e.g. Cronbach's α between .80 and .94) (Zimmermann et al., 2011, Blömker, 2016, Smith & Betz, 2000).

For the qualitative part of the study we chose two data collection points: In the week before the preliminary course began (first interview) and during the third and fourth week of studies in October (second interview, see Figure 1). Participation in the interviews was voluntary. In the first interview 14 students were interviewed, and out of this group 8 were chosen for a second interview according to their department, their results in both mathematics tests, and their school-leaving qualification, to generate a heterogeneous sample and thus gather a large range of positions present in the population. The interviews were conducted using a semi-structured interview guide. The questions focused on mathematics and social SE as well as the students' motivation for studying and learning. In addition to the qualitative data gained in the interviews, the results of the SE scales were analyzed using t-tests to find changes in SE during the course, and semi-partial correlations to assess the relationship between attending the preliminary course and changes in mathematics SE. The interviews were transcribed and analyzed using qualitative content analysis according to Mayring (2014). As the interviews should be used to understand the quantitative findings, our focus lay on mathematics and social SE. To find reasons for changes and the perceived level of SE, inductive category formation was used conducting a summarizing content analysis (Mayring, 2014).

Results

Self-efficacy scales

In 2015 the SE questionnaire was completed by 409 students in the pretest, by 243 students in the posttest and by 135 students in the follow-up test. The downturn in the number of participants from pre- to post- and follow-up test can be ascribed to the change from mandatory to voluntary participation. 167 students completed the questionnaires in pre- and posttest and 54 students completed all three questionnaires. Comparison of the results of students taking part in the pre- and posttest (n = 167) respective in all three tests (n = 54) shows no significant differences in general SE and significantly higher results in the post or follow-up test for social SE (small effect size: $d_{z_post} = 0.20$, resp. $d_{z_fu} = 0.40$) and mathematics SE (medium effect size: $d_{z_post} = 0.61$, resp. $d_{z_fu} = 0.79$). To measure the relationship between attendance in the course and changes in mathematics SE we compute a semi-partial correlation of preliminary course attendance (as stated by the students) and mathematics SE after three months. We assume that for students not taking part in the preliminary course, mathematics SE didn't change in the weeks of the course. As those students neither took part in the course nor in the associated e-learning, we suppose they didn't engage in mathematics during this time. As we did not have results from those students in the pretest, we compare the first result of each student (whether it is from the pre- or posttest) with their results in the follow-up test. We found a significant correlation of .26 (n = 112, p = .006) between preliminary

course attendance and mathematics SE in the follow-up test, after partialing out mathematics SE-results of the first test.

Interviews

In 2015 eight students taking part in the preliminary course were interviewed twice. They were of age 18 to 22 and studied chemical engineering (2), informatics (1), and electrical engineering (5). The qualitative content analysis resulted in six main categories: mastery experiences, vicarious experiences, verbal persuasion, goal setting, temporal effects, and type of teaching.

Pseudo-nym	School-leaving qualification		Last grade in mathe-matics	Test results		Mathematics SE		Social SE		Time since end of school
	Type	Grade		Pre	Post	Pre	Post	Pre	Post	
Dennis	ATC	1	1	81%	84%	80%	96%	59%	73%	0 years
Roman	A-level	3.3	4	68%	74%	72%	75%	53%	47%	1 year
Christoph	ATC	2.3	2	11%	61%	59%	72%	71%	80%	3 years
Leon	A-level	2.6	2.7	42%	95%	69%	80%	70%	72%	3 years

Table 1: Statistical data of the interviewees cited in the article with German grades ranging from 1 (very good) to 6 (inadequate) and German school-leaving qualifications "Abitur" (A-level) and "Fachgebundene Hochschulreife" (advanced technical certificate [ATC])

In the first interviews some students explained their perceived competence in mathematics by the time that has elapsed since their last studies of mathematics at school (Christoph, Leon) or by social comparison with students coming from more demanding schools (Dennis):

> Christoph: Well, my mathematics are a bit rusty. I had in the advanced technical certificate/ I was quite good. […] And now I haven't had any mathematics in the last three years apprenticeship. Really not a bit. (1st interview, translation by the author)
>
> Dennis: If I now/ If I consider, I've been at a main school (Hauptschule), so surely there is something missing in maths compared with those of a middle school (Realschule) or a grammar school (Gymnasium). (1st interview, translation by the author)

In some of the answers the changes in the preliminary course are quoted as reasons for confidence in social or mathematical settings. The groupwork at the beginning of each tutorial offered possibilities to meet people and to solve harder tasks together. In the second interviews differences in school-leaving certification were of lower importance than mastery experiences in social tasks (finding a study group) as well as in mathematical tasks, and in the latter case the mastery experiences were attributed to personal effort:

> Dennis: It worked out well. I met new people right after the preliminary course. It opened up with the groupwork. You have to talk to other people there. […] And then groups formed right away and why should it worsen?! (2nd interview, translation by the author)

Roman: […] I was first put off a bit by the exercises because I wasn't able to figure them out. But as I took the lecture notes, I understood it and I liked that a lot. Because I realized that I understand things I didn't understand before. And I could solve exercises I hadn't been able to solve and then I saw there is improvement. (2nd interview, translation by the author)

Christoph: That shows me that I'll be able to solve hard problems that I couldn't solve at first if I study for them. And it shows me concerning my studies that a 2.0 might be possible. Depending on how much I will knuckle down and then practice, practice, practice. (2nd interview, translation by the author)

The prediction of failure rates was mentioned by some students as a reason for their fear to fail in mathematics. These vicarious experiences didn't differ from those reported in 2014:

Interviewer: Okay. We get to the next statement: "I'm afraid to fail in my studies due to mathematics".

Leon: Well, I would say four. That's more like the case, there is a certain fear of course.

Interviewer: And can you explain why you put the cross at the four?

Leon: Well, if you look at the rate of the last year, there are many failing. And there is of course/ you fear a bit that you won't be able to make the cut somehow. (2nd interview, translation by the author)

Discussion and perspectives

In the study presented here we show that students' mathematics and social SE did rise during the preliminary course, and that this effect was stable over a period of at least three months.

As participation in the interviews and the posttests was voluntary, and participation in the pretest only mandatory for those taking part in the preliminary course, the sample is probably not representative of the population of first year students. For example, motivation for learning or studying might influence the students' decision whether or not to take part in the course, the interviews or the tests. This is especially important for the quantitative analysis of the SE scales. Although the qualitative analysis of the interviews doesn't need statistical representativeness, it should be considered that certain interesting types of students (for example those with low motivation) were not included in our sample. Besides this limitation of the study, the results presented show that even in the first weeks at university study-related (mathematics or social) self-efficacies change, while general SE remains relatively stable. For ethical reasons we had no control group of students wishing to take part in the preliminary course, and for organizational reasons we weren't able to collect pretest results of those students not attending to the preliminary course. Therefore, the quantitative data does not reveal whether the perceived changes in SE are caused by the preliminary course or not. At that point the qualitative data help interpret the quantitative results, as their analysis suggests that experiences in the preliminary course influence changes in SE. Those experiences resulted partly from the redesign of the preliminary course in 2015, for example, the forced group work at the beginning of each tutorial that fostered social SE.

We did assume that for students not attending the preliminary course, no changes in mathematics SE occurred during the time of the course. However, some students might have used other resources to

prepare for their studies or just reflected on their competencies in comparison to the requirements of the University of Applied Sciences Münster. Again, the qualitative data help us justify our conclusions, as none of the students taking part in post- or follow-up tests stated other mathematical activities as reasons for not attending the preliminary course in the open ended questions of the survey. We feel therefore confident that our conclusion is valid for at least the bigger part of the students. Nevertheless, the correlation between course attendance and development of mathematics SE has to be treated with caution.

With regard to our first research questions we can say that in contrast to the findings of Fischer (2014), SE beliefs did change in the first months at university at least in the part of the population that attended the preliminary course. Social SE rose slightly and mathematics SE rose moderately during the preliminary course. These results are in contrast to the findings described in the introduction that predict a decline of self-efficacies during transitions (Schunk & Meece, 2006). The differences may be a result of different ages of the observed students, effects of the preliminary course, or other reasons. Another possible explanation is that the decline of SE takes place even before the preliminary course and is therefore missed by our survey. Further research could clarify this by assessing SE well before the move to tertiary education, as well as before and after a preliminary course. With regard to the second research question, we did find clues that there is a relationship between the changes in SE and the preliminary course attendance. Those students who reported mastery experiences (social or mathematical) in the second interview showed an increased value for the related SE in the second test as well. Further research should investigate changes in SE during a preliminary course in an experimental or quasi-experimental design to investigate whether the changes reported here are indeed caused by the preliminary course.

Of the students cited in this paper, those who had been out of school for a longer period showed lower mathematics SE than the others. As they also stated the elapsed time without mathematics training as a reason for their perceived mathematics ability, it might be interesting to evaluate the effect of time on mathematics SE.

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Q^2 game used in a task design of the double quantification

Thomas Lecorre

University of Grenoble-Alpes, department of mathematics, France; thomas.lecorre@wanadoo.fr

This article deals with student's interpretation of multiply-quantified statements. The efficient interpretation of such statements is useful, for example, for those who try to understand and use the formalism of the definition of limit of a function. Literature points out some students' difficulties in the interpretation of double quantified statements. I am using the scientific debate methodology to design and implement tasks by means of two fundamental steps. In the first step, I draw on the results of a questionnaire to identify students' difficulties related to the order of quantifiers in the interpretation of such statements. The aim of the second step is to design the targeted tasks by considering the results of the first step; these tasks are based on a game called Q^2. The implementation of the design shows that students understand that there are at least two kinds of interpretations, and that conventions of interpretations are needed.

Keywords: formalism, quantification, limit, scientific debate in class, game Q^2.

Quantified statements problem

Mathematics formalism uses massively quantification and specifically multiply-quantification. Research shows that students have difficulties with the interpretation of multiply-quantified statements (Dubinsky & Yiparaki, 2000; Chellougui, 2009; Piatek-Jimenez, 2010). EA statements corresponding to "Exists…for all…" sentences must be distinguished from AE ones corresponding to "For all…there exists…". Dubinsky and Yiparaki (2000) study the impact of two main variables on the interpretation of double quantified statements. The first variable is the order of quantifiers (AE or EA) and the second one is the kind of statement, mathematical or non-mathematical. They show that the interpretation of non-mathematical statements is essentially correct but the interpretation of mathematical ones is difficult for students. For AE mathematical statements, the students' interpretation is very efficient, whereas their interpretation of EA ones seems to be done through an inversion: the EA statement is very often interpreted as the AE corresponding statement (the variables remain with their own quantifiers but the order of quantifiers are changed). Moreover, it is noticed that "the students did not appear to care of the syntax of a statement to analyze it […] the student did not appear to be aware of having engaged in interpreting the questionnaire statements." (Dubinsky & Yiparaki, 2000, p. 53). The inversion of interpretation of an EA statement is also noticed by Chellougui (2009) when she asks students to define the upper bound M of a set A and when almost all of them answer by what she calls "a strange definition": « $\forall\, x \in A, \exists\, M \in \mathrm{IR}, x \leq M$ ». Piatek-Jimenez (2010) confirms the asymmetric perception and interpretation of those two kinds of double quantification in the mathematical field. There seem to be two different problems: the use of "strange" conventions and the unawareness to interpret statements.

To overcome these difficulties Dubinsky and Yiparaki (2000) presented a game based on the dialogical logic (explained below) and used it with students to make them aware that two kinds of interpretation can be used and that this depends on rules of interpretation linked to the places of quantifiers. Results have shown that the game seems to help sometimes students to understand such statements but in some cases, it does not, and in a few cases this has created more confusion.

I want to pursue this research by designing tasks to overcome the difficulties of the interpretation of double quantified statements using the methodology of scientific debate. This implies two related goals. The first one deals with the identification of students' difficulties related to the order of quantifiers in a double quantified statement. I specifically explore the role of some other potential variables: the set of quantification (familiar or not, finite or infinite), the semiotic representation of the quantified variables (formal or usual language) and the kind of relation involved in the predicate (familiar or not). The second goal is to design and to implement tasks by considering the results of the study related to the first goal. A new game called Q^2 based on the interpretation of double quantified statements to determine a winning strategy will play a fundamental role in this design.

The aim of this paper is to study the following question: is there a domain of values of variables where students take into account the order of quantifiers and if yes how to take advantage of this domain to enlarge it? Specifically, how to make students aware that they are interpreting these statements and that they need rules for that?

The methodology of scientific debate: A tool for designing tasks

The scientific debate (Legrand 2001) is a socio-constructivist approach to learning and teaching mathematics based on two main principles: 1) the need of new knowledge can be obtained by making one realize that his/her previous conceptions may lead to contradiction; 2) the organization of appropriate debates among students permits firstly to express and share previous conceptions about the targeted subject, secondly to encounter the limits of these conceptions, and thirdly to be able to understand the institutionalisation related to this knowledge that is made by the teacher. Three fundamental steps shape the design of tasks by means of the scientific debate methodology:

1) The first step consists on epistemological and cognitive studies of the targeted mathematical concept: the role of quantifiers in formal statements and the interpretation of double quantified statements that stands on the use of a convention of interpretation. For this interpretation, I have chosen the dialogical logic (Lorezen, 1967). In this paper, we mainly explore the cognitive aspects and we use a questionnaire for that purpose.

2) The second step concerns the design of the tasks, which is done by using mainly two kinds of questions to initiate the debate. The first kind of question concerns the truth of a conjecture: is it true or false? This conjecture can be given by the teacher or can come from the students after a call for conjecture. The second kind of question concerns the nature of an object for a conjecture or a property: is it an example, a counter example, or off topic (neither example nor counter-example) for this conjecture, and is it an example for a property? The example can come from the teacher or from a call for an example. A vote is made: do you think it is true, false or *something else* (an example, a counter example, off topic or *something else*)? The possibility of voting for *something else* is given to preserve the authenticity of the other votes (voting True or False must be a choice founded on convictions). Then a debate is organized by the teacher among those who have different view-points. The teacher never gives any opinion about what is debated but tries to maintain a level of interaction by emphasizing the contradictions between students.

3) The third step concerns the level of experimentation and its analysis: what actually happens is confronted to what was expected to happen. Specifically, this analysis leads to discuss the efficiency of the choices made in the two first steps.

The whole study is conducted according to the aforementioned three steps. In this paper, I show my findings from the study related to the first step and I give more details about the elaboration and the results of the two final steps. For the first step, I have chosen to use a questionnaire on a sample of 181 students in their last year of secondary school to identify students' difficulties with the interpretation of double quantified statements in formal and non-formal context. For the second step, the results of this questionnaire are used in a way that is aiming to make students aware that the lack of the rules of interpretation is not a problem in a certain domain but leads them to conflicts out of this domain. This design is based on a game, called Q^2, in which the interpretation of double quantified statements is crucial: the question that will initiate the debate concerns the way to win this game. For the third step, I experiment this design for students in their last year of secondary school in a scientific class composed of 29 students (experimentations at university are planned).

Task design

Background: The result of the questionnaire

I have mainly studied five variables in the interpretation of double quantified statements: place of quantifiers, kind of statements (mathematical or not), the set of quantification (familiar or not, finite or infinite), the semiotic representation of the variables (formal or informal language) and the kind of relation involved (familiar or not). In the questionnaire, words (for all, exists) are used instead of symbols because symbols are introduced after the game Q^2 as a solution to the conflicts that appear about quantifiers and associated conventions. A questionnaire has been given to 182 students. We will only give the findings here (for more details see Lecorre, 2016a). The first finding is that the rule of "correct interpretation for AE statements and inversed one for EA statements" seems not so obvious: some EA statements are perfectly interpreted whereas some AE statements are interpreted through an inversion of the quantifiers and the variables. The second finding is that some other variables are correlated with difficulties in the interpretation (e.g. width of the quantified set, formalization of variables...). The third finding is that there exists a domain of correct interpretation for both EA and AE. This domain is made of non-mathematical statements quantified on a "small" finite set (less than ten values) without any formal variable and with a familiar relation. These findings are then used to design the tasks aiming at discussing the interpretation of double quantified statements.

The game Q^2

The game presented by Dubinsky and al. (2000) is based on the dialogical logic of Lorenzen (1967) which gives a way to decide on the truth of quantified statements using a codified dialog between a proponent and an opponent. In this game, for example, if the sentence is "for All x there Exists y such that $R(x;y)$", the A-player chooses x and the E-player has to find a y such that $R(x;y)$ is verified. If he/she fails to find such y, the A-player wins, otherwise the A-player can give another x (same kind of rule for EA). I call this game a direct game: given a statement and sets of quantification, you have to decide the truth. The Q^2 game is an inverted game: given a statement and the truth, the players construct the set of quantification to make a statement true or false.

For the Q^2 game I choose values of the variables that make it an easy game to play: non-mathematical field, small set of quantification, familiar relation. This choice is made to permit students to get into the game and into the interpretation of associated statements.

This is a two players' game. This game is given by four elements: a starting rule, a winning rule, a statement and a grid. For example, red player has to start (starting rule) and plays for the falsification (winning rule) of the statement S: "For all red letters, there exists the same black letter" and the given grid:

H	U	H
L	U	L
U	H	T

Figure 1: A grid for the game Q^2

So red has, first, to circle one letter with his red pencil. Then the black player circles one letter and so on until all the letters have been circled. With the given elements of this example (starting and winning rule, statement, and grid) and the coloration of letters, if the statement S is false then the red player wins, but if it's true the black one wins. This game has, of course, many variants, beginning with the filling of the grid and the starting and winning rules. This game is the heart of a design which aims precisely to enlarge the domain of good interpretation. I am going to show that a smart use of Q^2 in the design has the potential to reach such a goal by emphasizing the lack of the convention of interpretation.

"The Q^2 situation"

The principle of the Q^2 situation is to provoke conflicts of interpretations that will lead students to the need of the convention of interpretation. The situation Q^2 is divided into four periods:

- The first period aims an appropriation of the game Q^2 by playing.
- The second period deals with the concept of winning grid for the game Q^2.
- The third period targets a conflict of interpretation, in a way that students feel the need of conventions of interpretation. At that stage, the conventions are given.
- The fourth period is just an application of these conventions on the unsuccessful domain where the values of the variables lead students to difficulties of interpretation.

In the first period, a paper is given that contains eight games of Q^2 to play (each game is defined by a statement, a starting rule and a grid). These games are designed for the two players to have opportunities to win and to start to have ideas on how to play to have good chances of winning. In fact, with such a game, with "good players", the winner depends only on the statement (EA or AE), the starting rule, and the grid given.

The second period aims at the definition of winning grid. The students are asked to give the winning grids for a given rule, then a debate is organized about these propositions: are they winning grids, or not? The contradictory opinions about the propositions should lead students to identify the lack of a definition of winning grid. A winning grid is, in fact, quite difficult to define in a mathematical way for pre-university students (double recursive definition). Here, a definition such as "a grid is a winning grid for red if when red plays "cleverly", he is sure to win, even if black also plays cleverly" is largely sufficient for this design. When the students show a need for a definition, the above definition is given.

The third period aims to highlight the lack of convention of interpretation. Once again, for a given rule, the students are asked to give winning grids. There should be no more conflicts about what is a winning grid in general, but new conflicts should appear about the propositions: is this grid really a winning grid with this rule? This should happen because the winning grids depend on the interpretation of the rule which is a double quantified statement. And the need for a convention should appear with the impossibility to find a common agreement (is it a winning grid or not?). The didactical principle which is used here is the following: it seems very difficult to organize a direct confrontation of the different rules used by students to interpret a double quantified statement, because this problem, taken as a general one, is too theoretical and depends on too many variables (findings of the questionnaire). On the contrary, it is much easier to create a conflict on concrete consequences of the interpretation of such statements. Here, the conflict holds about the question ”is the grid a winning one or not?” Then, trying to understand each other, and trying to convince their peers, students are going to explain their own interpretation. And then it will appear that the implicit conventions used by students are contradictory. Students can realize that without common conventions, no agreement is possible. Deciding if a grid is a winning grid or not is possibly more complicated than the logical principles of deciding who won, but it leads students to materialize their own conventions through these grids and permit to confront these conventions.

Then these conventions are given in the manner of the dialogic logic (Lorezen, 1967), which is described above with the game of Dubinsky and Yiparaki (2000). At this stage, the game Q^2 plays as a preparation to such rules by simulating a game between a proponent and an opponent. The design here aims much more the awareness of the need of convention than the “right rules”. The aim is to make students aware of the necessity to check the validity of their interpretation relatively to the adopted conventions.

The fourth period consists in verifying, still using debates, that the given rules can lead to agreements, and even in the domain where students used to fail: the rules are helpful, efficient.

Results

The first period of the situation Q^2 (playing the game) shows a good appropriation of the game: the winning grids were almost always won by the one for who the grid was a winning one, which means that the students were playing “cleverly”. Some strategies seemed to begin to be used. And, above all, almost all the decisions about who is the winner were correct. All this is coherent with the results from the questionnaire in terms of domain of interpretation.

The second period led to the question of the definition of a winning grid. The definition is given.

The third period begun with a question of the interpretation raised by students in a debate. I present an extract of the script of this debate to explain this unexpected acceleration. Student are asked to give winning grids for red for the game “There exists a red square such that all black squares have the same symbol” where red starts and plays for true. The given statement does not specify if black squares shall have the same symbol as the red one or not. So, after six winning grids had been given by the students in the second period, this lack of information was intensely discussed.

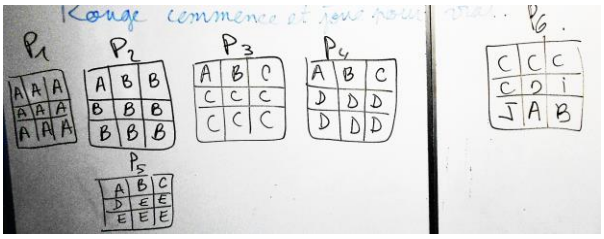

Figure 2: The winning grids for red given by the students

The grid P1 was put into debate and everyone agreed that it was winning grid for red. Then P2 was put into debate (28 votes for a winning grid for red and 1 for "something else"). Loïc who had voted "something else" changed his mind and explained why, for him, it was a winning grid for red:

Loïc: Because red starts and as he plays to win he takes the square A and…

Teacher: You're saying that "red plays A first" yes and what?

Loïc: Then black takes only B squares.

Teacher: Black only takes B squares. Why, in the end, red wins?

Loïc: Because….

Hadrien: Because Black has only B squares.

But Quentin disagreed with this explanation:

Quentin: And because red has it also (One B square)

But Hadrien and Louis did not agree with this addition:

Hadrien: No, he has A squares and B squares.

Quentin: Yes!

Louis: I think that I should just say that there exists a red square.

The sequence above shows that these students do not need to disagree on the fact that it is a winning grid or not to begin to explain their own interpretation: "the same symbol as the red square" (Quentin), or "the same symbol for all black squares" (Loïc, Hadrien). Then Fabio, in the same way, explained that nothing must be added contrary to the sayings of Quentin:

Fabio: I do think that what Quentin added is not necessary.

Then Quentin proposed a grid to strengthen the differences of interpretations:



Figure 3: Quentin's grid (As in black and Bs in red)

Leaving aside for a while the problem of the winning grid, the teacher asked whether the statement S1 was true or not, according to the grid of Quentin. Twelve students thought that the statement was true, ten false, while six students voted something else. Some conclusions were then raised :

Fabio: There are some, like me, that can think that all the black squares have got the same symbol is enough and some other, like Sébastien, who are thinking that there must be a red square that has got exactly the same symbol as any black square.

Mickaël: I'm asking: if you who think that only black squares have the same symbol is enough, this red square exists such that what?

This made Hadrien change his opinion, but Louis did not agree with this change:

Louis: There are two opposite opinions just because we're not thinking the same.

Teacher: Ok, you are not reading the same way…That is what Fabio said…

Louis: Exactly!

Juliette and Maxime then explained why they voted something else:

Maxime: That is exactly why, from the beginning, I voted Other.

Juliette: So do I.

These two interpretations are not directly linked to an EA/AE inversion but to the interpretation of the predicate. It would be interesting to investigate the role that the vernacular language (the linguistic subtleties that may eventually vary in French or English) may have played in leading students to interpret the two variables in the predicate as bound variables ("the same as red squares"), as shall be done in AE statements, where as it isn't the case in such an EA statement. In any case, the discussion among students led them to work on these interpretations and to get aware of the complexity of having a unique interpretation. The teacher then gave the conventions of interpretation of EA and AE logical statements. These conventions were then used successfully for the applications of the fourth period. One month later, the same students had to face double quantified statements in a situation aiming at the definition of limit. They experienced, once again, the need of conventions when they encountered another conflict of interpretation of these statements, so they checked the conventions to decide on their own (Lecorre, 2016b).

Later, another experimentation was realized with some other students using a "complete" rule ("…such that all the black squares have the same symbol as the red square") and this led students, with a grid of eight squares filled with symbols appearing twice, to the predicted conflict (EA/AE interpretation). A new didactic variable is then identified, the constitution of quantified sets (how they are filled relatively to the predicate), which raises an unexpected question: is the students' choice of convention between EA and AE convention for the interpretation more guided by the constitution of the sets of quantification than by the statement itself?

Conclusion

The Q^2 situation, with the scientific debate methodology, were used to make the interpretation of the double quantified statement the main object of students' discussion. The contradictions that the lack of conventions of interpretation was bound to imply then emerged in that discussion. This lack which was invisible to students suddenly came into light with these contradictions.

More precisely, the questionnaire led to identify some variables playing a role in the interpretation of students. There is a domain of the value of those variables which gives a good interpretation.

This Q^2 situation led students to be ready to receive the conventions of interpretation of double quantified statements. Indeed, they have experienced the need of shared conventions. The described situation Q^2 mainly aims the recognition that there are two kinds of double quantifications that should be differenced according to the order of the quantifiers in the statement and the convention of interpretation of the predicate but the validation using variables defined in function of the other variable remains a difficulty.

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A framework for goal dimensions of mathematics learning support in universities

[1]Michael Liebendörfer, [1]Reinhard Hochmuth, [2]Rolf Biehler, [2]Niclas Schaper, [1]Christiane Kuklinski, [1]Sarah Khellaf, [2]Christoph Colberg, [2]Mirko Schürmann, [1]Lukas Rothe

[1]University of Hannover, Germany

[2]University of Paderborn, Germany

liebendoerfer@khdm.de, hochmuth@idmp.uni-hannover.de, biehler@math.upb.de, niclas.schaper@upb.de, kuklinski@idmp.uni-hannover.de, khellaf@idmp.uni-hannover.de, christoph.colberg@math.upb.de, mirko.schuermann@upb.de, rothe@math.uni-hannover.de

In university mathematics, support measures address diverse goals in terms of students' knowledge and abilities, motivation and beliefs, or institutional goals like the reduction of dropout rates. In order to facilitate the analysis of specific support measures' goals, their evaluation and comparisons with other forms of support, we aim at developing a taxonomy for these goals. To this end, we have analyzed documents of 44 innovative projects of mathematics support in Germany and conducted supplementary interviews with teaching staff. We present the method and intermediary results of this research and discuss its potential use for researchers, policy makers and teaching.

Keywords: Educational objectives, Mathematics support, Taxonomy, Tertiary education.

Background and aim of the paper

Mathematics support has become a common endeavor in many universities and both researchers and teaching staff are interested in identifying the "best way" of supporting their students. The research we present here is part of the ongoing WiGeMath project (Wirkung und Gelingensbedingungen von Unterstützungsmaßnahmen für mathematikbezogenes Lernen in der Studieneingangsphase; Effects and success conditions of mathematics learning support in the introductory study phase), which is a joint research project of the Universities of Hannover and Paderborn (Colberg et al., in press) led by Biehler, Hochmuth and Schaper. In a first phase, the research aims at identifying and structuring goals that have been pursued in selected Projects of Mathematics Learning Support (PMLS). This paper reports on the methodology and the research outcomes of this first phase, drawing upon the project proposal and the intermediate project report of the three project leaders.

In WiGeMath, we examine 44 PMLS in mathematics programs, teacher education and diverse engineering programs at 14 German universities. These projects all have in common that they are trying to help students acclimatize to university mathematics. We label them innovative as they aim at deviating from the formats of standard lectures or standard tutorials. The extent to which they can fulfill this deviation is part of the evaluation in WiGeMath. The PMLS are clustered into four types: bridging courses, mathematics support centers, redesigned lectures and support measures that parallel courses. In most German universities, bridging courses are offered shortly before the beginning of the first semester and aim to bridge the gap between school and university, dealing with school contents as well as university contents. Mathematics support centers are relatively new to German universities and have been implemented in rather few universities. They are designed as optional classes for students where they can go and seek help or work on their math problems under

the supervision of experienced tutors. Redesigned lectures are also relatively new and are offered particularly for preservice secondary teachers in order to support their transition from school mathematics to more abstract mathematical content. They focus more on study techniques like problem solving or reading and writing mathematical texts than on mathematical content. Support measures that parallel courses are diverse and include formats that provide online learning material as well as tutorials for special study groups (e.g. students at risk of failing the course), with a special focus (e.g., applications) or for a special purpose (e.g., exam preparation). Using a program evaluation approach (Chen, 1990), WiGeMath aims at evaluating different PMLS based on their own assumptions and comparing them. Therefore, we developed a taxonomy in the sense of a descriptive (non-normative), structured set of goals, features and conditions of the PMLS. In this paper, we focus on the taxonomy for the goals of PMLS. Here, the term "goal" is interpreted in the sense of any observable criterion that provides ground for the evaluation of a PMLS against its own initial conception. Existing taxonomies of educational objectives include cognitive and affective aspects of students' learning (e.g. Krathwohl, 2002). However, they only describe learning outcomes of the individual. For evaluating the PMLS' outcomes against their respective initial conception and comparing different PMLS, existing taxonomies have proved insufficient. We found, for example, that some PMLS have goals that cannot be assessed in terms of individual learning objectives, like the support of certain study groups.

Method

We constructed a new taxonomy using an empirical approach consisting of two steps. In the first step, a document analysis was conducted based on documents from the partners' PMLS. This step resulted in a first draft for the taxonomy. In a second step, this rough taxonomy guided interviews with teaching staff to check, refine and supplement the categories.

For the first step, we asked our partners to send us any document that might inform us about the projects, including self-descriptions and learning material. The kind of information greatly varied across the projects: some PMLS had been described in conference papers or posters; others sent us lecture notes or books, and in several cases, flyers or websites addressed at students were available. In the case of redesigned lectures, we also analyzed study regulations. Moreover, the depth of information varied as some projects had explicit descriptions of their goals, others only had learning resources and some PMLS were just in the making and could not give us any documents. We thus asked several partners to describe the goals of their projects either via email or in unstructured interviews and then included these emails or interview notes and transcripts in the analysis. In the following phase of the document analysis, we followed the typical steps of skimming the documents for relevant passages, reading them and interpreting them as an iterative process (Bowen, 2009). Following the principles of inductive category formation in qualitative content analysis (Mayring, 2015), we worked through the relevant passages line by line, either subsuming the goals under existing categories or constructing new ones. In this step, we also reconstructed goals that were not mentioned explicitly but were apparent in the documents. If, for example, learning material explicitly asked the students to consider mathematics as a process of trial and discovery, we assumed the change of students' mathematical beliefs to be a goal. Due to the high diversity of materials and goals, we worked through all the material before reorganizing the

categories in a rough taxonomy. Here, distinctions from the literature were implicitly taken into account, like the distinction of cognitive and affective outcomes.

A limitation of document analyses may lie in insufficient details and a bias in the document selection (Bowen, 2009). Specifically, the teaching staff could have had goals that were not reflected in the documents. Thus, in the second step the rough taxonomy was used to guide eight interviews. Two interviews were held for each of the investigated designs, i.e., bridging courses, mathematics support centers, redesigned lectures and support measures that parallel courses. The interview guide first asked for the goals of the PMLS in general and then used the subcategories for deeper inquiry of each specific aspect. In the interviews, we found that some staff members did not mention all goals immediately. In reaction to a general question, only some goals were named. Explicit questions mentioning specific goal categories, however, led to additional goals named by the staff members. These interviews, taped and transcribed, were coded using the rough taxonomy as a coding scheme in order to see if each goal that was mentioned by the interviewees fit into one of our categories. This led to the refinement or reformulation of subcategories, but generally, all goals were covered by the taxonomy.

Results

The goals we identified were split into two main categories of educational goals and system-related goals, each of which consists of several sub-categories. We proceed with a presentation and description of the emerged categories and sub-categories that are summarized in Table 1.

Educational goals

The first main category of educational goals comprises learning outcomes with a focus on objectives regarding the individual learner. These educational goals are subdivided into knowledge goals, action-oriented goals and attitudinal goals. As these goals are more or less covered in existing taxonomies (e.g. Krathwohl, 2002) and space is limited, we omit a description and discussion of these goals. We only shortly mention the last category of **learning and working conduct.** It refers to learning rhythm (i.e., when do participants study), learning expenditure (i.e., how much do participants study), learning materials (i.e., which resources do participants use when studying), learning environment (i.e., where and with whom do participants study) and use of the PMLS' provision. This goal differs from the ones mentioned so far as it does not represent a final goal of studying but rather functions as a mediating partial objective that facilitates the fulfillment of other goals at a later point in the course of studies. Staff members explicitly mentioned that they wanted their students to work in specific ways they assumed to be most efficient.

System-related goals

In our study, we found goals which do not focus on students as individuals but rather take into account the university as a broader organization and therefore decided to label them system-related goals. We specify how we understand each of these categories and which aspects they include. It is important to note that the system-related goals and the educational goals are not necessarily disjoint. When using the taxonomy to categorize the goals of a PMLS, these goals may fall into one category under educational goals and, at the same time, into another category under system-related goals. The difference lies in the focus on the individual learner versus the institution as a broader

organization which for example has to establish its own reputation, take into account questions of funding as well as maintain its societal position in providing studies that lead to certain certificates.

Educational goals	
Knowledge goals	- school mathematics knowledge and abilities - higher mathematics knowledge and abilities - the language of mathematics
Action-oriented goals	- mathematical modes of operation - university modes of operation - learning strategies
Attitudinal goals	- beliefs - affective features - perceived relevance for the future job - perceived relevance for future studies - mathematical enculturation
Learning and working conduct	
System-related goals	
Creation of prerequisites for knowledge/abilities	- improvement of school knowledge and abilities as a prerequisite for university studies - requirements for lectures that exceed school knowledge
Improvement of formal study success	- dropout rates - passing rates/achievements
Improvement of teaching quality	
Improvement of feedback quality	
Promotion of social contacts and connections relevant for the studies	
Making university study demands transparent	
Supporting of certain student groups	

Table 1: Categories of the WiGeMath taxonomy

As to the **creation of prerequisites for knowledge/abilities**, PMLS with this target would aim at qualifying students to participate successfully in subsequent university classes. This might be achieved via the *improvement of school knowledge and abilities as a prerequisite for university studies*, a category that we defined as a first sub-category. We refer to topics and methods that are not part of regular university lectures but should be familiar to students from their school background like doing arithmetic with fractions, sine, cosine, and solving systems of linear equations. The difference between this goal and the educational goal of fostering school mathematics knowledge and abilities is that the educational goal is only achieved when each and every student has gained the knowledge in focus, whereas the system-related goal is reached when the bigger part of students has gained this knowledge so that future teaching can take this knowledge as shared. In the latter case, it is inconsequential whether all students make use of the knowledge supply and integrate it into their own learning. It is well known, for example, that

students entering university often have gaps in their knowledge while lecturers want to give their lectures as if this knowledge was present. In one bridging course, asked for knowledge as a prerequisite in future lectures, a staff confirmed this (all quotations are own translations):

> Staff member: Due to the fact that we refresh the students' school knowledge, there is something they can build on later [in the lectures] and certainly will. […] If I look at a proof of continuity, I need absolute values and inequalities. If I do not know them, particularly now that they were removed from the school curriculum, then I have to learn them before.
>
> Interviewer: That means, you try to compensate some deficits, in particular after changes through a school reform where some topics have been canceled from the curriculum?
>
> Staff member: Yes, I would say.

Often, PMLS also aim at providing *requirements for lectures that exceed school knowledge*. In the innovative courses included in our study, examples may be seen in topics like groups, rings or fields that could be shifted from regular university courses, in which they would usually be discussed. Sometimes, bridging courses also cover topics that were considered school knowledge in the past but are no longer taught in schools today.

For some institutions, the **improvement of formal study success** also proved to be important. This relates to objectively measurable success criteria like *dropout rates*, defined as the number of students who withdraw from their studies, and *passing rates/achievements,* which can be measured via the final exam. For a support center, for example, this was central in the interview: "Clearly, I mean, the big credo was always to reduce the dropout rate." In German universities, dropout rates in mathematics and engineering programs exceed 30 % (Heublein, 2014). These criteria are of special interest as they may be used for an institution's quality evaluation. PMLS addressing dropout include tutorials for specific study groups with low success rates or students who already failed an exam twice and now make their final attempt.

The **improvement of teaching quality** is another goal of PMLS. The aim is to improve the teaching quality as perceived by the students and gain better evaluations from the participants. This may include changing the teaching styles and improving the communication with the students.

In our analysis, three more goals emerged, which all reflect the position that, the university on the one hand wants to provide an optimal environment for the students' learning, yet on the other hand does not take the responsibility for each individual student. The sub-category of **improvement of feedback quality** covers the aim to provide students with high-quality feedback on their state of studies that helps them improve their learning. Under **promotion of social contacts and connections relevant for the studies**, we subsume the support of social exchanges and conversations, technical aids, and stimulation to form study groups. For example, students may be offered specific hours for a specific lecture in a mathematics support center so that they may form learning groups on their own account. Also, the design of such centers may reflect this goal: "These workstations there have desks, where eight students may sit at a time. Students have increasingly come in larger groups. And it was clearly our goal, to foster this". A further category that some PMLS aimed at is **making university study demands transparent.** This includes giving insight

into demands of university education, especially with future orientation concerning preconditions and requirements in the course of further studies. Together, these aspects enable the students to make well-informed decisions on how to study according to their personal preferences.

Finally, some PMLS were designed to help certain students more than others, for example, students with a different language background, females, students with a sideline job or students with children. These aims would be encoded under **supporting of certain student groups**. For one support center, for example, the focus was on low and medium-achieving students, as we could see in the interview: "Our priority is on everyone but the high-performing students. So, from low-achieving to average. […] If we have resources left, then the high-performers are also welcome". In contrast, a redesigned lecture addressed the average-achieving students but not the low-achieving ones: "We have, so to say the very good students, a broad midfield and then the lower fifth part or so, where most of them will probably dropout from their studies, because they are not suitable after all. But they take part at the beginning and want to try it, anyway. And my objective during the development and conduction of this course was to address the broad midfield and in addition provide challenges for the very good students." Another specific group that might be mentioned consists of students without "Abitur". The German school system is split into schools for lower-achieving students where students attend nine/ten years and the so-called "Gymnasium" that adds another three years of school. Generally, only the students of the Gymnasium are allowed to attend university after they have passed the final exam called Abitur. Yet, there are exceptions where students without Abitur may attend university, a scenario that occurs particularly frequently in engineering programs. As engineering was one of the study fields we focused on, this category seemed especially relevant. This goal may reflect a special profile of the university or study program or societal goals and may come along with special funding opportunities.

Discussion

Methodological discussion

As mentioned above, the taxonomy at hand was developed in cooperation with university staff involved in innovative PMLS. Contrary to our expectation that the goals of PMLS were thought through and decided upon in advance, we found, in the course of our work, that many of their goals had remained implicit until our inquiry. This observation raises the question of whether important goals of innovative PMLS can be fully accessed from documents and interviews. Similarly, goals of traditional lectures might also be implicit, since some lecturers may be reproducing these long-established formats without further reflection. To be more specific, it seemed as if some PMLS aim at establishing a specific didactic contract (Brousseau, 1984). For example, students are to be offered a good learning environment in hope of thereby increasing success rates but the university does not take responsibility for the success of each individual. However, this aspect was not included in the taxonomy as we lacked clear evidence. The lack of reference to the didactical contract in the materials and the staff interview might be because of its implicitness in PMLS goals and to the less familiarity of staff to such theoretical terms.

Discussion of the taxonomy

The educational goals in our taxonomy show similarities to the objectives of other taxonomies (Krathwohl, 2002). Whereas these models focus on individual learning outcomes, the WiGeMath

model aims at comparing and evaluating innovative measures as a whole, in particular following the program evaluation approach (Chen, 1990). This reflects in the new category of system-related goals. The system-related goals reflect three purposes: Some of the goals ensure the preservation of the institution. The creation of prerequisites for knowledge/abilities, improvement of formal study success, and improvement of teaching quality would fall into this category. Another set of system-related objectives consists of improvement of feedback quality, promotion of social contacts and connections relevant for the studies, and making university study demands transparent. These goals improve the environment for students' self-directed learning. Supporting certain groups of students is a goal of a third kind as it represents the societal goal of creating equal study conditions in higher education for a wide range of students. The taxonomy thus reflects the institutional framing, in particular goals related to the institutions' preservation as well as the preservation of the innovative PMLS, which mostly had no regular funding but were financed by federal grants for innovative support. Related notions can be found in classification systems for higher education institutions. However, these categories are intended to be strictly descriptive and can therefore not be reformulated as goals. An early example is provided by the Carnegie Classification of Institutions of Higher Education, which was created in the 70s for US institutions. Initially, it was mainly concerned with structural and organizational characteristics of institutions but has undergone major changes in 2000, strengthening the emphasis on teaching-related institutional characteristics (Bartelse & Vught, 2009). A shortcoming of the model is that it lacks a consistent theoretical framework. In following the program evaluation approach, we based its development on the individual language and paradigms of the PMLS and did not question them in their data. From a practical point of view, the model helps to highlight their similarities and specialties. From a theoretical point of view, a next step could be the consistent re-interpretation of their goals from a specific theoretical and epistemological stance, e.g. clarifying basic notions of knowing and learning. We should keep in mind, however, that such a taxonomy might not only be used for the non-normative exchange of ideas, but also be turned into a normative model.

Implications for research, policy and teaching

It seems obvious that a model like the one developed above is never complete and could be expanded not only to fit more goals of innovative measures but also in order to serve regular courses or classes in other fields of study. So far in the first study phase of university mathematics education, we focused on innovative PMLS. We may thus have missed goals that are related to studies beyond the introductory year or to innovative measures integrated in the regular classes. An example for a possible system-related goal that was not mentioned in our study is the qualification of future staff or PhD-students.

Since many practitioners had to reconstruct their goals during the interviews, we believe that our taxonomy may prove beneficial to both teaching staff and developers of courses and support measures as a heuristic tool, helping them reflecting on their goals and teaching practice. It may also highlight ethical questions, e.g. by pointing out that some PMLS were designed to specifically support some students but not others. This taxonomy could also improve the communication about goals between students, teachers and institutions. It provides a common language and a frame of reference. In research, this framework is ultimately intended for the evaluation of innovative measures and so far proved useful in doing so in the ongoing WiGeMath project. For policy-

makers, a taxonomy may help in the evaluation of their decisions. In our study, we found that many practitioners had thought about various aspects of their work, but did not have a (shared) language to communicate these thoughts.

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A vector is a line segment between two points? - Students' concept definitions of a vector during the transition from school to university

Tobias Mai[1], Frank Feudel[1] and Rolf Biehler[1]

[1]University of Paderborn, Institute of Mathematics, Germany;

tmai@math.upb.de, feudel@math.upb.de, biehler@math.upb.de

The vector concept is an important concept students are confronted with in their first year at university. To be able to build on students' previous knowledge it is important to find out what they have learned about vectors from school. This study aims at exploring university freshmen's personal concept definitions of a vector. We therefore analyzed common German school textbooks to find out how vectors are introduced and what conception of a vector students might have developed at school. In addition, we administered a short pretest in which students were asked what a vector is and to explain vector addition and its properties. We ascertained that freshmen stated a lot of individual concept definitions. The majority of students stated geometric ones, which were mostly not fully adequate, i.e. improperly formalized to be embedded into the theory of abstract vector spaces. Furthermore, various misconceptions were identified.

Keywords: vector, concept definition, transition, textbook analysis.

Introduction

Many students face problems connected with the transition from school mathematics to university mathematics. To reduce students' difficulties during this transition, the Ministry of Innovation, Science and Research initiated the Studifinder project. Part of this project are the studiVEMINT learning materials. We have developed these learning materials (which are an e-learning based bridging course for mathematics) at the University of Paderborn since 2014 (Colberg, Mai, Wilms, & Biehler, 2017). The development was completed in summer 2016, although quality assurance still takes place. Students can use the course for several purposes: to fill gaps in their mathematical school knowledge, or to get used to elaborated forms of school content. In any case a focus lies on accurate language and notions and a mathematical discourse based on the definitions introduced, as it is expected at university.

An important concept students are confronted with at school and university (at a more abstract level) is the concept of vectors. In early 2016 we started the development of a chapter on vectors for the studiVEMINT learning materials by looking into school textbooks to explore how the concept could be introduced. While looking into many of these textbooks we observed the following:

1. The formal definition – that is either geometric as an infinite set of arrows with the same direction and length or symbolic as a triple of real numbers – is often not referred to again in the chapters following the introduction of the vector concept.
2. During the mathematical discourse related to vectors, several models (in the sense of Dörfler (2000)) of the concept are used without arguing about isomorphism.
3. The symbols labeled as vectors and used are not always conceptually coherent with the axiomatic definition of a vector space.

These observations led us to question which of the many representations of a vector that were labelled as vectors at school the students actually consider to be a vector. This question was examined by analyzing the common German school textbooks dealing with the vector concept and by administering a short test to university freshmen at the beginning of their university studies. The results are presented in this paper.

Theoretical background

Although every formalized mathematical concept has a precise definition, students need to give it a meaning by operating with the concept (maybe just mentally) in order to understand it. Tall and Vinner (1981) use the term *concept image* to describe all associations students may have acquired by operating with it. These include examples, counterexamples, visualizations as well as properties of the concept. In order to specify the concept with words it has a *concept definition*. This can either be the formal definition accepted by the mathematical community, or student's reconstruction of a definition of the concept from their *concept image* (more precisely from the parts of the concept image that were activated during this reconstruction process, which Tall and Vinner (1981) call *evoked concept image*). In the latter case Tall and Vinner (1981) call it *personal concept definition.*

The formal definitions of the vector concept the students might have learned at school are the following ones, which were discovered when analyzing German school textbooks:

1. A vector is an infinite set of arrows with equal direction and length (Bigalke & Köhler, 2012; Bossek & Heinrich, 2007; Brandt & Reinelt, 2009; Weber & Zillmer, 2014).
2. A vector is a triple of real numbers or a matrix with one row (Alpers et al., 2003; Artmann & Törner, 1984; Griesel, Andreas, & Suhr, 2012; Griesel & Postel, 1990).

However, students' personal concept definitions, which they reconstruct from their concept images, may differ depending on individual experiences with the concept.

In the following we present for each of the two formal definitions of the vector concept, how they are introduced in German school textbooks and what *possible personal concept definitions* university freshmen might have, assuming these introductions formed their concept image at school from which they reconstructed their personal concept definitions. Then we discuss how the formal definitions are referred to further in the books when operating with vectors and how this might again influence the students' concept definitions reconstructed from their evoked concept image.

Analysis of books using the geometric definition of a vector

The geometric definition as an infinite set of arrows with same length and direction is often motivated by translations (Bigalke & Köhler, 2012; Weber & Zillmer, 2014), and sometimes even defined by these (Brandt & Reinelt, 2009). The translations are then represented by arrows with the same length and direction. Afterwards, students are told that all of these arrows describe the same translation and can therefore be identified as the same object (e.g., see (Weber & Zillmer, 2014)). This path would lead to the adequate concept definition of a *vector as an infinite set of arrows with equal length and direction (D1)*. However, this identification step is rather difficult as is denoted in the literature, and may result in the incomplete conception that a *vector is considered as a single arrow (D2)* (Malle, 2005). The motivation of the formal geometric definition as a set of arrows with equal length and direction may also lead students to think that a vector is a translation. While a definition of a vector

as a translation mapping operation on the whole plane is consistent with its formal concept, literature shows that translations are often understood as the motion of an object (Yanik, 2011). So the students might think of a *vector as a translation of an object or translation of a point (D3)*. Yanik (2011) also found out that the connection between a vector and a translation is often not understood, and that many teacher students thought that a vector only gives the direction of a translation. This might lead to the following misconception: *vector as direction indicator (D4)*.

Besides using translations, some books also motivate arrows in space as a quantity characterized by length and direction in physical contexts like speed or force (Bossek & Heinrich, 2007; Weber & Zillmer, 2014). The recognition that two of these arrows can be considered as the same, since only the magnitude and direction matter (e.g., for the resulting movement of an object) leads to the adequate concept definition of *a free vector, which is a quantity characterized by length and direction and represented by a free movable arrow (D5)* (Watson, Spyrou, & Tall, 2003). But since forces are normally considered as dependent also on the point of origin (Watson et al., 2003), this approach can again lead students to the consideration that *a vector is a single arrow (D2)*.

The geometric definition requires not only a lot of effort in its introduction, it is also difficult to handle afterwards. In literature, this is denoted as a lack of operability of the definition (Bills & Tall, 1998). For example the geometric definition is difficult to handle when defining vector operations because for all operations the independence from the chosen representative of the vector has to be justified. In some books, this problem is discussed (Weber & Zillmer, 2014), others ignore it, and vectors are simply identified with arrows when defining vector operations geometrically (Bossek & Heinrich, 2007). This can again lead to the conception *of a vector as a single arrow (D2)*. Another option to deal with these difficulties is highlighted in Bigalke and Köhler (2012): the addition of vectors is defined via the addition of the components in the symbolic representation as an *n*-tuple (directly after its introduction) and from then on the geometric addition only serves as a visualization. This does not result in a misconception but becomes problematic when trying to embed the geometric vectors with operations defined between triples into the formal theory of vector spaces because in the formal theory the operations have to be defined on the set, whose elements will be the vectors if the axioms are satisfied. After the introduction of the vector operations and their properties, the definition as a set of arrows (or as a translation) is not referred to again (Bigalke & Köhler, 2012; Brandt & Reinelt, 2009; Weber & Zillmer, 2014). Instead, in the following chapters on analytical geometry, single arrows and their corresponding number triples are used to describe geometric objects. This can lead to a loose connection between the formal definition and students' concept image from which they might deduce their own personal concept definition (Vinner, 2002). The resulting personal concept definitions in this case would be: *vector as a single arrow (D1) or vector as a number triple (D6).*

In summary, if the vector concept was introduced geometrically as an infinite sets of arrows with equal length and direction, the following concept definitions can be expected: *vector as an infinite set of arrows with the same length and direction (D1), vector as a single arrow (D2), vector as a translation of an object or translation of a point (D3), vector as direction indicator (D4), vector as a quantity characterized by length and direction (D5), or vector as a triple of numbers (D6).* The personal concept definitions *D1* and *D6* correspond directly to possible formal definitions of the vector concept, *D5* is also an adequate conception, in which the equivalence of arrows with equal length and direction is realized by independence from the space, *D2* and *D3* are incomplete concept

definitions (*D2* does not take into account that vectors are equivalence classes, *D3* does not take into account that a translation is a mapping on the whole plane) and *D4* is a misconception.

Analysis of books using the symbolic definition of a vector as *n*-tuples

The symbolic definition is often motivated geometrically by translations or arrows (Alpers et al., 2003; Griesel et al., 2012) or as coordinates of the points in the space (Griesel & Postel, 1990). Sometimes the symbolic definition is introduced earlier in connection with the theory of systems of linear equations (Artmann & Törner, 1984). The symbolic definition of a vector has the advantage of allowing a flexible interpretation as a point or an arrow. This can avoid the discussion about the equivalence of arrows (e.g., see Alpers et al. (2003)). However, besides the already mentioned incomplete conception of a vector as a single arrow, this flexibility can lead to another inadequate conception: *vector as a point (D7)*. The identification of vectors and points becomes problematic in higher mathematics, e.g., in the theory of affine spaces, in which they are considered different objects (Henn & Filler, 2015).

The way the vector concept is introduced in Artmann and Törner (1984) can also lead to another adequate concept definition. Artmann and Törner (1984) restrict their visualizations of vectors on points and arrows starting at the origin. If students identify the number triples with these arrows starting at the origin, they might consider *a vector as an arrow starting at the origin (D8).* These arrows starting at the origin can serve as elements of a vector space (with suitable operations defined between them).Unlike the geometric definition of a vector as a set of arrows, the symbolic definition is operable when defining vector operations and justifying their properties like the commutative law. However, some books do not mention these properties explicitly (Alpers et al., 2003; Artmann & Törner, 1984; Griesel et al., 2012). One reason, which is also noted in literature, might be their self-evidence (Harel, 2000). However, the symbolic definition can also be difficult to handle in the case of the definition of geometric concepts related to vectors such as the norm of a vector. Purely algebraic definitions of these concepts seem unnatural without further explanation (e.g., see Alpers et al. (2003)). Geometric definitions of these concepts on the other hand (e.g., see Griesel et al. (2012)), have the danger that the vector defined as a triple is again identified with just a single arrow, which is an at least incomplete vector conception.

After the introduction of vector operations, the concept of a vector is mainly used in geometrical settings (describing lines and planes in the space). This might cause students to not identify vectors with the originally defined ‘triple’ but with its geometrical representations such as *points (D4)* or *single arrows (D2)* (students might reconstruct their personal concept definitions of vectors from these representations and not from the formal symbolic definitions).

In summary, the symbolic approach can lead to two further personal concept definitions besides the intended definition of *a vector as a triple (D6)*, which have not been mentioned yet: *vector as a point (D7)* or *vector as an arrow starting at the origin (D8)*. The identification of symbolic vectors with arrows starting at the origin is not problematic because the latter ones can truly serve as objects, which the vector operations can be defined upon. The identification of vectors with points, however, can cause conflicts later in the theory of affine spaces, where these two objects have to be distinguished.

Methodology of the empirical study

Research question

On entering university, what personal concept definition of the vector concept do students have?

Data Collection

In September 2016 a short test was administered to 103 university freshmen in a mathematics bridging course at the University of Paderborn. These students were either freshmen majoring in mathematics or in mathematics for teachers at grammar schools. The pretest consisted of three open questions:

1. What is a vector?
2. Explain how you add two vectors $\vec{a}$ and $\vec{b}$.
3. Explain, why for all vectors $\vec{a}$ and $\vec{b}$ the following is valid: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

The first question was asked to identify what the students' personal concept definition of a vector in the sense of Tall and Vinner (1981) is. We did not ask for a definition because we did not want the students to try to recall the formal definition they had learned at school, but rather to specify the concept in their own words. We also did not use the term "definition" because we suspected that many students might not be familiar with the term and therefore might get confused.

The other two questions were asked to further analyze if the students used the defined objects to explain vector operations and their properties. This is important for a proper embedding of the old vector concept into the abstract notion of a vector space, which is a set with operations defined on its elements. However, this problem will be investigated later.

Data Analysis

The answers to the first question "What is a vector?" were categorized by using possible personal concept definitions deduced from the analysis of the textbooks (see theoretical background, categories *D1,…, D8)*. Furthermore, four additional categories have been added. The first one, a *vector as an element of a vector space* was added before the analysis because, although this generalization is not taught at school, it may happen that some students had heard about it (e.g., in mathematical clubs at school). The other categories depict inadequate personal concept definitions that often showed up during the analysis: a *vector as a line segment, a vector as a line* and a category containing other inadequate concept definitions not yet mentioned.

The whole typology of 12 categories is shown in Table 1. The first five categories can be considered adequate, which means that objects described in the definition can serve as concrete examples of vectors in a vector space (if suitable operations are defined on them) or if vectors are already considered as elements of vector spaces. Categories 6 and 7 contain incomplete concept definitions, categories 8 to l2 contain inadequate concept image definitions, which can be considered as misconceptions.

Category	Prototypical Examples
1. Set of equivalent arrows	"An infinite set of arrows in the space, which are equal in their length and their direction."
2. Quantity with direction and magnitude	"A vector gives information about the length and the direction of an element in an at least two-dimensional space."
3. *N*-tuple	"A matrix with n rows and just one column."
4. Arrow starting at the origin	"A vector describes a journey to a point of $\mathbb{R}^3$."
5. Element of a vector space	"An element of a vector space."
6. Translation of a point or an object	"A vector is a translation of a point in the coordinate system to another point."
7. Arrow or line segment with direction	1. "Visually: an arrow." 2. "A vector is a line segment with direction. "
8. Direction	"A vector points from a point into a direction."
9. Connection between points or line segment	1. "A connection between two points in the 3-dimensional space." 2. "A line segment in the 3-dimensional space"
10. Point	"A vector describes a point in the coordinate system."
11. Line	"A vector describes a line in $\mathbb{R}^3$."
12. Other wrong definition	"A multidimensional object, which is located in a multidimensional space."

Figure 1: Answer categories to the question "What is a vector?"

Two of the authors separately coded the data from the questionnaire. The interrater-reliability coefficient, Cohen's Kappa, was κ=0.803, which is good. Afterwards, they discussed the answers they had coded differently and agreed on a categorization.

Results of the study

The students' personal concept definitions of the vector concept that were identified form the students' answers to the question "What is a vector?" are shown in Figure 2.

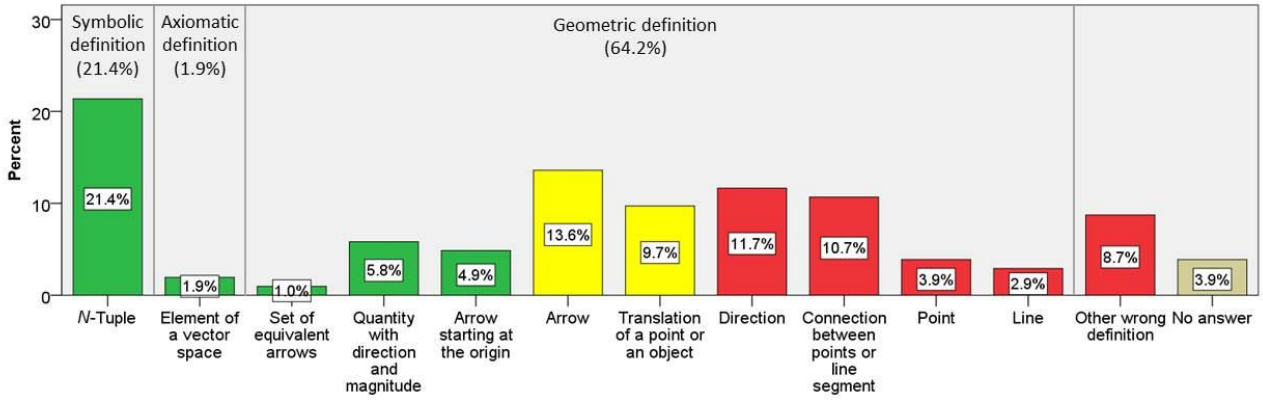

Figure 2: Students' answers to the question "What is a vector?" (N=103)

The bars of adequate personal concept definitions (which correspond roughly to possible formal concept definitions of models of the vector concept) are marked green, not fully adequate concept (i.e. they cannot be properly formalized or embedded into the abstract theory of vector space)

definitions are marked yellow, inadequate concept definitions are marked red. As can be seen in figure 2, the students had a variety of individual concept definitions of the vector concept when entering university. Most of them had a geometrical basis. However, in most cases these geometric concept definitions were either incomplete (the yellow bars, in which either the nature of a vector being an equivalence class was not mentioned or in which a vector was considered as a translation of points or objects and not as translations of the whole space) or inadequate (the red bars). Nevertheless, even the inadequate conceptions of a vector like “a direction”, “a connection between two points or a line segment” or “a point” have some properties of the adequate conceptions of vectors (e.g. if a vector is considered as a line segment, it has the property “finite length”, which is as basic property of the arrows, which represent a vector geometrically.

Conclusion and outlook on possible further research

Our study shows that the students have a variety of concept definitions of what a vector is when entering university. Thus we should keep in mind that freshmen do not come to university with a shared idea on what a vector is. The majority of students stated geometric definitions which were mostly inadequate definitions in the sense that they cannot be properly formalized or embedded into the common definition of a vector in mathematics. This indicates that it is difficult for students to fully grasp the concept of a vector. However, many students seemed to be familiar with the symbolic definition of a vector as an *n*-tuple and that it can be interpreted in manifold representations. This property of the *n*-tuple approach seems very appealing. Dealing consistently with equivalence classes including the independence from the chosen representative can be circumvented with this approach. Hence, we chose this approach for the studiVEMINT course. We utilized the connection between the symbolic and geometric representations as often as possible. However, we avoided using the geometric representations while introducing the mathematical discourse on vectors that we wanted to be consistent with the provided definition (Sfard, 2000), similar to what is required from the students in their upcoming lectures about linear algebra.

For further research we will look into the students’ answers to questions 2 and 3 more thoroughly with a more elaborated theoretical framework. We also intend to do a follow-up study to investigate the influence that the linear algebra course achieved on students at the end of the currently ongoing winter term 2016/17. Including a semiotic point of view and an analysis of textbooks from school as well as the introduction of vectors within the linear algebra course will improve the theoretical framework and provide further insights.

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What can calculus students like about and learn from a challenging problem they did not understand?

Ofer Marmur[1] and Boris Koichu[2]

Technion - Israel Institute of Technology, Faculty of Education in Science and Technology, Israel;
[1]omarmur@tx.technion.ac.il, [2]bkoichu@tx.technion.ac.il

This paper reports on part of a study regarding student learning-experiences and affective pathways in undergraduate calculus tutorials. The following question is pursued in this paper: How do the students' key affective states relate to the type of mathematical discourse conducted in class? We present and discuss two lessons where two similar problems were considered. The lessons were filmed and followed by stimulated recall interviews with nine students. Though the students in both lessons did not understand the solution to the challenging problem, they evaluated the lessons and subsequent learning experiences very differently. We suggest the difference was related to the type of discourse employed by the instructor. The lesson that evoked a negative reaction utilized only an object-level discourse. The lesson that evoked a positive reaction additionally utilized a meta-level discourse. We will call this heuristic-didactic discourse. Implications are drawn.

Keywords: Undergraduate calculus, emotional states, key affective events, discourse.

Introduction

Emotions have long been recognized to take an integral part in mathematical problem-solving activities, especially when coping with non-routine problems. However, relatively little is known on the role emotion plays in undergraduate student learning, and even more so in context of frontal tutorials. This paper is part of a wider research investigating student emotion and learning-experiences fostered by problem-solving explanations in calculus tutorials. In this paper we present and discuss 2 cases. The first case, which is at the focus of this paper, consists of students regarding a lesson containing a highly challenging problem rather positively, whilst not fully understanding the solution. Our interest in this case lies in the generally positive attitude towards this part of the lesson, accompanied by students admitting that key parts of the proof were incomprehensible, and showing disbelief in their ability to solve such problems on their own. This lesson will henceforth be referred to as Lesson-P (positive student attitude). Lesson-P especially stood out when juxtaposed with a lesson containing a similar challenging problem also not understood by students, yet their lack of understanding was accompanied by negative emotions of anger and frustration. This lesson will be briefly presented in the paper as a contrastive background and referred to as Lesson-N (negative student attitude). Thus, we were faced with the following question: how can it be that the students of Lesson-P described their learning experience in a rather positive manner, though not fully understanding the solution?

Theoretical background

Frontal teaching style of undergraduate mathematics

Undergraduate mathematics courses are typically comprised of lectures and tutorials. This paper focuses on large-group tutorials, which are lessons that present problems accompanying the theoretical material (presented in the lectures), and are taught in a traditional-frontal style (Marmur

& Koichu, 2016). The common practice of the frontal teaching style (henceforth referred to as FTS) in undergraduate mathematics education possesses pros and cons. On the one hand, there is evidence that FTS can be effective in modeling mathematical reasoning for students by "conceptual scaffolding through demonstration and worked examples" (Pritchard, 2010, p. 611). This modeling can be motivational to students, particularly when exposing the struggle that precedes the reaching of a solution (Pritchard, 2010). On the other hand, it has been argued that FTS at university level consists of a one-directional communication based on transmitting information (Biggs & Tang, 2011) and treating the students as "non-emotional audience" who are granted no room for individual difficulties (Alsina, 2002, pp. 5-6). It is not our intention to either support or oppose these claims. Rather, we recognize that FTS is widespread and will most likely not disappear in the near future. Therefore, it is vital to gain a better understanding of how students learn in this environment in order to be able to improve the system from "within", theoretically and practically, by identifying learning opportunities for students within the FTS paradigm. Lectures and tutorials comprise however only a certain percentage of the total time spent on an undergraduate mathematics course by students, and they are generally expected to spend many additional hours studying independently. Consequently, when we discuss the need to recognize and identify learning opportunities presented in the classroom, we mean not only those aspects related to the learning process in class, but also the aspects that support the learning that continues outside the classroom.

Emotions, learning, and discourse in the undergraduate classroom

In this paper we utilize Goldin's theory of local affect. Goldin (2000) defines *emotional states* as "the rapidly changing (and possibly very subtle) states of feeling that occur during problem solving" (p. 210). *Affective pathways* are regarded as a sequence of emotional states, and are linked by Goldin to mathematical cognition and heuristic processes students utilize at different stages of mathematical problem solving. Specifically, we choose to focus on what Goldin (2014) refers to as *key affective events* during mathematics learning, i.e., events "where strong emotion or change in emotion is expressed or inferred" (p. 404). Weber (2008) claims that emotional states may have a substantial impact on a student's failure or success in a high-level calculus course. In his paper, Weber demonstrates how a single and strong positive experience of success may alter a student's attitude and type of engagement with the material for the continuation of the course. Marmur and Koichu (2016) illustrate that also in a single lesson the creation of strong emotional experiences for students may significantly influence their level of focus, attention, and involvement in class.

Student emotions are examined in this paper in relation to the discourse led by the instructor in class. Theoretically, Evans, Morgan, and Tsatsaroni (2006) link emotions with discourse by regarding emotions as a "socially organised phenomena which are constituted in discourse" (p. 209). According to Sfard (2008), learning is perceived as a change in the mathematical discourse, while distinguishing between a discourse on mathematical objects, called *object-level discourse*, and a "discourse about this discourse" (p. 300), referred to as *meta-level discourse*. In relation to the FTS, the focus on the instructor's discourse finds additional support in Sfard's (2014) claim that this teaching style allows an expert to teach students how to "talk mathematics" and thus promote student learning through their introduction to a new mathematical discourse (p. 201).

Research question

The study reported on in this paper is part of a broader research on the link between student emotions and learning during calculus tutorials. This broader research focuses on characterizing classroom events students respond to during calculus tutorials, students' affective pathways and learning experiences during tutorials, and classroom learning-opportunities as reflected by the students' own point of view. This paper addresses these issues by concentrating on the following question: How do students' key affective states relate to the type of mathematical discourse conducted in class?

Method

Context and participants

The two lessons reported on in this paper were of two separate tutorial groups that were part of the same second-semester calculus course. The course was highly demanding and challenging, and was attended by students from the computer science faculty. Both lessons were attended by approximately 50 students. The instructors (henceforth referred to as Instructor-P and Instructor-N) were both experienced instructors with a good reputation at the university.

The problems

Lesson-P took place during the second half of the semester and was regarding the topic of the two-variable Riemann integral. For this lesson the students were asked to prove that the function below is Riemann integrable in two variables on [0, 1] × [0, 1] (and the value of the integral is 0).

$$f(x,y)=\begin{cases}\frac{1}{q}, & x\in\mathbb{Q}\ and\ y=\frac{p}{q}\in\mathbb{Q}, \frac{p}{q}\ in\ lowest\ terms, q>0\\ 0, & x\notin\mathbb{Q}\ or\ y\notin\mathbb{Q}\end{cases}$$

Lesson-N took place during the first half of the semester and was regarding the one-variable Riemann integral. The problem of interest was to prove that the "popcorn function" below (also known as "Riemann's function") is Riemann integrable on [0, 1] (and the value of the integral is 0).

$$f(x)=\begin{cases}\frac{1}{q}, & x=\frac{p}{q}\in\mathbb{Q}, \frac{p}{q}\ in\ lowest\ terms, q>0\\ 0, & x\notin\mathbb{Q}\end{cases}$$

Both instructors referred to the definition of a Riemann-integrable function. The problem in Lesson-P was planned as a follow-up two-variable version of the "popcorn function".

Data collection and analysis

Both lessons were filmed by the first author of this paper who also took notes during the lessons. Subsequently, individual stimulated-recall interviews were conducted with nine volunteering students: five on Lesson-P and four on Lesson-N, each student participated in only one of the two lessons. The interviews were conducted over a nine-day period after the lessons. Stimulated recall was the chosen methodology as it presents a non-intrusive method to help students "relive" the lesson and reflect upon their thought processes during its course (Calderhead, 1981). During the interviews, the students were presented with an approximately 20-minute video excerpt of the filmed lesson in which the problem of interest was taught. They were explained that the video served as a tool to help them "relive" the lesson, and were instructed to stop the playback whenever they had a particular recollection of what they thought or felt at that moment. During this part of the interview the interviewer occasionally asked clarifying questions, mainly in the form of "can you explain *why* you

thought/felt this way at that specific moment?" After watching the filmed episode, the students were asked follow-up questions regarding the problem, lesson, and course, the main ones being: a) Was the problem memorable for you, and if so, in what way? b) What were you pleased and displeased with during the lesson? and c) What is your general attitude towards the course? The interviews were audio-recorded and ranged in length from 40 to 65 minutes, depending on the level of detail shared by the student.

For the data analysis we utilized a "general inductive approach" (Thomas, 2006) that allowed us to coordinate the raw data into a brief summary that addresses and explains the "underlying structure of experiences or processes" (p. 238) most apparent in the data. The goal of the analysis was to identify: 1) students' key affective states as indicators of potential-learning or obstacle-for-learning episodes; and 2) types of mathematical discourse in the classroom. Specifically, we focused on: 1) episodes where all students stopped the video to reflect on the lesson; and 2) repeated statements or themes (whether within a specific interview or between interviews). Subsequently, we continued with a recursive process of going back and forth between the video observations and the student interviews in order to refine our conclusions. Although students were asked in each interview to express their emotional states *during the lesson itself*, it should be recognized that the accounts shared by the students might have been of their emotions *during the interview*. However, we considered this issue as a point of strength for the research, rather than a limitation. Such a selective recollection of emotions may shed light on the process, addressed by Goldin (2014), of how in-the-moment emotional states transform into longer-term attitudes and beliefs, and on how this process shapes the mathematical learning. Accordingly, while adopting Goldin's (2014) terminology of "key affective events", and in line with Marmur and Koichu (2016), we regarded: a) the most *memorable* emotional states of students as key emotional states that shape their overall learning experience; and b) student expressions of *strong* emotions as indicators of potential-learning or obstacle-for-learning episodes in class.

Findings

Due to the scope limitations of this paper, in the Findings section we will focus on: a) the instructional episodes most prominently addressed in the student interviews; and b) student thoughts and emotions regarding these episodes. Additionally, the Findings section will predominantly focus on Lesson-P as the main explored phenomenon, utilizing Lesson-N as a contrastive background to illustrate and emphasize certain aspects of the findings.

Lesson-P

After having presented the problem to the students, Instructor-P said: "Let us first try to understand what's going on here. [...] I want us to make some observations." The instructor reminded the students of the one-variable Riemann (popcorn) function, after which the following 10 minutes were focused on what was titled on the board as "Observation" and "Observation no. 2". The first "Observation" entailed that for a fixed x we get $\int_0^1 f(x,y)dy = 0$ and therefore the following iterated integral equals zero: $\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = 0$. After having written the title "Observation no. 2", Instructor-P asked the students: "What happens if I fix y?" The students participated in the discussion regarding a fixed $y \notin \mathbb{Q}$ and a fixed $y \in \mathbb{Q}$, the latter giving the Dirichlet function $D(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. This led to the

conclusion that the integral $\int_0^1 f(x,y)dx$ does not exist and therefore it is impossible to calculate the (opposite-direction) iterated integral $\int_0^1 \left(\int_0^1 f(x,y)dx\right)dy$.

These observations led the class to two conclusions regarding the function: 1) Its *double* integral exists, yet the *iterated* integral (in one of the directions) does not; 2) It demonstrates the necessity of the *continuity* assumption in Fubini's theorem which allows us to calculate the double integral $\iint_D f(x,y)dxdy, D = [0,1] \times [0,1]$, as the iterated integral $\int_0^1 \left(\int_0^1 f(x,y)dx\right)dy$. These conclusions were in fact surprising for the students, as conveyed in their interviews. It was only then that the instructor admitted that everything discussed so far "does not yet answer our problem". At that point he wrote: 'So how do we solve?', and said: "In such a case we need to follow the definition."

We interpreted the students' general attitudes towards that part of the lesson dealing with this problem, as rather positive in the following manner. At different levels of explicitness, all students claimed that the lesson and teaching were good, while mainly pointing at the problem presented above. For example, even before knowing what the interview was about exactly, Student A said: "You came to a very special lesson […] The instructor chose a non-standard problem [the discussed one] to convey his messages. I really loved it". Student B said: "You came to a good lesson, really!", and referred to the discussed problem as "*the* problem" of the lesson (emphasis in intonation). Student C said: "The lesson was interesting. The lesson was clear. The first function [the discussed problem] was different and new." Students A, C, and D called the problem beautiful. Students D and E claimed the problem was good since it prepared them for similar problems that may appear in the exam. However, all students admitted the problem was difficult and challenging, and the unexpected impression we got was that the students did not fully understand the solution, nor expressed confidence in their abilities to solve similar problems independently. For example: Student A admitted that a key line in the (actual) solution seemed to him like gibberish; Student C found the same line to be full of incomprehensible transitions which she referred to as "jumps"; Student B referred to the same line with: "What??? He said it, so it is probably true", and later in the interview admitted: "If this problem was in the exam, I wouldn't have succeeded solving it"; and Student E hoped such a problem would not appear in the exam and hoped the lecturer did not have such a "dark heart". However, these statements were not accompanied by any explicit expressions conveying negative emotions. Additionally, such statements barely appeared in other parts of the interviews, and did not even appear at all when the students were explicitly asked what they were displeased with during the lesson.

It is towards the opening "observations" part of the lesson that the students mainly expressed positive opinions on what had happened. Additionally, all students claimed that even though the "observations" part was not directly utilized in the actual solution, it was nonetheless an *indispensable* part in support of their learning. They supplied us with a variety of reasons: it exposed the thinking process of how to reach a solution; it allowed time to think about the problem; it included a counter-example for claims they thought were true; it imitated what they would actually do if they were to start solving the problem on their own (i.e., try to calculate the iterated integral); it helped them understand the problem through step-by-step analysis; it gave room for "mathematical play" where the goal was not merely to solve a problem; and it demonstrated that even if an attempt for a solution did not succeed, they should just try again in a different way. Some students clearly linked their

positive attitude with the following didactic aspect of the mathematical discourse led by the instructor in class. Student A shared that he really loved the approach taken by Instructor-P during the "observations" part. The student described that the instructor put himself in the position of a student, approaching the problem through their eyes, and instead of immediately solving the problem because he was already familiar with the solution, he started "playing" with it with the aim of seeing where this will lead them. A supportive angle is given by Student D who said that during the lesson Instructor-P really tried to give the impression that he did not already know the answer, but rather was trying to solve the problem with them. She said that only once she was convinced he was not "fooling" them, she started thinking with him. The instructor, however, was indeed familiar with the solution, and Student D admitted that only when watching the lesson again during the interview, she realized how planned and structured the lesson was.

Lesson-N

After having written what needs to be proven according to the definition of a Riemann integrable function, Instructor-N told the students: "At the beginning you may experience some lack of understanding. Once we reach the end [of the solution] you'll understand where I took the numbers from that initially might have looked a bit weird." Then he wrote the following: Choose n_0 such that $\frac{1}{n_0}<\frac{\epsilon}{2}$. This is a key moment where all 4 interviewed students stopped the video and expressed similar thoughts and strong dissatisfactions. The main criticism the students conveyed is expressed in the following interview excerpt: "It really bothers me that he reads the solution by the order of the proof and not by the order of how you think about the proof. [...] At the end it all works out. But it doesn't help me with how to solve a problem." The student then continues while expressing her anger: "It really pissed me off. He pulls the answer out of a hat, and I don't know how he got to it."

These negative opinions towards the lesson, while pinpointing the underlying reason to the key moment presented above, continued and repeated throughout all interviews. The students claimed that also at the end of the lesson they did not understand the solution, and that the promise made by the instructor at the beginning was left unfulfilled.

Discussion

While the case of Lesson-N demonstrates that students can possess negative emotions towards a solution they did not understand, the case of Lesson-P, containing a similar problem, demonstrates that a lack of understanding can still be accompanied by positive student emotions. Both lessons contained episodes focused on the *solving* of a challenging problem, which we suggest to regard as an object-level type of discourse. However, the positive emotions in Lesson-P were mainly directed towards that part of the lesson focused on *how* to approach a challenging problem, which we regard as a meta-level type of discourse. While other explanations for the students' positive attitude towards Lesson-P are certainly possible, our interpretation is based on what we found to be most prominently conveyed by students during the interviews. Furthermore, we suggest that not only did students appreciate this meta-level discourse, as expressed in their interviews, but that this discourse may have also had a neutralizing effect on the potential negative emotions related to not understanding the solution.

The meta-level discourse in Lesson-P revealed a heuristic approach on how to tackle a challenging problem. On the one hand, the discourse was planned and monitored from an expert's point of view,

which may be viewed as a teacher's learning goal (Simon, 1995) that did not coincide with the declared main goal of solving the problem. In the case of Lesson-P, the "observations" part did not constitute a directionless exploration, but rather led to the conclusions mentioned in the findings. On the other hand, as also regarded by the students themselves, the discourse was led by the instructor through a student's point of view, considering students' cognitive and affective needs, their ways of thinking, their assumed misconceptions, and the steps they would most likely take. We call such a discourse, presenting heuristics monitored from an expert's point of view yet derived from a student's point of view, a *heuristic-didactic discourse*. In the case of Lesson-P, the heuristic aspect of the discourse may be viewed in line with what Featherstone (2000) refers to as "mathematical play", which puts emphasis on the act of exploring rather than solving, and may support the creation of a zone of proximal development, giving guidance to the learning student. The didactic aspect of the discourse may be viewed in line with what Jaworski (2002) refers to as "harmony" between "mathematical challenge" and "sensitivity to students" (both their cognitive and affective needs) in order to help students make mathematical progress. This is one example of a heuristic-didactic type of discourse, and we call for further research on characterizing different types of heuristic-didactic discourses in the undergraduate classroom.

In practice, the presented study suggests that university students wish for a more heuristic-didactic discourse to be held in the undergraduate mathematics classroom. In simple terms this means that it is necessary for students to get "tools" on how to approach a challenging problem on their own. In the presented lessons, not only were students satisfied when a heuristic-didactic discourse took place, students also showed strong emotional responses of anger and frustration when this need was not fulfilled. Furthermore, even though Lesson-P could have been improved by the students also understanding the solution better, it clearly demonstrated that the learning induced by the heuristic-didactic discourse was perceived by the students as the most valuable kind of learning, even at the expense of not fully understanding a solution. Sfard (2008) regards meta-level learning as a change in meta-rules of the discourse, while claiming that this change is not likely to be initiated by students on their own. Accordingly, meta-level discourse in class may serve as an initial point of aid for students to continue a meta-level learning-process at home. All this implies that lecturers and instructors should consider paying more didactic attention in revealing to students *how* they came up with their solutions and proofs. This learning-opportunity may be implemented in the common undergraduate frontal teaching style and could supply valuable tools for the learning process that the students are required to continue independently.

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"What you see and don't see, shapes what you do and don't do": Noticing in first year mathematics lectures

Maria Meehan[1], Ann O'Shea[2], and Sinéad Breen[3]

[1]University College Dublin, School of Mathematics and Statistics, Ireland; maria.meehan@ucd.ie

[2]Maynooth University, Department of Mathematics and Statistics, Ireland; ann.oshea@nuim.ie

[3]Dublin City University, School of Mathematical Sciences, Ireland; sinead.breen@dcu.ie

We report on an analysis of 'decision points' that occurred during first year undergraduate calculus lectures. We analysed 135 accounts written by three lecturers concerning their own teaching; these accounts were written during a professional development project on employing the Discipline of Noticing (Mason, 2002). We classified the decision points in these accounts into eight categories. Furthermore, the triggers of these moments were identified and classified into seven categories; of these, the majority (58.2%) arose as a result of the lecturer monitoring either her own practice or the students' engagement.

Keywords: Teacher noticing, decision points, mathematics lecture, university mathematics.

Introduction

> Today I was happy with my 'performance' from a teacher-centred perspective as the lecture evolved: I felt I was coherent, explained and connected ideas well, used multiple representations of concepts and built on students' prior knowledge, However, I realised more than halfway through the class that my lecture was just that – very teacher-centred! I tried to rectify this but was not happy that my attempts were successful. (Lecturer C)

Lecturer C speaks about a tension inherent in large-group mathematics lectures between teacher-centered and student-centered methodologies, and this is probably familiar to anyone who has taught such a course. It also highlights the dilemma that faces a lecturer when it occurs to her mid-way through a lecture that she might deviate from her original plan. The change of plan can feel like a risky strategy, especially when working with very large groups of students; however research on mathematics teaching at school level indicates that rich learning opportunities can arise from decisions to change direction (Rowland and Zaskis, 2013). It is important then, that lecturers are aware of opportunities that present themselves, for the simple reason that the quote in the title, borrowed from Schoenfeld (2011, p. 228), suggests. It is also important that lecturers develop the skills to take advantage of these opportunities as they arise. In this paper, we explore the opportunities to make decisions that might arise in a mathematics lecture; we have named instances when a non-trivial choice between alternative courses of action could be taken 'decision points'.

The Discipline of Noticing developed by Mason (2002) consists of "a collection of practices which together can enhance sensitivity to notice opportunities to act freshly in the future" (p. 59). Mason notes that practitioners, of necessity, form habits in order to deal with issues that arise in their everyday practice. The practices described by Mason provide educators with techniques to assist them in developing the dual abilities to notice key events in the classroom, and secondly to have possible actions come to mind in-the-moment in order to respond to these in non-habitual ways.

Mason (2002, pp. 33-34) distinguishes between levels of noticing, from ordinary-noticing, through to marking, and finally recording. "Ordinary-noticing" is where a person's memory of something can be jogged if another person remarks upon it; "marking" is where someone has taken sufficient notice of something to "re-mark" upon it at a later stage; and, finally, "recording" is where one records or makes a note of something one has noticed, usually in writing. To do the latter, Mason (2002, p. 46) advocates the practice of writing "brief-but-vivid accounts". These are brief notes which give an "account-of" an incident, rather than an "account for" as Mason explains:

> To *account-for* something is to offer interpretation, explanation, value-judgement, justification, or criticism. To give an *account-of* is to describe or define something in terms that others who were present (or who might have been present) can recognize. (p. 41)

The data for this paper comes from a set of brief-but-vivid accounts written by the three authors on incidents that occurred during teaching over a two-year period when they took part in a project aimed at using the Discipline of Noticing to study their own teaching (see Breen, McCluskey, Meehan, O'Donovan & O'Shea (2014) for more details).

In this paper we will present an analysis of the accounts which relate to large-group teaching of first year modules; in particular we will endeavour to answer the questions: What types of decision points can occur during a mathematics lecture? What triggers these decision points?

Literature review

Teacher noticing has been receiving attention in the research literature recently, see for example the research studies included in the book edited by Sherin, Jacobs & Philipp (2011). In presenting an overview of these studies, the editors observe that while the conceptualisation of teacher noticing may vary, it is generally considered as consisting of one or both of two main processes. The first process, "attending to particular events in an instructional setting" (p. 5), relates to where the teacher does (and does not) place her attention in the classroom setting. The second process they describe as "making sense of events in an instructional setting" (p. 5) and note that some researchers conceptualise this process only as "interpreting" (p. 9) what is noticed, while others view it as "both interpreting and deciding how to respond" (p. 9).

Instances when a choice presents itself during teaching are labeled "contingent moments" by Rowland, Huckstep, and Thwaites (2005) in their work on the Knowledge Quartet. At these moments teachers have to think on their feet and possibly deviate from the planned lesson. Rowland, Thwaites and Jared (2015) identified three types of triggers of contingent moments in their study of mathematics teaching: responding to student ideas; teacher insight; and, responding to the availability of tools and resources. Teachers may deal with the first trigger in one of three ways: ignore; acknowledge but put aside; acknowledge and incorporate (Rowland and Zaskis, 2013).

Schoenfeld (2010) has developed a theory to explain what influences the decisions an individual makes when carrying out a particular task. He has applied his theory in particular to decision-making in mathematics classrooms. He proposes that decisions a teacher makes while teaching are a function of her resources, orientations and goals. Although "resources" is a broad term, Schoenfeld classifies the knowledge that a teacher possesses as being a key component of the resources she brings to the classroom. He uses "orientations" to encompasses one's "*dispositions, beliefs, values, tastes,* and

preferences" (p. 29). And while the term "goal" is self-evident, Schoenfeld notes that an individual may pursue a particular goal "simply in the service of other goals" (p. 20).

Most of the research studies discussed above in relation to noticing, contingent moments, or decision-making in the classroom have been conducted at the school level. An exception is the work of Barton, Oates, Paterson, and Thomas (2015) and colleagues in New Zealand who use Schoenfeld's (2010) theory to discuss taped video excerpts from participating mathematicians' and mathematics educators' lectures as a means of engaging in professional development on teaching practice. However in terms of noticing and the occurrence of decision points in lectures, there is little research. Indeed to many it might seem like traditional lectures provide few opportunities for contingency. However McAlpine, Weston, Beauchamp, Wiseman and Beauchamp (1999) report on a study of monitoring of student cues by university lecturers. The lecturers in this study were found to attend to four types of cues: student written, student verbal, student non-verbal, student state. McAlpine et al. (1999, p. 117) posit that the lecturers had a *corridor of tolerance* for these cues and a decision to change practice was only taken when the cue lay outside of this corridor. In contrast to the work of both Schoenfeld (2010) and McAlpine et al. (1999), we focus here on the opportunities for decision-making that arise in lectures rather than the process of decision-making itself.

Methodology

The authors are lecturers of mathematics at three different universities in Ireland. Each has a doctorate in mathematics or applied mathematics and has a minimum of fifteen years' experience of teaching mathematics at the tertiary level. Between them they have taught mathematics classes from first year undergraduate through to postgraduate level, and have experience of teaching students in class sizes ranging from single figures up to a few hundred students. In 2010/11, along with two other colleagues, they embarked on a project aimed at reflecting on their teaching using the ideas and philosophy described by Mason (2002). As part of this process, over the course of two years, they engaged in writing brief-but-vivid accounts of incidents or moments that occurred in relation to their teaching. See Breen et al. (2014) for further details.

The accounts of all five members of the group were collected and a general inductive approach (Thomas, 2006) was used to identify themes in a sample of the accounts. We noticed that many accounts described instances where the lecturer was faced with a decision about what to do next, or, instances where an opportunity to make a decision that might change the course of the lecture or discussion was implicit. We labeled these moments *decision points* (DPs). In order to focus specifically on these moments, all accounts that did not specifically deal with lecturing were removed. The first and second authors (AOS and MM) independently analysed all the accounts to both identify and code the DPs. After some discussion they agreed on the identification of DPs and the codes assigned. In addition, AOS identified the triggers. Then all five members worked through all the accounts to confirm their agreement or express disagreement with the DPs and triggers identified and the codes assigned. By the end of this process, the group had reached a consensus. AOS then grouped the codes into categories, and the third author (SB) examined these for consistency. In some of the accounts, the *action taken* by the lecturer as a result of a DP was recorded, however we will not discuss the identification or classification of these here.

In this paper we present findings on the DPs and triggers from accounts written by each of the three authors while lecturing a first year mathematics class. In total there were 135 accounts with 141 DPs identified, as some accounts contained more than one DP. These DPS were classified into 8 categories. In order to provide a context for the accounts, we note that Lecturer A taught Calculus to a group of approximately 200 students consisting of both mathematics and finance students in the first semester of both 2010/11 and 2011/12. Lecturer B taught mathematics to a group of over 200 first-year business students in the first semester of both 2011/12 and 2012/13, while Lecturer C taught Calculus to a first year class of approximately 50 mathematics students for the duration of the academic years 2010/11 and 2011/12. The format for each course consisted of either 2-3 lectures per week given by the lecturer, to which all students were required to attend.

While all three lecturers engaged in what might be considered *lecturing* - that is, the lecturer speaks to the whole class, and perhaps writes on a projector or board while the class is expected to remain silent - they also engaged in initiatives aimed at increasing student participation. These fall into two categories – *whole class question or discussion* and *class activity.* The former relates to where the lecturer asks the whole class a question or attempts to conduct a whole-class discussion, while by class activity we mean an activity that students are expected to engage in during class, usually in small groups. As a final part of the analysis MM classified each account containing a DP as occurring in either Setting 1 (S1) – lecturing; Setting 2 (S2) – whole-class question or discussion; and, Setting 3 (S3) – class activity. We now present the findings.

		Lecturer A			**Lecturer B**			**Lecturer C**			**Total**
	Decision Points	**S1**	**S2**	**S3**	**S1**	**S2**	**S3**	**S1**	**S2**	**S3**	
DP1	How to engage students?	5	0	3	1	0	0	4	2	0	15
DP2	How to respond to students' questions, answers, or comments?	8	0	1	0	4	0	3	0	3	19
DP3	How to ask questions to gather information?	4	0	2	1	0	0	1	1	1	10
DP4	How to deal with disruption?	1	0	0	18	1	1	2	1	0	24
DP5	How to conduct class activity or discussion?	0	0	3	0	2	0	2	2	7	16
DP6	How to deal with students' mathematical difficulties?	4	0	2	0	17	0	1	7	5	36
DP7	What to do next in the lecture?	5	0	0	1	2	0	4	3	2	17
DP8	Other	1	0	0	2	1	0	0	0	0	4
	Total	28	0	11	23	27	1	17	16	18	141

Table 1: Decision Points by Lecturer and Setting

Findings

In Table 1 we present the categories of DP with frequency by lecturer and setting. Lecturer A mainly wrote accounts about S1 and S3, Lecturer B about S1 and S2, while Lecturer C wrote about all three.

This is perhaps not surprising as she had a much smaller class. Over a quarter of all DPs identified belongs to the category "How to deal with students' mathematical difficulties?" (DP6, n=36, 25.5%). It is noteworthy that 31 of these occurred in Settings 2 and 3. The next largest category of DPs is "How to deal with disruption?" (DP4, n=24, 17.0%) with three-quarters of these attributed to Lecturer B in the lecture setting. The third largest category is "How to respond to students' questions, answers or comments?" (DP2, n=19, 13.4%) and while these DPs may be expected to occur in Settings 2 and 3, it is interesting to note that just under half are attributable to Lecturer A in the lecture setting. The category "What to do next in the lecture?" (DP7, n=17, 12.0%) contains DPs relating to opportunities for decisions that present themselves when moving from a whole-class question/discussion or class activity, back to the lecture setting. Four of the DPs did not seem to fit in any of the categories identified and were grouped as "other".

Each DP was found to have an associated trigger and in Table 2 we present the categories of triggers with frequency by lecturer and setting.

		Lecturer A			**Lecturer B**			**Lecturer C**			**Total**
	Triggers	**S1**	**S2**	**S3**	**S1**	**S2**	**S3**	**S1**	**S2**	**S3**	
T1	Lecturer monitors aims/goals	3	0	1	1	0	0	2	1	3	11
T2	Lecturer monitors practice	6	0	1	2	1	0	3	1	0	14
T3	Lecturer monitors student nonverbal	9	0	0	0	5	1	8	4	0	27
T4	Lecturer monitors absence of student verbal	0	0	6	0	0	0	0	0	4	10
T5	Lecturer monitors disruptive behavior	1	0	0	18	0	0	0	1	0	20
T6	Student question or comment or answer	8	0	3	2	19	0	6	9	9	56
T7	Other	1	0	0	2	0	0	0	0	0	3
	Total	28	0	11	25	25	1	19	16	16	141

Table 2: Triggers by Lecturer and Setting

The first five triggers listed in Table 2 (T1-T5) are as a consequence of the lecturer monitoring her aims/goals for the class, her practice, what the students were (not) doing, which students were not answering questions or contributing, and student behaviour. These account for 58.2% of triggers identified. In terms of the triggers identified when the lecturer monitored students (T3-T5) there are some similarities between the cues identified in McAlpine et al. (1999) and those described in the accounts relating to these triggers. The category labeled "Student question or comment or answer" (TP6, n=56, 39.7%) contains student-initiated triggers and is the largest of the trigger categories. The "Other" category relates to triggers that are neither lecturer- nor student-initiated and relate to issues such as a cold room or poor attendance due to bad weather. These findings are similar to those of Rowland et al. (2015) who in their study classified triggers of contingent moments as emanating from the teacher, the students, or resources and tools, with the latter category accounting for far fewer triggers than the first two. Exploring links between DPs and triggers, not surprisingly almost all DPs

categorized as DP2 (18/19) and most of those as DP6 (29/36) arose from T6 triggers. Over half of DP3 and most of DP4 (20/24) resulted from T1 and T5 triggers respectively.

We now present some examples of accounts featuring DPs and associated triggers that occurred in the lecture setting. We note that the account by Lecturer C at the start of the paper is an example of a lecturer attempting to make her lecture more student-centred in order to engage students (DP1) as a consequence of monitoring her practice (T2). In the following account by Lecturer A, the trigger for the decision point about how to engage students (DP1) is the lecturer monitoring what the students are (not) doing (T3) while she is lecturing.

> I continued the introduction to limits today. I was doing a lot of talking and I realized that people weren't taking anything down. I tried writing more explanations on the pictures I was drawing, so hopefully it will make more sense when they look at it again. (Lecturer A)

Similarly the account by Lecturer C describes how noticing students' expressions (T3) during a lecture, prompts her to explain what mathematicians do in an attempt to engage the students (DP1):

> On noticing students exchanging glances when I asked them why they would attempt to solve a particular problem in a particular way, I was prompted to reiterate what "doing mathematics" (at this level) entails. (Lecturer C)

The next account illustrates how a student asking a question (T6) during a lecture results in a decision point for the lecturer about whether to review material already covered (DP2).

> I found the limit of $\sin(2x)/\sin(3x)$ as $x \to 0$ when a mature student, who asks lots of questions, asked why you couldn't just cancel the sin's and the x's to get 2/3. We proceeded to discuss the meaning of the term $\sin(2x)$ and the difference between multiplication and composition. I had spoken about this before but felt talking about it here was useful. (Lecturer A)

When a lecturer monitors student behaviour (T5) during a lecture, decision points about how to deal with disruptions (DP4) may arise as illustrated in the following account by Lecturer B:

> Shortly into the lecture I ask a group of four students to stop talking. Minutes later I tell two other students to stop talking. Some minutes later, I ask the first group to stop talking again. I look at the class – most of them are staring straight at me and not moving. I realize I am nagging and stopping the lecture for the sake of a few "talkers". As I continue to write and talk, I hear whispering coming from various parts of the theatre and my explanation falters. I decide I can't get any more annoyed. I put up a question and ask the class to work on it. (Lecturer B, T5, DP4)

Unlike the accounts so far which occurred in the lecture setting, the following account takes place during a class activity. During the activity the lecturer experiences a decision point concerning how best to respond to student questions (T6 and DP2). But on completion of the activity another decision point arises about what to do next in the lecture (DP7).

> I put up an exercise for the students to work on. I remind them that the first part is revision – they have to find the profit function. As I walk around the theatre a student asks: "What are the fixed costs?" I remind him that this example is different to the one I did earlier in class. Another asks: "Do I multiply this function by q to get total revenue?" "No, you are given total revenue, you don't have to find it", I reply. "What is q?" another asks. I feel deflated – this is revision. I planned to

finish the exercise today, but instead, show them how to get the profit function and complete the exercise in the next lecture. (Lecturer B)

Discussion and conclusions

We wish to discuss three points in this section. The first concerns decision points that arise while lecturing. One might assume that in the traditional sense of a lecture, a lecturer delivers from a pre-prepared script and unless a student asks a question, the lecturer will not deviate from the script. However there is some evidence from the three lecturers' accounts on their practice that indicates that even while lecturing in the traditional sense, they monitor their aims for the class, their own practice, and how the student cohort are acting, and that this monitoring leads to the occurrence of many of the decision points. About one third of the triggers for decision points in our study were student initiated, (T6 in Table 2) in contrast to the findings of McAlpine et al. (1999), but in comparing one must be cognizant that our methodology differs from theirs. While we acknowledge that the fact that the three lecturers undertook a professional development project on using the Discipline of Noticing (Mason, 2002) to improve their practice may mean that they are not typical, it would be interesting to explore further what mathematics lecturers focus on while in a lecture as well as gathering more data to illuminate the relationship between triggers and decision points.

Secondly we observe from our findings that decision points relating to how to address students' mathematical difficulties usually emerged in the context of a whole class question or discussion or a class activity. This is perhaps not surprising but does highlight the importance of including such activities during a lecture in order to assess students' mathematical understanding. However it is also worth pointing out that over half of all the decision points identified in this study were as a consequence of such an activity, which may suggest an increased cognitive load for the lecturer who engages in such activities.

Our final point relates to the methodology used in this study. Our analysis of the settings seems to indicate the lecturers' individual preferences for pedagogical techniques and activities however we cannot use our data to draw conclusions about the proportion of class-time they each spent in the three settings, or make general claims about the relationship between the settings and the occurrence of decision points. This is because the lecturers in this study had complete autonomy over what incidents they chose to write about. They wrote about what mattered to them and many of the accounts relate to incidents where the lecturer felt unsure about what to do, or uncomfortable about a decision she had made. In any lecture, there may have been a multitude of more interesting moments worthy of noting, but either they did not notice them, or chose not to write about them. In this way there is a parallel with the findings of Barton et al. (2015). In choosing excerpts from a video-taped lecture for group discussion, the authors note that: "Counterintuitively, lecturers chose parts in which they felt less comfortable" and in group discussions "frequently chose to focus the group's attention on interludes in the lecture when unexpected decisions were made" (p. 152).

We return to the quote from Schoenfeld (2011): "Noticing is consequential – what you see and don't see shapes what you do and don't do" (p. 228). We suggest that the use of the Discipline of Noticing (Mason 2002) can help lecturers to identify opportunities for making (possibly different) decisions in their lectures. Individually the process also highlighted for each of us different aspects of our practice that we wanted to work on – for Lecturer A it was how to make her lectures more student-centered,

for Lecturer B the issue was how to deal with disruptive behavior, for Lecturer C, how to address mathematical difficulties effectively and sensitively. We also recommend that lecturers, as a professional development exercise, write a selection of brief-but-vivid accounts and discuss them in a group setting using Schoenfeld's theory (2011) to frame the discussion.

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To be or not to be an inflection point

Regina Ovodenko [1] and Pessia Tsamir [2]

[1]Center for Educational Technology, Tel-Aviv, Israel; reginao@cet.ac.il
[2]Tel-Aviv University, Tel-Aviv, Israel; pessia@post.tau.ac.il

This paper describes students' grasp of inflection points. The participants were asked to define inflection points, to judge the validity of related statements, and to find inflection points by investigating (1) an algebraic representation of a function, and (2) the graph of the derivative. We found that participants provided their own "filtering conditions" to determine or deny the existence of an inflection point In order to analyze participants' conceptions of inflection points, we used the lenses of Fischbein's theoretical framework.

Keywords: Inflection point, formal knowledge, algorithmic knowledge, intuitive knowledge.

Introduction

Functions receive considerable attention in secondary school, commonly in algebra and calculus lessons. Inflection point is one of the function-related notions addressed in high school and in further mathematics studies. In preliminary studies, we found some indications of common erroneous conceptions of the notion (e.g. Ovodenko & Tsamir, 2005; Tsamir & Ovodenko, 2004; 2013). These findings encouraged us to expand our research regarding the grasp of the notion of inflection point, and regarding possible sources of related common errors, while addressing a larger and diverse population who was given richer types of tasks (elaborated upon in the methodology section). In this paper, we present part of the findings from the extended study (Ovodenko, 2016). The research tools were designed and the findings analyzed with reference to a number of theoretical frameworks, including Fischbein's theory of algorithmic, formal, and intuitive components of mathematical knowledge (Fischbein, 1987, 1993a) and his theory of figural concepts (Fischbein, 1993b). Specifically, our research questions are: In the students' opinion, (1) When is a point an inflection point? (2) When is a point a non-inflection point?

What does research tell us about students' conceptions of inflection points?

Literature gives some indications of students' difficulties when using the notion of inflection point. Some researchers (e.g. Carlson, 1998) have reported that students tend to use fragments of phrases taken from earlier-learned theorems, such as "if the second derivative equals zero [then] it is an inflection point" when solving problems in the context of dynamic real-world situations.

Other researchers have reported that early experiences with the tangent to a circle contribute to the creation of the intuitive grasp of the tangent as a line that touches the graph only at one point and does not cross it (e.g. Vinner, 1982). This intuition was evoked when students were asked to identify and draw a tangent line to a curve's points that included non-differentiable and differentiable inflection points (e.g. Biza & Zachariades, 2010).

In a previous study, we examined students' conceptions of inflection points, in which we came across a novel tendency to regard a "peak point" as an inflection point (e.g. Tsamir & Ovodenko, 2004). We found tendencies to regard $f'(x) = 0$ as a necessity for the existence of an inflection point (Ovodenko & Tsamir, 2005), as well as tendencies to view $f'(x) \neq 0$ as a necessary condition and

$f''(x) = 0$ as a sufficient condition for an inflection point (Tsamir & Ovodenko, 2013). Consequently, we designed a large study to examine students' conceptions of the inflection point when solving a rich variety of problems. Here we report on part of the findings (Ovodenko, 2016).

Methodology

The research population included 223 participants from different educational levels of mathematics: high school students studying mathematics at the intermediate level, high school students studying mathematics at the advanced level, university students and university graduates (the latter majoring in mathematics-rich subjects, such as mathematics, computer science, and electronic engineering). All participants had studied the notion of inflection point during their calculus lessons.

According to the Israeli mathematics curriculum for secondary schools, an inflection point is defined as a point on a curve at which the curve changes from being concave up to concave down or vice versa, usually relating to functions that are at least twice differentiable in a small neighborhood of the point. It is important to note that the first encounter with inflection points occurs before the term is defined. It happens when students start investigating functions: they solve $f'(x) = 0$ to find possible x-s of extreme points and accidentally encounter cases where $f'(x) = 0$, but there is no extreme point because the function is monotonic in the interval that includes this point. In such cases – $f'(x) = 0$ but the point is non-extreme – students are first guided to label these points as inflection points for purposes of communication and to distinguish them from extreme points. Afterwards, useful theorems are introduced, e.g., a necessary condition for x_0 to be an inflection point is $f''(x_0) = 0$; a sufficient condition may be: (1) $f''(x_0^+)$, and $f''(x_0^-)$ have opposite signs in the neighborhood of x_0; or, (2) $f'''(x_0)$ exists and $f'''(x_0) \neq 0$.

In order to widen the scope of the gathered data, two types of tasks were designed: Produce-a-Solution (production) tasks (i.e., solve mathematical problems), and Evaluate-a-Solution (identification) tasks (i.e., examine the correctness of given solutions). Tasks in each of the two types were presented in verbal, graphic, and algebraic representations (see Figure 1).

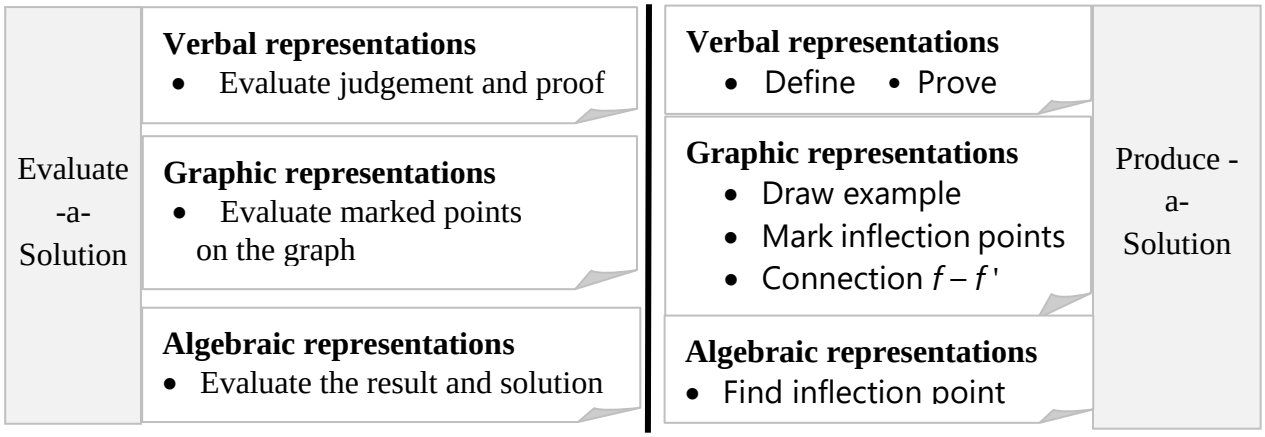

Figure 1: Structure of research questionnaire

This systematic structure of tasks provided insight to the participants' ideas and reasoning. It was developed and empirically validated (during preliminary pilot studies) as a tool that allows students to explain related formal, algorithmic and intuitive components of their mathematical knowledge (Fischbein, 1987, 1993a). The contribution of these structured tools may go beyond the broad exploration of students' conception of inflection points; such structured tools could be useful to

reveal students' conceptions of additional mathematical notions. That is, the research tools offered in this study may serve as a model when designing research tools aimed at investigating students' conceptions of other mathematical notions.

In the following, we present four tasks (Produce-a-Solution) from the questionnaire.

Task 1: Define: What is an inflection point?

Task 2: True or false? Prove:

Statement 1: f: R → R is a differentiable function.
If $f'(x_0) = 0$, then P(x_0, $f(x_0)$) is an inflection point.
Statement 2: f: R → R is a continuous, (at least twice) differentiable function.
If $f''(x_0) = 0$, then P(x_0, $f(x_0)$) is an inflection point.
Statement 3: f: R → R is a continuous, (at least twice) differentiable function.
If $f'(x_0) = 0$ and $f''(x_0) = 0$, then P(x_0, $f(x_0)$) is an inflection point.
Statement 4: f: R → R is a continuous, (at least twice) differentiable function.
If $f''(x_0) = 0$ and the function is monotonically increasing (decreasing) in the neighborhood of x_0, then P(x_0, $f(x_0)$) is an inflection point.

Task 3: Find (if possible) the inflection points of the functions:

(1) $f(x) = x^4 + 2x^3 - 1$; (2) $f(x) = x^4 + 32x$; (3) $f(x) = |x^3 - 1|$;

(4) $f(x) = \begin{cases} x^3, & x \le 0 \\ -x^3, & x > 0 \end{cases}$ (5) $f(x) = \begin{cases} x^2, & x \le 0 \\ -x^2, & x > 0 \end{cases}$

Task 4: Find (if possible) the inflection points of $f(x)$, $g(x)$, $t(x)$, based on the graphs of $f'(x)$, $g'(x)$, $t'(x)$ – see Figure 2.

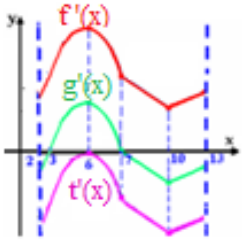

Figure 2: Graphs in Task 4

Task 1 and Task 2 are in a verbal representation, addressing participants' formal knowledge, Task 3: in an algebraic representation, addressing what we expected to be algorithmic knowledge; and Task 4 in a graphic representation, addressing participants' figural conceptions. Intuitive knowledge may be expressed in all four tasks.

Before giving our high school participants the tasks, we asked their teachers whether these tasks would be familiar to them. We learnt that Task 3 was expected to be most familiar – students usually investigated algebraic expressions of functions. They seldom analyzed graphs of the derivative, as required in Task 4; and were rarely asked to determine and prove the validity of a statement, as required in Task 2. Students' modest experience with such tasks and the impossibility of applying routine algorithms in the related solutions led us to assume that students' knowledge might be challenged.

Based on the analysis of their solutions, 20 participants were invited to individual, semi-structured, follow-up interviews, where they were asked by the researchers to elaborate on their written solutions. Interviewees were chosen according to their solutions in the questionnaires, focusing on interesting, correct, and incorrect ideas, while aiming to understand their reasoning. Interviewees were asked, among other things, to explain their solutions and to analyze solutions proposed by other participants. The interviews took 30-45 minutes and were audiotaped and transcribed.

Results

In this section, we answer the research questions: In the students' opinion (1) When is a point an inflection point? (2) When is a point a non-inflection point? We discuss each of the ideas as presented in relevant tasks. More specifically, we present each of the conceptions found as expressed in the four tasks, in descending frequency of the phenomenon.

When is the point an inflection point?

Our data indicates six sets of conditions that in the participants' opinions guarantee the presence of an inflection point.

Passage point from convex to concave or vice versa ⇒ Inflection point – This conception was mainly expressed (55%) in Task 1 (Define): "Inflection point is a point where the graph shifts from concave to convex (or vice versa)". However, no reference was made to characteristics of the function (domain, continuity or differentiability).

***f ″*(*x*₀) = 0 and a passage from convex to concave ⇒ Inflection point –** This conception prevented participants from finding the non-derivative inflection point of the functions, based on the graphs of derivatives in Task 4 (24%). In addition, it was also expressed in solutions to the algebraic representation of function $f(x) = |x^3 - 1|$ (16%) in Task 3.

***f ″*(*x*₀) = 0 ⇒ Inflection point** – This conception was evident in the participants' solutions to three tasks. In Task 3, $f''(x) = 0$ considerations were erroneously used as sufficient for inflection points of polynomial functions and even of piecewise functions. For example, participants correctly found A(0, –1) and B(–1, −2) as inflection points of $f(x) = x^4 + 2x^3 - 1$ by examining only $f''(x) = 0$ (30%). Similarly, the point (0, 0) was erroneously claimed to be an inflection point of $f(x) = \begin{cases} x^3, & x \le 0 \\ -x^3, & x > 0 \end{cases}$ by examining only $f''(x) = 0$ (15%). About 30% incorrectly claimed that Statement 2 is correct (Task 2). Some of them added: "That is the definition: P is an inflection point if and only if $f''(x_P) = 0$". Others provided algorithmic considerations, mentioning solutions to investigate-the-function and stopping after solving $f''(x_0) = 0$, e.g.: "We find inflection points when looking for extreme points. If $f''(x_0) = 0$, then the point is not minima or maxima, and that is why it is an inflection point". In their reactions to Statement 3 (Task 2) participants (10%) incorrectly answered that the statement is valid and explained that "$f''(x_0) = 0$ is a sufficient condition for the existence of an inflection point".

***f ′*(*x*₀) = 0 and *f ″*(*x*₀) = 0 ⇒ Inflection point –** This conception was mainly expressed in Task 2; about half of the participants incorrectly claimed that Statement 3 was valid. Most of them provided the methods that they used to solve investigate-the-function tasks. They wrote: "[We] always find inflection points when looking for extreme points, thus starting this search with $f'(x) = 0$", then: "An inflection point is a point where $f''(x) = 0$". In addition, in Task 4 (Investigate the graphs of derivative), about 20% incorrectly found an inflection point only at $x = 6$, exhibiting an erroneous assumption: "We see that $t'(6) = 0$, that is, the slope of the tangent of $y = t'(x)$ at $x = 6$ is zero, so, $t''(6) = 0$, therefore at $x = 6$ there is an inflection point."

***f ″*(*x*₀) = 0 and monotonicity in the neighborhood of *x*₀ ⇒ Inflection point –** Participants (42%) incorrectly claimed that Statement 4 was valid (Task 2). They explained: "Those are sufficient

conditions for an inflection point", or gave a supporting example, like $f(x) = x^3 + 5$. In their interviews, several of the latter explained their solution in terms of: "If the second derivative is zero and the function continues to increase when increasing and to decrease when decreasing, there is a change of convexity-concavity". The combination of these two conditions is likely to determine an inflection point. However, this answer can be refuted by a counter example, like $f(x) = x^4 + 32x$ (this function appeared in Task 3, but it was not used to refute this statement), $f''(0) = 0$ and the function is increasing monotonically in the neighborhood of $x = 0$ but (0, 0) is not an inflection point. Evidence about this set of insufficient conditions that seem to be "allegedly certain" for ensuring an inflection point, to the best of our knowledge, was found for the first time in this research.

$f'(x_0) = 0$ and monotonicity in the neighborhood of x_0 ⇒ Inflection point – Participants (16%) wrote, for instance: "A point where $f'(x) = 0$ and the graph keeps increasing (or decreasing) before the point and after it is an inflection point" (Task 1). In reaction to Statement 1 (Task 2), participants correctly answered that the statement is false, but their explanation was: "$f'(x_0) = 0$ is necessary but not a sufficient condition for an inflection point. If in addition to $f'(x_0) = 0$ the function increases (or decreases) before and after the point, only then is the point an inflection point" (10%). It should be noted that a combination of these conditions define a particular type of notion – a horizontal inflection point. This grasp of inflection point probably ignores non-horizontal inflection points (Task 3), like the inflection point of the function $f(x) = x^4 + 2x^3 - 1$ (10%).

When is the point a non-inflection point?

We found three conditions that deny the existence of an inflection point:

No differentiability ⇒ No inflection point – In reactions to Task 3, investigate the function $f(x) = |x^3 - 1|$, most participants (63%) found an inflection point only at $x = 0$, providing algorithmic considerations of solutions to investigate-the-function, such as: "$f'(0) = 0$, $f''(0) = 0$, before $x = 0$, $f''(x)$ is positive, so $f(x)$ is convex and after $x = 0$, $f''(x)$ is negative, so $f(x)$ is concave, thus $x = 0$ is an inflection point". Some of them added a correct graph to their investigation and, with relation to $x = 1$, wrote that "although at this point the function changes from concave to convex, it is a non-inflection point, because the function is not differentiable at $x = 1$". In their interviews, several participants explained: "The function is non differentiable at $x = 1$ and therefore there is no inflection point", or: "No differentiability, no inflection point." Following this, a quarter of the participants defined "inflection point" (Task 1) as requiring differentiability. For example: "A point where $f''(x) = 0$ and the function turns from concave to convex or vice versa", or: "The slope of the tangent of the function at this point is zero, and the function is either increasing on both sides of the point, or decreasing on both sides of the point."

No second derivative ⇒ No inflection point – This conception, that is consistent with the condition $f''(x_0) = 0$, is used as a filter to reactions to Task 3, Investigate the function $f(x) = \begin{cases} x^2, & x \le 0 \\ -x^2, & x > 0 \end{cases}$, included explanations (23%) such as: "When there is no second derivative – there is no inflection point. All inflection points must satisfy the condition $f''(x_0) = 0$. Otherwise there is no infection point". It was also found in reactions to Task 4 (15%), Investigate the graphs of $f'(x)$, that only the point $x = 6$ that satisfies the condition $f''(x_0) = 0$ was identified as an inflection point. The other point ($x = 10$), where the second derivative is not defined, was ignored.

$f''(x_0) = 0$ and $f'''(x_0) = 0 \Rightarrow$ No inflection point – expressed in the investigation of the function $f(x) = x^4 + 32x$ (7%). Note that this function really has no inflection points. Yet, this correct judgment was based on a wrong consideration. A counter-example is $f(x) = x^5$ has an inflection point, yet both $f''(0) = 0$ and $f'''(0) = 0$.

Commonly studies report of students' conception of mathematical notions by reporting on criteria that lead to regard the notion as defined. Here we show a new angle of criteria that regards the notion as undefined.

Discussion

We discuss the findings by using Fischbein's (1993a, 1993b) theoretical framework for analyzing students' errors and for examining possible related sources.

What are possible sources of students' mathematical errors?

Fischbein studied broad aspects of students' mathematical reasoning, claiming that an analysis of students' performance has to take into account three basic aspects: algorithmic, formal and intuitive (Fischbein, 1987, 1993a). The algorithmic aspect includes knowledge of (a) "how" to solve a problem, and (b) "why" a certain sequence of steps is correct. The formal aspect includes knowledge of axioms, definitions, theorems, proofs and knowledge of how the mathematical realm works. The intuitive aspect of mathematical knowledge is an immediate and self-evident, though not necessary correct, knowledge, accepted with certainty. Fischbein's three components of mathematical knowledge and their interrelations play a vital role in students' mathematical performances. However, "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures", causing inconsistencies in students' solutions (Fischbein, 1993a, p. 14). Fischbein further addressed the impact of drawings (e.g., in geometry) on learners' mathematical reasoning by explaining that the figural structure may dominate one's reasoning instead of being controlled by the corresponding formal constraints (Fischbein, 1993b).

What are the possible sources for students' errors with the concept of inflection points?

We found tendencies to determine or deny the existence of an inflection point under certain conditions. It is important to note that during the study we did not ask directly: Under which conditions, does or does not one get an inflection point? Participants provided "filtering conditions" by their own initiative. So, if one of the following sets is true: (1) convex-concave; (2) $f''(x_0) = 0$ and convex-concave; (3) $f''(x_0) = 0$; (4) $f'(x_0) = 0$ and $f''(x_0) = 0$; (5) $f''(x_0) = 0$ and monotonicity in the neighborhood of x_0; (6) $f'(x_0) = 0$ and monotonicity in the neighborhood of x_0; on the other hand, if there is (7) No differentiability; (8) No second derivative; (9) $f''(x_0) = 0$ and $f'''(x_0) = 0$ – then there is no inflection point. During the interviews, students reinforced these views.

An initial evaluation of the reasons underlying erroneous conceptions suggested two main causes: algorithmic experience with investigations of functions, and the impact of the drawing. Students tended to explain that: "This is how I find an inflection point when I investigate a function", or, [in relation to Task 4] "According to the graphs, each function has one inflection point at $x = 7$ where the graphs shift from concave down to concave up". Thus, it seems that the answers may intuitively evolve from the participants' mathematical, algorithmic experiences (Fischbein, 1993a) and from their figural concept of inflection point (Fischbein, 1993b).

Four of the six sets of conditions that participants presented for "being an inflection point" do not necessarily lead to inflection points (sets 1, 3-5); the other two sets determine inflection points only for a limited family of functions (sets 2, 6). For example, in set 1, the participants provided intuitive definitions, without reference to the type of the functions (e.g., continuous or differentiable). In set 4, participants exhibited slope-zero figural concepts (Fischbein, 1993b) in their reactions to Task 4, when they incorrectly found an inflection point "where slope of the tangent is zero...", or, "where the first derivative and second derivative cross the *x*-axis". In set 5, the necessary condition $f''(x) = 0$ was presented as a critical step in the algorithmic offering (Fischbein, 1993a), but in combination with the condition of monotonicity, that at first sight seems "sufficient" for an inflection point, surprisingly this does not necessarily lead to an inflection point (as presented in the results section).In set 6, the unnecessary condition $f'(x) = 0$ was possibly used as a result of the "primacy effect". That is, it might be the case because these are usually the first inflection points addressed in calculus lessons (Fischbein, 1987); but it was presented with the condition of monotonicity, and thus defined a particular type of notion – a horizontal inflection point.

The three sets of "denying conditions" can shed some additional light on students' conception of inflection point from two perspectives: (1) types of functions that are usually investigated, and, (2) logical constraints of their knowledge. Requesting the necessity of differentiability (set 7) can be related to functions that are usually investigated or presented graphically in textbooks – most of these are differentiable at the inflection point. Thus, an intuitive image of a "smooth inflection point" was created. Here, as in many other cases, students recognize the concept "by experience and usage *in appropriate contexts*" (Tall & Vinner, 1981, p. 151; their emphasis). The necessity of twice differentiability (set 8) might be rooted in intuitive ideas that interfere with students' formal knowledge (Fischbein, 1987; 1993a). That is, from the theorem, "If $f(x)$ is twice differentiable in some neighborhood of x_0, and if x_0 is an inflection point, then $f''(x_0) = 0$" students erroneously conclude that "if no second derivative then there is no inflection point". Here, this answer can be refuted by a counter example, like $f(x) = x^{5/3}$. In set 9, the inadequate declaration: '$f''(x_0) = 0$ and $f'''(x_0) = 0 \Rightarrow$ No inflection point', might be rooted in intuitive ideas that interfere with students' algorithmic knowledge (Fischbein, 1987). That is, from the theorem "$f''(x_0) = 0$ and $f'''(x_0) \neq 0 \Rightarrow$ Inflection point" students erroneously create the rule "if ... then..., if not … then not…".

This study considerably enriches the existing body of knowledge regarding high school students', university students', and university graduates' conceptions of inflection points. Only a small number of studies have dealt with students' conceptions of inflection points directly (e.g. Rivel, 2004; Tsamir & Ovodenko, 2004) and indirectly (e.g. Biza & Zachariades, 2010; Vinner, 1982). Reported studies usually addressed a limited population and dealt only with specific conceptions. The current research offers a broad collection of related correct and incorrect conceptions found among participants with suitable mathematical backgrounds (as specified in the methodology section), with reference to the type and the representation of the given tasks.

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Math teaching as jazz improvisation: Exploring the 'highly principled but not determinate' instructional moves of an expert instructor

Alon Pinto

University of California, Berkeley, USA; allalon23@gmail.com

When engaging students in genuine mathematical problem solving, how can instructors maintain a productive learning environment? In this paper, I examine a series of improvised instructional moves of Alan Schoenfeld, a renowned teacher of mathematical problem solving, and investigate his dilemmas, considerations, and in-the-moment decisions. I use the TRUmath framework to unpack the conflicts that underlie Schoenfeld's dilemmas, and to propose a tacit teaching heuristic that help explains his hard-to-justify moves. I conclude that Schoenfeld's in-the-moment decision making is tacitly oriented towards maintaining certain kinds of balances between his pedagogical principles. On the basis of this analysis, I recommend exploring further the use of TRUmath as a framework for analyzing in-the-moment decision making in the context of conflicting pedagogical principles.

Keywords: University math teaching, teaching dilemmas, decision making, problem solving.

Introduction

> I think this kind of teaching is highly principled by not determinate. What I was thinking of is jazz improvisation. It's anything but random; there are moves that the musician will say would or would not be right; but, there may not be a sound justification for any particular in-the-moment move other than 'it just felt right'.

In this quote, Prof. Alan Schoenfeld reflects on a series of in-the-moment teaching decisions he made during a problem solving session. Schoenfeld is an expert teacher of mathematical problem solving (MPS hereafter); he has been studying and teaching MPS for more than three decades now. After so many years, Schoenfeld's instruction seems anything but improvised. For this study, he has reflected on numerous teaching moves that he made during his MPS course, and he was typically able to provide a sound and detailed rationale for his decisions. However, there were certain decisions that Schoenfeld found hard to fully justify, as a key ingredient in their making was a tacit sense of where the class is and how different decisions could work out. In the quote above, Schoenfeld argues that this kind of hard-to-justify decisions makes the instruction of MPS a lot like jazz improvisation, in the sense that both activities are 'highly principled but not determinate'. In this paper we investigate Schoenfeld's jazz-like teaching moves through a case study of three hard-to-justify decisions in one MPS session. In this session, Schoenfeld faced a typical dilemma in MPS-oriented lessons: how should a teacher react when a student comes up with a beautiful and original idea that opens the door to a mathematical exploration that seems worthwhile for some of the students, and a step too far for other students? The aim of this paper is to unpack Schoenfeld's conflicting pedagogical considerations in this case, and to provide insights into his decision making.

The lesson examined in this paper was part of Schoenfeld's MPS course. Schoenfeld's teaching in this course has been studied in several papers. For example, Arcavi, Kessel, Meira and Smith (1998) studied Schoenfeld's teaching in relation to the establishment of classroom norms and MPS heuristics. Schoenfeld's in-the-moment decision making in this course, which is the focus of this paper, has not been studied so far. This MPS course was given to education graduate students and

prospective teachers and comprised of paper reading, a small scale research project, and engagement in authentic MPS. In terms of goals and pedagogy, the lesson described in this paper is similar to lessons that Schoenfeld has taught in earlier years to undergraduate mathematics students. Therefore, the dilemmas and instructional moves discussed below are not specific to teacher-education courses, and should be viewed in the context of MPS-oriented instruction at university.

There are various approaches for explaining why teachers make the decisions they make as they teach. One approach, which has been gaining much attention in recent years, is to explain teaching decisions in terms of knowledge, goals, and orientations (Schoenfeld, 2010). This approach has been used at the university level in empirical studies (e.g. Pinto, 2013) and also in professional development programs, as an organizing framework for instructors' self-reflections on their teaching (e.g. Schoenfeld, Thomas, & Barton, 2016). However, a notable limitation to the explanatory power of this approach is that instructors' self-reflections are oriented towards what instructors notice in their teaching and have words for. Therefore, there is a need for an organizing framework that would draw attention to various important facets of the work of teaching. One candidate framework is the Teaching for Robust Understanding of Mathematics framework (TRUmath) (Schoenfeld, 2015). In this paper we analyze Schoenfeld's reflections on his teaching from a TRUmath perspective, and examine the use of TRUmath as an organizing structure for instructors' reflections on their dilemmas and decisions that attends to all the major contributors for productive learning environments.

Setting and methods

This paper examines a lesson taught by Alan Schoenfeld in a "Mathematical Thinking and Problem Solving" course at the Graduate School of Education at UC Berkeley. The lesson took place during the 11th week of the semester. The class comprised of 21 students – graduate students in the school of education and students from teacher preparation programs. The class met once a week for a 3-hour lesson and every lesson included an MPS part where students worked alone or in small groups on a list of problems and then reconvened to share ideas and solutions. The author videotaped the lessons and took notes. After each lesson, Schoenfeld wrote down some reflections on his dilemmas, his instructional moves and decisions, and their impact on the lesson. In addition, the author conducted three 1-hour interviews with Schoenfeld at different stages of the semester that focused on where the class is with respect the learning trajectories for the course.

The analysis in this study is based in part on the TRUmath framework, which seeks to characterize the main contributors for productive learning environments. This framework was derived through a comprehensive literature review by distilling the factors that shape learning in classrooms into a small number of "equivalence classes". These classes are represented through five dimensions: (1) the richness of the mathematics, (2) cognitive demand and opportunities for "productive struggle", (3) equitable access to content for all students, (4) students' opportunities to develop agency, ownership, and positive mathematical identities; and (5) formative assessment. According to the TRUmath framework, these five dimensions are both necessary and sufficient for studying learning environments in the sense that instruction needs to do well along these dimensions in order to produce mathematically proficient students. Figure 1 provides a brief account of each dimension.

The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Formative Assessment
The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.	*The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development. There is a happy medium between spoon-feeding mathematics in bite-sized pieces and having the challenges so large that students are lost at sea.*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class. No matter how rich the mathematics being discussed, a classroom in which a small number of students get most of the "air time" is not equitable.*	*The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency (the capacity and willingness to engage mathematically) and authority (recognition for being mathematically solid), resulting in positive identities as doers of mathematics.*	*The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings . Powerful instruction "meets students where they are" and gives them opportunities to move forward.*

Figure 1 – The five dimensions of the TRUmath framework (Schoenfeld, 2015)

In this paper, we explore TRUmath's explanatory power on Schoenfeld's own instruction, and use the five dimensions as an organizing structure for the discussion of his dilemmas and considerations. Schoenfeld is one of the leading developers of the TRUmath framework, and therefore it is particularly suitable for exploring his decision making. We examine a sequence of three hard-to-justify decisions, first from an outside observer perspective based on the videos of the lessons; then from Schoenfeld's inner perspective based on his post-lesson written reflections and the interviews; and finally, from a TRUmath perspective, where the dimensions are used to organize and compare Schoenfeld's various considerations, to unpack the pedagogical conflicts that underlie his dilemmas, and to investigate what kinds of balances he achieved in his in-the-moment decisions.

The Square-ness task

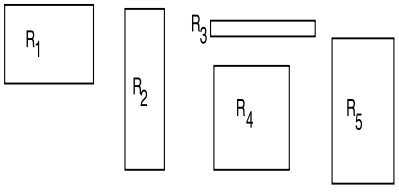
The discussion analyzed in the paper revolves around the Square-ness task, given in the figure below, which was designed by Judah Schwartz in cooperation with members of the Balanced Assessment Group at the Harvard Graduate School of Education. In the lesson, Schoenfeld framed this task as "*an introduction to the game mathematicians play*", and directed students to start with their intuition as to what it means for a rectangle to be more 'squarish', and then mathematize this intuition by "*coming up with a mathematical characterization that would enable anyone to perform some sort of operations on a rectangle […] and obtain a number that would tell, in some sense or other, how close to being a square that rectangle is.*"

Below you will find a collection of rectangles.

(a) Define a mathematical measure that allows you to tell which rectangle is the "most square" and which rectangle is the "least square".

(b) Define a different measure that achieves the same result.

(c) Is one measure "mathematically superior" to the other?
Argue why, and be prepared to defend your choice to the class



Analysis

> It's not the note you play that's the wrong note – it's the note you play afterwards that makes it right or wrong. (Miles Davis)

Taking a cue from a renowned jazz improviser, I maintain that improvised teaching decisions should not be examined in isolation but rather as part of the flow of instructional moves that teachers make during a lesson. Accordingly, this investigation of Schoenfeld's improvised teaching moves examines three hard-to-justify decisions in the context of Schoenfeld's instruction throughout the MPS session. The analysis is presented as a narrated description of the whole session that comprises of three threads: an outside-observer description of Schoenfeld's moves; a synthesis of Schoenfeld's reflections on his moves, as explicated in the interviews; and a TRUmath perspective on three challenging teaching dilemmas and their hard-to-justify resolutions.

The discussion of the Square-ness task began with a short introduction by Schoenfeld, after which he invited students to present their candidates measures. One of the students, Sophie, approached the board and suggested that the square-ness of a rectangle with side-lengths a and b will be defined by: "The ratio a/b ought to be close to 1". Two students objected to this definition, arguing that it is not well defined and that it should specify that a is the shortest side length. Sophie disagreed at first, claiming that "it does not matter", but was eventually persuaded. She added "where $a \leq b$" to the written definition, and walked back to her seat. At this point, Schoenfeld intervened:

> Ok, I love it when the class takes over and raises mathematical objections. The question is, if you're characterizing the square-ness of this figure and you're getting a number, shouldn't we get the same number if we happen and bring it down this way instead? (Draws a figure of a rectangle rotated by 90 degrees) It's the same rectangle, so whatever measure you have should bring you the same number. If you have say, a 1 by 3 rectangle, then you get 1 over 3 which is not the same distance from 1 as 3 over 1; so, it begins to be problematic unless you lay it down so that 'a' is the smaller one of the two [side lengths], and then take 'a' over 'b'.

After making this remark, Schoenfeld leaned quietly against the wall and waited for the students to react. In his reflections, Schoenfeld noted that up to this point the discussion took off just as he intended, as the students were engaged in defining, comparing and criticizing measures, and by doing so, expressing their implicit expectations from a measure. Schoenfeld noted that it is quite typical that the first candidate measure is based either on the ratio or on the difference between adjacent side lengths of the rectangle. The measure Sophie suggested has the nice property that it can be defined in a way that makes it invariant under rotations and scaling. The students' debate on whether Sophie's measure was well defined did not address explicitly the properties of the measure. The students seemed more occupied with figuring out the exact routine for computing the measure. Nevertheless, Schoenfeld explained in the interview that this debate provided him with an opportunity to acknowledge, respond to and build on students' ideas, while rephrasing these ideas in a way that fit his goals – to engage students in discussing the desired properties of measures.

Emmy was the first to comment on Sophie's example: "I think that I can probably find a quadrilateral that is not a square, but would have that, hmm… would be a square under that measure of square-ness, but it is not a rectangle, is that OK?"

Schoenfeld's immediate response was to the entire class: "Do you want to take a vote? Is that OK?" The class seemed divided with some students wondering whether considering quadrilaterals other than rectangles is allowed, while others expressing interest in seeing Emmy's example. Schoenfeld agreed with the students that the problem is stated just for rectangles, but as several students responded in disappointment, he paused to make a quick evaluation and to decide how to proceed.

In his reflections on this moment, Schoenfeld noted that Emmy's example came as a surprise, and too soon with respect to where he felt the class was. He explained that he was more expecting students to propose another candidate measure, which he could compare to Sophie's measure; or to point out that Sophie's measure is invariant under scaling, which would have provided another opportunity to discuss properties of measures. Schoenfeld reflected on his dilemma: on the one hand, the class seemed eager to see Emmy's example; Emmy's comment was well aligned with his own agenda of discussing the desired properties of measures, as it pointed out the fact that Sophie's example might not be generalizable to parallelograms. He also considered this comment as an authentic and beautiful example of 'doing mathematics' and he wanted to acknowledge this; and, coming from a student, this generalization felt natural and organic rather than an artificial teaching move, making it even more appealing. On the other hand, Schoenfeld noted that he was worried that Emmy's example might steer the discussion towards arbitrary quadrilaterals, and he was not sure that the class was ready for this level of abstraction. He wanted more students to participate in the discussion and considered putting Emmy's example on hold so other students could present their candidate measures for rectangles. He also noted that he had examined some of the student work on this problem and that there were a few important insights that he wanted to draw from that work.

In the classroom, Schoenfeld responded almost instantly:

> Now, before I throw [Emmy] out of class (laughing), let's examine what [she] said. One of the properties one might like for, hmm... any definition is to ask the question: what class of objects this definition applies to? […] So, this measure (points at the Sophie's definition on the board) works for rectangles, but being my psychic self, I think the figure Emmy had in mind was a family of rhombuses (draws two rhombuses on the board), all of which have a measure 1 according to the definition, if we think in terms of side lengths; but they don't look like a square!

Note that Schoenfeld's response opened the door for Emmy's example, but that he presented the example himself, as he understood it, rather than letting Emmy present it in her own words. Looking back at his response, Schoenfeld noted that it is hard to justify, claiming that on one hand, it is highly principled in the sense that it is consistent with his goals and orientations, as explicated in his reflections on his dilemma. On the other hand, this decision was not determined by principles, as it was based in part on a tacit sense of where the class was, and how well things could work out.

We now turn to analyze Schoenfeld's hard-to-justify decision from a TRUmath perspective. One option Schoenfeld had was to invite Emmy to present her example in her own words. This option is well aligned with the Formative Assessment and Agency dimensions. Moreover, Schoenfeld considered Emmy's idea to be "a beautiful example of doing mathematics", and since his goals were to discuss properties of measures, he considered this option to also be well aligned with the Mathematics dimension. However, Schoenfeld's reflections suggest that he found this option less appealing from the perspectives of the Access and Cognitive Demand dimensions. He explained that

it is essential that students understand and relate to the goals of the exploration. Emmy, who might still be struggling to formulate her idea, could end up leading a discussion that the rest of the students could not engage with productively. Another option Schoenfeld considered was flatly rejecting Emmy's example, or putting it on a back burner. This option would have given Schoenfeld more control over the lesson, which has merits in terms of the Mathematics, Cognitive Demand and Access dimensions. However, Schoenfeld considered this route potentially harmful in terms of Authority, Agency and Identity, and Formative Assessment. Schoenfeld's response represents an alternative to these two options. He acknowledged and built on Emmy's idea (Formative Assessment), lowering the risk of being perceived as rejecting her thinking (Authority, Agency and Identity). However, he did so by proposing two visual examples of rhombuses, making the discussion more concrete and accessible (Cognitive Demand and Access). Moreover, Schoenfeld provided a crisp outlining of the topic of the discussion: "what class of objects this definition applies to" (Access), orienting the discussion towards the properties of measures (Mathematics). To summarize, Schoenfeld considered the first two options to be potentially beneficial as well as potentially harmful; his reaction chose a middle ground that he still considered beneficial, and safe.

The lesson continued with Sophie, Emmy and a few other students discussing how to modify Sophie's measure to make it 'more square'. This discussion led to a new candidate measure for square-ness: the product of the ratio between adjacent side lengths and the ratio between adjacent angles. However, Emmy criticized this measure, claiming that while this measure has the nice property that squares are separated from other shapes, she can no longer see what kind of ordering this measure induces on parallelograms, and whether this ordering has anything to do with her initial intuition as to what square-ness should mean. Several students endorsed this criticism, and the class abandoned this measure. One student suggested that it might not be possible to find a measure that works for both rectangles and rhombuses, and the whole classroom discussion started to break up into several concurrent discussions. At this point, Schoenfeld intervened:

> I'll point out that what we're doing right now is exactly the business mathematicians are engaged in. […] We start with rectangles and see candidate measures for rectangles; then the question is, what about parallelograms? Trapezoids? Arbitrary quadrilaterals? Is it possible to find a measure that could characterize square-ness for all of those? We only got one definition of square-ness of rectangles so far, and I want to see a few more. It is possible that if we are just looking at rectangles any of the candidate measures will do, although some might be easier to calculate, some might correspond more to your intuition in terms of how square something is. And then, as we move on, only some of those definitions work for more objects. That's the game mathematicians play. So, we have two directions to go. We have this definition (wipes the board clean and writes 'a/b closest to 1 where $a \leq b$'). We can ask, are there any other characterizations, or reasons to like them more or less; and we can ask do they generalize and how much, which can also get us to a discussion about just what properties of definitions in general do we want, and what properties do we want in this particular case. The floor is open.

Reflecting on this intervention, Schoenfeld noted that this was a point where he sensed the class was indeed not ready for the exploration they initiated, as he anticipated might happen; in his message to the students, he was trying to steer the discussion back to rectangles, while making sure he is still giving due credit to the exploration the students were engaged with, framing it as the 'game

mathematicians play'. Schoenfeld considered this decision as essentially based on a tacit evaluation of where the class is. In the interview, he used TRUmath terms to make this evaluation somewhat more explicit: He explained that he was reading from the students' facial expressions that some students were getting disconnected, signaling Access was becoming an issue; he also noted that the fact students starting to question whether the problem could be solved at all was for him a signal that the Cognitive Demand of the task might be too high. Schoenfeld concluded that in his intervention he was implicitly trying to attend to the Access and Cognitive Demand dimensions.

At this point, Emmy suggested: "I have an idea, but I don't know how to turn it into a measure […] I have a measure that would split out squares, but I don't know how to make it order everything else. Should I share it? (Schoenfeld nods) OK, my theory is that if you have a given a perimeter for a quadrilateral, the square will have the maximum area. So, I want something that takes perimeters and determine whether or not, hmm… determines whether or not that's the maximum area for that circumstance and then order everything else according to how not maximum it is, or something."

Emmy's suggestion led to a rapid exchange between her and Sophie, while the rest of the students remained quiet. In the interview, Schoenfeld referred to this moment as another challenging dilemma that led to a hard-to-justify decision. While he considered Emmy's comment to be mathematically inspiring, it also impeded his attempts to lead the discussion back to rectangles. Schoenfeld explained that he guessed Emmy's idea is intuitively based on the isoperimetric theorem, and he estimated that forming a measure for arbitrary quadrilaterals on the basis of this intuition might prove too difficult for most students, potentially reducing their confidence and sense of efficacy even further. Thus, even though following up on Emmy's comment was very appealing from the Mathematics and Formative Assessment perspectives, this option seemed very risky from the perspectives of Access, Cognitive Demand and Identity. However, Schoenfeld found that while his principles directed him to object to exploring Emmy's idea, his sense of the class suggested otherwise: he sensed that the students were quiet but not passive, that they were actively listening to Emmy and Sophie. Consequently, Schoenfeld explained he decided to try and provide the class with just enough scaffolding to enable more students to engage productively in the new exploration:

> Ok, let's take what we do know and see if we can turn this to a measure. Hmmm… you may have heard […] of this thing known as the isoperimetric theorem … the general theorem is that if you take any figure whatsoever for a fixed perimeter, the circle is the figure with the largest area. If you limit yourself to quadrilaterals, to rectangles, it turns out that for any given perimeter the square is the figure with the largest possible area. So, the question is whether we can turn that into a measure we can use, and then think about abstracting this into some of these other figures.

In TRUmath terms, Schoenfeld's decision can be expressed as an attempt to amend the level of Cognitive Demand so to increase Access. The intervention paid off. Four more students joined Sophie and Emmy and participated in the exploration. It took just a few minutes of discussion for Sophie to come up with a measure that works: *"The perimeter over four, squared, over the area of the shape".* The class enthusiastically picked up on this suggestion, and eventually endorsed it.

Discussion

In this paper, we examined a sequence of three in-the-moment decisions. Schoenfeld's first decision was to open the door to Emmy's original idea, but present it in his own words; when the exploration

of Emmy's idea seemed too challenging for the class, Schoenfeld's second decision was to try and steer the discussion back to the original problem; and finally, a quick evaluation of where the class was led Schoenfeld to reverse his second decision and allow an even more challenging exploration. Schoenfeld considered his decision making to be highly principled in the sense that his decisions were well aligned with his explicit orientations and goals; however, in his reflections, he also observed that some of these decisions were hard to fully justify since they were strongly influenced by a tacit sense of where the class is and how things could work out. This sense of the class is a resource Schoenfeld developed over years of teaching the course; his reflections suggest that this resource has a crucial role in his decision making when faced with challenging dilemmas: it helps resolve pedagogical conflicts that rise from tensions between competing goals and orientations. The TRUmath framework proved to be useful for unpacking these tensions by providing an organizing structure for the different considerations and the conflicts they present. For example, in the context of Emmy's original idea, when examining three alternative options, we found that Schoenfeld considered two of the options to very well aligned with some of the dimensions, but also potentially harmful from the perspective of the other dimensions. The TRUmath analysis suggested that Schoenfeld chose a path that he considered more moderate across all five dimensions in terms of potential gains and risks. This analysis led Schoenfeld to suggest a teaching heuristic that may have tacitly guided him: keep the lesson productive from the perspective of each dimension, and avoid the temptation to excel in just one or two dimensions at the expense of the other dimensions.

This paper illustrates the theoretical potential of TRUmath as a framework for explaining decisions made in light of conflicting goals and orientations, and the practical potential of TRUmath as an organizing structure for teacher reflection that highlights the gains and risks entailed in different instructional moves. As Schoenfeld is both the subject of this study and one of the developers of TRUmath, further research is required to assess TRUmath's explanatory power for other instructors.

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Identifying discussion patterns of teaching assistants in mathematical tutorials in Germany

Juliane Püschl

University of Paderborn, Germany; jpueschl@math.upb.de

Mathematical tutorials play an important role in tertiary teaching of mathematics in Germany. However, we do not know a lot about what actually happens in these tutorials. This paper reports from a small study in the context of a PhD project which investigates the work of teaching assistants (TAs). The result of this study is a typology of discussion patterns of tasks in mathematical tutorials. In this study typological analysis, including a hierarchical cluster analysis, was used to identify different ways in how TAs work on mathematical tasks in classroom discussions. The findings suggest that there are five main patterns for the discussion of tasks, differing in focus, length and support of students' learning.

Keywords: Teaching assistants, tutorials, problem oriented instruction, classroom discussion.

Introduction

In Germany, teaching assistants (TAs) play a vital role in the learning of mathematics at universities. A mathematical course usually consists of three parts: the lecture, homework assignments and small group tutorials. The lecturer delivers the lecture and assigns three to five tasks for the homework assignments. After working on these assignments for one week, the students hand in their solutions to the tasks, TAs correct them and later discuss these tasks in small group tutorials. Usually, the lecturer and TAs meet every week for planning the tutorial session. TAs do not assist the lecturer in his teaching activities, but they are on their own in the tutorials and free to discuss the tasks as they like. TAs in Germany are undergraduate students who are just a few years ahead of the students they teach and they are usually employed for one semester. By leading the tutorials, TAs are close to the students' actual learning and may function as link between students and lecturer, e.g., they shall connect the contents of the lecture to the tasks, they ought to get feedback from the students and report this to the lecturer. TAs are expected to be accurate and to facilitate students' learning by also sharing their own experiences as successful students.

Although TAs play such an important role in the learning of the students, there is hardly any research on German TAs' practices. Internationally, research on TAs has increased in the last 20 years. However, as their role and practices differ in many ways from that of German TAs, it is uncertain in what respect the results of these studies can be transferred to the German context. Therefore, the aim of my PhD project is to understand how German TAs organize their tutorials in order, later, to be able to provide a basis on which we can develop quality criteria for German tutorials and use them to support TAs in weekly meetings and tutorial trainings.

Theoretical background

In the last years, research on mathematical teaching practices has increased and there have been a few studies that compare characteristics and teaching styles of instructors in different levels. Weber (2004) showed that one lecturer might use different teaching styles, depending on his learning goals and the content he is teaching. Weber could identify three different ways of teaching proofs in a

traditional lecture format: logical-structural, procedural and semantic. This suggests that, at least advanced lecturers, change their teaching styles intentionally.

There have also been studies on comparing teaching styles of different instructors. One example is Pinto's (2013) study on two TAs that are even supposed to follow the same lesson plan. Using Schoenfeld's resources, orientation and goals theory (Schoenfeld, 2011), Pinto illustrates how both TAs prepare and hold their tutorials quite differently according to their research background and experience in teaching this particular topic. Also Mali (2015) could accord to the findings of Pinto. In their study on two tutors in small group tutorials, they found out that teaching seems to be closely linked to research practices of the tutors: the mathematician "uses the graph to make fundamentally mathematical ways of thinking transparent to students", whereas the mathematics educator uses the graph "as an alternative to explain the mathematics" (p. 2193).

Of course, looking at TAs from the German context, they do not have a research background that might influence their teaching. However, their experience from their own tutorials or from teaching tutorials in former semesters as well as their learning goals, beliefs about learning or the mathematical contents might influence the way they discuss tasks. Therefore, the main research question for this paper is: What different "discussion patterns" do TAs use when working on mathematical tasks in tutorials? This includes sub-questions like: Do TAs focus on the problems they identified when correcting the students' work? Do they always discuss the whole task or only parts of it? How much do they support the students in their learning process? For this paper I combine the answers to these questions by trying to find overall patterns for classroom discussions.

Methods and design

The study is being conducted at a German university. Students of mathematics for pre-service teachers are expected to attend lectures and additional small group tutorials of 10 to 30 students. Both, lectures and tutorials are weekly sessions of 90 minutes, the students are supposed to work on tasks out of class which are assessed by the TAs and discussed in tutorials. The videos for this study were generated in tutorial sessions containing 78 task discussions in total (2-3 discussions per tutorial). 32 TAs have taken part in this study, all of them pre-service teachers. The data set includes task discussions in tutorial sessions from different semesters of the years 2010 to 2015 and the discussion covered a range of topics from analysis, arithmetic and geometry. This great variety of tutorials was chosen to get a representative sample of task discussions, thereby reducing other influences (style of lecturer, difficulty of mathematical content, etc.).

The students in the tutorials were all bachelor students studying teaching mathematics in secondary schools. Students were not directly videotaped unless they presented parts of the solution in front of the class. Their questions, comments and other contributions to the task discussions are audible in the videos.

In a previous part of the PhD study, these 78 task discussions were analyzed for different aspects: how TAs start and end the task discussions, what methods they use, how they try to support the students using visualizations, references to the lecture, etc. Using qualitative content analysis, different categories were generated to investigate each aspect in more detail (Püschl, 2016).

This part of the study focus on the task discussion as a whole. As we want to find out different types of discussions, an empirical typification method, namely the "typological analysis" by Kuckartz (as

described in Kluge, 1999), was used for analyzing the data. An important part of typification methods is to identify features which characterize the different discussion patterns. These features for the classification of task discussions are based on some categories from the previous part of the PhD project (see Püschl, 2016). The following four features were considered for the typological analysis: *use of didactical elements*, *completeness of discussion*, *focus on problems* and *focus on strategies*.

The *use of didactical elements* is a key feature for the discussion, because it indicates to what extend the TAs support the students in their learning process. In the previous analysis of the PhD study, ten different didactical elements were identified: "use of visualization", "highlighting common mistakes", "reference to lecture, other exercises, school", "giving structural advice", "clarification of expectations", "solving in several ways", "recapitulation of main results", "clarification of student questions", "generating cognitive conflicts", "returning students' questions" and "asking advanced questions". Some were only used in a few discussions (like "returning students' questions") while others could be found frequently (e.g., "use of visualization"). Some didactical elements are rather pedagogical (e.g., "returning students' questions"), others demand more mathematical skills (e.g., like "asking advanced questions"). For the purpose of this paper instances in which a TA used a didactical element were counted, ranging from a minimum of 0 to a maximum of 13 elements per task discussion. The feature *completeness of discussion* tells, whether the TAs discuss the whole task or only parts of it. Therefore, this is a binary feature, 0 standing for an incomplete discussion. The other two features give insight into the focus of discussion and relate to categories from prior analysis. The focus could be on the mathematical problems the students had when working on the task. The *focus on problems* feature consists of three categories from the previous part of the study: "giving feedback on the work of students", "highlighting common mistakes" and "telling the students to review specific parts of the task". The *focus on strategies* feature is quite different: here, the TAs do not discuss mathematical difficulties, but rather pass on strategies that students need to solve a specific type of tasks. It contains the following three categories: "pointing out the task difficulty", "giving structural advice" and "summarizing of task". For both features, the elements in each discussion are counted, ranging from 0 to 4.

Based on these four features, agglomerative hierarchical clustering was used to find patterns in the 78 task discussions. Agglomerative clustering starts with all 78 cases, each building one cluster. The algorithm then merges a selected pair of clusters into a single cluster, so that after 77 steps only one big cluster is being left (Hastie, Friedman, & Tibshirani, 2001, p. 472). To equal the relative influence of the four features in the cluster algorithm, they were scaled between 0 and 1. The Squared Euclidean distance was used as a metric to calculate the distance between each cluster. To decide which clusters are joint in each step, the Ward's Method was chosen as linkage criterion. The Ward's Method tries to minimize the total distance from centroids by joining two clusters. This method facilitates the construction of clusters with similar sizes and is frequently used because it has often provided better results than other linkage criterions (Bortz, 2005, p. 573). As the clusters become more heterogeneous in every step of the algorithm, it is often recommended to stop at a number of clusters before the greatest increase of distance (Bacher et al., 2010, p. 241). The statistical analysis was facilitated by the software SPSS (version 23).

The data has been analyzed regarding the four features mentioned above. In addition, other factors have been taken into account as they might influence the results (Kuckartz, 2012, p. 125). Therefore, the discussion patterns identified by the cluster algorithm were analyzed in regard to a variety of other

factors. Three factors are presented in this paper: time spent on the discussion, the TAs' individual approaches and their experience in leading tutorials.

Results

The results from the cluster analysis suggest five clusters from the 78 discussions. Using four or less clusters would have resulted in a high increase in distance between the cases in the cluster. Three cases of the 78 discussions were eliminated, because they did not fit one of these clusters for several reasons[1]. One case, for example, which showed a great distance to all of the other clusters, was a task discussion in which the TA was really ambitious and wanted to discuss the whole exercise with a lot of student participation, focusing on problems as well as strategies. However, in this way she ran out of time and was not able to discuss even half of the exercise in more than 40 minutes. As there is no similar task discussion in the data and it would have influenced the cluster algorithm too much, this case was eliminated according to these qualitative and quantitative considerations.

Cluster		***completeness*** **(1 complete, 0 incomplete)**	***number of didactical elements*** **(0-10 elements)**	***focus on problems*** **– number of elements (0-4 elements)**	***focus on strategies*** **– number of elements (0-4 elements)**
1	median	1	4	0	1
	N	18	18	18	18
2	median	1	2	0	0
	N	22	22	22	22
3	median	1	6	1	0
	N	10	10	10	10
4	median	0	5	2	0
	N	6	6	6	6
5	median	0	2	0	0
	N	19	19	19	19
overall	mean value	1	3	0	0

Table 1: median of four features distributed into five clusters (N=75)

Table 1 shows that most of the discussions fall into clusters 1, 2 and 5. Only 16 of the 75 discussions have a focus on problems. Cluster 2 and 5 make up more than half of the discussions (55%), both not focusing on either problems or strategies. Cluster 1 is the only cluster with a focus on strategies.

Taking into account the five clusters from cluster analysis five different discussion patterns could be identified in the material:

[1] For further explanations on this process of analysis see Hastie, Friedman & Tibshirani (2001, p. 473).

Heuristic discussion (Cluster 1)

The TA discusses the complete task by focusing on strategies the students have to acquire in order to solve tasks from this specific type. The heuristic discussion is the only pattern with this focus on strategies and used 24% of all task discussions.

Pragmatic discussion (Cluster 2)

The TA discusses the complete task without focusing on strategies or problems and with minimal use of didactical elements to support the students in their learning processes. The pragmatic discussion is the most frequently used discussion pattern (29%).

Student-oriented discussion (Cluster 3)

The TA discusses the complete solution of the task while focusing on the specific problems the students might have had. This pattern is called student-oriented, because the TA satisfies the students' request for a model solution, but also tries to help them to overcome their difficulties. The student-oriented discussion is only used in 13% of the task discussions.

Problem-oriented discussion (Cluster 4)

This type of discussion can be characterized by a focus on problems. The TA highlights the difficulties in the solution process, using many didactical elements to support the students. This pattern is similar to the student-oriented discussion, only differing in the completeness of the discussion. Only 6 task discussions fall into this pattern.

Minimalistic discussion (Cluster 5)

TAs using the minimalistic discussion just discuss parts of the solution without a specific focus. The TAs hardly use didactical elements to support the students. This pattern is quite similar to the pragmatic discussion except for the completeness of the discussion. About 25% of the task discussions fall into this cluster.

Duration of discussions

The average discussion time of one task is about 19 minutes long, ranging from a minimum of 2 to a maximum of 53 minutes. The boxplots in Figure 1 show how long the discussions last in each discussion pattern.

Although the differences in average discussion time between the different patterns is not statistically significant (one-factor ANOVA at a 5% significance level) there are some interesting observations to be made: The pragmatic discussion and the minimalistic discussion are about 10 minutes shorter than the discussions in the other clusters. Especially for the pragmatic discussion, this result is quite surprising as the TAs are going through the complete solution. However, both discussion patterns do not have a specific focus. TAs who discuss problems or strategies probably need some time to concentrate on this focus and need more didactical elements to support the students. Time might be an important factor for TAs to choose a discussion pattern like the pragmatic or minimalistic patters as they often run out of time in the tutorials.

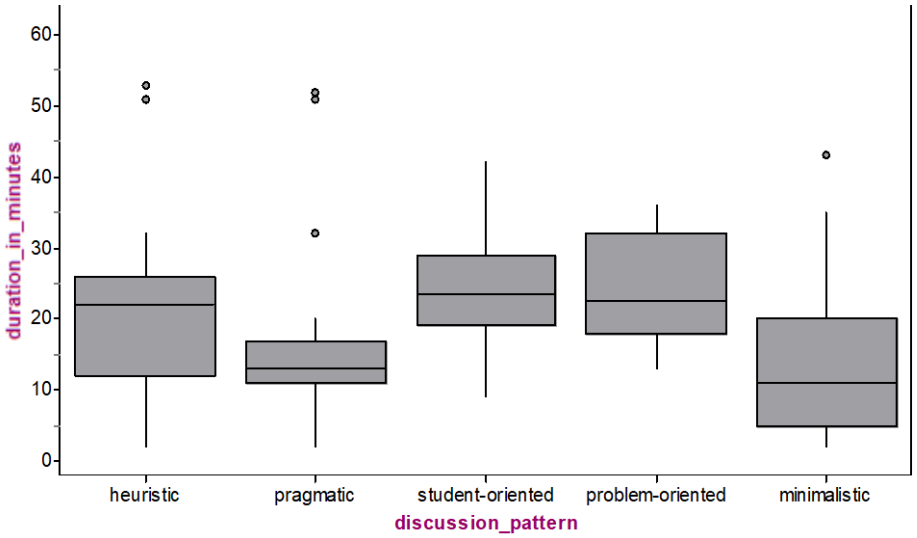

Figure 1: duration of discussion in minutes each discussion pattern (N=75)

Individual approaches of the TAs

Another factor which might influence the choice of discussion pattern might be the individual approaches of the TA. Depending on their beliefs on learning, some TAs might prefer learning from mistakes and usually choose discussions in the problem-oriented or the student-oriented pattern. Others rather want to hand on the model solutions and therefore choses the pragmatic discussion pattern. One interesting aspect to investigate might be whether TAs usually choose the same pattern or whether they switch between the patterns when discussing different tasks.

As most TAs in the data only discuss one or two tasks, the data for this analysis is quite small. The following table shows the distribution of discussions from two TAs who discuss seven different tasks:

TA	**discussion pattern**					**overall**
	heuristic	pragmatic	student-oriented	problem-oriented	minimalistic	
Andrew	1	5	0	0	1	**7**
David	2	2	2	0	1	**7**

Table 2: distribution of clusters on example of two TAs

Andrew tends to use complete discussions with no specific focus. Although he discusses seven different tasks, he never focuses on strategies. David's discussions fall in four different patterns. He does not seem to have a preference for any of the different types. This result suggests that some TAs have individual approaches while others might be rather flexible in their choice of discussion pattern.

Both, Andrew and David, are experienced TAs who have led tutorials for several semesters. However, not all of the TAs are as experienced as David and Andrew. Like expert and novice teachers differ in some aspects of their teaching, rather inexperienced TAs might also choose different ways of discussing tasks. Only 11 TAs of the 32 TAs in the data have led tutorials before the semester of this study and are therefore labeled as "experienced" while the other 21 TAs are called "inexperienced".

experience of TAs	discussion pattern					overall
	heuristic	pragmatic	student-oriented	problem-oriented	minimalistic	
experienced	11	10	3	1	10	35
inexperienced	7	12	7	5	9	40
Overall	**18**	**22**	**10**	**6**	**19**	**75**

Table 3: distribution of discussion patterns depending on experience of TAs (N=75)

Table 3 suggests that there are some differences between experienced and inexperienced TAs. The experienced TAs, which have lead tutorials in the previous semesters, seem to prefer the heuristic, pragmatic and minimalistic discussions. The inexperienced TAs have a greater focus on problems. They use the student-oriented and problem-oriented patterns more frequently than the experienced TAs. This result is quite surprising as you would rather expect experienced TAs to have a specific focus in the discussion. However, the number of cases is quite small, it would be very interesting to analyze this aspect for a bigger set of data.

Final remarks

This paper presents five different patterns for the discussion of tasks in mathematical tutorials. The "heuristic discussion" focuses on strategy teaching which is very important for learning. This type consumes more teaching time than the other types and is rather used by experienced TAs. As Brophy explains, this kind of instruction is not "only demonstration of and opportunities to apply the skill itself but also explanations of the purpose of the skill (what it does for the learner) and the occasions on which it would be used" (2000, p. 25). Therefore, the students can hopefully gain more than just another solution from this kind of discussion.

In the "student-oriented discussion" and the "problem-oriented discussion" TAs focus on aspects the students struggle with and help them to overcome these difficulties. These types of discussion take some more minutes than the average discussion. Interestingly these patterns are seldom used by experienced TAs. One explanation might be that inexperienced TAs are more aware of the students' problems as they still remember their own problems. It could also be the case that experienced TAs do not believe that these patterns help the students in their learning process. However, this result might only due to the data.

More than half of the discussions have no specific focus and few didactical elements to support students' learning. The "pragmatic discussion" and the "minimalistic discussion" consume the least teaching time, so time pressure in the tutorials might be a reason for the frequent use of these two discussion patterns. However, the TAs might have other motivations for using discussion patterns without focus. Possibly, TAs might pursue learning goals that we are not aware of in this study. This shows that further research on this aspect is needed.

According to the findings of Weber (2004), some TAs seem quite flexible in using the different discussion patterns. This result gives rise to possibility that we can support TAs in using the appropriate pattern for a specific learning goal. Especially, the heuristic and the problem-oriented

patterns could be practiced in tutorial training. Apart from that discussion patterns could be a topic in the weekly meetings, helping the TAs to plan their tutorials more goal-oriented.

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French engineers' training and their mathematical needs in the workplace: Interlinking tools and reasoning

Pierre-Vincent Quéré

CREAD, Université de Bretagne Loire, Rennes, France; pierre-vincent.quere@ac-rennes.fr

This paper deals with the theme "mathematics in the workplace" in the context of engineering work in France. In the continuity of recent research, it draws results from a two-step enquiry (questionnaire and interviews) with 237 French engineers. Using the Anthropological Theory of the Didactic (ATD), I study questions concerning the praxeological mathematical needs encountered by these engineers in their daily work ("in the workplace") and about their mathematical training and its adaptation to these needs depending on the training institution. This article shows that math should not only be considered as a "tool", because engineers sometimes need to have an accurate understanding of what they use. Furthermore, it shows that the two first years (called Preparatory Cycle) have a great impact on the future of these engineers' mathematical abilities.

Keywords: Engineering schools, mathematics in the workplace, praxeologies, preparatory cycle.

Introduction - Context

The field of "Mathematics in or for the Workplace" has recently received an increasing interest especially at Tertiary Level (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). Worldwide researchers have contributed to think about and beyond dichotomies such as "school versus work" maths (Bakker, 2014). In the case of engineering apprentices, Ridgway (2002, p. 189) shows that "mathematical challenges of engineering differ from the mathematical taught in school. In particular, great precision is required, applied to a variety of mathematical techniques; a good deal of practical problem solving is necessary". Hochmuth, Biehler and Schreiber (2014) go further considering differences between mathematical practices in higher mathematic lectures and in advanced engineering lectures. They highlight the idea that for "solving a specific task, (engineering) students have to make specific decisions regarding the relevance of knowledge" (p. 697). Kent and Noss (2002, p. 1) have identified "a pattern of mathematics-in-use in which mathematics of school (are) transformed in something rather different, […] part of a social practice", and Romo-Vázquez (2009, p. 37) adds that "their most advanced dimensions tend increasingly to be supported either by experts or by software" and that "the needs of non-specialists seem to move towards the ability to manipulate these mathematics as a tool for communication through specific languages" (p. 37). All these works evidence that the usual training received by future engineers is not always adequate and depends on the kind of training institution. They also evidence that their mathematical needs are complex.

In the following, I investigate similar issues in the context of engineering education in France. In this country, "engineering schools" are independent institutions, not inserted within universities. To become an "engineering apprentice" in an engineering school, students first have to follow two years of "Preparatory Cycle" after the baccalaureate. These two-year studies can take place in different kinds of institutions:

- CPGE (Preparatory classes, Classe Préparatoire aux Grandes Écoles): This is a demanding training that concerns 50% of French future engineers. It takes place in "Lycées" (upper secondary schools) and has historically been created to allow students to enter the most prestigious engineering schools. The curriculum is rather generalist, and the admission very selective.
- CPI (Integrated preparatory cycle, Cycle Préparatoire Intégré): for nearly 25% of future engineers, this training takes place directly in engineering schools. The curriculum is more adapted to the specialty of the school (Mechanics, Chemistry, and so on); the admission is also selective.
- University: the remaining 25% of French future engineers follow their Preparatory Cycle in classical Universities (no selection for admission).

In this paper, after a presentation of the theoretical framework and my research questions, I explain the methodology and the details of my enquiry. It comprises two elements: an online questionnaire submitted to working engineers, and semi-structured interviews with some of the respondents. I analyze the answers to selected questions of the questionnaire and then the interviews. Finally, I discuss these results and present some perspectives.

Theoretical framework and research questions

Mathematics practices in the workplace are conducted by the needs of the workplace itself. The diversity of existing tasks added to the particular tools and resources used in each workplace tend to make a research generalization difficult. Moreover, it is recognized that "school mathematics are often obscured by the production goal, technology, artifacts and established routines of workplace activity" (LaCroix, 2014, p.158). Furthermore, speaking of "school mathematics" requires making a difference again between the institutions where the training has taken place. For these reasons I have chosen an institutional perspective, provided by the Anthropological Theory of the Didactic (ATD) (Chevallard, 2006).

I use in particular the concept of praxeology that is a system $[T, \tau, \theta, \Theta]$ designed to model every human activity (i.e. a certain subject's activity in a certain institution). Among the four elements of this organization, we can first meet the type of tasks T. The observed type of tasks T is associated to a technique τ to create the "practical-technical block" (so-called a *know-how*). The second block is called "technological-theoretical block" formed by a technology θ (meaning a rational discourse that justifies the technique that is used) and a theory Θ, whose role towards the technology is the same as the one of the technology towards the technique. I am interested in "mathematical praxeologies", which means here praxeologies where mathematics intervenes in one or several components of the praxeology. In Chevallard's theory, praxeologies can moreover be adapted from a very general to a very precise point of view, following "codetermination levels" that I do not detail here (Chevallard, 2006). Using this approach, the two research questions I study in this paper are the following: (1) Which mathematical praxeologies live in the "workplace" institution for French engineers? (2) In which institution did they learn the mathematics they use in the workplace?

Methodology

The first step of my enquiry is an online anonymous questionnaire addressed to active engineers. I have sent it to institutional mailing lists (more than 20) of former French engineering students. To be as relevant as possible, I have tried to spread this questionnaire in schools with different domains of specialty such as data processing, electricity, electronics, agronomy, finance, chemistry, mechanics, materials, etc. In fact, I was not able to know in advance the number of engineers that would receive the invitation to participate, nor how many of them would answer.

The questions mostly deal with the training engineers have received in maths and the questionnaire is divided in four parts. Only the first three ones will be analyzed in this work:

- The first one concerns personal and professional elements.
- The aim of the second part is to precise what kind of praxeologies they have encountered during their training in mathematics in their engineering school.
- The third part concerns their effective use of mathematical praxeologies: Is maths a real need for their job? For what type of tasks do they need maths more frequently? For what professional objectives? Have they had in-service or self-training after their engineering school? What difference with the techniques of the initial training? What kind of tools: software, books, community, lectures notes, MOOC, etc.?

The second step of my enquiry consisted of semi-directive interviews with 6 engineers selected according to their responses to the questionnaire and representing different classes according to the following variables: age, gender, institution of preparatory cycle (I've invited some ex-CPI students but none of them have unfortunately answered) and domain of specialty (see Figure 1).

	John	Peter	George	Matthew	William	Alice
Age/Gender	25/male	27/male	35/male	29/male	35/male	30/female
Qualification	Computer	Computer	Materials	Chemistry	Electricity	Materials chemist
Domain of work/job	Signal (audio) processing	Data security	Consultant	Control process engineering	Entrepreneur in financial analysis	Motorcars development engineering
Preparatory Cycle	CPGE	University	CPGE	CPGE/University	CPGE	CPGE

Figure 1: The six engineers interviewed

I describe here briefly the four parts of the interviews: The first one concerns the opinion of the engineers about their own training (preparatory and engineering curricula) regarding their current specific mathematical needs: what seems to them well adapted or not and why? Based on the same idea, the second part asked them to give indications of content that should be or should have been taught in their training, how and why. The third part concerns their view about student's autonomy; I do not use it in this paper. The fourth part concerns their self-training for learning useful specific mathematical praxeologies: which devices or resources? What difference with their initial training?

Analysis of the answers to the questionnaire

237 engineers from all over the country filled this questionnaire, some of whom are currently working abroad. In part 1, I observe that the predominant represented domains of activity are Chemistry, Physics Materials and Energetic, Computer, Electrical and Electronics, Production and Mechanics, Generalist, Agronomy and Economy. The repartition according to the principal variables is as follows (Figures 2):

Age	Min	Med	Max	Avge
Years	24	29	61	32

Gender	Women	Men
%	38	62

Preparatory cycle	CPGE	CPI	Univ
%	68	12	20

Figure 2: Age, Gender and Preparatory Cycles repartition

In part 2, question 10 (have you received a training in mathematics in your engineering school?), 183 engineers amongst the 237 (77%) answered yes. Among the other 23%, we note that 83% are chemistry engineers. This may indicate that the mathematical training depends on the precise orientation of the studies.

Question 12 (During your training in engineering school, the main mathematical contents taught were…) concerns the mathematical contents mostly taught in the engineering schools, for which I proposed a list of main mathematical themes. I chose those themes according to groups of chapters mostly found in maths literature for engineers: the results are in Figure 3.

Contents	Scientific computation	Analysis	Algebra	Probability	Statistics	Modelling	Logic	Set Theory	Graphs
%	40.4	47.5	44.8	68.9	84.7	27.9	23.5	17.5	16.9

Figure 3: Mathematical contents taught in engineering schools

In Figure 3, I notice the score of Statistics and Probability: it seems to be the most common mathematical theme taught in the engineering schools in France, followed by Analysis.

In part 3, question 19 (Would you say that you encounter (or have encountered) a real need of mathematics in your job as an engineer?), 53% declare that they do not have a real need of maths. In the next question (question 20), like in question 12, I proposed a list of main mathematical themes used in the workplace; the results are presented in Figure 4.

	Scientific computation	Analysis	Algebra	Probability	Statistics	Modelling	Logic	Set Theory	Graphs
%	69.4	44.1	25.2	37.8	55.9	49.6	54	9	18

Figure 4: Main mathematical contents needed

In figure 4, the Scientific computation domain reaches the highest level. Then comes Statistics but with a far lower result compared with Figure 3; we observe the same for the Probability, Algebra and Set Theory domains. On the contrary, according to those percentages, the domains of Scientific computation, Modelling and Logic seem to represent important needs although they are not taught widely. In the answers to question 21 (For what kind of professional tasks?), the engineers explain

the practical use of these contents. The tasks mentioned are simulation, modelling, data analysis, software or algorithms development, basic calculus for estimations, budgets, chemical dosing…

Analysis of the interviews

In this section I try to observe, drawing on sections 1 and 4 of the interview, the mathematical praxeologies present at the workplace, according to the interviewees. I recall that I consider as a "mathematical praxeology" a practice, and a discourse commenting/explaining this practice, where mathematics intervene. I propose a classification of these praxeologies, and I also try to identify in which institution the mathematics involved were met.

Transversal types of tasks and mathematical technologies

I classify in this category praxeologies of the workplace where the types of task is general, not necessarily linked with mathematics (as we see below, it can range from "problem solving" to "communicating"); and the engineers mention mathematical techniques, and even more importantly technologies in the corresponding praxeology.

Some engineers identify, in the workplace, "reasoning" or "problem solving" type of task directly linked or not with mathematics (e.g. making an estimation of costs). Those coming from CPGE declare that, for such tasks, techniques and technologies they learned during this preparatory cycle are useful. The techniques and technologies they cite are linked with proof, testing hypotheses or logic. Obviously these techniques and technologies have been met in CPGE for very different types of tasks, but these engineers have transferred them to the workplace. For instance, John says that proof, seen as a method in CPGE, is very important to him in his job because it makes him understand the utility of mathematical rigor. George explains that, as a project manager, he has to understand the mathematical thinking hidden behind a phenomenon more than the phenomenon itself. William says that the prominence of hypotheses verification in reasoning is what sometimes makes the difference between him and some of his colleagues, as well as being able to rigorously check the result of this reasoning at the end. Finally, Alice tells us the importance of logic in her everyday job. She gives the example of the contraposition: when she had been taught this kind of logical reasoning in CPGE, she thought it would be useless for her. Years later, when she had to work on "experience plans", she realized that it is very important to master it when trying to show that an implication is true or false.

According to the declarations of the interviewees all the mathematical contents corresponding to these daily needs are taught especially in CPGE more than any other institution.

Another kind of transversal mathematical praxeology is what John, Matthew and William refer to as "basics" – that we identify with the term 'basic skills' used by Ridgway (2002). The corresponding types of task in the workplace are situated in many domains like cryptography (Peter), resolution of recursive problems in computing (John), and actuarial science (William). Because of the variety of tasks, it is also difficult to identify comprehensively all the techniques (integrating, solving equations or differential equations, etc.) and technologies (functions of several variables, geometry, matrices) in use. One important type of task appearing in the interviews can be formulated as: "Meeting and understanding new concepts". For this type of task, having a good general knowledge

in Analysis and Algebra, including theoretical aspects, is mentioned as very helpful. This can be seen as an evidence of the theoretical bloc of praxeologies in action.

In a similar way, I identified in the interviews the type of task: "communicating about or with mathematics". George declares that, thanks to his training in CPGE, he feels at ease to communicate about maths subjects with the people he works with. In this case the type of task is directly related with mathematics, and the techniques for presenting mathematics have been learned in preparatory classes. Another type of task cited by George is "Exploring new domains" like, for instance, static physics. I observed the same type of task for Matthew and William in other domains like computation or finance. For this type of task their initial training in mathematics is not sufficient, and brings to "searching on the Internet"(forums, specialized websites). Sometimes they have a look into their old lecture notes or in books as mathematical references that they need anyway to be able to enter the field. For this way of learning, they say that they feel satisfied to find the right information by themselves.

Types of tasks in specific domains and mathematical techniques

In the interviews the six engineers also describe types of tasks met at their workplace but belonging to scientific domains, like physics; the techniques in the corresponding praxeologies include mathematics. In these praxeologies I did not clearly identify technologies. This is the second type of mathematical praxeologies I observe in my analysis.

First, I would like to highlight the fact that basic mathematical skills are also mentioned as providing techniques for many specific types of tasks in various domains, like for example the task "modeling the ageing performance of a material" (Alice). Nevertheless, the principle of use of the techniques and technologies differs: the aim is to be able to use some results (like theorems or formulae) without trying to understand them mathematically. Most of those basic skills are taught in the Preparatory Cycle, but the techniques (and technologies) they provide for the workplace are taught in the engineering schools. In fact these types of tasks are well known by them since many years; the same holds for the associated techniques.

Amongst these basic skills, the case of Statistics and Probability seems specific because this domain is mostly not taught in the various Preparatory Cycles in France. Each engineering school provides its own specific training adapted to its needs. According to the interviewees, once confronted in the real world of the workplace, sometimes a statistics formula becomes useful (they mostly remember having learnt at the engineering school a lot of theory which does not intervene in their work).

Reasoning + Using = "Reasusing": a concept for a personal and new mathematical experience

A last category of mathematical praxeology I found in the interviews combines mathematics in the techniques, in the technology and even in the theory. This seems to be linked to specific types of task, requiring the development of original techniques – almost a research work. John cites a type of task that can be formulated as: "outperforming competitors in the design of new software". He explains that he has to know which theorem he must use, but not exclusively: he also has to have a deep understanding of the proof of this theorem to be able to understand which parameters will allow him to obtain a result in a smarter way than other colleagues. To illustrate this, he gives the example of audio latency that is one of the most important qualities for the client of music production software. The type of task here could be "Reduce the latency". It corresponds to a short

period of delay between when the musician plays and when he can hear the sound through the sound system (e.g. headphones). When the competitors offer a 20 milliseconds latency, John has to put his efforts to find in the theorems or in their proofs (mostly based on Fourier Analysis) how to minimize it to 6 ms. This will make the commercial difference and it requires that he really understands what is happening "inside" the theorem. This corresponds to the technique "analyze a theorem proof". I consider this as a third type of mathematical praxeologies with a type of task requiring some innovation.

Discussion - Conclusion

Drawing on the results exposed in this paper, I now come back to the two research questions presented above.

Regarding the mathematical praxeologies that live in the "workplace" institution for French engineers, the primary result in this study is that only 47% declare they have a real need of maths in their everyday job. Concerning the mathematical needs, I have encountered three different kinds of praxeologies: A first one with a general type of task, like "solving a problem" or "communicating"; techniques, and mostly technologies involving mathematical elements like reasoning and proving, and also some elementary mathematical skills. Rigor, logic and an amount of maths basics (sometimes considered as useless at first sight, because lacking of concrete sense to them) are necessary for the everyday work of these engineers, and also allow them to communicate more easily with other people in their working environment. The second kind of praxeology that lives in the workplace comprises specific tasks (simulating, modeling, data analyzing, calculating, etc.) associated with mathematical techniques: here again, the maths basics are considered as very important but they are seen as providing techniques. The last and rather interesting kind of praxeology is the mix of reasoning and using (I call it "reasusing"): for an engineer, it means to interlink a technology or even a theory to make them become an integrated part of a technique for a specific kind of mathematical type of task (such as a logical analysis of a situation, understanding a theoretical mathematical concept).

For the second research question about the institution where they learn the mathematics they use in the workplace, I notice that the praxeologies developed in all types of Preparatory Cycles are mostly concerned with teaching basic mathematical skills. To end this analysis, I must highlight that the engineers who declare needing the first kind of praxeologies (thinking, reasoning and problem solving) that where taught during their Preparatory Cycle are all coming from the CPGE institution.

Finally, my study certainly has some limitations. It cannot be considered as fully representative of the whole population of French engineers (in terms of age, gender, domains of work, and Preparatory Cycles). Moreover a large part of it is based on what the participants *say* about the mathematics they have learned and use, but it is not clear that they all have in mind the exact same interpretation of things. I will work on this issue in my future research.

But the results that I expose can lead us to think that even if an important part of the engineers do not really need mathematics daily, they do not consider them exclusively as providing techniques. Receiving a training of the type "maths as a toolbox" is not satisfactory for them because they sometimes need to understand the precise functioning of the tools. It is possible for them thanks to their own mathematical "culture" (or background) and also their will to investigate by themselves

some new concepts. I interpret this as the need for "complete" praxeologies (Bosch, Fonseca & Gascón, 2004): the engineers do not only need the *praxis* (basically taught in engineering schools), but also the *logos* (essentially depending on the Preparatory Cycle training). Moreover, several interviewees declared that they did not perceive the usefulness of the theoretical aspects when they were students. We interpret this as a need to motivate the praxeologies when taught.

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Approaches to learning of linear algebra among engineering students

Ragnhild Johanne Rensaa

UiT the Arctic University of Norway, Norway, ragnhild.rensaa@uit.no

The present paper investigates engineering students' own descriptions of what they mean by learning of linear algebra and how they know that they have learned something. I seek to extract keywords from engineering students' descriptions of learning of this discipline by drawing on grounded theory techniques and classifying the answers in conceptual and procedural approaches. By this, both detailed and more meta perspectives on learning are obtained. Results indicate that when explaining their learning of linear algebra, conceptual more than procedural approaches are emphasized. However, in order to know that they have learned something, many engineering students need to know that they are able to solve relevant tasks in the discipline.

Keywords: Approaches to learning, linear algebra, engineering students.

Introduction

Students' learning of mathematics is a main interest within the community of researchers in didactics of mathematics. We seek to know how students learn, what they learn, but also how they perceive their own learning (Sfard, 2007). Learning may be defined according to which point of view one has in an investigation, but also by taking into consideration what is relevant for the particular individuals of a study. A classical definition is given by Hiebert and Lefevre (1986), distinguishing between conceptual and procedural knowledge that may yield conceptual and procedural learning. Conceptual knowledge is defined as "knowledge that is rich in relationships" (ibid.1986, p. 6), which means that it cannot exist in isolation. Procedural knowledge includes sequential relationships or step-by-step instructions. Engelbrecht, Bergsten and Kågesten have found conceptual and procedural notions valuable in their research of engineering students (2009), and because the target group of the present investigation is engineering students, these constructs will be utilized.

The present paper focuses on engineering students' interpretation of their own learning in a linear algebra course. Such reflections are beneficial because the students then have to reflect on how they see their mathematical knowledge and for what purposes they study the discipline. Thus, asking questions about learning is valuable and frequently done by researchers. An immediate example is the present data collection, in which questions asked to the students were picked from a research investigation of a related group of students in a mathematics and physics foundation program for students going into an engineering program (Marshall, Summers, & Woolnough, 1999). Based on data from a longitudinal study over an academic year, they derive conceptions of learning held by these students. In my study the setting is somewhat different as the students are experienced engineering students, their reflections about learning are confined to a particular domain in mathematics, and it identifies students' reflections at the end of the course. In this particular setting the following research questions are asked: Which approaches do engineering students include in their description of learning in linear algebra and how do they explain their knowing that they have learned something?

Theoretical background

The study reported on here investigates engineering students' description of their learning approaches rather than the cognitive processes of learning itself. As will be argued for, such approaches are adequately split in two main categories: approaches connected to conceptual and to procedural knowledge. The definitions were originally given by Hiebert and Lefevre (1986) and are widely used. In this framework, conceptual knowledge is pieces of knowledge connected together or, as explained by Kilpatrick, Swafford, and Findell (2001), "an integrated and functional grasp of mathematical ideas" (p. 118). Procedural knowledge, on the other hand, includes familiarity with symbols but also representation systems in mathematics along with knowledge of rules and procedures that can be used in task solving strategies in mathematics (Hiebert & Lefevre, 1986, p. 6). However, conceptual and procedural knowledges are partners and the interplay between them is valuated, emphasizing how one knowledge may lead to the other (Rittle-Johnson & Alibali, 1999). Indeed, they are increasingly regarded as interrelated and inseparable, but also object for extensions to superficial and deep qualities of the knowledges (Baroody, Feil, & Johnson, 2007). Such relationships are multifaceted, and researchers move towards more integrated views in which determining the dynamics between the two is the objective (Engelbrecht et al., 2009).

Students often perceive linear algebra as difficult. This stems from three sources of difficulties (Dorier & Sierpinska, 2001). It is about the pedagogical approach, as proofs are found difficult (Rogalski, 1990). It is also a matter of difficulty with grasping the theoretical concepts and mathematical language; the 'obstacle of formalism' (Dorier, 1997). Finally, linear algebra demands a 'cognitive flexibility' as one has to move between different languages, both theoretical and practical forms. Students tend to think in practical terms (Sierpinska, 2000), and lack of connection to theoretical structures may hinder their learning (Dorier & Sierpinska, 2001).

Engineering students recognize mathematics as a foundation of their education (Khiat, 2010). Still, they consider the discipline as a routine practice of their profession (Steen, 2001) and expect to be exposed to real-world engineering problems in mathematics (Hjalmarson, 2007). With such an approach, the formalism of linear algebra may be especially hard to get a grip of. Engelbrecht and colleagues (2009) found that engineering students uphold mathematics as procedurally founded. As part of their investigation, the authors created tailor-made working definitions to focus on engineering students, thus these are adopted in the present study:

"Procedural approach: Use and manipulate mathematical skills, such as calculations, rules, formulae, algorithms and symbols.

Conceptual approach: Show understanding by e.g. interpreting and applying concepts to mathematical situations, translating between verbal, visual (graphical) and formal mathematical expressions and linking relationships." (Engelbrecht et al., 2009, p. 932).

Methodology

The present investigation is part of an ongoing study dealing with engineering students' views about the learning of linear algebra. The teaching format in the course which was taught in English was 'traditional', with large group lectures followed by task solving sessions where students worked in groups. The 'untraditional' part was that a well-functioning video recording system recorded all

lectures and published them in-time. The linear algebra course was scheduled in the students' fourth year of studies to become master engineers, postponed in accordance with Carlson's recommendations (1993). However, some basic tools in linear algebra had been introduced in a mathematics course in their first year of studies, since these are necessary for use in the professional disciplines. All together 59 students attended the course this year, and data was collected as I was the teacher and arranged for a questionnaire to be answered at the end of the course. The open questions picked from (Marshall et al., 1999) discussed in the present paper were: "What do you mean by *learning* in linear algebra? And how do you know that you have *learned* something?" Due to experiences from a previous investigation (Rensaa, 2014), the questionnaire was made mandatory but anonymous to increase truthfulness, and the response rate was very good; 93% (55 out of 59).

Data analysis was done in phases. Initially, grounded approaches were used (Strauss & Corbin, 1998) to obtain codes that embrace engineering students' approaches to learning. Next, these codes were related to the definition of conceptual and procedural approaches as described by Engelbrecht and colleagues (2009) since this definition is tailor-made for engineering students. It offers a meta-perspective on the analysis results from coding, and this provides answers to the research questions about engineering students' approaches to learning.

Analysis and results

The development of codes was done in steps. Initially, I wrote down headwords in each student's description which was given in English. By comparing these, some seemed to describe similar things, e.g., 'utilize for own goals' and 'use in gps' [Global Positioning System], both which could be interpreted as 'learning as applying mathematics'. Because I was working back and forth between statements and codes with an aim of reducing the number of codes without deteriorating their meanings, each time two replies were interpreted within the same category had to be put down as a criterion for the category. For instance, for descriptions of obtained learning, 'know the whole picture' and 'associate theory to applications' were both interpreted as being able to relate the different aspects of linear algebra to each other, thus crystalizing a category called 'ARel' (able to relate). The importance of emphasizing relation in this category was helped forward by a statement that did not fall into this category: 'use different theorems to achieve solutions to practical problems'. The emphasis here is on obtaining solutions more than the relation, thus crystalizing a category called 'ASol' (being able to solve problems). Going back and forth between statements and codes resulted in a final reduction to 8 categories for what learning is and 6 categories for what is meant by learning of linear algebra.

Next, the original data set and my developed codes were sent to another researcher for validation purposes. This researcher used the codes to independently code the data. Then, we met for comparison of results and refinement of codes. A main refinement was deepening the meaning of *applications*. Students had referred to applications when trying to describe learning in linear algebra, but we agreed that students should express that applications were actively studied in a mathematical connection in order to be coded as 'Study Applications' (SAp). An example of a statement where the coding was adjusted by this interpretation is the following:

> Student 30: For me, learning is knowing the practical use of theory and how to execute said theory. As a computer engineer student specializing in games development, linear

> algebra is central in the programming I perform. I only know I have learned something if I can associate theory to a problem I encounter.

We agreed that this student is not stating that he is studying applications, but rather that he is actually taking advantage of knowing applications from other disciplines as part of his learning process. Thus, 'Utilize Theory' (UTh) is a closer category as the statement points to how theory may be utilized for practical purposes. The other refinement of codes that was needed was a specification of *relations,* originally named 'Rel'. It was unclear which types of relations this was referring to. The category had derived from students' answers as relating back to previous knowledge, thus the category needed to be adjusted to 'RelB' (relating to background).

Two additional codes were agreed on: the categories 'NoAns' (no answer) and 'Other'. All blank responses could be categorized as 'NoAns', while 'Other' refers to answers that responded to something else than what was asked about. The 'Other' category developed from cases in which divergence in our separate coding appeared. We both encountered problems because none of the codes actually fit with some of the particular answers. An example is 'It really gives the knowledge of different engineering mathematical problems'. One researcher had interpreted this statement as 'Study Applications' (SAp), the other as 'Able to understand why/what is going on' (AUn), but the student does not seem to be actually describing his learning. Thus, the final coding for this response was 'Other'. This joint coding process showed that the codes were adequate and could be used to code all statements. However, we experienced that coding statements together often resulted in finding more information in a reply than what we had done individually.

Ending the process, the following codes crystallized for engineering students' description of what they mean by learning in linear algebra: **SAp** (Study Applications), **GUn** (Gain Understanding), **UTh** (Utilise Theory), **ForM** (Grasp Formalism), **SimP** (Simplify), **SoL** (Solve problems), **RelB** (Relating to Background), and **ToO** (Use Tools). Analytical results for this question are given in Table 1, presenting both the number of students in each category and percentage (rounded off) of the total number of 55 students. The category 'No Answer' consisting of 17 replies is left out, while a number of explanations covered approaches in more than one category. Thus, the sum of percentages does not add up to 100.

	SAp	**Gun**	**UTh**	**ForM**	**SimP**	**SoL**	**RelB**	**ToO**
Number/%	8/15%	11/20%	10/18%	2/4%	2/4%	11/20%	2/4%	2/4%

Table 1: Responses to what engineering students mean by learning in linear algebra

Coding responses to engineering students' description of how they know that they have learned something gave the following codes: **ASol** (Able to Solve), **AExp** (Able to Explain), **AUn** (Able to Understand Why/What is going on), **AAp** (Able to Apply), **ARel** (Able to Relate), and **ARem** (Able to Remember). Analytical results for this question are given in Table 2, including responses coded as **Other** (answering something else). The table presents both the number of students in each category and percentage, and again multiple codes were found in some answers.

	ASol	AExp	AUn	AAP	ARel	ARem	Other
Number/%	15/27%	3/5%	6/11%	9/16%	1/2%	2/4%	5/9%

Table 2: Responses to when engineering students know that they have learned something

When the codes and categories were set, I assigned the codes in conceptual and procedural parts. As the codes had developed based on engineering students' own descriptions, they were aligned with Engelbrecht and colleagues' working definition (2009) for conceptual and procedural approaches of engineers. This was done by linking the description of codes to statements given in the definition. Some codes were easier to categorize, like GUn. Gaining understanding was classified as a conceptual approach as this is necessary to be able to expose mathematical understanding. Other classifications were harder. An example is ASol. Problems may be complex, theoretical and demand deep argumentations, and solving these should classify as a conceptual approach. On the other hand, problems may as well be 'standard', connected to a set of skills that are more like a routine part of a learning process. Such dual interpretations of an activity highlight the complexity involved in interpreting conceptual and procedural knowledges in a praxeology. However, engineering students tend to 'proceduralize' problems, even those of a conceptual nature (Engelbrecht et al., 2009). Considering this, I deduced that ASoL ought to be categorized as a procedural approach, but highly interdependent upon conceptual approaches

By going back and forth between the definition and codes, a final classification of codes was obtained. For what is meant by learning in linear algebra, the following codes were classified as *conceptual*: SAp fits with 'applying to mathematical situations'; GUn is about 'showing understanding'; UTh may be interpreted as 'translating between verbal and formal mathematical expressions'; and RelB is about 'linking relationships'. The remaining categories were classified as *procedural*: ForM is about 'manipulating' linear algebra expressions; SimP is simplifying by 'calculations'; SoL refers to a way of 'using mathematical skills'; and ToO is to use tools like 'rules, formulas and algorithms'. About knowing that something is learned, the following codes were classified as *conceptual:* AExp is about 'interpreting concepts'; AUn is about 'showing understanding'; AAp is about 'applying concepts to mathematical situations' and ARel is ability to 'link relationships'. The remaining codes were classified as *procedural*: ASol is knowing how to 'use and manipulate mathematical skills'; and ARem may be a part of the manipulation of mathematical skills by recalling how to do this. Drawing on these interpretations, Table 1 and 2 may be organized in conceptual and procedural approaches. Gray coloring of conceptual cells and white coloring of procedural cells indicate the appropriate classification. In many cases, an interpretation of a student's reply comprised more than one of the codes given. An example is the following statement with three codes of a conceptual type and one of a procedural type, codes included in parenthesis:

> Student 6: Generally, I mean that learning is to study something until you understand (GUn) the theory (UTh), and is able to use it in both theoretical and practical problems (SAp and SoL).

A statement could be coded in a mix, as illustrated by the last part of the above statement. Interpreted as being '*able* to use it,' this may be about studying applications as a way of utilizing knowledge in problem solving – SAp, a conceptual approach. Interpreted as being 'able to *use* it' this would be more about the solving process itself – SoL; a procedural approach. Thus, a statement could be coded in both procedural and conceptual categories, again illustrating the close relationship.

Discussion

The analysis results summed up in Table 1 and 2 give some indications of engineering students' conceptions of learning. In many cases, an interpretation of a student's reply comprised more than one of the codes and one phrase could be coded in a mix as illustrated by Student 6's explanation. Engelbrecht and colleagues emphasize that the distinction between conceptual and procedural approaches are complex and not absolute (Engelbrecht et al., 2009). Thus, mixed coding may be expected. Brought together, however, the frequencies of codes give a meta perspective on which approaches (procedural or conceptual) are most appreciated by engineering students. In this perspective, Table 1 shows that engineering students emphasize conceptual approaches more than procedural ones when explaining what learning in linear algebra means to them.

Table 1 shows that 'Gain Understanding' (GUn) is important to students, having the highest response rate. However, understanding is often – like in the above example – connected to knowing how to *apply* this understanding. Only when being able to apply their knowledge the students think they have understood linear algebra. This result is in line with the fact that these students are engineering students, busy with relating to the use of mathematics (Hjalmarson, 2007). To some students, however, solving of problems becomes the main issue and the scale by which they measure their learning. Lower interest is given to understanding, as the main objective is to obtain a correct answer. An example is the following:

> Student 34: in my opinion, linear equations are some kind of tool (ToO) to solve the problems (SoL) in real industrial areas such as factories and… (AAp).

Not all replies coded as describing learning in a procedural way focus on solving problems. Grasping formalism, which is an aspect of difficulty for students when learning linear algebra (Sierpinska, 2000), may also be interpreted as a procedural approach in terms of manipulating the linear algebra language. This is illustrated in the following student's description:

> Student 5: the meaning of learning linear algebra is actually learning a mathematical language (ForM), a language you can use to solve big questions with many variables (SoL).

Responses to the question about engineering students' knowing that they have learned something, summed up in Table 2, are more equally distributed between procedural and conceptual approaches. This is mainly due to the category 'Able to Solve', which takes all together 27% of the responses. An example of a statement coded within this category is:

Student 35: The simplest way to know that I have learned something is that I can solve some problems (ASol), when I am faced with some practical problems using this method.

This student indirectly says that he seeks to apply the mathematics in practical situations but knowing that he has learned something is concentrated to the solution process itself.

Altogether, a rough answer to the stated research questions may be that the present engineering students emphasize conceptual more than procedural approaches when explaining learning of linear algebra, but in order to know that they have learned something a noteworthy amount need to know that they are able to solve relevant tasks in the discipline.

Conclusion

A result of the present analysis is that the engineering students emphasize conceptual aspects like understanding and utilizing theory as most important in their learning of linear algebra. This may be an anticipated result when dealing with students in general, but engineering students' expectations towards mathematics are slightly different. They consider mathematics more as a routine practice (Steen, 2001) and procedurally founded (Engelbrecht et al., 2009). Thus, the result is noteworthy. However, to know that they have learned something, the same students seek confirmation in terms of being able to solve problems; a more expected procedural approach. An explanation to this result may be that the mathematics course is one in linear algebra. This course is more theoretical framed than the initial calculus courses, thus students are somewhat new to proofs and proving when coming to the course. Students find such approaches difficult (Dorier, 1997; Dorier & Sierpinska, 2001; Rogalski, 1990), and engineering students may therefore put particular attention on these aspects in learning of linear algebra. Their consecutive measure of knowing that they have learned something in terms of ability to solve problems then shows that the connection between theory and task design is particularly important. Tasks should offer opportunities to engage in conceptual arguments on the preferred premises of solving tasks. However, as assessment guides students' ways of studying, task design in exams is the most vital part. Thus, an investigation of engineering students' learning approaches related to design of exam tasks will be an important follow-up of the present project.

Even if students in the present study were asked to reply in writing – which naturally reduces the richness of the replies compared to responding orally – interesting responses were given. The following is an illustration of this, concluding the paper:

Student 9: To learn does not necessarily mean to remember something, but to understand it in depth (GUn) and be able to utilize that information for your own goals (UTh). When one has truly learned something, one can easily explain it to someone else (AExp).

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Access to conceptual understanding – Summer courses for linear algebra and analysis after the first semester

Kirsten Schmitz[1] and Ingolf Schäfer[1]

[1]University of Bremen, Germany;

kschmitz@math.uni-bremen.de, ingolf.schaefer@uni-bremen.de

The transition from school to university mathematics is known to present a major challenge to many students, resulting in poor performances and high dropout rates during the first semesters. In this paper we present the preliminary results on summer courses in Linear Algebra and Analysis after the first semester, which were designed based on the Abstraction in Context framework (Dreyfus & Kidron, 2014) and on the self-determination theory (Deci & Ryan, 2000). In particular, we investigate the potential of this course design to contribute to the motivation of the students and to their ability to engage in concept construction. These qualitative preliminary results will be used in further research to quantitatively assess the effect of these courses as well.

Keywords: Mathematical concepts, advanced mathematical thinking, tertiary level mathematics.

Introduction

A recurring problem in undergraduate mathematical education is students' difficulty in coping with the transition from school mathematics to the higher paced, more demanding and more formal mathematics at university level. This can be seen in the high dropout rates of math students in the first semesters and the poor exam results. In particular, some students require more time to learn than the time span determined by the lecture course, and find it hard to catch up with their peers for the next course. The previous approach of the mathematics faculty at the University of Bremen had been to install weekly extra tutorials for the courses in Linear Algebra and Analysis, parallel to the corresponding lecture courses. Even though these were conducted by senior student tutors, they yielded poor results. Few students attended, and there was no perceived positive effect on motivation or performance with regard to concept building. Limited time during the semester was stated as the main obstacle to continuous attendance.

In response to this, a new approach consisting of summer courses in Linear Algebra and Analysis was developed and implemented by the authors. These were voluntary courses that took place within two weeks between the first and second semester, after the students had not only had the weekly feedback from their homework groups, but also some individual feedback on their progress by the end-of-semester test. In addition to the goal of enabling the students who had failed (or scored low in) the first test to pass the retake, the courses were intended to foster the students' motivation and help them create appropriate concept images (Tall & Vinner, 1981) of the core concepts of each course. This is assumed to be of vital importance, in order for them to be able to profitably take part in the follow-up courses in the second semester. From the experience gathered in the first implementation of these courses, we intend to give a preliminary evaluation with respect to these two factors. This will be used to make some adaptions to the following implementations, which will then be evaluated more thoroughly.

In this paper, we will focus on the description and evaluation of the Linear Algebra additional course given by the first author. The analysis course was designed analogously by the second author and the results were similar.



Figure 1: Structure of the Linear Algebra module

In figure 1 the structure of the whole Linear Algebra module in Bremen is displayed. The lectures extend over 14 weeks with 4 hours of lecture, 2 hours of exercise in groups and 2 hours of plenary exercises each week. The test at the end of the first semester takes place about a week after the lectures end and does not contribute to the final grade. The students nevertheless have to achieve a certain number of points in order to complete the module. The pass rate is set at a low level compared to the final exam, so the students know what to expect on the final, but are not impeded by having to retake the test, unless their score is very low.

Theoretical background

In their extensive study on teaching Linear Algebra in the first university semester, Dorier et al. (2002) identified and described a main obstacle for students to learn the subject, which they called the *obstacle of formalism*. This obstacle is found to be a conglomerate of formal reasoning, abstraction and extracting ideas from concepts, due to the vast abstractions, simplifications, and unifications in the subject's history.

We tried to give students additional support to master this difficulty by a course design based on the *Abstraction in Context* framework, AiC, (Dreyfus & Kidron, 2014). This framework merges the idea of the *vertical mathematization* process by Freudenthal with Davydov's *method of ascent to the concrete* (ibd., p.87), in order to explain how abstraction processes can occur. Abstraction is assumed to happen in a "three-stage process: the need for a new construct, the emergence of the new construct, and the consolidation of that construct". The corresponding (observable) epistemic actions are *recognizing* (R) the relevance of certain known constructs in a given situation, *building with* (B) these constructs to achieve (local) solutions and *constructing* (C), i.e., integrating the previously known constructs into a new whole. This model is referred to as the RBC-model.

Since performance at challenging and creative tasks is dependent on intrinsic motivation, we make use of the *self-determination theory of motivation* (SDT), which postulates the existence of three basic psychological needs (Deci & Ryan, 2000). When these needs for autonomy, competence and sense of relatedness are fulfilled at a satisfactory level, intrinsic motivation is likely to occur. Low achieving students, in particular, may benefit from an environment that fosters positive experiences with regard to these needs in relation to mathematics (Um, Corter, & Tatsuoka, 2005; Rakoczy, 2006). Hence, both in the course design and during the course hours themselves, special attention was paid to these factors of SDT.

As in the methodology of design research according to Gravemeijer and Cobb (2006), we plan to develop the design of the additional courses in experiment-reflection-cycles. Gravemeijer and Cobb identify three main stages: (1) preparing for the concrete design experiment, (2) experimenting in the classroom (in this case the actual implementation of the course), and (3) analyzing the

experiment retrospectively. At this point of our research, we consider ourselves to have obtained enough information from the initial design and implementation, as to be able to proceed to the first genuine design cycle within this framework.

Research questions

The main research questions we want to deal with are the following:

1.) How does the participation in the course change the perceived fulfillment of the basic psychological needs and, thus, intrinsic motivation?

2.) What is the effect on the epistemic actions with regard to the relevant concepts for the participants in the following course?

In addition, a minor question can be seen to be "Is there a beneficial short term effect on the retake test?", as this is the main motivation for the students to take part and also an institutional concern. We are, however, more interested in the long term effects.

Description of the course design and design principles with examples

The Linear Algebra course consisted of five days of activity spread out over two weeks (alternating with the corresponding Analysis course). Each day was divided into a morning and an afternoon session of three and two hours, respectively, with a lunch break in between. The sessions themselves were each devoted to a central topic of the Linear Algebra lecture, e.g., bases, linear maps or different interpretations of matrices, and were split into smaller working units of varying types and content, which were occasionally adjusted spontaneously according to the students' needs.

The instructional design of the course was based on the observation that low achieving students often do not have the mathematical resources to occupy themselves for a long time span with a given (in general more complex and open) task, but need to be guided to acquire these resources. Thus, the course proceeded from very short and clearly defined tasks of varying nature (but with immediate feedback) to longer and more open and self-determined ones. For example, the first task of the whole course consisted of a lecturer-guided group discussion of very small exercises concerning relevant geometric objects (lines, planes), while the very last task was a guided session, in which students were asked to create individual exercises to given topics by themselves and then solve each other's exercises, including negotiations on the wording of the task and different approaches and solutions. In this way, the possibility of increasing the levels of fulfillment of the basic psychological needs was provided.

With respect to the subject specific aims, on the one hand the focus was on developing the core ideas of Linear Algebra and important techniques (Gauss' algorithm, proofs relating to algebraic structures, etc.). On the other hand, great emphasis was placed on the creation and discussion of a zoo of examples and counter-examples to the relevant notions, as this is known to be very effective in the initial understanding of new concepts (see Dahlberg & Housman, 1997).

We illustrate how the AiC framework was used in the task design by the example of the concept of "linear independence" of vectors, which was comprehensively dealt with in the afternoon session of the first day and then later referred to throughout the whole course.

To aid the construction of viable concept images of "linear independence", the students were given geometric situations, where this concept is relevant, e.g. describing a plane in a three-dimensional space as the span of two vectors and characterizing the pairs of vectors where this description fails. Thus, the students had to recognize (R) that this particular knowledge of a linear relation between vectors is relevant to particular geometric problems (note that at this point the students were already aware of the existence of the definition of linear independence). Afterwards, the students were given tasks, where linear independence appeared in different situations and in relation to other concepts (such as basis, coordinate system, etc.) as to integrate the concept into their mathematical views and work out means to examine and apply the concept locally (B). Finally, by discussions among the students and with the whole class, the students were encouraged to express and evaluate different views on linear independence, which we hope has helped the students in building an (at least preliminary) concept image of "linear independence" (C).

Since the students had already encountered the definitions during the lecture and found themselves naturally confronted with extracting meaning from them, we did not incorporate a component of guided reinvention in the sense of Freudenthal into the course. Students were not encouraged to create models of the key concepts themselves as proposed in RME (Gravemeijer, 1999), but instead the "models-of" were already given (e.g. in the form of the definition of linear independence) and had to be realized as "models-for" in corresponding applications.

Field notes including students' actions and reactions during the course were collected by the first author.

Implementation and observations using AiC and SDT

We will illustrate the implementation of the course and relate it to the relevant theory by the help of two examples. In the first sessions of the course, the students were given explicit exercises of varying type and difficulty involving some recognizing of, but mainly building with, concepts. As a first example, we will therefore report on one specific exercise at this stage of the course.

Problem 5

(a) Find a basis of $\mathbb{R}^3$, such that every entry of every vector is stricly negative.

(b) Find four vectors in $\mathbb{R}^3$, such that every choice of three of them is a basis of $\mathbb{R}^3$.

(c) Let V be an arbitrary $\mathbb{R}$-vector space and (v_1, v_2, v_3, v_4) a basis of V. Is

$$(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1)$$

a basis of V?

(d) Assume that (v_1, v_2) is a basis of a K-vector space V. Is $(v_1 + v_2, v_2)$ a basis of V? (Attention, there is one special case here, where you have to watch out. Where?)

(e) Assume we have an $\mathbb{R}$-vector space V with a basis $B = (v_1, v_2)$. Then $B' = (2v_1 + v_2, v_1 - v_2)$ is also a basis of V (can you prove this?). Find a method that can calculate the coordinates of any $w \in V$ with respect to B' from its the coordinates with respect to B. Test your method with examples!

Figure 2: A problem from the course in linear algebra

The exercise shown in figure 2 was given to students on the second day, after the concept of basis had been established and been related to the concepts of generating system and linear independence. It was part of a set of exercises, on which the students worked in small groups of two or three or individually according to their own choice. The students took to the exercises well, and most of

them found access to part (a) quickly (possibly after short exchanges with other groups or with the lecturer). Part (b) was claimed to be impossible by some students, but this was resolved mainly among the students themselves without much intervention of the lecturer. The remaining parts were difficult. Part (c) was attacked by an attempt to prove that the vectors are linearly independent, and linear independence was claimed persistently. For time reasons, part (d) and (e) were only dealt with by some students, who all implicitly assumed K to be the real numbers, V to be the real plane and whose arguments were of a geometric nature. It is noteworthy, that the students refrained from trying to manipulate the expressions according to some formal rules they had not understood, but rather tried to give meaning to the statements, albeit not always succeeding. Similar effects could be observed throughout the course.

With regard to the epistemic actions in AiC, in this exercise the students were mainly building with the concept of basis, while recognition of the relevance and the role of linear independence was required. The students' reaction (i.e. the persistent claim of linear independence) to part (c) indicates that the concept of linear independence had not yet been consolidated, although it had been extensively covered before. The geometric approach to the tasks, however, seems to show that the students already had acquired a basic concept image and tried to use this, as opposed to merely manipulating the concept definition. There was no particular observation concerning the motivation for this exercise.

The second example to illustrate the implementation is taken from the last day of the course, where the students were given the task of creating exercises themselves. Each student was given an index card, which was labeled with one of the concepts that had been dealt within the course. He or she was then asked to make up an exercise together with a solution involving the corresponding topic and write up the exercise on the index card. Students who had quickly completed this task were given additional index cards, until everyone had made up at least one exercise. The cards were then redistributed randomly, and the students were asked to solve the exercise given to them. Once finished, they were told to check their solution with the creator and if both agreed, they repeated the process with another exercise.

There were very different approaches to this task by the students. Some students chose to model their exercises on the ones they had been given during the course with only slight changes (different numbers, different number of variables/equations, etc.), while others tried to produce an original task by combining things they knew. Both approaches were encouraged by the lecturer. During the creation of the exercises, the most striking observation was that the students suddenly felt the need for certain insights, which had been difficult to stir beforehand. E.g., one student wanted to create a linear system of equations, where a row of zeros would appear at some point in the process of Gaussian elimination, and was confronted with the need for a practical criterion to achieve this. Even though this issue had been dealt with during the course, she seemed to recognize the relevance of linear independence of the rows only at this point, when it was explicitly needed by her. Additionally, the participants were aware of the need to communicate their mathematical problem well enough for someone else to make sense of it and there seemed to be a genuine effort to achieve this. In the following session of solving each other’s exercises, a dynamic of interaction was observed, where the randomly allocated pairs of creator and solver got together to sit down and negotiate the wording of the exercises and the validity of different approaches (in particular,

students who had not closely worked together before). There was very little intervention from the lecturer.

Concerning the epistemic actions in AiC, there was much activity of recognizing and building with concepts. Moreover, the proposition that there has to be a need for a concept for an abstraction process to occur was confirmed in some cases, e.g., in the case of the student described above. We do not know, however, at this stage, whether this led to the construction and consolidation of the relevant concepts, which we consider to be a long term process and not measurable in this short time span. With respect to self-determination theory, after a short period of orientation a high level of motivation was observed during this task. The students seemed to feel a strong sense of autonomy, as they were given the freedom to create a problem of their choice with only the general topic prescribed. During the period of solving, many students displayed a boost in perceived competence, particularly when they were in the creator-role, and experienced their exercise and their comments on it to be of value for someone else. Furthermore, the students appeared to build new, largely positive, relationships with each other via the random and varying allocations of creator-solver pairs. This interaction seemed to be of natural importance to the students (the extensive meetings of these pairs were their spontaneous creation and had not been suggested by the lecturer). Hence, the conditions for intrinsic motivation according to SDT were largely fulfilled, which might help to explain the unusually high motivation. We believe, however, that this exercise would not have worked out well if the participants had not been prepared for it in advance by the preceding days of the course.

Results

One of the institutional measures to evaluate the success of the course was the pass rate of the test and retake test, which was compared to the one of the previous year (as the style of the lecture and of the exams were largely the same, this comparison seems justified). In 2015, the pass rate for the regular test was about 65%, while the retake test was passed by 50% of the participants. In comparison, the regular test in 2016 was passed by 78% and the retake by 95%. Although there are of course various factors, which play a role in these results, an effect of the additional course seems likely.

In addition, the course was hoped to have a positive lasting effect both on the motivation as well as on the ability of the students to achieve concept construction by themselves with the means of the relevant epistemic actions. This has not been quantitatively assessed yet and will be the subject of further research. However, there are many indications that such an effect might indeed be observed.

During the sessions of the course, the students appeared to be (at least extrinsically) motivated and confident of benefiting from the course. Confidence and perceived competence seemed to increase, as the students advanced from the passivity of merely carrying out tasks imposed onto them by the instructor to more self-determined action. This seemed to be accompanied by a shift from mere extrinsic motivation (to pass the retake) to at least some intrinsic motivation, e.g. some students were observed to carry on discussing tasks during the break. The actual mathematical competence was seen to increase accordingly, as students gathered experience and perceived deeper mathematical insights, which they worked to develop. By the analysis of the two examples above, it

can be inferred that in the framework of AiC and of SDT there have indeed been positive effects on motivation and the ability to perform epistemic actions with regard to the relevant concepts.

These positive effects reached far into the next semester. Most participants felt that they had acquired a basic knowledge and techniques, which were necessary for the following courses, and were repeatedly observed by the authors to use methods they had picked up on in the additional course during the exercise classes of the following course. In almost all cases this was bolstered by the fact that the retake had been passed. Many students expressed their conviction of having profited to a great extent from the additional courses, both immediately after the course and about half way into the next semester.

Conclusion and perspectives

Compared to the previous approaches of the faculty of mathematics and to other approaches described in the literature (e.g. remedial courses as discussed by Di Pietro (2014), which were shown to be largely ineffective) to help the students in the transition from school to university mathematics, the additional courses described here seemed to be more effective with regard to the research questions posed above. Various factors are assumed to be of importance for this.

As this was a voluntary course aimed at students having failed the end-of-term test, it was made clear beforehand that it should be seen as a chance to acquire skills not yet developed rather than an obligatory task (as in Di Pietro, 2014). The course was announced at the beginning of the semester, giving the students the opportunity to take it into account in their planning. In general, the allocated time was met with approval, both because it meant no conflict with the workload during the semester and because the end-of-semester test provided some feedback for the students whether attendance was advisable (due to other time restrictions regarding school and computer practica for the students, however, the course should not exceed two weeks). In addition, the common goal to prepare for the next semester and pass the retake seemed to ensure a sense of belonging to a peer group with a common aim. Hence, the general framework of the course proved to be more feasible than the previous ones, and will be kept for the time being.

The design of the course with sessions devoted to central topics, which were each split into smaller units of varying tasks, were also perceived to have a positive impact with regard to the above questions. In particular, the general progress from rigidly set tasks to more self-determined ones seemed to be appropriate in this setting. The exercises themselves were developed using the AiC framework, and their construction will be refined, based on the observation of how the different tasks affect the perceived concept building in the students. For example, in the exercise given in figure 2, part (c) has already been modified to ask for linear independence of the differences of the given vectors as opposed to their sums. This allows for the generalization of ideas from lower dimensions to higher ones, giving the students the possibility to use previously built knowledge more directly. Moreover, part (d) was simplified to ask only for real vector spaces, as the additional difficulty of an arbitrary field seemed to be ignored or actually hindered the building with in the previous course. This will be reflected on further, using the gathered experience, and systematically revised for the next implementations. After this first preparatory cycle, we hope to be able to contribute to the research regarding the mediation of students' problems in the first year of university by carefully designing and evaluating the next cycle of the described additional courses.

We are going to evaluate the two main questions in the next cycle as follows. To assess the effect with regard to SDT, we propose to use a pre-post-test according to the items presented in Rakoczy (2006) with the pre-test set at the end of the lectures of the first semester and the post-test in the middle of the second semester. We want to check for correlations of these results with the performance in the end-of-first-term test. With regard to concept building, we are going to evaluate some abilities concerning the relevant epistemic actions at the end of the additional course itself with a free form questionnaire, supplemented by two or three interviews with students to reach theoretical saturation and to clarify the data.

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Navigating through the mathematical world: Uncovering a geometer's thought processes through his handouts and teaching journals

Sepideh Stewart[1], Clarissa Thompson[2] and Noel Brady[1]

[1]University of Oklahoma, Department of Mathematics, Oklahoma, USA; sstewart@math.ou.edu

[2]Kent State University, Department of Psychological Sciences, USA; cthomp77@kent.edu

[1]University of Oklahoma, Department of Mathematics, Oklahoma, USA; nbrady@ou.edu

In this case study, we examined a mathematician's thought processes as he taught a course on Algebraic Topology. The mathematician shared his teaching-related journals over an entire semester and discussed them in depth during weekly meetings with the research team comprised of a mathematics educator, a cognitive psychologist, and a postdoctoral fellow in mathematics. Concurrently, one of his students took detailed journals on most lectures. The team employed Tall's three worlds of embodied, symbolic, and formal mathematical thinking as various lenses to gain insight into the mind of the working mathematician as he taught a course on Algebraic Topology. Although the analysis of the data from the instructor's journals and the in-depth discussion of the journals during the team meetings revealed his thought processes, the 35 handouts that he prepared, aligned with students' needs, provided the most insight into his way of thinking.

Keywords: Embodied, symbolic, formal, Tall's three worlds, Algebraic Topology

Introduction

Communicating advanced mathematical ideas to university students is a challenging endeavor. It is a common and accepted practice for many mathematicians to write definitions, theorems and proofs on the board and make comments as they introduce mathematical ideas to students. Thurston (1994, p. 162) asked the question: "How do mathematicians advance human understanding of mathematics?" In interviewing 70 research mathematicians, Burton (1999, p. 31) found that "intuition, insight, or instinct" was seen by most mathematicians as a necessary component for developing student knowledge. Although we have some literature on examining mathematicians teaching practices (e.g. Fukawa-Connelly, 2012; Stewart, Schmidt, Cook & Pitale, 2015), research on what takes place in the minds of mathematicians and their students is still scarce (Speer, Smith, & Horvath, 2010). Dreyfus (1991) believed that, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician" (p. 29). In this study, we examined a mathematician and one of his students' daily thoughts on Algebraic Topology. The overarching goal of this research was to investigate the way mathematicians and students think about mathematics and the possible pedagogical challenges that they may face.

Theoretical framework

In this study, we employed Tall's (2013) three-world model of conceptual embodiment, operational symbolism, and axiomatic formalism in order to describe an expert geometer's ways of mathematical thinking. In Tall's view, the embodied world involves mental images, perceptions, and thought experiments; the symbolic world involves calculation and algebraic manipulations; the formal world involves mathematical definitions, theories and proofs. Tall (2008) asserts that, "all humans go through a long-term development that builds through embodiment and symbolism to formalism" (p.

23). Bridging between the embodied and symbolic worlds is of critical importance. Tall emphasizes that "a curriculum that focuses on symbolism and not on related embodiments may limit the vision of the learner who may learn to perform a procedure, even conceive of it as an overall process, but fail to be able to imagine or 'encapsulate' the process as an 'object' (p. 12).

Tall and Mejia-Ramos (2006, p. 3) declared that the word 'world' is carefully chosen and has a 'special meaning' in order to represent "not a single register or group of registers, but the development of distinct ways of thinking that grow more sophisticated as individuals develop new conceptions and compress them into more subtle thinkable concepts". As Dreyfus (1991, p. 32) declares "One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take… Teaching and learning this process of switching is not easy because the structure is a very complex one." Duval (2006) claims that many students do not have the cognitive framework to perform the switch. The overarching goal of the first author's research program is to investigate the ways in which mathematicians move between modes of thought and facilitate their students' movements among these modes. Tall's theoretical framework accounts for movement between the worlds of mathematical thinking and is a suitable scaffold for this research. Through our collaborations, we are beginning to understand how the minds of working mathematicians operate. Thus, we hope to evolve Tall's theory and use it to analyze rich data from many mathematicians. We endeavored to investigate the following research questions: (a) How did the instructor and student move between the formal, symbolic, and embodied worlds? (b) How did the instructor use handouts in order to help students move between the worlds?

Viewing Homology Theory through three lenses

The mathematician appreciated the developmental aspect of Tall's framework in which one begins with a very embodied view of the world around them and then moves with increasing age and experience to a symbolic view as one matures. However, he took issue with the "formal" viewpoint as the ultimate destination of this progression, especially since formal from a math perspective (i.e., set theoretic axioms, definitions, and formal deductions from such a system) is not the way mathematicians think. One can program a computer to generate (i) statements and (ii) formal proofs of these statements within an axiomatic system. In what sense can we say that the computer is discovering a mathematical theory? Humans use a lot more when they discover/develop a mathematical theory. Among all the myriad of possible statements that could be true in this formal theory, mathematicians choose certain ones (usually as a result of intuition and metaphors possibly supported by symbolic computations to garner evidence for the particular statements) called conjectures, and they try to prove them. Instead, the mathematician made sense of Tall's worlds by thinking of them as three lenses that allowed him to view a mathematical reality/world. Figure 1 illustrates his views of Homology Theory through these lenses. The embodied lens allows the mathematician to see cycles as geometric objects, and similarly for chains and various topological spaces. The symbolic lens allows the mathematician to use symbolic computation tools such as the Mayer-Vietoris sequence and produce symbolic computations (e.g., the homology of the 2-torus). The formal lens allows the mathematician to work with the Eilenberg-Steenrod axioms and results which can be derived formally from these axioms. The geometric side of topology spans the embodied and symbolic lenses. Algebra, primarily in the form of Homological Algebra, spans the symbolic and formal lenses.

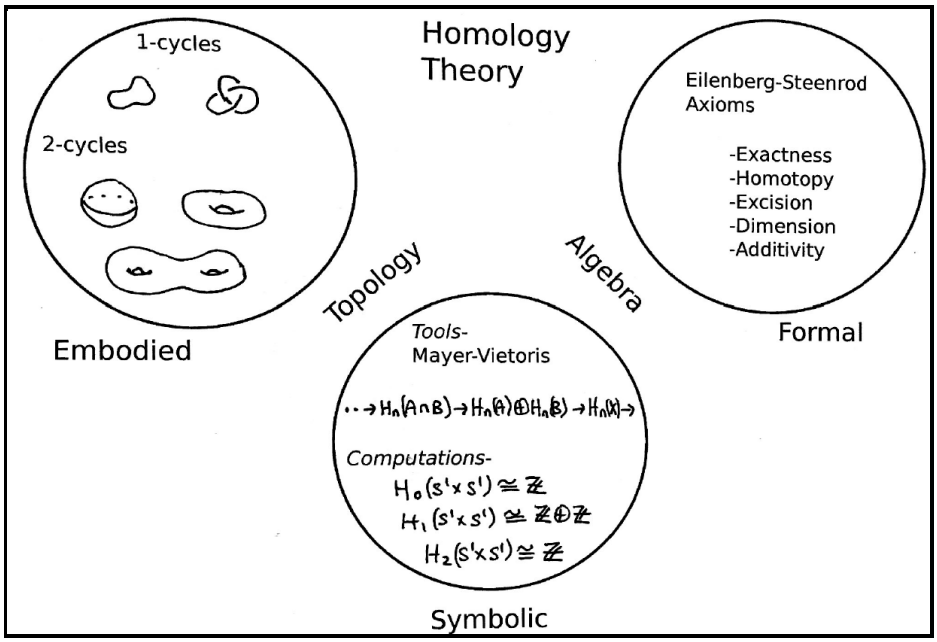

Figure 1: The three-lens view of Homology Theory

We can think of similar lenses, for example, in medicine. One can look at a patient with one's eyes, take an x-ray or an MRI of the patient, view the patient through an infrared lens, listen to the patient's heart and lungs etc., talk to the patient about their symptoms, and draw blood and perform tests. These are different modes of gathering information to give a practitioner a more complete picture of the patient.

Method

The participants. Our qualitative narrative study investigated the ways an expert mathematician navigated among Tall's worlds of mathematical thinking. The research team consisted of four members: a mathematics education researcher; a geometer, Noel Brady (the course instructor); a cognitive psychologist; and a mathematics postdoc familiar with both Algebra and Topology.

The course. The Algebraic Topology course was the first in a two-semester sequence of graduate courses. There were eight graduate students enrolled in the course. During class meetings, Noel often passed out handouts to help students follow along with the topic of the day. He believed some topics covered in the chosen textbook (Hatcher, 2001), needed to be handled in a more detailed fashion. "Hatcher is a bit fast and loose with all of this". Students actively solved problems together in groups, or individual students were called to the board to complete problems. Noel also helped to revive an extracurricular, student-led topology seminar.

Data and procedures. In this study, we analyzed a geometer's thought processes and actions while he taught Algebraic Topology over the entire Fall 2014 semester. One source of data was a series of teaching journals that contained Noel's reflections on his preparations for class, what happened during class, as well as some descriptions of the events that took place during office hours and a student-led topology seminar. The research team read his daily journal entries and discussed them during weekly research meetings. During these meetings, we asked Noel further clarification

questions, and he often drew additional pictures as he described the course content. These meetings were audio recorded and later transcribed and will be used as a source of data. Another source of data came from one of Noel's graduate students who also wrote daily journals. These student journals provided an additional perspective into the events that took place in class. In addition, further data came from 35 handouts that Noel provided.

Coding scheme. The data were analyzed thematically, meaning we mainly considered the key issues that emerged in this study. The main themes and their sub-categories were identified and coded (see Figure 2). In addition to assigning codes for the three worlds of mathematical thinking, we also created codes for movement between the worlds (e.g., embodied-symbolic). While coding Noel's journals, at times we assigned multiple codes for a particular instance. For example, an excerpt could be coded with both the "Teaching" and "Tall's Worlds and Movements" codes.

In the following section, we give a glimpse into the analysis of Noel's journals, as well as instances from the student's journals to illustrate how the student perceived movement among the worlds.

Results and discussion

Figure 2 shows the percentage of total qualitative codes that were applied to excerpts from Noel's teaching journals.

Main Theme	Sub-Category	Percentage Use Main Theme	Percentage Use Sub-Category
Teaching		46%	
	Pedagogy		26%
	Examples		22%
	Handouts/notes		20%
	Proofs		11%
	In-Class Activities		10%
	Hatcher/textbook		7%
	Theorems		3%
	Homework		2%
Tall's Worlds and Movements		25%	
	Symbolic		25%
	Embodied (Intuition)		21%
	Topology - Algebra		20%
	Embodied-Formal		18%
	Formal		10%
	Embodied-Symbolic		7%
	Symbolic-Formal		0%
Reflections		20%	
	On students: understanding/preparedness		47%
	On self: Careful/precise		31%
	Pacing		22%
Students		9%	
	Igor (one of the students)		26%
	Students ask questions outside of class		26%
	Student seminar		26%
	Students ask questions in class		22%

Note: Percentage Use Main Theme adds to 100%
Percentage Use Sub-Category adds to 100% for each sub-category

Figure 2: Qualitative coding scheme

The main theme of Tall's three worlds of mathematics comprised 25% of the total codes. Teaching was the main theme that was coded the most (46%) in Noel's journals. Reflections included 20% of codes, and codes pertaining to students involved 9% of the total codes. Analysis of the data revealed ample evidence that Noel repeatedly navigated between the three worlds of mathematical thinking. Below, we provide examples from our analysis of his teaching journals and a student's journals to illustrate movement between worlds.

Moving between embodied (intuition) and formal worlds

According to Noel, this may have been the type of movement that the students found the most challenging: "There were a lot of questions about how to pass from an intuition to a formal proof (many of these examples used techniques/results from quotient spaces)."

The analysis of the student's journals showed his concerns regarding the proofs. This excerpt was taken from one of his journals at the beginning of the semester:

> Dr. Brady's way of proving results that come from concepts we're already supposed to have come across before his class is nice, I think. He gives a detailed outline verbally, which is helped along visually by his pictures and hand gestures. For the most part I'll watch without writing almost anything, but I definitely get a lot out of reviewing concepts in this way. I'm a little worried, however, that when we get to brand new material Dr. Brady's way of proving results might remain in the same verbal/hand-waving/picture-drawing style and that this won't be enough for me to follow the proof right there and then. He tends to speak and write very quickly, which is fine when we're reviewing. But since I can either copy furiously what he writes on the board or listen to him, but not both, this could become a problem.

Noel refused to give students proofs that were pre-packaged. More specifically, he wanted to provide students with intuitions/pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it. In one of the research meetings Noel said:

> I mean I can give verbatim proofs of things or give them more detailed proofs where Hatcher leaves stuff out, but that will just waste time and I'll reproduce a book and nobody will get anything out of it. So I've given them intuitions, enough of an intuition that they can tag that together with a formal proof.

Later in the course the student wrote: "I've seen van Kampen's theorem before, but Dr. Brady's from-the-ground-up approach was very nice in that it showed us through comprehensive diagrams just where exactly the theorem comes from."

Movement between embodied and symbolic worlds

Noel discussed moving from embodied demonstrations (e.g., rope trick) to having students complete symbolic examples (e.g., right-angled Artin group (RAAG) complexes and the torus knot spine):

> More of the same. I connected back to several examples from the first week and from the intro to π_1. The pair of circle links in S^3 example (a.k.a. the rope trick) and the RAAGs. This seemed to go ok. Mentioned again that RAAGs are deceptively simple looking groups, but that their subgroup structure is surprisingly rich. In particular, Bestvina-Brady (1997) and Agol-Wise (2012) contain very surprising results about subgroups of RAAGs. Told them that the story is still ongoing. Left off with an example of a torus knot spine (Hatcher).

The handouts

Analysis of the 35 handouts that Noel created illuminated the motives behind some of his thought processes and movement between worlds. These handouts gave the team a more authentic glimpse into the mind of the mathematician than the teaching journals that Noel regarded as self-critical (self-aware). Figure 3 shows the first two pages of a handout Noel created on barycentric subdivision. The

start of the handout contains the formal definitions of "barycenter" and of "barycentric subdivision." These definitions build on a previous definition (and square bracket notation) of an n-simplex. The definition of "barycentric subdivision" is recursive (i.e., defined in terms of lower dimensional versions of itself). The rest of the two pages is devoted to building students' intuitions for these definitions. At the bottom of the first page, two embodied examples are provided which demonstrate how to unwrap the recursive definition to determine the barycentric subdivision of a 1-simplex (a line segment) and of a 2-simplex (a triangle). This is followed by an exercise which asks the student to add another layer of recursion and describe the barycentric subdivision of a 3-simplex (a triangular-based pyramid). This is a very embodied example. At this stage, Noel hoped that the student should be gaining confidence working with the recursive definition and should be developing an intuition that the symbolism will work in higher dimensions where one's embodied intuition fails. The second exercise asks the student to iterate the barycentric subdivision process for a 2-simplex. Again, this is very embodied and can be drawn easily in the plane. Noel pointed out that developing an intuition about iterated barycentric subdivisions is important since they will form the heart of the proof of the "locality result" and the proof of the "excision theorem" for singular homology later on in the course. The two Roman-numeral-labeled observations at the end of page 2 build on the student's embodied intuition of the behavior of iterated barycentric subdivisions in dimension 2 (obtained from doing exercise 2). They motivate the statement of the theorem that will be given and proven on subsequent pages of the handout. They also alert the student to the fact that some care will have to be given to the proofs on the subsequent pages. This is particularly so, since these proofs will hold in arbitrary dimensions.

Noel pointed out that, from a textbook perspective, one can skip straight from the definitions of barycenter and barycentric subdivision to the statement and proofs of the theorems about the behavior of the diameters of simplices under iterated barycentric subdivisions. Nothing in the logical progression and framework would be lost. However, students' intuitions would be lacking (save for the rare student or two who can do some mental exercise equivalent of the examples, exercises and observations of these two pages.). This handout is one of a sequence of three handouts. These handouts get increasingly symbolic and abstract. Eventually, the results contained in the last handout are just what are needed in the formal proof of the "locality theorem" (and the "excision theorem") of singular homology. At this stage, the proofs are very symbolic and far removed from geometry. It is good that students have developed an embodied intuition about iterated barycentric subdivisions, so that they have concrete models in their mind for how excision works on the geometric level of chains.

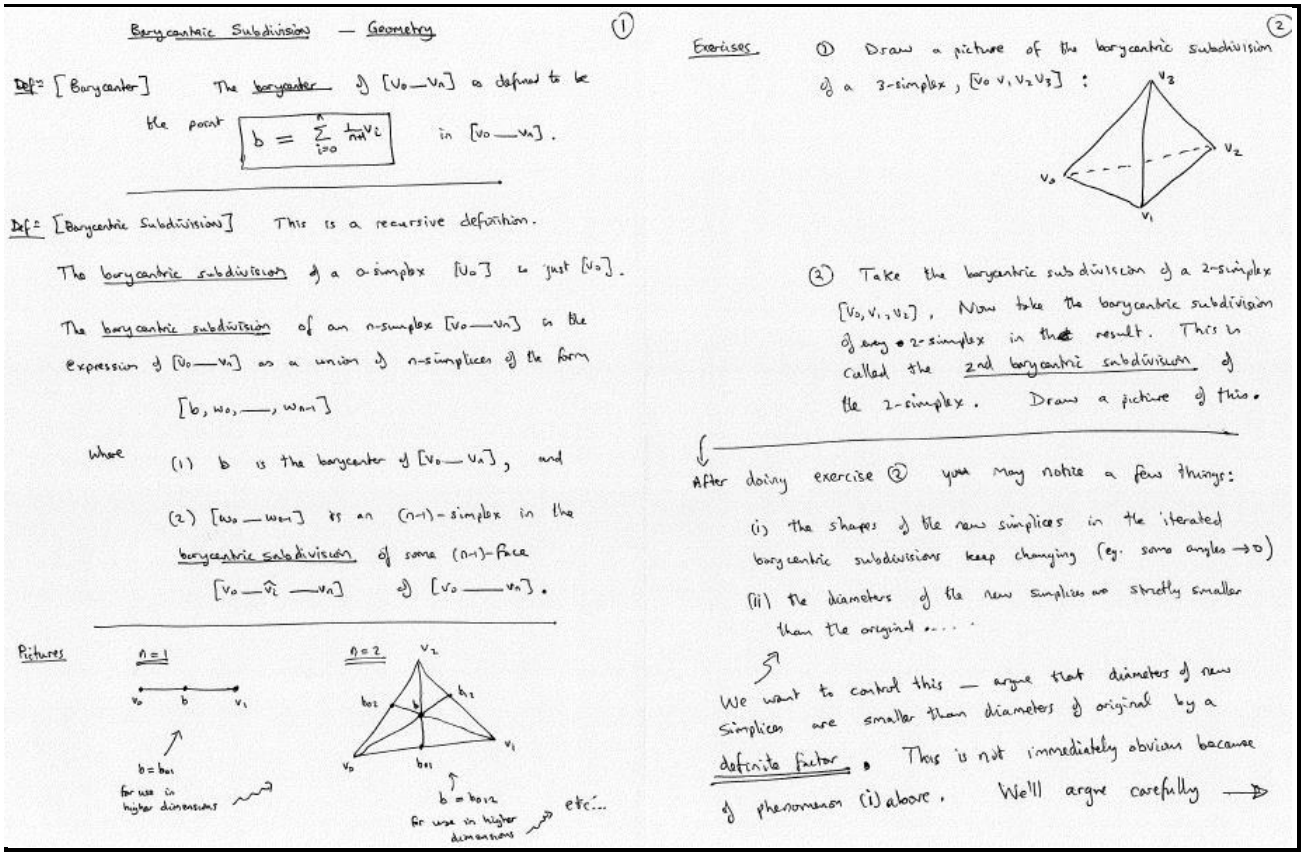

Figure 3: An excerpt from Noel's handout

Concluding remarks

This study revealed that Noel viewed Algebraic Topology through all three mathematical lenses (embodied, symbolic, formal), and his handouts provided his students with opportunities to view the course material through these different lenses as well. In one of the research meetings, Noel mentioned:

> When I think of the mathematical world of algebra I have examples in my mind, many of which are very embodied, and many of which are symbolic, I also know the axiomatic definitions of concepts in this world like "group," "ring," "field" etc. So, when I think of the world of algebra all three lenses (embodied, symbolic, formal kick into gear. Likewise, for the mathematical world of topology.

Our research team, comprised of a mathematician, a mathematics educator, and a cognitive psychologist, are working together to apply and evolve Tall's theoretical framework by analyzing the teaching journals of mathematicians and their students. We have come to realize that the embodied, symbolic, and formal worlds blend together as applied to Algebraic Topology; it is often not clear where one world starts and another world ends. In addition to thinking about problems from the ESF perspectives, mathematicians often translate a problem from one area of mathematics (e.g. Topology) to another (e.g. Algebra). This translation is achieved using mathematical constructs called *functors*.

Noel used the analogy of a translator to describe the mathematical notion of a functor. When a statement of a problem is translated from one language to another, some of the details may get lost in

the translation. Perhaps this loss of information has an unexpected benefit; the simpler formulation of the problem in the new language might allow for new insights or intuitions to be gained, and perhaps even for a solution to the original problem.

Noel talked about functors in his journals, and described how they are used to solve problems in topology by first translating then into algebra problems:

> We introduced some other situations where Algebraic Topology functors might help solve topology problems, and mentioned that the homology functors would be introduced and studied in the course.

We are using analogies and metaphors to communicate with one another as we attempt to understand the pedagogical decisions of the working mathematician. As Thurston (1994, p. 168) asserted: "we mathematicians need to put far greater effort into communicating mathematical ideas. To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking...we need to appreciate the value of different ways of thinking about the same mathematical structure".

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Discursive shifts from school to university mathematics and lecturer assessment practices: Commognitive conflict regarding variables

Athina Thoma and Elena Nardi

University of East Anglia, United Kingdom; a.thoma@uea.ac.uk

We report part of an ongoing study that aims to characterise lecturers' assessment discourse, especially on closed-book examinations. We focus particularly on lecturers' discourses that concern the transition from school to university mathematics, and we do so through highlighting one commognitive conflict regarding the use of variables in a task from an examination paper for a Year 1 module on Sets, Numbers and Probability taught in a UK mathematics department. We show evidence that the lecturer's assessment practices aim to facilitate students' avoidance of errors that are occurring because of said conflict. Here, we focus on students' scripts which illustrate that, nonetheless, students make errors and do not draw on the discourse of integers when deciding the domain of the variables used in the task. We conclude with a brief discussion of students' experience of commognitive conflict in the transition from school to university mathematics.

Keywords: Undergraduate examinations, assessment routines, commognitive conflict, variables.

Introduction

Studies in mathematics education have focused on students' transition from secondary school to university (e.g. Gueudet, 2008). Part of how students experience said transition is evidenced in their engagement with examinations during their first-year undergraduate modules. The nature of tasks in examinations has been studied using different theoretical frameworks (e.g. Tallman, Carlson, Bressoud & Pearson, 2016). Researchers have also examined lecturers' perspectives on examination tasks (Bergqvist, 2012; Tallman et al., 2016). In our study, we take a discursive approach in analysing examination tasks and lecturers' perspectives focusing on aspects of the transition from secondary school to university mathematics. This theoretical approach allows a characterisation of the mathematical discourse the students engage in when solving the tasks, and provides insight into lecturers' assessment practices and their expectations from students' responses.

In this paper, we analyse a task from a first-year module on Sets, Numbers and Probability offered in a UK mathematics department. We build on previously reported work from this study (Thoma & Nardi, 2016) in order to delve into lecturers' assessment practices facilitating students' transition to university mathematics in more detail. Specifically, we take the case of variables and the way these appear in a Number Theory task of the module's examination paper. In choosing this particular case, we take cue from previous works (e.g. Epp, 2011) which note that variables have diverse uses in mathematics, some of which often create difficulties for students' transition to algebra and other advanced topics. Of particular relevance in this paper is the discussion by Biehler and Kempen (2013) about the difficulties with variables that students face.

In the part of our study reported here, we focus on a commognitive conflict concerning the number domains in the secondary school and university mathematics discourses. First-year mathematics undergraduate students' errors regarding variables, when engaging in a Number Theory task, provide evidence of this unresolved commognitive conflict. In what follows, we present briefly the theoretical framework of the study, the examination task and the study's participants. We then analyse the task

and the interview data with the lecturer who posed the task. The interview data illustrate the lecturer's ways of assisting the students to avoid the errors. Finally, we highlight the errors evidenced in the student scripts despite this assistance, we present the case that these errors stem from aforementioned commognitive conflict and conclude with a discussion of findings and how these are embedded into the larger study.

Commognitive conflicts and assessment routines facilitating discursive shifts

Sfard's (2008) theory of commognition is a discursive approach that is being increasingly used in mathematics education (Tabach & Nachlieli, 2016), as well as specifically in university mathematics education (Nardi, Ryve, Stadler & Viirman, 2014). Mathematics in this approach is a discourse that can be described in terms of the following four characteristics: *word use* (e.g. divisor), *visual mediators* (e.g. algebraic symbols), *endorsed narratives* (e.g. definitions) and *routines* (e.g. proving). The routines are distinguished in *deeds* ("an action resulting in a physical change in objects"; Sfard, 2008, p. 236), *rituals* ("creating and sustaining a bond with other people", p. 241) and *explorations* ("producing endorsed narratives", p. 259) with the explorations further categorised in *recall*, *substantiation* and *construction*. Of particular relevance to our analysis here is the construct of *commognitive conflict "*the phenomenon that occurs when seemingly conflicting narratives are originating from different discourses – from discourses that differ in their use of words, in the rules of substantiation, and so forth.*"* (p. 257). For example, a commognitive conflict may occur between the different relationships that the number domains have in school and university mathematics discourses. In school, number domains are introduced progressively. They are used for some time; and then subsumed in the next number domain. Positive integers are introduced first. Then, as the students learn about division, rational numbers follow. After a while, the discourse about unsigned rational numbers (which includes integers and rational numbers) together with negative numbers constitute the discourse on rational numbers (p. 121). The discourse of real numbers is introduced in later stages of the secondary school. In the university discourse, the number domains play a different role. They are presented as crucial abstract structures, the ring of integers and the fields of rationals, reals and complex numbers. In particular modules, the focus of study are those abstract structures and that is the case for Number Theory, where the domain of the variables is restricted in the discourse of integers.

In our study, we examine students' participation in the university mathematics discourse taking also into account the lecturers' perspectives, particularly their rationale for the choices of the examination tasks and the wording of the tasks. Our previous analysis of examination tasks and lecturers' assessment practices (Thoma & Nardi, 2016) highlighted the following assessment *routines*: giving directions to the students regarding the steps their response to a task may take; structuring the tasks and subtasks in ways that allowed students to secure and optimise marks as they progressed from one part of a task to another; and, providing guidance regarding expected justifications in the students' responses. Overall, these routines aim at assisting students' shifting from school to university mathematics discourse. Here, we aim to extend our previous analyses, also taking into account lecturers' assessment routines, which aim to avoid expected errors. We will make the case that unresolved commognitive conflicts are responsible for those errors. We are, therefore, starting to look in tandem at aspects of students' experience (here: commognitive conflicts relating to variables in a Number Theory examination task) and lecturers' perspectives on – and intended practice relating to

– this experience. In the following, we outline the larger study our paper originates in; and, introduce the examination task and a brief commognitive analysis of it. We then offer an analysis of the lecturer's perspectives on the task, highlighting those assessment routines that aim to help students avoid errors relating to variables. Finally, we present the student examination scripts, which illustrate these errors and examine whether, and how, the unresolved commognitive conflict in the different relationships with number domains at school and university, may be seen as responsible for these errors.

The examination task and the participants of our study (lecturer and students)

The data of our study consists of examination tasks from different modules, lecturers' interviews on those tasks and students' scripts corresponding to these examination tasks. The focus of this paper is on one task from the module Sets, Numbers and Probability. This is a first-year module and has two parts: Sets, Numbers and Proofs taught in the autumn semester and Probability taught in the spring semester. The final examination includes six tasks: the first two are compulsory and the other four optional. One of the compulsory and two from the optional tasks are on Numbers, Sets and Proofs and the others on the Probability part of the module. At the final examination, the students have to solve both the compulsory tasks and three from the optional tasks. The total grade of the examination is 100 marks and the pass grade is 40 marks. This paper focuses on the compulsory task from the Sets, Numbers and Proofs part of the module (Figure 1). More specifically, in this part of the module, the topics covered are: Set Theory (notation, operations, cardinality and countability), Functions (introduction to functions, injection, surjection), Proofs (direct proof, proof by induction, proof by contradiction, proof by counterexample), Number theory (greatest common divisor, prime numbers, modular arithmetic) and Equivalence relations. The topic examined in this task is proof by induction and Number theory. Our analysis will focus on students' responses to the Number theory part of the task, task (ii). The model solution for part (ii) created by the lecturer for departmental use is in Figure 2. We note that this solution is not made available to the students.

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2.$$

(ii) (a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

Figure 1: Compulsory task on Sets, Numbers and Proofs

(ii)(a) d is a *divisor* of a means that there is $k \in \mathbb{Z}$ with $a = kd$. **[2 marks]**

If d is a divisor of a and of b then there exist $k, l \in \mathbb{Z}$ with $a = kd$ and $b = ld$. Then for all $m, n \in \mathbb{Z}$ we have

$$ma + nb = m(kd) + n(ld) = (mk + nl)d.$$

As $(mk + nl) \in \mathbb{Z}$, it follows that d divides $ma + nb$. **[2 marks]**

(ii)(b) Following the method in lectures, let $a = 123$ and $b = 45$. Carrying out the Euclidean algorithm we obtain:

a	$=$	123			
$2b$	$=$	90	45	$=$	b
$a - 2b$	$=$	33	33	$=$	$a - 2b$
$6b - 2a$	$=$	24	12	$=$	$3b - a$
$3a - 8b$	$=$	9	9	$=$	$3a - 8b$
		9	3	$=$	$11b - 4a$
		0			

We conclude that $\gcd(123, 45) = 3$ and that

$$3 = 11b - 4a = 11 \cdot 45 + (-4) \cdot 123 = 123m + 45n$$

where $m = -4$ and $n = 11$. **[8 marks]**

(ii)(c) No. Since 3 divides both 123 and 45 it follows from (ii)(a) that 3 divides $123s + 45t$ for all $s, t \in \mathbb{Z}$, but 3 does not divide 7. **[2 marks]**

Figure 2: Model solution of part (ii) of the compulsory task

Fifty-four students took part in the final examination and the marks of their responses in this task ranged from 4 to 20, with the mean being 16.85 marks. The scripts of 22 students were selected by the first author to represent a variety of marks (Figure 3). The errors based on what we see as an unresolved commognitive conflict regarding variables were observed in 6 students' scripts. Here we report: first, analysis of the task and from the lecturer interview data; then, a sample from the analysis of the six students' scripts.

Task analysis and the lecturer interview

In part (i) of the task (Figure 1), the students are asked to engage in a substantiation routine (proof by induction). The wording of the task directs the students to this type of proof. In part (ii), the students are directed to engage first in a recall routine, giving the definition of a divisor, and then in a substantiation routine of a relationship describing the connection between the linear combination of *a* and *b* and the divisor *d* of *a* and *b* (iia). The students are then directed toward using the Euclidean Algorithm in (iib) and, in the last part (iic), they are expected to engage in a proof by contradiction (not explicitly mentioned in the wording of the task) in order to prove that the linear combination given is not divisible by 7 – see (Thoma & Nardi, 2016) for more detailed analysis of the task. For the purpose of this paper, we focus on the lecturer and student data corresponding to the second part of the task, (ii).

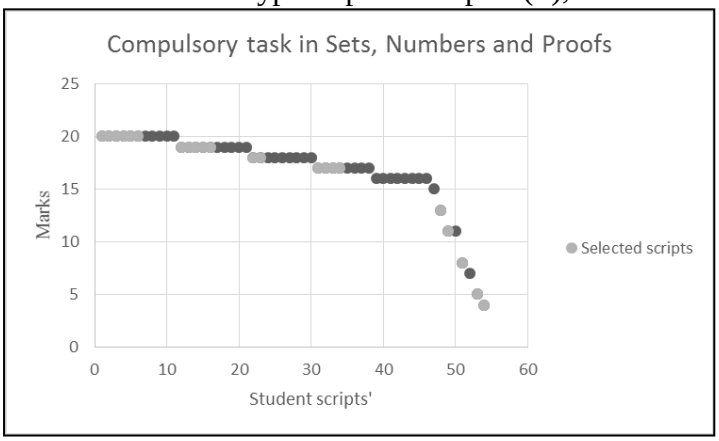

Figure 3: Marks from all the students' scripts

During the interview, the lecturer said:

> Lecturer: (…) my memory of school mathematics is that there was a lot of doing things but not necessarily a lot of formally defining things (…) And of course they came to university thinking that they knew what that [the definition of the divisor] meant but in this situation it really matters that they are restricting themselves to the-to the ring of integers (…) and all the symbols represent integers so what it means to divide is very different than if they were working with fractional numbers or something where they could write *a* over *b* and things like this.

Our commognitive analysis highlights the differences in the lecturer comments between the school discourse and the university discourse and, more specifically, with regard to the routine of defining and the importance of understanding that "all the symbols represent integers".

He highlights the differences between what the students are used to and what they are expected to do at university level. Our analysis sees this as the differences between the two discourses: on the one hand on the focus of the routines; on the other hand, on the constraints of the different discourses that exist within the mathematical discourse at university level. More specifically, the students, working with this definition have to restrict their work on integers – and not on rational or real numbers. The lecturer, then, speaks about the nature of the symbols involved. We recognize this comment by the

lecturer as foreseeing students' errors in the case that they treat the divisor as a rational instead of an integer, drawing on the discourse of rational numbers instead of the discourse of integers. Integer numbers are rational numbers, and making the distinction between the two – and then opting for working within the discourse of integers – is not something that these students have been routinely working with in school. This shows that the lecturer expects students to use division in the way that they were taught in school, instead of considering the abstract structure that this task is asking them to restrict their activity in. In the excerpt that follows, the lecturer explains his assessment routines which aim to assist students with avoiding errors that are happening because of what we see as a commognitive conflict: subtask (ii) is gradually structured as first asking the definition, then, substantiating a narrative that draws on this definition, engaging with the Euclidean algorithm and, finally, combining all the above to engage in a proof by contradiction. He comments on the purpose of this gradual structure as follows:

> Lecturer: (…) what's being tested here is their ability to write down something formally and correct. And I would worry that if I didn't prompt them to write down formally the definition of what it means for one integer to divide another in the exam, in the pressure of the exam and so on, then their answers could start looking very 'creative' at the second part and they might start writing down fractions.

Therefore, he aims that the gradual structure aids students towards achieving the expected solution. This can be thought of as a way of helping students avoid experiencing errors stemming from what we labelled as commognitive conflict, where a and b, would be treated by students as rational numbers, instead of integers: this gradual structure serves as a reminder that they should restrict themselves in the discourse of integers. Additionally, the analysis shows that the lecturer stresses the routine of justification and the rigor of the university discourse compared to the school discourse, a further staple of the transition that these students are at the moment experiencing (Gueudet, 2008).

> Lecturer: (…) the only challenging part would be the last part, the part that requires some thought and they need to-to sort of understand or remember that somehow it relates to what happened up here [shows parts (iia) and (iib)] (…) to remind them that I want them to explain why they are answering what they are saying.

In the excerpt above, the lecturer comments on the challenge of the (iic) part of the task and the purpose of the prompt "Explain your answer carefully". In this part of the task, the students have to engage in a substantiation routine that is based on the endorsed narratives that they have created for parts (iia) and (iib). The lecturer suspects that the students may omit justifying their response regarding the substantiation of the given relationship and aims that this prompt will help them do so.

From the above, we see that the lecturer has identified students' difficulties with the nature of the variables being used in this task. Our commognitive analysis sees this as evidence that the lecturer appears alerted to this difficulty as a difference between the school and the university discourse. The students, during their school years, gradually moved from the discourse of the natural numbers, to the one of the integers, then to the rational numbers and finally to the reals. Now, in this task, they are asked to endorse the discourse of the integers, which is subsumed in the discourse of rational numbers, within which they have been performing division of numbers in school. We now turn to students' responses, which evidence that, despite aforementioned aid provided by the lecturer, errors

illustrating this commognitive conflict were not avoided. Of the twenty-two student scripts analysed, six contained said evidence ([01], [03], [06], [11], [16], [17]).

The students' scripts

Student [03] first communicates the relationship between the divisor *d* and *a* using written verbal

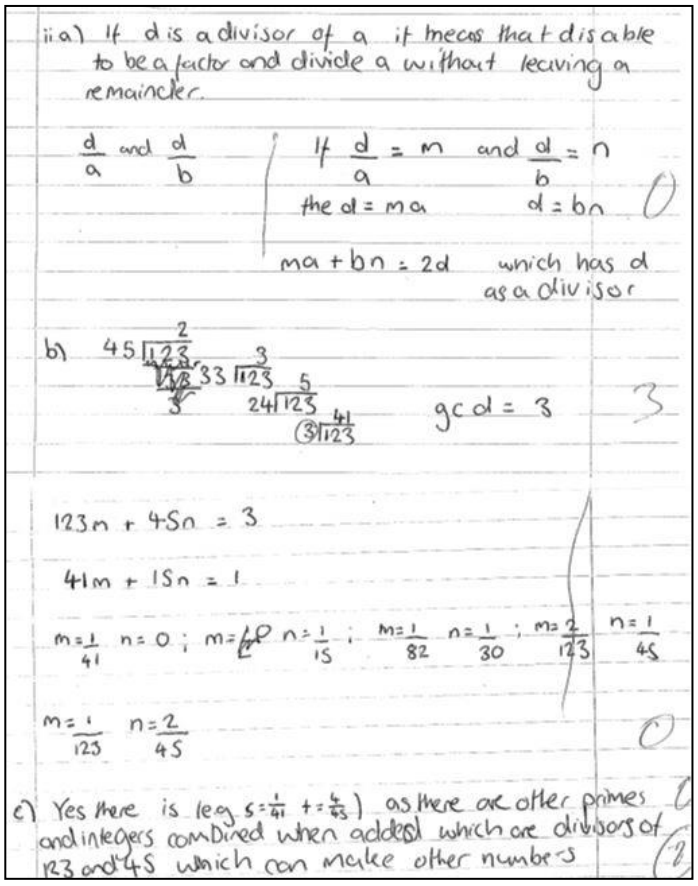

Figure 4: Student [03]'s script to (ii)

visual mediators. In the second part of (iia) the student writes, using symbolic mediation, that *d* is a divisor of *a* and *d* is a divisor of *b*. However, in the symbolic realisation of the divisor, the student deploys fractions, with *d* being the numerator and *a* and *b* being the denominators. This way of writing that *d* divides *a* can be seen as a translation of the written verbal mediator into a symbolic mediator without taking into account that the fraction line means that the denominator divides the numerator. In the case of the task, this division would result in a non-integer number. This way of writing signals that the student may see *m* and *n* (written on the right hand of the script) as numbers and not as integers. Then, the student writes the relationship between the symbolic mediators *m*, *d* and *a* and concludes that *d* is equal to the product of *m* and *a*. The student used all the symbols given in the wording of the task to produce a narrative that involves fractions. Fractions could be part of the discourse of rationals, and this task asks the students to restrict their activity within the discourse of integers. We see the appearance of fractions and the absence of the constraints regarding the variables as evidence of the commognitive conflict regarding the relationships between the number domains in school and university mathematics discourses. Unclear meaning making regarding the object of a divisor is evidenced as the student starts by explaining that *d* is a factor of *a*, then engages in the discourse of rationals concluding that $d=ma$ but then saying that the product $2d$ has *d* as a divisor. Having concluded that the greatest common divisor is 3 (Figure 4) using long division - and not the Euclidean algorithm - the student writes $123m + 45n = 3$. Then s/he divides all the terms of the equality by 3 and takes different cases where the new equality is true.

In doing so though, [03] does not take into account the nature of the variable symbolic mediators and the variables become rational numbers. Also, there are multiple values in the rational numbers that satisfy this equality as can be seen in the response (Figure 4). Finally, in the last part of the task, the student responds affirmatively that there are "integers*" s* and *t*. However, the *s* and *t* s/he gives are

rational numbers chosen to result in 7. This signals a ritualised use of the word "integers": the student uses the word integer, seemingly repeating the wording of the task but in fact providing numbers that are not necessarily integers.

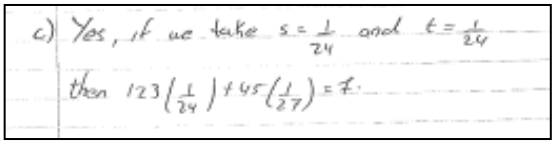

Figure 5: Student [16]'s to (iic)

Two more students [16] and [17] give similar responses. Student [16] (Figure 5) does not give a definition of the divisor, attempts the substantiation of the relationship describing the connection between the linear combination of *a* and *b* and their divisor *d*, finds the greatest common divisor and uses only integers similarly in the identification of the integers *m* and *n* which give the linear combination of the greatest common divisor. However, when responding to (iic), [16] finds two non-integer rational numbers for *s* and *t* and confirms that the expression results in 7. Similarly, in the scripts from the students ([01], [06], [11]) there are instances where errors occur as the number domain of the variables is not clarified and the students work in a different number domain than the integers, which is what the task calls for.

Symbolic visual mediation and the transition to university mathematics

Looking at the model solution produced by the lecturer (Figure 2) and the wording of the task (Figure 1), we can see that there are four different instances where the students have to define the symbolic mediators they use: first, in the definition of the divisor where an integer is introduced to illustrate the relationship between *a* and *d*; then, in the narrative which connects *a, b* and their divisor *d*; next, in the substantiation of the relationship between the linear combination of *a* and *b* and their divisor *d;* and, finally, in order to prove by contradiction that a linear combination of *a* and *b* is not divisible by 7. In the last instance, the symbolic mediators on both sides of the equality have to be integers and, as 7 is not divisible by 3, the contradiction occurs. The wording of the task, where the lecturer stresses that all the variables in this part of the task are integers, and the structure of the task aim to signal to the students that they have to be working within the domain of integers. Our analysis suggests that lecturers design the tasks being aware of the students' potential errors regarding the variables belonging to different number domains and thus adds to Bergqvist's (2012) results on what lecturers take into account when designing assessment tasks. The data from the students' scripts shows however, that, in six out of the twenty-two analysed responses, students' errors signal an unresolved commognitive conflict that relates to making a distinction about the nature of the variables. Engaging with this distinction is not a routine that students are typically engaged with, at least in the UK secondary classrooms where the students participating in this study have been educated in: number domains are introduced progressively and students are simply expected to always work within the latest introduced number domain. However, in the university, the numerical context of a task may differ from task to task and the students are expected to be able to swiftly identify the domain that is appropriate for each task and work within it. We approach this issue therefore as a non-negligible aspect of the students' transition from school to university mathematics. Our results resonate with those in Biehler and Kempen's (2013) study. They found that, frequently, their participants would use symbols without providing information regarding the domain of the variable; in our case, not attending to such information results also in leaving parts of the task practically not answered – especially in the cases where the explicit request for "integers *s* and *t*" in (iic) receives non-integer responses.

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From ritual to exploration: The evolution of biology students' mathematical discourse through mathematical modelling activities

Olov Lui Viirman[1] and Elena Nardi[2]

[1]University of Agder, Kristiansand, Norway; olov.viirman@uia.no

[2]University of East Anglia, Norwich, UK; e.nardi@uea.ac.uk

We report analyses from a collaborative, developmental research project between two Norwegian centres of excellence in higher education (MatRIC and bioCEED) in which biology-related mathematical modelling (MM) activities are introduced to biology students as a means to motivate their appreciation for, and competence in, mathematics. This phase of the project involved four sessions with 11 first-semester students. We report data and analyses from two activities: Yeast Growth and Digoxin. Our commognitive analyses trace the evolution of the students' mathematical discourse in two episodes, revealing a scaffolding story about the gradual transition from ritualized to exploratory engagement with MM, and pointing to the crucial role played by the teacher in this process. We conclude with discussing some implications of our analysis for the design and use of MM activities for students of Biology, and other non-mathematics specialists.

Keywords: Theory of commognition; mathematical modelling in biology; mathematical discourse; routines; rituals and explorations.

Teaching mathematics to biology students through mathematical modelling

Research into the mathematical needs of non-mathematics specialists is by no means new (e.g. Kent and Noss, 2003). Participants in many university-level studies are often non-mathematics specialists (e.g. engineers or pre-service teachers), but their specialism often remains a mere part of the study's backdrop (Biza, Giraldo, Hochmuth, Khakbaz & Rasmussen, 2016). The relatively small but growing number of studies in this area (e.g., Gould, Murray & Sanfratello, 2012) have touched on issues such as: the double discontinuity between school, university and workplace mathematics; the challenges of teaching mathematical modelling at school and university levels; issues of confidence in and appreciation for mathematics; and, embeddedness of mathematics into other disciplines.

Within biology, mathematics is becoming increasingly important, placing new demands on the education of future biologists. In the US, for example, the recognition of these demands has led to two national projects focusing on developing undergraduate biology education (Brewer & Smith, 2011; Steen, 2005). A potential problem with placing greater emphasis on mathematics in biology education is that "biology education is burdened by habits from a past where biology was seen as a safe harbour for math-averse science students" (Steen, 2005, p. 14). The project that we draw on in this paper aims to improve student appreciation for mathematics through helping them experience the relevance of mathematics to their field of study. It does so through exploring the suggestion made by several authors (e.g. Brewer & Smith, 2011; Steen, 2005) for greater integration of mathematics and biology in the curriculum. MM, as Brewer and Smith (2011) point out, is a basic skill within the 'core competencies and disciplinary practices' (p. 17) of biology – and a vehicle for improving student appreciation for the role that mathematics can play in scientific research.

Studies which have investigated the use of MM in university biology education (e.g. Chiel, McManus & Shaw, 2010) indicate that engagement with MM activities can contribute to more positive attitudes towards, and self-perceived competence in, both biology and mathematics. Concerning an integrated approach to mathematics and biology, Madlung, Bremer, Himelblau and Tullis (2011) investigated whether such an approach might have adverse effects, such as breadth at the expense of depth, or mathematics anxiety problems. Two versions of a bioscience module, one of which contained a computational statistical element, were developed and offered to an introductory and an advanced biology class. Results showed no detrimental effects of an integrated approach but indicated that advanced level students were more able to benefit from it.

To examine the evolution of biology students' appreciation for, and competence in, mathematics as they engage with MM activities we espouse a discursive perspective – particularly that of the theory of commognition (Sfard, 2008; Nardi, Ryve, Stadler & Viirman, 2014, p. 183-5) – according to which learning is change in one's participation in well-defined forms of activity (discourse). In what follows, we introduce those components of the commognitive perspective pertinent to the data analysis we present in this paper; we then present a sample of our data and analysis (two episodes from students' engagement with two MM activities, *Yeast Growth* and *Digoxin*).

The commognitive construct of *routines*: *Explorations*, *deeds* and *rituals*

According to the commognitive perspective, 'it is by reproducing familiar communicational moves in appropriate new situations that we become skillful discursants, and develop a sense of meaningfulness of our actions' (Sfard, 2008, p. 195). A *routine* is a set of meta-rules that describe a repetitive discursive action. Sfard defines three types of mathematical routines: *explorations*, *deeds* and *rituals*, with *deeds* and *rituals* presented as predecessors of *explorations*. A routine is called an *exploration* 'if its implementation contributes to a mathematical theory' (p. 224) (e.g. equation solving, defining and proving). *Explorations* involve the *construction*, *substantiation* or *recall* of narratives about mathematical objects. *Routines* that involve practical action (action resulting in change in objects, either primary or discursive, p. 241) are called *deeds*. *Deeds* are therefore different from *explorations*, which aim to effect change on narratives. Often, however, there are *routines* that "begin their life as neither *deeds* nor *explorations* but as *rituals*, that is, as sequences of discursive actions whose primary goal […] is neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people' (p. 241).

Sfard claims that *rituals* are a 'natural, mostly inevitable, stage in this development process' (p. 245) and that the road to *exploration* often leads through *ritual*. The data and analysis sample we present in this paper examines this claim with a particular focus on the following research question: "What characterizes the use of *routines* by Y1 Biology students as they engage in MM activities?"

Aims, methods, data and participants of the study

The research design of our study comprises cycles of developmental activity (planning, implementation, reflection, feedback) which are theoretically informed, contribute to the emergence of theory and take place in a partnership between teachers (in this case, a university mathematician) and didacticians (Goodchild, Fuglestad & Jaworski, 2013). This ongoing project is a collaboration between two Norwegian centres of excellence in higher education – the Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC) and the Centre for Excellence in

Biology Education (bioCEED). The aim of the project is to improve biology students' motivation for, interest in, and perceived relevance of mathematics in biological studies through the use of MM. The teaching took place at a well-regarded Norwegian university where biology students take one compulsory mathematics course, taught in the first semester, designed not specifically for the biology undergraduate programme but for students from about twenty different natural science programmes. Typically, in this university, there is little collaboration between the mathematics and biology departments, and few opportunities for focusing on issues specific to biology in the mathematics course. The data for this paper originate in four three-hour sessions with twelve volunteer students, nine female and three male. Activity during the sessions was video and audio recorded, both from whole-class and small-group work. Also, all written material produced by the students was collected. The teaching was conducted in English, but all student group work and most student contributions to group discussions were in Norwegian. The first session began with an introduction to the basic ideas of MM and to the modelling cycle. Students were then asked to work in smaller groups on modelling problems of varying complexity, but requiring only pre-calculus mathematics. The structure of the three remaining sessions was similar, but the initial exposition instead introduced specific types of models relevant to the problems given in that session. The data we draw on in this paper are taken from sessions two and three, and concern one group of four female students as they work on two different, but related, tasks, *Yeast Growth* and *Digoxin*. In the analysis, we examined the discourse of the students looking for recurring patterns that could be described as routines, for instance graph construction. Furthermore, we looked for signs indicating the type of routine use. For instance, we aimed to discern the motives (if any) students provide for their activity. Since the changes in discourse that we aim at charting in this paper take place gradually and over extended periods of time, they are difficult to exemplify through data excerpts within this short paper. Hence, in presenting the data analysis we have opted for offering instead a condensed, selectively detailed narrative account of key incidents illustrating these changes.

Mathematical modelling for biology students: *Yeast Growth* and *Digoxin* tasks

A large part of the first session was spent on a very open task where the students were asked to estimate the density of a rabbit population based on the number of roadkill rabbits along a stretch of highway. Reflecting on the session, the lecturer felt that the students had not been able to work productively enough on this task, and he decided to make the second session more structured. The first 45 minutes of that session were spent first on a follow-up of a homework task given at the end of the previous session, followed by a brief lecture on "steady-state box models" and, related to this, a very short task on pollution in a lake. Then the focus shifted to modelling change, introducing a task concerning the growth of a yeast culture in a petri dish (*Yeast Growth*). Contrary to the first session, however, the task was broken into subtasks that the students worked on for 10-15 minutes each, with whole-class summaries in between.

For the **first *Yeast Growth* subtask**, the students were given a first part of a table of data, taken from an old research paper (Pearl, 1927), with three columns (time, amount of biomass, change in biomass) describing the growth of a yeast culture. The students were asked to: analyze the numerical data in the table; plot the data and analyze the graph; suggest a simple model based on a difference equation of the form $\Delta p_n = k_1 p_n$, where p_n is the size of the yeast biomass after n hours,

$\Delta p_n \overset{def}{=} p_{n+1} - p_n$ is the change of biomass between two measurements, and k_1 is a positive constant; and, explain what their expectations would be regarding the predictive power of the model they constructed. The initial plan for the **second subtask** was to give students the second part of the table and ask them to: analyze this new data (noting the change in population per hour becomes smaller as the resources become more limited); plot the population against time, explore the shape of the graph and state what they would expect in the long run; and, calculate the expected value for "carrying capacity" in this case (noting that, based on the graph, the population appears to be approaching a limiting value, known in biology as "carrying capacity"). However, in the actual session (due to limitations of time) the students were instead given a non-linear model based on incorporating the carrying capacity: "We may estimate carrying capacity to be 665 (this value is not precise and your value may differ a bit). As the number $665 - p_n$ gets smaller and smaller as p_n approaches 665, we may adjust our simple linear model replacing it with a nonlinear model $\Delta p_n = k_2 p_n (665 - p_n)$ or alike, if you have chosen 664 or 666. Test a new model by plotting Δp_n against $p_n (665 - p_n)$ to check whether a reasonable proportionality is observed. Then, estimate the proportionality constant k_2. What is your value?" For the **third and final subtask**, the students were asked to use the new model, with k_2 =0.00082, to compute values and compare them with the actual data ("Compute twelve values of p_n using the formula and starting with the initial value $p_0 = 9.6$. ").

Digoxin was the first task of session 3 and also concerned the modelling of change, in this case the decay in the body of *Digoxin*, a drug used in the treatment of heart disease: (**a**) For an initial dosage of 0.5mg in the bloodstream, the table shows the amount of digoxin a_n remaining in the bloodstream of a particular patient after *n* days, together with the change Δa_n each day. Plot Δa_n versus a_n and explore the graph. Suggest a simple model based on a difference equation of the form $\Delta a_n = k_3 a_n$, where k_3 is a positive constant. What is your choice of k_3? (**b**) Now our objective is to consider the decay of digoxin in the blood stream to prescribe a dosage that keeps the concentration between acceptable levels so that it is both safe and effective. Design a simple linear model describing the following scenario: we prescribe a daily drug dosage of 0.1mg and know that half the digoxin remains in the system in the end of each dosage period. (**c**) Consider three different options where the initial one-time dose of medicine received by the patient is a_0 = 0.1mg, 0.2mg or 0.3mg. What are your conclusions? What would you recommend if you were this patient's GP?"

In *Yeast Growth*, the students were expected to find an approximately linear relation between the change and the amount of biomass, estimate the proportionality constant, and conclude that this rate of growth cannot continue indefinitely. With the additional data then provided, they were then expected to conclude that the growth decreases and the amount of biomass stabilizes at the carrying capacity of the petri dish, in this case 665. The students were then given a suggested non-linear model and were expected to check the validity of the model by finding the proportionality constant. Finally, they were expected to use the model to generate values that could be compared with the actual data. To do this, they needed to solve the equation $\Delta p_n = p_{n+1} - p_n = k_2 p_n (665 - p_n)$ for p_{n+1}.

In *Digoxin*, in part (a) the students were expected to find a linear relationship between the change and the amount of digoxin remaining, and estimate the proportionality constant from the graph. In

part (b) they were expected to construct a model of the form $a_{n+1} = 0.5a_n + 0.1$, and then, in part (c), use this model with the different initial conditions to realize that, in all cases, an equilibrium of 0.2mg will eventually be reached, leading to a recommended initial dose of 0.2mg.

In what follows we highlight two critical incidents, one from *Yeast Growth* and one from *Digoxin*.

Yeast Growth: Ritualized engagement with mathematical modelling

The group ignores the first question in the subtask, about analyzing the data in the table. Instead, their initial efforts concern the practical details around graph construction and data plotting: choosing the right scale for the axes, and the like. They do all work in parallel, constructing one graph each, on millimetre grid paper, but they still work collaboratively, discussing their work at every turn. The routines they are using seem familiar to them, but there is no evidence of any reflection concerning the purpose of the activity they are engaging in. The task requests of them to plot the data, and since this is something they know how to do, they do it. We see this as suggestive of ritualized routine use. After about ten minutes, however, they seem confused about how to interpret the data in the table: what does Δp_n actually mean? They start discussing how to fit a straight line to the data, but the relative inefficiency of their working method – putting a lot of effort into the design of the graph and all drawing their own copy – means that, in the end, they do not have the time to do this, let alone find the proportionality constant. In the first whole-class follow-up, the students quickly agree that the problem concerns exponential growth, but none of the groups have succeeded in finding the constant k_1. It turns out that that they have constructed the wrong graph: plotting change against time, not against amount. We see this as evidence of ritualized routine use. Had the students engaged with the first question in the subtask, and reflected about the interpretation of the data, this mistake might have been avoided. Instead, the students resorted to a well-established routine for data plotting, using time as the independent variable. After this mistake has been clarified, the students are given additional data, and start discussing the validity of the model: is unlimited growth reasonable? The need for a revised model is established.

The work on the second subtask still mostly revolves around plotting the data, but now the group only constructs one plot. There is, however, some remaining confusion regarding the nature of the data: does Δp_n represent change or the actual amount? One of the students interprets the decrease in Δp_n as evidence of a population crash (a catastrophic decline in population), but the other group members point out that the decrease is in change, not actual amount: “But this is just the change, this is not the number of living cells.” Thus, when engaged in biological discourse, they are able to reason in a meaningful manner about the interpretation of the mathematical symbols. However, the formulation of the task creates additional confusion. It explicitly mentions a nonlinear model, but at the same time asks for proportionality. Finding proportionality between the more complexly presented quantities in this task seems unfamiliar to the students – and, since this is something not normally done in school, it probably is. Following the recent whole-class discussion, but contrary to what is written in the formulation of the subtask, the students do what they were expected to do in the first subtask, plotting the change Δp_n against p_n instead of against $p_n(665 - p_n)$. They thus struggle with fitting a straight line to the data, since their plot does not describe a linear relationship. In the whole-class follow-up, it turns out that, yet again, none of the groups have been able to compute the constant k_2, and, in the end, the lecturer provides the students with an estimated value

and asks them to use the model they now have to compute a number of values of p_n and to check the predictive value of the model. This turns out to be very confusing for our group, who are at a loss as to how to proceed: "I don't have a clue. I feel so stupid." The work they have been doing in both sessions so far has been geared towards constructing models, not validating them, leaving them unprepared for this way of using models. Furthermore, the routines they have been using have all concerned graph construction and plotting, and now they are supposed to compute values. After some initial confusion, they start doing computational work, but their nervous laughter and exclamations of surprise suggest that they have little faith in that what they are doing makes sense. Indeed, the different numbers they are juggling around suggest that they are making various computational errors. Also, they spend quite some time plotting the values that they obtain. We see this as indication that their routine use is still highly ritualized: they do certain things because they feel that it is expected of them, without having any clear rationale for why they are doing so.

Looking at the way the students engage with the *Yeast Growth* task, we conclude that what was intended by the lecturer as scaffolding – dividing the task into clearly delineated, smaller subtasks – in fact amounted to restricting student agency. We propose that this restricted agency is connected to ritualized routine use. The formulations of the subtasks state explicitly what the students are supposed to do, and even suggest what specific routines to invoke (plot the data; estimate the constant). This decreases the need for reflection about what routines to use and why, thus inviting ritualized routine use. This interpretation is further supported by how they struggle when asked to perform a different set of routines, using a given model for substantiation purposes, rather than constructing a model from given data. This indicates to us that they are not yet using the construction routines in an exploratory manner.

Digoxin: Towards exploratory routine use

Although there seems to be a connection between the highly scaffolded format of the *Yeast Growth* task and students' ritualized routine use, we do not intend this to be seen merely as a cautionary tale. Indeed, looking at the students' work on the *Digoxin* task in session 3 four weeks later, there is evidence of progress towards making the discourse of growth model construction their own. The *Digoxin* task was presented as a whole, without the same amount of scaffolding as the *Yeast Growth* task. As in the second session, the group focuses their effort on constructing the graph, but has some problems interpreting the task because of unfamiliar terminology (e.g. difference equation). Contrary to *Yeast Growth*, in *Digoxin* time is not included as a column in the table of data, thus minimizing the risk of students resorting to the "plotting against time" routine. Still, one of the students suggests using *n* as the independent variable, in an attempt to fall back on the familiar routine. After some discussion, they decide not to resort to the earlier default option of using time as the independent variable, and, using the graph and the table, they manage to find the proportionality accurately. This might be interpreted as an indication of what Sfard (2008, p.251) calls "thoughtful imitation". Having failed at constructing the requested plot in subtask 2 of *Yeast Growth*, and then being shown by the instructor what should have been done, they are now able to engage more fruitfully with this similar, but less complex, task. There is some additional confusion due to the formulation of the task (even though we are dealing with decay, the task still prescribes that k_3 should be positive). Here we see signs that the group have still not made the discourse fully their own, but rather are emulating the discourse of the teacher. Rather than trusting their own

reasoning, they handle the problem in a manner familiar to many students – they adapt the answer to fit the teacher's expectations: "Let's just drop the minus sign." As for parts (b) and (c) of the task, they (as well as the other two groups) run out of time before managing to make much headway. Still, it appears as if the ritualized routine use when working on the *Yeast Growth* task has supported the students' pathway towards handling the *Digoxin* task in a more exploratory manner.

The path to exploration passes through ritual: Conclusions and ways forward

In this paper, we examine a case (Y1 Biology students' engagement with MM) of how new routines evolve, and particularly how discursants experience a step from ritualized to exploratory routines. The analysis points to the crucial role played by the teacher in facilitating this process. For instance, through the tasks presented to students, he influences their routine use, not only in the obvious way of suggesting *what* routines to use, but also *in what way* to engage with these routines. We have seen how a highly scaffolded task, which explicitly states what routines to invoke, might in fact invite ritualized routine use, whereas a less strongly scaffolded task might necessitate reflection about what routines to invoke and why, thus inviting a more exploratory engagement. At the same time, our analysis suggests that perhaps the ritualized routine use suggested by more scaffolded tasks might be a necessary step on the route towards exploratory routine use.

Per Sfard (2008), rituals are a 'natural, mostly inevitable, stage in this development process' (p. 245) and, recognizing this as so, recognizes fully the 'inherently social nature of human thinking and learning' (p. 245). Our claim here resonates with Sfard's: the road to exploration must sometimes pass through ritual. There is an inherent circularity in this evolutionary process: a learner 'could not possibly appreciate the value of the new routine until she was aware of its advantages; such appreciation, however, could only emerge from its use' (p. 246). Furthermore, 'the deed-enhancing mathematical explorations would sometimes involve new abstract objects, objects that can only emerge through implementation of this very routine' (p. 247) and this holds for the evolution of an individual's mathematical discourse as well as that of the field of mathematics as a whole. Discursive researchers – Sfard herself as well as Bakhtin – posit that thoughtful imitation can be a transitory phase in transforming ritual into exploration (where imitation is meant as a non-trivial process that involves evaluation, assimilation, reworking and re-accentuation). Indeed, in the students' work on the *Digoxin* task, we have shown signs of such "thoughtful imitation".

Deritualization results in consolidated discourse, namely a 'well-developed network of interlacing, partially overlapping routines' (p. 254). In this trajectory of growth there are at least two 'basic conditions for effective mediation': the principle of the continuity of discourse ('introducing a new discourse by transforming an existing one', p. 254); and, the principle of commognitive conflict ('the situation in which different discursants are acting according to different metarules' (p. 256) – a potential source of discourse change, and thus of learning). In this paper, we sample evidence mostly of the former principle. Our scrutiny of the entire dataset is now gearing towards the identification of evidence of the latter. Further, we anticipate that rolling out more MM activities to a new cohort of Y1 Biology students will lend corroborative power to the conjectures we explore here. It may also provide an opportunity for a more extended testing out of using the commognitive framework towards analyses that inform pedagogical practice.

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Meta-representational competence with linear algebra in quantum mechanics

Megan Wawro[1], Kevin Watson[2] and Warren Christensen[3]

[1]Virginia Tech, USA; mwawro@vt.edu

[2]Virginia Tech, USA; wk223@vt.edu

[3]North Dakota State University, USA; warren.christensen@ndsu.edu

In this report we analyze one student's meta-representational competence as he engages in solving a quantum mechanics problem involving the linear algebra concepts of basis, eigenvectors, and eigenvalues. We provide detail on student A25, who serves as a paradigmatic example of a student's power and flexibility in thinking in and using different notation systems. This case study, which lays the groundwork for future analysis, provides evidence that meta-representational competence (MRC) is beneficial to a student's ability to make sense of and use concepts from linear algebra while solving quantum mechanics problems.

Keywords: Linear algebra, meta-representational competence, physics.

Introduction

The National Research Council's (2012) report, which charges the United States to improve its undergraduate Science, Technology, Engineering, and Mathematics (STEM) education, specifically recommends "interdisciplinary studies of cross-cutting concepts and cognitive processes" (p. 3) in undergraduate STEM courses. It further states that "gaps remain in the understanding of student learning in upper division courses" (p. 199), and that interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines … and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202). Our work contributes towards this need by investigating student understanding of linear algebra in quantum mechanics. Two research questions that guide us in this paper are: what are the various ways in which students reason about and symbolize concepts related to eigentheory in quantum physics, and in what ways might meta-representational competence impact how they make sense of linear algebra concepts in quantum mechanics?

In this paper, we focus on one student's reflection on symbolizing choices he makes while solving a quantum mechanics problem that involves linear algebra. In particular, we analyze his reasons for how and why he chooses a specific symbol system – either Dirac notation or matrix notation – for solving an expectation value problem. We align our analysis with the frameworks of meta-representational competence (diSessa, Hammer, Sherin, & Kolpakowski, 1991) and of structural features of algebraic quantum notations (Gire & Price, 2015). This case study, which lays the groundwork for future analysis, explores in what ways MRC might aid a student's ability to make sense of and use concepts from linear algebra while solving quantum mechanics problems.

Background and theoretical framework

In this section, we give an overview of research conducted on student understanding of symbols and representations in mathematics and physics, as well as our theoretical orientation. We conclude with a brief introduction to eigentheory in Quantum Mechanics and Dirac notation.

Student understanding of symbols and representations

The recognition of the importance of students' understanding of symbols used in mathematics and physics has grown over the past few decades. Arcavi (1994, 2005) coined this as "symbol sense," which includes aspects such as being "friendly" with symbols, engineering symbolic expressions, choosing which aspects of a mathematical situation to symbolize, using symbolic manipulations flexibly, and sensing the different roles symbols can play in various contexts. Other research along this vein include: an explication of how different perspectives, such as cognitivist, situationist, and social-psychological, provide vastly different ways to understand how students make sense of and use inscriptions and symbols (Kaput, 1998); a study of how students mathematize their language from a Vygotskian perspective (Van Oers, 2002); and an exploration of how notational systems can serve as a mediational tool which triggers and sustains mathematical activity (Meira, 2002).

Research into students' competence with symbols and representations is not limited to primary and secondary school studies. For example, Hillel (2000) described three modes of description (abstract, algebraic, and geometric) of the basic objects and operations in linear algebra and pointed out that "the ability to understand how vectors and transformation in one mode are differently represented, either within the same mode, or across modes is essential in coping with linear algebra" (p. 199). Thomas and Stewart (2011) found that students struggle to coordinate the two mathematical processes captured in $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where A is an n x n matrix, $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$, and λ is a scalar, to make sense of equality as "yielding the same result." This interpretation of the "equals" symbol is often novel and nontrivial for students (Harel, 2000). Harel also posits that the interpretation of "solution" in this setting, the set of all vectors $\boldsymbol{x}$ that make the equation true, entails a new level of complexity than does solving equations such as $cx = d$, with each taking values from the reals. Thomas and Stewart (2011) conjecture that this complexity may prevent students from progressing symbolically from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, which is particularly useful when solving for the eigenvalues and eigenvectors of a matrix A.

Research into students' understanding of quantum mechanics also investigates student use of symbols, such as how students make sense of and use a novel notation, called Dirac notation (explained in the subsequent section). Most closely related with this current study, Gire and Price (2015) looked at structural features of three different notation systems used in quantum mechanics (Dirac, matrix, and wave function) and how students' reasoning interacts with these features. The features identified by the authors are: (a) individuation, or "the degree to which important features are represented as separate and elemental" (p. 5); (b) externalization, or "the degree to which elements and features are externalized with markings included in the representation" (p. 7); (c) compactness; and (d) symbolic support for computation. Using problem-solving interviews with students as insight into these features, Gire and Price found that students readily used Dirac notation, and that the structural features vary across the different notations and among contexts.

Relatedly, diSessa et al. (1991) importantly discovered that students have a great deal of knowledge about what good representations are and are able to critique and refine them, which the authors defined as Meta-Representational Competence (MRC). diSessa and Sherin (2000) explained that MRC includes inventing and designing new representations, judging and comparing the quality of representations, understanding the general and specific functions of representations, and quickly learning to use and understand new representations. Furthermore, diSessa (2002, 2004) offered a

variety of critical resources students possess as part of their MRC for judging the strength of representations, such as compactness, parsimony, and conventionality. Two particular resources encompassed by MRC that we focus on in our data are "critique and compare the adequacy of representations and judge their suitability for various tasks," and "understand the purposes of representations generally and in particular contexts and understand how representations do the work they do for us (diSessa, 2004, p. 94).

In this study, we align ourselves with the theory that representations are a sense-making tool, in that "the construction of representations on paper during problem solving mediates and organizes one's understanding of mathematical concepts" (Meira, 2002, p. 101). We couple this with a framing of MRC, specific to two particular notational systems, to investigate a student's reflection on his own notational preferences in quantum mechanics and what that may reveal about his understanding of change of basis and eigentheory in that context.

Brief introduction to eigentheory and Dirac notation in quantum mechanics

In quantum mechanics, certain physical systems are modeled and made sense of using eigentheory. To a physical system we associate a Hilbert space (such as $\mathbb{C}^2$), to every possible state of the physical system we associate a vector in the Hilbert space, and to every possible observable (i.e., measurable physical quantity) we associate a Hermitian operator (usually given in its matrix form). The only possible result of a measurement is an eigenvalue of the operator, and after the measurement the system will be found in the corresponding eigenstate.

Dirac notation, also known as bra-ket or just ket notation, is a commonly used notational system in quantum mechanics. A vector representing a possible state is symbolized with a ket, such as $|\psi\rangle$. Mathematically, kets behave like column vectors, such as $|\psi\rangle \doteq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, a_1, a_2 \in \mathbb{C}$, and are usually normalized. The complex conjugate transpose of a ket is called a *bra*, which behaves mathematically like a row vector, such as $\langle\psi| \doteq [a_1^* \quad a_2^*]$. In addition, the eigenvalue equations for observables are central to many calculations. For example, the eigenvalue equations for S_x (the operator measuring the x-component of intrinsic angular momentum) of a spin-½ particle are $S_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle_x$, where $|+\rangle_x$ and $|-\rangle_x$ form an orthonormal eigenbasis of S_x, and $\pm\frac{\hbar}{2}$ are the two possible measurement results of the observable. When symbolized in terms of this eigenbasis, the matrix representation of S_x is $\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}$. One can also measure spin along other directions, such as z; similarly, the eigenvalue equations are $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ (it is common for no subscript to be used for the z-direction). Thus, "within its own basis," the matrix representation of S_z would be identical to the aforementioned diagonal one for S_x. It is often beneficial to change between bases; for example, $|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ and $|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$, so S_x in the "z-basis" is $\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}$. Finally, inner products are involved in computing the expectation value of observable A for state psi, $\langle\psi|A|\psi\rangle$. These calculations require the bra and ket expansion to be in the same eigenbasis as the matrix representation of A. As such, expectation value problems present a rich setting for investigating students' symbolizing of eigentheory and change of basis in a physics context.

Methods

Participants for this study were third year undergraduate physics majors at a large, public, research-intensive university in the Pacific Northwestern United States. They were drawn on a volunteer basis from a class of 35 students in a Spin and Quantum Measurements course; this course met for 7 class-hours per week for three weeks and involved many student-centered activities and discussions. The data for this report come from individual, semi-structured interviews (Bernard, 1988) conducted with 8 students at the end of the course. The goals of the interview questions were to learn how students reasoned about linear algebra concepts (e.g., normalization, basis, and especially eigentheory), how they reasoned with these concepts as they discussed quantum mechanics concepts and solved quantum mechanics problems, and how they symbolized their work.

To begin our analysis, we viewed the video and observed how students navigated the interview problems, while we kept in mind the overarching research questions regarding students' reasoning about and symbolizing eigentheory in quantum physics. We noticed some students were particularly fluent in how they talked about and worked with both matrix and Dirac notations. This compelled us to investigate the literature about student use of symbols and notations, the most relevant of which were discussed above. Our analysis draws most heavily on the work of diSessa and colleagues regarding MRC, and that of Gire and Price (2015) regarding structural features of algebraic quantum notations. In particular, we coded for instances of students mentioning structural features of the mathematics or students making explicit meta-commentary on the representations they chose to use. This allowed us to integrate our analysis of students' MRC with Gire and Price's types of structural features in a way novel to the physics and mathematics education fields.

In this report, we focus on one student: A25, a double major in physics and nuclear engineering who had completed two 10-week courses in linear algebra. The reason we chose to focus on participant A25 was his demonstrated ability to articulate his thinking. During the interview, he exhibited flexibility in reasoning about the concepts we were probing, and through his explanation a great deal of MRC seemed visible and analyzable.

Results

In the beginning of the interview, student A25 volunteered that he sometimes explicitly chooses between doing calculations in matrix notation or in Dirac notation:

I: So how do you feel like, using eigenvectors and eigenvalues, in spins has been similar to and different from how you've experienced those in other classes?

A25: Uh, well, it's very similar because you're doing a lot of the same math …the difference especially in physics, you're looking at kets. In, in at first I was kind of jarring, like to- to try to do the math in kets. But now, it's kinda- it's kinda easier, there's problems, there certain problems…where there's two ways to do them, they're kind of parallel, you can do it and you can expand the- the state in- in like as a- and expand them as kets in a different basis, or you can write that state as a- as a, as a vector, in that basis, and you can either do the matrix math for the like expectation values for example, you can do the matrix math or you can do the ket math, and sometimes it's, I'm finding that I, rather expand something in the ket.

From the transcript we see that A25 was aware multiple legitimate ways exist to solve the problem, seemingly understanding the various mathematical nuances and implications of his notational choices. His brief explanation highlights sentiments that are consistent with Arcavi's characteristics of symbol sense, such as being "friendly" with symbols and using them flexibly. Also, A25's self-reflection on his symbol usage adds a metacognitive aspect to the symbol sense characterization.

Because A25 volunteered expectation value problems as a situation in which he could use either notation, the interviewer had him work on such a problem right away, even though it was prepared to be at the end of the interview: "Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system. Calculate the expectation value for the measurement of S_x." A25 immediately worked on the problem using Dirac notation, saying, "basically to find the expectation value… it's like denoted that way [writes $\langle A\rangle$] but really what you're doing is you're taking the, the bra of the state, and then you're putting the operator [writes $= \langle\psi|A|\psi\rangle$] in the middle of the inner product." He continued to explain his work as he proceeded, with statements such as "you know that S_x is just going to um, like apply it's eigenvalues to these, so, so like the eigenvalue corresponding to plus x is going to be $+\hbar/2$ and the, the eigenvalue corresponding to $-x$ is going to be $-\hbar/2$, so you end up with this equation that looks like this [points to the second half of line 2 in Figure 1a]. Note that his work in Figure 1a, which led him to the correct answer of $7\hbar/50$, involved the state's expansion and use of eigenvector equations for S_x in ket notation. He did not need to physically write the expansion of $|\psi\rangle$ in the x basis kets, nor did he write out the eigenvector equations; however, his verbal description of his process relied on his understanding of both basis and the eigenvector relationships at play. Furthermore, this notation was novel to the students during this course; as such, A25 was clearly quick to use and understand this representation (a quality of MRC, diSessa & Sherin, 2000).

After discussing his work and solution, the interviewer asked: "Before you were telling about bra-ket versus matrix notation, you brought up an expectation value as an example of like, either or both, so can you, now that you had this problem, kinda revisit that?" A25 immediately solved the problem completely within matrix notation. He began by saying "if we're strictly in the plus and minus x basis" and wrote the column vector $\begin{bmatrix} -\frac{4}{5} \\ i\frac{3}{5} \end{bmatrix}$ associated with the given ket $|\psi\rangle$. He then said, "and then the bra would be, um, minus 4 over 5 and then minus i 3 over 5," writing out the row vector $\begin{bmatrix} -\frac{4}{5} & -i\frac{3}{5} \end{bmatrix}$ as he spoke (see Line 1 in Figure 1b). He then said, "and so what you do is take this [copies the column vector]…and then you have the operator in the middle [writes an empty 2x2 matrix], and then you have the bra here [copies the row vector], and the operator in this case is S_x and we're in the x basis so it's just $\hbar/2$ and -$\hbar/2$, 0, 0" [fills in the 2x2 matrix values] (see line 2 in Figure 1b). Impressively, he was able to fluidly move from his original ket notation to matrix notation, flawlessly making translations from the bras, kets, and operators in ket notation to row vectors, column vectors, and matrices in the matrix notation, further evidence of his strong MRC. Next, he explained his process for computing the matrix times the column vector before he did the computation, noting that "you're gonna get a vector." Again in line 3 he explained "then I do it again, so, um, this time you're gonna get a number out," meaning he anticipated that a row vector times a column vector would be a number. This shows two aspects of A25's strong understanding: first, a fluency in the calculations and computations within matrix notation similar to his ease in working in ket notation, including the ability to anticipate results before actually carrying out a computation (as in anticipating the result of

a matrix times a vector); and second, an ability to compare the two notations as well as an understanding that the two notations represent two ways to conceptualize the quantum physical calculation of expectation value. We see this as flexibly using symbolic manipulations (Arcavi, 1995) and an anticipation of results.

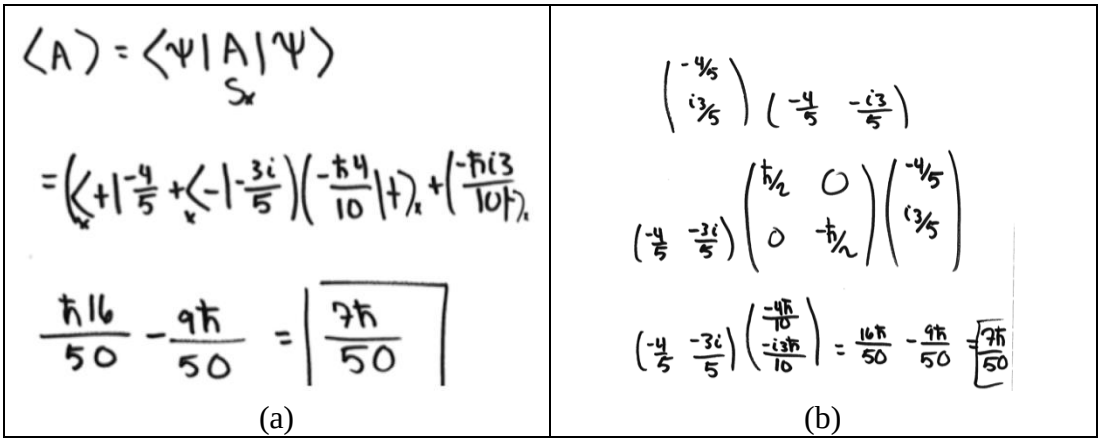

Figure 1: A25's expectation value problem, in ket notation (a) and matrix notation (b)

The interviewer then asked A25 to reflect on any preference between the two notations:

A25: Uh...To be honest, I don't really, I don't really know why I prefer this [Figure 1a], I think it's just because, um, I like this notation. This specific notation [Figure 1a line 1] like this to me is like a cleaner way of writing that [Figure 1b line 2] because that- I mean this and that [touches Figure 1a line 1 and Figure 1b line 2 simultaneously] I feel like are your starting points, so you, you start here with this nice, like, looking thing [traces one finger under $\langle\psi|A|\psi\rangle$], or you start here with this big array of numbers [puts two open hands around Figure 1b line 2], and I prefer this [Figure 1a line 1], even though you have to expand this into basically the same amount of information [Figure 1a line 2]. And also, the nice thing about, about this [Figure 1a line 1], is it—actually this is really why it's better—is because you can, you can say ok S_x works- acts directly on these kets, you can just get rid of the matrix altogether...

We see his use of "nice looking thing" and "big array of numbers" in comparison to one another are an example of compactness. He also compares Figure 1a line 1 and Figure 1b line 3 regarding the "amount" of information they convey, which involves reflection on the physical and mathematical content expressed in the compared notations. Finally, acting directly on the expansion in terms of the eigenstates of the operator allow him to forego the matrix calculation entirely, which speaks to A25's view of compactness, parsimony, and symbolic support for ket notation for this problem.

When asked about his notation preferences if the basis expansion of a given state vector and the operator "didn't match," A25 recalled a problem from his last homework that was "actually easier…to do the matrix multiplication," stating "you don't want to have to change these kets into different bases all over the place 'cause they're already all written in the same basis and you know what the operator is in that basis so you might as well just, do the matrix multiplication." Here we see how strong A25's understanding is of the important linear algebra concepts of bases and change of bases, and how they relate to the matrix multiplication within expectation value quantum mechanics problems.

Furthermore, we see another aspect of his MRC, namely his understanding that different notations have different strengths and weaknesses, and his ability to leverage these strengths and weaknesses depending upon the particular quantum mechanics situation. This speaks to his awareness of symbolic support as well as using symbols flexibly. Finally, when asked if the concepts of basis or eigenvectors/eigenvalues come up more in one notation than the other, A25 stated, "certainly…every time you write down a ket you're, you're very conscious of what basis you're in. In this one [points to Figure 1b] it's just kinda implied…all this [is] in the same basis, so you're just, you're just writing out numbers, an arrays of numbers, but here [in Figure 1a] you're thinking ok, this is the S_x operator, this is the x plus ket, this is the x minus bra…so I think that you're definitely more aware of what basis you're in when you're using this, because you have to be." This explanation is consistent with externalization (Gire & Price, 2015), in that the ket notation allows features of the problem, namely basis, to be externalized in a way that matrix notation did not provide for A25. This again attests to his understanding that notations have different strengths and weaknesses, an element of MRC that seems particularly important within quantum mechanics.

Conclusion

In this report we analyzed one student's MRC and his understanding of change of basis and eigentheory as he solved an expectation value problem in quantum mechanics. This case study lays the groundwork for future analysis by being a paradigmatic example of a student's power and flexibility for thinking in and using different notation systems. In addition, it provides evidence that MRC seemed to positively impact this student's ability to make sense of and use concepts from linear algebra while solving quantum mechanics problems. In addition to analyzing the other students from our data set, future research includes investigating how classroom interactions may have influenced students regarding their notational choices, what aspects of MRC seem most important to success in using linear algebra when solving quantum mechanics problems, and what that implies regarding students' understanding of the mathematics and physics content involved.

Acknowledgment

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The impact of the teacher's choices on students' learning: The case of Riemann integral in the first year of preparatory classes

Inen Akrouti and Sassi Haddad

Virtual University of Tunis, ISEFC, Tunisia; akroutiinen@yahoo.fr

University of Carthage, Tunisia; haddad.sassi@icloud.com

Keywords: Riemann integral, action and formulation, and students' work.

The teacher's practice plays a major role in the learning process and influences students' behavior. Moreover, teacher's choices affect students' self-perception and guide their work. Ghedamsi (2008) also established that mathematical activities of students depended on mathematical organization and teachers' designs. Robert and Rogalski (2002) also found that the mathematical activity of students such as calculus was affected by the way an exercise was organized. The work presented in this paper continues along the same lines and analyses the impact of the teacher's practice on students' learning of the Riemann integral concept in the first year of university.

Theoretical considerations and method

Studying the impact of the teacher's choices on students' learning in the case of the Riemann integral in the first year of preparatory studies led me to considering a tool developed by Ghedamsi (2015) for analysing the teaching and learning process in a regular lesson at the first year of university. This tool includes two dimensions: teacher management and students' work. In this paper, the students' work consists of two levels. The first is based on action and formulation by which students formulate questions concerning specific knowledge; spontaneously express knowledge by changing semiotic setting, making examples, or linking several notions; and formulate a view on knowledge. The second level is based on validation when students indicate technical methods; perform their own validations; and discuss patterns proposed by peers or by the teacher.

Students participating in this study were in first-year preparatory classes at IPEIT (Preparatory institute for engineering school) in Tunisia. They had all obtained a mathematical baccalaureate. The method used in this research consists of three phases. In the first phase, we prepared a preliminary test for 25 students to be taken before studying the concept of Riemann integral, aimed at analysing and understanding their background. This test proposed tasks found at the end of the secondary school. Then, we observed some regular lessons (18 students were present in these lessons). In preparatory classes, the courses are organized into lectures and tutorials that each lasts two hours. Tutorials represent an opportunity to apply the definitions and theorems taught previously during lectures. We developed a detailed analysis of the two tutorials on the Riemann integral. Observation of these lessons allowed us to see the interaction between the teacher and the students. Finally, we prepared a second test after the teaching of this concept; 15 students participated in this test. The questions; which were presented in the second test primarily involved problems concerning Riemann integral concepts.

Results

Based on an analysis done using the tool of Ghedamsi (2015) cited above, we can conclude that the Riemann integral is used in the calculation of limits of some sequences that refer to Riemann sum. The teacher had the intention to invite debates and give students the opportunity to express themselves about the knowledge at play but the contract he established limited the students' opportunity for interactions, thus hindering the students in developing their analytical skills and improving their critical thinking.

The intervention of the teacher seemed to point out the importance of the integral as a tool for calculations.

The classroom observation allows us to see more clearly the impact of the teacher's choices on the quality of students' learning. We can make the following observations:

- Only the teacher offers the solutions and the different techniques required. The mathematical activity of the students is reduced to simple applications of the procedures proposed by the teacher. Thus, the practical implementations of knowledge are limited to the technical level.
- The organization and structure of the tutorial sequences conducted by the teacher do not encourage student's autonomous work. The types of tasks proposed do not invite reflective thought that mobilizes the supposed acquired knowledge, but rather invite algorithmic work. Students' work is limited to a few memory techniques to solve most often stereotyped questions.
- The majority of tasks can be treated in the algebraic register. There is no recourse to other registries such as graphic register.

Conclusion

The analysis of the data collected suggest that is possible to develop teaching sequences for the Riemann Integral concept which take into consideration students' autonomous work and encourage them to create their own self-perception. Hence, a Didactic Engineering can be elaborated in order to surmount the problems identified in this analysis and to propose another alternative improving the teaching-learning process of the Riemann Integral concept.

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Interactive mathematical maps for de-fragmentation

Anna-Maria Schwarz[1], Matthias Brandl[2], Tobias Kaiser[3] and Andreas Datzmann[4]

University of Passau, Faculty of Computer Science and Mathematics, Germany;
[1]annamaria.schwarz@web.de; [2]Matthias.Brandl@Uni-Passau.de; [3]Tobias.Kaiser@Uni-Passau.de;
[4]Andreas.Datzmann@Uni-Passau.de

Keywords: Technology uses in education, maps, visualization, joined-up thinking, transition.

Transition and fragmentation

Transition problems from school to university in mathematics

Tall (2008) presented a theoretical model which he called the three worlds of mathematics to describe the transition problem from school to university in mathematics. He postulates the difference within the existence of a so-called axiomatic-formal world at university and a separated conceptual-embodied respectively proceptual-symbolic world at school. This model helps to account for some difficulties of many students' transition to university in mathematics, resulting in missing common threads and not knowing connections. Consequently, one of the ideas of our visualization project is the attempt of following "known" conceptual-embodied and proceptual-symbolic truths – where possible through development in time – into the axiomatic-formal world in order to see their genetic connection, the desired and meaningful so-called "golden thread".

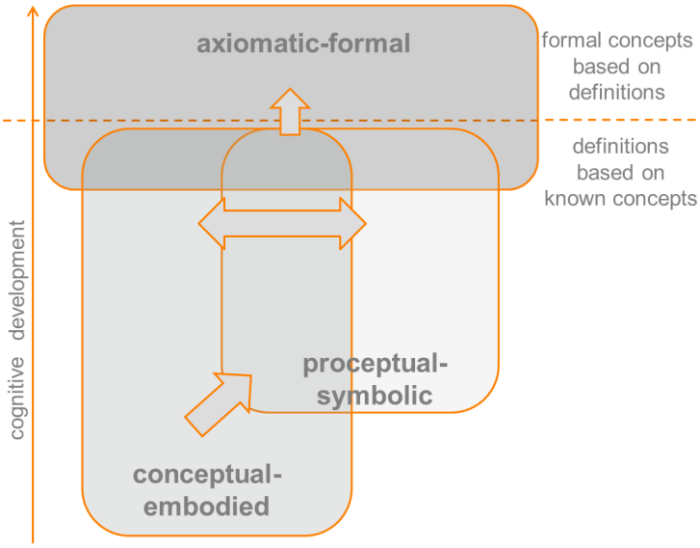

Figure 1: The three worlds of mathematics (Tall, 2008)

Interactive mathematical maps

The concept of "mathematical maps" was introduced by Brandl (2008) as a didactical tool in the form of a virtual tree or net, which shows interrelation between topics (horizontal dimension) as well as the development of a subject matter – starting from an initial problem – in time (vertical dimension). A structural model can be seen in Figure 2. This concept offers several opportunities to foster joined-up thinking and will allow the student to follow the development of an initial problem in time. For example, the visualization in three dimensions allows for an ideal transparency of the interdependencies or the connection of single nodes which additionally offer contents from other platforms by link (Brandl, 2008, pp. 106–109). Another concept, especially for the horizontal orientation, is made by Acevedo (2014) with the "OpenMathMap" which organizes different subjects of

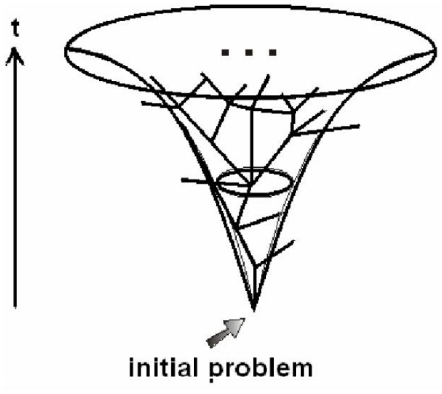

Figure 2: Structural visualisation of the development of an initial problem in time

mathematics by size according to the amount of published papers and by closeness representing the relationship of two subjects (pp. 6–7).

Design-based research methodology

The new teaching format for students in mathematics and mathematics teacher education at the University of Passau will be developed via design-based research methodology. Interactive mathematical maps will be used as a didactical tool to show the interrelationship between different mathematical topics as well as between mathematics at school and at university level, connected by the development of the subject matter in time.

The current status of our work-in-progress is ongoing conceptualization of the teaching format and development of the code for the interactive mathematical maps. Figure 3 shows a screenshot of a first attempt (via JavaScript) to produce content knots which are connected to related content knots by lines in an automatically evolving 3D-visualisation. In order to link the interactive mathematical maps with contents, ILIAS will be used as an e-learning tool (which is the common tool for online courses at the University of Passau). Reasons for the use of e-learning formats are given by constructivism, improvement of quality of teaching, motivation of students and preparation for lifelong learning (Kreidl, 2011, p. 15).

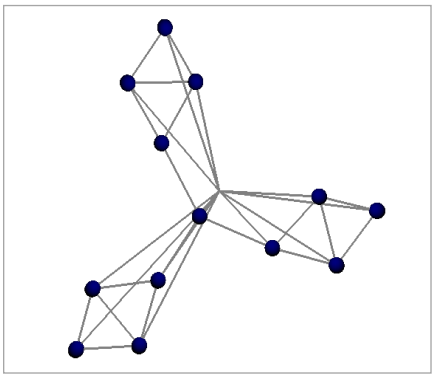

Figure 3: Screenshot of the work-in-progress ICT-tool

The final product will be implemented in mathematics teacher education for (higher) secondary schools (i.e. Gymnasium) to serve for a de-fragmentation process of mathematical contents learned at school and at university. We intend to improve teaching by using e-learning – particularly blended learning – in connection with these interactive mathematical maps. Furthermore, there will be special courses connecting courses in mathematics (first: geometry) and didactic of mathematics (first: didactics of geometry) using the presented ICT-tool in mathematics teacher education.

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In what ways do lecturers receive and use feedback from large first year mathematics classes?

Anthony Cronin and Maria Meehan

University College Dublin, Ireland, School of Mathematics and Statistics;

anthony.cronin@ucd.ie; maria.meehan@ucd.ie

Keywords: Feedback, large class instruction, mathematics support.

We report on a study aimed at examining the ways in which lecturers of large mathematics classes receive and use feedback from their students, with a particular focus on how they use feedback received via the Maths Support Centre (MSC). Three separate interviews with each of thirteen lecturing staff over the course of one semester were conducted. We discuss the ways in which these lecturers receive and use ten modes of feedback from their students and examine where the feedback received by the MSC sits in the general context of this feedback. We conclude that MSC feedback is one of the most valuable to lecturers and state the reasons given for such a claim.

Research questions

- How do lecturers receive and use feedback from large first year mathematics classes?
- In what ways, if any, do lecturers find the feedback provided by the MSC on students' visits, useful? How do they use it?
- Where does MSC feedback sit placed in the general context of formative feedback?

Methodology

Thirteen lecturers from a research-intensive university in Ireland volunteered to participate in this study. Lecturing experience varied from two to seventeen years and two lecturers were teaching their particular module for the first time. The modules' sizes ranged from 66 to 550 students in subjects including Calculus, Statistics, Linear Algebra, Computer Science and Applied Mathematics. Twelve of the 13 classes we examined were mathematics/statistics modules taught to non-mathematicians, in particular the cohorts consisted of agriculture, computer science, engineering, business, science and applied mathematics students.

Thirty-seven semi-structured interviews consisting of three interviews with each lecturer were conducted in semester one of 2014/15 (interview one was not conducted with one lecturer as there was no MSC feedback to discuss at that time and the final interview for one module was conducted with both co-lecturers of that module simultaneously). This feedback, on the content of each students' visit, is generated by the attending MSC tutor and electronically uploaded (anonymously) in real time via the MSC software system where it can be viewed at any time by the lecturer (Cronin & Meehan, 2015). Interview 1 was an exploratory interview conducted in week 4 of the teaching term where lecturers were asked to review the MSC feedback collected from their module to see if they could identify the topic and mathematical difficulty being reported. The mathematical content of this feedback is discussed in the PhD of Nuala Curley and more information is available in Curley & Meehan (2016). Interview 2 was conducted in week 8 where lecturers were asked to comment on the various ways in which they receive feedback from their students and to comment on the usefulness of each. The third interview, conducted three weeks after teaching had finished, invited lecturers to

summarise their experiences with the MSC feedback mechanism throughout the term and discuss the value associated with each of the feedback forms they received from students throughout the module. Interviews were analysed using thematic analysis (Braun & Clarke, 2006).

Findings

Lecturers reported nine ways in which they receive feedback from large classes. These are: in-class questions, after-class questions, continuous assessment (e.g. quizzes), midterms and final exams, module tutors, online activity (Blackboard, Moodle, WebWork etc), the institution's Module Feedback system, staff-student fora and MSC feedback (Figure 1). Lecturers identify MSC feedback as one of the most valuable forms of feedback from a large mathematics class. In particular it is *specific*, *detailed* and lecturers reported that it aligns closest to in-class questions as it is *content based*, *formative* and in *real time*. It is *mathematically accurate* being the MSC tutor's interpretation of the student's difficulty. Lecturers stated that it is *reassuring* and *confirms what is been asked* at (and after) lectures. Many instructors stated that reviewing MSC feedback has had impact on their practice including; revising lecture content, writing midterms and revision classes, omitting material and delaying (or bringing forward) continuous assessment components.

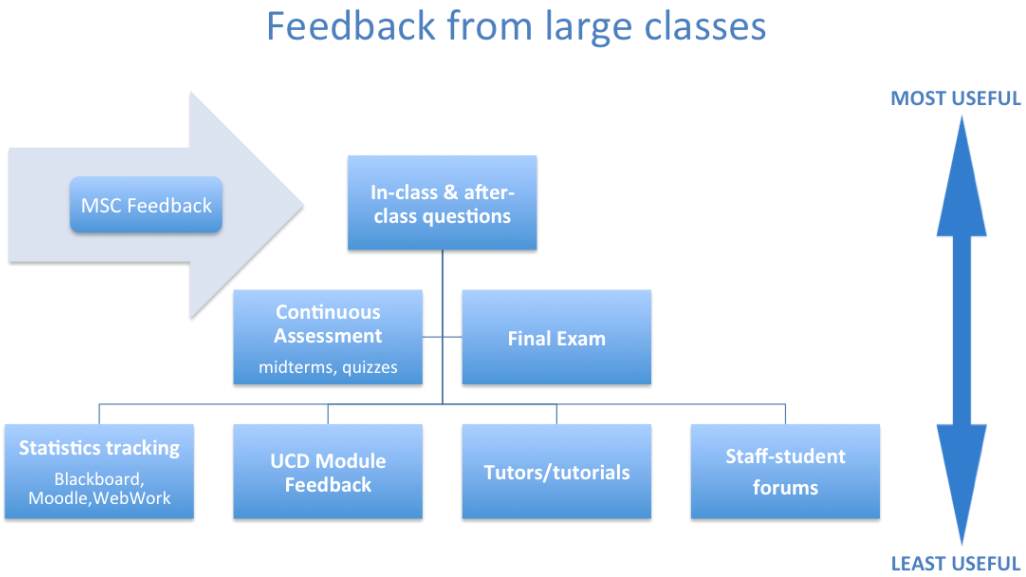

Figure 1: Usefulness of feedback forms to lecturers

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Measuring gains in knowledge within algebra instruction at community colleges: Assessing the efficacy of the Algebra Pre-Calculus Readiness (APCR) test

Irene M. Duranczyk[1], Vilma Mesa[2], Nidhi Kohli[3], April Strom[4] and Laura Watkins[5]

[1, 3]University of Minnesota, Minneapolis, Minnesota, USA; duran026@umn.edu, nkohli@umn.edu; [2]University of Michigan, Ann Arbor, Michigan, USA; vmesa@umich.edu; [4]Scottsdale Community College, Scottsdale, Arizona, USA; april.strom@scottsdalecc.edu; [5]Glendale Community College, Glendale, Arizona, USA; laura.watkins@gccaz.edu

Keywords: Mathematics, postsecondary instruction, algebra, classroom-based research.

The measurement of student learning for a federally funded project, EHR #831882, (Watkins, Duranczyk, Mesa, Ström, & Kohli, 2016) will be used to investigate the connection between instructional practices and student learning in algebra courses at six community colleges in three states of the United States. The poster focused on measurement issues faced in identifying students' learning gains in the pilot data.

Although we like to think that teaching causes learning, the truth is that such connection has not been established empirically (Hiebert & Grouws, 2006). As a first attempt to establish this connection, we investigate the extent to which there is a correlation between what occurs in the classroom and what students learn in a one semester course. Whereas there is some research documenting how individual and institutional characteristics (e.g., prior achievement, family support, financial aid, learning support and tutoring centers, and ratios of full- to part-time instructors) factor into failure rates and other performance measures (Bradburn, 2002; Feldman, 1993), there is little information about the fundamental work of teachers in the classroom, and the interaction that occur between instructors, students, and the mathematical content. The Quality of Mathematics Instruction (QMI), a video analysis tool used in P-12 settings (Learning Mathematics for Teaching Project, 2011), was adopted to measure faculty and student interaction at the community college. Research in K-12 and four-year colleges documents that the association between quality of instruction and student outcomes can be moderated by instructors' knowledge and attitudes towards innovative teaching practices, knowledge of algebra for teaching, and their beliefs about mathematics, its curriculum, and students' learning. The association is also moderated by students' attitudes, beliefs, and confidence about mathematics, their patterns of adaptive learning orientations, and the perceptions they have about their instructors' behaviors in the classroom and by the personal characteristics of instructors and students. The Algebra and Precalculus Concepts Readiness (APCR) test (Madison, Carlson, Oehrtman, & Tallman, 2015) was used to measure student learning.

The first phase, presented here, is the pilot testing of the APCR instrument to measure learning gains. The algebra instruction captured for analysis focused on three key algebra topics: linear, rational, and exponential equations. The APCR tests these topics and was administered in the second week of the semester before the topics were introduced and then two weeks before the end of the semester after the three focal topics had been taught.

The analysis based on the APCR data from 6 community college faculty and 161 students in beginning, intermediate and college level algebra courses lead to questions about the suitability of

the instrument for our work. Examining the descriptive statistics, item analysis, and reliability measures from classical test theory (Thorndike & Thorndike-Christ, 2010) revealed problematic point-biserial correlation and poor item discrimination. Ten item on the test fell well below acceptable levels (point-biserial correlations below 0.20).

Reliability measure gives information about the extent to which the scores produced by such measurement procedures are consistent and reproducible. The APCR had a Cronbach's alpha of 0.676 for the pretest data and 0.784 for the posttest data. The test-retest coefficient reliability coefficient (Crocker & Algina, 1986) for the APCR data was equal to 0.597, another low indicator.

Conversations were held with the authors of the APCR and the instrument is undergoing major revisions to be re-tested in April 2017. Working collaboratively with the APCR team on the newly revised version hopefully will generate an instrument that performs at a higher standard for greater reliability and discrimination. The poster presentation provided an excellent venue to review the testing results and confer with colleagues in possible next steps which were taken in March, 2017.

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Reflection on didactical design for action: The cases of the convergence of sequences and of complex numbers

Fatma Fattoum[1], Raja Tanazefti[1], Imène Ghedamsi[2]

[1] Virtual University of Tunis, ISEFC, Tunisia; fatmafattoum@yahoo.fr, tanazeftiraja@yahoo.fr

[2]University of Tunis, IPEIT, Tunisia; ighedamsi@yahoo.fr

Keywords: design for action, convergence of sequences, complex numbers.

The dilemma of the didactical design

This poster reports on a large study dealing with the issue related to the implementation of didactical designs in the institutional context. This presentation is an attempt to specify some methodological principles for planning didactical design that could be used as a resource by the institution. We build on the Theory of Didactic Situations (TDS) construct of didactical engineering (DE) (González-Martín, Bloch, Durand-Guerrier, & Maschietto, 2014) to first address the issue concerning the use of DE both as a fundamental research tool and as a tool ready to be used for action (Artigue, 2016). We then briefly introduce preliminary results of two studies which empirically investigate this issue in the case of the teaching and learning of convergence of sequences and complex numbers. DE is a research methodology usually associated to TDS, that "*consists of designing, regulating and making controlled observations of experimental situations where certain mathematical knowledge appears as the optimal way to address a mathematical problem."* (González-Martín et al., 2014, p.120). Three global steps shape the design of DEs as a research tool: 1) Epistemological and cognitive analyses which deal with the mathematical specificities of the targeted knowledge and their impact on the cognitive process of learning; the results are supposed to define the didactical variables – namely the parameters that influence students' work, which should be taken into account to design the projected situations; 2) The phase related to the *a priori analysis* leads to the identification of the values of the didactical variables that are used to build the experimental situations; these situations are thus analyzed in terms of milieu – "*namely the set of material objects, knowledge available, and interactions with others*" (p. 119) including the interactions with the teacher; 3) The results of the *a posteriori analysis* (by comparing with the a priori analysis) permit to assess the relevance of the experimental situations and the validity of the theoretical model. The aim of the final analyses is to give feedback on the theoretical frame and its efficiency; the research does not actually consider the conditions for the implementation of the DE in the institutional context. In the last decades, research dealing with didactical design has increased its focus on the issue concerning the "*application role when didactical design is seen as a way for organizing the relationships between research and practice or, in other words, for developing educational actions inspired by research and incorporating its results*." (Artigue, 2008, p. 9). Yet, this issue is still pending and the didactical actions are not defined neither theorized. In the following section, we describe the methodological tool that we have elaborated to plan efficient DEs for action.

The methodological tool for empirical investigation: the cases of the convergence of sequences and complex numbers

The fundamental principle of the two DEs we have constructed is based on two essential ideas: 1) the necessity to negotiate and to plan such DEs with the actors of the educational system (teachers,

trainers, policy makers, etc.); 2) the experimentation should provide some flexibility to the teacher in order to make adjustments depending on the class context. We rely on the research version of the design methodology (DE) and go forward to provide three empirical phases used as a method to plan and to experiment DE for action (Figure 1).

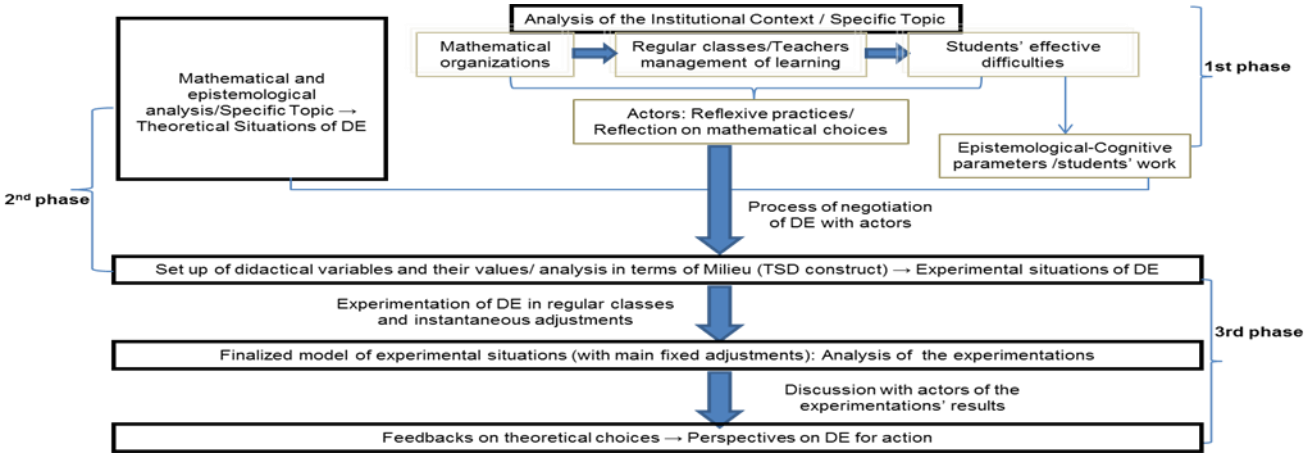

Figure 1: A methodological tool - DE for action

Its use with more than one mathematical topic gives more legitimacy to our methodological tool. In spite of their mathematical differences, these two studies revealed the cornerstone role of the institutional mathematical organization in the teaching and learning processes and how to manage it for more efficiency, for instance: 1) regarding the convergence of sequences: improve the neglected role of approximation and numbers; 2) regarding complex numbers: rehabilitate the operational level of these numbers by emphasizing the role of their several representations. For both cases, the results of the implementation of the first step corroborate the necessity of such adjustments.

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Investigating students' difficulties with differential equations in physics.

Diarmaid Hyland[1,2], Paul van Kampen[1] and Brien Nolan[2]

Diarmaid.hyland3@mail.dcu.ie

[1]CASTeL and School of Physical Sciences, Dublin City University, Ireland;

[2]CASTeL and School of Mathematical Sciences, Dublin City University, Ireland.

We have investigated the difficulties encountered by undergraduate physics students when studying differential equations, and how these are best addressed. We developed a survey to identify these difficulties that was administered to a pilot cohort of students. The results were used to develop an instructional intervention, informed by APOS Theory, that seeks to address the difficulties uncovered by the survey. The intervention comprised fourteen one-hour tutorials. The tutorials were trialled for the next iteration of the module in question and the survey was given to the students who had completed the module. Applying a design based research approach, the results from these surveys were used to improve the intervention which is being evaluated using a combination of pre- and post-testing and interviews with students in the coming academic year. The interviews will be analyzed using the APOS framework, which acts as the overall conceptual framework for this research

Keywords: Mathematics education, physics education, APOS theory.

Research questions

1. What is the precise nature of the difficulties encountered by physics students in using (identifying the need for/setting up/solving) differential equations?
2. How may these be addressed?

These questions lead to the following specific aims for the project which are: (1) identify the areas students struggle with and excel in during their study of differential equations; (2) develop a set of tutorials based on the data from the survey; and (3) evaluate and improve the tutorials that were developed.

Conceptual framework

In 1984, Dubinsky began developing a theory of how mathematical concepts may be learned. This eventually became known as APOS Theory and is a constantly evolving model developed and refined through application in research and instructional design. As described by Arnon et al.

> APOS Theory focuses on models of what might be going on in the mind of an individual when he or she is trying to learn a mathematical concept and uses these models to design instructional materials and/or evaluate student successes and failures in dealing with mathematical problem situations. (2014, p.1)

As explained by Dubinksy (1991), APOS Theory describes the mental structures and mechanisms an individual constructs and applies when trying to understand a problem in mathematics. This project primarily uses APOS Theory as an analytical, evaluative tool. The language of APOS Theory is used to describe the level of understanding displayed by students during their interviews.

Methodology

Achieving the aims outlined in the opening sections will require the combination of both qualitative and quantitative data. The design diagram below shows how the project is structured.

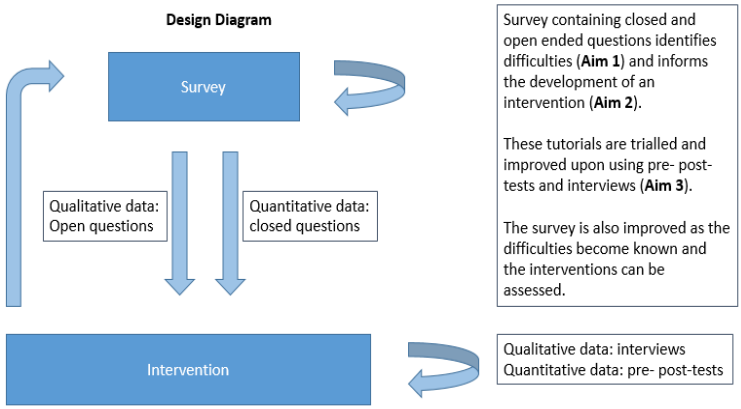

Figure 1: Design diagram

The first aim, identifying difficulties, gave rise to both quantitative and qualitative data, gathered using a survey. The Diagnostic Survey was divided into four separate sections assessing different aspects of the students' learning in differential equations: prior mathematical learning; conceptual issues in the study of differential equations; transfer issues; and modelling. To evaluate the tutorials a series of pre- and post-tests and interviews are being used to assess the effectiveness of the intervention. These results feed into the evaluation and improvement of the tutorials.

Results

The results obtained from the Diagnostic Survey show that of the four sections contained in the survey, students struggle most with conceptual understanding, indicating (in terms of APOS Theory) shortcomings in their ability to encapsulate Actions and Processes as Objects. Comparing to the results of the students who completed the tutorials, we saw a dramatic increase in their conceptual understanding, in addition to improvements in the three other sections of the survey. The pre- and post-test data were also used to amend the tutorials for the second research cycle.

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A practical study of a design of university mathematics courses for humanities and social sciences students

Mitsuru Kawazoe[1] and Masahiko Okamoto[2]

[1]Osaka Prefecture University, Japan; kawazoe@las.osakafu-u.ac.jp

[2]Osaka Prefecture University, Japan; okamoto@hs.osakafu-u.ac.jp

Keywords: Mathematical literacy, decontextualization, structuralization.

Problems in university mathematics education: in the Japanese context

It is becoming more important for all students regardless of educational level to acquire the ability to use mathematics in a variety of contexts, especially in real world situations (Niss & Jablonka, 2014). From this viewpoint, mathematics education for non-mathematics students at university is becoming an important issue; however, it is still under-investigated (Artigue, 2016). In Japan, it is an emergent issue that we have no successful mathematics curriculum designed especially for humanities and social sciences students. Many of those students have math anxiety and difficulty in doing and learning mathematics, and do not understand how mathematics is used in the real world. Hence mathematics education for those students is a challenging issue.

Design principles of mathematics courses for humanities and social sciences students

Continuing empirical studies from our previous research (Kawazoe et al., 2013), we have developed the following design principles of mathematics courses for humanities and social sciences students: (1) Design lessons according to mathematical modelling processes; (2) Choose topics and contexts by considering which mathematical knowledge students will encounter in real life and in which situations they will encounter it; (3) Present problems in different contexts associated to the same mathematical knowledge; (4) Connect different areas of mathematical knowledge by using different *mathematizations* of the same problem or mutually related contexts; (5) Explain mathematical concepts and operations in both mathematical language and everyday language; (6) Engage students in group activities rather than individual activities; (7) Design worksheets based on hypothetical cognitive processes of students' understanding and use them as tools for formative assessment.

The above principles originate in the four perspectives of learning environments developed in learning science (Bransford, 2000, Chapter 6). The first four principles (1)-(4) also originate in discussions on mathematical literacy (cf. Sfard, 2014) and mathematical modelling (cf. Kaiser, 2014). Especially, (3)-(4) are aimed at making students' knowledge *decontextualized* and *structuralized* respectively, because it is often claimed that teaching mathematics in real world contexts makes students' knowledge restricted to the learning contexts (cf. Sfard, 2014). In a previous study (Kawazoe et al., 2013), we showed that mathematics courses designed according to the above principles are successful in reducing students' math anxiety and motivating them to learn math. However, their effectiveness for decontextualization and structuralization was not examined. To examine this effectiveness is the research objective of the present study.

Evaluation of decontextualization and structuralization

To evaluate the effectiveness of our design principles for decontextualization and structuralization, we analyzed the performance of examinations and the free descriptions in the self-report questionnaires conducted at the end of each semester in *Basic Math I* and *Basic Math II*, which are successive one-semester mathematics courses for humanities and social sciences students designed according to the above principles. The data were collected in the academic year 2012, when 300 and 244 students took the courses respectively, with students divided into four classes.

The examinations mainly consisted of problems in real world contexts, but the contexts were different from the learning contexts. The mean scores of the exams of Basic Math I & II were 85.9 and 75.4 out of 100, respectively. The results of the exams indicated that students were able to use mathematics in contexts different from those they had studied before, suggesting that they acquired abilities to use mathematics in real world situations.

In the qualitative analysis of the self-report questionnaire, we found evidence of structuralization of mathematical knowledge only in a small number of descriptions. Here we show an example:

> Student: In this course, it seemed that mathematical formulae, which were just isolated pieces of knowledge, had been changing into meaningful ones. I felt that the knowledge base of mathematics that I had became harder and stronger; in that respect, it was a meaningful time.

In sum, the design principles can be considered to be effective for decontextualization, however, the structuralization is still a challenge. More study is needed for structuralization.

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Connections between approximating irrationals by a sequence of rationals and introducing the notion of convergence

Hichem Khechine[1] and Alejandro González-Martín[2]

[1]Université virtuelle, Isefc, Tunis, Tunisia; hichem.khechine@gmail.com

[2]Université de Montréal, Département Didactique, Canada; a.gonzalez-martin@umontreal.ca

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Keywords: Approximation, decimals, convergence, irrationals, praxeology.

Introduction and research problem

Progressing from the study of rational numbers to irrational and real numbers can prove challenging for students. The way irrationals and reals are introduced in secondary school textbooks does not seem to promote the development of ideas that allow students to adequately grasp convergence or density in their tertiary studies (González-Martín, Giraldo, & Souto, 2013). Difficulties with irrational numbers also have been reported in university students (Kidron, 2016). However, we believe that certain activities, such as studying the approximation of irrational numbers by sequences of rationals, could help students grasp, informally, the ideas behind the formal $\varepsilon - N$ definition of convergence at university. We have not found works in the literature that address the connections between the teaching of these two notions. For this reason, we seek to investigate how they are presented in textbooks and study the connections that are made (or the lack thereof). Our research is guided by the following questions: 1) how do pre-university textbooks organise and present the notions of limit and of approximation of a real number by a sequence of rationals?; and 2) how do textbook tasks help students develop connections between the two notions?

Theoretical framework

Our research uses tools from Chevallard's (1999) anthropological theory of the didactic (ATD). ATD acknowledges that every human activity generates a *praxeology* or *praxeological organisation* identified by the quadruplet $[T/\tau/\theta/\Theta]$, where T is a type of task, τ is a technique used to complete this task, θ is a discourse (technology) that justifies and explains the technique, and Θ is a theory that includes and justifies the given discourse. The couple $[T/\tau]$ is the practical block (or *praxis*) and $[\theta/\Theta]$ is the theoretical block (or *logos)*. We focus on the *institutional relationship* with the notions of real number (and activities of approximation) and limit, as reflected by the textbooks.

Methodology

Our study uses the tools provided by ATD to analyse the textbooks and recommendations of the Tunisian Ministry of Education's official programme. Teachers in Tunisia use a different textbook for each school level; all are published by the Ministry. We chose two textbooks for our analysis: the first is used in the ninth level of the basic cycle (14-15 year-old students) and the second is used in the third level of secondary school (17-18 year-old students), which is the penultimate year before university. In examining the first textbook, we analysed the content related to real numbers, paying special attention to tasks concerning the approximation of an irrational by a sequence of decimals. We focused on whether such tasks implicitly use a theoretical block based on the notion of limit. For

the second textbook, we looked at two chapters, "Sequences of Real Numbers" and "Limits of Sequences of Real Numbers", to determine whether those chapters develop *praxeologies* that use (implicitly or explicitly) any of the elements present in the first textbook.

Main results and discussion

We identified three types of tasks in the first textbook (geometrical, algebraic and numerical). These are: T_{pr} (prove that a given figure is a square and calculate its area), T_{cl} (calculate the square of rational numbers using a calculator) and T_{bn} (determine lower and upper bounds of a square root). All these tasks seek to introduce routine techniques, such as the algorithm for bounding $\sqrt{2}$ between two decimals. The institutional relationship with this notion is characterised by an institutional void with regard to the theoretical block, which includes two technologies: θ_{pr} (the diagonals of a square are perpendicular and isometric, and they intersect at the same midpoint) and θ_{cl} ($0 < a < b < c \Rightarrow \sqrt{a} < \sqrt{b} < \sqrt{c}$), both of which are derived from Euclidian Geometry and elementary algebra. This institutional void may have an impact later on in the third level of secondary school, during the introduction of limits and convergence. Concerning the second textbook, it proposes three types of tasks (calculus, geometrical and numerical) focusing on bounding square roots and π. With respect to calculus, the textbook presents an activity related to the Fixed Point theorem and, essentially, the application of the algorithm of Newton's method. This task, "determine upper and lower bounds for $\sqrt{37}$," proposes a technique that breaks the exercise down into three sub-tasks: T_{res} (solving of an equation), T_{rep} (representation of a sequence), and T_{bnd} (bounding of $\sqrt{37}$). The technology is implicit in this activity because the students have not yet studied the Fixed Point theorem. The two geometrical activities concern the approximation of π using Archimedes' polygonal approach. Finally, the numerical activity asks students to determine the rank n of a recurrent sequence (x_n) that verifies $\left|\sqrt{a} - x_n\right| \leq 10^{-p}$. Our results indicate that all tasks presented in the two textbooks are routine. Both textbooks insist on the approximation of square roots and π. This could affect students' learning of real numbers when the latter are formally introduced in university. We also found that both textbooks contain praxeological organisations that insist on the practical block and that implicitly present the theoretical block. Furthermore, the activities are presented using algorithms, without discussing the useful application of the knowledge gained.

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Unconventional didactical choices for the benefit of course instruction: A glimpse into a lecturer's decision-making

Igor' Kontorovich

The University of Auckland, New Zealand; i.kontorovich@auckland.ac.nz

Keywords: Decision making, root concept, university lecturers.

The Mathematics community often does not agree on definitions of concepts and on the meanings of their symbols. For example, in the topic of complex numbers some Israeli textbooks propose that $\sqrt{a+bi}$ represents a single value, while others maintain that two values are indicated (in the field of real numbers there seems to be a consensus around $\sqrt{a^2} = |a|$). When introducing such a debatable concept to students, teachers usually choose a particular approach among mathematically acceptable alternatives, and stick with it. In this poster, I am focusing on the question "what does a university lecturer take into account when making such a choice?".

I embark on the question with Schoenfeld's (2011) theory of decision making. According to this theory, in-the-moment decisions that teachers make in classroom situations can be modelled with three explanatory constructs: *resources* (consisting of teacher's knowledge inventory, social and material resources), *goals* that are set to be achieved (either consciously or unconsciously) and *orientations* (including beliefs, values and preferences). Schoenfeld suggests that the model can be further used for structuring developmental trajectories and promoting teacher expertise.

For an experienced university lecturer, the choices among mathematical alternatives are hardly "in-the-moment", in the sense of spontaneity. Instead, I argue that these choices are systematically reproduced by some decision-making mechanisms that a lecturer brings to bear in a variety of teaching situations. Indeed, after the choice has been shared with the students (i.e. in one 'moment' during a course), for the benefit of course coherence, a lecturer is expected to maintain the choice through consequential inferences. Accordingly, making sense of these choices might illuminate some aspects of one's epistemological perspectives on *mathematics didactique*. I showcase this idea with the case of Elza and direct interested readers to Kontorovich (2016), where additional illustrations were analyzed with a different theoretical lens.

Elza is a highly-reputed lecturer in a technological university in Israel, who holds a master's degree in mathematics and a PhD in mathematics education. Elza has more than thirty years of teaching experience, and she specializes in teaching linear algebra. These facts suggest that her knowledge inventory is rich and solid. Elza's specialization determines a significant component of her instructional setting; typically, students in Elza's university learn a single course in linear algebra, which is a pre-requisite for many other courses. In this way, didactical choices that Elza makes shape students' knowledge development in her course and in the courses taught by her colleagues.

An interview with Elza revealed a variety of interesting didactical choices that she makes, some of which are barely conventional. To name a few, it turned out that in the field of real and complex numbers, she defines $\sqrt[n]{a}$ to be multi-valued - all solutions to the equation $x^n = a$. Elza explained that this is her way to connect roots of numbers, roots of equations and roots of polynomials, all of which are discussed in her course. When a singular positive root is required, she uses the symbol '$+\sqrt{\ }$'. For example, when computing a module, Elza writes $|a+bi| = +\sqrt{a^2+b^2}$. With regard to

an apparent conflict between her multi-valued approach to real roots and a single-valued approach which is used in the calculus courses, she indicated that, "[I]n calculus they have functions and [here] we are dealing with values in linear algebra. Every branch of mathematics works with its own premises". It also turned out that Elza avoids using complex numbers, the Cartesian form of which contains roots (e.g., $\sqrt{3}+i$) because "they look like a single thing but are actually two numbers". Lastly, Elza explained that she has just a few hours to cover the foundations of complex numbers, and then she does not go into "nuanced details". She would have expanded the scope of the course if it was intended for pre-service teachers only.

Several observations can be made based on Elza's choices. On the one hand, a multi-valued approach to the root concept and radical symbol entails consistency between several topics in her linear algebra course. For instance, real roots of a number are preserved and possibly extended with non-real numbers in the field of complex numbers. Then, for Elza's students a transition from the field of reals to complex numbers is mostly a matter of concept extension rather than redefinition. On the other hand, Elza's choices create 'monsters' (cf. Lakatos, 1976), such as inconsistency with mathematics studied in other courses, ambiguity and unconventional symbols. When monsters are necessary for linear-algebra purposes, Elza conducts "monster-adjustment" to familiar categories (e.g., '$+\sqrt{\ }$'), otherwise, the "monster-barring" method is used to expel the problematics from the course scope (e.g., $\sqrt{9}+i$).

In terms of Schoenfeld (2011), Elza's didactical choices can be explained with a complicated intertwining of resources, goals and orientations. A limited resource of time promotes setting pragmatic goals, the scope of which does not go beyond a particular course. In some cases, the choices lead to unconventional consequences and artificial limitations of the course content. In other cases, the choices seem to benefit the course instruction through tightening the connections between the topics. Evidently, the choices align with Elza's orientations about the nature of mathematics (e.g., different branches can have incompatible approaches to the same concept) and students' academic needs (e.g., roots should be taught differently to pre-service teachers). In this way, the choices seem to reflect an 'equilibrium' that Elza achieved in her didactical decision-making.

Are Elza's choices really beneficial for her students? Are students aware of non-chosen alternatives? How unique is Elza's decision-making? I invite the community of university mathematics education to join me in pursuing these interesting questions through further research.

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Polynomial approximation: From history to teaching

Rahim Kouki[1] and Yassine Hachaïchi[2]

[1]University of Tunis El Manar, LAMSIN-ENIT; Tunisia; rahim.kouki@gmail.com

[2]University of Tunis El Manar, LAMSIN ENIT; Tunisia; hachaichi.ens@gmail.com

Keywords: Taylor approximation, dialectic tool-object, syntax, semantics.

Introduction and research problem

Our motivation for engaging in this research project relies on recurrent students' difficulties in the appropriation of the concept of Taylor approximation. Despite the richness of this concept as a tool in many fields of application (e.g calculation of limits, local study of a function and the study of physical phenomena), for what we know, there is little research on this topic in mathematics education. Beyond the lack of comprehension of the notion of convergence for Taylor series (Martin, 2013), we hypothesize that the articulation between syntax and semantics is not clear for students, creating a fuzzy area that prevents them from being able to apply Taylor approximation as a tool in applied mathematics. In this paper, we summarize the main results of our epistemological investigation and the first element of a didactic study of teaching material on this topic, in order to identify paths for improving its teaching and learning.

Theoretical framework

Throughout our epistemological and didactic investigations, we refer to the dialectics between semantics (graphic interpretation, dynamic interpretation and numerical approximation) and syntax (the different formulations of the Taylor approximation), including their articulation with the *tool-object dialectic* (Douady, 1991) and the *registers of semiotic representations* (Duval, 2006). (this approach is developed and used in Kouki, to appear).

Epistemological investigation

We outline the evolution of the mathematical object known as "Taylor approximation" through a historical and epistemological investigation. Our study emphasizes the co-existence of different types of techniques mainly with geometric, algebraic and dynamic origins that contributed to the emergence and development of this object (Kouki, Belhaj Amor, & Hachaïchi, 2016).

Our research identifies key moments in the development of infinitesimal calculus and local approximations which led to the calculus as it is taught nowadays. In particular, the works of Roberval, Euler, Fermat, Leibniz, Newton, until Taylor and Weierstrass are significant in the development of Taylor approximations.

Didactic investigation

Based on our epistemological investigation, we have conducted a didactic study of the Tunisian curriculum, textbooks and course handouts on Taylor approximation. The Tunisian curriculum, three handwritten courses and four handbooks were analyzed. Through this study, we confront the various dimensions of this concept revealed by the epistemological investigation with those that are present in teaching materials.

This study supports the hypothesis that there are gaps in the teaching of Taylor approximation. In particular, some of them are due to insufficiencies of graphical representations and numerical approximation, corresponding to a deficit of semantic work, the emphasis being put on the syntax.

Conclusion

The epistemological and didactical investigations summarized in this paper are part of a larger project aiming at developing propositions for enhancing the teaching and learning of Taylor approximation. This enhancement can help to foster an appropriation of the concept allowing to use it in the various applications undergraduate students could meet in further studies, in particular in applied mathematics.

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The flipped learning model in teaching abstract algebra

Tore Alexander Forbregd and Magdalini Lada

Norwegian University of Science and Technology, Norway; magdalini.lada@ntnu.no

Keywords: College mathematics, algebra, flipped learning.

Introduction

Flipped learning is a relatively new model of instruction currently growing both in popularity and success. In a flipped classroom the elements of typical lecture and homework are reversed. Students are introduced to new material at home, mostly through videos prepared by the teacher, while the classroom time is reserved for solving problems, group work, discussions and other activities that help students control, deepen and extend their understanding and knowledge.

The board of the Flipped Learning Network has defined Flipped Learning as

> … a pedagogical approach in which direct instruction moves from the group learning space to the individual learning space, and the resulting group space is transformed into a dynamic, interactive learning environment where the educator guides students as they apply concepts and engage creatively in the subject matter (FLN, 2014)

The number of college and university instructors who practice Flipped Learning has increased over the past two years and has expanded in all subject areas. Research in Flipped Learning has shown positive impact both on students' achievement and engagement (Overmeyer, 2015).

Methodology

During our study we collected data from the students, the lecturer and the class. We conducted a semi-structured interview with the lecturer in the middle of the semester at the university campus and had several informal discussions through Skype and emails at the end and after the end of the semester. The students were given evaluation forms to fill out both in the middle and in the end of the semester, the latter being enriched with open questions where the students were asked to describe their experience with the flipped course. In addition, we visited the class and observed how in-class time was spent and how the students interacted with each other and with the teacher.

The researchers in this study have direct experience with the course material and the way it was previously taught, as they attended it as students. The second author has also taught the course.

The structure of the course

The lecturer-informant recorded videos, using a mobile phone, while writing down on a blanc A4 paper what he would otherwise write on the blackboard in a traditional lecture. The videos were then made available to the students through the course's webpage. The problem sheets, which are, except from small changes, the same sheets that were used in previous years, were also put out on the web. The students were instructed to watch the videos at home and work with the problems when they met in class.

Ten to twelve students on average were normally attending the class, which is an expected number for this course. In a relaxed and informal environment, students formed groups freely and worked with the problems, while the lecturer was walking around answering questions and guiding them

through the problems. The groups were let to progress through the problem sheets at their own tempo and already in the middle of the semester different groups were working with different parts of the curriculum. Thirteen students took the exam at the end of the semester.

Findings

Even on this primitive form the Flipped Learning Model turned out to be beneficial to the learning outcome of the students. The students spent more time working on computational problems, concretizing difficult abstract concepts such as for example the radical of rings and modules, projective and injective modules, exact sequences, resolutions and dimensions, clarifying in this way the connection between theory and concrete problems on special classes of algebras.

Some more advantages pointed out by the lecturer include, among others:

- Differentiated guidance according to the level of each student
- Students forced to work more actively with the course, during the whole semester
- Better balance between learning the theory and working on problem solving

What the students found more beneficial was, among others,

- Watching the videos at their pace. Pause it, think, watch it again.
- Thorough explanations from the lecturer
- Working the problems in groups, learning from each other.

There was a slight improvement on the average of the grades, compare to the previous years the lecturer-informant had taught the course, though the class is too small to make any safe conclusions.

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Mathematics and medicine: The socioepistemological roots of *variation*

Angélica Moreno-Durazo[1] and Ricardo Cantoral[2]

[1]Center for Research and Advanced Studies of the National Polytechnic Institute (Cinvestav-IPN), Mexico; gamoreno@cinvestav.mx

[2]Cinvestav-IPN, Mexico; rcantor@cinvestav.mx

Keywords: Variation, prediction, shortened variation, medicine.

Introduction

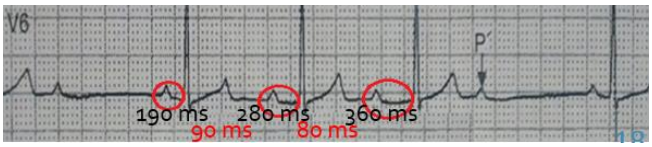
Variational Thinking and Language (PyLV) is a line of investigation developed at Cinvestav, whose objective is the study of forms in which individuals deal with change mediated by their culture for predictive purposes. It is a form of methodology for the elaboration of teaching proposals based on the investigation. PyLV is founded in the Socioepistemological Theory of Educational Mathematics (Cantoral, 2013), where mathematical knowledge is recognized as part of human wisdom, that is to say, of the articulation of knowledge of diverse nature (scientific, technical and popular knowledge); this requires a *decentration of the mathematical object*, which leads to the analysis of the practices that accompany the construction of the object; we call this process the social construction of mathematical knowledge.

This way, we assume change and variation as a substantial part of the scientific and technical work areas and the daily experiences of individuals and social groups in non-school situations. In these areas, the prediction is socially constructed by the development of normed practices that we call social practices (Cantoral et al., 2006; Tuyub & Cantoral, 2012). Derived from the projects developed, strategies and fundamental variational arguments were characterized for prediction. Therefore, a series of teaching sequences was developed for the improvement of education both in and outside school; moreover, the results of PyLV have had an impact in the educational reforms in Mexico. The prediction of the phenomena that have been investigated under PyLV involve mathematical models whose resolution needs the convergence of the Taylor series in a given domain (deterministic nature) as the future state, the value of f (x+h), depends only on the starting values of h, f(x), f'(x), etc. through the expression: $f(x+h) = f(x) + \frac{f'(x)h}{1!} + \frac{f''(x)h^2}{2!} + \cdots$

Now, based on this research, the goal of our project is to characterize the nature of the predictions made by mathematizing phenomena not governed by the analytical study of a formal mathematical expression, as described above. We have taken interest in the analysis of the ways in which change and variation are used in the estimates made by doctors in their professional practice. In this scenario, the dynamics of the system are not deterministic but rather chaotic, since the conclusion of the trajectories that patients follow after a certain treatment may be divergent - sensitive to initial conditions. The development of these objectives requires an in-depth study of the dynamics followed in cardiac functioning, through the analysis of original works and medical specialized books (Castellano, Et al., 2004; Harvey, 1994), the analysis of the medical practices, and its articulation in previous investigations in the PyLV.

First findings

We found that medical practice requires more than assuming that something changes, it is necessary to recognize how fast the change is in order to diagnose patients. Our first finding was that, in the case of the interpretation of electrocardiograms, the recognition of how, how much and why the heart rhythm behavior changes, is done by the practices of comparison and seriation; reported as variational strategies in the study of optimization problems and calculation of the derivative of a function (Caballero, 2012). Hence, we claim that the basic practices of the study of change and variation are shared by medical professionals and students or teachers in Calculus courses. In addition, we identified the use of different orders of variation in the location of different types of blockages in electrical conduction in the heart. For example, in the figure, we can see that type II Mobitz I block requires the identification that the time invested in the PR segment has a progressive extension and a progressive decrease in that increase (Lobelo et al., 2001, p. 2126); that is, it is necessary to analyze the change in the change in the PR segment between beats.



Final thoughts

The next phase in the research corresponds to an ethnographic study where the theoretical construction on the study of variation is contrasted with the practices developed by medical professionals. It is important to mention that the future intention of our research is the impact on the educational system, particularly in higher education. Concerning this, the characterization of basic forms of reasoning is fundamental for theoretical proposals of sociocultural nature, especially for the socioepistemology that bases the redesign of the school mathematical discourse on the assumption that knowledge is constructed based on practices.

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Developing conceptual knowledge by using ICT on mathematics lessons

Mária Slavíčková

Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Bratislava, Slovakia; slavickova@fmph.uniba.sk

Keywords: course evaluation, diagnostic teaching, active learning, Bloom taxonomy

Theoretical background and research question

According to our previous study (Slavíčková, 2013) we state the following research questions: "*How to design a course of calculus to help students obtain higher level of understanding of the topic? How to implement an ICT into this process? Will be using of and ICT helpful?*" To find the answers we used mathematical and didactical software in the lectures for presentation of mathematical terms, definitions, properties etc. We also created an e-learning course in the Learning Management System Moodle (LMS). We used this LMS as a primary communication channel between the teacher and the students, we published there all the materials from the lectures, as well as some extra tasks for the students (solving of these tasks was not obligatory).

Activities implemented into the educational process were based mostly on constructivism. Our goal was prepare such activities which help students obtain cognitive dimension "Apply" according to the RBT (Revised Bloom Taxonomy) according to Anderson & Krathwohl (2001) and Web-link Bloom taxonomy (2015). Most of the activities were supported by ICT and we used mostly the works of Jonassen (2000) and Kadijevic (2006).

The important notes concerning the learning/teaching process according to RBT:

- Before you can understand a concept, you must remember it.
- To apply a concept you must first understand it.
- In order to evaluate a process, you must have analyzed it.
- To create an accurate conclusion, you must have completed a thorough evaluation.

Intervention

We used Derive, Graphic Calculus, MS Excel and GeoGebra software in the lessons to demonstrate mathematical properties of functions, sequences, limits of a sequence, a function etc. We prepared small environments for observing, modeling and exploring.

Organization of the teaching - learning process

Lectures: lectures were lead by us; we used the computer and data projector to project mathematics formulas, theorems, some parts of the proofs and the most important – to demonstrate the relationships between the theory and the praxis.

Seminars: The main work was on the students. We use data projector and computer to project the tasks, questions, some interesting schemas etc. Students can use computers to solve them or to help themselves to get deeper view into the problem. The materials from the activities were uploaded to the LMS Moodle, so students can use them at home preparation.

The Table 1 shows the focus on the lessons and test – on the lessons we focused every field marked with "x". On the test were only task highlighted by red.

	Remember	Understand	Apply	Analyze	Evaluate	Create
Factual	x	x	x	x		
Conceptual	x	x	x	x	x	
Procedural	x	x	x	x	x	
Metacognitive						

Table 1: Cognitive and the knowledge dimension according to RBT

Results

The implementation ICT and using different type of software we obtained the good results. The most important results from the observations in our group are:

- students take an active part of the teaching/learning process,
- students wanted to discuss the interesting mathematical topics on the lessons,
- students started modelling of the situation without a teacher's command,
- students interest in the topics was higher than in the other group (taught by our colleagues – it could be also by the personality of the teacher, but previous years showed us, that in our and colleagues group were similar result),
- some students started studying more outside the classroom so they can follow the topics without the troubles (we avoided the troubles which still resist in colleague group, like if we change the task a little, students do not know what to do)

Discussion

The preliminary results shows that there is potential in using ICT on calculus lessons to obtain deeper knowledge and better understanding of the topic. Using different kind of software shows students different approaches to the different issues. The next step in our research will be to enlarge the sample and to prepare more technology oriented materials for the research.

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Heuristic strategies in mathematics teacher education

Peter Stender

University of Hamburg, Germany; Peter.Stender@uni-hamburg.de

Keywords: Heuristic strategies, teacher education, tutorial in mathematics.

Here a tutorial is described that focuses on mathematical methods that implicitly appear in the mathematics shown in a Linear Algebra lecture, and makes these methods explicit for the students. Interviews with students at the end of their first semester show some effects.

Theoretical framework

Mathematics, as all sciences, comprises content (e.g. definitions, theorems, finished proofs, …), but also mathematical methods.

Pólya (1945) underlined the importance of the mathematical methods in the subtitle of "How to solve it -A new aspect of mathematical methods". We use a broad approach of heuristic strategies as a central part of the mathematical methods shown in the following list:

- *Organise your material / understand the problem*: change the representation of the situation if useful, try out systematically, (Pólya, 1945) use simulations with or without computers, discretize situations,
- *Use the working memory effectively*: combine complex items to supersigns, which represent the concept of 'chunks' (Miller, 1956), use symmetry, break down your problem into sub-problems,
- *Think big*: do not think inside dispensable borders, generalise the situation (Pólya, 1945),
- *Use what you know*: use analogies from other problems, trace back new problems to familiar ones, combine partial solutions to get a global solution, use algorithms where possible (Pólya, 1945),
- *Functional aspects*: analyse special cases or borderline cases (Pólya, 1945), in order to optimise you have to vary the input quantity,
- *Organise the work*: work backwards and forwards, keep your approach – change your approach – both at the right moment (Pólya, 1945).

Other mathematical methods are e.g. "proof strategies" which make extensive use of formal language such as mathematical induction, proof by contradiction, proof by exhaustion, the invariance principle or others. Another kind of methods are the use of mathematical language e.g. in depth reading mathematical texts or writing down mathematical proofs correctly.

Tutorial example

In the tutorial we made mathematical methods implicitly used in the lecture explicit. We also used these methods giving additional explanations and reflected this use afterwards in the tutorial. The concept of quotient spaces in linear Algebra was established on rings. One example given in the lecture was $\mathbb{Z}/m\mathbb{Z}$. In the tutorial we reduced this example to the case $\mathbb{Z}/4\mathbb{Z}$ and gave additional representations of this concept, some of them shown in figure 1.

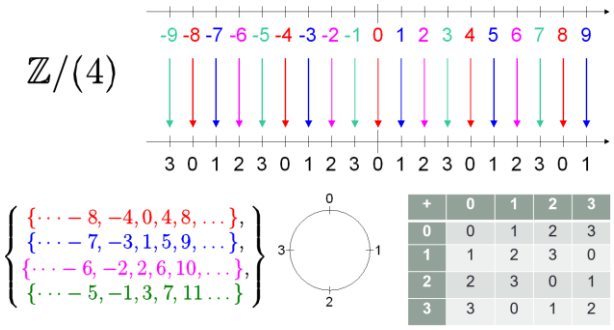

Figure 1: $\mathbb{Z}/4\mathbb{Z}$ in several representations.

Here the four residue classes are shown in several representations, together with the way of calculating on that structure. The similarity of the calculation on a common clock was stressed as a possible link to school mathematics and further explanations were given. Calculating on the equivalence class that are supersigns from the heuristic point of view was difficult for the students and the similarity to calculating with fractions that are equivalence classes and supersigns as well was discussed. Furthermore, moving between representations was stressed as a method to deepen the understanding of the content or to find a way to solve a problem.

Students' feedback

At the end of the first semester we interviewed five students who participated in the tutorial over the whole semester. The interviews were transcribed and analyzed using quantum content analysis.

All students remembered well the connection to school mathematics based on mathematical methods that appear in university and school as well. The students also emphasized that visualized explanations with less formal language were very helpful. These explanations using "a change of representation" were remembered well.

The input related to the heuristic strategies was remembered by the students. The concept of "heuristic strategy" was remembered and the students could describe examples given in the tutorial, connecting the examples to the appropriate heuristic strategy.

Conclusions and looking ahead

Supporting students by stressing mathematical methods, and particularly by stressing heuristic strategies, seems to have a positive impact, but is not realized easily. For this we will continue with this tutorial in the third and fourth semester bringing in heuristic strategies wherever possible.

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Supporting university freshmen – an intervention to increase strategic knowledge

Thomas Stenzel

University of Duisburg-Essen, Germany; thomas.stenzel@uni-due.de

Keywords: Secondary-tertiary transition, problem solving, metacognition.

Rationale

> At school, we actually only had exercises where we could calculate with numbers and formulas. Now, it was really abstract what we had to do. [...] We didn't really know: Where should we begin? How does this work? (Aylin, mathematics student, 1st semester)

The quote above shows that some strategic knowledge is needed for problem solving at university level. Additionally, a necessary condition to deal with those problems is to be familiar with the related content of the lecture. The literature provides a lot of successful strategies both for problem solving (Pólya, 1945) and learning (Ramsden, 2003), but it is also known for a long time that it is insufficient for improving students' skills to provide them with a list of those strategies (Schoenfeld, 1985). Therefore, I am cyclically developing an intervention program to help students act strategically.

Research interest

This study aims to test how an intervention may influence the students' problem solving and learning processes and how the induced changes affect their success in university mathematics.

Design of the intervention

The intervention is embedded into the classic problem sessions of a first-semester lecture of B.Sc. students and future secondary school teachers (weekly sessions, 14×90 minutes). So far, there have been two pilot studies accompanying the Calculus 1 lecture. The classic purpose of the problem session, to discuss homework already submitted by the students, has not been altered. However the main objective of the intervention is to foster students' self-regulation by engaging them in metacognitive processes. Therefore the students' actions are systematically reflected. This is done in three steps:

1) From a constructivist point of view it is of great importance that students develop their own strategies rather than using a checklist. Therefore students work on a few problems every week at home and write down used strategies.

2) To support the individual development of strategies, different approaches are collectively reflected. In particular, those students who were unable to find a solution themselves get the opportunity to explicate their strategies and can benefit from the experience of their more successful fellow students. A lot of different ideas, especially those that might not lead to a solution, are reviewed and evaluated. Lastly, at least one solution chosen by the students is completely expanded upon. That way, students who were not able to solve the problem by themselves can have a deep insight into that process.

3) During a reflection phase at the end of each class, the utilized strategies are discussed once again and those that could be helpful for more than one particular problem are written down on a "strategy board". Usually those strategies resemble those postulated by Pólya (1945) and Schoenfeld (1985).

For instance, when dealing with sequences or series, one firstly needs to clarify affiliated concepts like convergence and divergence. Then different strategies might be helpful, for example to generate different representations (a sketch, tables, etc.) or look at special cases. Considering related problems might also be a helpful strategy.

Evaluation

The program evolves cyclically based on student interviews, observation of solving processes, evaluation sheets and systematically documented experiences. At the end of the first pilot phase, the students were asked how often they use certain strategies (4-level likert scale). It showed that those displayed on the strategy board had significantly higher values than others. Especially self-reflection and clarification of concepts are frequently used. Nevertheless a qualitative analysis of videographed solving processes showed that the students still had a lot of difficulties to clarify concepts on their own. Therefore, in the following phases, we tried to increase their activity level concerning this task. We also integrated one unknown problem per week so we could experience a whole problem solving process together.

In the second pilot phase, the design needed to be altered since the composition of students was very different. In summer most of the students have already failed at least one exam in Calculus 1.–In addition to possible lower mathematical skills, their academic self-concept might be affected. We observed a much lower activity level than in winter, so we had to add introductory tasks to encourage the students to actively take part in the sessions. This could also apply to single weaker students that in winter would be unnoticed.

Perspective

As the order of courses was changed in the academic structure, the main study, starting April 2016 will take place in the context of Linear Algebra. The necessary adaption of the concept will be an interesting task. For the first time, we will also be able to treat a statistically relevant number of students, so in addition to ongoing qualitative research, quantitative evaluation will be possible.

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Teaching mathematics to non-mathematicians: The case of media technology undergraduate students

Evangelia Triantafyllou and Olga Timcenko

Aalborg University, Department of Architecture Design and Media Technology, Copenhagen, Denmark; evt@create.aau.dk, ot@create.aau.dk

Over the past years, a number of engineering programs have arisen that transcend the division between technical, scientific and art-related disciplines. Media Technology at Aalborg University, Denmark is such an engineering program. In relation to mathematics education, this new development has changed the way mathematics is applied and taught in these disciplines. This paper discusses a doctoral dissertation that investigated and assessed interventions to increase student motivation and engagement in mathematics among Media Technology students. The results of this dissertation have been used to assess and improve practice in Media Technology and they may inspire interventions in other trans-disciplinary engineering programs.

Keywords: Mathematics education, non-mathematicians, media technology.

Introduction

This poster presents the main contributions of a doctoral dissertation, which aimed at investigating mathematics teaching and learning for Media Technology students (Triantafyllou, 2016). Media Technology at Aalborg University is a program that focuses on research and development, which combines technology and arts and looks at the technology behind areas such as advanced computer graphics, games, electronic music, animations, interactive art and entertainment, to name a few. This dissertation investigated and assessed interventions to increase student motivation and engagement in mathematics among Media Technology students. These interventions focused on two directions: a) teaching methods and b) ICT-based learning environments. As far as teaching methods are concerned, this project has applied the flipped instruction model (or the flipped classroom). Regarding ICT-based learning environments, a game engine (Unity) has been introduced as a domain for mathematical learning. Since many studies have indicated that the attitude towards mathematics influence the achievement of learning goals, Media Technology students' attitudes towards mathematics were also investigated.

This dissertation employed several mixed method studies. Observations and a survey study were employed for gathering information on student attitudes towards mathematics, student approaches on mathematical problem solving and student competences (Triantafyllou, Misfeldt, & Timcenko, 2016). In regard to research on ICT-based learning environments, a use case study was conducted exploring development of student mathematical knowledge and effect on student motivation, when mathematics is being taught by programming in a game engine (Triantafyllou, Misfeldt, & Timcenko, in press). As far as the flipped classroom approach is concerned, two use case studies and a statistics course redesign and assessment took place (Triantafyllou & Timcenko, 2014; Triantafyllou & Timcenko, 2015; Triantafyllou, Timcenko, & Busk Kofoed, 2015).

This dissertation has provided insights in student attitudes towards mathematics in Media Technology. It was found that these students often lack mathematics confidence and they consider

mathematics a difficult subject that they do not like but value. The adoption of the flipped classroom instructional model revealed that students perceive learning with online resources on their own pace as contributing to their understanding and they reported that they could adjust the learning process to their own needs. This dissertation has also proposed the use of a model of reflection for designing activities that promote experience-based learning in flipped classrooms. As far as ICT-based learning environments are concerned, the study on the use of a game engine for mathematics learning provided insights on how students apply knowledge from a mathematical model to implement a physical model. This study shed light on students' misconceptions and difficulties but also on their opportunities to challenge their understanding. This dissertation contributed to the discussion of the theoretical foundation of the flipped classroom and discussed aspects of ICT-based mathematics learning for Media Technology. These results can be furthermore used to assess and improve practice in Media Technology and other trans-disciplinary engineering programs.

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Getting into university: From foundation to first year engineering

Stephanie Treffert-Thomas

Loughborough University, UK; s.thomas@lboro.ac.uk

Keywords: Engineering mathematics, foundation degree, transition, motivation, activity theory.

I present a longitudinal study with engineering students taking a (science) foundation course (FC). The aim of the research is to explore the reasons why students decided to take the FC going beyond the rather obvious reason, "because they did not have the necessary qualification for direct entry". Secondly, I explore students' mathematical progression into first year engineering. Many UK universities have addressed students' lack of mathematical preparedness (e.g. Hawkes & Savage, 2000) by establishing drop-in centres or one-to-one support with specialist tutors; some offer a one-year FC as an alternative route into higher education, primarily aimed at students wanting to study a STEM subject. While the FC is often seen as aimed at students who 'missed' (perhaps narrowly) their target grades at A-level, it became clear to me that students came from varied backgrounds and qualifications, work experience, and included students with health problems that impacted on their progression. Thus I became interested in the motivational factors that led students to take a FC and the mathematical progression that students make. The FC at my institution consists of a number of modules, of which mathematics and physics are compulsory for engineering students, the largest group in the cohort. Topics studied include indices, logarithms, differentiation, integration, matrices and complex numbers. I pose two research questions: (1) Why did students take the FC? (2) What is students' mathematical experience when moving into first year engineering study?

Methodological considerations

So far two cohorts of students have been interviewed at the end of their FC; one cohort has been re-invited for interview at the end of the first year of their engineering course. Ten to eleven students were interviewed each year. I, therefore, pursue a case study methodology within an interpretive paradigm. The study is longitudinal in design with further interviews planned over the next years. Data analysis is qualitative with a focus on the reasons that students gave for taking the FC. I take an activity theory perspective since identifying reasons means identifying the motive of activity, hence closely linked to characterising activity (Leontiev, 1981). Action-goals can be used to discuss students' mathematical goals, and actions taken in pursuit of these, a focus in future analyses. Research was with students I had taught. Interview questions were communicated in advance and followed a fixed order. All interviews were recorded and subsequently transcribed and analysed.

Research findings and conclusions

I report on two separate findings. First, in interviews in 2015, students were asked why they had chosen a FC. Students replied in a variety of ways, giving one or more reasons. As part of analyses, these were categorised using an open coding procedure and summarised in Figure 1. Most students cited career advancement and gaining entry to their engineering course when A-level grades had not been good enough. Some did so strategically, i.e. there was no need to re-apply through UCAS, the UK body overseeing university applications, since passing the FC guaranteed entry. Also represented were six students who took the view that it gave them an advantage over other students when entering the first year of engineering. Thus students' motives were nuanced, going beyond

Reasons given	Freq
Change of career or advance current career	6
A-level grades were not good enough	6
Wrong subject taken at A-level/change of mind	3
Interest/passion for the subject	5
Foundation year gives an advantage in year 1	6
Foundation year as orientation course	1

Figure 1: Reasons for choosing the FC

"because they did not have the necessary qualification for direct entry". Most (not all) students reported having difficulties with the mathematics content of the FC, and some students said they were overwhelmed by them. Second, I report on three of seven students who were re-invited for interview and asked to compare current experience with how they recalled their experience during the FC. The second and third column (Figure 2) relate to students' experience and achievement during the FC (as recalled by the student one year later); the last column to students' current experience. Clearly there are some differences in how students perceived their transition.

	Foundation Year	FC Maths grade	Year 1 Engineering degree
Student A2-14	Maths is not strong		Student feels settled
	GCSE is highest qualification	40 to 60%	Student achieved 40 to 50% (Maths)
	Student is not confident		Degree programme is not heavily reliant on mathematics
Student A2-12	Maths is not strong		Student is struggling
	BTEC is highest qualification	60 to 70%	Student has re-sit exams in mathematics
	Student is not confident		Degree programme is heavily reliant on mathematics
Student A2-15	Maths is strong		Student does not feel settled
	IB is highest qualification	80 to 90%	Student achieved 40 to 50% (Maths)
	Student is confident		Degree programme is heavily reliant on mathematics

Figure 2: Results from three interviews

The FC is well established at my university and students are thought successful in progressing into their engineering courses. Students cited different motivational factors for taking a FC. While some were not surprising (e.g. change of career, A-level grades not good enough) others were, and related to students employing a more strategic decision (e.g. to gain an advantage later on in their studies). Considering students' progression it is clear that this is far from linear. Both mathematically strong and not so strong students cited struggling with the mathematical demands placed on them, while a mathematically weaker student reported feeling settled and coping well. This raises some questions to explore further, e.g. to what extent can the FC provide a good mathematical basis for all students.

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TWG15: Teaching mathematics with resources and technology

Introduction to the papers of TWG15: Teaching mathematics with resources and technology

Alison Clark-Wilson[1], Gilles Aldon[2], Iveta Kohanová[3], Ornella Robutti[4]

[1]UCL Institute of Education, University College London, United Kingdom, a.clark-wilson@ucl.ac.uk

[2]Ecole Normale Supérieure de Lyon, France

[3]NTNU, Department of Teacher Education, Trondheim, Norway, iveta.kohanova@ntnu.no

[4]University of Turin, Italy

Keywords: Technology uses in education, mathematics instruction, teaching mathematics, teacher competencies, teacher behavior, computer assisted instruction.

Introduction

For the second successive CERME two groups addressed mathematics education research concerning technology. TWG15 focused on issues concerning teaching, teacher education and professional development, whereas TWG16 focused on students' learning with technologies and software and task design issues (see Drijvers, Faggiano, Gerianou & Weigand, Introduction to TWG16 in this volume).

TWG15 engaged in work that was stimulated by contributions in the form of 19 research papers and 6 posters that had responded to the call, which had highlighted the following themes:

- The specific knowledge, skills and attributes required for efficient/effective mathematics teaching with technologies and resources.
- The design and evaluation of initial teacher education and teacher professional development programmes that embed these knowledge, skills and attributes – to include programmes that involve teachers' working and learning in online communities.
- Theoretical and methodological approaches to describe the identification/evolution of teachers' practices (and of effective practices) in the design and use of technology and resources in mathematics education.
- Theory and practice related to the formative/summative assessment of mathematical knowledge in a technological environment.

The work of TWG15 drew upon research from 17 countries: Australia, Austria, Denmark, Faroe Islands, France, Israel, Italy, Lebanon, Germany, Greece, Norway, Palestine, Spain, Sweden, Turkey, United Kingdom and USA.

TWG15 themes

The contributions to TWG15 were grouped according to the following themes: large-scale professional development through online courses; technology-mediated assessment of students' mathematical learning; establishing quality criteria for digital mathematics tasks; understanding teacher perspectives on technology use; in-service teachers' knowledge and practice; pre-service teachers' knowledge and practice; and the advancement of theories on technology use in mathematics education.

The TWG15 was organised as follows:

1. A research paper by Kimeswenger was selected to be the focus of a single TWG session as it highlighted a new issue for the TWG, which was the development of quality criteria to support the selection of (dynamic) digital resources for teaching mathematics.
2. Two symposia, to address the themes: on-line large-scale professional development courses; and digital assessment of students' mathematical learning. These included selected papers that were presented by the main author, followed by an invited reaction by one participant.
3. The remaining papers were grouped by theme and presented as individual short presentations by the main author, followed by individual reaction by another invited participant.

In all cases, the discussion was opened to the whole group (in small groups of 6-8 participants), which provided the opportunity for explicit links to be made with the topics of the poster submissions by both the TWG leads and participants, and to encourage all participants to share their own knowledge and experience during discursive work. Brief feedback from these small groups was collected at the end of each session.

Large-scale professional development: Online courses

The papers by Hohenwarter et al., Taranto et al. and Panero et al. focused on the design and early evaluation of three large-scale professional development online courses that had been designed for participants from Austria-Romania-Turkey, Italy and French-speaking countries respectively. Central to all three courses was the objective to offer practicing teachers an opportunity to develop their uses of technology in mathematics classrooms. These courses were described as either 'open online' (OOC) or 'massive open online' (MOOC), where the word 'massive' implied that there were no geographical boundaries nor limits to teachers' registration and participation, although the language of the course was a limiting factor.

The three courses used theoretical frames in different ways. Hohenwarter et al. adopted Koehler and Mishra's Technological, Pedagogical and Content Knowledge model (TPACK, Koehler & Mishra, 2005) to inform their course design. Panero et al. and Taranto et al. sought to network the theories of meta-didactical transposition (Arzarello et al., 2014), documentational genesis (Gueudet & Trouche, 2009) and communities of practice (Wenger, 1998) to understand the collaborative work of teachers in the online context as seen through their productions.

The invited reaction given by Bretscher stimulated a discussion that raised the following issues: defining and understanding 'participation' within open online courses; specific design features of (M)OOCs for teachers of mathematics and the balance between technological and mathematical content; the appropriateness of (M)OOCs for the (large-scale) professional development of teachers; and how research methodologies might need be developed to assess the impact of (M)OOCs on teachers' classroom practices.

Assessing students' mathematical learning

The papers by Sikko et al., Chenevotot-Quentin et al. and Olsher & Yerushalmy were centered on the use of technology in classes by teachers and students, for activities and assessment. Even if referring to different school levels, they focused on the technology as a means to support teachers' assessment activities. In the first case, Sikko et al. presented the use of motion sensors in the Norwegian primary

school classroom to support pupils' construction of meanings for functions and their graphs. This work was set in the context of a large European project (Formative Assessment in Science and Mathematics Education, FaSMEd) aimed at researching the use of technology for formative assessment. In the second case, Chenevotot-Quentin and colleagues showed the use of a technological tool for the assessment of lower secondary school students' learning of numbers and equations. Their technological tool is applied in a way that is consistent with an epistemological analysis of the topics and with the theoretical perspective of the Chevallard's Anthropological Theory of Didactics (ATD, Chevallard 1985). Olsher & Yerushalmy presented a platform where students respond to geometrical tasks using a dynamic geometry environment, which are then classified within the platform according to their geometrical dynamicity. From the teacher's perspective, the three papers presented in this session engaged the participating teachers with professional considerations in diverse and deep ways: as designers of tasks for assessment, as teachers while teaching; and as observers of students. The papers highlighted the importance of teachers' and researchers' collaborative work in the design and evaluation of such resources for the classroom.

The discussion at the end of the presentation, stimulated by the invited reaction given by Yerushalmy focused on: the potential impacts of online formative assessment on teaching; the nature of online mathematics tasks and their formative/summative assessment; possible theoretical frameworks to support design and evaluation; automatization of students' responses and subsequent feedback to students/teachers; high stakes testing; and issues of design.

Quality criteria for digital mathematics tasks

The paper by Kimeswenger problematized the existence of online platforms that host many thousands of user-generated digital resources for teaching mathematics, which presents a particular challenge for (other) teachers as they seek to locate suitable resources that meet their individual requirements. The author described a project in its early stages that seeks to develop a research-informed set of criteria to support different methodologies for user-review. The research focused on the views of 'experts' with respect to the existence of quality criteria alongside their personal descriptions of the 'educational value' of digital mathematical resources, concluding eight quality dimensions. This has led to an exploration of the possible correlation between resources that are highly rated as other users have decided that they have a 'high-quality author' and those that are identified by users as containing 'high quality material'.

As anticipated, the TWG15 participants were most interested by, and animated to discuss, the issues raised by this paper, given that many had themselves been involved in the design of open educational resources or worked alongside teachers to try to support them to make thoughtful resource selections. This discussion concerned: the authors of quality criteria and the mathematical cultures/content/values on which such criteria might be based; the role of a consumer-led approach (i.e., 'likes' by teachers?) or a community-led approach; and, given the vast number of available resources, the usefulness of new algorithms that might automatically score 'quality', based on developed criteria

Technology integration: Understanding teacher perspectives

The two papers by Abboud & Rogalski and Bretscher both addressed aspects of technology integration into 'ordinary' secondary mathematics classrooms in France and England respectively.

Whilst Abboud & Rogalski analyzed videos of lessons at distance using an 'ergonomic' theoretical approach (Robert & Rogalski, 2005) that highlighted tensions and disturbances in the observed practice, Bretscher used classroom observation and interviews to research aspects of a teacher's mathematical knowledge for teaching with technology. These two papers instigated a critical discussion within the working group that was revisited several times during the conference as TWG15 sought to understand, and question the use of the word 'ordinary' to describe teachers (and their classrooms) within research studies. For some this referred to experienced teachers who are dependent on their own ability to (re)design lesson with technology (as in Abboud & Rogalski). For others, it referred to teachers who are required to adapt their teaching to their situation and institutional constraints in a world of changing digital tools. There was a general agreement that teachers who are involved in research studies/projects/communities concerning the use of technology in mathematics were rarely 'ordinary'. One helpful description that was offered described the set of teachers who were not yet aware of their own instrumental genesis with new technologies (or that of their students), which seemed to resonate with many of the researchers in the TWG. The TWG15 participants concluded that 'ordinary' was an unhelpful descriptor and this highlighted the importance that researchers describe teachers' contexts more fully (i.e. country, teacher background, school system, school curriculum, etc.) to enable deeper and more critical insight into each other's research settings.

The paper by Kolovou & Kynigos differed from the two previous papers by focusing specifically on the learning processes of the designers of dynamic digital resources to foreground students' and teachers' mathematical creativity, which is fully described in their paper. By focusing on a 'community of interest' (which included teachers) that had been formed around the design of a particular creative book (c-book), the authors show how the participants' learning was stimulated by the boundary objects (Fischer, 2005) in the design process.

A focus on pre-service teachers

The papers in this theme offered different approaches to pre-service teachers' training and the different interpretations of their required knowledge about technology.

Prodromou investigated the usefulness of a flipped classroom approach in tertiary education in Australia. The theoretical frame was that of the 'four pillars' that define a flipped classroom, which take account of the flexible environment; a shift in the learning culture; intentional content; and the role of educators (Flipped Learning Network, 2014). The analysis of an experiment with pre-service teachers was presented with a particular focus on the role of the lecturer in a flipped classroom. The study by Herrelko tracked the implementation of technology in a mathematics methods course for pre-service teachers in the USA. The method, based on the Apple Classroom of Tomorrow framework (ACOT, Dwyer, Ringstaff, Haymore & Sandholtz, 1994), sought to describe the necessary conditions for the development of pre-service teachers who are knowledgeable about instructional technology. Baya'a et al. focused on pre-service teachers' TPACK (Koehler & Mishra, 2005) and provided analyses of the impact of teacher preparation courses that had been shown to develop teachers' TPACK.

The main questions that these presentations highlighted, and were discussed by the TWG, are linked to understanding the pre-service teachers' perspectives in the design and implementation of

mathematics with technology. Leading on from this, there is a need for deeper understanding of the required technological, mathematical, pedagogical, and epistemological knowledge that is essential for future teachers in order to prepare them to use digital tools effectively in their teaching.

A focus on in-service teachers

This theme concerned in-service teachers, their professional engagement in the various activities related with teaching: planning lessons; using technologies; working in communities; orchestrating different devices in laboratory activities for students; balancing laboratory activities and more traditional teaching. The presenters of the papers showed various aspects of the ways that teachers work with technologies, both related to their teaching practices and to the design and management of educational materials.

Tamborg shared research on the use of a platform in Denmark, Meebook, for planning mathematics lessons in accordance with the teachers' pre-determined learning objectives, teaching approaches and curriculum. The framework used for the study is the 'instrumental approach' (Gueudet & Trouche, 2009), along with the 'documentational approach' (Gueudet et al., 2012) to describe the teachers' collective processes in their use of the platform to plan their lessons. Kayali's study centered on an investigation into the uses of mathematics education software by English secondary mathematics teachers, to understand why some software is used more or less than others, in which ways and for which reasons. Again, the instrumental and documentational approaches are the adopted frames alongside the 'teaching triad' (Jaworski, 1994), for the collection and analysis of data on teachers' considerations when implementing tasks in mathematics lessons. Zender and colleagues showed a motivating way to support students' learning with technologies outside of German classrooms using MathCityMap, a geo-located application for smartphones, which is used as an instrument for a range of situated mathematical tasks. The collaborative professional development of teachers in Lebanon on the use of GeoGebra in mathematics classes was the theme introduced and discussed by Kasti et al.. They based their research on the frame of Valsiner's three zones (Valsiner, 1997) and the TPACK theory (Koehler & Mishra, 2005), using questionnaires and interviews to investigate how GeoGebra is introduced in various mathematics activities.

Advancing theories on technology use in mathematics education

The papers by Gustafsson and Grønbæk et al. both focused on advancing theories on technology use in mathematics education. Gustafsson investigated the potential of Ruthven's Structuring Features of Classroom Practice framework (2009) as a tool to analyze empirical data to conceptualize and probe teachers' rationales for technology integration in the mathematics classroom. Gustafsson's results showed that, whilst the framework captures most aspects of their rationales, it did not fully encompass teachers' justifications with respect to their students' attitudes and behaviors. Hence, he suggested the addition of a new (sixth) structuring feature that relates to teachers' craft knowledge of the use of technology to manage different types of student behaviors or attitudes.

Grønbæk et al. suggested the addition of the concepts of *out-* and *in-sourcing* to Chevallard's (1985) Anthropological Theory of the Didactics. These concepts, taken from the field of business economics, are used as metaphors within the dialectics of tool and content in the planning of teaching. Their addition offers a 'production model' to support teachers' reflection on crucial choices between instrumented and non-instrumented praxeologies. This highlights the need for teachers to be able to

identify core activities (with potential for in-sourcing) and non-core activities (candidates for out-sourcing) based on the learning goals and the possible praxeologies when planning their use of technology in mathematics lessons.

TWG15 participants' reflections

During the final TWG session, the participants were invited to reflect upon, and record on paper, their own insights and learning during the conference. A textual analysis of the 31 responses highlighted the following aspects of knowledge exchange:

- **Broader appreciation of theoretical frames and their uses**: exposure to new theories; consideration of the limitations of theories; discussions of networking theories; and reflections on personal interpretations and applications of theories.
- **Deeper understanding of international contexts**: theoretical traditions; institutional constraints (i.e. curriculum and examinations); the rejection of the concept of an 'ordinary' teacher, which seems difficult to define or to establish a common meaning across countries.
- **Widening of knowledge on emerging themes:** opportunities afforded for large-scale teacher development through MOOCs and OOCs; the need for 'quality' criteria for digital resources to support their selection/uses by teachers; and the role of technology within formative and summative assessment that relates strongly to the mathematics curriculum, its values and traditions.

Some participants reflected upon the unintended consequences of the division of the technology group for the two different foci (teachers and students), highlighting that an opportunity was missed to explore 'How to develop a good framework to capture the interplay between mathematics content, technology (tools/resources), teachers *and* students/learners?'.

Finally, a few participants commented on the collegiality of TWG15, highlighting they had also been 'inspired' as they learned about the CERME spirit: 'humility - and how to express both orally and in writing with humility'.

Conclusions

The broad range of papers and posters presented at CERME 10 highlighted the diversity of research interests in the participating countries. However, many common concerns prevail. The design and implementation of programmes and courses for future and practicing teachers was one such challenge. Debate over the exact content of such courses in order to address the knowledge and skills to integrate technology into future mathematics teaching practice was paramount, alongside the modes of delivery and the integration of teachers' classroom experimentations. This highlighted the different views and perceptions of the simplicity and complexity for teaching mathematics with digital tools and the dilemma between technology appearing to make a teacher's role easier (e/g/ by automatically marking students' productions), whilst at the same time introducing new teaching challenges (e.g. by introducing new representational forms and related interactions). The TWG discussed ways to face such challenges through the development of research-informed teachers' collaborative professional development models that integrated coaching, face-to-face and online communities, often conducted or sustained over a time period of years, rather than months, drawing on the outcomes of the recent ICMI Survey on this theme (Robutti et al., 2016).

A common challenge is the scaling of such professional development models, for which (M)OOCs might offer some solution, although substantial research is needed to evaluate the 'best designs' to respond to the many different cultural contexts and requirements. The networking of theories proposed by Panero et al. and Taranto et al. provided some potential theoretical and methodological tools to this effect.

The topic of the automated assessment of students' digital work led the group to question deeply the nature of mathematical activity (and mathematics itself) that warranted such assessment. This raised a general concern over the ease with which closed mathematical questions can be posed and digitally assessed and the much greater technical challenge to design automated assessment that privileged mathematical processes such as reasoning, justification and proof. The poster by Recio and the two papers by Olsher & Yerushalmy and Chenevotot-Quentin et al. respectively contributed greatly to this debate.

TWG 15 critiqued advancement in theories concerning teachers' uses of technology in mathematics education. In particular, the notions of 'tensions and disturbances' as a theoretical construct to support analyses of teachers' practices (Abboud & Rogalski), alongside extensions to Ruthven's 'Structuring Features of Classroom Practices' (Bretscher & Gustafsson, and the poster by Simsek & Bretscher).

Looking ahead to CERME11, TWG15 concluded the following questions, which might inform individual and collaborative research efforts over the next two years:

- Which theoretical frames and methodological approaches focus on aspects of the collaborative work of researchers and teachers within the context of the use of technology for teaching mathematics?
- What approaches might be fruitful to raise teachers' awareness of the mathematical-pedagogical decisions concerning the design and use of technology for learning and its assessment?
- How do we create opportunities and approaches that support teachers to appreciate and plan for the process of students' instrumental genesis?
- In the design of technology-focused professional learning for mathematics teachers (pre- and in-service), what is the balance between professional needs across generic technologies and mathematics-specific needs? and how can this be achieved?

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Real uses of ICT in classrooms: Tensions and disturbances in mathematics teachers' activity

Maha Abboud[1] and Janine Rogalski[2]

[1]LDAR, University of Cergy-Pontoise, France; maha.blanchard@u-cergy.fr

[2]LDAR, University Paris Diderot; rogalski.muret@gmail.com

This paper presents an extension of approaches of the teacher technology-based activity, articulating the Double Approach alongside with the Instrumental Approach within the overarching frame of Activity Theory. Tensions and disturbances are defined for analysing the dynamics of the teacher's activity when ICT tools are mediating both teacher's and students' activity. The approach is illustrated throughout a comparative study of two "ordinary" teachers using dynamic geometry. Various tensions related to the temporal, cognitive and pragmatic dimensions were observed, differently managed depending on personal, material and social determinants. Tensions are inherent to the dynamics of the situation. Together with disturbances, they are lenses contributing to a fine-grained analysis of teachers' activity.

Keywords: Teachers, technology, activity, tensions, disturbances, dynamic environment

Introduction

The activity of "ordinary" teachers integrating technology into teaching is constrained and depends on several determinants, namely personal, institutional and social. The work of researchers such as Ruthven (2009), Drijvers et al. (2010) and Abboud-Blanchard (2014) emphasize the need to study the practices of these teachers, often not technology experts and practicing in non-experimental conditions(i.e. ordinary practices). One of the aims of such studies is to better understand what happens in the classroom and thereby to address professional development issues (Clark-Wilson, 2014). The aim of the present paper is to contribute to this research line by introducing two new theoretical concepts, *tensions* and *disturbances.* These concepts were developed within a model of instrumented activities of teacher and students and were actually used as complementary resources within ICT teacher education programs.

We consider the teacher as managing an "open dynamic environment" (Rogalski, 2003), and we focus on both the relationship between the lesson preparation and its actual implementation (anticipation, adaptation); and also on the management of the inherent uncertainty within such an environment. Indeed, the use of technology adds a "pragmatic" dimension emphasizing the "open" character of the environment that constitutes the classroom activity. Monaghan (2004), stresses that this use leads to an increased complexity in teachers' practices and also that the uncertainties related to students' mathematical activities with technologies bring teachers to modify their objectives *during* the lesson in progress, leading them to focus on new "emergent goals". The concepts we introduce enable an

analysis of the impact of the dynamics of students' interactions with technology tools on the management of the planned (by the teacher) *cognitive route* (Robert & Rogalski, 2005), and the possible divergences from this during the lesson. In this paper, we provide an example of the comparative analysis of the activity of two "ordinary" teachers' uses of dynamic geometry with their (6th grade) students to describe the methodology and associated analytical tools and to highlight their usefulness. We selected this particular example from our research data as it is relatively easy to present in a short paper.

Theoretical and methodological approaches

The ergonomics theoretical perspective considers teaching as a case of dynamic management of the teaching environment (Rogalski, 2003). This environment is "open" as it contains many uncertainties due to the fact that the students' activity cannot be completely predicted and the teacher is often in an improvisation mode. The teacher's conceptions of the mathematical domain to be taught, and of the relation students have to it, are subjective determinants of his professional activity. These conceptions condition the "didactical process" he wants his students to follow i.e. the planned *cognitive route*, alongside the management of the processes developed during the lesson (Robert & Rogalski, 2005). Although the didactic scenario is familiar, the students' diversity and the specific context of the class introduce a factor of uncertainty. This uncertainty is exacerbated when students are working with a technological tool as the teacher may encounter difficulties to control the tool's feedback due to students' manipulations and to identify their emerging interpretations. Teachers often have to deal with tensions due to the presence of the tool and its role in the student's activity, and also its interaction with the mathematical knowledge at stake.

Following Rabardel's Instrumental Approach (2002), technological tools can be viewed from both the teacher's and the students' perspectives. In both cases, the subject-object interactions are mediated by the tool. As Rabardel states:

> Beyond direct subject-object interactions (dS-O), many other interactions must be considered: interactions between the subject and the instrument (S-I), interactions between the instrument and the object on which it allows one to act (I-O), and finally subject-object interactions mediated by an instrument (S-Om). Furthermore, this whole is thrown into an environment made up of all the conditions the subject must take into consideration in his/her finalized activity (Rabardel, 2002, p.42-43).

Nevertheless, the object of teacher's activity is the students' learning, whereas the object of the students' activity is the content of the task given by the teacher; their instruments based on the same tool are thus different. Figure 1, presents how these two instrumented activities are articulated within the dynamics of class preparation.

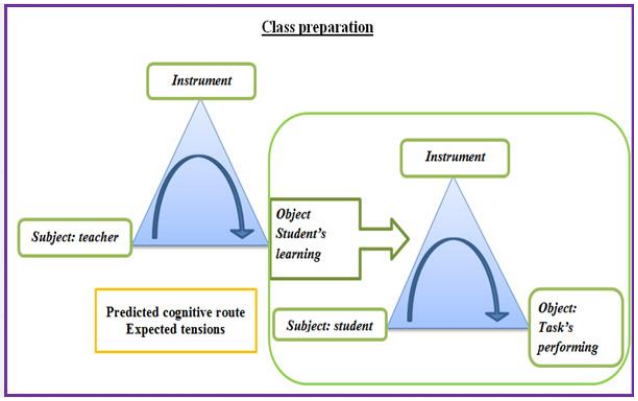

Figure 1. Teacher's and students' instrumented activities within the preparation phase

We now consider the classroom environment and present how the two instrumental situations are articulated within the dynamics of class management, indicating possible tensions and disturbances.

Tensions and disturbances

In our approach, we depart from the way Kaptelin & Nardi (2012) introduced the terms "tension" and "disturbance" when presenting the concept of contradiction central in Engeström's framework of analysis for how activity systems develop (Engeström, 2008). These terms appear in their familiar use; emphasis being put on the analysis of contradictions in activity systems as main learning sources.

We do not define tensions as conflicts or contradictions. In the teacher's activity *tensions* are manifestations of "struggles" between maintaining the intended cognitive route and adapting to phenomena linked to the dynamics of the class situation. Some of these tensions might be predicted by the teacher and so he/she plans how to manage them. Others are unexpected and constrain the teacher to make decisions, in situ, that direct his/her actual activity.

Disturbances are consequences of non-managed or ill-managed tensions that lead to an exit out of the intended cognitive route. Disturbances happen when a new issue emerges and is managed while the current issue is not completely treated or when the statement of a new issue is not part of the initial cognitive route.

We consider here only tensions and disturbances related to the local level of a class session; while some tensions are or might be managed at a more global level (i.e. over several sessions). Figure 2 illustrates how tensions can be related to different poles of the system of teacher-and-student activities; they can be shaped differently along three dimensions (previously introduced by Abboud-Blanchard (2014)): temporal, cognitive, and pragmatic.

Tensions related to the cognitive dimension appear in the gap between the mathematical knowledge the teacher anticipated would be used during task performance and the knowledge that is actually involved when students identify and interpret feedback from the instrument. Tensions related to both the pragmatic and cognitive dimensions are produced by the illusion that mathematical objects and operations implemented in the software are sufficiently close to those in the paper-and-pencil context

(we refer to Balacheff (1994) analysis of the "transposition informatique"). Tensions related to a temporal dimension are frequent in ICT environments and are linked to the discrepancy between the predicted duration of students' activity and the actual time needed to perform the task. Teachers are generally aware of such tensions; they often manage them by taking control of the situation, either by directly giving the expected answer or by manipulating the software themselves. Finally, a tension non-specific to ICT environment may concern the didactical contract: Students cannot identify the type of answer the teacher is expecting. ICT environments may amplify this type of tensions when students are uncertain of the goal of the activity i.e. is it about a mathematical object to manipulate with the software or about the use of the software itself.

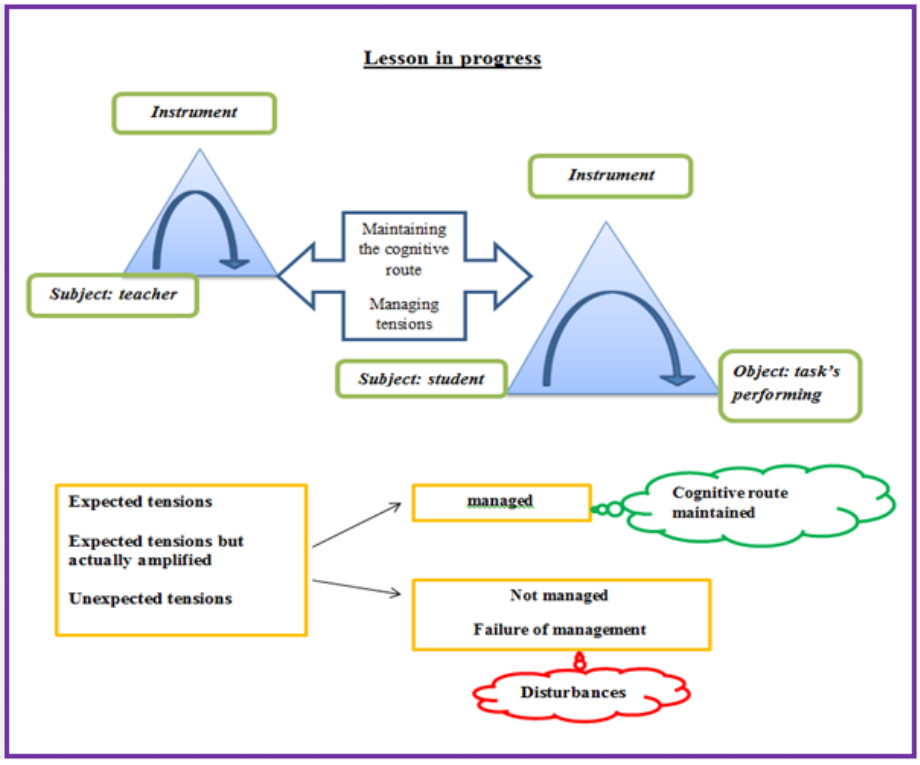

Figure 2. Tensions and disturbances within the dynamics of class management

Illustrating the theoretical approaches through a comparative case study

We present how tensions and possible disturbances appear in the case of two teachers, Alan and Colin, using dynamic geometry software (Geoplan) with 6th grade students to introduce the notion of perpendicular-bisector. They are both "ordinary" teachers who use technological tools occasionally, and willingly, in ways that are in line with the institutional expectations, that is to introduce students to an experimental approach. The two teachers designed the same cognitive route based on the succession of two tasks: moving several points (eight) on Geoplan screen in order to place each of them at the same distance from two fixed points, M &N, (ICT task) and then similarly drawing 8 points with the same condition in a paper-and-pencil context (p&p task). Each teacher's final goal was to: give the definition of the perpendicular-bisector as a set of points equidistant from two given points; and establish an efficient associated construction method using compasses. Alan's school is

in a low-income socio-economic zone, while Colin is in a middle-class zone. Their working environments are different: Alan had access to a traditional classroom and a computer room that lacked either a video projection device or a black board, while Colin worked in a classroom equipped with laptops.

The sessions included in our analysis were video recorded by the teachers themselves. Our choice of data collection approach is to reduce as far as possible the impact of researchers on the teacher's and students' activity in the class. The analysis of the teachers' preparation documents and deferred interviews enable the identification of some personal and social determinants. We then compare the observed succession of episodes in the video alongside the planned cognitive route, to enable us to detect tensions and disturbances.

Results

A somewhat surprising result is that both Alan and Colin managed the session without temporal tensions despite a number of "unfavourable" material and social determinants. In Alan's case, these could have resulted in strong tensions, e.g. the time needed to move from classroom to computer room and students' prior cognitive difficulties. In fact, Alan took into account the social determinants of his class and the material constraints by anticipating and avoiding tensions that could have produce disturbances through a threefold organisation: temporal, pragmatic and cognitive. Indeed, Alan closely supervised his students and organised their activity by structuring the cognitive route as a succession of well-defined sub-tasks. This mode of guidance has been identified previously as a common approach that teachers use to manage such experimental approach in order to avoid students' erratic behaviour (Abboud-Blanchard, 2014). The rhythm of sub-task completion is also strictly planned and guided. This is probably linked to Alan's personal determinants that led him to establish strong routines to discipline students in all moments. Indeed, not all teachers with this type of students are able to establish such routines and to be at ease when implementing them. Colin avoided temporal tensions in quite different ways. He started by presenting the task both with ICT and p&p. During the session, he used the IWB for sharing elements of the task outcomes with the whole class. He particularly drew students' attention to where they should look on the screen, and by doing so, avoided some pragmatic and cognitive tensions. Colin's open attitude may be related to a personal determinant of "compliance" inherent in his relationship with his students.

Regarding tensions related to the cognitive dimension, an important result is the shared illusion of transparency: Implicitly, Alan and Colin took for guaranteed that after completing the ICT task, all students *would* have detected the existence of a straight line on which all equidistant points are situated. This was clearly not the case. This tension was not managed as indicated by the absence of any collective comment concerning the point of transition between the ICT task and the p&p one. For some students, this fact led to a divergence from the intended cognitive route: a "local" disturbance. These students persisted conscientiously throughout the whole session to draw equidistant points without appreciating the notion of a straight line as the set of such points. A tension related to the

didactical contract was also observed in both classrooms when, during the p&p task, some students tried to place the 8 points at the same positions they occupied on the screen. What may have triggered this students' interpretation of the task differs for the two teachers. Alan had introduced the p&p task, by saying "now we will do the same task but without the computer". However, for Colin, the computers were not shut down and thus students may have continued to refer to what they saw on the screen. During the session, both teachers succeeded in managing this tension by explaining the differences of the two situations.

Finally, a pragmatic tension that was not managed was an implementation issue that could be related to a shared belief among teachers that students are skilled with technology using a trial and improvement approach. At the beginning of the ICT task, for any given point (P) on the screen, the students *could* read the relative distances to the point M and N and, when moving this variable point, they *could* observe the numerical changes. However, dragging the point P to maintain the equality involved two degrees of freedom on the plane. Therefore, an efficient approach relied on both students' awareness of this constraint and their development of an adequate concept-based strategy to "maintain a constant dimension when moving along the other" or "anticipate the curve on which the point is moving" (as shown in Abboud-Blanchard, 2015). In Colin's class, some students continued to drag points without any strategy even when the teacher asked them to engage in the p&p task. As a consequence, they could not easily be aware of the efficiency of using compasses instead of rulers when switching to the p&p context. For Alan's students, we can infer that some of them succeeded to place only few points with limited opportunities to notice their alignment.

Overall, we have identified a set of tensions in the activity of two ordinary teachers use of dynamic geometry and have illustrated the dimensions they may affect. By predicting many tensions, these teachers avoided some of these tensions by anticipating and organising the students' activity relative to the temporal, cognitive and pragmatic dimensions. The teachers managed others *in situ* mainly via individual interactions with, and support for the students. Their approach to classroom management depends on personal, material and social determinants. Nevertheless, the analysis also shows that even though they succeeded to maintain the essence of their intended cognitive goal, some unrecognised or ill-managed tensions led some students to diverge from the planned cognitive route.

Discussion and conclusion

In this paper, we have presented how and why we developed the notions of tensions and disturbances to analyse the dynamics of teachers' activity in ordinary contexts when they are using ICT in line with the institutional demands. We first schematised how ICT occupies two different positions as an instrumental tool for the teacher and for the students. The first schema is based on the postulate that there exists a crucial difference in the object of activity of the teacher (e.g. students' understanding/learning of the mathematical content involved in the task), and of the students, which is essentially to complete the task. Within the second schema, we added and defined several types of

tensions that may appear within the teacher's activity along three dimensions: temporal, cognitive, and pragmatic, and at the level of the didactical contract. The cases of Alan and Colin are examples of ordinary, experienced teachers investigated within a larger study, who use technology regularly. In addition, they are convinced that dynamic geometry enables students to make pertinent observations through the immediate feedback (this belief is widely shared by teachers). We identified a set of tensions: some of which were managed through anticipation; others in real-time (depending on different personal and social determinants); some were not detected or detected and not managed by the teachers. If teachers are able to identify and manage tensions, they can maintain the intended cognitive route for students or, when disturbances occur, modify this route to different effect. Several issues remain open for further research. First, the present study focused on the "local level" - an analysis of a specific classroom session. It will be necessary to extend our concepts and methodological tools to analyse tensions and disturbances at the more global level of a sequences of tasks on particular mathematical topics. Secondly, we analysed cases with "simple" ICT-based students' tasks; other cognitive and pragmatic tensions could appear when the tasks involve objects and operations that are more deeply modified by the "transposition informatique". Finally, at the deeper level considered in Artigue (2007), what different kinds of tensions would teachers encounter when students are engaged in mathematical activity involving mathematical objects (an epistemic orientation) compared with more specific computer-based tasks (a pragmatic orientation). Finally, we consider that the concepts of tensions and disturbances have enriched the range of theoretical tools to study teachers' instrumental activity, in particular for the identification and analysis of critical aspects of the dynamics of this activity. We conjecture that this could also inform approaches to teacher education.

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Out- and in-sourcing, an analysis model for the use of instrumented techniques

Henrik P. Bang[1], Niels Grønbæk[2] and Claus R. Larsen[3]

[1]Christianshavns Gymnasium, Denmark; hb@cg-gym.dk

[2]Department of Mathematical Sciences, University of Copenhagen, Denmark; gronbaek@math.ku.dk

[3]Christianshavns Gymnasium, Denmark; cl@cg-gym.dk

We outline a model for analyzing the use of ICT-tools, in particular CAS, in teaching designs employed by 'generic' teachers. Our model uses the business economics concepts "out-" and "in-sourcing" as metaphors within the dialectics of tool and content for the planning of teaching. Out-sourcing is done in order to enhance outcomes through external partners. The converse concept of in-sourcing refers to internal sourcing. We shall adhere to the framework of the anthropological theory of the didactic, viewing out- and in-sourcing primarily as decisions about the technology component of praxeologies. We use the model on a concrete example from Danish upper secondary mathematics to reveal what underlies teachers' decisions (deliberate or spontaneous) to incorporate instrumented approaches.

Introduction

Has use of computers in schools resulted in better education? With the steadily growing take-up of technology throughout the world, this question is as important as ever. The role and importance of technology has undergone phases from initial excitement to, more recently, a mixture of cautious optimism, moderate skepticism, and the stance that the use of computers might forfeit the true values of educational discipline. A recent, rather extensive international report (OECD, 2015) indicates countries' improvements in learning by a number of measures against their investment in ICT. The foreword summarizes the implications for educational policy:

Mere embracement of ICT in itself is at best harmless. Access to ICT does not automatically improve learning, "The results also show no appreciable improvements in student achievement in reading, mathematics or science in the countries that had invested heavily in ICT for education".

In person teacher-learner contact is essential, "One interpretation of all this is that building deep, conceptual understanding and higher-order thinking requires intensive teacher-student interactions, and technology sometimes distracts from this valuable human engagement".

There is a need for alignment of technology and learning: "Another interpretation is that we have not yet become good enough at the kind of pedagogies that make the most of technology; that adding 21st-century technologies to 20th-century teaching practices will just dilute the effectiveness of teaching".

A deeply rooted trust in the progressive power of ICT (but with a somewhat unimaginative scope to traditional learning material), "Why should students be limited to a textbook that was printed two years ago, and maybe designed ten years ago, when they could have access to the world's best and most up-to-date textbook?".

A tall order on teachers to meet the expectations (by pedagogy rather than by subject discipline), "Perhaps most importantly, technology can support new pedagogies … it is vital that teachers become active agents for change" (OECD 2012).

Other meta-studies (Higgins, Xiao, & Katsipataki, 2012; MBUL, 2015) point to similar conclusions, "There is at most a weak positive correlation between bulk use of computers and learning outcome."

In contrast, there are numerous reports on very fruitful and insight-giving use of computers (Böhm, Forbes, Herweyers, Hugelshofer, & Schomacker, 2004; Heid, 2003; Nabb, 2010). These are often the result of computer focused teaching designs that are part of didactic research or teaching development, carried out by dedicated teachers. Therefore, the question is not how much, but how, about what, and by whom.

The Danish landscape

In the OECD report Denmark is ranked second in use of computers. From the mid 1990's there has been rapidly growing CAS-use in Danish high schools, starting with graphing calculators and accelerated through the extensive use of PC's from around 2005. The situation now is that most high schools use Maple, TI-Nspire, Geogebra, and/or a CAS-tool specially developed for Danish high schools (WordMat, a CAS engine integrated within Microsoft Word). Students bring their own PC to the classroom, and use of PC is required at examinations. Initially, the transformation was carried through by progressive and CAS-curious teachers, many of whom were inspired by reform pedagogic ideas that supported a shift from abstract mathematics towards applications and more intuitive conceptual understandings. There was (and is) also an element of believing in diffusion: If CAS helps advanced (university) students to solve advanced problems, we might as well use CAS to help less advanced students solve less advanced (but to them difficult) problems. The educational system eagerly supported the development. Mathematics has been a vehicle for use of PC in other subjects, and examinations using CAS could (possibly) help more students achieving higher levels in math. Since 2005, CAS has gone from being a tool for enthusiastic teachers to a tool for everyone, including teachers with less interest and less competency in CAS. There has been no essential change in the standard curriculum (only minor ones allowing time for, say, χ^2-test) – and standards for CAS use have not been introduced. On the contrary, the curriculum endorses the use of CAS in mathematical modeling and concept building, but without any indication of how, and in connection with what topics, to carry this out. In this landscape, many teachers have developed templates that students are allowed to use in exams, and the preparation of students to use these has become an important activity during normal lessons. Students of teachers, who for one reason or another disfavor such, may find templates on the internet or borrow from friends. Most of such templates have little *epistemic value* and a rather narrow *pragmatic value* in the sense of (Artigue, 2002) towards solving (standard) problems. With CAS at the national tests these tendencies of trivialization are even more pronounced, as problems must be formulated to be equally solvable on different CAS platforms.

Denmark's extended use of computers in education reflects of course a trend in society but is also as described above to a large degree the result of explicit educational policies. Hence, a teacher has to find his/her pathway through the affordances, constraints, possibilities etc. stipulated by official guidelines, curriculum and instruction plans. As indicated, successful use of computers does seemingly not scale up (MBUL, 2015). In order to understand the reasons for this better we propose

an analysis model to help understand teachers' decisions on use of computers in mathematics teaching.

Theoretic framework

As our proposal aims at elucidating teaching in an institutional context we find the anthropological theory of the didactics (ATD, Chevallard, 1999) well suited. We start by briefly recalling the most important concepts of ATD that we shall use. Mathematics as an enterprise (educational as well as scientific) is seen as human activity composed of two blocks each with two parts, a *praxis block* comprising types of problems, *tasks*, with *techniques* to accomplish these and a *theory part* comprising *technology* and *theory*. Together such two blocks are termed a *praxeology* (praxis + logos). Tasks are the immediate goals of the activity, i.e. finding the slope of a graph of a function at a given point. A task can be accomplished by several techniques, i.e. plotting the graph on a computer, zooming in on the point in question until the graph appears linear and reading off the slope. The technology part concerns the discipline discourse of the technique and its relation to the tasks, i.e. the scope and limits of computer rendering of graphs (in relation to variation of functions). The theory part is a discourse on the technology part and its relation to the praxis block, i.e. on the concept of linear approximation that the sketched approach leads and on how it is related to a larger body of mathematical knowledge and practice, for instance that of the theory of differentiation.

We would like to stress a couple of points. A given task can be unfolded in many praxeologies. To choose, detail and organize such unfoldings is the essence of teaching design. Any praxeology has underlying praxeologies, i.e. praxeologies aimed at slope of a linear function, and is itself related to/part of other praxeologies. A praxeology always comprises all four parts. This is one key point of the analysis in (Barbé, Bosch, Espinoza, & Gascon, 2005).

Praxeologies take place within organizations of mathematical practice and knowledge. In ATD such organizations are formed by two components, a mathematical body consisting of a totality of objects, concepts, statements, interrelationships, procedures, etc., termed the *scholarly knowledge* and an *institution* of society within which this body is taught, manifesting possibilities and constraints for acquisition of learning. The passage from scholarly knowledge to its institutional version (which has more components than indicated) is in ATD conceptualized as *didactic transposition.*

In (Artigue, 2002, p. 271) it is noted that didactic transposition in its first version was described with respect to rather traditional scholarly mathematics. In order to underline a computerized setting Artigue has singled out the term *instrumented techniques.*

The model

In ATD the teacher is considered as the director of the learners' didactical processes (Barbé et al., 2005), that is, responsible for the establishment of relations between learners and organizations of knowledge within institutions. We shall take this a step further, viewing it as production of learning outcomes through production activities, the praxeologies. For this production, the teacher has at his/her disposal a palette of resources, typically in terms of techniques (along with their theoretical block) to solve tasks. The 'employees' who use the techniques towards the production are the students.

This setup is very similar to a business economic model of the production of a corporation. In order to enhance the outcome a corporation director makes sourcing decisions on the allocation of resources. In modern terminology, one speaks of outsourcing, insourcing, backsourcing[1], "outsourcing involves allocating or reallocating business activities (both service and/or manufacturing activities) from an internal source to an external source" (Schniederjans, Schniederjans, & Schniederjans, 2005, p.3). Insourcing can be viewed as an allocation or reallocation of resources internally within the same organization. Any business activity can be outsourced or insourced (dichotomy), but this decision is crucial to the success of the corporation. The basic idea of outsourcing is old, essentially, it is the dictum 'buy or make'. However, in the last few decades, outsourcing has grown almost explosively. A main reason for this is the development of ICT. But outsourcing is risky. It is reported (Schniederjans et al., 2005, p. 12) that half of all outsourcing agreements fail due to lack of appropriate analysis, and the necessity of *strategic planning* has become evident. There seems to be general acceptance (Schniederjans et al., 2005, p. 9) that such starts with an analysis to identify the strengths of the corporation, in terms of *core activities* ('core competencies' in (Schniederjans et al., 2005). Loosely, a core activity is what the corporation does better than its competitors and possible outsourcing providers. Core activities must be insourced, non-core activities are candidates for outsourcing and a balanced decision to achieve the strategic goals must be made. Key advantages of outsourcing of inspiration for didactic equivalents are: focus on core activities, gain of outside technology, enhancement of capacity and lower cost, whereas some key disadvantages are loss of control, increased costs, negative impact on employees' morale and difficulties in managing relationships with outsourcing provider.

In the didactic version, the *client* is a didactically transposed knowledge organization along with the teacher(s) to direct the didactic processes. The *outsourcing provider* is an external knowledge organization, typically within a CAS. In the business model, external/internal refers to ownership. For our purpose the fundamental feature of ownership is that it allows for control of processes, i.e. outsourcing implies loss of control. We shall take this as the defining property. Hence *outsourcing a mathematical activity means allocating it to a resource at the price of giving up control of processes*. A blunt example could be a teacher encouraging students to find solutions to homework on the internet; a more elucidating example is employment of instrumented techniques in the form of black-box applications of CAS. As pointed out, any activity can be outsourced or insourced, that is full praxeologies, be it punctual, local or regional, or just parts of praxeologies, typically the praxis block. To be more precise, the starting point of CAS outsourcing is typically a problematic task to be solved by the outsourcing provider's technique thereby creating a transformed or new praxeology. We stress that a CAS such as Maple is not solely a provider. To the extent that a teacher exercises control over CAS processes, these are considered insourced. Outsourcing to CAS is a more restrictive concept than mere use of CAS. (Teacher control must be distinguished from student control as the latter is the result of the first, and perhaps of other competencies, acquired without the influence of the teacher.)

[1] Backsourcing means reallocating tasks from external sources to internal. This could be in order to regain control of the production process but could also be imposed by outside regulatives. In an educational context such could be new stipulations of use of CAS at national tests.

A simple example (an object of many controversies) illustrates the concepts. Arithmetical computations require a careful analysis of what are core activities that accordingly should be insourced. Depending on (long-term) learning goals, these could be the systematics of paper and pencil algorithms, skills of mental arithmetic with "nice numbers", etc. On the other hand, multiplication of many-digit numbers is hardly a core activity and is therefore a candidate for outsourcing to calculators[2]. This does not mean that tasks, which can be solved by mental computation, should not be insourced by calculator techniques. The point is that the core activity of mental computing may give control also of some calculator computations. Note that a calculator praxeology is completely different from its non-instrumented equivalents, for instance its theory part may involve representation of numbers in a finite memory.

The very decision to use CAS (or other *instrumented techniques*) involves, regardless of its specific use, outsourcing. The teacher has no control of the coding that underlies the CAS, the syntax, the defaults, the library of routines, etc. Most CAS-tools are developed with teaching in mind, at least partly. Perhaps most importantly, the CAS design may have didactic intensions, which the teacher may surrender to if not disable. Maple's 'clickable math' is a good example of this. The relationship between non-instrumented mathematics and computerized mathematics resembles that of a *strategic partnership* with mutual outsourcing. This relationship is dialectic in nature. The potential of CAS in mathematical praxeologies needs non-instrumented mathematics to be redeemed.

There is of course nothing new in the very idea of strategic planning. Mathematical activity has at all times involved use of 'non-controlled' components and didactic considerations have always had this as a condition. The modern aspect of CAS is the magnitude of impact, calling for a much clearer elaboration of such planning. The addition of the concepts of out- and insourcing to ATD offers a model for reflection on crucial choices between instrumented and non-instrumented praxeologies on basis on insight in the CAS-tool and in possible mathematical activities. On one hand, the model gives a framework for investigation of ordinary teachers' undertakings and perhaps more importantly, of what is not undertaken. On the other, it provides a strategic planning scheme for the teacher cf. (Schniederjans et al., 2005, Figure 1.3), where II+III are the crux of the matter:

I. Establishment of content and learning goals of the mathematical organization to be taught
II. Detailed analysis of subject matter and activities of possible praxeologies.
III. Identification
    a. Core activities
    b. Non-core activities
IV. Sourcing decisions
    a. Core activities are insourced
    b. Non-core activities are candidates for outsourcing.

[2] A business equivalent of the historically initial excitement about the freeing potential of calculators and the afterthought concerning (permanent?) loss of core activities: The reservation system of the flight company TWA was superior to those of its competitors, i.e. a core activity, but was outsourced in the 90's. TWA never regained its market share and went out of business in 2001.

How do teachers decide on what is a core activity? The dialectics of pragmatic and epistemic value (Artigue, 2002) seems inevitable, but is not directly reflected in the dichotomy of out- and in-sourcing. The computational power of several thousand digits, obviously to be outsourced, may have epistemic value in relation to approximation by decimal expansions. The pragmatic value of graphing of polynomials may be an asset of outsourcing in order to study whether polynomials have desired properties, which are considered epistemic of certain mathematical models. In other praxeologies graphing, by hand or by CAS, may be insourced.

Methodology for prospective work with the model

We aim at a fully-fledged model to give a general description of ordinary teachers' implementation of CAS and through this, an insight in the scaling-up question mentioned previously. Our first step is to analyze a rather extensive material of reports on teaching designs with CAS, succeeded by reflections on further development, modification and refinement of our model. These reports are produced by project participants at Center for Computer based Mathematics Education (CMU[3]), University of Copenhagen, as the last step of a reflective practitioner process. The mission of CMU is to support use of CAS in Danish high schools respecting core mathematics qualities in order to reverse the trivialization tendencies described above. The only condition for participating is a moral subscription to this mission. Thus, teachers have been free to choose subject, CAS platform (within CMU's coaching expertise), design of teaching, etc. This first round of analysis data is collected in contemplation of dissemination, rather than evidencing answers to research questions, but in a systematic way that allows for a grounded theory approach.[4]

Having an elaborated model, we intend a large-scale investigation on Danish mathematics high school teachers' choices and rationale for outsourcing to CAS.

A sketch of an analysis: a praxeology of finding derivatives

The so-called 3-steps method of finding the derivative, $f'(x_0)$, of a function is the canonical approach to differentiation in Danish high schools, explicitly mentioned in official guidelines. We recall: (1) With $\Delta y = f(x_0 + \Delta x) - f(x_0)$ form the fraction $\frac{\Delta y}{\Delta x}$; (2) reduce the fraction to make $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ accessible (3) find the limit $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (if it exists). These are tasks in three praxeologies with rather separate theory blocks involving algebra, topology and geometry. In a CMU – project (Differentialregning, tretrinsreglen), the teacher, in the sequel L, wants to improve on students' understanding of the method by CAS-outsourcing "to give the students hand-on experience of secant and tangent slope and limits through experimentation with CAS sheets" (our transl.). An outsourcing strategy like this is rather common in Denmark. L has worked on the teaching problematics of the 3-steps method for many years 'without really understanding why students find it so difficult' (our

[3] The Danish Industry Foundation, Department of Mathematical Sciences at University of Copenhagen, The Danish Ministry of Education, and Maplesoft Inc. sponsor CMU.

[4] For further details about the CMU material, we refer to the CERME 10 poster of TWG 15 (Bang, Grønbæk, & Larsen, 2017)

translation from Danish taken from the project report). This time L starts with a thorough analysis of prerequisites ending in 12 points. L decides to use CAS in the case of $f(x) = x^2$ on three points of the 12: '(5) computing slope of a straight line; (7) understanding what tangent and slope are; (12) understanding (the concept of) limit' (our transl.). A few observations: (A) L is by his very wish to understand reasons for learning difficulties led to in- and outsourcing considerations. (B) There is a tendency to regard pragmatic and epistemic values as separate features: (5) is pragmatic and (7) & (12) are (by L phrased as) aiming at epistemic value. (C) Some core activities are recognized as such and insourced, i.e. algebraic reduction of polynomial expressions such as Δy for $y = a\,x^2 + b\,x + c$ - partly insourced to paper and pencil, partly to CAS. Other core activities are outsourced, i.e. use of sliders on the graph $y = x + 2$ to find $\lim_{x \to 0} x + 2$, as last step in the 3-step method with x replacing Δx; (D) non-core activities are not spelled out. What is it that sliders can do for secant-tangent considerations without sacrificing core activities? (E) Affective aspects are outsourced: 'CAS tools should … activate students and challenge their desire to … explore mathematical problems' (our transl.) From L's reflections, it appears that the outsourcing (D) is indirectly motivated by the textbook treatment of the subject. Textbooks rarely have core activity considerations, but rather bold instigations to CAS use. This risk of dilution of mathematical competency is pinpointed by the concept 'outsourcing core activities'.

Further use and development

L is an example of a teacher with neither desire nor reputation to be a front-runner, but navigating resourcefully and dedicatedly under post-modern circumstances of mathematics teaching. Our observations (A), … (E) apply to many teachers (CMU, 2015; CMU, 2016) so the sketch of an out-/in-source analysis of L's project is testimony that our approach may have potential for shedding light of the kind of decisions, with shortcomings and potentials, that ordinary teachers make. The business metaphor seems confluent with natural praxis of resource considerations, thus providing a framework for large-scale investigations much similar to studies of business economics forces that govern trade and production. One may fear a risk of introducing yet another business corporation model to education. Outsourcing is growing in business due to the incitement of fierce competition. While perhaps tempting, a flat educational interpretation of this is misleading. The rooting in a business model is motivated within a local or regional mathematical organization through its level of didactic co-determination. Even though mathematics may be seen as a productive force, learning outcome is not a commodity. It cannot be bettered simply through optimization tactics.

Our sketch has focused on director decisions, i.e. the teacher's planning. Further development must include employees, i.e. students, that is, the last step of the didactical transposition: matter taught → matter learned.

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The development of pre-service teachers' TPACK for the use of digital tools

Nimer Baya'a[1], Wajeeh Daher[1,2], Rawan Anabousy[1] and Ahlam Anabousy[1]

[1]Al-Qasemi Academic College of Education, Baqa-El-Gharbia, Israel; bayaan@qsm.ac.il

[2]An-Najah National University, Nablus, Palestine

The ministry of education is launching a national project to implement the use of ICT in the Israeli education system. To prepare pre-service teachers with whom we work for this kind of implementation, we designed a model, which supports them to learn to use digital tools effectively while integrating a particular pedagogy for teaching a specific mathematics or science content. The goal of the present research is to study the development of these pre-service teachers' Technological, Pedagogical and Content Knowledge (TPACK), attitudes toward computers and their ICT proficiency. For this purpose, we used and adapted questionnaires from different sources. The research results show significant improvement in the TPACK level and ICT proficiency, but no significant effect of the preparation on most of the components of the teachers' attitudes toward computers, being positively high before and after the preparation.

Keywords: Pre-service teachers, TPACK, digital tools, professional development.

Introduction

Shulman (1986) suggested the PCK (pedagogical content knowledge) model to represent the interaction of two types of teachers' knowledge: content knowledge and pedagogical knowledge. He proposed that this interaction be considered in order to understand teachers' expertise in teaching a particular subject matter. Various researchers (for example Koehler and Mishra, 2009; Niess et al., 2009), built on Shulman's PCK to describe the interaction of teachers' understanding of educational technologies with their PCK that results in effective teaching with technology. Specifically, they talked about the technological pedagogical and content knowledge of teachers (TPACK), where this model describes the interactions between and among the three main components of teachers' knowledge: content, pedagogy, and technology. These interactions result in new types of teachers' knowledge, namely PCK, TCK (technological content knowledge), TPK (technological pedagogical knowledge), and TPACK. In this paper, we describe the development of pre-service teachers' TPACK as a result of preparing them in the use of digital tools over one academic year.

Technological, Pedagogical and Content Knowledge

Though some researchers consider TPACK too blunt an instrument (e.g., Clark-Wilson & Hoyles, 2016; Thomas & Palmer, 2014), other researchers refer to it when studying mathematics teacher's professional development (e.g., Balgalmis, Shafer, & Cakiroglu, 2013; Bowers & Stephens, 2011). Generally speaking, TPACK is the knowledge of how to integrate technology in teaching the subject matter. This knowledge also includes the appropriation between a specific technological tool, the teaching of a specific topic and being aware of the difference between various technological tools in teaching a specific topic. Further, this knowledge means being aware of students' problems of the subject matter that could be overcome by using specific technological tools. It also includes the

awareness of students' difficulties of the subject matter that result from using specific technological tools and how to overcome these problems (Koehler & Mishra, 2009).

Robova, and Vondrova (2015) studied mathematics teachers' awareness of the specific technological skills needed for their teaching (making functions visible on the screen, changing visual appearance of graphs, interpreting numerical results, using dynamic features of a tool) and their ability to design teaching which takes the specific skills into account. Furthermore, Koh and Divaharan (2011) described an instructional model for developing pre-service teachers' TPACK. We follow the previous attempts to suggest a preparation model for developing pre-service teachers' TPACK in utilizing digital tools in their teaching.

Pre-service teachers' attitudes toward computers

Fishbein (1967) defined attitude as a learned tendency to respond to an object in a consistently favorable or unfavorable way. Other researchers (Zan & Di Martino, 2007) defined attitude in terms of emotions: a positive or negative emotional reaction toward a specific situation. These definitions show the possible influence of attitudes on behavior in general and on pre-service teachers' behavior in particular. Attention to attitudes has risen when ICT started to emerge as a possible tool for the improvement of teaching and learning. In this context, researchers found that these attitudes have major influence on the success and meaningful use of the ICT in their teaching (Albirini, 2006).

In our research, attention was given to pre-service teachers' attitudes toward computers, together with the development of their TPACK and ICT proficiency, as a consequence of their preparation in the use of digital tools. We used the 'teacher's attitudes toward computers' questionnaire (TAC) as it probes teachers' attitudes toward ICT use in teaching and their intention to do so (Baya'a & Daher, 2013). We were also interested in the pre-service teachers' proficiency level in ICT as an indicator of their intention to use ICT in their teaching as the proficiency variable is reported to affect teachers' readiness to use ICT in their teaching (Granger, Morbey, Owston & Wideman, 2002).

The research questions

The main research question is: How will the preparation of pre-service teachers in the use of digital tools, according to the model that we designed, affect their TPACK level, ICT proficiency and their attitudes toward computers?

Research context, participants and the preparation model

This current research accompanies the preparation of pre-service teachers to study how to use effectively digital tools in the mathematics or science classroom. This knowledge is the core of the TPACK model. We administered questionnaires to measure the advancement of the TPACK levels and attitudes toward computers of the pre-service teachers who implemented the model, as well as their ICT proficiency. Approximately 55 students majoring in mathematics and science teaching in intermediate schools completed the questionnaires at the beginning and end of the preparation. These students were in their third year of training alongside two courses that provided a background in the use of ICT for teaching mathematics and science.

The preparation model aimed to improve the pre-service teachers' selection of a suitable digital tool for a specific pedagogy and subject. It also tried to improve the integration of digital tools to teach some specific content. This preparation model concentrated on two aspects. First, knowing the tool technically and being able to adapt it to teach some specific content. Second, developing the ability to select and integrate suitable digital tools for some specific content and pedagogical method. In more detail, each pre-service teacher worked independently to learn to use at least two digital tools and to prepare user guides (as PDF file or digital book) that included descriptions of the most significant functions of these digital tools. Furthermore, the pre-service teachers were required to record video clips of screen shots while performing operations in these digital tools as explanations for another user. The pre-service teachers were asked to select the digital tools from a catalog of general digital tools prepared by the ministry of education in Israel. This catalog includes various digital tools that could be adapted for use in various subjects and levels, such as: Flipsnack for creating online digital books, Linoit for creating collaborative bulletin board, Socrative for personal and class assessment and Mindomo for creating mind maps.

Moreover, each pre-service teacher was required to prepare pedagogical materials on how to use the digital tools that she was engaged with in teaching mathematics or science, and then present the materials in the training workshop to receive comments from her peers and the pedagogical supervisor. Following that, the pre-service teacher reflected on her developed materials, adjusted it and uploaded all the materials to an internet site that was constructed by the pre-service teachers and the pedagogical supervisors. This internet site constituted a data bank for digital tools. In addition, each pre-service teacher was requested to prepare at least two lessons for teaching mathematics or science and pick three digital tools from the catalog (including one that she was engaged with) to use them in her teaching. These lessons had to involve also collaborative investigations that encourage the use of higher order thinking skills. Finally, each pre-service teacher picked a subject from within a digital textbook for teaching mathematics or science, and added connections to pedagogical activities based on using digital tools from the data bank site.

During the first semester, the pre-service teachers had two options: to start from the digital tool and integrate it for teaching some specific content, or starting from the content and selecting a suitable digital tool to teach that content. In the second semester, each pre-service teacher was asked to experiment with the prepared materials and lessons in her training school with at least one of the chosen tools, and reflect on the experience. This reflection was on the actual implementation of the digital tool in the classroom environment, and it was posted in the data bank for digital tools for other pre-service teachers to consider as they selected digital tools for their own use.

Research instruments

The research instruments included three questionnaires as follows: Questionnaire 1: Technological, Pedagogical, and Content Knowledge (TPACK) (revised) questionnaire, constructed on the basis of the TPACK instrument for pre-service teachers developed by Schmidt et al. (2009).

Questionnaire 2: Teachers' Attitudes toward Computers (TAC, v. 6.1) questionnaire: This questionnaire was tested by Christensen and Knezek (2009) who concluded that it is a well-validated and reliable instrument for teachers' self-appraisal of their attitudes toward computers.

Questionnaire 3: The Use of ICT in Colleges of Education (UICT): This questionnaire was developed by The MOFET Institute (A Center for the Research, Curriculum and Program Development in Teacher Education in Israel) to track the professional development of pre-service teachers' use of ICT. We used the ICT proficiency part of the questionnaire.

The validity of the questionnaires was considered by giving the Arabic translations to a group of pre-service teachers who were requested to examine if the questionnaires' statements were clear to the reader. As a result, some items of the questionnaires were rephrased to clarify their meaning.

The pre-service teachers' scores in the overall constructs and their categories, before the preparation and after it, were examined for internal reliability using Cronbach alpha. The results showed high Cronbach alpha (above 0.85 for all the categories and for the overall construct) indicating adequate internal reliability for the questionnaires and their categories. These results were expected due to the extensive use of these questionnaires in the literature.

Data processing

Data was analysed using paired-samples t-test to determine if there were significant differences between scores of pre-service teachers in the various questionnaires before and after the preparation. Cohen's d (the ratio between the difference of the means and the average of the standard deviations) (Cohen, 1969) was used to compute effect sizes to assess the practical significance of results.

Results

Pre-service teachers' ICT proficiency

Table 1 shows the proficiency level of the pre-service teachers before and after the preparation (values between 1 to 5), as well as paired sample t-test between the two observations.

	Before Preparation		After Preparation			
Outcome	M	SD	M	SD	*t*	*d*
Score of ICT proficiency in UICT	3.80	0.56	4.20	0.59	4.17***	0.70

*** $p < 0.001$

Table 1: Means, standard deviations and t-test for pre-service teachers' ICT proficiency level (n=54)

As displayed in Table 1, the results show that the pre-service teachers' ICT proficiency level differs significantly before and after the preparation. Large positive effect size of 0.70 was derived for the preparation on the pre-service teachers' ICT proficiency level. This advancement was mainly the result of the major improvement in their 'multimedia tools proficiency'.

Pre-service teachers' TPACK level

The TPACK level comprised the total score of the TPACK questionnaire and six other scores for each partial type of knowledge for technology, pedagogy, content and intersections between them. Table 2 shows the TPACK components' scores of the pre-service teachers before and after the preparation (values between 1 to 5), as well as paired sample t-test between the two observations.

	Before Preparation		After Preparation			
Outcome	M	SD	M	SD	*t*	*d*
TPACK	3.93	0.53	4.50	0.46	8.19***	1.15
TK	3.90	0.63	4.44	0.59	6.31***	0.89
PK	4.04	0.52	4.57	0.46	6.35***	1.08
PCK	3.81	0.57	4.57	0.49	8.49***	1.43
TCK	3.78	0.76	4.45	0.52	7.02***	1.05
TPK	4.02	0.67	4.50	0.63	4.85***	0.74
TPCK	3.88	0.78	4.51	0.68	5.19***	0.86

*** $p < 0.001$

Table 2: Means, standard deviations and t-test for pre-service teachers' TPACK level (n=54)

As displayed in Table 2, the pre-service teachers' scores in the components of TPACK differ significantly before the preparation and after it. Large positive effect sizes of 0.74 and more were derived for the preparation on the pre-service teachers' TPACK and its components.

Pre-service teachers' attitudes toward computers

Attitudes toward computers were assessed using 9 categories. Table 3 shows components' scores of the pre-service teachers' attitudes toward computers before and after the preparation (values between 1 to 5, except perception 1 to 7), as well as paired sample t-test between the two observations.

	Before Preparation		After Preparation			
Outcome	M	SD	M	SD	*t*	*d*
TAC General	3.86	0.47	3.99	0.53	2.09*	0.26
Interest	4.37	0.64	4.32	0.74	-0.42	-0.07
Comfort	4.11	0.99	4.17	1.11	0.38	0.05
Accommodation	4.65	0.59	4.38	1.04	-1.86	-0.33
Interaction	3.82	0.90	4.19	0.87	2.53*	0.41
Concern	2.54	0.69	2.82	0.85	2.55*	0.36
Utility	4.29	0.51	4.39	0.74	1.01	0.16
Absorption	3.45	0.91	3.82	1.07	2.28*	0.37
Significance	4.18	0.67	4.27	0.78	0.88	0.12
Perception	5.60	1.06	5.79	1.19	1.25	0.16

* $p < 0 .05$

Table 3: Means, standard deviations and t-test for pre-service teachers' TAC level (n=54)

As displayed in Table 3, the pre-service teachers' scores in the categories of attitudes toward computers differed significantly before the preparation and after it in the categories: interaction, concern, absorption and TAC general. In these constructs, a small effect size of 0.26 was derived for the preparation on the pre-service teachers' general TAC score, and moderate effect sizes of 0.41,

0.36 and 0.37 were derived for the preparation on the interaction, concern and absorption respectively.

Discussion and conclusions

The research aimed to examine how the preparation course affected the pre-service teachers' ICT proficiency, TPACK level and their attitudes toward computers. The research results indicated several significant positive effects of the preparation model used in that preparation that related to the pre-service teachers' abilities and knowledge regarding the integration of digital tools in teaching.

Pre-service teachers' ICT proficiency

The research results indicated significant improvement in the pre-service teachers' ICT proficiency as a consequence of the preparation, especially in multimedia tools proficiency. The mathematics and science pre-service teachers usually have high ICT proficiency, but the requirements in the preparation model led to significant improvement particularly in their multimedia proficiency. These results are due to a consideration of the technology knowledge related to the digital tools in the preparation process. This resulted in the pre-service teachers increased competence in their use of digital tools for personal and professional purposes, which caused them to feel confident to utilize new digital tools independently and individually (Prestridge, 2012), and thus improved significantly their ICT proficiency. This suggests that pre-service teachers need to be given the opportunities to work with technological tools in order to improve their ICT proficiency and their readiness to integrate ICT in their teaching (Muir-Herzig, 2004).

Pre-service teachers' TPACK level

As a result of the preparation, the general TPACK level of the pre-service teachers, as well as its six partial types, were significantly improved. These results could be due to the attention of the pre-preparation model to the ability of the pre-service teachers to appropriate the digital tools pedagogically to teaching a specific content, and vice versa. It could be said that the pre-service teachers' diverse experiences in the workshop improved their knowledge in different types of knowledge related to their teaching mathematics or science. Thus, the preparation model provided the pre-service teachers with opportunities to maintain and shift their instructional approaches enriched with innovative educational technologies (Martin, 2015). This preparation model could be implemented worldwide when taking into consideration the particular background and conditions of the pre-service teachers involved.

Pre-service teachers' attitudes toward computers

The results of this research show that following the preparation process, no significant improvement was detected in the pre-service teachers' attitudes toward computers for most of the TAC components, with exception of TAC general, interaction, concern and absorption. We should note that in both cases, before and after the preparation, the attitudes were very favorable toward computers.

As for the positive change in some attitudinal categories, such as absorption, the pre-service teachers had, during the workshop, the chance to be actually involved and improve their knowledge in computers and ICT. This might have improved their ability to solve problems related to the

computer use in the classroom; which encouraged them to insist to solve these problems, even the hard ones. This influence of teachers' experience in technology on their ability to solve technological problems is supported by DeLuca (1991) who claims that technological knowledge overcomes technological problems in the classroom. This could improve pre-service teachers' attitudes toward computers.

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Beyond a positive stance: Integrating technology is demanding on teachers' mathematical knowledge for teaching

Nicola Bretscher

UCL Institute of Education, London, UK; n.bretscher@ucl.ac.uk

Research on technology in mathematics education highlights the importance of teachers having a positive stance towards technology for successful integration into classroom practice. However, such research has paid relatively little attention to teachers' knowledge of specific mathematical concepts in relation to technology. This paper examines the innovative use of technology by a teacher, Robert, as a critical case study, to argue that the significance of mathematical knowledge for teaching using technology should not be overlooked nor underestimated.

Keywords: Computer uses in (secondary mathematics) education, knowledge base in teaching, TPACK, situated abstraction.

Introduction

Seeking to understand teachers' integration of technology, research on technology in mathematics education (e.g. Zbiek et al., 2007) has documented the important role teachers' beliefs and conceptions play in their integration of technology into classroom practice. For example, Zbiek et al (2007) identify the constructs of pedagogical fidelity and privileging as useful in understanding the extent and nature of technology integration in a teacher's classroom practice. Pedagogical fidelity is described as the degree to which teachers' beliefs about the way a digital technology allows students to act mathematically coincides with their beliefs about the nature of mathematical learning (Zbiek et al., 2007). Privileging is a notion developed by Kendal and Stacey (2001) to describe how teachers, consciously or unconsciously, frequently use or place a priority on certain things in their practice, for example, types of representation, skills or concepts and by-hand or by-technology methods (Zbiek et al., 2007). Both these constructs relate to teachers' conceptions of mathematics as a discipline (Thompson, 1992), their beliefs about the nature of teaching and learning mathematics and how these interact with their beliefs about technology.

Such studies have in common a focus on teachers' global conceptions of mathematics as a discipline and on teachers' beliefs about the nature of teaching and learning mathematics with technology. They do not tend to focus on teachers' knowledge of specific mathematical concepts in relation to technology. This is an important omission since the documented shifts in teachers' views suggest a move towards models of teaching aimed at developing conceptual understanding. Such models may require a great deal of knowledge for successful implementation and inconsistencies between teachers' professed beliefs and practices may be the result of lacking sufficient knowledge and skills necessary to implement them (Thompson, 1992).

Whilst highlighting the role of teachers' conceptions in technology integration is important, this paper argues that the significance of mathematical knowledge for teaching using technology should not be overlooked nor underestimated. For example, Bowers and Stephens (2011, p. 290) assert that the set of (teachable) knowledge and skills for teaching mathematics using technology may be empty, emphasising instead that teacher educators should seek to nurture a favourable conception of

"technology as a critical tool for identifying mathematical relationships". Whilst it may be that teacher educators should seek to nurture favourable conceptions towards using ICT in their trainees, this paper argues the knowledge required to put such conceptions into practice should not be neglected.

Theoretical framework

The central Technology, Pedagogy and Content Knowledge (TPACK) construct of Mishra and Koehler's (2006) framework is useful in highlighting mathematical knowledge for teaching using technology, by emphasising technology as a knowledge domain alongside pedagogy and content knowledge (Bretscher, 2015). Whilst space does not allow for a full description of the framework, the central TPACK construct serves to highlight the situated nature of such knowledge. In particular, in this paper, mathematical knowledge for teaching using technology is viewed as a *situated abstraction* (Noss & Hoyles, 1996), that is, 'abstract' mathematical knowledge simultaneously situated in the context of teaching with technology.

Borrowing from Shulman (1986), mathematical knowledge for teaching using technology is assumed not only to be a matter of knowing how – being competent in teaching mathematics using technology - but also of knowing what and why. That is, although much of teachers' knowledge may be tacit, craft knowledge (Ruthven, 2007), at least some of their know-how is underpinned by articulated knowledge that provides for "a rational, reasoned approach to decision-making" (Rowland et al., 2005, p.260) in relation to teaching mathematics using technology. In other words, mathematical knowledge for teaching using technology, as defined in this study, is when know-how or knowledge-in-action is underpinned by and coincides with the teacher's articulated knowledge. This intersection between articulated knowledge and knowledge-in-action is important because it is this type of knowledge that initial or in-service teacher education programmes focus on developing.

Method: Robert as a critical case

Four teachers were selected from a group of English mathematics teachers who took part in a survey of secondary school mathematics teachers' use of ICT (n=183) and who further agreed to be contacted as case study teachers (Bretscher, 2011; 2014). The four case study teachers were chosen along two dimensions of variation likely to be associated with mathematical knowledge for teaching using technology, based on their responses to survey items. Firstly, the case study teachers were chosen to be two of the most student-centred and two of the most teacher-centred in their approach to mathematics teaching in general (not limited to ICT use) of those who volunteered. Secondly, two teachers were chosen to be from schools with a high level of support for ICT and two with a low level of ICT support. In addition, the four case study teachers had described themselves as being confident with ICT. As technology enthusiasts, the case study teachers were likely to display mathematical knowledge for teaching using technology; the variation in case selection aimed to highlight such knowledge – making it more 'visible'.

Each case study teacher was observed teaching one lesson in a computer suite where pupils were given direct access to ICT. These observations provided opportunities to infer the case study teachers' knowledge-in-action in a situation involving the work of teaching mathematics with technology. Post-observation interviews then provided an opportunity to infer the case study

teachers' articulated knowledge and hence, triangulated against their knowledge-in-action observed in the lesson, provide evidence indicating mathematical knowledge for teaching using technology.

Robert was selected as a case study teacher because he was one of the most student-centred teachers in the survey sample. In addition, his school appeared to be generally supportive of ICT use compared to the other schools surveyed. He stood out, even amongst the case study teachers, as being a critical case of a teacher likely to display mathematical knowledge for teaching using technology for two main reasons. Firstly, Robert showed a favourable conception of technology, as described in the following section, in relation to mathematics teaching and in line with Bowers and Stephen's (2011) description of viewing "technology as a critical tool for identifying mathematical relationships". Secondly, Robert's lesson appeared to be exceptional: he used GeoGebra software to affect his pupils' learning in an innovative way that would not be easy to achieve without digital technology, in comparison to the other lessons observed where software was used to replicate and enhance paper-and-pencil activities. He had 4-6 years of teaching experience, held a management position within the mathematics department and had completed a Masters in Education degree. Robert was also the most technologically proficient of the four case study teachers: his undergraduate degree was a Bachelor of Engineering in Computing.

Analysis and discussion

Robert's favourable conception of technology use in mathematics teaching

For the first part of his lesson, Robert had created a series of maze activities, embedded in GeoGebra files, designed to take advantage of his 12-13 year old pupils' tacit understandings of reflection as a means of making them explicit and thus leading towards a more formal understanding of reflection. Using the mouse to direct the movement of a point, coloured in blue, the pupils had to guide the blue point's reflection, shown in red, successfully through a maze (see Figure 1).

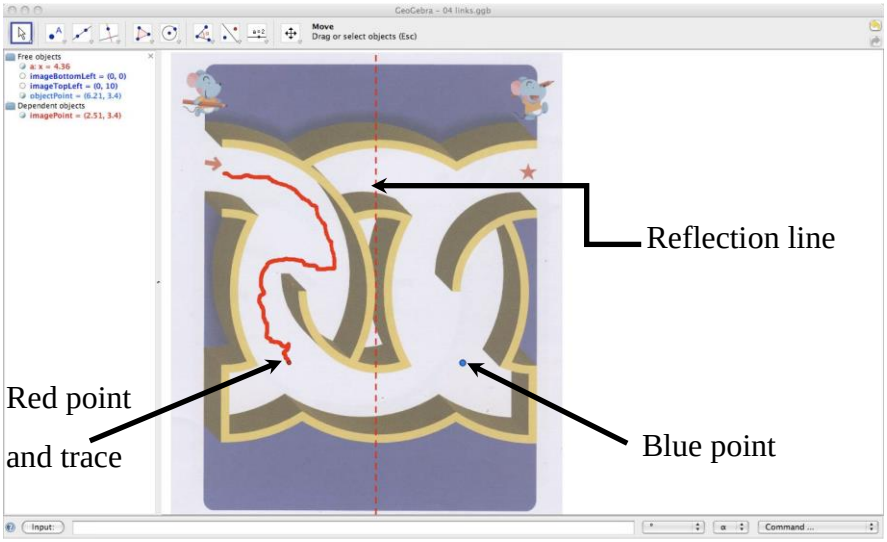

Figure 1: One of Robert's GeoGebra maze activities - by dragging the blue point, guide the reflected red point through the maze

The reflection line was super-imposed on the maze diagram and the path of the red point was traced. Robert hoped that the activity would encourage pupils to predict how the reflected red point would move in relation to movement of the blue point as a means of increasing their chances of

completing the maze successfully. By predicting the movement of the red and blue points, he hoped his pupils intuitive understandings of reflection would be made more explicit.

In the post observation interview, Robert explained what inspired him to create the maze activities. He provided a critique of similar GeoGebra activities as lacking an impetus to focus attention on and articulate tacit understandings:

> Robert: I had a look on the GeoGebra wiki and most things tended to be 'Here's a mirror line, here's a shape, if you drag this, what's happening?' just kind of ... and say what you see. And I could imagine them sitting there with that and basically just dragging the mouse a bit and seeing it happen and ... and then where does it go from there?

He also described a pedagogic strategy of predict-then-test that he aimed to use in the lesson to make pupils' understandings of mathematical relationships explicit:

> Robert: just you know introduce that 'pause' of what do we think is going to happen and then let's test that it's going to happen

and how he intended to formalise these understandings during the lesson by introducing mathematical vocabulary:

> Robert: So one of the things I wanted to talk about was that if you're moving that point parallel to the mirror line, the point moves in the same direction, whereas as soon as you're moving it in a direction that's not parallel, the point doesn't move in the same way.

Summarising at the end of the lesson, he did introduce mathematical vocabulary during class discussion, in a similar way to the intention described above, describing the movement of the red and blue points. Thus Robert's design of the maze activities, his use of them in the lesson and his comments about the lesson in the post-observation interview demonstrate the strong emphasis he placed on the use of technology to explore the mathematical relations behind the mathematical phenomenon of reflection, consistent with Bowers and Stephens' (2011) description of a favourable conception of technology.

Robert's mathematical knowledge for teaching using technology

Using the series of maze activities successfully to meet the aims of the lesson depended on transforming students' strategies for completing the mazes into more formal understandings of reflection that could be used as strategies for constructing the image given an object and line of reflection. As indicated above in excerpts from the post-observation interview, Robert recognised his interventions with individual pupils and directing whole class discussion as being critical to effecting this transformation.

The maze activities potentially addressed two complementary strategies for using geometric properties to construct the image given the object and line of reflection: 1) using the local geometry of the object together with the properties of reflection, namely, preservation of length and of direction parallel to the line of reflection and reversal of direction in the axis perpendicular to the line of reflection, to construct the image; and 2) using the geometric property that the line of

reflection is the perpendicular bisector of line segments connecting corresponding points on the object and image.

The first strategy was addressed through the maze activities by the necessity of considering how to drag the blue point, i.e. in what direction and how far, to guide the reflected red point through the maze. In particular, the main challenge in completing the maze is derived from the reversal of direction caused by the reflection. Less obvious perhaps is that length is preserved: dragging the blue point causes the red point to move the same distance. The second strategy was addressed in later maze activities by the addition of the line segment connecting the blue and red points as a possible aid to maze completion.

Robert was not satisfied with his interventions during the lesson. In the post-observation interview, he pointed to technical difficulties, his desire to let the students enjoy the maze activities and his rush to move onto the second activity as contributing to the result that he did not spend as much time as intended on discussing the geometric implications of the pupils' maze-solving strategies. Timing was certainly a factor and technical difficulties meant that he was unable to direct a whole class discussion juxtaposing the identical mazes with and without the line segment joining the red and blue points. As a result, Robert was unable to address the second strategy outlined above involving recognition of the line of reflection as the perpendicular bisector of the line segment joining the red and blue points. However, he did have two opportunities during the lesson to elicit the geometric properties of reflection that underpin the first strategy through whole class discussion.

The first opportunity came when Robert brought the class back together after some time engaging with the maze activities. He displayed one of the early maze activities with a vertical line of reflection and asked pupils to give instructions to a pupil-volunteer to direct their movement of the blue point. Robert summarised their responses, drawing attention to the relative direction of movement of the red and blue points i.e. that when the blue point was dragged up or down the red point moved in the same way but that dragging the blue point left or right caused the red point to move in the opposite direction. Whilst drawing their attention to the direction of movement, Robert did not mention that dragging the blue point causes the red point to move the same distance, thus he did not draw his pupils' attention to the geometric property that length is preserved under reflection.

Robert then displayed a maze with a horizontal line of reflection and, employing his predict-then-test strategy, asked the pupils to predict whether the relative direction of movement would be the same or different. The pupils correctly predicted it would change: now, dragging the blue point left or right would result in the red point moving in the same way but dragging the blue point up or down would cause the red point to move in the opposite direction. Contrasting these diagrams made the point that the relative direction of movement of the red and blue points was connected to the orientation of the line of reflection. At this juncture, Robert could have introduced the mathematical terms parallel and perpendicular to specify the nature of the connection between the relative direction of movement and the orientation of the line of reflection, thus generalising to state the effect of reflection on direction. He could also have noted that in both maze diagrams, independent of the orientation of the line of reflection, dragging the blue point causes the red point to move the same distance, hence length is preserved under reflection.

Robert did not introduce the mathematical terms parallel and perpendicular at this point nor did he note the geometric property that length is preserved under reflection. Instead, apparently on impulse, he offered his pupils a new challenge: to find out whether turning the mouse back to front would help them to complete the mazes, presumably by double-reversing the direction of movement. This challenge risked distracting from the aims of the lesson, since turning the mouse back to front involves a rotation of 180 degrees and not a reflection. Later in the post-observation interview, Robert dismissed it as "just a silly question to get a few of them thinking". However, in asking this question, he missed an opportunity to capitalise on his pupils' correct predictions to generalise their maze-solving strategies towards a shared, formal understanding of the geometric properties of reflection. In particular, Robert's challenge highlights the situated nature of mathematical knowledge for teaching using technology in terms of weighing up the pedagogical value of interpreting how the mouse movement relates (or not) to the geometric properties of reflection.

The second opportunity occurred at the end of the lesson. Due to the shutdown of the computer system, the students were unable to begin the second GeoGebra activity Robert had prepared. After spending some time wrestling with the technology, Robert gave up and gathered the pupils to summarise the lesson. In this moment of contingency, Robert was inspired to ask his pupils to imagine the join between two rectangular tables, where they met along their longest edge, was a mirror. One of the pupils sitting at the table was holding a ball: this became the de facto 'blue point'. Robert discussed moving the 'blue point' close to the mirror, through the mirror (which he noted you can't do in reality), and finally parallel to the mirror. He did not have another chance to discuss what happens when the 'blue point' moves perpendicular to the mirror nor to discuss the preservation of length under reflection because, at that point, the bell rang for the next lesson.

Although his second opportunity to elicit the geometric properties of reflection was cut short, in the post-observation interview, when asked what he wished to do had there been more time, Robert did not articulate that he meant to discuss what happened when the blue point moved perpendicular to the line of reflection and to note that distances remained the same under reflection. These missed opportunities, together with the post-observation interview, suggest that Robert had not planned precisely what and how he would use mathematical terminology in his interventions to support his pupils' interpretation of controlling the red and blue points via the mouse, thereby transforming his pupils' maze-solving strategies into more formal understandings of reflection to connect with the aims of the lesson. In addition, when asked what he would have done differently in preparing the lesson, he focused solely on planning to prevent the technical difficulties arising rather than suggesting he could have been more precise in his use of mathematical terminology. Although Robert did not have much time to deliberate over the lesson (as the author has) and it is understandable that the technical difficulties that were so disruptive were uppermost in his mind, this suggests his experience during the lesson did not prompt Robert to recognise the need to plan his interventions more precisely to connect his series of maze activities with the mathematical aims of the lesson. In particular, Robert appeared to lack a frame of reference to help him identify what his mathematical difficulties were in using technology to make his pupils' tacit understandings explicit and, as a result, why his interventions appeared unsatisfactory. However, such a frame of reference can be seen as part of mathematical knowledge for teaching using technology, since in this study such knowledge is assumed not only to be a matter of knowing how – being competent in teaching mathematics using technology - but also of knowing what and why (Shulman, 1986, p.13).

Conclusion

Despite his favourable conception of technology, using the maze activities in practice was not trivial and Robert did not entirely succeed in making explicit the mathematical relationships the pupils were exploring using the GeoGebra software. His difficulties, in supporting his pupils' mathematical interpretation of controlling the red and blue points via the mouse to elicit the properties of reflection, appear at once *mathematical* and yet simultaneously *situated* in the context of teaching using technology. In particular, the strength of Robert's maze activities lay in the real difficulty of controlling the direction of movement of the reflected red point via the mouse. This difficulty focused attention on how the direction of movement changes under reflection, which Robert drew to his pupils' attention through his interventions, albeit without making use of precise mathematical terminology. However, dragging the blue point using the mouse results in the red point moving the same distance unproblematically. Thus the maze activities did not draw attention to preservation of length in the same way, underlining the need for teacher intervention to highlight this property of reflection. The strain placed on his mathematical knowledge for teaching using technology was most evident perhaps when Robert included a challenge relating to rotation, finding out what happens when the mouse is turned back to front, which distracted from his stated lesson aims regarding reflection. This challenge again highlights the situated nature of mathematical knowledge for teaching using technology in terms of weighing up the pedagogical value of interpreting how the mouse movement relates to the geometric properties of reflection.

This suggests that a positive stance towards technology, in terms of global aspects of teacher knowledge (e.g. Bowers & Stephens, 2011; Zbiek et al., 2007), may not be sufficient to ensure a teacher's use of technology enhances mathematical instruction. The missed opportunities to transform pupils' maze-solving strategies into more formal statements of the geometric properties of reflection, using precise mathematical terminology to make connections between the maze activities and the aims of the lesson, suggest that mathematical knowledge for teaching using technology has a significant role to play in successful technology integration. Thus, whilst highlighting the role of teachers' conceptions in technology integration is important, this paper has argued that the significance of mathematical knowledge for teaching using technology should not be overlooked nor underestimated.

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What influences in-service and student teachers use of MathCityMap?

Iwan Gurjanow[1], Matthias Ludwig[2] and Joerg Zender[3]

[1]Goethe Universität-Frankfurt am Main, Faculty of Mathematics and Computer Science, Germany; gurjanow@math.uni-frankfurt.de

[2]Goethe Universität-Frankfurt am Main, Faculty of Mathematics and Computer Science, Germany; ludwig@math.uni-frankfurt.de

[3]Goethe Universität-Frankfurt am Main, Faculty of Mathematics and Computer Science, Germany; zender@math.uni-frankfurt.de

At the Goethe University Frankfurt am Main a new digital tool was developed to easily create mathematics trails as a mathematical outdoor activity aimed at school education, called MathCityMap. Following the articles and studies of many others, the usage of a new tool is quite an issue for teachers. Following some teacher training activities offered by our team, we offer interim results of user behavior based on data from an online survey. Our results can be useful for the implementation other digital tools.

Keywords: Mathematics activity, handheld devices, computer uses in education, teacher education.

Introduction

In their publication "Learning Outside the Classroom" the English and Welsh Department of Education and Skills strongly recommended that more lessons should take place outside of the classroom. They listed many benefits "nurture creativity, develop skills, improve attitude to learning, stimulate and improve motivation" just to name a few. (DfES, 2006)

The advantage is quite obvious, going outdoors means to encounter real life objects. For mathematics education, it is possible to create authentic tasks such as: What is the height of a certain building? how many stones have been used to build that wall over there? how much water is in that pond? and so on. Tasks such as these immediately require many process competences such as problem solving, reasoning and proof, communication, connections and representations. In the early 1980s Blane and Clark proposed the idea to connect those kind of tasks to form a *mathematics trail*. This requires a map on which to find the tasks, a description of the tasks (both together is called the *trail guide*) and then you can start to walk around and solve mathematical problems. (Blane & Clark, 1984)

In short, a mathematics trail is a set of mathematical outdoor tasks in walking distance. To solve the tasks you normally will need tools like a measuring tape and so on, which should be listed in the trail guide.

Although mobile devices and computers are widely used in every aspect of our daily lives (especially among pupils), they play a small role in education (Chen & Kinshuk, 2005). Going on a mathematics trail could be greatly enhanced by the use of mobile devices, since they allow learning to occur in an authentic context and extend to real environments. At the Goethe University of Frankfurt am Main we started the MathCityMap Project (MCM) which combines traditional mathematics trails with the opportunities of new technologies. In 2013 the first ideas were made concrete

(Ludwig, Jesberg, Weiss, 2013), but it took until 2016 to finally launch an accompanying web portal and mobile application. These have been released mainly for teachers to use in class, but are openly available to anyone who wishes to use it.

In spring 2016 we started to promote mathematics trails in combination with MCM by providing in-service teacher training and student courses at the university. Although the feedback on the training and courses was highly positive, the real usage of the MCM tools falls short of our expectations. In this article, we investigate reasons for this phenomenon we have encountered.

Theoretical background

Challenges creating a mathematics trail

Many mathematical tasks today are contextualized and appear to be realistic. But are they authentic? Following the definition Vos (2011) has given, an object is authentic, if it is clearly not created for educational purposes. Consequently, it is not easy to find authentic tasks. The objects in the tasks of MCM can be described as real-life objects, however, the authenticity of the tasks depends on the creators. We provide assistance by offering training alongside best-practice examples.

Usually the creation process of a mathematics trail consists of designing appropriate tasks and the trail guide or trail booklet (Cross, 1997). On the one hand, creating the tasks can be challenging for teachers as studies have shown (Jones & Pepin, 2016). On the other hand, manually putting the tasks together into a trail guide which should also contain a map overview and a title page, may be time consuming.

Difficulties integrating new technologies into mathematics classes

Given the availability of new technology in schools, questions have always arisen such as, do teachers work with the new tools? how do they use them? and so on. Drijvers made a study in 2012 about the factors for successful use of new technology amongst teachers. One of the three important factors is the role of the teacher (Drijvers, 2012). In Germany, a majority of teachers report to have not enough time alongside their daily tasks at school (Schneider, 2015 p. 20). Consequently, the time a new tool needs to be set up is an important issue. The MathCityMap project tries to simplify the creation process of designing tasks and trails to make it less time consuming for teachers.

In addition, Kuntze, Siller and Vogl (2013) have shown that both pre-service and in-service teachers self-perception towards mathematical modelling is mainly negative. Especially the in-service teachers lack of knowledge about new technologies and modelling. They feel unprepared for modelling by their university education. Pre-service teachers on the other hand feel a lack of diagnostic pedagogic skills and feel unable to give good hints to the pupils. There is a difficulty to integrate modelling into classes, especially with new technologies.

GPS-based applications in mathematics education

Two examples of applications in mathematics education, that already successfully use mobile GPS-data, are Wijers, Jonker & Drijvers (2010), who developed a game which allows students to walk along the shape of geometric objects outside the school, and Sollervall and de la Iglesia, who have developed a GPS-based mobile application for embodiment of geometry (Sollervall & de la Iglesia, 2015)

The MathCityMap project

The intention of the MathCityMap (MCM) project is to automate many steps in the creation of the mathematics trail booklet/guide and to provide a collection of tasks and trails that can be freely used or just viewed to get inspiration for own tasks. Furthermore, it gives users (e.g. groups of pupils) the possibility to go on a mathematics trail more independently by using mobile devices' GPS functions to find the tasks location, by giving feedback on the users answer and by providing hints in the case that one got stuck at a particular task. The core of the MCM project can be divided into two parts, the MCM web portal and the MCM app.

MCM web portal - www.mathcitymap.eu

The web portal is a mathematics trail management system. After a short registration, the user can view public trails and tasks or create his own tasks and trails by typing in the necessary data (e.g. position, the task itself, the answer, an image of the object etc.) into a form (see Figure 1). For every mathematics trail, the mathematics trail booklet can be downloaded as PDF or accessed via the MCM App (see Figure 2). It contains all task information, a map overview and a title page.

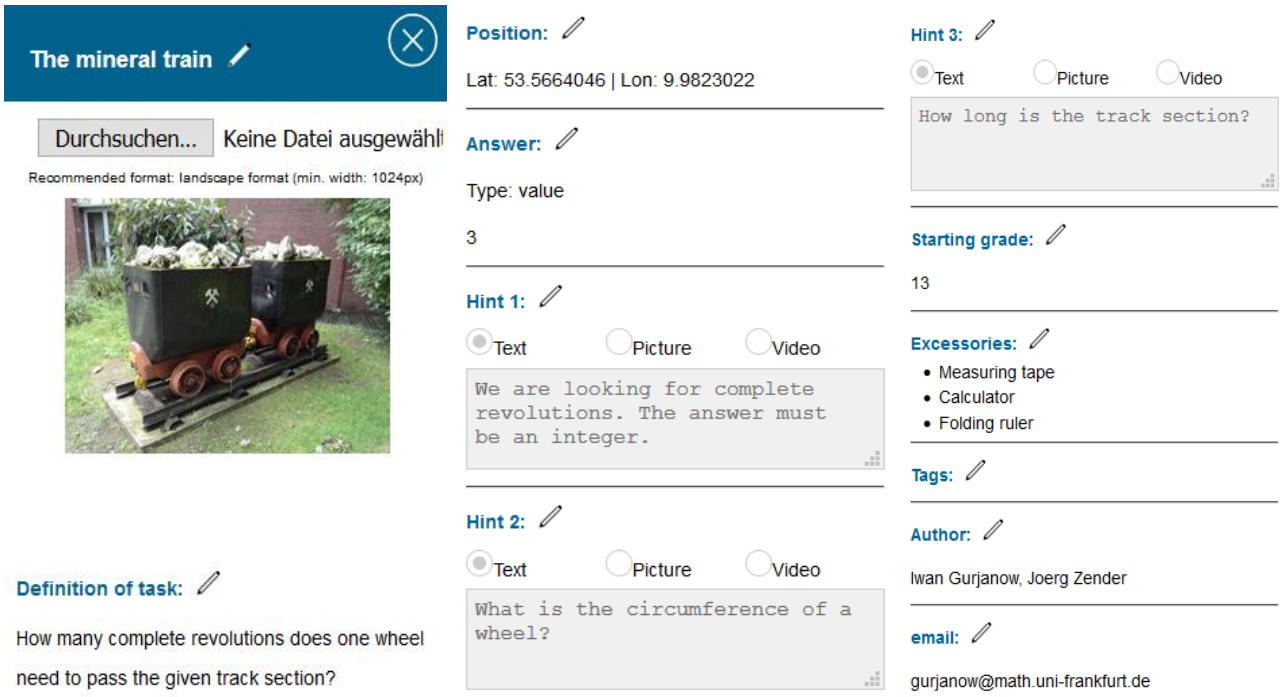


Figure 1: The MCM web portal form for tasks

MCM app for mobile devices

The MCM app allows the user to access mathematics trails created within the web portal. The trail data, such as images and map tiles, can be downloaded to the mobile device. After this procedure, it is possible to use a trail without an internet connection (see Figure 2). This design decision minimizes technical issues when using the app without mobile internet or in an area with low connectivity. Furthermore, the app offers an open street map overview for orientation purposes, feedback on the entered answers and a stepped hint system. The hint system enables pupils to solve the tasks independently and additionally has a positive impact on learning performance, learning experience and communication (Franke-Braun, Schmidt-Weigand, Stäudel, & Wodzinski, 2008).

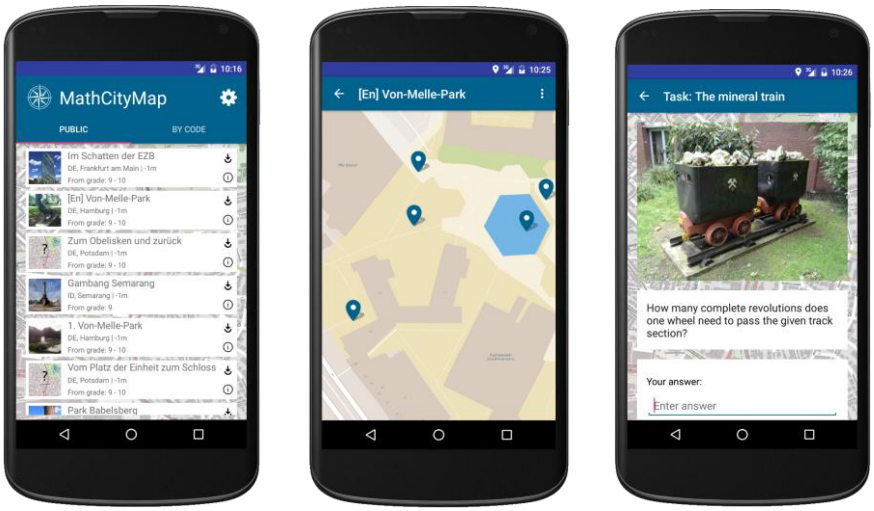


Figure 2: Screenshots of the MCM App

To describe the pedagogic functionality of MCM, we use the model by Drijvers, Boon and Van Reeuwijk (2010). It divides digital technologies into three groups of didactical functionalities: (a) do mathematics, (b) practice skills, (c) develop concepts. MCM offers mathematical tasks at real life objects where the user mainly can practice his skills.

Research question

Following the teacher training events, we had expected more teachers to become active by creating own mathematics trails with MCM. This leads us to the research question:

Why do (and don't) in-service teachers and student teachers use MathCityMap? By this question we follow Drijvers study of the usage of digital tools by teachers (Drijvers, 2012).

Methodology

To promote MathCityMap as a digital tool (and therefore the usage of mathematics trails in school) we have implemented three teacher trainings with 143 participants and two university student courses with 30 students during spring/summer 2016. To evaluate the trainings and gather further information for future improvements of the MCM tool, an online questionnaire was created. Additionally, we have analyzed the usage statistics.

Teacher training

The training is a half-day intensive training for in-service teachers. Since they have already studied mathematics and have a lot of teaching experience, we keep the theoretical parts on outdoor mathematics and task design rather short and prefer to go out on a prepared mathematics trail so they can experience this kind of activity. Later on, we also let them find tasks and focus more on the handling of the web portal and the app. After this course every teacher will have experienced doing mathematics outdoor with MCM, but also how to create new tasks in the web portal.

Student courses

The student courses took place at Goethe-University in Frankfurt (11 students) and the University of Potsdam (19 students) in the summer semester of 2016. The following topics formed part of the seminar: Theoretical introduction to mathematics trails, introduction to the MCM App and going on

a mathematics trail with the app, aspects of outdoor task design, creating new tasks and setting them up in the MCM web portal, arranging a mathematics trail, testing the trail with a school class (grade nine), reworking the trail, testing the trail with another class (grade eight). Compared to the teacher trainings the students had to really engage themselves in mathematics trails with MCM.

Online survey

About 200 people (143 participants of the teacher trainings plus registered users of the web portal), who have agreed to receive e-mails about MCM, were invited to take part in the survey. Twenty (eight students and twelve teachers) of them completed the questionnaire.

The online survey consists of 27 items, from which twenty are closed questions or statements and seven are open text fields. The questionnaire is divided into five sections:

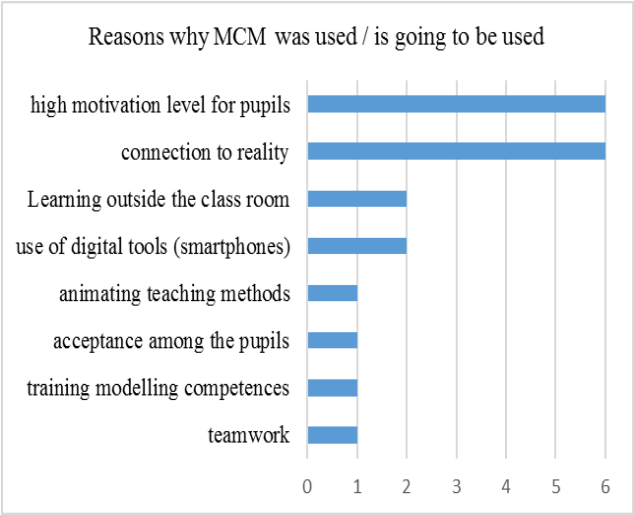
1. General Information (Five closed questions)
   Sample item: How did you hear about MathCityMap?
2. Usage of the MCM web portal (Seven mainly closed questions)
   Sample item: Do you already have created a task in the web portal?
3. Statements about the MCM web portal (Seven 5-point Likert scale items: I do not agree – I agree)
   Sample item: The interaction between web portal and app is easy to understand.
4. Feedback on MCM (Four mainly open questions)
   Sample item: Which are the reasons for you to use MathCityMap?
5. General use of digital tools in mathematics classes (Four closed and open questions)
   Sample item: What are your requirements for using a digital tool in mathematics classes?

Results

Questionnaire

Four of twelve teachers stated that they had created their own tasks. Two of them had already used the mathematics trail with a class. All eight students had created tasks and went on a mathematics trail, because it was part of the seminar.

Due to the low number of participants we report the reasons why MathCityMap was used or is going to be used and the reasons why it was not used yet in a qualitative way by forming categories. The answers were collected by an open text field, so multiple reasons could be given. The following categories are sorted by the number of mentions.



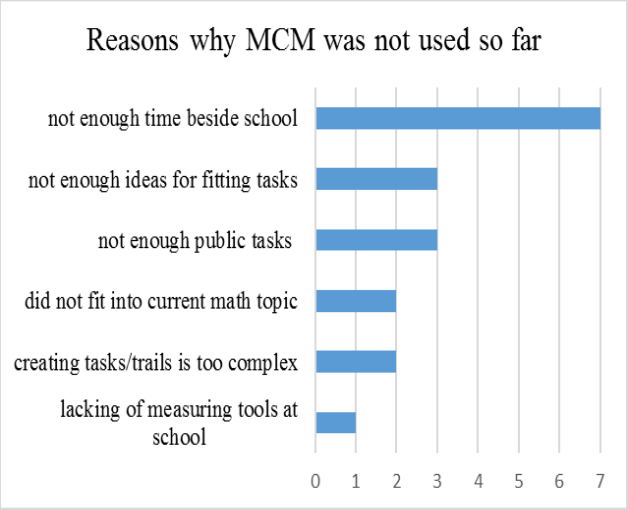


Figure 3: results of the survey about the reasons for using / not using MCM (13 / 8 persons in total, open text answers)

If we take a look at the things teachers do require from a digital tool to be used by them, MCM is doing quite fine. MCM is easy to get and free to use. It is not time consuming to learn it and some of the teachers already have positive experiences (see Figure 4).

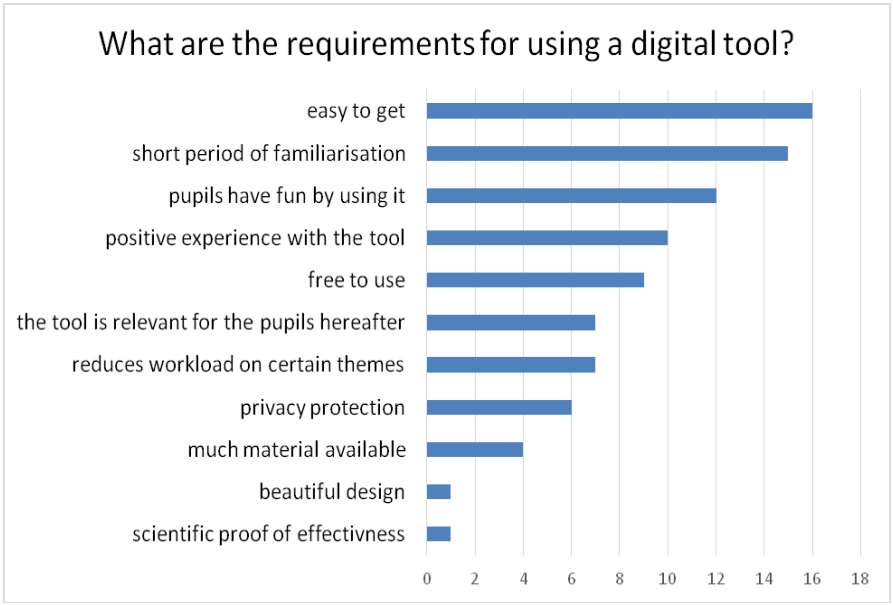


Figure 4: results of the survey about the requirements to use a digital tool in class (17 persons in total, preset answers, several selections possible)

Usage statistics

Independent of the online survey, we also analyzed the statistics relating to the web portal and the app to describe the current state. In September 2016 74 users were registered in the web portal. Thirty of these were in-service teachers who participated in the trainings, about 20 were students who were part of the student courses. The other users were not part of the trainings or courses. In total 33 unique users (45%) created 140 tasks. About 25 mathematics trails were created by 22 unique users (30%). The app has been installed 210 times which means that there must be some people who use only the app, without being registered in the portal (e.g. pupils).

Discussion

MathCityMap as a digital tool seems to be mainly used as the mathematics trail idea is considered positively (high motivation for students and connecting mathematics to the reality). Hereafter the integration of digital tools in mathematics classes is another reason to use MCM (see Figure 3).

The lack of time, difficulties in creating appropriate tasks and the integration into the current lessons are the most mentioned reasons for why MCM had not yet been used. However, the findings also suggest that the task and trail creation processes in the web tool might be too complex at its current state (see Figure 3). All of these reasons could be interdependent. If one has difficulties in finding tasks or difficulties in integrating the mathematics trail into the lessons, it will take more time to solve these problems. Since many teachers report that they are short of time, this might lead to not using MCM (Kuntze, Siller, Vogl 2013, Schneider 2015).

Conclusions

In our case the reasons for not using the tool (web portal and app) were mainly identified not in the tool itself, but in the mathematics trail concept (creating tasks, implementing the trail in classes). The teacher training events and student courses need adjustments so that they pay more attention to the following identified difficulties:

1. Higher focus on task design – guidance, best practice examples, blueprint tasks which can be easily adopted to the participants' school surroundings.
2. Creating a teaching concept – concrete example of how to integrate mathematics trails in combination with MCM into mathematic lessons for a particular topic.

In addition, 'doing mathematics outdoors' could be integrated into the school curriculum to increase its significance. On the technical side, further research is needed on how to improve the usability of the MCM web portal to make the creation process more intuitive.

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Exploring a framework for technology integration in the mathematics classroom

Patrik Gustafsson

Mälardalen University, Sweden; patrik.o.gustafsson@mdh.se

The aim of this paper is to investigate the potential of Ruthven's (2009) framework the Structuring Features of Classroom Practice (SFCP) as a tool to analyze empirical data to conceptualize and analyze teachers' reasoning about technology integration in the mathematics classroom. The framework is tested on interview data from a Swedish design research project seeking to develop design principles for Classroom Response System (CRS) tasks. The results show that the framework captures a large part of teachers' ways of reasoning, while the parts it does not capture are related to students' attitudes and behaviors. If the SFCP framework aims at capturing key features of classroom practice, and is to be built on a system of constructs closer to the 'lived world' of teacher experience and classroom practice, it would benefit from an extension.

Keywords: Technology integration, framework, classroom response system.

Introduction

Recent years have seen a remarkable increase in technology investments in education, and nowadays many teachers and students have constant access to computers or tablets in the classroom (OECD, 2015). The reasons for these investments are likely related to expectations that digital technology can enhance students' learning, and there are several studies that suggest this (e.g., Cheung & Slavin, 2013; Li & Ma, 2010; Lynch, 2006). However, the mere presence of digital technology in the mathematics classroom does not guarantee improved student learning. For instance, a report on PISA 2012 (OECD, 2015) showed that increased time spent with the computer at school can decrease students' learning in mathematics. It may be possible to explain these ambiguous findings by studying how the technology was used in the classroom (Drijvers, 2013; Hattie & Yates, 2014). Nevertheless, integrating technology in the classroom seems to present a challenge, and one of the most important factors influencing successful integration is the teacher's expertise (e.g., Drijvers, 2013; Ruthven, 2013). Hence it is important to learn more about how to successfully integrate the technology within mathematics education. There is a need for practical analytical research tools and frameworks that offer the potential to analyze teachers' technology integration in mathematics. A commonly used framework, derived from Shulman's (1987) pedagogical content knowledge, is Koehler and Mishra's (2009) Technology Pedagogical and Content Knowledge (TPACK), which focuses on the aspects of teacher knowledge that are needed for the effective use of technology in the classroom. Other researchers have used the theory of "instrumental orchestration" as an interpretive framework for analyzing technology-mediated teaching and learning (cf. Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Trouche, 2005). This theory focuses on a process of "instrumental genesis" whereby a tool evolves into a functional tool and, simultaneously the teacher evolves into a proficient user. Another recently developed framework for analyzing and identifying critical aspects of technology integration in the mathematics classroom is Ruthven's (2009) Structuring Features of Classroom Practice (SFCP). TPACK and "instrumental orchestration" are commonly used frameworks for analyzing technology

integration in the classroom, but as Ruthven (2009, 2013) stresses, the SFCP framework includes aspects such as the complexity and importance of a teacher's "craft knowledge", which other frameworks largely overlook. This is the main reason I have chosen to explore the SFCP framework. A second reason is that the SFCP framework is relatively new and needs to be tested using empirical data from other contexts (Ruthven, 2009). Thus, my contribution to research is that I have investigated the potential of this framework using interview data from a Swedish design research project seeking to develop design principles for classroom response system[1] (CRS) tasks in a multiple-choice format. Hence, the aim of this paper is to investigate the potential of the SFCP framework as a tool to analyze empirical data to conceptualize and understand teachers' reasoning about technology integration in the mathematics classroom**.**

Participants, context and data

The framework was tested on interview data from two cases of CRS integration in mathematics classrooms within grades 6-9 in lower secondary school in one of Sweden's largest municipalities. One teacher participated in Case 1, and six teachers in Case 2. The reason for working with only one teacher in Case 1 was that this case was a pilot study, which prioritized the depth of the intervention and analysis in the beginning of this design research project. Further, the choice of teachers at these particular schools was partly due to the fact that during these academic years, I was a mentor to mathematics teachers at these schools. In addition, the schools were one-to-one schools, where all students had access to their own computer. These teachers were not explicitly chosen for the research project, they just represent ordinary Swedish teachers in ordinary schools. The reason for working with six teachers in Case 2 was that all of the mathematics teachers at that particular school wanted to improve their teaching and asked me to guide them. Further, the teachers had no (or little) experience in utilizing a CRS, and had received training in how to use the digital resource in practice. In both cases, CRS supported with specific tasks was used to engineer mathematics classroom discussions that could both elicit evidence of learning and also give the teacher an opportunity to advance the students' mathematical thinking. These tasks were often used in the beginning of the lessons or after a short lecture on the topic. Additionally, in Case 1, tasks were also used to evaluate the lessons and obtain information about the students' knowledge at the end of lessons. Based on the teachers' own wishes in Case 2, the teachers also used flipped classroom method to gain more time for classroom discussions, and Peer Instruction method as support for orchestrating the discussions. Based on the teachers' lesson goals and a pilot of the design principles, the researcher constructed and supplied suggestions for tasks to be used with the CRS. In both cases the topic of fractions was chosen, determined by the timing of the study along with the teachers' wishes. The teachers used and evaluated a total of 31 tasks. Figure 1, which follows, shows an example of one evaluated task type with different multiple defendable answers (Beatty, Gerace, Leonard, & Dufresne, 2006).

[1] Using a computer or smartphone, students can answer their teacher's question and the teacher can instantly see the results compiled in a chart in the software program and display this for all the students on a shared screen.

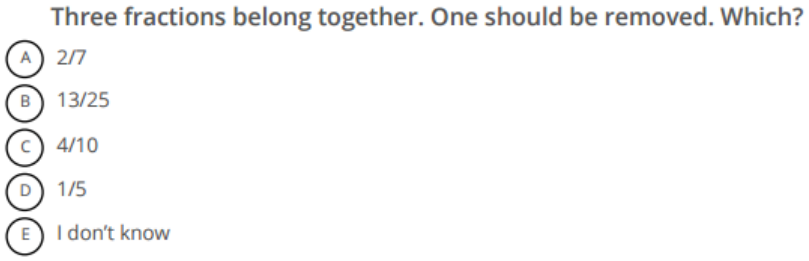

Figure 1: A constructed CRS task with multiple defendable answers

All CRS tasks were built on the idea that tasks that produce a spread in students' answers, are more likely to prompt a mathematical classroom discussion (e.g., Crouch, & Mazur, 2001). This particular task was developed to be used as a repetition of some important properties of fractions that students had already encountered. Semi-structured interviews were conducted to support one phase of the evaluation of the intervention. In order to explore the SFCP framework, I chose to test the framework using the data from one interview with the teacher in Case 1 and one group interview from Case 2. The interviews were audio-recorded and then transcribed and analyzed in NVivo 10.

The SFCP framework

The idea of the SFCP framework is to support the identification and analysis of certain crucial features of technology integration (Ruthven, 2009). The framework was developed by synthesizing and extending concepts and constructs from earlier research on classroom organization, interaction and teacher craft knowledge, which resulted in five crucial features (Ruthven, 2009). These features of classroom practice shape the ways in which teachers integrate new technologies (Ruthven, 2013). Ruthven's own summary of the framework is presented in Table 1 (Ruthven, 2013, p. 12).

Structuring feature	**Defining characterization**	**Examples of associated craft knowledge related to incorporation of digital technologies**
Working environment	Physical surroundings where lessons take place, general technical infrastructure available, layout of facilities, and associated organization of people, tools and materials	Organising, displaying and annotating materials Capturing or converting student productions into suitable digital form Organising and managing student access to, and use of, equipment and other tools and materials Managing new types of transition between lesson stages (including movement of students)
Resource system	Collection of didactical tools and materials in use, and coordination of use towards subject activity and curricular goals	Establishing appropriate techniques and norms for use of new tools to support subject activity Managing the double instrumentation in which old technologies remain in use alongside new Coordinating the use and interpretation of tools
Activity structure	Templates for classroom action and interaction which frame the contributions of teacher and students to particular types of lesson segment	Employing activity templates organised around predict-test-explain sequences to capitalise on the availability of rapid feedback Establishing new structures of interaction involving students, teacher and machine, and the appropriate (re)specifications of role
Curriculum script	Loosely ordered model of goals, resources, actions and	Choosing or devising curricular tasks that exploit new tools, and developing ways of staging such

	expectancies for teaching a curricular topic, including likely difficulties and alternative paths	tasks and managing patterns of student response Recognising and responding to ways in which technologies may help/hinder specific processes and objectives involved in learning a topic
Time economy	Frame within which the time available for class activity is managed so as to convert it into "didactic time" measured in terms of the advance of knowledge	Managing modes of use of tools so as to reduce the "time cost" of investment in students' learning to use them or to increase the "rate of return" Fine-tuning working environment, resource system, activity structure and curriculum script to optimise the didactic return on time investment

Table 1: The SFCP framework components

Method of analysis

The interview data was used as a means to explore the potential of the SFCP framework. In this exploration, the framework was used as an analytical tool to capture teachers' reasoning about utilizing a CRS. To support the exploration of the framework's potential, I used two analytical questions: 1) How much of teachers' reasoning ends up in the various categories in the SFCP framework? and 2) Are there parts of teachers' reasoning that do not fit the categories of the SFCP framework? If so, does a new theme emerge? To answer these questions, I conducted a content analysis with systematic quantification (Kvale & Brinkmann, 2009), with text segments in the transcriptions of the interviews coded in NVivo 10 based on the categories in the SFCP framework. I then compiled the text segments from every category and wrote an accompanying narrative.

Summary of the content analysis

Due to space limitations, the outcome of this content analysis is not presented in detail here; instead, some of its main findings are discussed.

Working environment

The teacher in Case 1 pointed out that when the projector screen is pulled down it blocks a large part of the whiteboard surface. This can constrain the usage of the CRS. When the teacher wants to write students' solutions to or explanations of CRS tasks on the whiteboard she has to pull up the projector screen and blacken the computer projection, and then pull the screen down again to continue the CRS tasks. This may constrain the possibility to conduct a classroom discussion. Further, in both cases the teachers declared that students sometimes do not bring their computer, and sometimes do not have access to the internet. Students without a functional computer or internet access constrain the work in the classroom. The teachers solved this by letting students work with a peer who had a computer.

Resource system

The teacher in Case 1 emphasized the importance of combining CRS tasks aiming at engineering a discussion with a demonstration of methods. This suggests that the teacher needed to coordinate these two curricular elements to achieve the lesson's goal. The teacher also mentioned that students seemed to be reluctant to work out solutions to the CRS tasks on paper before submitting an answer in the software program. According to the teacher, this constrained her opportunity to identify and see students' reasoning behind their answers before the discussion. A teacher in Case 2 did not

believe students needed access to paper and pencil before responding to a task, but thought this could be useful afterwards if they were to proof their own or others' answers. Further, teachers in Case 2 told of struggling with the software and launching tasks in the wrong mode. This gave all the students access to all the tasks at once, which resulted in the teachers decision to shut down the CRS work for that particular lesson. They then discussed the possibility of trying out the different modes before the lesson.

Activity format

The teacher in Case 1 said that the CRS tasks were a great way to get all of the students focused. The students interacted with their computer, and were forced to contribute with an answer to the tasks. They then interacted with their peers through peer and whole-class discussions. The activity formats used in both cases were: first alone, then peer discussion, and finally a whole-class discussion; and also first alone and then a whole-class discussion. One teacher mentioned that it was hard to decide whether to orchestrate a whole-class discussion or a group discussion in tasks with multiple correct answers when applying peer instruction, which holds that students benefit from a group discussion if 30-70% of the students responding correctly. Further, the teachers also discussed the importance of allowing time *before* the students are to respond to the CRS task. Several of the teachers let the students take as much time as they needed, which led to some students having to wait a couple of minutes.

Curriculum script

Teachers mentioned that the CRS tasks made them aware of some student misconceptions, and gave them an opportunity to deal with them. Further, the teacher in Case 1 pointed out the improvement of feedback, both the possibility to use instant feedback through the computer in CRS tasks and the feedback in the peer and whole-class discussions related to the discussion tasks. The technology and tasks 1) gave the teacher information about students' knowledge, and 2) added a new form of feedback resource which, together, developed the teacher's curricular script. Moreover, in Case 2, several teachers identified and talked about different types of CRS tasks and their characteristics, and how they had succeeded in engineering a discussion. One teacher realized that you could not always have tasks with several correct answers, because the students quickly realize this. The teachers also stressed that it is hard to conduct whole-class discussions on CRS tasks, and one teacher mentioned the importance of having a clear teaching strategy for every CRS task to improve the whole-class discussion.

Time economy

The teacher in Case 1 believed that having CRS tasks at the end of the lesson makes students more focused on mathematics for a greater part of the lesson. These tasks improved the "rate of return" in two ways: firstly, students worked with mathematics for a larger part of the lesson; secondly, the software program automatically gave students instant feedback on their answers. One teacher said she would continue using CRS, although it takes time to prepare. Thereafter, she mentioned that *"it's worth the time because it activates every student…when I activated one student who usually doesn't participate she said 'ahaaa' in front of the whole class. It was amazing"*. Several teachers pointed out that the discussions take time, and that it is a challenge to decide how long to work on each task and how many tasks to use in one lesson.

Results

In this section I present the results of the analysis of the framework's potential according to the analytical questions.

How much of teachers' reasoning can be categorized within the SFCP framework?

Table 2 shows the coverage of the different categories in the transcription of the interviews regarding teachers' reasoning in Cases 1 and 2. Some text fragments were coded in several categories. I have also rounded the figures. All features captured some parts of the teachers' reasoning, and a total of 90% of the interview in Case 1 and 65% of the group interview in Case 2 were captured by the framework.

	Working environment	**Resource system**	**Activity format**	**Curriculum script**	**Time economy**
Case 1	5%	20%	25%	20%	20%
Case 2	5%	15%	15%	25%	5%

Table 2: The SFCP frameworks coverage of teachers' reasoning in the interviews

Are there parts of teachers' reasoning that do not fit the categories of the SFCP framework? If so, does a new theme emerge?

Approximately 10% of teachers' reasoning in Case 1 and 35% in Case 2 did not fit the SFCP framework categories, and when the parts the framework did not capture were analyzed a clear theme emerged. Almost all reasoning that the framework did not capture was related to students' attitudes and behaviors. I will continue with a summary on this theme.

All teachers reasoned about their students' attitudes and behaviors concerning the lessons. Some classes and students greatly enjoyed working with CRS tasks in mathematics. As one teacher said, "*they think it's fun to discuss things*". Another teacher reported that "*the students were crazy about the CRS tasks*", and another talked about how the students want very much to respond correctly to the tasks and demanded to do it again in the next lesson if they failed the first time. In some classes the students were eager to discuss the CRS tasks; the teacher commented that "*the students want to hear their peers' opinion and they want to tell the class about their own perception*". Further, teachers also mentioned that some students did not want to participate, especially in the discussions, during which they simply sat quietly. Some teachers had difficulty in handling students who wanted to respond quickly and could not wait for others to think and respond to the tasks. All the teachers in Case 2 talked about the difficulty of getting students to do the homework and to be prepared for the work with the CRS tasks in the classroom. One teacher mentioned: "*In one of my classes, only one student had done the homework and watched the flipped movie at home.*"

Conclusions and discussion

The exploration of the SFCP framework showed that it captured a large part of teachers' reasoning about technology integration in the mathematics classroom. Most of the teachers' reasoning was related to features of activity format, curriculum script and resource system. My conclusion is that the SFCP framework could be useful as an analytical tool for conceptualizing and analyzing

teachers' reasoning about technology integration in the mathematics classroom in the context of Sweden and CRS technology. However, the framework did not capture all the teachers' reasoning about important aspects of technology integration. Almost all of the reasoning that did not fit any category was related to students' attitudes, and students' behavior. According to Ruthven (2009), the SFCP framework aims at identifying and making key structuring features of classroom practice analyzable for the integration of technology into a classroom. Further, Ruthven (2009) states that the benefit of the SFCP "is in providing a system of constructs closer to the 'lived world' of teacher experience and classroom practice" (p. 145). This study's results indicate that students' attitudes and behaviors are an important factor that teachers reason about when discussing the implementation of technology in the mathematics classroom in Sweden. Like all five features of the SFCP, I suggest that students' attitudes and behavior are also important factors for successful technology integration in classroom practice. Research on CRS points out that students' attitudes and behaviors are a challenge that teachers face (e.g., Kay & LeSage, 2009; King & Robinson, 2009; Lee, Feldman, & Beatty, 2012). If the SFCP framework aims at capturing key features of classroom practice and is to be built on a system of constructs closer to the 'lived world' of teacher experience and classroom practice, it would also benefit from taking into consideration students' attitudes and behaviors. This could be done by adding a new, sixth construct to the framework relating to teacher craft knowledge for managing different types of student behaviors or attitudes.

The main contribution of this paper is that it investigates the potential of the SFCP framework with empirical data from a new context and new types of data. It was partially tested on data from group interviews in the context of CRS integration in mathematics at Swedish lower secondary schools. Further, this study and the conceptualization of teachers' reasoning about CRS integration can contribute to the knowledge regarding challenges involved with utilizing a CRS in the mathematics classroom. This conceptualization may also be useful for teachers intending to integrate CRS into their practice. For instance, they could gain knowledge about different activity formats and common challenges, as well as how to deal with these challenges. Finally, the results from this study need to be further investigated with empirical data from similar or other contexts.

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Preparing preservice teachers to use instructional technology: How much development can happen in one semester?

Janet M. Herrelko

University of Dayton, Dayton, Ohio, USA; jherrelko1@udayton.edu

Abstract: Knowledge of instructional software programs that can meet mathematical curricula objectives, motivate and engage students in problem-based learning/inquiry is essential for teachers. This is the first of a series of studies tracking the implementation of instructional technology in a mathematics methods course. Data were collected from surveys, power point presentations of an instructional technology lesson, and the reflections written post lesson presentations. The data were used to classify where preservice teachers were on the five steps Apple Classroom of Tomorrow (1994) inclusion of instructional technology in the classroom. Of the 24 preservice teachers, 21 were solidly on Step 2 – limited use of technology. There were 3 who stood at Step 3 creating their lesson plans to use technology on a daily basis.

Keywords: Instructional technology, preservice teachers, mathematics education.

Introduction

Teaching mathematics in 2016 requires far more than a deep understanding of mathematics. Multiple pedagogical methods and strategies are needed to address student learning needs. Today, teachers need to use instructional technology that applies scientific processes and stored knowledge to solve practical tasks (Earle, 2002). Knowledge of computer software that can meet curricula objectives, motivate students, and engage students in problem-based learning and inquiry is essential. The ground work for applying technology with intent and purpose should be part of the responsibility of teacher preparation programs. Defining instructional technology in education has been evolving since the American Educational Communications and Technology group produced a broad definition in 1963 that matched the elements of pedagogical courses of the time (Ely, 1963, p. 18-19). While the technology sections of learned societies grappled refining the definition of instructional technology, the Association of Mathematics Teacher Educators (AMTE) created standards for the preparation of preservice mathematics teachers for grades Pre-Kindergarten to grade 12 (PK-12). These standards devoted a section of the Adolescence to Young Adult (AYA) grades (C.1.6. Using Mathematical tools and technology) identifying the types of mathematical software preservice teachers should master. AMTE explained C.1.6. with the following statement:

Well-prepared beginning teachers of secondary mathematics must be proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and in supporting student learning of mathematics. In particular, they should develop expertise with spreadsheets, computer algebra systems, dynamic geometry software, statistical simulation and analysis software, and other mathematical action technologies, as well as other tools such as physical manipulatives (AMTE, 2017, p.133). The AMTE elaboration aligns with the goals of this research to help mathematics preservice teachers become competent and frequent users.

Integration of instructional technology into curricula issues

Ertmer, Conklin, Lewandowski, Osika, Selo, and Wignal (2003) found that preservice teachers needed specific ideas and examples of how to put instructional technology into their mathematics instruction. Wang's (2004) research noted that goal setting increased self-efficacy. Dexter and Riedel (2003) identified clinical educators use of instructional technology as a key to helping preservice teachers increase their classroom technology self-efficacy. For this research, instructional technology was practiced in the methods course providing teaching ideas and strategies to the preservice teachers. A teaching assignment goal required use of a software program following Wang's (2004) finding.

Introduction of technology to pre-kindergarten to Grade 12

In the 1970's, Apple® created the Apple®II for classroom work. By the 1980s, computers became part of some PK-12 classrooms. School districts developed technology plans to implement the use of computers at each grade level. However, the districts were missing measurable objectives to track and identify the educational impact of computer use by the teachers and student. By the 1990s, schools had a computer on every teacher's desk and computer laboratories. The 2000s students worked with personal devices such as Chromebooks and iPads. Graphing calculators span the decades since Demana and Waits (1988) noted the importance of creating multiple graphs to grasp a mathematical concept.

Preservice teachers using technology for knowledge production

Preservice teachers are well versed in the use of electronic devices as are today's PK-12 students. Applying that understanding beyond word processing, communications, and gaming to using technology for knowledge production should be part of every teacher preparation program. Doering, Hughes, and Huffman (2003) did a five-year study that provided the hardware and software for their preservice teachers and content faculty at the University of Minnesota. Initially, they found that preservice teachers had solid knowledge of technology use, but integrating technology into daily instruction and problem-based learning was not a skill they had. By the end of the study, integrating instructional technology into content and pedagogical classes, preservice teachers became productive users of instructional technology in their field experiences. Franklin (2004) reported on the attitudes of University of Virginia elementary level teacher graduates. The participants noted a clear understanding of classroom technology to foster student curiosity and construct ideas. These teachers had a deep comprehension of electronic pedagogical content knowledge as the reason for their smooth transition to classroom implementation. The use of Web 2.0 tools in the classroom by preservice teachers was examined by Sadaf, Newby, and Ertmer (2016). Use Web 2.0 tools to increase learning, preservice teachers needed: support of their clinical educator; easy access to those tools; and to hold a high level of self-efficacy regarding their ability to help student learning. Only when these elements were met did the preservice teachers use instructional technology in their classrooms.

Assessment scales for instructional technology implementation

A long term study by, Dwyer, Ringstaff, Haymore, and Sandholtz (1994) working with Apple Classroom of Tomorrow (ACOT) examined how teachers adapted their classrooms and pedagogy to using technology when provided with multiple computers, an abundance of software, technical support and training. The researchers identified a five step progression of how teachers developed technology-based pedagogy naming it the ACOT stages of classroom change. Step 1 - Entry. The

teachers are acquainted with the basic tools of the computer and classroom programs. Step 2 – Adoption, the teachers adopted the computer programs for limited use (defined as practice not knowledge building). Step 3 – Adaptation, the teacher thoroughly integrated the use of computers into the curriculum. This step resulted in students learning more, being engaged with the content, and producing better knowledge products. Step 4 – Approbation, teachers who cannot teach without computers. Step 5 – Invention, teachers created their own programming that enhanced student learning. The ACOT (1994) study noted that teachers' development was not done in leaps, but moved forward in increments over time. As the teachers embraced technology, their pedagogical strategies shifted from being teacher-centered to student-centered.

Theoretical framework

The researcher selected the ACOT (Dwyer et al., 1994) steps to serve as the theoretical framework for this study to judge how preservice teachers developed using technology. The ACOT instrument focused on the changes in teacher practice whereas, other instruments focused on the partnership of the teacher and the students. The case study descriptive quality lends itself to using the ACOT descriptions to define advancement on these steps. This research is to learn how far preservice teachers can grow using instructional technology in one semester.

Method

This article is the first report of a long term descriptive case study following American preservice mathematics teachers in a mathematics methods course that required a lesson using instructional technology to be taught during a 90-hour field experience. A case study format fits this research as it describes the conditions necessary to produce knowledgeable preservice teacher's regarding instructional technology. The research question is: How far can preservice teachers develop using instructional technology on the ACOT Steps in one semester?

Participants

The 24 preservice teachers in this study were enrolled in a course entitled Secondary Mathematics Methods, which was required for state licensure to teach. Eleven majored in Adolescence to Young Adult Mathematics Education (AYA). Thirteen were Middle Childhood Education (MC) mathematics. There was one male in each licensure group with 10 AYA females and 12 MC females.

Setting

The university is a private, non-profit school in the south-western, urban section of a Midwestern state in the USA. There are approximately 8,529 undergraduates and 3,117 graduate students. The Teacher Education department conducts classes at the undergraduate and graduate levels.

Procedure

In the methods course, a survey was given asking how frequently students used: 1) word processing; 2) spreadsheets; 3) power point presentations; 4) photomath; 5) Wolfram Alpha; 6) DESMOS; 7) GoogleSketchUp; 8) Polling apps; 9) GeoGebra; added in 2016 10) Kahoot. Likert scales from 0-never used to 5-used all the time were the choices. The course introduced freeware mathematical programs, demonstrated them and provided practice teaching. During the clinical experience, the preservice teachers were required to create and teach an instructional technology lesson.

Study history: Equipment survey of partnership schools

One of the elements needed for preservice teachers to use instructional technology was easy access to Web 2.0 tools in the classrooms (Sadaf et al., 2016). This study interviewed the university's partnership school districts to learn what technology was in their classrooms. The schools reported that they had invested in computers for all teachers and individual laptops, iPads, or Chromebooks for the students. The more frugal districts had multiple computer carts with 30 individual devices for classroom use. Regarding educational software programs, freeware was the programming of choice. The availability of Web 2.0 tools and software programs allowed the researcher to create an assignment goal requiring the use of instructional technology that Wang (2004) recommended.

The search for mathematical freeware and program criteria

The criteria used to evaluate the appropriateness of the instructional technology were the eight Common Core State Standards Mathematical Practices (CCSSM) (National Governors' Association & Council of Chief State School Officers, 2012) and the eight National Council of Teachers of Mathematics (NCTM) Teaching Practices (Leinwand, 2014). Any mathematical program had to require students to perform six of the eight Mathematical Practices and the teacher to use all eight NCTM Teaching Practices. At an NCTM affiliate meeting, in a Skype session by DESMOS creator Eli Luberoff taught the participants to use the program in minutes. At the NCTM Interactive Institute 2015, the program focused on freeware of GeoGebra, DESMOS, and polling programs to engage students in discourse to evaluate the solutions of others. These polling programs required smart phones rather than clickers. Wolfram Alpha could be used for higher levels of mathematics. This program had many more options for teachers to integrate other content areas. This search for instructional technology was not exhaustive. Once these major programs were found, the researcher stopped the search. The study included: DESMOS, GeoGebra, Wolfram Alpha, PollEverywhere.

Data collection

The data collection began with a survey at the start of the mathematics methods class asking preservice teachers how frequently they used software programs. The instructional technology lesson plans with preservice teacher reflections were collected after the six weeks of field experience. The reflections served as a record of the preservice teachers' comfort level, frequency, and self-efficacy using technology. The preservice teachers presented their instructional technology lesson in a power point presentation that included video clips of their teaching with instructional technology, their classes discussing and completing the mathematical work, and voting on the most elegant solutions. The video tape clips verified what was stated in the lesson plans and reflections. Data were recorded regarding the frequencies that the preservice teacher used: instructional technology; had students use that software; and the issues that arose while teaching a technology-based lesson.

Data analysis

Analysis of survey

The Survey of Classroom Technology for Knowledge Production was conducted to learn how familiar the preservice teachers were with use of instructional technology. The Likert scale scores were totaled and measures of central tendency were calculated.

Assessment of class assignment

The researcher created an assignment with a goal matching the findings of Wang (2004) to learn what level the preservice teachers reached on the ACOT scale of proficiency – could they use and integrate technology smoothly into their teaching? Each preservice teacher created and video-taped an instructional technology lesson plan using a real world problem for their classes to solve by using a mathematical software program. Once the students had solutions, they were grouped and each student explained their solution to the group. The group debated the solutions and modified their work to create their best solution. Each group presented their work to the class. PollEverywhere was used by students to vote on the elegant solution. The projects were graded using the rubric found in Table 1.

16 points of the total possible of 21 points is the minimum passing grade

Elements	1	2	3	Score
1. Completed UD lesson plan format	Hard copy of the lesson passed in with the incomplete reflection. The lesson plan lacks the sections and requirements of the AYA/MC UD Lesson Plan format.	Hard copy of the lesson passed in with the completed reflection. The lesson plan follows most of the section requirements of the AYA/MC UD Lesson Plan form	Hard copy of lesson passed in with written reflection. The lesson plan format follows all the requirements of the AYA/MC UD Lesson Plan format. The reflection provides clear and thoughtful responses.	
2. Use of DESMOS, GeoGebra, or GoogleSketch-Up	Your presentation shows your attempt to use DESMOS, GeoGebra, Google-Sketch-Up to present the problem	Your presentation shows your use of DESMOS, GeoGebra, Google-Sketch-Up posing the problem	Your presentation clearly shows your mastery of DESMOS, GeoGebra, or GoogleSketch-Up as the presentation mode for posing your problem	
3. Problem solving with engaging real world problem	Evidence of the students solving a mathematical problem.	Evidence of the students solving a real world math problem.	Evidence of solving an engaging, real world problem. The problem is posted and easily read.	
4.Math discourse: explain,defend, challenge the ideas of others	Evidence of students engaged in classroom discourse, but not on topic.	Evidence of engaged classroom discourse. Types of discourse are not clear.	Evidence of classroom discourse that includes explaining, defending, and challenging the solutions/ideas of others.	
5. Student presentations	Evidence of presenting solutions with no reasoning.	Evidence of solutions with little explanations.	Evidence of presentations explaining/defending their solution.	
6. Student use of polling	No use of polling devices	Some students use polling devices.	Evidence of the students using electronic polling	
7. Clear video of the voted solution.	No clear result to the voting. Or the solution is not clear	Problem solution is correct, but not the voting.	Evidence of the problem solution selected by the class can clearly be read.	
Total Score				**_____/21**

Table 1: Technology and mathematics project rubric

During the power point presentations, notes were taken by the researcher regarding the frequency of use and the issues that the preservice teachers had when implementing this lesson. Lesson reflections were reviewed for common themes and attitudes.

Results

The results of the Survey of Classroom Technology for Knowledge Production revealed that the AYA preservice teachers used mathematics instructional technology more than the MC preservice teachers prior to the methods course. All 24 preservice teachers used word processing, 11 used spreadsheets, and 19 used power point presentations. When the preservice teachers reached the specific mathematical instructional technology questions, many scores were zero. See Table 2 for median scores, standard deviation and standard error measure.

Table 2: 2015 Survey of Classroom Technology for Knowledge Production, N=24

Question	1	2	3	4	5	6	7	8	9	10
Median	5	1	2	0	2	2	0	0	1	0
S.D.	0	0.65	0.79	0.2	1.57	1.31	0	0	1.4	0
S.E.M.	0	0.13	0.16	0.04	0.32	0.27	0	0	0.29	0

The preservice teachers were able to implement the instructional technology lesson to varying degrees. The AYA preservice teachers were able to create a real-world problem around which they built their lesson using instructional technology. The preservice teachers taught the students how to use their selected computer program on one day and the lesson the next day.

The MC preservice teachers created their real-world problems, but those who taught in grades 4 and 5 were not able to use any program. Their cooperating teachers believed that this technology was not developmentally appropriate for the students. Since a majority of the students in these grades did not own smart phones, many of the preservice teachers used Kahoot. The preservice teachers used this software program to project questions for voting.

Table 2a: 2016 Survey of Classroom Technology for Knowledge Production Pre-Field, N=15

Question	1	2	3	4	5	6	7	8	9	10
Median	5	3	4	0	1	0	0	2	2	2
S.D.	0.52	1.26	1.39	0.60	1.21	0.85	0 .28	1.72	1.34	1.59
S.E.M.	0.14	0.35	0.39	0.17	0.34	0.24	0.08	0.48	0.37	0.44

Table 2b: 2016 Survey of Classroom Technology for Knowledge Production Post Field, N=15

Question	1	2	3	4	5	6	7	8	9	10
Median	5	2	4	0	0	1	0	1	0	2
S.D.	1.3	1.35	1.06	0.52	1.10	1.22	0 .74	1.20	1.10	1.46
S.E.M.	0.34	0.35	0.27	0.13	0.28	0.32	0.19	0.31	0.27	0.38

The 2015 scores from the project rubric were reported in the following data for Year 1:

Table 3a: 2015-Data Results from Technology + Mathematics Methods Project Rubric Scores.

Rubric #	1	2	3	4	5	6	7	N=24	
Median	3	2	3	3	2	3	3	S.D.	0.64	0.63
0.51	0.52	0.51	0.26	0.74						
S.E.M.	0.17	0.16	0.13	0.13	0.13	0.07	0.19			

Table 3b: 2016-Data Results from Technology + Mathematics Methods Project Rubric Scores.

Rubric #	1	2	3	4	5	6	7	N=9	
Median	3	3	2	3	3	3	3	S.D.	0.0	0.0
0.71	0.44	0.0	0.0	0.0						
S.E.M.	0.0	0.0	0.24	0.15	0.0	0.0	0.0			

Over all, the 2015 and 2016 rubric scores were 2 or 3. There was one student who did not do the assignment. The preservice teachers were successful having the students vote on the elegant solution to their problem. The high mean score with the lowest standard deviation and standard error measure confirm the attention the preservice teachers paid to this element of the assignment. The power point presentations with video clips demonstrated the ease with which the preservice teachers acquainted their classes with instructional technology. This assignment matched the attributes needed for success by preservice teachers learning instructional technology found by Ertmer et al., (2003) specific ideas, Sadaf et al., (2016) available Web 2.0 tools, and Wang (2004) providing goals. The overall theme in the preservice teachers' reflections found that using instructional technology was a positive experience. They solved technology issues including: the internet working only on laptops in half of the classroom; no internet accessible; reserving the cart of tablets then found another teacher took the cart without regard for the reservation list. Observing the power point presentations and reading the lesson reflections for Year 1, the preservice teachers noted the number of times they used instructional technology beyond the assignment. Given the one preservice teacher who did not do the assignment, 20 preservice teachers were solidly on ACOT Step 2 – limited use of technology. Three of the 24 used some version of instructional technology almost every day placing them on Step 3 – Adaptation where they built more lessons implementing instructional technology. For Year 2, there was a wider variation in the levels achieved. Three pre-service teachers were at Step1entry use of technology. They only used technology for this unit. Ten pre-service teachers achieved Step 2 which is defined as limited use of technology. Five pre-service teachers (four AYA, one MC) achieved Step 3 where they incorporated technology as a major teaching tool on a regular basis.

Discussion

Implications and connections to mathematics teacher education

Adding program elements for instructional technology into the curriculum for preservice teachers is not a simple fix. Instructional technology needs time to present, model, and practice. The research by Carlson and Gooden (1999) suggests that the responsibility for teaching preservice teachers to integrate technology be done not only in education courses, but also in mathematics classes. If these two departments can collaborate sharing this responsibility, the preservice teachers would witness the power and benefits of teaching with technology. Preservice teachers need to learn how to use instructional technology in order to create student-focused classrooms that engage their students in the learning process from their first day of teaching mathematics.

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Maths teachers' adventure of ICT integration: From an open online course towards an online teacher community

Judith Hohenwarter[1] Selay Arkün Kocadere[2] and Markus Hohenwarter[3]

[1]Johannes Kepler University, Linz, Austria; judith.hohenwarter@jku.at

[2]Hacettepe University, Computer Education and Instructional Technology, Ankara, Turkey; selaya@hacettepe.edu.tr

[3]Johannes Kepler University, Linz, Austria; markus.hohenwarter@jku.at

The purpose of this paper is to introduce an ongoing Erasmus+ project "Maths Teachers' Adventure of ICT Integration (MTAII)" and its outputs. The main aim of the project is to provide professional development for mathematics teachers to integrate Information and Communication Technologies (ICT) into their lessons. To achieve this, three intellectual outputs have been designed; an open online course (OOC), open educational resources (OER), and an online teacher community (OTC). In the scope of the OOC teachers should gain insights about ICT integration into mathematics classrooms. Through development and dissemination of OER we aim to help overcome the scarcity of resources. Through support of peer teachers and researchers at participating organizations we hope to establish an OTC including the teachers participating in the OOC. This paper focuses on the design of the three outputs.

Keywords: Erasmus+ project, online learning, open educational resources, online community, professional development

Introduction

Numerous research studies (Li, & Ma, 2010; Cheung, & Slavin, 2013) have shown that technology integration can play an effective role in tackling the challenges of teaching mathematics. In their study Hew and Brush (2007) have determined several barriers of technology integration for teachers: resources, institutional constraints, subject culture, attitudes and beliefs, knowledge and skills, and assessment. They also describe strategies to overcome these barriers such as: having a shared vision and technology integration plan; overcoming the scarcity of resources; changing attitudes and beliefs; reconsidering assessments; and conducting professional development. Bingimlas (2009) and Kopcha (2012) have also pointed out that professional development activities for teachers play an important role concerning technology integration in education and, in relation to this, several studies indicate the potential of active professional communities (Arkün, & Aşkar, 2013; Duncan-Howell, 2010; Vrasidas, & Glass, 2004).

MTAII

The project "Maths Teachers' Adventure of ICT Integration"[1] (MTAII, www.mtaii.com) aims to help overcome the barriers of knowledge and skills in relation to ICT integration (Hew and Brush, 2007) into mathematics teaching by creating a professional development environment for teachers that

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includes several of the abovementioned strategies. To achieve this, three intellectual outputs have been designed, focusing on a different strategy each, and combined with each other, forming a professional development environment for mathematics teachers from different countries.

Addressing the need for direct professional development opportunities, an open online course (OOC) is designed as the first output of the project. The goal is to help teachers gain some insight into the potential benefits of using technology for education, particularly for learning and teaching mathematics. As part of the OOC teachers are guided towards developing interactive instructional materials and integrating them into their own classroom teaching. The second output addresses the scarcity of high quality, yet ready-to-use educational materials through the development and dissemination of open educational resources (OER). The project's third output will provide the infrastructure and expertise to establish and foster an online teacher community (OTC) that is expected to support the participating teachers beyond the duration of the project. Social, instructional, and technical support will be provided to teachers through the OTC by peers (other teachers) and researchers from the participating organizations.

In addition, the project partners will provide a series of face-to-face workshops, with different content targeting the specific needs of mathematics teachers in each of the participating countries. These events will also serve to promote and disseminate the project's outputs.

Output 1: Open online course

The MTAII OOC is underpinned by three modules that address different aspects of ICT integration for teaching mathematics: (a) Module 1 – ICT use for learning and teaching, (b) Module 2 – Design and development of instructional materials with GeoGebra, (c) Module 3 – Implementation, and evaluation of ICT integration.

After a general introduction of ICT integration and its potential benefits for teaching and learning mathematics (Module 1), participating teachers will be guided through the process of analyzing, designing and developing their own instructional materials based on the interactive mathematics software GeoGebra (Module 2), before they are encouraged to implement their designed lesson in their classrooms and evaluate their experiences supported by experts and peers (Module 3).

As Hew and Brush (2007) have highlighted the necessity for teachers to be able to have easy access to technology, the GeoGebra Math Apps (www.geogebra.org) have been selected for the project. This set of educational mathematics software applications has been developed for teaching and learning mathematics and are freely available all over the world. In addition to the apps being available in a multitude of different languages, the accompanying GeoGebra Materials platform offers additional support and features related to the creation and dissemination of interactive educational materials. The sharing of such materials with students is achieved by the use of GeoGebra Groups, a simplified Learning Management System (LMS) for mathematics educators and their students.

The Technology Integration Planning (TIP) model (Roblyer and Doering, 2014) was selected as the theoretical framework for the OOC. Organized in three phases, the TIP Model provides a practical approach to lesson planning by guiding teachers towards methods and strategies of using ICT for their teaching in an effective way. It also helps them to identify and address the potential challenges involved in this process. Thus, Module 1 of the OOC is based on Phase 1 of the TIP model, the analysis of learning and teaching needs, whilst Module 2 focuses on Phase 2, planning for integration.

Finally, Module 3 will conclude the course by implementing Phase 3, post-instruction analysis and revision, of the TIP model.

Module 1: ICT use for Learning and Teaching

The main goal of Module 1 is to illustrate the purpose and benefits of using ICT for the teaching and learning process, by introducing a variety of suitable technology applications, as well as sharing best practice examples with the participating teachers, in order to demonstrate effective strategies of technology integration. In addition to providing their expertise in ICT integration, the project partner HU (Hacettepe University, Department of Computer Education and Instructional Technology, Ankara, Turkey) will also guide the participants of the OOC through phase 1 of the TIP Model (Roblyer and Doering, 2014). In the second step, teachers are led towards assessing their own technological pedagogical content knowledge necessary for teaching a self-selected topic in their own classroom, based on the Technological Pedagogical Content Knowledge (TPACK) framework suggested by Koehler and Mishra (2009).

TPACK is a model to emphasize the types of knowledge needed by a teacher for effective instructional practice in a technology based learning environment. This framework suggests that ICT integration for teaching specific content like mathematics, requires understanding of the three components Technology (T), Pedagogy (P) and Content (C), as well as of the relationships between them, resulting in seven different knowledge (K) areas: CK, PK, TK, PCK, TCK, TPK, and TPCK. According to Koehler and Mishra (2009) teachers should have technological pedagogical content knowledge in order to be able to effectively integrate ICT into teaching.

In order to support the TPACK framework, each of the project partners contributes according to their area of expertise, with HU providing TPK, JKU (Johannes Kepler University, Department of Mathematics Education, Linz, Austria) contributing PCK and AIGB (GeoGebra Institute Association of Botoşani, Romania) adding TCK. By creating an interdisciplinary OOC, the participants of the OOC should be able to form TPC Knowledge, which represents the intersection of all three knowledge areas. Roblyer and Doering (2014) suggest teachers assess the knowledge they already have and either use this knowledge or identify what they need to learn and broaden their technology-based teaching methods.

Module 2 – Design and Development of Instructional Materials with GeoGebra

Following a general introduction of ICT integration into mathematics teaching and assessment of their TPC Knowledge in Module 1 of the OOC, Module 2 provides an opportunity for the participants to plan their own technology-supported lesson, deepen their technology content knowledge by learning about the basic use of the GeoGebra Math Apps, as well as create their own instructional materials and integrate them into their lesson plans. Both AIGB and JKU will combine their expertise and experience concerning the introduction of teachers to mathematics software, its use for teaching and learning, as well as the development of interactive instructional materials, while developing Module 2.

The main aims of Module 2 of the OOC are: to raise participants' awareness of pedagogical aspects of integrating ICT into their teaching (Bingimlas, 2009) by providing best practice examples for different methods of successful ICT integration into mathematics teaching; and to guide them towards planning and developing their own technology-supported lesson tailored to their own classroom

teaching. The participating teachers are first encouraged to select a mathematical topic relevant for their teaching. They are then guided through the planning process of a technology-integrated lesson by implementing the following steps: (a) deciding the objectives of the lesson and selecting effective assessment strategies to evaluate the success of the lesson; (b) analyzing and preparing their technological teaching environment and inquiring about potential technical support available during the lesson; (c) selecting appropriate instructional strategies and planning how to implement and adapt them to their students' needs; and finally (d) designing and creating appropriate instructional materials and activities that will help their students reach the objectives of the lesson.

Awareness of the different technology-related skills of the course participants, as well as of their potential previous knowledge of using the GeoGebra Math Apps, Module 2 of the OOC will also provide the opportunity to broaden the participants' technology content knowledge by learning about the basic use and features of certain GeoGebra Math Apps, as well as introducing online tools to create instructional materials on the GeoGebra Materials platform.

By providing a series of different tutorial components to introduce the GeoGebra Math Apps, participants will have the option to select the app most relevant to the mathematical topic of their lesson (e.g. geometry, function graphing, manipulation of equations). As suggested by Preiner (2008), the content and structure of the tutorial components are carefully selected, taking into account the potentially different technical abilities and diverse backgrounds of the OOC participants, as well as the difficulty level of mathematical content and the potential complexity of the introduced features of the software. Thus, the content of each tutorial component will be partitioned into a series of interactive worksheets containing one task each, that can be solved quite easily and enable participants to progress steadily through the chosen content. Being aware of the different technological abilities of the course participants, basic tasks will be optional, but will guide the participants towards gaining the skills necessary to also solve more complex, mandatory tasks of the respective tutorial component.

In order to allow for an individual learning pace and the option of selecting content relevant for each of the course participants, each tutorial component will provide automatic and immediate feedback to the user's work on the provided interactive tasks. However, expert course moderators will be available throughout Module 2, providing feedback or assisting with potential technology-related problems participants might encounter.

Furthermore, the participating teachers will learn how to create their own interactive instructional materials by using the online editors for interactive worksheets and online books provided on the GeoGebra Materials platform. Thus, they will explore the option of creating new interactive worksheets relevant for their lesson 'from scratch', as well as experience the possibility of searching the platform for suitable ready-to-use interactive online worksheets of other authors and collecting them in a so called GeoGebra Book. In this process, the participants will be able to decide themselves whether to share their developed materials with their peers or keep them private, only sharing them with the course moderators for feedback purposes. During the entire planning and lesson preparation process, course moderators and experienced GeoGebra material authors from JKU and AIGB will be available to support the participating teachers on a pedagogical and mathematical content level, as well as providing technical support, an aspect that Bingimlas (2009) identified as being a potential barrier for effective ICT integration if lacking. In addition, the course team will offer constructive

feedback about the newly developed materials and lesson plans, increasing the likeliness of a successful and effective implementation of the lesson in the teachers' classrooms.

Module 3 – Implementation, and Evaluation of the ICT Integration

By the end of Module 2, the course participants are expected to have finished the planning stage of their technology-supported lesson, which should be ready to be implemented in their classrooms at the beginning of Module 3. Subsequently, Phase 3 of the TIP Model (Roblyer and Doering, 2014) will be applied in order to guide teachers through analyzing and reflecting about their lesson, as well as to help them to revise and improve their initial lesson plan. During this process, participants will be encouraged to share their lesson plans, as well as reflection about the implementation with their peers, allowing for further revisions and improvements of their instructional materials, based on the expertise of the experts and their peers.

After completion of the OOC, i.e. by the end of Module 3, each of the course participants is expected to have designed and carried out an effective technology-supported lesson, which can be shared among and reused by peer teachers, contributing to an online pool of ready-to-use interactive instructional materials that foster ICT integration into mathematics teaching.

Implementation of the OOC

As the project is a transnational cooperation of the three countries Austria, Romania, and Turkey, the participating teachers are expected to be from diverse backgrounds with different native languages. Consequently, a nurturing and meaningful online environment for the participants is needed. The following guidelines for increased participation in online communities suggested by Çoban and Arkün-Kocadere (2016) have been taken into account for the implementation of the developed OOC: limiting the number of participants and forming communities from small groups; giving the opportunity to interact in their mother tongue; focusing on participants' direct needs; giving feedback; gamifying the online environment; limiting the workload of the participants; expressing the aim of community and expectations from participants explicitly.

Being aware of the local needs of teachers to support communication, feedback and development of educational materials in their native languages, as well as to encourage participation in the accompanying online discussions (Çoban, & Arkün-Kocadere, 2016), the OOC will take place in four language branches - English, German, Romanian and Turkish - as opposed to offering one course requiring all participants to use English for communication. Each of the four language branches will be supported and moderated by the project partners, as well as additional experts fluent in the respective languages. Also, limiting the number of course participants per language will allow for smaller discussion groups as well as individual feedback by moderators and peers, who will be able to directly address the needs of each of the participating teachers, keeping in mind the different educational backgrounds and teaching methods in the respective countries. Each of the four language branches of the OOC will use a different GeoGebra Group as the platform for communication and sharing of materials, providing valuable insights into the needs of different language groups, which will allow us to repeat the OOC in the future and make it available to the educational community in even more languages.

Some gamification elements will be integrated into the OOC to encourage teachers' continued involvement and active participation, as well as to attempt to minimize the drop-out rate during the

course. Gamification can be defined as using game elements in a non-game context. Literature shows that gamification has the potential to solve engagement, motivation, and especially participation problems in online courses (Çağlar & Arkün Kocadere, 2015). In their applied study, Borras-Gene, Martinez-Nunez, Fidalgo-Blanco (2016) found that participation in online courses can be increased by developing online communities and gamification methodologies, in addition to providing support for students' learning and participation, by increasing their motivation. Being aware of the potential benefits of gamification elements for online courses, like rewards, badges, leaderboards, progress bars, and levels, the project team is currently planning a potentially gamified environment for the different Modules of the OOC, including the subdivision of each Module into levels, allowing participants to keep track of their progress in each level, as well as awarding the successful completion of each level. Furthermore, the course participants might be able to compete among each other, taking into account their current level of progress, their activity level of participating in the course, as well as their readiness to support their peers throughout the duration of the course. Finally, successful completion of each of the tutorial components in Module 2 might result in awarding a badge on the respective participant's GeoGebra Profile page, while MTAII certificates will be awarded for a successful completion of the entire OOC.

As a professional development opportunity for in-service mathematics teachers, the time-frame and duration of the OOC were planned carefully and took account of the limited time available to teachers during the school year in general, as well as the potentially different schedules of the school year in the countries of the participating teachers. Thus, the OOC will be implemented over 6 weeks with an expected workload of about 3 hours per week for the participants (M 1 - one week; M 2 - three weeks; M 3 - two weeks). Since the implementation of the developed lesson plans involves students, the participating teachers will be informed about the general expectations of the course and the required classroom teaching at the beginning of the OOC to allow for sufficient time for organizational issues related to technology-supported teaching methods.

Output 2: Open Educational Resources (OER)

After the OOC, the materials developed by the participating teachers will be reviewed by the GeoGebra experts at JKU and AIGB and selected materials will then be transformed into OER, to help overcome the aforementioned scarcity of resources for ICT integration (Bingimlas, 2009; Hew, & Brush, 2007). The project partners will develop best practice examples and make lesson plan suggestions for different teaching methods that can be added to the interactive GeoGebra materials developed as part of the OOC. By translating these materials into multiple languages involved in this transnational project and publishing them on the GeoGebra Materials platform, a large community of teachers will have access to these high-quality instructional materials. In this way, teachers (and especially those who are new) to the concept of ICT integration in their everyday teaching, can benefit from the outputs of the project by getting access to a variety of ready-to-use interactive instructional materials, as well as to the experiences and expertise of teachers, who are expert ICT users for teaching and learning mathematics.

Output 3: Online Teacher Community (OTC)

The third output aims to establish an online teacher community to complement the OOC and ensure sustainability of the project outcomes. Literature shows the success of online teacher communities

for professional development in general, as well as for supporting the ICT integration into teaching and learning process in particular (Arkün, & Aşkar, 2013).

The OTC will be designed and conducted with the support of the 3 project partners (JKU, HU, AIGB) who developed the OOC and will use GeoGebra Groups as the underlying platform. Again, four different OTC branches will be provided, allowing participants to communicate in their native languages, while the moderators of the OTC groups will also communicate in English across these language groups. In addition, high quality interactive instructional materials developed in either of the languages might be translated and adapted to the teaching methods of the other language OTC branches, making them available to a larger community of teachers across the different countries.

Throughout the lifespan of the project, the OTC will provide a platform for teachers to discuss their experiences with technology-supported teaching and learning, exchange their lesson plans and educational materials, and share their reflections and improvement suggestions. The participating teachers will receive support from peer teachers as well as from the experts in the partner organizations through the teacher community. The OTC is expected to have a long life cycle as it will be supported by the project partners as well as GeoGebra experts, with the goal of becoming self-sustaining community groups by the end of the project, being complementary to other efforts of ICT integration and sharing of high-quality interactive instructional materials. Furthermore, as opposed to the teacher communities on some other subjects, the project's OTC will be a math teacher community focused on ICT integration into teaching and learning, developing ready-to-use high-quality instructional materials in different languages tailored to the diverse needs of mathematics teachers and students in different countries.

Conclusion

The main aim of the project described in this paper, is to combine the diverse expertise of the project partners for the design and development of a professional development environment for mathematics teachers consisting of an open online course, open educational resources and an open teacher community, and focusing on overcoming a selection of different barriers of ICT integration into teaching and learning of mathematics.

The online course is designed to support the participating teachers throughout the entire process of planning and developing materials, while also supporting them during the implementation of their technology-integrated lessons and subsequent revision of their materials. Through the development of high-quality and ready-to-use resources in different languages, many mathematics teachers will be able to benefit from the project's efforts by joining an online community of mathematics teachers willing to integrate ICT into their everyday teaching. By building a community that combines the expertise of researchers and actual classroom teachers from different countries and languages, the project aims to develop a network providing continuous support for teachers at all stages of ICT integration into the teaching and learning process, that we hope will become self-sustaining and outlast the duration of the actual project.

While the content of this paper focuses on the design and implementation of the described project, further studies will describe the accompanying research design and analyze its outcomes.

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The effect of GeoGebra collaborative and iterative professional development on in-service secondary mathematics teachers' practices

Houssam Kasti[1] and Murad Jurdak[2]

[1]Lebanese University, Lebanon; houssamkasti@gmail.com

[2]American University of Beirut, Lebanon; jurdak@aub.edu.lb

Integrating technology in education is still not an easy task, teachers' adoption of technology in their teaching is even more problematic and the wide availability of technology made things more challenging. This research is a multiple case study that aims to study in depth the effect of a GeoGebra (a free mathematics software) intervention on the teaching of in-service mathematics teachers in secondary schools who follow the Lebanese curriculum. The type of the study is Design-Based Research that focuses on working closely with practitioners in collaborative and iterative manner in the real context to add principles to theory and practice. Results showed an increase in the extent teachers use GeoGebra in their student-centered teaching approach.

Keywords: Technology integration, professional development, in-service secondary teachers, GeoGebra, design based research.

Introduction

When new technologies appear in medical or industrial fields, there is often a rush to replace obsolete tools with new ones, the staff get immediate training on their use and the adoption level is high and quick. Why does this not happen in the education field? Answering this question is not an easy task due to the multiple factors are involved in adopting technology and the rate of change in the education field, which is known to be slow.

Literature review

Research has extensively focused on the problem of technology integration in general and in mathematics in particular. First, research in many countries has shown that technology still plays a marginal role in mathematics classrooms and that access to technology resources, educational policies, and institutional support are insufficient conditions for ensuring an effective integration of technology into teachers' everyday practices (e.g., Cox, Abbott, Webb, Blakely, Beauchamp, & Rhodes, 2004; Cuban, Kirkpatrick, & Peck, 2001; Goos & Bennison, 2008). Second, research studies in general focused on some aspects of the integration problem such as lack of teachers training (e.g., Law, 2008; Tondeur et al., 2008) or lack of theory (Mishra & Koehler, 2006). Others suggested certain solution(s) such as conducting professional development of specific characteristics, working with mentors (Kratcoski, Swan, Mazzer, 2007), working in a community-based inquiry environment (Lavicza, Hohenwarter, Jones, Lu, & Dawes, 2010), or working based on a theoretical framework such as TPACK, but most of these suggestions "have crashed on the hard rocks of the classroom" (Herrington, McKenney, Reeves, & Oliver, 2007, p. 9). Third, in most studies the methodology used is not sufficient for such a complicated multi-faceted problem, and this partially explains why research has had limited impact on practices (Herrington, McKenney, Reeves, & Oliver, 2007). A key factor is that teachers should be able to actively participate in the process of technology integration (Voogt et al., 2011). To summarise, this research aims to study how a collaborative and

iterative work with in-service mathematics teachers affects their level of GeoGebra integration in their teaching to answer the following research questions:

1. How does a cooperative and iterative intervention affect in-service secondary mathematics teachers' practices regarding the integration of GeoGebra in their teaching?
2. How do participants' Valsiner's three zones mediate the impact of the intervention on teachers' practices regarding the integration of GeoGebra in their teaching?

In this study we have used the Valsiner's zone theory, which states that the factors that affect teachers' use of technology can be categorized into three zones: (1) Zone of proximal development (ZPD) which includes skill, experience, and general pedagogical beliefs; (2) Zone of free movement (ZFM) which includes access to hardware support, curriculum and assessment requirements, students (3) Zone of promoted action (ZPA) which includes pre-service education, practicum courses and professional development (Goos et al., 2010).

Methodology

Three iterations of a design based research (DBR) methodology were used in this study across two stages (Figure 1).

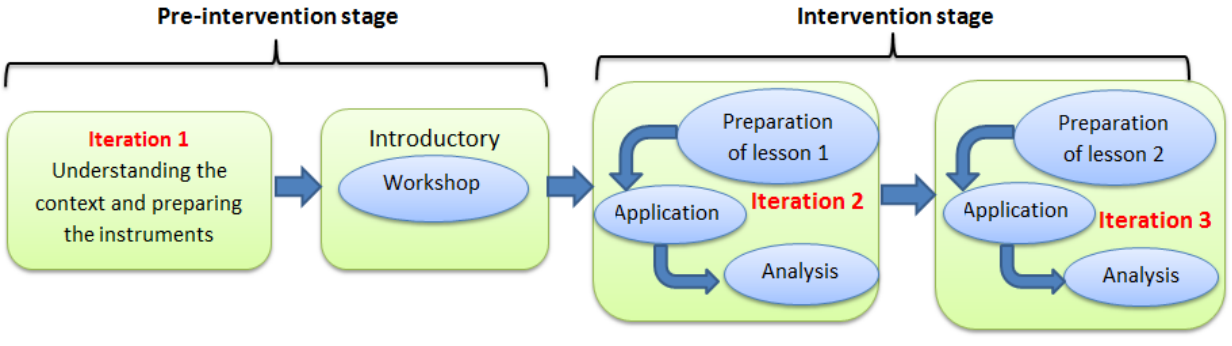

Figure 1. **The stages of the study**

The first pre-intervention stage was dedicated to understanding the situation of integrating GeoGebra in the Lebanese curriculum, piloting the GeoGebra activities and testing the instruments. Six workshops were conducted over two years and a pilot study with two teachers. At the end of this stage four teachers (other than the ones in the pilot study) were selected as cases for the study. After selecting the participants, a 3 hour-workshop was conducted by the researcher with the four participants to ensure that all participants had acquired the basic features of the software (GeoGebra). In addition, we discussed as a group the topics in the secondary mathematics Lebanese curriculum that could be better taught with the use of GeoGebra. The second stage was the intervention stage, which comprised two iterations. In this stage collaboration was one-to-one between the researcher and each of the participants. In the first iteration, the participating teachers decided which lesson they wanted to teach with GeoGebra in accordance with their school mathematics scope and sequence. They were provided with a ready-made GeoGebra activities (made by the researcher) to be implemented in their classes. In the second iteration, teachers adapted already made GeoGebra activities and/or made their own GeoGebra activities. Three visits were conducted with each participant at his/her own school and according to his/her available time. The first visit was to prepare for the first lesson. The second visit was to evaluate the first lesson and prepare for the second lesson.

Analysis of data collected from the instruments was done before starting the second iteration as required by a design based research methodology. The last visit was to evaluate the second lesson and give a general overview of the whole experience.

Instruments

For the pre-intervention phase, three questionnaires were administered by the participating teachers: (1) Demographics questionnaire, (2) Instructional Practices in GeoGebra Questionnaire IPGQ (Form 1), (3) Barriers (grouped in zones) in Using Technology Questionnaire BUTQ (Form 1). The purpose of these questionnaires was to measure teachers' current (before the intervention) integration practices of the GeoGebra software in their teaching and the barriers (grouped in three zones) that affect their technology integration. After conducting the first lesson, a semi-structured interview parallel form was used (IPGSI (Form 2) and BUTSI (Form 2) to measure the impact of the intervention on teachers' practices and to find out to what extent the zones could mediate that effect. In addition, another instrument was used to assess the GeoGebra activity itself, the Lesson Assessment Criteria semi-structured Interview (LACI), which is based on instrument by Harris, Grandgenett & Hofer (2010).

The analysis was done in general for the four participants and later individually. The general analysis looked for the general impact of the intervention and for the dynamicity of change in the extent of use in each category of the practices and its subcategories. For the impact of the intervention we were interested in the change in the extent of use of GeoGebra at the end of implementation, whereas for the dynamicity we were interested in the pattern in the extent of use of GeoGebra of change happened in between the implementation stages. The dynamicity could be: (1) static: there was no change in extent of use in between the implementation stages or (2) dynamic: there was a change in extent of use in between the implementation stages.

Participants

In the sixth (last) workshop conducted by the researcher attendees were given the pre-intervention questionnaires mentioned above. Based on the answers, for the practice instrument, the values were 0 (never use GeoGebra), 1(sometimes use GeoGebra), and 2(most of the time use GeoGebra). The average of all the questions was calculated. Similarly the average for each zone was calculated in the zone questionnaire that consists of 27 questions. Based on these results, four cases were selected (Pseudonyms: Tima, Sara, Amani, and Hazem) in a way that they differ among themselves in practice level and/ or in at least one barrier level. Table 1 represents the characteristics of each participant.

Name	Age	Highest degree	Teaching experience	Practice level	ZFM	ZPA	ZPD
Amani	50-55	BS	25 years	Low	Moderate	Moderate	Low
Tima	23-26	Masters +TD	2 years	Moderate	Low	Moderate	Not*
Sara	33-40	BS	7 years	Moderate	Moderate	Low	Not
Hazem	41-50	Masters	31 years	High	Moderate	Not	Not

Table 1. Participants domographics, practice and zones level

****Not: the zone is not considered as a barrier to GeoGebra integration***

GeoGebra modules

The criteria used for lesson selection are based on those identified by Angeli & Valanides (2009) called ICT-TPCK. The GeoGebra activities were prepared by the researcher and tested on both students and teachers. The activities were designed based on the following criteria: Each activity: 1) should be student centered, 2) can be conducted by students in a computer lab or elsewhere (classroom or at home), 3) allows student to discover the concept or theorem under study, 4) includes immediate application of the concept under study, 5) does not require prior knowledge of the software.

Each teacher selected an activity according to his/her scope and sequence, so each teacher applied a different GeoGebra activity. Table 2 shows type and place of activities applied by each teacher.

	Activity 1	Place	Activity 2	Place
Amani	Sign of quadratic polynomials	In class	Derivative	In lab
Tima	Vectors	In lab	3D	In class
Hazem	Equation of a straight line	In class	Thales Theorem	In class
Sara	Translation of functions	In lab	Vectors	In lab

Table 2. **The intervention activities conducted by participating teachers**

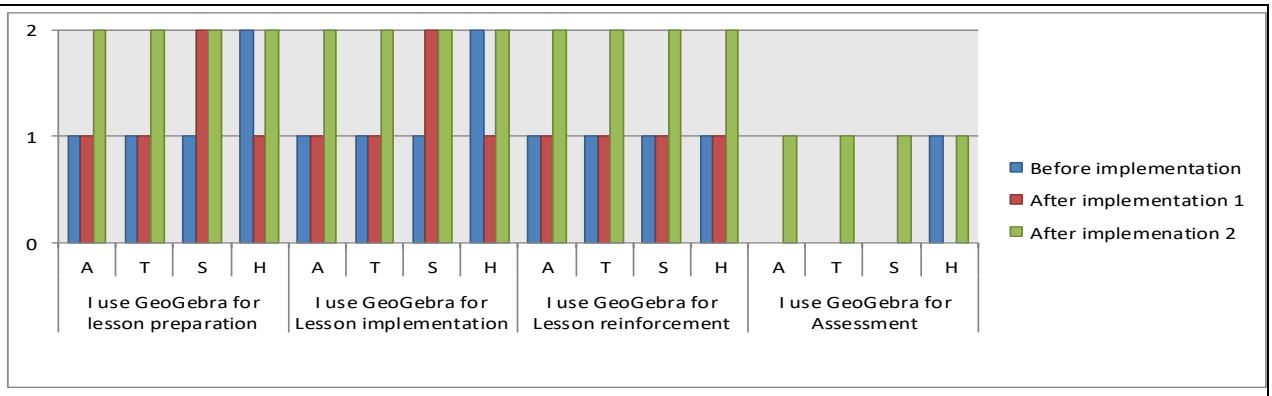

Figure 2. The extent of using GeoGebra by the participating teachers over the three stages: Before the intervention, after implementing the first lesson , after implementing the second lesson. 0: Never; 1: sometimes; 2: Most of the time A: Amani; T: Tima; S: Sara; H: Hazem

Results

Stage of Use of GeoGebra

Figure 2 shows that the pattern of impact was the same for using GeoGebra in *lesson presentation*, *lesson implementation*, and *lesson enhancement* but different for *assessment*. For *lesson presentation*, *implementation*, and *enhancement*, in general, participants started with '*sometimes* use GeoGebra' and ended with '*most of the time'* after the second lesson. For assessment, there was a slight breakthrough from '*never use* of Geogebra in assessment' to *'sometimes use'* for each of the four

participants. For all the stages of using GeoGebra, in general, the change was static then dynamic. Probably more time was needed for the change to happen prior to the second implementation, which was due to teachers' need to: become more confident in using the software; be more knowledgeable; and have more free movement.

Concerning the stage of teachers' use of GeoGebra the intervention resulted in: (a) an increase in using GeoGebra in most stages mediated positively by teachers' ZPD, and (b) an increase in teachers' appreciation of GeoGebra as a teaching tool due to the characteristics of the activities. There was interdependence between confidence and the extent of using GeoGebra in each stage. When teachers applied the activities, this led in an increase in teachers' confidence which in turn led to an increase in the extent of GeoGebra use in each of their teaching stages. There was a low impact on using GeoGebra in assessment mediated by teachers' ZFM. Three particular ZFM factors mediated negatively the impact of the intervention on assessment, these factors were: (a) *Lebanese national curriculum* which is so demanding with little time left for discovery, (b) *Lebanese national assessment policies* which assess mostly procedural knowledge, and (c) *school assessment policies* which are mainly set by the school administration and teachers have little impact on changing them. The characteristics of the GeoGebra activities that made impact of the intervention more effective were: (a) the effectiveness of the GeoGebra activity, (b) the ease of operating the software by students, (c) the strong alignment between the activity and the curriculum, and (d) lastly the strong fit of the activity with the instructional strategies each teacher uses.

Method of use

It is important to use GeoGebra, but what is more important is how to use it. In this category of practices the intervention had, in general, *no* to a slight increase in the extent of use in most subcategories and the general pattern of change was static with minimum dynamicity. For example the intervention did not affect Amani's use of GeoGebra for '*presenting a lesson'* or for '*conducting an activity with the help of students'*. Amani used for the first time GeoGebra for '*discovery activity done by students'* or for '*students to present their work'* but that change was static (never use) then dynamic. The impact of the intervention on Amani's *method of use* was a change in her teaching method to become more student-centered (*activity done with the help of students)* mediated positively by her ZFM and her ZPD. A second example is Tima, despite her ZFM factors that mediated Tima's extent of use of GeoGebra in her methods of teaching she applied for the first, time discovery activities done by students in the computer lab and/or in class. The collaboration between Tima and the researcher increased her self-confidence, skills and knowledge and that mediated positively her GeoGebra application. A third example is Sara. Before the intervention Sara was a moderate user of technology in general, and GeoGebra specifically, but the *lack of a computer lab* in her school and the *lack of hardware* in her class were the main barriers to increase technology integration. Sara used to show her students some applets using her class LCD connected to her own laptop but for the first activity she made a huge effort to take her students to the computer lab to apply discovery activities and she said:

> After this experience (applying GeoGebra activity) for the first time and in a lab I will change a lot of things (in my teaching) now I have a lab for secondary. Frankly I will not use that with an LCD in the class to show students such things, there is nothing called to show (not effective)

> showing them is like treating them as babies not capable of applying and concluding results, when they do it, it is different even for me I felt different. (Interview 2, November 7, 2015).

The intervention had an important effect on increasing the use of *conducting discovery activities done by students* in the computer lab and that change was not the same dynamicity for all teachers. The barriers teachers faced in this part of the practices were the *accessibility to the computer lab* and *curriculum requirements* (ZFM) but these barriers minimally mediated the impact of the intervention.

Place of use

Similar to *method of use* category there was *no* to slight effect of the intervention on the extent of use of GeoGebra in their classroom or at home. There was a noticeable impact on the use of GeoGebra in the computer lab since three out of the four teachers tried one or both of the GeoGebra intervention lessons for the first time in the computer lab. This was not a surprise because to use GeoGebra in class or in the computer lab is related to availability of equipment and the way of using GeoGebra. This change was not the same dynamicity for all of the teachers. Amani's change was static then dynamic, Tima's change was dynamic then static, Sara's change was dynamic, and Hazem's change was static then dynamic.

An example is the case of Sara, her first student-centered discovery activity was the activity she applied in her first lesson of the intervention. In this lesson she sensed the importance of discovery activities and how this students motivated the students and she said:

> I gave them four cases with aim of acquaint to GeoGebra trace, animation, and sliders. They liked a lot so and got their attention and interest. Gave them the function act printed and they started working, one student volunteered to help me… Students enjoyed a lot the activity and attained all the required objectives. They could see things (Interview 2, November 7, 2015).

A second example is Hazem' case, the intervention did not affect the place where Hazem uses GeoGebra. He mentioned *availability of a computer lab and/or the accessibility to the laptops* (ZFM) to be the only barriers to more extent of using GeoGebra in his teaching. He did overcome that barrier by asking every student to bring his own device mainly tablets. Since his first interview Hazem affirmed his continuous use:

> I am willing to use GeoGebra if it is related to my lesson, I consider working with GeoGebra as 'clean work' contrary to board drawing (draw, redraw…). I encourage my students to use it; I already introduced them on its features and how to use. (Interview 1, November 7, 2015)

In his second interview he said: "all students contributed [in the activity discussion], to a certain extent, according to their motivation. If they bring their own device things would be more beneficial." (Interview 2, February 11, 2016)

Summing up, due to the intervention the extent of using GeoGebra for discovery by students in the computer lab increased. For all categories of the practices the *accessibility* and *availability of hardware* were the main negatively mediating factors to higher levels of practices for all participants. The general pattern of change in the practices was more from static to dynamic in the *stages of use*, static in the *method of use* and in the *place of use*.

Discussion

It seems that unlike the medical or the industrial fields, the educational field is more complex in integrating technology in terms of social and psychological factors of all the stakeholders.
In the medical field for example the instrument for measuring blood pressure is one tool that is used for all people, young or old, under-weigh or over-weight… To use this instrument or an updated version of it does not require social acceptance or/and making the medical staff believes of its importance. On the other hand, in the educational field there is no technology that fit all ages, abilities, and intelligence levels… Deciding to use any instrument in a certain class needs to pass many filters in order to be an integral part of the teaching-leaning process.

Recommendations

To see change in mathematics teachers' extent of using GeoGebra in particular and technology in general it seems one day workshop is not the perfect choice according to this study. Maybe with such professional development teachers' knowledge might change quickly but more has to be done in order to change their practices. How should universities prepare their pre-service teachers to be ready to use technology most of the time in their teaching? How should professional development be designed to make sure teachers' practices are changed regarding integrating technology in teaching? Maybe this study answers some of these questions but more work still needs to be done to solidify them.

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"One of the beauties of Autograph is … that you don't really have to think": Integration of resources in mathematics teaching

Lina Kayali and Irene Biza

School of Education and Lifelong Learning, University of East Anglia, UK; l.kayali@uea.ac.uk

This paper introduces part of a larger study on the use of technology, specifically mathematics-education software, by secondary mathematics teachers. It presents some of the data collected with the aim to investigate teachers' use of mathematics-education software: why are certain settings used, or underused, how are they used, and what are the reasons behind such use? The findings will be discussed by drawing on the documentational approach (Gueudet & Trouche, 2009) and teaching triad (Jaworski, 1994). The data comprised one interview and one lesson observation with a secondary mathematics teacher. While the documentational approach provides an overview of the set of resources being integrated to achieve a specific goal, the teaching triad offers a lens to observe teachers' considerations when implementing a task in a mathematics lesson.

Keywords: Tasks, documentational genesis, teaching triad, mathematics education software.

Introduction

The complexity of the teaching profession imposes several factors that impact upon teachers' classroom actions, that include not only their beliefs and knowledge but also their experiences and the educational context in which they act (Biza, Nardi, & Joel, 2015; Speer, 2005). Teachers, while planning and teaching, consider the contexts they work within: their students, school environment, curriculum, etc. In other words, any study of teachers' practices should take into account the different contextual conditions in which these practices develop to include personalities, institutions, circumstances, epistemology, time issues and materials (Herbst & Chazan, 2003). The study presented in this paper forms part of the PhD research of the first author and it investigates mathematics teachers' ways of balancing the different elements in their working environment, especially when using technology (in this study mathematics education software, i.e. software designed for mathematics teaching and learning purposes), by looking at their practices or intended practices within specific contexts. Furthermore, our work examines any gaps between intended technology use in mathematics classrooms and actual teachers' practices. To this aim we invite teachers' views on hypothetical classroom situations that involve teaching with technology in written responses and follow-up interviews. Then, we observe teachers' use of technology in their classroom. In this paper we present preliminary analysis from one participant, Adam, by drawing on two theoretical perspectives: the *documentational approach* (Gueudet, Buteau, Mesa, & Misfeldt, 2014; Gueudet & Trouche, 2009) and *teaching triad* (Jaworski, 1994).

The documentational approach

The documentational approach looks at teachers' interactions with resources where a resource is defined as "anything that can possibly intervene in [a teacher's] activity", it can be an artefact (e.g. a pen or a mathematical technique), a teaching material, or even a social interaction (Gueudet et al., 2014, p. 142). Adler (2000, p. 207) adds that "resource" can be also "the verb re-source, to source again or differently". During their interaction with resources, teachers develop schemes. A scheme

is a set of organised procedures carried out on an artefact (Gueudet & Trouche, 2009). It consists of "the *goal* of the activity; *rules of action*; *operational invariants*; and *inferences*" (Gueudet et al., 2014, p. 140, italics in original). Here, operational invariants are cognitive concepts established throughout the activity to be used in comparable situations (Gueudet et al., 2014). For teachers, these constitute "professional knowledge" (Gueudet et al., 2014, p. 142). The documentational approach describes a two-way influence between a resource and a teacher: a resource affects the teacher's actions and knowledge; and the teacher's perceptions and experiences impact on the way the resource is used (Gueudet et al., 2014, p.140). A management process, which Gueudet & Trouche (2009) call instrumental orchestration, is required in order to organise the learning environment (e.g., space, time, dialogue) by the teacher, whose responsibility is to manage the process according to the requisites of the task (Gueudet et al., 2014). When a teacher uses a set of resources according to a specific scheme for a specific goal, s/he creates a document. Such a development of a document is called *documentational genesis.* Thus, the documentational genesis is the process of a teacher developing schemes for adapting different sets of resources to achieve a specific target (Gueudet et al., 2014). The documentational approach studies the development of "structured documentation system[s]" that represent teachers' work and progress as a result of influencing and being influenced by different resources (Gueudet et al., 2014). In this study we have conjectured that teacher's schemes are dynamic and are being re-adapted from one situation to another and that the teaching triad (Jaworski, 1994) can help the exploration of those schemes.

The teaching triad

Jaworski's (1994) teaching triad (TT) addresses classroom management as an act of harmony between three domains of activity: sensitivity to student (SS), mathematical challenge (MC) and management of learning (ML). These domains are evident when a teacher plans a lesson and starts to think of how to consider teaching a specific mathematical idea (MC), particular students' needs (SS), the best way to work on the task with the students (ML) (group work, individual work or classroom discussion). The same domains will be in play during lessons, but within a different context as this time the interactions with the students are happening and the teacher should respond on demand, in many cases by diverting from what was planned.

As Jaworski (1994) and Potari and Jaworski (2002) suggest from the "macro analysis" of classroom interactions, alongside the TT domains, teachers' plans and practices are also influenced by social factors, such as: time pressure; having to complete a set syllabus; the requirement that students know specific things for exam purposes; expectations from the teacher; school ethos; and the training provided for teachers. Such factors seem to be at the centre of teachers' considerations and they include students' social culture, teaching resources and materials, syllabus, assessment schemes, time restrictions, room constraints, and cultural considerations of what constitutes good teaching practices (Goos, 2013, p.523). Hence, the TT domains, along with these factors, reflect the range of considerations mathematics teachers have to balance. The TT can be "used as an analytical device (by researchers) and as a reflective agent for teaching development (by teachers)" (Potari & Jaworski, 2002, p. 351). We are conjecturing that the teaching triad domains are related to schemes' development regarding the use of resources. Sustaining the goals of the resource use, depends on how teaching is balanced by the teacher. Operational invariants can be derived from artefacts, mathematical concepts or social environments. All the above can be used to satisfy a specific goal

and produce a proper usage and inferences, but these are flexible techniques of balancing and rebalancing of the TT domains from one lesson to another. The analysis presented here aims to investigate this conjecture.

Technology in mathematics teaching - A view through the lenses of the documentational approach and the teaching triad

Technology software and hardware devices are "artefacts" (Gueudet et al., 2014, p.141) and can be adapted to provide access to formal mathematical knowledge. They afford "opportunities for additional student actions, such as the manipulation of on-screen objects and the ability to make a range of mathematical inputs, which places an additional demand on teachers as they strive to make sense of a diversity of student activity in real-time" (Clark-Wilson & Noss, 2015, p. 95). Thus, interactions with technological resources influence teachers' documentational geneses by developing the ways they organise their classroom activities and manage learning situations with impact on the shape of the teaching process and on the way knowledge is communicated. When employing technology, resources become more complex, and so do the TT domains. Sensitivity to student becomes more evident (e.g. if students know more about technology than their teacher). Tasks can be more challenging for teachers to design, and the management of learning becomes more complicated with the higher chances of distraction. We also re-emphasise the importance of social factors when technology is used based on several premises. First, the technology use dependency on the teacher training provided (Gueudet et al., 2014, p.144; OECD, 2015, p.69). Second, the availability of hardware and internet connection (Bretscher, 2014, p. 66; OECD, 2015, p.61 & p. 146). Third, the national curriculum obligations (OECD, 2015, p.70). Fourth, and most importantly, the education policies that aim to embed technology (OECD, 2015, p.50).

Methodology

This paper reports from the first phase of a project that looks at secondary mathematics teachers' work with technology that involves participants' written responses to *situation-specific tasks* alongside follow up interviews and classroom observations. Situation-specific task methodology has been suggested by Biza, Nardi, & Zachariades (2007, p. 301) where tasks are given classroom situations that "are hypothetical but grounded on learning and teaching issues that previous research and experience have highlighted as seminal; are likely to occur in actual practice; have purpose and utility; and, can be used both in (pre- and in-service) teacher education and research through generating access to teachers' views and intended practices". The study is conducted in England and participants are secondary school mathematics teachers with different levels of experience and training. The work is based on providing qualitative findings established on an interpretative research methodology (Stake, 2010, p. 36).

In this paper we discuss the written response to the situation-specific task presented in Figure 1 (we call it the 3D Task), the follow-up semi-structured interview and the lesson observation (75 minutes) of one participant, Adam. Both interviews and observations were conducted by the first author. The situation described in the 3D Task regards an open investigative question to be given to the students. It did not suggest any specific use of technology, and left that to be decided by the teacher. The situation regards a geometrical problem with a potential consideration of the affordances of software available at the school where the data were collected, such as Autograph

(http://www.autograph-math.com/) or Geogebra (https://www.geogebra.org/). Adam was invited to offer a written response to the 3D task and, then, he was interviewed, to clarify the answers and offer more elaboration where needed. Then, a lesson was observed of a Year 12 class (17-18 year-old 4 female and 5 male students) and was audio-recorded. The focus of the observation was Adam's use of resources, especially his use of Autograph or Geogebra, which he said he frequently used, and his classroom management. At the time of the data collection, Adam was a mathematics teacher with four years' experience, during which he taught students aged 12-18 years. He held degree in economics, a postgraduate certificate for teaching mathematics at secondary level, and was about to finish his master's degree in educational practice. The school had interactive whiteboards, a computer lab, and Geogebra and Autograph software installed on all computers.

3D Task

A group of Year 11 students are asked the following question:

Design a milk container with capacity of 1L. What dimensions and which design uses less materials? Why?

- What are the mathematical ideas and activities addressed in this question?
- Would you use this question in class? Why or why not? What are the learning objectives for which you would use this question? Would you modify it?
- Would you use technology with this question? If yes, what type of technology? If no, why?
- If you were to use technology, how would you use it?
- What teaching approaches and resources would you suggest for this question?
- Do you anticipate any problems or challenges (either with students or resources)?

Figure 1: The 3D Task

A preliminary analysis of teachers' comments and interactions during interviews and lessons was performed for the transcripts (May, 2001). This was coded according to the Teaching Triad (Jaworski, 1994) into SS, ML, and MC. During each interaction, we explored the teacher's interactions with the resources, according to the documentational approach (Gueudet et al., 2014). We then reviewed the results from the interview and observation, and offered a discussion according to TT and documentational approach together.

Adam's responses to the 3D Task and follow up interview

Adam identified mathematical ideas involved in the 3D Task, such as "volume", "surface area" and "calculus". In his response to the task, he frequently repeated the word "scaffold" to state that he would try to adapt the task according to the students' needs and "prior knowledge". He emphasised that he would not use the problem as it is because it needed a lot of scaffolding and it included "too many variables". During the interview, he explained that the scaffolding would include giving hints and examples and even values to work on for weaker students. He said he will not use 1L in this problem, but would use a bigger number:

> I think straight away students having to think of a length width and height that times to get 1 will be quite difficult for students… They might be able to go 1 1 1 and they might be able to go 2 ½ 1 or something like that. That will be it, they'll really struggle.

He wrote that he would use technology with the task for "gradient of curves on Autograph" and "to visualise the shapes". During the interview, when asked how and when he would use it during a lesson, Adam suggested "I think as a group activity. It wouldn't be the focus of the lesson though it would just be almost the point at the end". When asked to elaborate he said:

> I think it is because one of the beauties of Autograph is that it means that you don't really have to think... I want the student to be thinking about problems and how to approach problems. I think

> almost Autograph gives you too much, too much help and then you don't have to think about the shape of the graph because you can just plot it in Autograph. And then other, obviously other reasons a lot of my students have never used Autograph even at key stage five[1]. So, to start understanding it, it will take quite a long time and a lot of effort just to get the students to understand it to start with.

Adam said he will not use the task as it is because although it works well in "an ideal world", it does not go well with the way the syllabus is set. He anticipated problem with keeping track of calculations, prior knowledge and many involved variables (e.g. Adam suggested if a student chose to design a cylinder container, the case would be very confusing because s/he would have to think of adjusting the radius and height of the cylinder in order to find the minimum surface area).

Adam's teaching observation

The observation was on a revision lesson about solving simultaneous linear and modulus equations (i.e. equations that include absolute value). Adam started by moving a stick in the air in order to draw a specific graph, and asking the student to recognise the graph. One of these graphs was the *sine* graph, but the students seemed to be confused about what graphs were being drawn. Then, Adam asked his students to solve some problems that were displayed on the board. All the problems apart from one (which was designed by Adam) were chosen from the textbook. During the lesson, Adam used Autograph to check the answers given by the students, he entered the functions and the graphs were projected on the board. Then a discussion/demonstration of the algebraic solution was led by him on the white board. For example, for the simultaneous equations: $y = |x + 2|$ and $y = 3$, he asked students to draw the graphs on their notebooks and see the solution before solving it algebraically: "You will get two points, you can see this graphically". Then, he started to write one of his student's algebraic answers on the board: "$3 = x + 2$ or $-3 = x + 2$". He then commented on the student's answer: "So, math says $x = 1$ or -1... What text book would say is $y = (x + 2)$ or $y = -(x + 2)$. Textbook would just say that, I'll probably do it this way". Later, with the problem that followed, he commented that: "This is GCSE grade C[2] […] This is mark C in C1[3]". He repeatedly encouraged the students to solve another problem he displayed on the board by saying that it is an "exam question". When the students asked Adam why they should learn modulus equations, he went to his computer and googled "when to use modulus equations in real life" and gave the answers accordingly "Distance, currency exchange…". Two of the students finished with the problems on the board earlier than the rest of the class and Adam gave them an extension problem which might have been suggested spontaneously in response to the need of extra work. The extension problem was in two parts, the first asked for two different modulus functions that do not intersect, the second asked for two that intersect once. "Is that possible? Can you give me two that intersect once?", Adam asked the class, and the dialog below followed:

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[1] Key stage five is post-16 school education in England i.e. for students aged 16-18.

[2] GCSE stands for the General Certificate of Secondary Education. It is the qualification taken by school students aged 14–16 in the UK (except Scotland). Its exams are graded on a scale of A* to U, with A* being the highest grade and U the unsatisfactory. A grade/mark C reflects an average progress (pass).

[3] C1 stands for Core1 and it refers to one of the mathematics textbooks, used at Adam's school, for students aged 16-18.

Student A: $y = |x|$ and $y = 2|x|$, shift across
Adam: Oh, ya it is.
Student A: Ya, you've translated it.
Student B: $y = |x - 4|$ and $y = 2|x|$.
Adam looked at the graphs on Autograph and nodded in what seemed like a hesitant agreement
Student C: Change the slope.
Adam amended the equations as student C suggested and wrote $y = 2|x - 4|$ and $y = 2|x|$ without commenting on student's B answer

Adam did not follow up student's B response or student's C correction, but moved straight to a completely different activity by which he concluded the lesson.

Analysis

From his responses to the task and the observation we notice that Adam's resources were the textbook used at his school, help cards, a computer, Autograph, Excel, Google, interactive whiteboard, the stick he used at the beginning of the lesson observation, information about exam grades and questions, past experiences with students along with the mathematical concepts and methods. Adam's appreciation of Autograph's ease of use as a tool for visual representation was evident, so he used the software to check students' work, and present graphical solutions before going for algebraic ones. So, he would ask his students to solve graphically, check that their graphical solutions are right according to the answers on Autograph, and then ask them to find the same answers algebraically. However, Adam seemed being confused by the Autograph when it came to student B's answer on which he seemed to hesitantly agree. This might be because only one intersection point was visible within the displayed part of the graph. In this case, Adam missed the opportunity to use the full affordances of Autograph in order to improve student's B answer and to explain the correct answer to the rest of the class. There was no evidence that the rest of the class, apart from student C, realised where the problem was and how it was amended.

In terms of the TT, Adam indicated sensitivity to students "they don't know Autograph", "prior knowledge", "scaffold", "weaker students" (SS). In his teaching choices, he also showed consideration of the syllabus he had to follow, exam questions and the timeframe he had to adhere (MC and ML). The way he intended to use resources showed an attempt to balance mathematical challenges (MC) (e.g., change 1L, exercises from the textbook) with students' needs (SS) (e.g. students do not know how to use Autograph, providing extension question when needed), and management of teaching (ML) (e.g., use technology at the end of the lesson as a group activity, encouraging pair work when solving textbook exercises, graphing the equations to see the answers and then doing the algebraic solution because "Putting it in a graph might be easier") with attention to management of learning with technology (e.g. technology takes a lot of time).

Now, we will look at how Adam used the available resources to design and implement his teaching. Along with the textbooks that are being used at his schools, he mentioned he would also use help cards with hints or examples. These will help him "scaffold" and build on "prior knowledge", these terms seem to be adopted during Adam's teaching practice or teacher's education courses for reflection on students' needs (SS). Also, he drew on his teaching experience (as a resource) when he mentioned in the interview that students would struggle to "keep track of their calculations" (ML and SS). In terms of Autograph as a resource, Adam would use it as a graphing software that helps

visualise graphs and shapes and shows answers (ML and MC). The data showed the two-way influence between Adam and the resources. For example, Adam's belief that students do not think when using Autograph (SS) was influencing the way the resources were used, so he used Autograph to show or check answers (MC). Also, the resources available influenced the teacher's decision, so in this instance he used Autograph to show the graphical solution and then asked the students to do the algebraic solutions keeping the answers from Autograph in mind. Additionally, Adam frequently used the textbook as a source for exercises, so the textbook influenced which mathematical challenge he gave the students (MC). At the same time, Adam used the textbook exercises along with Autograph and, by doing so, Adam's way of managing the teaching affected the way the textbook was used. Adam's management of the teaching situation led us to conjecture that his use of resources is connected with a potential scheme developed in order to properly use the resources available and achieve a specific goal, which in the lesson observation was revising the topic of linear and modulus simultaneous equations. We have identified some operational invariants in Adam's schemes, for example the use of Autograph as a class activity managed by the teacher on the board. However, we believe that more data and observations are needed to further investigate Adam's schemes.

Discussion and summary

The preliminary findings we present in this paper derive from a study on mathematics teachers' practices/intended practices in relation to the used resources and especially mathematics education software.

Adam's attempt to balance the different domains of activity described by the teaching triad was evident in the interview and during the lesson observation; and his interactions with resources were influenced by considerations of these domains as well as considerations of exams' questions and grades, time management, and the syllabus. His use of technology resources was led by him on the board, because of his concern that he would be teaching mathematics and technology use if his students were to work independently or in pairs on computers. Although the teacher used Autograph frequently, his use was mainly for checking answers and displaying visual representations. This is due to his concern that Autograph offers excessive help and would stop students from thinking about the mathematical problems. The pilot observation proved that more clarification about the teacher's actions should be sought from future observations along with pre- and post- lesson interviews. This is because the pre- and post- lesson interviews will give more space for the teachers' interpretations of their classroom actions. More observations are also needed to explore the teacher's documentational work and investigate how the teaching triad helps clarify teachers' considerations when working with resources.

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Identifying and assessing quality criteria for dynamic mathematics materials on platforms

Barbara Kimeswenger

Private University College of Education of the Diocese of Linz, Austria; barbara.kimeswenger@ph-linz.at

This paper describes a PhD research project addressing the issue of the quality aspects of dynamic materials. Platforms with user-generated Educational Resources for mathematics teaching show a wide variety in terms of the quality of the materials. The presented project investigates possible quality criteria for dynamic materials based on the opinions of experts in electronic resource development, who describe their views on educationally valuable use of dynamic materials. The relevance of the findings that have emerged is examined through a further quantitative study. Results of this project offer new inputs and ideas for designing manual and/or automatic review systems for dynamic material platforms such as the GeoGebra Materials platform.

Keywords: Technology, quality assurance, mathematics materials, empirical study, GeoGebra.

Introduction: Quantity vs. quality of resources

Numerous online platforms provide a large number of Open Educational Resources (OER) for teaching mathematics (e.g. GeoGebra Materials, 2016; LearningApps, 2016; I2Geo, 2016). The enormous quantity and variability of quality make it difficult for users to quickly find appropriate resources for their teaching (Trgalova, Jahn, & Soury-Lavergne, 2009). The problem of inconsistent quality particularly appears on platforms with user-generated Educational Resources, not supported by a dedicated editorial team. They are often free or low-cost materials, which are created and shared by different types of users (Camilleri, Ehlers, & Pawlowski, 2014; Ott & Hielscher, 2014).

An example of a large repository of dynamic mathematics materials – GeoGebra Materials

One example of such a platform is GeoGebra Materials (2016) that already offers more than a half million public dynamic materials (as of November, 2016). Since dynamic worksheets – created using the dynamic mathematics software GeoGebra – can be uploaded, copied, edited and organized into collections by every user, this platform is subject to the aforementioned problem of inconsistent quality (Kimeswenger & Hohenwarter, 2014, 2015). According to interviews with GeoGebra users, it is not always easy to find high-quality resources on this website that comply with users' own quality standards. Thus, it might be desirable to reconsider the review and ranking systems of a platform that might influence the appearance and order of search results.

Quality and assessment of mathematics materials on platforms

Several platforms with materials for mathematics teaching have different mechanisms to assess the quality of their resources with a similar aim: influencing the search results and ranking the high-quality materials first (Libbrecht et al., 2008; Ott & Hielscher, 2014). For instance, the project Intergeo, Trgalova et al. (2009, p. 1163) characterized nine "relevant indicators" of quality of dynamic geometry resources on their platform I2Geo: "metadata, technical aspect, mathematical dimension of the content, instrumental dimension of the content, potential of the DG, didactical

implementation, pedagogical implementation, integration of the resource into a teaching sequence, [and] usage reports." For example, one criterion according to the indicator "content" is "validity" with the question "Are the activities in this resource correct from the mathematical point of view?".

Intentional vs. non-intentional reviews

To facilitate the development of a new review system for the GeoGebra Materials platform I considered combining an intentional and non-intentional review system. Under the project Intergeo, a questionnaire was developed based on the above-mentioned quality indicators. The assessment of the quality of a particular resource on the I2Geo platform requires users to respond to 9 broad statements, which can be extended optionally to 59 questions – using a scale from 'I agree' to 'I disagree' (Kortenkamp et al., 2009; I2Geo, 2016; Trgalova et al., 2011). Alternative ways for users to intentionally contribute to the evaluation of a resource are through 'likes', comments or star ratings, which are also often implemented on platforms with a large number of resources for teaching mathematics – for instance, on CK-12 (2016) or LearningApps (2016). These review possibilities are highly depend on the willingness of individual users to contribute to the review process of certain materials.

In many cases, only a small number of the materials that have been viewed have also been reviewed by a user. For example, on average, only 0.22% of viewed resources on the video-sharing website YouTube have been reviewed by likes or comments (Siersdorfer et al., 2010). Therefore, Ott and Hielscher (2014), who investigated the issue of the quality of interactive exercises on the platform LearningApps, considered assessing quality in an automatic way. For instance, one quality criterion is related to the communication with other users. It turns out that authors of exercises of LearningApps with an average well-rated content (4-5 stars out of a maximum of 5 stars) communicate more than the average authors. Thus, the absolute number of authors' messages could be used as an evaluation criterion for their created content according to Ott and Hielscher (2014).

In summary, I am interested to identify suitable methods for reviewing dynamic materials that are different to traditional practices, such as solely by questionnaire. It is possible that elements of both approaches, where users assess quality intentionally or non-intentionally/automatically, might be combined within a new conception of a review system for dynamic instructional materials on the GeoGebra Materials platform.

OECD results – Does technology provide benefits?

The recent OECD results highlighted the necessity to regard the quality of mathematics teaching materials supported by technology. According to the report of OECD (2015) there is no evident improvement of students' achievements in countries that invested heavily on educational ICT (Information Communication Technologies) concerning their performance in PISA. Drijvers (2016) reflected to the state of the art, and questioned OECD's rather generally formulated claims. Assuming that digital technology is generally good or bad is not a proper approach according to Drijvers (2016). He emphasized the necessity to ask HOW technologies should be used to benefit mathematics learning and encourage high-quality teaching. Higgins, Xiao, and Katsipataki (2012, p. 3) also highlighted the urgency to think carefully about HOW technology should be used for teaching mathematics:

> We need to know more about where and how it is used to greatest effect, then investigate to see if this information can be used to help improve learning in other contexts.

Drijver's interpretation of the OECD results and the above quote of Higgins et al. (2012) also indicate that technology use does not always result in good teaching. The interpretation within the OECD (2015, p. 3-4) report emphasizes that solely using technology without educational considerations is clearly insufficient. Nabb (2010) highlighted that the availability of different available devices has forced a fundamental question: "How should such devices be used in the teaching and learning of mathematics?". In my research, I investigated the quality of technology-supported teaching materials and their 'valuable' uses in education. I believe that my study will assist to fill the gaps suggested by OECD and Drijvers by offering possible guidelines for identifying high-quality technology materials for teaching and learning of mathematics. In particular, I aim to address how to recognize and create high-quality materials, which leads me to the research questions of this project.

Research questions

Q1: What quality criteria for dynamic materials exist according to experts?

Q2: How do experts describe the educationally valuable use of dynamic materials?

Q3: How could the conclusions from research questions 1 and 2 contribute to the conceptual design of a new review system and the further development of platforms, e.g. "GeoGebra Materials"?

Research design

I began by investigating the complexity of quality aspects of dynamic materials using qualitative research based on Grounded Theory (Strauss & Corbin, 1996). Experts, in particular mathematics teachers and mathematics educators, were interviewed to enquire about their perspective of quality materials. I selected international participants who were deeply involved in different projects and had been working on the development of instructional materials with GeoGebra for many years, thus they can be named as "GeoGebra experts". I considered different nationalities and cultures to reflect on different perspectives concerning decisive criteria for a high-quality material for mathematics teaching, because the GeoGebra Materials platform is also used by wide-range of users from all over the world. Consequently, I interviewed experts from Hong Kong, Uruguay, England, Austria, Hungary and Germany.

After analyzing the interviews, I created a category system that described core dimensions that contributed to the quality of a dynamic material. In addition, I expressed a list of quality criteria for dynamic resources in a "theoretical and detailed quality catalog" obtained from the expert interviews. These considerations about quality criteria and the educational value of the use of dynamic materials should provide new ideas for a conceptual design of a review system for platforms like the GeoGebra Materials website. and that might combine different elements of existing review systems used by other platforms. Based on the initial results of this stage, I also conducted quantitative research and received responses from 84 Italian and Austrian mathematics teachers using an online questionnaire and investigated the relevance of the emerged results.

Examples of quality criteria – The orthocenter of a triangle

The following example offers an idea what quality aspects were mentioned by experts, in this case by a highly-experienced teacher and user of GeoGebra, who referred to a specific dynamic resource on the GeoGebra Materials platform focused on the orthocenter of a triangle (see Figure 1).

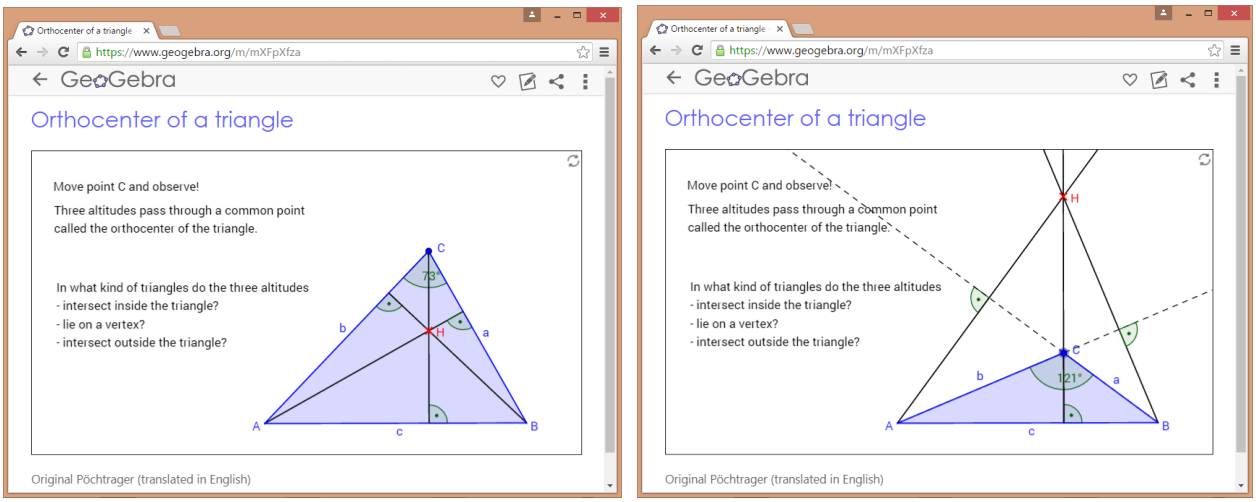

Figure 1: Dynamic worksheet, Interview 2014-12-11, https://www.geogebra.org/m/mXFpXfza

The interview with the expert revealed one of many quality criteria for a dynamic resource is "supporting the learning of mathematics". A related question that could be asked of users concerning the resource shown in Figure 1 is, "Does the dynamic material support the learning of mathematics?". An answer, from the perspective of the expert, might be, according to the instructions next to the construction, students should move point C and observe the effect of its position to the triangle orthocenter's shape and position. The potential of the material is from the expert's point of view that the dynamic worksheet allows students to explore through the dynamic construction. Depending on the location of the vertices of the triangle, the position of the orthocenter changes. For instance, students could discover that the orthocenter lies inside an acute triangle and outside of an obtuse triangle. Such materials are intended to encourage students to come up with their own assumptions and formulate insights, as in Table 1 summarized.

Quality criterion	Question	How?
"Supporting the learning of mathematics"	"Does the dynamic material support the learning of mathematics?"	Allows students to explore with the dynamic construction
		Allows students to discover mathematics
		Encourages students to make their own assumptions
		Encourages students to formulate insights

Table 1: Quality criterion "Supporting the learning of mathematics", Interview 2014-11-12

This example shows that often many different quality aspects come together to influence the overall quality of a particular material. Experts were asked about their opinion and perspectives on the potential of instructional dynamic materials in order to investigate the complexity and different facets of the issue of quality of dynamic materials. Table 1 summarized the aspects that concerned the quality criterion "Supporting the learning of mathematics" considering the dynamic worksheet

about the orthocenter and showing that many aspects could contribute to the quality of this dynamic worksheet. Additionally, I would like to highlight the difficulty to express guidelines or criteria defining the quality of a dynamic material:

> [T]he issue of how to maximize the benefits of the integration of technology is hard to capture in overarching guidelines. (Drijvers et al., 2010, p. 86)

Nevertheless, in this paper, I summarize and describe dimensions developed in this study that could contribute to the quality evaluation of dynamic materials.

Eight quality dimensions of dynamic materials

The analysis of the expert interviews revealed eight core "quality dimensions" as crucial factors: (i) author, (ii) mathematical content, (iii) resource type, (iv) supporting the learning of mathematics, (v) integration into teaching, (vi) advantages of dynamic material, (vii) design and presentation, and (viii) technical aspects.

These dimensions were compared to the literature and could significantly influence the quality of a dynamic material created for mathematics teaching. The "author" (i) can be considered as a main quality dimension influencing all of the other above-mentioned items (ii-viii) and is therefore listed first. This importance is due to the fact that the creator has a considerable effect on the resource that she or he has developed and can assist in the decision of what mathematical content is presented to support learning. The creator should consider how to integrate technology to benefit classroom activities and to exploit the potential of the dynamic material. Depending on these considerations and on the available technology in the classroom, the author adopts the dynamic material's design to be as user-friendly as possible and considers teaching aspects related to this kind of material. Next, I present examples showing the importance of the author and how certain conditions – such as available technology and the author's view on learning – effect the development of GeoGebra resources.

Importance of the author of a dynamic material

The majority of experts stated that there is a strong correlation between the quality of the author and the created material (see Figure 2).

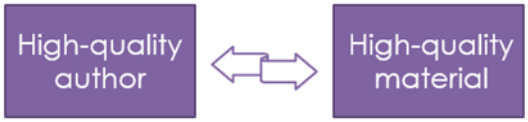

Figure 2: Correlation between quality of author and dynamic material

When I asked experts to describe their search strategies to find high-quality materials, the importance of the author was pointed out several times. It seems nearly impossible to decide in general, whether a specific resource is of "high quality" or not: "A given resource can be 'good' in one context and 'poor' in another." (Trgalova et al., 2009, p. 1162). Nevertheless, there seem to be certain strategies for searching for "good" dynamic materials on the GeoGebra platform and certain "authors" were often named. For instance, a British expert mentioned an author, whose: "materials are brilliant and if you see something of [him] then it is a guarantee of quality." (Interview 2015-07-15) Expert users of GeoGebra seem to search particularly on profile pages of already known "high-

quality authors", which they expect to produce "good" dynamic materials according to their own standards of quality:

> If you get to know people who produce quality materials, they don't tend to produce quality materials by accident. Once, you find one or two things by somebody which is good, you can expect pretty much more materials with high quality. (Interview 2015-07-15)

As mentioned in the beginning, inconsistent quality especially occurs, when users with different quality standards share their dynamic materials online. However, this can be regarded not only as a disadvantage, but also as an advantage. These user-generated Educational Resources provide a vast number of instructional materials ready to be used in classrooms created by different authors considering varying circumstances such as diverging curriculums, technical requirements or quality standards.

Available technology

An author's creation of a dynamic material depends on the available technology and the individual classroom situation. In a computer lab, pupils can work independently and actively on dynamic materials, but they require clear instructions and questions complementing the applets. In contrast, dynamic materials do not necessarily contain instructions if it is the intention for the teacher to demonstrate the concepts using a projector. In this case, the teacher can explain the purpose of the resource to students during the presentation. Another example of the influence of available technology on the design of a dynamic material could be derived from mouse driven approaches as opposite to working on touch-sensitive devices. On tablets and mobile phones, learners may directly use their fingers to work with a dynamic resource, while on non-touch devices, dynamic materials are manipulated with a mouse. In addition, displays of these devices are usually smaller than computer monitors and this should be considered within the design of the resource (Kimeswenger & Hohenwarter, 2014). In summary, the classroom situation, especially the available technology, strongly influences the design and use of dynamic material's design to include task instructions and the usability.

The author's views on learning – learning theories

Authors' views on learning and on the acquisition of knowledge considerably affect the use of the created materials. The structure and design is influenced by teachers' intentional or non-intentional views on learning theories such as behaviorism, cognitivism or constructivism.

The value of students' own constructions has been often discussed in educational research papers. Mercat, Soury-Lavergne, and Trgalova (2008) mentioned that principles that draw on a constructivist approach to teaching and learning are commonly accepted in mathematics education. Nevertheless, the development of an instructional material depends on the author's view on learning theories (behaviorism, cognitivism or constructivism). A platform should draw users' attention to high-quality authors who have similar views on learning. On the one hand, this would simplify finding "good" materials within a vast number of resources in repositories. On the other hand, these high-quality authors should be recognized and honored for their effort and top-quality materials.

Identification and recognition of high-quality authors

Experts suggested to allow users to follow particular authors on the website. Based on these interview results, the "Followers" Badge was released on the resource-sharing platform GeoGebra Materials (March 2016). It seems to be important that a review system of a platform enables users to find materials of specific authors quickly who adhere to similar quality standards. It should also allow following these authors and support finding of these dynamic materials by giving resources of the 'followed' authors a higher priority among the search results.

The "Followers" Badge could help identifying high-quality authors chosen by other users. It represents some kind of recognition outlined by a mathematics educator of Hong Kong:

> If the material is good [on the platform GeoGebra Materials], I think the designer has paid a lot of effort. He or she need more encouragement or appreciation. (Interview 2015-07-13)

Caprotti and Seppälä (2007, p. 7) also emphasized that authors should be respected and recognized for their high-quality resources. Users will share more materials if their work is recognized – "Credits to creators".

Conclusion

This study addresses the key issue: What is a high-quality material for mathematics teaching supported by technology? The internet offers an immeasurably large number of diverse mathematics teaching and learning resources proving it difficult to navigate for ordinary teachers and students. Therefore, it would be beneficial to identify and assess the quality of dynamic materials before using them in the classroom. It seems important that a review system influencing the search result order of a platform enables users to quickly find materials of specific authors who adhere to similar quality standards as well as to allow following these authors. Additionally, the platform should give resources of followed authors higher priority in search results, but it is also important to offer chances for new authors to be represented among the highlighted search results. Further analysis of different expert interviews and 84 responses of an online questionnaire will examine the complexity of the quality of dynamic materials in greater detail resulting in a detailed catalog of quality criteria. With this background knowledge, additional suggestions for intentional and non-intentional review systems for dynamic material platforms such as GeoGebra Materials could be devised. Beyond the PhD project described in this paper, software developers may implement further results of this study on the material sharing platform GeoGebra to improve finding of high-quality materials for mathematics teaching and learning. Further research will be necessary to investigate these new releases and support their continued improvement.

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Teachers designing e-books to foster creative mathematical thinking: the case of curvature

Angeliki Kolovou and Chronis Kynigos

University of Athens, Educational Technology Lab and CTI & Press Diophantus, Greece;
angkolovou@ppp.uoa.gr, kynigos@ppp.uoa.gr

This study focuses on the design of a novel genre of e-books incorporating dynamic constructionist artefacts-widgets that aim to induce mathematical creativity allowing students to interact with their content in significant ways (called 'c-books', c for creativity). The design of the c-books is addressed through collectives of educational professionals with a diversity of expertise. The analysis of the design process of a c-book on Curvature shows that the interactions fostered by the socio-technical environment allowed diverse practitioners to learn from, work with and collaborate across their boundaries supported by collectively evolving artefacts as boundary objects. The end-product of this social creative process, the c-book on Curvature, evolved through the constant versioning of a narrative intertwined with malleable dynamic constructionist artefacts.

Keywords: Curvature, design of digital resources, social creativity, documentational approach of didactics, theory networking.

Introduction

Curvature is weakly addressed as a conceptual field (Vergnaud, 2009) with respect to its potential to generate environments rich in opportunities for mathematical meaning making by students. In traditional curricula it lies disparately in Euclidean Geometry sections, in Algebra and Calculus depicting systematic co-variation, in 3D Geometry but in simple applications like conic sections. The dynamic and diverse representational repertoire provided by digital media allows us to approach curvature anew with a disposition to re-structure (to use Wilensky & Papert's term, 2010) the ways in which mathematics is conceptualized in education in the quest to make it more attractive to students, affording meaning making, creative mathematical thinking (through problem-solving/posing and constructionist activity) and engagement. Previous research has shown that dynamic digital environments and especially 3D spatial environments support students in constructing meanings about challenging conceptual fields (Kynigos & Psycharis, 2003; Zantzos & Kynigos 2012). In this study, we look at a digital medium affording the meshing of narrative with malleable constructionist artefacts such as *half-baked microworlds* (Kynigos, 2007) and study the ways in which a diverse community of professionals jointly design narratives around curvature with an explicit interest to afford creative thinking in their prospective students. We perceived of the medium which we called the 'c-book', as potentially affording thinking out of the box, allowing for restructuration of mathematical conceptual fields. We also began from the premise that collaborative design by educational professionals with diverse expertise would generate a socio-technical environment more likely to produce creative ways to enhance creative mathematical thinking (CMT) for students. In this study, the designed c-book eventually focused on a story involving comparisons between Archimedes' and exponential spirals. In addition, by involving in-service teachers in all stages of the design process, we aimed to induce their reflection on the affordances and pedagogies incorporated in the tools, the tasks and narratives under

development, as well as the changes to the mathematical content and the classroom practices that the presence of technology brings about.

Theoretical background

The design of digital educational resources for fostering mathematical creativity and meaning generating in a mathematically rich conceptual field like curvature is a complex task, which brings the issue and study of creativity in the design to the fore. In the design, process and product are inextricably linked in the sense that the former draws its very existence on the pursuit of the creation of some novel products, and that a creative product acquires its substance as the end-result of some design processes. However, the role of creativity in instructional design, or specifically in educational resource design, has been only recently acknowledged (Clinton & Hokanson, 2012). In this study, we looked at the process of designing such a resource collaboratively, focusing on the emergence of Social Creativity (e.g., Fischer 2005; 2014) which we employed as a theoretical frame to both understand and foster creativity in collective design. Our designer communities were engineered to include diverse expertise and personal histories of educational professionals (we borrowed Fischer's term 'communities of interest' or CoI) working together and using digital tools specially designed to amplify the outcome of their collaborative efforts. We hypothesized that social creativity builds on the wealth of diverse individual perspectives brought in by different stakeholders in addressing a design problem of common concern and focuses on the interactions occurring in socio-technical environments (Fischer, 2014) i.e., among the individual members of a community and between them and particular technologies and artefacts.

The diversity within a CoI although being a source of discontinuities and breakdowns in communication, can be also a source of new ideas, insights and artefacts. According to Akkerman and Bakker (2011) boundaries are defined as the "sociocultural differences that give rise to discontinuities in action and interaction" (p. 139), which can be overcome through boundary crossing processes, i.e., efforts made by individuals or groups 'at boundaries' to establish or restore continuity in action or interaction across practices, leading to learning, identity development and re-conceptualization of practice. These efforts are facilitated by boundary objects (Fischer, 2005) which are externalizations of ideas that help to establish and maintain a common ground supporting communication and shared understanding. They come in the form of artefacts (such as specially designed computer tools), discourses (as a common language), or processes that allow the coordination of actions. Thus, computational support for CoIs should enable the creation, discussion and refinement of boundary objects that allow different knowledge systems to interact (Fischer, 2005).

In our study, we worked with educational designers and thus needed to also employ a framework to help us understand the context particularly of teachers as resource designers. Thus, we adopted the documentational approach of didactics (Gueudet & Trouche, 2009), which focuses on the interactions between mathematics teachers and resources and their consequences for professional growth. Teachers 'learn' when selecting, transforming, implementing and revising resources in the course of their teaching. The documentational approach proposes a specific conceptualisation of this learning as a *documentational genesis*, which jointly generates a new resource and a scheme of utilisation of this resource in an ongoing process. In relation to collectives instead of individual teachers, community documentational genesis describes the process of gathering, creating and

sharing resources to achieve the teaching goals of the community. The result of this process, the community documentation, is composed of the shared repertoire of resources and shared associated knowledge (Gueudet & Trouche, 2012). A collaborative design activity, and more particularly an activity involving teachers as designers of creative educational resources, is thus a process that is expected to trigger collective documentational genesis. The present study aims to unfold social creativity, located in and nurtured by the boundary crossing encounters among the CoI members, and collective documentational genesis processes in the design of digital creative mathematical resources for curvature, which takes place in a socio-technical environment consisting of a community of diverse educational professionals and a digital environment specially designed to allow them coordinate their efforts in designing these resources.

Method

The Community of Interest (CoI)

A wide range of expertise was brought together in the design of the c-book "Curves in Space". The seven CoI members participating in this joint design were practitioners in different levels of education (from primary to tertiary education) specialized in mathematics, mathematics education, creative writing, computer mediated communication and the design of digital tools for mathematics education. This diversity in knowledge domains, perspectives and cultures was meant to enhance the CoI's creative potential.

The Computational environment

The C-book environment provides the 'CoICode workspace', a tool for asynchronous online discussions allowing designers to choose between a threaded forum discussion organised in a tree-like structure (see Figure 1) and a mind map view. When posting a contribution, CoI members have to state its nature (i.e., alternative, contributory, objecting, off task or management) by using a specific icon, and can attach and refer to objects like online resources, texts or widget instances that reside in the c-book under construction. In addition, the environment contains a platform which is the space for authoring (the C-book authoring tool) and the space where students interact with the c-book (the C-book player). The platform is designed to incorporate pages with dynamic and configurable widget instances accompanied by corresponding narratives (see Figure 2). In this case, MaLT+, a 3D Logo-Based Turtle Geometry tool affording dynamic manipulation of variable values was used (http://etl.ppp.uoa.gr/malt2). Spirals are generated by either constant or incremental curve and torsion changes to a turtle respectively repeating very small displacements.

Data and analytical approach

Our data were the 124 contributions uploaded in the 'Curves in Space' workspace from the outset of the design process (6/4/2015) until the final version of the c-book was released (23/7/2015). The analysis of the contributions posted in CoICode involved the selection and analysis of critical episodes, i.e., relatively brief and uninterrupted periods in CoICode discussion, shedding light on some important aspect of the social creativity processes and/or products developed, by focusing on the interactions among the CoI members and with the C-book technology. Furthermore, we traced paths of socially creative ideas, which stretch over longer periods of time and include several critical episodes, in terms of the critical moments in their evolution from the initial to the final idea (i.e., an

idea implemented and incorporated into some part of the c-book). The emphasis was on unveiling the social nature of the processes involved in the development of ideas and in the examination of the C-book environment features which added to the formulation, elaboration and cross-fertilisation of the CoI members' ideas.

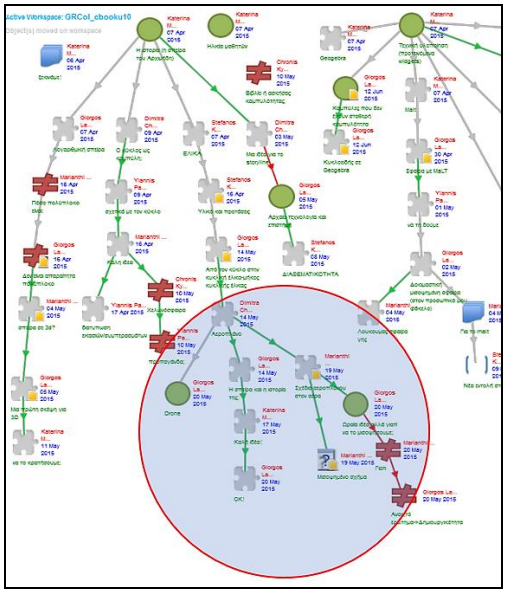

Figure 1: Excerpt from the CoICode workspace depicting critical episode 1

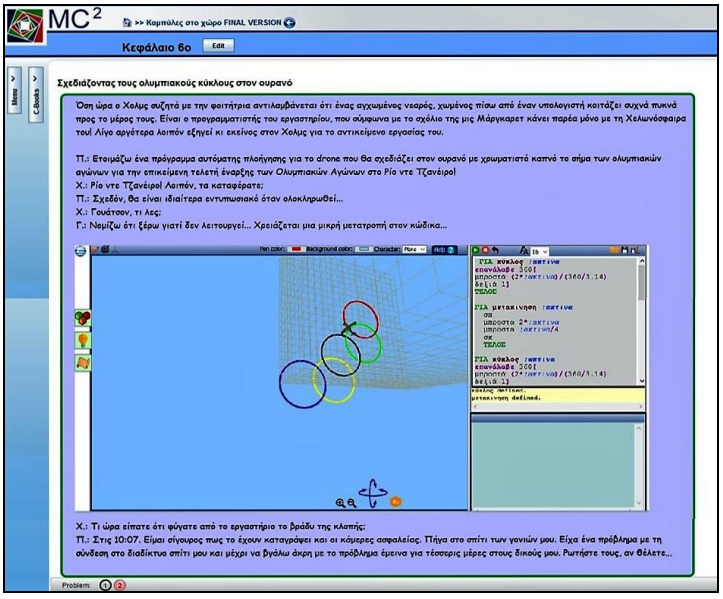

Figure 2: A 'Curves in Space' c-book page asking students to fix the code for designing the Olympic rings

Results

Critical episode: The design of a widget instance

The episode selected (see Figure 1) started one month after the outset of the design process, it lasted 8 days (14/5-20/5/15) and the participants in it were three CoI members: George, Mathematics teacher and graduate student in Mathematics education, Dimitra, Literature teacher and graduate student in ICT in Education, specialised in creative writing and Marianthi, MA ICT in Education graduate and developer. At that time an exchange of resources was taking place on the mathematical idea of Helix-Spiral between a senior mathematician (Stefanos) and George. The discussion is initiated by Dimitra (14/5) who, inspired by the airplane functionality in MaLT+, suggests that students calculate the distance covered by airplanes performing spiral movement during air shows. George (14/5) responds enthusiastically, elaborates on Dimitra's idea and provides a Wikipedia link on jets streams. Marianthi then puts forth a suggestion on a *half-baked* widget instance (i.e., a 'buggy' procedure where students are asked to experiment, figure out what is wrong or superfluous in the code and correct it):

> Marianthi (19/5): […] in MaLT+ I created a procedure where the airplane movement forms the Olympic rings. I am sending you the complete code so that we can half-bake it, e.g. it can turn by a 45-degree angle in the last two turns so that the rings do not come out straight (attaches 'Olympic_correct.txt')

> Marianthi (19/5): I am sending you the one I wrote with the wrong angles (attaches 'Olympic_wrong.txt')

George (20/5): I like it a lot! I suggest not to half-bake it, but ask students to create it by themselves by looking at an image of the Olympic rings […]

Marianthi (20/5): […] I think if it's half-baked it will be more challenging for students to correct it than create it from scratch. Also, we can focus on specific mathematical topics like the turn angle or the distance of cycles.

George (20/5): […] Since the unit addresses senior students it would more creative to allow them work without such restrictions. If we half-bake it though, wouldn't it be better to use variables for the angles?

George (20/5) refers to drones as a more innovative alternative to airplanes and designs two alternative versions of the widget instance in which he adopts Marianthi's proposal. Finally, one of his versions was incorporated in the c-book without further negotiation including his suggestion of imprinting the traces of a drone instead of a plane (see Figure 2). This episode shows how the collective resource system is enriched through the sharing, reflection and transformation of individual resources to boundary resources. What is more, boundary crossing interactions between CoI members allowed the cross-fertilization of diverse perspectives: mathematics, digital tools development and creative writing. Dimitra, having studied existing resources is inspired to articulate the airplane idea stating in what ways it deviates from what has been heard before. Marianthi turns Dimitra's idea into a 'tangible' object, i.e., a widget instance, while George expresses considerations initiating an interesting exchange on the pedagogical affordances of different types of activities. He brings CMT to the fore and poses the challenge to other members to directly argue on specific pedagogical and technical affordances of the proposed activities. The final version of the instance appears in the c-book as a result of the *coordination* of George's and Marianthi's ideas. Social creativity is thus enhanced by exchanging, discussing on and modifying half-baked curve designs acting as boundary objects, allowing the communication and coordination of diverse perspectives. Mathematical resources thus take a mediational role between diverse perspectives undergoing several transformations and revisions until they are reified as widget instances in the c-book. As teachers negotiate over an emergent mathematical construction, they are challenged to reflect on and reconsider their beliefs and practices as well as their meanings of mathematical objects and relationships, thus expand their learning.

The evolutionary path of the narrative

The path presented below is related to the evolution of the narrative of the c-book. The respective path includes 52 contributions and stretches along the entire workspace. Early on in the design process, CoI members were concerned with devising a narrative that, together with appropriate widget instances, would provide opportunities for mathematization and meaning making around curvature. The mathematical affordances of various digital tools also became an early topic of discussion so that tools, narrative and mathematical concepts were interrelated in the design of the c-book. Below we provide decisive contributions from individual CoI members and stress the social nature of the processes involved in the development of the scenario from its first appearance to its incorporation in the c-book. At that time a number of widget instances designed to afford creativity and meaning making in curvature took the role of boundary objects by evolving through multiple cycles. However, a cohesive narrative that would incorporate and join together these elements was pending, despite the fact that some interesting ideas had been already suggested. The path sheds light into how the CoI members'

conceptions about their productions in terms of didactical design (widget instances and corresponding learning activities) intertwined with their ideas about the narrative of the c-book. Stefanos (22/6) presents a -rather loose- synthesis of his own and other members' ideas on the c-book narrative integrated in a new version of the c-book: the history of curves, two detectives working to solve a crime, a 3d printer laboratory, and solving riddles related to spirals. George (24/6) reacts enthusiastically and attaches an elaborated version of Stefanos' story incorporating Sylvie's comments on enhancing the story: two renown detectives (Hercule Poirot and Sherlock Holmes) try to solve a mysterious robbery in a laboratory, which is connected to constructions related to curvature. Sylvie, a teacher and creative writing specialist who joins the discussion at that time, presents a totally different idea on the structure of the scenario relying on contemporary characters, which fuels an intense debate. Stefanos (25/6) objects to Sylvie's suggestion on the grounds that the storyline should blend with the widget instances so that students follow a learning trajectory working with tools of gradual increase in complexity. He also posts a document in which he justifies his rationale for building his own version in which mathematical concepts are presented in a coherent and meaningful way. George and Katerina (computer mediated communication specialist) react to Stefanos' post:

> George (26/6): Very insightful comments, especially in relation to the way the current narrative supports the smooth integration of the learning sequence on curvature. […] Wouldn't it be better to make some corrections without discarding what we've done until now? […] (He attaches 'What a strange morning in the laboratory.doc' where he expands his previous version to include logarithmic spirals).
>
> Katerina (27/7): […] I don't think that the new version rejects previous constructs and ideas […] but rather promotes them by organically binding them with a fresh, creative story.

Up to this point there are two opposing views on the scenario; a mathematics oriented strict, structured and robust learning scenario mainly supported by a senior member who is an experienced Mathematician, teacher and researcher (Stefanos) and a more innovative one which embodies a set of characters and situations of contemporary culture. This tension is released when George (28/7) replying to Katerina, posts a new synthetic version of the scenario. In the next two versions of the scenario Sylvie, Katerina and George collaborate so that Sylvie presents a more robust synthetic version that organically integrates the designed widget instances. The coordination of the two prevalent perspectives in the design of the c-book, i.e., the mathematics and the creative writing perspective, made possible the infusion of creative elements in the narrative, while not losing sight of its mathematical focus on curvature. It is noteworthy that these two perspectives are not only gradually reconciled after the catalytic intervention of George, they also enrich each other; the senior mathematician later proposes two additional ideas on the scenario much more innovative than his initial ones, while the creative writing specialist, after closely collaborating with mathematicians, comes to adjust her story in a synthetic version. These reflective processes are essential for the interweavement of widget instances with the narrative into a concise whole. Social creativity is thus facilitated by the meshing of the Sherlock Holmes narrative with curvature, which can only have emerged because of the diversity in the CoI. Furthermore, the process of story versioning boosts social creativity as it allows for the generation of new ideas which capitalize on, object to and finally synthesize previous ones. It is an ongoing process where ideas are adjusted, adapted and combined to produce new documents.

Conclusions

The analysis of social creativity in the design process of a c-book on Curvature focused on the boundary crossing interactions between the CoI members and the role of the narrative and the widget instances as key resources for the development of social creativity. Two important boundary crossing processes, *coordination* and *reflection*, have enhanced social creativity establishing communication between different communities of practice: Mathematics, Literacy/creative writing and Digital tools development. Reflection, on the other hand, is the process which gave ground to the fertile synthesis of different views. Moreover, the story and its versions as a key resource was paramount to social creativity within the CoI. The story versioning process allowed for warm debate and idea exchange to take place: it created common ground for all CoI members to unfold their expertise, as well as the meshing of narrative with constructionist artefacts-widgets on curvature. As a result, a collective document, that is the c-book, was developed, associating various shared resources (activities, widget instances, text, and CMT representations) and a scheme for interweaving all these elements in a coherent whole. The issues that emerged during the construction of successive c-book versions challenged teachers' perceptions with respect to the teaching and learning of curvature resulting in innovative approaches fostering creativity and meaning-making. Embedding the comparison of constant to incremental turn and torsion changes to generate spirals in space within a Sherlock Holmes 'who dun it' story involved stepping out of curriculum structures for curvature and making a new conceptual field available to students connecting curvature with functions and 3D geometry.

The use of different theoretical perspectives, i.e., Social Creativity, Documentational Genesis and Boundary Crossing, has helped us gain a deeper insight in the phenomena in question. In our framework of analysis, the Social Creativity perspective provides the lens thought which the social dimension of teachers' documentational genesis process can be approached. Moreover, resources take the role of boundary objects allowing the coordination of diverse perspectives leading to the generation of creative products (the c-book as an end-product). Thus, by coordinating these theoretical approaches, we seek to develop a networked understanding (Prediger, Bikner-Ahsbahs, & Arzarello, 2008) of the collective design of a c-book as a novel digital medium to foster students' creative mathematical thinking; a single theory would not suffice for understanding such a complex process. Even though drawing connections between theories is not a trivial task, such networks are potentially powerful and useful for the further development of mathematics education as a scientific field. (Prediger et al, 2008).

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Planning geometry lessons with learning platforms

Andreas Lindenskov Tamborg[1]

[1]Metropolitan University College, Department of School and Learning, Copenhagen, Denmark/ Aalborg University, Department of Learning and Philosophy, Copenhagen, Denmark, alta@phmetropol.dk

*This paper investigates how mathematics teachers plan lessons with a recently implemented Danish learning platform designed to support teachers in planning lessons in line with a recent objective-oriented curriculum. Drawing on data from observations of and interviews with teachers, three mathematics teachers' joint planning of a lesson in geometry with a learning platform called Meebook is analyzed using the instrumental approach. It is concluded that the interface in Meebook orients the teachers work toward what the students should **do** rather than what they should **learn**, although the latter is a key intention behind the implementation of the platform. It is also concluded that when the teachers succeed in using learning objectives actively in their planning, the objectives support the teachers to design lessons that correspond with their intentions. The paper concludes with a discussion of the dialectics between learning objectives and planned activities.*

Keywords: Planning lessons, objective-oriented curriculum, learning platforms.

Introduction

Teachers' planning of lessons is an important aspect of teaching as the decisions made at this stage shape students' opportunities to learn (Superfine, 2008). Planning is especially important for mathematics teachers as techniques and tools are closely linked to mathematical conceptualizations (Haspekian, 2005). It is therefore essential that teachers' choices of resources and tasks resonate with the teachers' intentions of what the students should learn. Currently, an increased number of technologies are becoming available that support teachers to plan lessons (Johnson, Adams Becker, & Hall, 2015) by giving teachers access to new resources and allowing them to design their own materials (Gueudet, Pepin, Sabra, & Trouche, 2016). Although such technologies bring new opportunities, they also bring challenges as new resources and materials often require mathematics teachers to reconsider how environments that give the students the right opportunities to learn can be designed (Haspekian, 2005).

In Denmark, learning platforms are currently being implemented that are an exemplar case of new technologies that support teachers in planning lessons. Among other things, the learning platforms serve to give students, parents and teachers access to plans for students' learning progression, and the platforms are designed to support teachers in planning and sharing lessons (KL, 2014). The learning platforms share a number of characteristics with learning management systems (LMSs; see, for example, Watson & Watson, 2007), but learning platforms also integrate affordances that are not typically associated with LMSs. Although LMSs typically are designed to handle all aspects of student learning, the learning platforms also support teachers to design lessons by giving the teachers access to online curriculum materials and enabling the teachers to create their own. Previous research about platforms that support teachers' planning has identified a need to support teachers to design lessons and choose resources that are in line with the teachers' intentions for students' learning (Hodgson, Rønningen, Skogvold, & Tomlison, 2010). Danish learning platforms

were implemented in the wake of a recent curriculum reform that foregrounds learning objectives, and the idea is that learning objectives will support teachers to make choices that reflect the teachers' intentions for student learning. Although the learning platforms are already used widely in Danish primary schools, there is yet little research on *how* teachers plan lessons with these platforms. This paper investigates how Danish mathematics teachers plan lessons with one of the most widely chosen platforms, called Meebook (https://meebook.com/). It derives from a small-scale pilot study in an ongoing PhD-project. The paper contributes to the literature as it offers the first empirical analyses of how Meebook mediates teachers' planning and discusses the consequences of this planning for the orientation of their planning and of the foundation on which teachers build their choices of resources and tasks. The data in the study consists of a case of three teachers' joint planning and individual interviews with the same teachers. I begin the paper by explaining the Danish context and some of the key ideas behind the implementation of the learning platforms. I then introduce the instrumental genesis framework and my methodological approach and analyze a case of three teachers' joint planning with Meebook. I conclude with a discussion about the dialectics between learning goals and planned activities in which I draw on a concept of rational and relation modes of planning (John, 2006; Superfine, 2008).

Context

In 2014, as part of building a national digital infrastructure, the national government decided that all municipalities in Denmark would be required to purchase and implement a learning platform during the 2016/2017 school year. Instead of developing a common, national learning platform, the Government and Local Government Denmark (KL) allowed municipalities in Denmark to choose a platform that best meets their needs. As some degree of uniformity was needed, KL stated 64 functional requirements that the learning platforms must fulfill in order to be approved (KL, 2014). Among other things, these requirements included that the platforms should support the implementation of an objective-oriented curriculum reform and that they should support teachers to define the learning objectives for each lesson (KL, 2014). The idea was that teachers would begin their planning by defining a learning objective and then design or choose activities and resources that will enable the students to attain the objectives. Currently, six platforms are available that fulfill the 64 functional requirements of Local Government Denmark. These platforms differ in design, the amount and type of support that teachers are offered in planning lessons and how the national curriculum is considered as part of teachers' planning.

The school in which the present study took place is in a municipality that has chosen Meebook, one of the most widely chosen platforms. In contrast to some of the other available platforms (for example, https://minuddannelse.net), Meebook is characterized by an interface that allows teachers to choose how and when to integrate learning objectives in their planning. For example, MinUddannelse requires teachers to define a learning objective as the initial step of planning a lesson. The school had begun a gradual implementation of Meebook in December 2014 when the teachers initially were encouraged by school leaders to experiment with the platform. In August 2015, school leaders made it mandatory for teachers to use Meebook to plan lessons in both mathematics and native language education.

The Meebook interface

Figure 1 illustrates Meebook's interface to create a course and add a chapter, text, picture, video material, a PDF document, a hyperlink, a task or activity, e-textbook material or a student reflection. In this interface, the teacher defines what should happen in the lesson and which resources should be integrated. The learning objectives are in a separate tab that is illustrated in Figure 2 and can be accessed at individual teachers' convenience. However, a learning objective must be defined before the course can be saved.

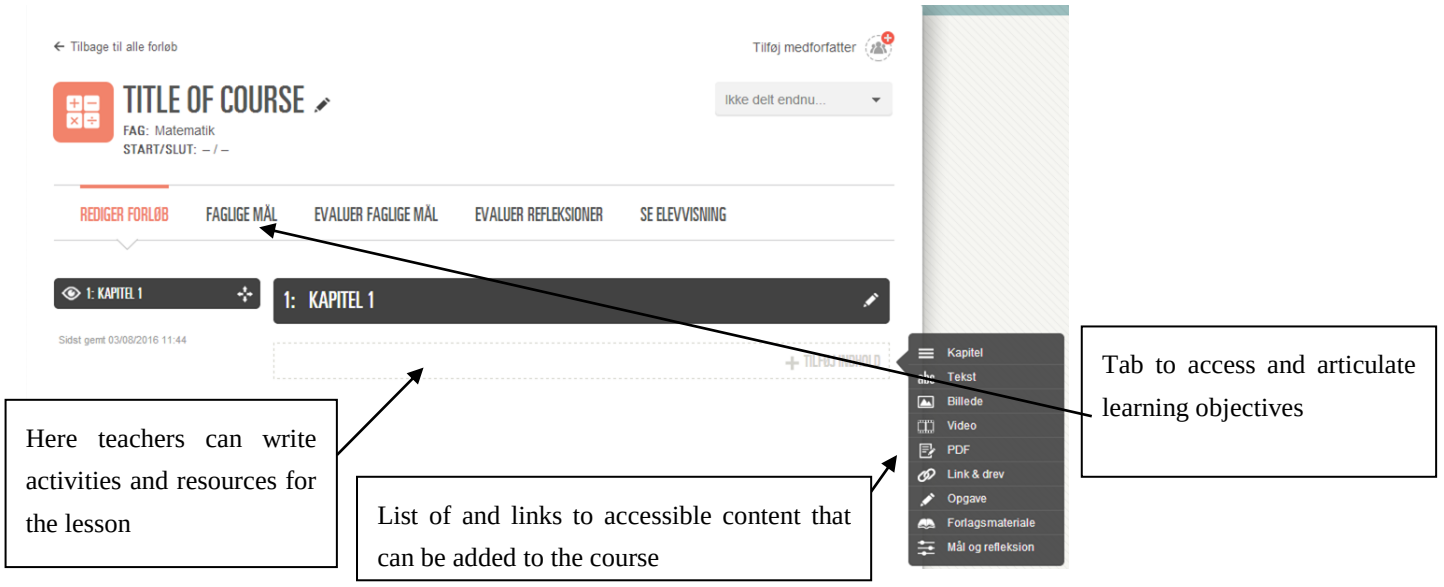

Figure 1: Meebook's interface for teachers to plan a course/lesson

Figure 2 illustrates the interface in Meebook where teachers can define learning objectives for the course. Here, the teachers can also access the learning objectives from the national curriculum through a link and select those addressed by the course or lesson. Teachers can also define their own objectives.

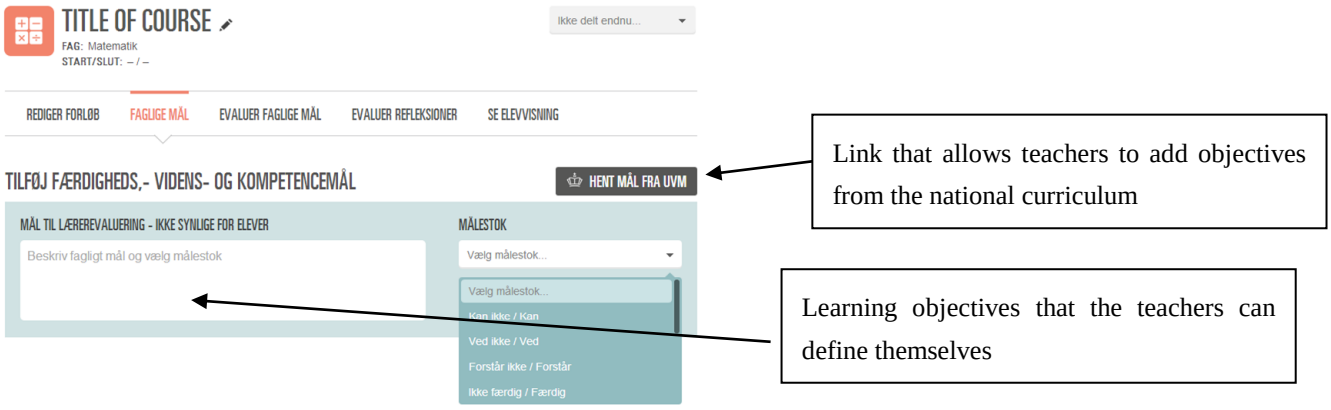

Figure 2: Meebook's interface in the tab called "Add skill, knowledge and competence objectives"

Theoretical framework and research questions

In this paper, I draw on the instrumental approach (Guin, Ruthven, & Trouche, 2005). The instrumental approach is a framework developed to study the consequences of different kinds of tools, technologies and software for learning and teaching mathematics (Gueudet, Buteau, Mesa, &

Misfeldt, 2014). A key aspect of this approach is the assumption that the relation between design and use is dialectic rather than one-sided (Haspekian, 2005). When a subject uses an artifact in an activity with a specific objective in mind, the artifact can shape the appearance of the activity or even force the subject to redefine the objective of the activity. The subject's use of the artifact can, however, also exceed the intended uses of the artifact. The latter is referred to as design that continues in use (Ejsing-Duun & Misfeldt, 2015).

The instrumental approach distinguishes between artifacts and instruments. An artifact is defined as a cultural social construct that offers mediations of human activity, and an instrument is defined as the product of a subject's use of the artifact for certain activities with a certain objective (Gueudet & Trouche, 2009). An artifact therefore *becomes* an instrument when the artifact is used by a subject. This process is called instrumental genesis and results in a change in the mediating artifact and in the activity the artifact is used for. These two opposite processes (the shaping and the being shaped) are referred to as instrumentation and instrumentalization (Haspekian, 2005). Instrumentation is the process in which the subject's use of an artifact shapes the artifact, while instrumentalization is the process in which the artifact shapes the subject's activity (Gueudet & Trouche, 2009). The approach also distinguishes between pragmatic and epistemic mediations (Rabardel & Bourmaud, 2003). Pragmatic mediations are the use of technology to perform a task (Rabardel and Bourmaud use the hammer as an example) while epistemic mediations are the use of technology that allows the subject to learn about the object through the use of the technology (Rabardel and Bourmaud use the microscope as an example). Finally, the framework distinguishes between different orientations of instrumentations and proposes three main orientations: orientations toward the object of the activity, toward other subjects or toward oneself (Rabardel & Bourmaud 2003). I use this framework to address the following research question: Which mediations of the teachers' work occur as they plan lessons with Meebook, what are the consequences for the orientation of their work and for the foundation on which they build their choices of resources and tasks for the lesson?

Method

The data in this study comprised observations of two 2-hour meetings during which three teachers collectively planned lessons, and of individual interviews with the three teachers. In general, the teachers expressed a positive attitude towards Meebook, although none of them had previous experience of using LMSs. The observations focused on: 1) the materials and task that was chosen by the teachers; 2) whom or what their planning was directed towards; and 3) what the foundation of their decisions seemed to be. The meetings, which took place at the school where the teachers worked, were documented by video recordings and field notes taken during the session. The video was recorded with a high-resolution camera that showed how the teachers maneuvered in Meebook. All video recordings were subsequently transcribed as closely to the spoken word as possible and supplemented by the notes taken during the observation.

The interviews were carried out after the observations and supplied data about what the teachers found important to consider when planning lessons—in general and related to the observed sessions. The interviews also collected data about the teachers' educational backgrounds and their seniority.

For this paper, I use a single case (Yin, 2014) that was selected as it gives insight into the relation between the teachers' planning practices and their use of the different interfaces in Meebook.

Although the amount of data in this study is sparse, the case reveals important problems and prospects associated with supporting mathematics teachers' planning with technologies, such as learning platforms. The case also identifies issues that future research in this area could consider.

Case

This case took place during three teachers' joint planning of a lesson on geometry in middle school (students aged 10–11). The three teachers were Karen, Miriam and Gina. At the time this session took place, the teachers were two weeks into a three-week course on geometry. Karen is 29 and has 2 years of teaching experience, Gina is 40 years of age and has 5 years of experience and Miriam is 46 and has 22 years of experience teaching.

During the meeting, the teachers alternately discuss how to plan the lesson and write their decisions in Meebook in the tab illustrated in figure 1. While working in this tab, they decide that the students should work in groups and categorize the geometric figures they had been working with for the last two weeks (rectangles, squares, trapeziums, parallelograms and rhombuses). The teachers agree that each group should be given cardboard figures in the shape of these five figures and that the students should categorize the figures by placing them on an A2 piece of paper. Gina then openly poses the question whether the students should categorize the figures 'freely' or whether they should follow certain instructions. As the teachers discuss this matter without immediately reaching an agreement, Karen turns to the tab in Meebook where they have written the learning objectives for the course (illustrated in Figure 2). Karen reads the objectives aloud to her colleagues: "According to the objectives, the students should be able to distinguish between the five figures and categorize different types of figures according to their side lengths and angle sizes." Miriam argues that if these objectives should be addressed, then the students should identify the figures *from* their properties and that they therefore should be given instructions to do so. The two other teachers concede. Gina then comments: "If we give them figures to categorize, how do we make sure that they actually talk about the properties of the figure?" This comment makes the teachers aware that there is a risk that the students will categorize the figures from what they spontaneously believe the figures look like. The teachers find this likely, as the students have been working with the same five figures for two weeks at this point. This method of categorizing the figures would not target the objective for the lesson. The teachers therefore agree to hinder this from happening by cutting the cardboard figures into shapes that are unlike the figures the students have been exposed to during the last two weeks (for example, a 'crooked' trapeze, as Miriam calls it). They believe that this will make it difficult for the students to recognize the figures and that this will prime the students to actually investigate the properties of the figures and to do their categorization from this. The teachers also decide to instruct the groups to take turns picking a figure from a pile of cardboard with the figures facing downward, then place the figure in the category on the A2 paper where they believe it belongs and explain to the rest of the group why they believe it belongs there.

Results

The teachers' planning of the lesson initially takes place in the interface illustrated in Figure 1. This interface in Meebook displays an overview of the resources available to the teachers and presents a blank field for them to fill. This blank field refers to the content of the lesson: which resources they will draw on and which activities they will include in the lesson. Meebook's visualization of the

content as the first aspect of the lesson to consider seems to be reflected in the teachers' initial decision that the students should categorize laminated geometric cardboard figures and that this activity should be carried out in groups. At this stage, the teachers' activity is oriented toward the object (the lesson to be planned), and their objective seems to be to decide which resources and task to include in the lesson. This priority of the content contradicts all three of the teachers' statements in the interviews where they emphasized the importance of beginning their planning by defining the objectives for the lesson. Miriam expresses it in the following way: "We always begin our planning with the learning objectives. That way, we can find or design the resources and tasks that fit the objectives. That's the whole starting point when we plan lessons." This suggests that the teachers' use of Meebook leads to a shift in the orientation of their activity from being oriented toward learning objectives toward being oriented toward defining the content. A consequence is that the choice of the cardboard figures does not reflect considerations about which specific geometric learning the students should obtain. This choice seems rather to reflect that the current topic is geometry. In the interviews, Karen and Miriam stated that they found it important that students have the chance to verbally express themselves in mathematics, as they believe that this creates good opportunities to learn. It is possible that the teachers' choice of organizing the lesson in groups is a reflection of this belief. The teachers' choice of using cardboard figures and that the students should categorize the figures, however, rather seems to reflect an objective of deciding what the students *should do* than what the students should *learn*. As previously stated, a subject's use of an artifact in an activity can shape the appearance of the activity or even force the subject to redefine the objective of the activity. In this case, Meebook's visualization of the activity 'planning lessons' seems to instrument the teachers' activity and orient it toward deciding the content for the course instead of prompting discussions about what the students should learn and which resources and tasks would enable this learning to occur. The case does not clearly illustrate an epistemic nor a pragmatic mediation. The case however illustrates that the teachers' use of Meebook does not lead to a new understanding of how the lesson could be planned according to their intentions.

As the teachers' meeting continues, they discuss whether the students should categorize the figures freely or whether their categorization should be guided by specific instructions. This decision requires a basis, and to find this basis, Karen turns to the tab in Meebook's interface where the teachers previously have written the learning objectives for the course. By turning to the objectives, the teachers become aware that the setting requires certain instructions if the learning objectives should be addressed. In this manner, the teachers use their knowledge about the students to anticipate how they would engage in solving a task and what learning in which this would result. This can be seen as an instrumentation of Meebook as the teachers merge two otherwise separate interfaces in Meebook. This results in the opportunity for an epistemic mediation of their activity that did not occur when the teachers worked in Meebook's *content* interface. As the learning objectives in Meebook become available for the teachers to use, they are enabled to explore their design of the lesson and modify it according to their intentions. At this point, the teachers' activity is characterized by a shift in orientation from the content of the lessons toward other subjects: the students, and more specifically, the students' learning. In other words, the teachers' activity shifts from being oriented toward what the students *should do* toward designing a lesson that creates opportunities for the students to *learn* something specific.

Discussion and conclusion

One of the main ideas behind implementing learning platforms in Denmark is the assumption that integrating learning objectives in the platforms will support teachers in choosing resources that correspond to the teachers' intentions for student learning. However, this pilot study suggests that it is not sufficient that learning objectives are integrated as a part of teachers' planning in the platforms, but that *the ways* the learning objectives are integrated in the design of technologies are important. In the case presented here, Meebook's interface separates the objectives from the content of the course that in this case implies that the teachers' choice of resources is separate from the learning objectives. Considering the importance of techniques and tools for mathematical conceptualizations, it is crucial that the choice of resources and tasks is carefully considered. This does not seem to be the case here. The case demonstrates that learning objectives can be valuable assets and work as epistemic mediators for teachers when they plan lessons. The teachers' use of the learning objectives enables the teachers to explore their lesson design and modify it so it corresponds with their intentions. However, it is remarkable that this opportunity arises as a consequence of the teachers' instrumentation of Meebook rather than of Meebook's instrumentalization of their activity. In addition, the initial choices (that the students should categorize cardboard figures in groups) are not changed or reconsidered during the session. In the case presented here, the teachers succeed in building a lesson with the cardboard figures and group organization a way that reflects the teachers' intentions. However, it remains to be known whether the teachers would have changed or discarded the cardboard figures or not if it turned out that these resources were incompatible with the learning objectives. This point suggests that this is an issue to be aware of in future research.

Previous research in mathematics teachers' planning distinguished between a rational and a relational mode of planning (John, 2006; Superfine, 2008). The rational mode views education as a linear input–output relation in which the planning begins by defining the objectives and resources/activities then are decided. The relational mode is planning focused on how students encounter each other, the mathematical content and with the teacher in a specific setting and the opportunities to learn arising thereof (John, 2006). The rational mode has been criticized for resulting in lesson plans that overlook and fail to anticipate the complexity and contingency of educational contexts while the relational mode is often referred to as a 'better alternative' (John, 2006). The results in this paper challenge that these modes should be separated sharply. It is exactly when the teachers foreground the learning objectives that they are able to engage in a relational mode of planning and design tasks and resources in ways that reflect the teachers' intentions. Through combining these modes, this potential is exploited, and neither the rational nor the relational mode in itself would enable this process. This result suggests that future research in this field could benefit from considering *how* learning objectives are integrated in technologies that support teachers' planning and *what kind* of planning modes these objectives en- or disable.

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What can we assess using multiple choice problems that involve learner generated examples?

Shai Olsher[1] and Michal Yerushalmy[2]

University of Haifa, Faculty of Education, Haifa, Israel; [1]olshers@edu.haifa.ac.il, [2]michalyr@edu.haifa.ac.il

Multiple choice (MC) items are the natural choice for automated online assessment. Ideally, making a choice should be based on knowledge and reasoning. Nevertheless, studies demonstrate that often various techniques (e.g. guessing) are the common practices. In the last decade technology has been employed to support real-time feedback as formative assessment for teaching and learning. This study examines whether and how learner generated examples, when required as support to the choice made in MC task, could be automatically identified to give insights into learners' understandings. Results show discrepancies between chosen correct statements and their supporting examples. Other automatically assessed characteristics are related to learner's approaches and strategies.

Keywords: Reasoning with examples, geometry, multiple choice questions, automatic online assessment, formative assessment.

Theoretical background

Multiple choice (MC) tasks are the most well-known tasks when it comes to automatic assessment (Farrell & Rushby, 2016; Sangwin & Kocher, 2016). They are used for testing various topics of study as well as different levels of abilities from basic through high order thinking. They hold several advantages including objectivity of scoring, and availability of more items in each assessment due to short solving times (Farrel & Rushby, 2016). However, MC tasks are often criticized for being biased (Hassmen & Hunt, 1994), for sometimes being measurements of how fast a student could make an educated guess or use elimination techniques, and not necessarily assessing what the MC tasks were designed to assess (Lau, Lau, Hong, & Usop, 2011).

Recent use of technology has made it possible to automatically assess responses by not only using MC tasks (Stacey, & Wiliam, 2012; Sangwin & Kocher, 2016). Immediate presentations of information on tasks performed in a technological environment are used as means to formative assessment: serving as feedback to modify teaching and learning activities (Black & Wiliam, 1998). One of these avenues is by automatically assessing learner generated examples (LGE) in a dynamic geometry environment (DGE) (Leung & Lee, 2013). Example generation tasks may serve as possible means to show conceptions of mathematical objects, or concept images (Vinner, 1983), informing about possible difficulties and inadequacies (Zaskis & Leikin, 2007). Another use of examples is for determining the validity of mathematical statements (Nagari-Haddif & Yerushalmy, 2015), in which systematic analysis of LGE could shed light on the evolving understanding of the status of examples in proving or refuting a mathematical claim (Buchbinder & Zaslavsky, 2009). Combining the accessibility of MC tasks in assessment with the potential reasoning abilities and demonstration of understanding by generating examples in a technological environment has the potential to enhance formative assessment in the mathematics classroom.

Methodology

This study is part of a long-term project aiming to explore ways in which automatic assessment could give more insight about student understanding in mathematics (Olsher, Yerushalmy, and Chazan, 2016). Specifically, we ask whether the requirement to provide examples to support a chosen answer gives the assessor additional insight into students' understanding in MC tasks. The participants were 32 secondary Israeli geometry teachers, from all over the country who taught different levels and ages ranging from 7th to 12th grade. The study was conducted as part of a broader national professional development effort to expose teachers to innovations in mathematics education.

Tasks

The study included three tasks and the context was mathematical similarity. The tasks were designed as interactive diagrams describing a geometrical context, using the STEP platform[1]. The interactive diagrams were constructed using GeoGebra and they enabled the participants to drag a set of elements in the diagram, according to the predefined characteristics determined by the designers of the task. The context was described in the task, and several statements were provided for the participants to consider. The participants were required to select the statements that are correct in regards to the diagram. More than one statement could be correct in relation to the interactive diagram in any single task. The tasks were similar to conventional multiple choice tasks accompanied by an interactive diagram, with one main difference: In these tasks the participants were expected to experiment and manipulate the interactive diagram into a state that fits one or more of the statements. In order to add another layer of reasoning, we asked the participants to attach a screen capture of the applet in a state that exemplified each of the statements they have chosen, thus requiring the participants to add an example instead of just select a statement as in traditional multiple choice tasks.

Automatic checking of tasks

The STEP platform enabled an automatic analysis of the predefined mathematical properties of the submitted solutions in order to characterize these solutions pedagogically and mathematically (Olsher, Yerushalmy and Chazan, 2016). The tasks in this study were designed so that the system would indicate if the corresponding example fits the criteria in the relevant statement[2], and enable the teacher to immediately have access to filtered answers accordingly. Yet, it is important to state that at the present time technology cannot determine correctness on its' own for these types of rich tasks. For each task, a well-defined set of conditions should be applied in order to determine the type of feedback affording formative assessment. Meeting the conditions of the checking algorithm does not mean correctness. It just means that this is what was automatically checked, and any

[1] Seeing the Entire Picture - STEP – is a formative assessment platform developed at the University of Haifa's Center for Mathematics Education Research and Innovation (MERI). For more detail about this platform, see www.visustep.com.

[2] When automatically checked, a margin of accuracy was determined by the teacher in which solutions are considered sufficiently accurate. For example, in this case, parallel lines, or coinciding points.

interpretation about correctness is purely suggestive, and should be carefully examined by both the assessor and the assessee.

Analysis

Our unit of analysis was the task. The first stage included locating discrepancies between a correct choice and the accompanying interactive example. We checked which participants chose the set of correct statements and compared it with the number of participants to correctly attach examples for all of the statements. The second step included a refinement of the analysis. We counted how many correct statements were chosen per-task (more than one choice could be correct for a single task), and compared it with the number of incorrect examples that do not meet the required answers' conditions. The third stage included the coding of the discrepancies according to pre-set categories (e. g. *familiar mistakes* or *additional reasoning*) in order to study the characteristics which could be subsequently assessed automatically.

Results

Table 1 shows the distribution of answers (consisting of chosen statements and supporting examples submitted by the participants (n=32) to the 3 different tasks.

Task (number of correct statements)	No. of participants which submitted an answer (n=32)	Sum of correct statements chosen by submitting participants	Sum of correct examples attached to correct chosen statements	No. of participants with all correct statements chosen (n=32)	No. of participants with correct answers and correct supporting examples. (n=32)
1 (3)	30	66	49	13	5
2 (2)	28	40	32	11	7
3 (3)	21	39	27	7	5
Total	N.R*	145	108	N. R.*	N. R.*

* N. R = Not relevant

Table 1: Answers submitted, statement choices, and examples provided for 3 tasks

As can be seen in Table 1, a total of 145 correct statements were chosen and submitted. For 108 of them (74.5%) correct examples were submitted. The remainder (25.5% of the correct choices) were submitted with incorrect or no examples. We now investigate the work related to two statements of task 3 in order to learn the nature of the examples that did not seem to be coherent with the choice of statement. In this task (Figure 1), the topic is the recognition of similar triangles, and ratios between areas of similar triangles. The context of the dynamic figure is presented to the participants in multiple representations: a verbal description in the task description, starting with: "point D is the midpoint …", a symbolic representation in the digital geometry environment (DGE): ED⊥AB,

AD=DC, and a DGE construction: A draggable triangle ABC with point D and E. Measurement tools and numerical feedback are not supported in this task.

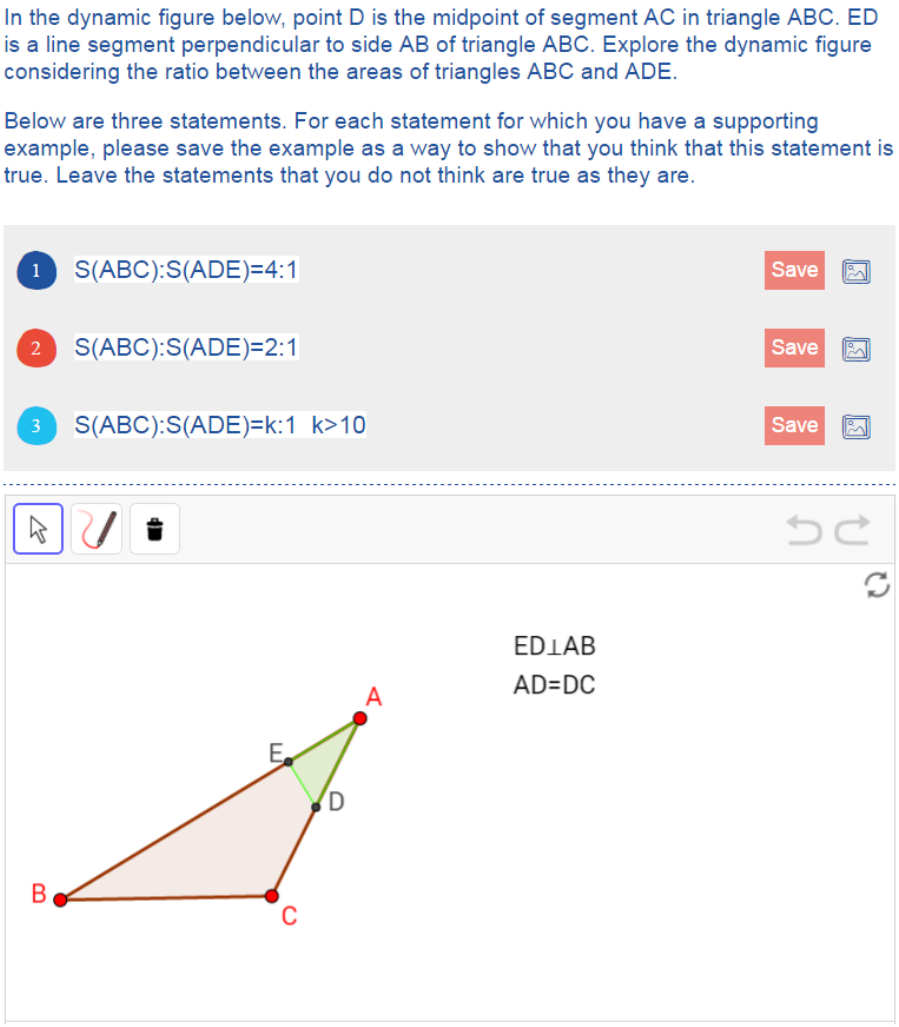

Figure 1: Multiple choice with supporting example task

In terms of correctness, the red points in the diagram could be dragged to create an example for any of the three statements in this task, making all the choices potentially correct ones. In order to construct a supporting example for the first statement, the lines ED and BC should be parallel[3], in order to construct a supporting example for the second statement points E and B need to coincide. A supporting example for the third statement would be any position where AB>5AE. But in order to construct such examples (mostly for statements 1 and 2) participants are required to have some understanding regarding similarity and ratios of areas of similar triangles.

Automatic assessment of this task was performed with the STEP platform, which is designed to present the submitted examples in several representations, including a visual representation of all examples attached to each of the statements (Figure 2), while enabling the assessor to automatically

[3] There is also another option to construct this where E is outside ABC and AE=AB.

filter the results according to mathematical and pedagogical criteria (Olsher, Yerushalmy, and Chazan, 2016), as will be demonstrated for this task.

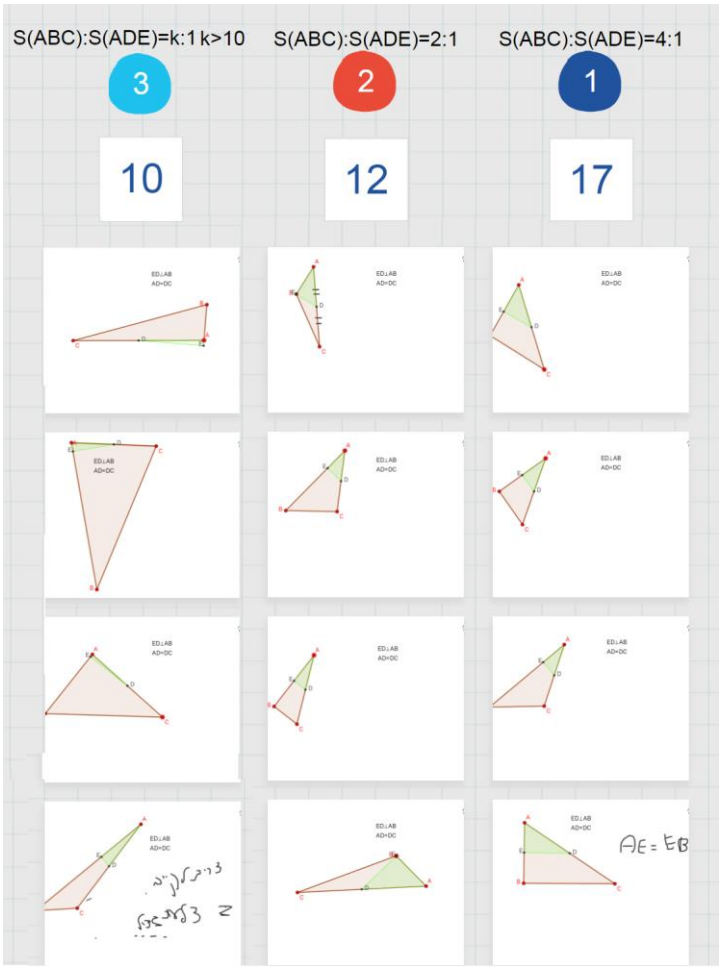

Figure 2: A sample of supporting examples for statements presented on the STEP platform

Incorrect examples that do not meet the required answers' conditions

Analyzing the collection of examples per-statement suggest a finer categorization. Statement 3-1 (the first statement in task 3) stated that the ratio between the area of triangle ABC and the area of triangle ADE is 4:1. Triangles ABC and ADE are similar as ED is perpendicular to AB. In addition, AD has the same length as DC. Thus, any example in which ED is parallel to BC, which means AE has an equal length to EB and vice versa provides a supporting example. There were a total of 17 examples submitted for this statement. 13 of which met the criteria for correctness. In Figure 3 appear the 4 submitted examples that were automatically marked as incorrect, as the automatically calculated ratio between the relevant triangle areas was not approximately 4:1.

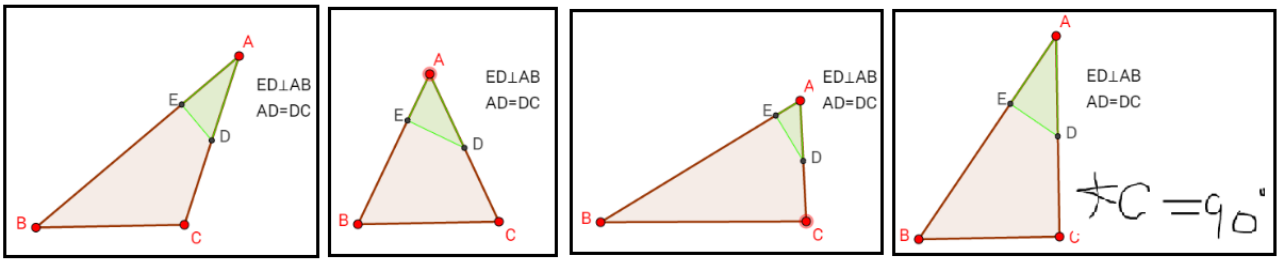

Figure 3: Incorrect submitted examples for task 3-1

The main characteristic that could be automatically assessed with this representation is the possibility that these participants did not address the characteristics relevant for this statement in their submitted examples - ED is not even approximately parallel to BC.

Incorrect examples in line with familiar student mistakes

Statement 3-2 stated that the ratio between the area of triangle ABC and the area of triangle ADE is 2:1. Triangle ABC and ADE are similar. Thus, any example in which points E and B coincide provides a supporting example. There were a total of 12 examples submitted for this statement. 8 of which met the criteria for correctness. In Figure 4 appear the 4 submitted examples (a, b, c and d from left to right) that were automatically marked as incorrect, as the automatically calculated ratio between the relevant triangle areas was not approximately 2:1.

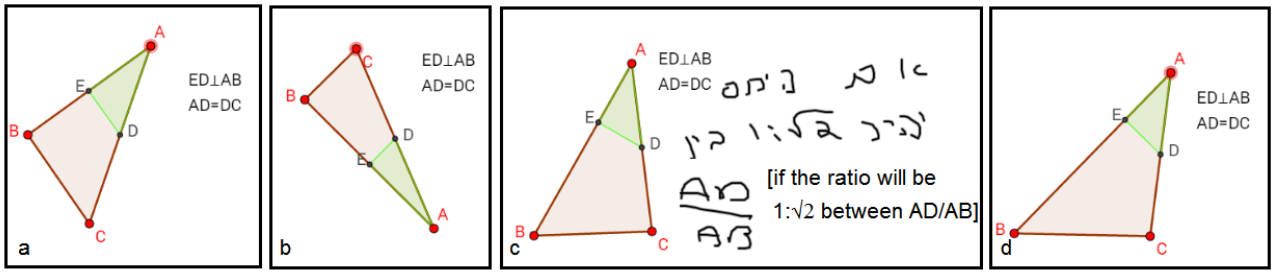

Figure 4: Incorrect submitted examples for task 3-2

In Figure 4, we see two incorrect examples (4a, 4b) that were further automatically categorized as "familiar mistakes". In these examples, the ratio between the lengths of BC and DE is approximately 2:1. These examples are a possible indication of holding the image of “linearity” between ratio of lengths and ratio of areas, a familiar phenomenon from the literature and teacher practice.

Examples with additional verbal, symbolic or free-hand graphic reasoning

Figure 4c shows an example that includes a correct verbal explanation but without any matching change to the dynamic diagram. One of the functionalities of the STEP platform is a free-hand pen tool which can be used for making annotations and marks, or any other use that the participant might find helpful. The participants were aware that the platform automatically checked their figures, and that text or graphical markings, if submitted, are presented for the teacher to review but not analyzed automatically. The example above is one of 18 submitted examples (across all three tasks in this study) that were accompanied with free-hand markings on the interactive diagram. Apart from verbal explanations, the examples also included annotations in the form of either symbolic writing or in graphical marks on the diagram. There are numerous possible explanations

for such responses. The participant might not have been able to construct the example, but thought about its mathematical properties, and wanted to demonstrate his knowledge. Annotation could also indicate that the participants needed to justify their example in a more robust, mathematical fashion, not fully accepting the diagrammatic example alone as a valid justification for a statement, which is closer to Israeli standard classroom practice.

The example in figure 4d was not automatically categorized beyond its' incorrectness, as it did not fit the predefined filter for a familiar student mistake.

One other aspect of the automatic assessment of MC tasks is the correctness of the entire task (e. g. choosing all of correct statements *and* providing them with correct supporting examples). In this study 13, 11, and 7 participants made a correct choice of statements in the three tasks respectively (Table 1). The number of participants who chose both the relevant statements and also provided a correct corresponding supporting example is lower: 5 (of 13), 7 (11), and 5 (7) (Table 1). This might be because the tasks were not clear enough, not specific about the relevant conditions; or perhaps ill-defined in terms of level of accuracy required. In order to enable efficient formative assessment these analyses are presented to the assessor in various graphic (e. g. Figure 2) and analytic (e. g. Venn diagrams) representations for further investigation.

Conclusions

This study provides initial information about discrepancies between choosing a correct statement, which could be a result of a guess or good examination tactics (Hassmen & Hunt, 1994; Lau, Lau, Hong, & Usop, 2011; Sangwin & Kocher, 2016), and providing an example to support this statement, which requires the translation of the conditions into a DGE context.

Many of the automatically assessed characteristics of submitted examples were not related to the correctness of the example in supporting the claim, but to other aspects such as student approaches and strategy (e.g. construction of prototypical figures, unexpected solutions). Although, due to space limitations, this report has focused on the limited analysis of discrepancies between chosen *correct* mathematical statements and their supporting examples, it has provided several additional insights into the MC tasks. Some of the solvers did not attend to significant characteristics required to support the answer (e. g. a line that needs to be a mid-section therefore to connect mid-points of two sides of the triangle and to be parallel to the third one), or the fact that they construct an example in line with a familiar student mistake (e. g. the ratio between the areas of similar triangles is the ratio of their sides squared, not the exact same ratio as reflected in the submitted example). These types of phenomena could help teachers better assess the performance level on these types of tasks, in the relevant mathematical topic, providing meaningful real-time analysis in the service of instruction. The automatic analysis and categorization alongside the visual representation methods played a key part in discussions of the results with teachers. This practice is well aligned with what Olsher et al. (2016) claim that the viability of this assessment in the mathematics classroom lies within the ability to automate the assessment process as much as possible, and to provide the teachers with suggestive insights as part of a better picture of their classroom example space and answers.

Acknowledgment

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Analysing MOOCs in terms of their potential for teacher collaboration: The French experience

Monica Panero[1], Gilles Aldon[1], Jana Trgalová[2] and Luc Trouche[1]

[1]University of Lyon, ENS de Lyon-Institut Français de l'Education, S2HEP-EducTice, France; monica.panero@ens-lyon.fr, gilles.aldon@ens-lyon.fr, luc.trouche@ens-lyon.fr

[2] University of Lyon, Claude Bernard University Lyon 1, S2HEP (EA4148), France; jana.trgalova@univ-lyon1.fr

The main aim of this paper is to analyse the experience of a MOOC for mathematics teacher training implemented in France, in parallel with a similar experience in Italy[1]. The study focuses on teacher collaboration within such an online learning environment, in terms of co-working and co-learning. The Italian and the French teams outlined a common starting point for the research and some common research concerns. Each team then reformulated the research questions and sought to answer them through specific theoretical lenses. In the case of the French MOOC eFAN Maths, we study the trainees' collaborative design of a pedagogical resource, by focusing on the efficiency of an evaluation grid designed by trainers to be used within a global process of peer evaluation. A comparison with the results of the Italian experience is shown in the conclusion.

Keywords: MOOC, teachers' professional development, meta-didactical transposition, community, collaboration.

Introduction

Internet communication tools provide the opportunity to develop new types of teaching methods that combine online courses, resources and discussions. MOOCs (Massive Open Online Courses) were conceived in 2008 at the initiative of prestigious American Universities (MIT, Harvard, Stanford, ...) that sought to enlarge the courses they offered to a wider audience. Since then, the MOOC phenomenon has been growing steadily and worldwide, the number of MOOCs doubled between 2014 to 2015 (Shah, 2015). Although there is a wide choice of many different topics, when looking specifically for a MOOC aimed at teacher training, the range is limited, especially in mathematics. Nevertheless, there is a growing interest in MOOCs involving mathematics teachers as participants, as shown by the TSG44 work during the 13th ICME[2]. In particular, from our experiences, there is a need for designing and implementing a MOOC for teacher training in mathematics education with a particular focus on the development of communities of practice (Wenger, 1998) and collaborative work among teachers as the basis for their professional development. Indeed, when people co-work (work together collaboratively) they can also co-learn (learn together collaboratively), as highlighted in the ICME survey of Robutti et al. (2016). The authors found that teachers can learn through discussion, conversation and reflection on their own teaching, on students' learning and the teaching of others. The methodology of the French MOOC eFAN Maths aims to create collaborative contexts

[1] See Taranto et al. (2017). Since the French MOOC and the Italian MOOC were delivered at the same period of time, even if the contents were designed independently, our teams had the opportunity to discuss and exchange about them.

[2] For more information, see http://www.icme13.org/files/tsg/TSG_44.pdf

for teachers' work, where they can learn from these kinds of practices. Taking into account the necessity for teachers to be supported in exploiting the affordances of technology, the shared objectives of both the French and Italian MOOCs are shared, namely: accompanying teachers in the production of teaching resources, by examples of activities and reflection on their ongoing resource design; fostering a reasoned use of technology, encouraging teachers to choose appropriate digital tools for the classroom. Such aims are related to the interest in the design and the implementation of teacher professional development programmes to include the role of teachers working and learning in communities (Wenger, 1998; Jaworski & Goodchild, 2006).

The originality of our research based on the data collected from two MOOCs (in Italy and in France) that share similar aims and objectives, is twofold. First, it focuses on the specific dynamics of online interactions – among trainees and between trainees and trainers – to study the trainees' use and appropriation of a tool (an evaluation grid) designed by trainers, for supporting peer evaluation of collaborative projects. It is topical and urgent to analyse the efficiency of such tools and interactions in the context of distance learning, because of the increased interest in this approach in recent years. Second, it analyses such dynamics according to the cultural constraints that shape MOOC design and development. French and Italian school environments have some remarkable differences and one of the most palpable is a wider freedom that institutional regulations traditionally give to the Italian teachers, compared to the major institutional constraints met by the French teachers. The Italian *Indicazioni*[3] (guidelines) highlight for each discipline the fundamental learning goals that students have to achieve at the end of each cycle of instruction (two or three scholastic grades). These guidelines have the character of general didactic guidelines and require teachers to take the responsibility to choose and link the specific mathematical contents to be developed in the classroom in order to reach the required learning goals. The French *Programmes*[4] (syllabus) are also based on competences for a given cycle of instruction, but they appear to be more detailed and normative: for each mathematical content, they provide some examples of activities. Moreover, they are accompanied by several additional resources intended to support the curriculum implementation in the classroom.

In this paper, and in parallel with the Italian contribution to the symposium, we draw on the common theoretical element of the Meta-Didactical Transposition (Arzarello et al., 2014) to highlight how the concepts of community and of collaborative work evolve, taking new and different forms, and how these processes impact on teachers' professional learning. As members of the French team, we are working together with members of the Italian team to compare data from the two MOOCs. In our conclusion, we will discuss the relevance of cultural and institutional aspects to the specific dynamics of the two experiences.

[3] http://www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf

[4] Links to the French curriculum and supporting material are available at http://eduscol.education.fr/

Description of the MOOC eFAN Maths

The MOOC eFAN Maths[5] was delivered on a French national platform (FUN, France Universités Numérique) and its target was the world-wide French-speaking community, namely mathematics teachers and teacher educators willing to improve their practices in using technology in their classrooms. The second season of the MOOC, which is reported in this paper, lasted from early March to mid-April 2016 over a period of five weeks. The MOOC eFAN Maths is part of the *Stratégie Mathematiques* of the French Ministry of Education, which stresses the relationship of mathematics with other sciences and with the world, and aims at training teachers in this perspective in order to give students a refreshed image of mathematics. More specifically, this season of the MOOC was created with a dual institutional aim: to support teachers and teacher educators to understand and implement the new French curriculum (introduced in September 2016 in all French primary and secondary schools); and to promote collaboration within the French-speaking community. Courses, activities and discussion were especially focused on the new themes involved in the French curriculum, namely algorithmics and interdisciplinary work. The MOOC was grounded on a project-based pedagogy, aimed at the design and the analysis of a classroom activity involving the use of digital tools. Every week, trainees took courses on specific topics of mathematics education from three video-based lessons, answered the related quizzes and worked on specific activities. The courses were constructed to provide trainees with elements to develop their reflections and projects about teaching and learning mathematics with technology. They showed brief episodes of classroom observation and their analysis, or were based on existing resources, showing and commenting animations or programs created with technology. Collective work was strongly encouraged among trainees. For this purpose, they were invited to join the "MOOC eFAN Maths 2016" group, created on Viaéduc[6], a professional social network specifically designed for teachers' exchanges. Viaéduc essentially allows members to post comments, to create subgroups, to create and publish documents and to comment/recommend/share them. Group members can work collaboratively either asynchronously, being authors of the same online document, or synchronously, writing on the same online collaborative board (*padlet*).

During week 0, on Viaéduc, trainees were invited to propose a theme, an idea or a project to work on and to start to establish relationships with others. Week 1 was devoted to the characterisation of digital or non-digital resources that can support teachers' and teacher educators' work. On Viaéduc, trainees gathered together around a project constituting public subgroups of the main "MOOC eFAN Maths 2016" group, so that any trainee could read the work of any subgroup and follow any discussion. Week 1 activity involved the selection of resources to constitute a "toolkit", deemed relevant to the group project. Week 2 was devoted to the analysis of students' activity using technology in mathematical situations. On Viaéduc, each group had to design a mathematical situation and to analyse it from the students' point of view using an analysis grid proposed by the trainers. Week 3 was devoted to the analysis of the teacher's role in the designed mathematical situation. An analysis grid that focused on instrumental orchestration was presented through the courses and trainees were

[5] eFAN Maths stands for *Enseigner et Former avec le Numérique en Mathématiques* (Teaching and Training in Mathematics with technology). See https://www.fun-mooc.fr/courses/ENSDeLyon/14003S02/session02/about

[6] See www.viaeduc.fr

invited to apply it to their situation. During week 4, trainers organised a process of peer evaluation of the different projects (submitted as versions 0) and proposed an evaluation grid grounded on the theoretical frames presented in the course, underpinned by the analysis of digital resource quality (Trgalová & Richard, 2012). Finally, every group was supposed to use the feedback received to refine and revise their project, submitted as version 2.

Theoretical framework

The MOOC eFAN Maths was analysed using a combination of three main theoretical frameworks: the Meta-Didactical Transposition (Arzarello et al., 2014), the documentational approach to didactics (Gueudet & Trouche, 2009) and the concept of communities of practice (Wenger, 1998).

The Meta-Didactical Transposition (MDT) model and the documentational approach to didactics, which places a major emphasis on the collective aspects of teachers' work, both required a theory to support the analysis of the development of teachers' (and researchers') collective work. Both approaches adopted the theory of communities of practice (CoP), mainly because these communities are structures where learning occurs. CoP are formed by people who engage in a process of collective learning in a shared domain of human endeavour. For Wenger (1998), a condition for the development of such communities is to balance participation and reification, where reification means producing resources, symbols, stories etc., recognised by the whole community as common products. Such communities may develop by themselves, or be "cultivated" (Wenger et al., 2002), i.e. encouraged, supported by an organisation. Indeed, organisational knowledge develops in a constellation of CoPs, and each CoP plays a specific role in this organisation.

The MDT model captures the dynamic interactions between teachers' and researchers' practices when these two communities interact, typically in training contexts, and in particular in the case of training programmes in mathematics education. Using the MDT lens, we can address research questions that infer the influence of the practices of one community on the other. Such practices are described through the notion of praxeology (Chevallard, 1999): a praxeology for a given task consists of a practical block comprising techniques to accomplish the task, and a theoretical block justifying and supporting these techniques. In a teacher education programme, trainees and trainers bring into play the components of their respective praxeologies. The objective of the programme is to transpose, in the sense of Chevallard, some components of the trainers' research practices into the teachers' practices, taking into account the classroom reality and teachers' expertise for effectively enacting such components. Thus, the two communities together contribute to creating a shared praxeology, which both communities can adapt in their future practices. This occurs through the phenomenon of *internalisation*: a community internalises a component of the praxeology of the other community, that was previously external to it, entailing an evolution of practices.

The documentational approach to didactics (Gueudet & Trouche, 2009) analyses teacher professional development through the interplay of practices and resources. This interplay is modelled as a *documentational genesis*, extending the notion of instrumental genesis introduced by Verillon and Rabardel (1995) between artefact and instrument. A documentational genesis involves different steps, such as looking for resources, selecting/designing mathematical tasks, planning their sequence, managing available artefacts, etc., to achieve a given teaching goal. This genesis gives birth to a document, which is a mixed entity composed of the revised and recombined resources and the

associated usage schema. Each documentational genesis is then a means to trigger teacher professional development. The genesis of a document combines two processes: *instrumentation*, when the affordances and constraints of the resources influence the subject's activity, and *instrumentalization*, when the subject shapes the resources that he/she appropriates. The documentational approach to didactics, from its beginning, and even more in its recent developments (Pepin et al., 2013), gives a major importance to the collective aspects of teachers' work with resources, evidencing the importance of interactions within teacher collectives for spurring documentational genesis and teachers' professional development.

The combination of these three frameworks supports our analysis of what happened during the MOOC, seen as a constellation of cultivated CoPs. These communities are not created at once, they emerge in the dynamics of a shared project. First, the community of trainers emerges as they design and implement a new teacher education programme. Then the communities of trainees emerge, each one developing through the advancement of the individual projects. The elaboration of a project involves the design of a pedagogical activity from selected existing resources that are subsequently adapted, modified and combined by the group members. Thus, this process can be seen as a collective documentational genesis. In such a process, we are interested to analyse the efficiency of the tools designed by the trainers, as elements of their meta-didactical praxeologies, to foster both collaboration and project development. We study the phenomenon of internalisation in interaction with the process of documentational genesis as a reification process, addressing the following research questions: How does each trainees' CoP emerge through the process of documentational genesis according to CoP criteria? How do the CoP of trainers and each trainees' CoP interact through the MDT lens?

Methodology of the study

In this paper, we focus on the final week of the MOOC and in particular on the trainees' use of the evaluation grid. This tool had been constructed by the trainers to include all of the phases of the pedagogical design, developed in the MOOC week after week. Analysing the way trainees used it can provide insights into the way they have understood and internalised the principles underpinning each phase of design. Moreover, the final week revealed an interesting dynamic between individual activity (the evaluation of another project through the grid) and collective work (of each group on the delivered version 0 for improving it). The evaluation grid was structured around the following four criteria: 1. Accuracy of the definition and description of the project; 2. Relevance of the mobilised digital resources with respect to the educational goal of the designed mathematical task; 3. Relevance of the analysis of the students' activity; 4. Relevance of the analysis of the teacher's role. For each criterion, aimed at evaluating the work done by a group during a specific week, some guiding questions were proposed with a double objective: to foster the production of justified feedback; and to deepen the reflection carried out in the previous weeks of the MOOC. The grid finally asked for a brief global feedback on the project and some suggestions to improve the work. Trainers provided this tool to support trainees in the process of peer evaluation, with the implicit aim to facilitate the internalisation of the evaluation criteria. Each trainee was invited to use the grid individually to evaluate the project of another group, by answering each guiding question with an appreciation: very good, satisfactory, weak or insufficient, accompanied by a justification. Trainers gradually collected feedback and comments in a table and shared it in a specific space on Viaéduc, called "Project evaluation", so that all the trainees could access them.

Data analysis

2500 people were enrolled in the MOOC, and more than 700 registered on Viaéduc, of which approximately 11% contributed to the work of a group, generating a large amount of data from multiple sources. In this paper, we analyse some discussions on Viaéduc related to how trainees used the evaluation grid and illustrate the main finding of our study: the trainees' internalisation of the evaluation criteria, through the tools provided and the process established by the trainers.

The evaluation grid became a resource for trainees. It was used both to provide feedback to other projects (instrumentation) and for reflecting on and refining their own project (instrumentalization). We illustrate this double action, that of the grid on the trainees and that of the trainees on the grid, through some excerpts of Viaéduc discussions. The comments, written by some trainees (CC, CM) on the wall of the "MOOC eFAN Maths 2016" group, show a formative value of the peer evaluation and specifically refer to an introspective use of the grid.

CC It is clear that the peer evaluation of the projects is also an exercise [...] I think that the aim is not marking "very good" everywhere, so I try to use the grid with its criteria that I start to understand [...]. A difficulty that I encounter is that, when I perceive a small flaw in one of the aspects of the whole structure of the project, this flaw seems to impact on several items of the grid...? [...]"

CM Indeed, by evaluating another project, you discover much better how to improve yours. The evaluation grid is a great support but I join CC on the domino effect of some points.

In particular, CC's and CM's comments show a well-thought-out use of the grid, especially the awareness that the evaluation criteria are interrelated. The double process of instrumentation and instrumentalization of the grid shows that trainees internalise the evaluation criteria using the grid also for reflecting on the quality of their own document.

Moreover, the remote collaboration on Viaéduc allowed trainees to evolve version 0 of the document into the version 1, taking into account both peers' external feedback and each member's introspective feedback. An example of this step in the collective documentational genesis is represented by the use of the padlet "TO DO List" within one of the groups. In this padlet, the group members organise the different tasks to be done in order to make the common document (version 0), seen as a resource, evolve into a new document (version 1). When a comment is ticked off and an author and a date are specified, this is the sign that the task has been done. This to-do list consists of feedback coming from peers but also of some personal comments, such as "For the resources I think that we must orient the reading of the first ones according to the soma cube activity". We can reconstruct the story of this proposal to reorganise the project resources, due to the parallel discussions that had occurred in the group. Such discussions show that trainees (JP in this case) benefited from both peers' feedback and introspective feedback, coming from the action of evaluating other projects.

JP [on the wall of the group] Hello, after having read several projects I actually expected that someone "criticises" a little bit our profusion of resources. […] Perhaps, we could prune it in the v1 of the project by keeping only those that are actually usable in the SOMA cube activity. What do you think?

In terms of remote collaboration, it is worth noting that some groups used such collaborative tools for organising their remote work. This organisation guided the transformation of the version 0 into the version 1 of the project, as a reification of the collaborative participation within the CoP.

Discussion

Each community of practice benefited from the feedback of others and from the introspective reflections that the members, who were engaged in the design process, made during the evaluation process. Crossing the external and the introspective feedback allows the trainees to work collaboratively on the refinement of their version 0 and to produce the version 1 of their project. This new version of the document is both a stage in the documentational genesis and the result of the internalisation of the evaluation criteria which occurred through the participation in the evaluation process. On the one hand, the evaluation grid is a technical tool of trainers' meta-didactical praxeologies, based on the theoretical concepts tackled in the courses and justified within the global process of peer evaluation. On the other hand, the trainers' choice of collaborative tools, seen as a trainers' technique, is grounded on the trainers' objectives to foster the emergence of communities of practice among trainees. Our analysis shows how these praxeological choices influence the trainees' work when they improve the version 0 of their project into the version 1. In return, this analysis influences the trainers' meta-didactical praxeologies in the perspective of a re-design of the new season of the MOOC. In particular, the organisation of the MOOC schedule will be modified taking into account the emergent question of time. Communities of practice need time for establishing an effective remote collaboration, so that participation and reification equilibrate as much as possible. At the same time, the evaluation process and the documentational genesis need time for being effectively carried out. In that sense, we can observe the double phenomenon of internalisation: from trainers to trainees as well as from trainees to trainers.

Comparison with the French experience and conclusion

As French team, we observed local communities of practice. We studied the phenomenon at a micro level, intervening in the groups' discussions to support and encourage the development of the collaborative work. The Italian team (Taranto et al., 2017), instead, observed a general community. They studied the phenomenon at a macro level, that is to say not intervening in the interactions between trainees.

During MOOC Geometria (the Italian MOOC) local groups are generated "emerging from chaos" (Siemens, 2004), namely they are subject to a spontaneous generation. During MOOC eFAN Maths trainers induce the generation of local groups and regulate peer relationships. Despite the fact that the cultural aspects affect these differences for sure (as we underline in our similar introduction), for both MOOCs there is an affinity that relies on the fact that trainees' learning is often generated by self-feeding discussions and instrumentalization processes.

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Using a flipped classroom approach in the teaching of mathematics: A case study of a preservice teachers' class

Theodosia Prodromou

University of New England, Australia; theodosia.prodromou@une.edu.au

This study investigates the usefulness of a flipped classroom approach in tertiary education. This exploratory study was conducted to understand the efficacy of the flipped classroom approach that was implemented by a lecturer teaching mathematics education. The study was conducted with the lecturer and her 185 pre-service teachers who attend her online course. Data collection instruments included a survey designed to investigate the dynamics of the flipped approach, semi-structured interviews for the pre-service teachers and the lecturer. The study adds to the literature related to the flipped classroom approach and the role of the lecturer in a flipped classroom.

Keywords: Digital technologies, flipped classroom, flipped learning, mathematics education, tertiary education

Introduction

The integration of digital technologies is becoming central in our mathematics classrooms. Some schools use digital technologies to replace or supplement teaching resources (Geiger, Goos, & Dole 2015). Multiple studies have been conducted that show that teachers and schools find it challenging to integrate such resources within mathematics lessons (Gueudet, Pepin, and Trouche 2012; Clark-Wilson, Robutti, and Sinclair 2014). Flipped classrooms integrate digital technologies allowing the teacher to implement the restructuring and re-organisation of teaching materials in both synchronous and asynchronous modes. The facilities of the classroom approach can be accessible from anywhere, in both real time or otherwise. How to best take advantage of digital technologies implemented in flipped classrooms is a challenge while attending at the same time to the dynamics of a flipped classroom and what is entailed in flipped learning. Given that research on the flipped classroom approach is in its infancy, there is limited research that has examined the approach under a pedagogical microscope.

Flipped classroom and flipped learning

Bergmann and Sams (2012) distinguished between the terms 'flipped classroom' and 'flipped learning', pointing out that they are not synonymous and that flipping the classroom does not necessary lead to flipped learning (FLN 2014). There are different interpretations of this approach and associated variations in implementations strategies. According to Bergmann, Overmyer and Wilie (2013), the characteristics of flipped learning are:

- increased interaction and personalized contact time amongst students and teachers;
- students take responsibility for their own learning;
- the role of the teacher is not the 'sage on the stage', but the 'guide on the side';
- a blending of direct instruction with constructivist learning;
- students who are not able to attend the class due to illness or extra-curricular activities such as athletics or fieldtrips, don't get left behind;
- content is permanently achieved for review or remediation;

- all students are engaged in their learning;
- all students can receive a personalised education.

Abeysekera and Dawson (2015, p. 3) defined the following typical characteristics of a flipped classroom within higher-education settings:

- a change in use of classroom time;
- a change in use of out-of-class time;
- doing activities that were traditionally considered as 'homework' in class;
- doing activities that were traditionally considered as in class work out of class;
- in-class activities that emphasise active learning, and problem-solving;
- pre-class and post-class activities;
- use of technology, especially video;

For this article, I will use the term 'flipped classroom' as defined by the Flipped Learning Network (FLN) to refer to the mode of teaching and learning 'in which direct instruction moves from the group learning space to the individual learning space, and the resulting group space, and the resulting group space is transformed into a dynamic interactive environment where the educator guides students as they apply concepts and engage creatively in the subject matter' (FLN 2014 para. 1). The FLN propose the 'Four Pillars of Flip' that recognise the significant features that are essential for learning to occur in a flipped classroom: (1) flexible environment, (2) a shift in the learning culture, (3) intentional content, and (4) professional educators (FLN 2014). This framework was considered to account for diverse learning modes, and its implementation necessitates the arrangement of flexible learning environments that may include, for example, the rearrangement of learning spaces and the use of digital technologies.

The Four Pillars of Flip framework is appropriate for gaining a better understanding of how the flipped classroom approach is implemented in practice. This framework is appropriate when analysing data about the teacher's role in organising the teaching materials in the flipped classroom. Abeysekera and Dawson (2015) proposed a theoretical model for the flipped classroom (see Fig. 1) that identifies the capacity of the flipped classroom to help students have a sense of competence, relatedness, and autonomy that will lead students to increased extrinsic and intrinsic motivation.

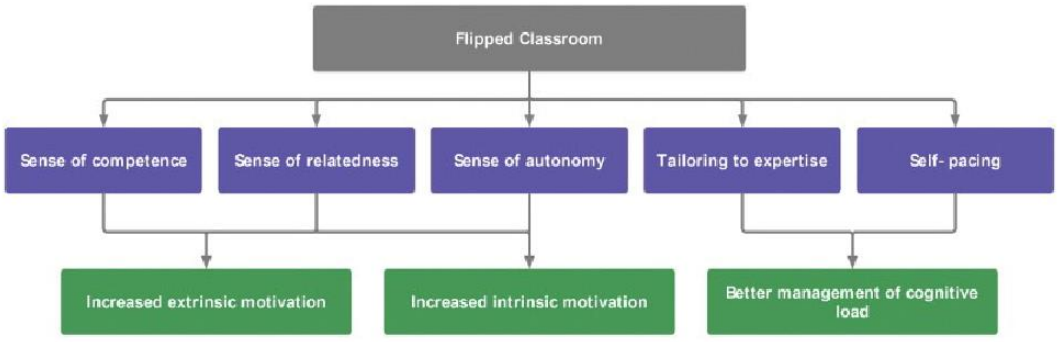

Figure 1 Theoretical model for the flipped classroom

Moreover, a flipped classroom approach is characterised by tailoring teaching material and activities to students' different expertise that allows students' self-pacing of pre-recorded lectures that may reduce cognitive load and help learning in a flipped classroom.

The theoretical model for the flipped classroom was considered appropriate for interpreting data pertaining students' engagement and motivation while it is required measurement of the cognitive load and motivation that are useful mechanisms for learning. The flipped classroom approaches are being adopted with enthusiasm despite the lack of specific evidence about their efficacy. However, substantial research questions remained unanswered. This article is particularly interested in university lecturer's implementation of the flipped approach in her practices. A small-scale localised intervention, including an experimental study, was conducted to understand the significant features of the lecturer's role against the essential criteria for learning to occur in a flipped classroom approach.

Methodology

An exploratory case study methodology was chosen as it involves a detailed study of a group of pre-service teachers' experiences. The data collected for the case study was both qualitative and quantitative. Analysis methods were employed (Creswell 2003) to provide richness and depth to the empirical investigation of a single university unit within its real context using multiple sources of evidence such as the lecturer, the entire cohort of preservice teachers who enroll in and attend the unit of study, and the teaching and learning resources that were used by the pre-service teachers.

The choice of the lecturer and her class was purposive in that the lecturer had indicated a strong desire to improve aspects of her online teaching related to student engagement, motivation, and self-pacing. The lecturer used a flipped classroom approach, and the appropriate technological infrastructure and digital technologies was in place to provide students with access to all teaching resources.

The study was conducted with the lecturer and her 185 pre-service teachers who attended an online unit that was required for a bachelor in Primary education or master in teaching Primary Education. The unit covered the content and pedagogy appropriate to teaching primary school students at stage 3 (year 5 or Year 6) in the strands of Data, Chance, Patterns and Algebra, and Number (numeracy). Students were asked to demonstrate their personal content knowledge in these strands, discuss associated teaching strategies, and create developmental learning sequences. The prescribed textbook, (Siemon, D., Beswick, K., Brady, K., Clark, J. and Faragher, R., 2015) was used to guide decisions about the sequence of mathematics topics to be taught. The textbook was used in combination with the Australian Curriculum (ACARA, 2016).

The lecturer prepared lectures recorded by Echo360, and offered two interactive online tutorials that were offered in real time by Adobe Connect. Each online tutorial was scheduled for two hours and students were able to ask questions and complete the tutorial activities with the help of the lecturer. Students who were not able to attend the online tutorials had access to the recording of the tutorials (in asynchronous mode). Additionally, demonstrations, electronic resources and relevant readings were available for students' use on the Moodle learning platform, which served as an online learning space were students interacted with each other, posted questions and engaged with collaborative activities (i.e. students were constantly experiencing feedback).

All students had access to the internet, and the online materials were also available for downloading in their own computers. The students were encouraging to assess the recordings of the online lectures, online tutorials, resources and assessment tasks for each topic.

Participants included one lecturer (Dr. April, pseudonym) and 185 students who attended the online unit. 142 students completed the online survey and 5 (3 females and 2 males) volunteered to participate in the semi-structures interviews. Dr. April, the lecturer had been teaching tertiary education for 7 years and was qualified to teach mathematics education to pre-service teachers and in-service teachers within primary and secondary and post-graduate programs. She had experience in using features of flipped classrooms for 7 years.

Data collection instruments consisted of an online survey that contained 18 questions about the use of the lecturer-prepared online resources. Responses were recorded against a five-point Likert scale. Semi-structured interviews were developed by the researcher to allow the lecturer to probe the pre-service teachers' experiences of the flipped classroom approach in this discipline. Pre-service teachers' online activities, postings, pre-service teachers' participation per day and the semi-structured interview of the lecturer were used to triangulate the data collected from pre-service teachers. All interviews were audio-taped and transcribed. Pre-service teachers' responses were coded and ascribed to five thematic categories as identified by Abeysekera and Dawson (2015). Lecturer's responses were also analysed and coded in four thematic categories based on the four Pillars of FLIP framework.

Results

In this section we present an analysis of the data collected from interviews with the lecturer, which was analysed using the Four Pillars of FLIP as a framework to recognise the significant features of the lecturer's role against the essential criteria for learning to occur in a flipped classroom: (1) flexible environment, (2) a shift in the learning culture, (3) intentional content and (4) professional educators. The data is supplemented with survey and interview data from the pre-service teachers.

Flexible environment:

When the lecturer was asked about the flipped teaching environment in an interview, she commented:

> Dr April: Students are expected to view the online lectures and the online resources (videos) any time that is best for them because they are mature students who study this online unit and they prefer working during night hours.
>
> Dr April: I am trying to extend classroom into their home where students cover the weekly teaching content and work the routine tasks and examples at home prior to the weekly lecture.

Students reported to making very good use of the online resources prepared by the lecturer and the recordings of the online tutorials at home, indicating that both were helpful for their mathematical learning and the completion of their assessment. As stated by two students, George and Sarah:

> George: I do not worry if I missed the real-time online tutorial due to my family commitments. I have the capacity to access the recordings of the online tutorials and lectures at any time I wish.
>
> Sarah: It is fantastic because I can watch the entire online tutorial or lecture without pausing it. But sometimes, I will watch it and pause it to write my notes about it. If I would like to repeat a part of it, I will simply go back to the part I need to listen again. I refer to certain parts many times because they are very essential for the online course.

A shift in the learning culture

Dr April relegated the more procedural demonstrations of solving a mathematical problem or constructing a mathematical application using mathematical software for the pre-service teachers to watch outside of the online tutorial time. As she commented:

> Dr April: It is fantastic because it frees up teaching time instead of spending half of the time of the tutorial to show students the procedure of solving a mathematical problem with or without the use of mathematical software. I am trying to maximise the tutorial time by covering the key aspects of the weekly content taught and the teaching techniques of teaching school students a target content.

The students reported that they make good use of Dr April's video based procedural presentations and demonstrations. For example, John and Jill (pseudonyms) said during an interview:

> John: I first listened to the online tutorials. I do find the online tutorials very helpful because the lecturer explains the mathematical content by applying the mathematical concepts in real life situations and solving mathematical problems. If you first watch the online tutorial very carefully it explains the mathematical content, application of mathematical content, primary students' difficulties in understanding specific concepts and pedagogical approaches and appropriate to teaching these primary students.
>
> Jill: Dr April's video based resources are very helpful to understand the content at home on my own without attending traditional classrooms.
>
> George: The online tutorials allow me to interact in real time with a group of other students and the lecturer, so I do not feel lonely during my studies. I post my questions on Moodle platform when I am stuck or unsure of the correct process.

Intentional content

Dr. April selected the content, decided on ways to present this content, and what resources (e.g., interactive whiteboard activities, or interactive games) would be appropriate for the online tutorial. As Dr April said:

> Dr April: In the video-based resources that students are expected to view before the tutorial, I demonstrate how to use technology when they teach, for example I demonstrate how they insert data in a table using Tinkerplots and create various graphical representations of data and analyse data. Or I explain the use of interactive board activities or educational games that would help pre-service teachers to teach students how to compare fractions using computer based interactive tasks.

Dr April's approach included the use of technology to transform the teaching content from the textbook into video format that allowed her to unpack mathematics in more depth during the online tutorials and lectures. She also prepared materials and videos that related mathematics more directly to cross curriculum priorities such as Aboriginal and Torres Strait Islander history and cultures, Asia and Australia's engagement with Asia, and sustainability.

Moreover, pre-service teachers commented that the digital technologies employed by their lecturer to teach mathematics had provoked rich discussions amongst the pre-service teachers, allowing them to access the step-by-step instruction as the lecturer intended.

Professional educators

As Sarah noted, when asked if there was anything about the online teaching of Dr April that helped her particularly to develop her mathematical learning:

> Sarah: The technology she uses helped me to visualise abstract mathematical concepts … the presentations help me to visualise the graphical presentations as she manipulates the graphical representations to help me to observe the impact of dragging the graph to the algebraic equation of the function. She is explaining very well and clearly all the step-by-step procedures and their dynamic behaviours. I am really amazed by the animations that she provided to show the dynamic behaviour of mathematical concepts.

John also pointed out:

> John: I feel that Dr April's videos are giving mathematics life. They explain mathematics better than the textbook does. I really prefer the recordings of the online tutorials because Dr April explains in detail what she is doing step-by-step using a very simple language.

George and Sarah also mentioned that:

> George: the presentations are very professional and the quality of the sound is perfect. The recordings of the online lectures and online tutorials were easy to follow and although mathematics was not my strongest subject of study, I enjoyed my study. I found those recordings very engaging.
>
> Sarah: I had all the online resources on my laptop and I study them while I am writing my assignment. I listen also to the online tutorials and lectures to write my assignment, but I do not use the prescribed textbook.

As indicated, the online resources were more appealing to the pre-service teachers and preferable to the prescribed textbook. The preparation of the online materials used for the pre-service teachers to access on their own exemplifies the enactment of the first, second, and third pillars of the flip framework. There was a shift from a static to a dynamic representation of the content and an intentional selection of the aspects of direct instruction that could be assessed in students' individual spaces.

Summary of 142 pre-service teachers' responses to the Likert items

Pre-service teachers indicated that they found the online resources prepared by Dr April to be very helpful for their mathematical learning (Q1, 96%). They also indicated that the online resources prepared by the lecturer are easy to follow and flowed well (Q2, 97%) and they agreed that they are engaging and interesting (Q3, 100%). The resources prepared by their lecturer were favoured over other online resources (Q4, 89%). Responses to the survey also indicated that pre-service teachers did not use the internet resources to help them with their mathematics learning (Q5, 73%). Pre-service teachers indicated that they found the online lectures relevant and meaningful, and they assist them with their mathematics learning (Q6, 96%). They also indicated that the online lectures were easy to follow (Q7, 99%) and they were about the right length (Q8, 95%). Significantly, respondents also found that the online tutorials offered them a great opportunity to answer questions about the mathematical content (Q9, 95%), fitting nicely into their schedule (Q10, 87%), being interesting and engaging (Q11, 89%; Q12, 2% boring). Interestingly, responses indicated that they make use of the

'Online tutorials' to explore mathematics (Q14, 5%) and they rarely use the 'Online tutorials' as a last resort when they were stuck on problems (Q15, 12%). However, they seemed to believe that they did better on the assessment because they watched the online tutorials (Q16, 82%). Importantly, pre-service teachers also believed that the step-by-step 'Demonstration of Mathematics software' was simple to use and it was beneficial (Q17, 93%), with only 1% of the responses indicated that the 'Demonstration of Mathematics software' was lacking and it required more technical tools (Q18, 1%); a possible consequence of the selection of the intentional content that Dr April expected students to cover before attending the online lectures and tutorials.

Conclusion

Dr April's attempt to implement aspects of a flipped classroom approach to her online course was made possible within a highly structured teaching and learning content because of the robust technological infrastructure in the university. Nevertheless, the pre-service teachers' engagement with their study of the online course was a consequence of the online resources developed by the lecturer to support her students' online learning. It would appear that pre-service teachers appreciated the opportunity they were offered to progress through materials at their own pace exercising a degree of autonomy in developing their own mathematical competence.

The four Pillars Framework afforded insights into the specific features of Dr. April's implementation of the essential features of flipped learning. The analysis of the data showed that Dr. April has provided students with a *flexible environment* that provided students with a *sense of autonom*y and *self-pacing*. Reflection on lecture's practice indicates a *shift in the learning culture* by transferring direct instruction from the classroom environment to home. The flipped classroom approach was implemented without removing the synchronous interaction of pre-service teachers with other students and the lecturer. It is noteworthy to point out that the interactive online tutorials aided in preventing students from being alienated from other students, the lecturer and the teaching materials. The selection of the intentional content was challenging in order to *tailor* the content to different *students' expertise.* Pre-service teachers appeared to be motivated to access the online resources prepared by Dr. April stressing the importance of *relatedness* between students and materials prepared by the lecturer to foster a version of an online flipped classroom. There was evidence that pre-service teachers engaged in rich mathematical discussions about the content of the teaching materials, showing that a flipped classroom can increase motivation and interaction amongst students and learning of ideas aimed by the provided teaching materials. The fourth Pillar of the flipped learning, the *professional educator*, appeared to be the agent of the flipped classroom approach. A *professional educator* (as Dr. April demonstrated) could foster the implementation of the flipped classroom approach in teaching and learning practices without making radical change to their current pedagogical approaches— a finding consistent with research conducted by Muir and Geiger (2016) who found that a flipped classroom approach could be implemented without radically reforming a teaching practice. Future studies could examine the suitability of flipped mathematics classroom approaches in different teaching contexts, such as inquiry learning contexts, teaching school mathematics, and whether or not it would be effective in improving students' learning outcomes.

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‘Walking a graph’ – primary school students’ experimental session on functions and graphs

Svein Arne Sikko, Øistein Gjøvik, Jardar Cyvin, Maria I.M. Febri and Ragnhild L. Staberg

Norwegian University of Science and Technology (NTNU), Norway; svein.a.sikko@ntnu.no, oistein.gjovik@ntnu.no

In this paper we report an experimental activity involving working with functions and graphs in a grade 6 class in a Norwegian primary school. We argue that working with graph loggers in the form of an echo sound system enhances student conception of mathematical graphs.

Keywords: Mathematical concepts, mathematics activities, mathematics education, educational technology.

Background and literature review

The research reported in this paper was part of the EU FaSMEd project[1], which brings together seven European countries and South Africa, researching the use of formative assessment and technology in mathematics and science education. Part of each country’s work has included case study interventions in ordinary classrooms in close cooperation with school teachers. In this paper, we focus on a particular set of lessons concerning functions and graphs, and how one particular type of technology was used to increase student activity and engagement, leading to enhanced learning.

The function concept is generally regarded as a difficult concept for students to grasp (Dreyfus & Eisenberg, 1982; Sajka, 2003; Sierpinska, 1992). Dreyfus and Eisenberg (1982) point out that the function concept is not a single concept by itself, but has several aspects and sub-concepts associated with it. DeMarois and Tall (1999) connect this to the learning of functions, saying that “for many students, the complexity of the function concept is such that the making of direct links between all the different representations is a difficult long-term task” (p.264). Also in Norway, national and international tests have shown that mathematical functions is a problematic topic area. It took humanity several thousand years of mathematical activity until functions were introduced in the 17th century; and even then it took time to place functions on a solid foundation within mathematics. It is therefore not surprising that pupils struggle with functions and how to respond to this struggle has been addressed in various ways. During the “New Math” movement in the 1960-70s, it was believed that school mathematics should resemble research mathematics, and attempts were made to introduce functions during the first years of primary school. Eicholz, Martin, Brumfiel and Shanks (1963) did exactly that. This American textbook was translated into several languages, including Norwegian. The pendulum turned away from this with the “back to basics” movement, and, in Norway, functions were moved to the secondary school. As a consequence, most research on students’ understanding of functions and graphs is conducted with secondary school students. In this paper, we report on work done in primary school.

[1]FaSMEd = Improving Progress for Lower Achievers through Formative Assessment in Science and Mathematics Education, see https://research.ncl.ac.uk/FaSMEd/

According to Duval (2006) we can only gain access to mathematical objects by semiotic representations. Janvier (1984) distinguished between four representations of functions: situations; graphs; tables; and formulae; and how to work with transitions between these. Duval (2006) stresses that "What matters is not representations but their transformation" (p. 107). That is, when learning about a mathematical concept, students deal with a representation of the object, and the main difficulty is to change between different representations of the same object. Duval distinguishes between conversions and treatments. Here, treatments take place between the same registers (e.g., changing $y = 2x$ from one particular form to another), while conversions take place between registers (e.g., reading a table and using the numbers within it to interpret a situation). The latter seems to be far more difficult, while the former is the most common activity format in school. Consequently, Janvier's framework helps teachers to focus on the change of registers rather than algebraic manipulation alone. To support the learning of functions, many different kinds of digital tools have been developed and used. Regardless of any particular view on learning outcomes from using technological tools, it is important to realize that use of technology is more than the introduction of new tools. In a survey on mathematics teachers' use of technology in England, Bretscher (2014) found that while ICT might contribute to change, the direction of this change was as likely to be towards "more teacher-centered practices rather than encouraging more student-centered practices" (p. 43). The tools used in our study are a particular type of data loggers. These are mainly used in science, but we claim that inclusion of these types of tools may be beneficial also in the mathematics classroom, and may contribute to more student-centered practices. According to Newton (2010) "data-logging methods involve the use of electronic devices to sense, measure and record physical parameters in experimental settings." (p. 1247). Measurements and results of the logging can be displayed on a computer screen, either subsequently or simultaneously. We used this type of technology to study students' early understanding of graphs. The use of motion experiments in the learning of functions has been studied by several authors. Nemirovsky (2003) conjectures that "mathematical abstractions grow to a large extent out of bodily activities" (p. 106). Arzarello and Robutti (2004) claim that students can grasp mathematical concepts through meaningful sensory-motor experiences if they are encouraged to communicate and have the necessary support (p. 308). Arzarello, Pezzi and Robutti (2007) point out that teachers can use new technology to design experiences for students "where graphs can be presented in a dynamical and genetic way" (p. 135). Robutti (2009) conducted research on time-distance graphs with kindergarten children using motion sensors and calculators, finding that even very young children were able to make connections between the movements they made in front of the sensor and the graph sketched by a calculator.

The research question addressed in this paper is: How can a primary school teacher use data logger technology to enhance primary school students' engagement and conceptual knowledge about function graphs?

Method

The teaching experiment was carried out in a grade 6 class (students around 11 years old) in a primary school in Norway. The number of students in the school is close to 600, and the number of teachers around 35. The participating teacher has background from general teacher education, with specialization in mathematics and history. At the time of the experimental sessions, he had been working as a teacher for 7 years, the last three years at the school in question. He had been teaching

mainly mathematics, and also some science. During his participation in the FaSMEd project he was teaching the same group of students, which began in grade 5 and continued with the same group of students into grade 6. There were 31 students in his class, 15 girls and 16 boys.

The theme of the teaching sessions was time-distance graphs. Several technological tools and software had been introduced to the participating teachers at FaSMEd meetings at the university. Teaching material from the FaSMEd toolkit that addressed time-distance graphs had also been introduced. The planning of the experimental lessons began at this meeting. The teacher would give one lesson introducing the students to graphs and to the connection between graphs and real life situations. Working with mathematical graphs connecting situations and graphical representations is usually not done in Norwegian primary schools. According to the national curriculum, functions and graphs is not a specified learning goal for students until after grade 10. This would therefore be the first time this teacher had worked with students in primary school on graphs. Because of this, he wanted to first pilot the lesson on a small group of students that he considered high achieving and with an interest in mathematics. Subsequently the lesson was repeated with a group of students considered to be lower achievers.

The technology used was two echo sounder devices developed by Pasco. This was chosen as the entry level for using it was not too high, and therefore the teacher considered it could be experimented with grade 6 students. It facilitated students to walk back and forth in relation to a logging device, such that a graph was immediately drawn on the computer screen indicating how near they were the device during a time lapse of ten seconds. The immediate live update of the graph distinguishes this activity from most regular science data logging activities. The computer was loaded with an app with premade tasks that were presented to the students. When students walked in front of the echo device, the computer would give a live display of graph in a time – position coordinate system. The tasks[2] were a mix of practical tasks: "Walk a graph", and open-ended questions about interpretation of the graphs from the "walks". All the results were saved and could be used by the teacher for assessment and feedback to the students. These data were e.g. used by the teacher at the end of the sessions to determine which student groups should present their work in a plenary. Students were chosen deliberately to give good examples of graphs made and how to interpret them.

Data sources collected during the experiment include a) observation sheets from two sessions; b) audio recordings from two sessions, from teacher pre- and post-interviews, post-lesson reflections, interviews and q-sorting activities with students; c) video recordings from two sessions; d) transcriptions of audio and video recordings; e) photos taken at sessions and of student work; f) files and screen shots from PC during student activity; g) teacher lesson plans for two sessions.

After the "walking a graph" activity, students were interviewed in a q-sorting[3] activity; i.e. they were presented with a set of statements printed on cards and asked to sort the cards according to

[2] The echo sound activity used some tasks taken from the software bundled with the Pasco software. Instructions for using the software and tasks were translated into Norwegian by the FaSMEd team. Some additional tasks were added.

[3] https://en.wikipedia.org/wiki/Q_methodology

whether they agreed, disagreed or were undecided about, the statement on the card. This activity was carried out in groups of 3 to 4 students.

Findings

The echo sound activity made this lesson stand out from an ordinary mathematics lesson. One student said,

> Student A: It was very different (…) In maths lessons we never move, we sit at our seat; except sometimes we go out to do measurements, but that is always during summer.

The tasks were also considered different to normal mathematics exercises on two accounts. First, students were not used to doing mathematics tasks using computers. Second, in the classroom they usually have to compute things, whereas in these lessons

> Student B: There were word problems and we had to do things.

The q-sort revealed that students generally agreed to statements that connected mathematics to real life. E.g., students agreed to the statements "Mathematics helps us to understand our surroundings" and "Mathematics is used in everyday life", whereas they disagreed with the statements "Mathematics is only for the classroom, not for real life outside school", "I can do without mathematics" and "Mathematics is not relevant for my future life". The q-sorting activities were completed around two weeks after the time-distance graph lesson. We may therefore claim that there is some evidence indicating that the lesson had made students aware of, or strengthened their awareness of, connections between mathematics in school and real life situations that can be described by mathematics or where mathematics is used. Students agreed that "Mathematics is important", claiming that

> Student C: We use it all the time. Everywhere. In the shop. (…) On trains. Airplanes. The bus.

It seems that these groups of students held positive attitudes towards mathematics, and that they were able to see mathematics as relevant for themselves and for real life situations. During the echo sound activity, the students had to relate what they were doing, i.e. the way they were walking in front of the sensor to the graph the software would display on the PC screen. We can see evidence that students were able to connect the pace of their walking to the slope of the graph:

> Student E: It rises earlier because you walk faster.

This relates the time (horizontal axis), distance from the sensor (vertical axis) to speed (how steep the curve is), a fundamental relationship in understanding time distance graphs, and a fundamental relationship in physics, and of course in everyday life. There were several student utterances showing the same kind of understanding:

> Student F: It will be more slanted the faster you walk. So you start slow, then you walk faster.

The teacher asked another student how you can find from the graph where you walked faster. The student said that

> Student H: You can see, because, first it is quite slanted, and then it goes straight up.

Students also developed understanding of the fact that a graph does not have to start at the origin. When trying to walk in a way that would produce a W as graph:

Student I: You have to start far away [from the sensor] because then it goes downwards and then it goes upwards and then it goes downwards.

We see here that they understand that a graph can cross anywhere on the vertical axis, and the relationship between distance from the sensor and time passed. Their descriptions and discussions did not use mathematical vocabulary or concepts. Rather, they described what they saw in everyday terms, which shows that they are able to change registers and not only operate within the same register. These examples show how such an activity helps students in the process of conversion. The activity offers two aspects of working with graphs. On the one hand, students had to translate a given situation into a movement in front of the echo sound device, observe the graph being plotted on the computer screen and adapt their movement to change the graph as needed. On the other hand, students would interpret a graph plotted on the screen into what kind of movement that this would correspond to. In the interviews, students said the tasks in this lesson were more challenging than the mathematics tasks they normally work with, e.g., in that they had to explain how they did things. Being challenging is not really a bad thing, and students said they found the sessions to have been great fun and exciting. They claimed that they had learned a lot about graphs. During q-sorting, students who agreed to the statement "I can better understand when I use the technology tools in our mathematics lessons" also agreed that the statement referred to learning about graphs:

Student E: I learned a lot about graphs and how they change with the computers

When the lesson was repeated with students that were considered to be lower achievers, the setup remained unchanged, making it more relevant for comparison. Notably, we found that it was not possible to distinguish any big differences between the first session with higher achievers and the second session with lower achievers.

Teacher: It was indeed very similar (…) maybe these were a bit slower. And I would be tougher, push the others more. (..) But I think they were clever, they were good at cooperating, learning on each other. (…) It shows that if you have open and good tasks, you have a lot of differentiation included.

The activity prompted student communication and discussion. The teacher found that students who normally keep quiet were engaged in discussion.

Teacher: In particular, some of the girls in the last group, they were talking, usually they are very quiet. Now they talked, without me having to point at them, prompting them; now they gave their opinion (…) And I was positively surprised at how easy it was for them, to listen to each other's arguments.

It was obvious that, even if these types of activities with graphs are common in Norwegian primary schools, it had not been too difficult and that this is a topic that could easily have been done with the whole class. The teacher said that

Teacher: I think, interpreting graphs, it could have been done quite easily. (…) I think this might be more fun in primary than in lower secondary school. They still find it exciting with graphs (...) they are more curious and less biased.

In the interviews, all students said that they had enjoyed taking part in the project and performing the lessons with graphs.

Student: In my opinion everything was good (…) We learned a lot about graphs.

Discussion and conclusions

The echo sound graph plotting activity was very useful to give the students hands-on experience in using modern technology and use their own physical movements to create something to talk about mathematically. Acquiring experience with new technologies can be an educational goal in itself, and in particular, echo sound technology is not common in the classroom, but it is well known in other aspects of life. In the interviews, students claimed this was an important part of what they had learned and which distinguished these lessons from ordinary mathematics lessons. In traditional data logging experiments, students might see the data collection and the data analysis as two separate entities as these are separated in time (Barton, 1997). In our experiment, the gap between the collection of the data and the displayed graph is narrowed down to practically zero. In this respect, this activity also resembles working with dynamical graph tools, like GeoGebra[4]. These software tools allow students to explore graphs by manipulating parameters within designated bounds, while walking a graph changes freely the look of a graph only limited by the range of the echo sound device. This is more in line with work by e.g. Arzarello and Robutti (2004) and Robutti (2009).

When looking at the Janvier table we see that what the students had engaged in was making a transition between a situation and a graph. However, the typical sketching activity proposed by the Janvier framework when working with functions usually has a different feel than in this experiment. Not only is the sketching part of the activity itself done in a kinesthetic manner. There is also a dual aspect in that the students continually interpret the graph whilst the graph is sketched by the program on the screen. This way we can say that students work simultaneously with two elements of Janvier's framework, giving further evidence that changing registers is a difficulty that can be overcome by giving appropriate teaching materials.

The kinesthetic part of the activity, the walking, is in itself an important aspect of the experiment. As it turned out, the designated low achievers were able to perform well and display great enthusiasm during the session. This can be related to the way learning through movement can be an alternative approach to put students in a receptive state, ready for learning. Learning through actually moving your body is rarely an aspect of mathematics lessons, but can certainly encourage engagement, as seen in this experiment.

The type of activity exemplified in this experiment is completely devoid of focus on algorithms or procedural performance in the form of computations. Students do not know in advance how to solve the problems presented, and so focus is on developing conceptual knowledge about function graphs. From their statements, we also see that they relate mathematical concepts, like slope, to real world experiences, like speed. This is similar to findings in Robutti (2009, p. 68). A well founded conceptual understanding of functions and graphs in a time-distance setting will contribute to better understanding of functions on a more general level. When students encounter functions at higher

[4] http://www.geogebra.org

grades, their conceptual foundation will make it easier to grasp other aspects and algorithmic approaches to functions.

The literature suggests that teachers need support of different kinds in order to conduct teaching with new technologies in a meaningful manner. For example, building on a large teacher survey in Singapore, Tan, Hedberg, Koh and Seah (2006) suggest that teachers need support from laboratory technicians, data logger training, and instructional material to use data loggers effectively. In our case, none of these were present. We do however acknowledge the collaborative effort between teacher and researchers as instrumental to the success of the sessions. It is also important to stress that learning is not an automatic outcome from playing with technological tools, no matter how sophisticated the tools are. The role of the teacher is instrumental in bringing about learning, as highlighted by Clark-Wilson, Robutti and Sinclair (2014, p.396).

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PÉPITE Online automated assessment and student learning: The domain of equations in the 8th grade

Stéphane Sirejacob[1], Françoise Chenevotot-Quentin[2] and Brigitte Grugeon-Allys[3]

[1]Laboratoire de Didactique André Revuz (LDAR), Université Paris Diderot, Paris, France; stephane.sirejacob@etu.univ-paris-diderot.fr

[2]Laboratoire de Didactique André Revuz (LDAR), Université d'Artois, Universités Cergy Pontoise, Paris Diderot, Paris Est Créteil, Rouen, France; chenevotot.francoise@neuf.fr

[3]Laboratoire de Didactique André Revuz (LDAR), Université Paris Est Créteil, Universités d'Artois, Cergy Pontoise, Paris Diderot, Rouen, France; brigitte.grugeon-allys@u-pec.fr

"Pépite" is an online automated assessment tool for elementary algebra for students in secondary education (12-16 years-old) in France. Pépite was initially developed for students at the end of compulsory schooling in France (16 years-old). At CERME9, we presented its transfer at different school levels and illustrated it with the design of Pépite test for grade 8th students. Information provided by Pépite allows identifying students' consistent reasoning and calculation in order to organize teaching corresponding to students' learning needs. In this paper, we focus on the use of Pépite test for grade 8th students to learn the domain of equations. We defined an epistemological reference of the algebraic domain that allows us not only to build the tasks selected for the test and to analyze students' responses but also to propose suitable courses adapted to students' learning needs.

Keywords: diagnostic assessment, Information and Communication Technology (ICT), elementary algebra; equations, student's profile, teaching suggestions.

Context of the study

This paper deals with the issue "Digital assessment of and for learning" of TWG16 "Students Learning Mathematics with Technology and Other Resources". Since the 1990s, our team has developed several multidisciplinary projects (Delozanne & al., 2010; Grugeon-Allys & al., 2012) concerning the design, development and use of online tools for diagnostic assessment and student learning. One of these tools, named "*Pépite*", is relevant for learning elementary algebra for students of secondary education (12-16 years) in France. We have disseminated *Pépite* online tool on platforms[1] largely used by teachers and students.

In this paper, we deal with the use of *Pépite* online assessment for learning the domain of equations for grade 8th students. First, we revisit the theoretical foundations of *Pépite* online assessment. Then, we illustrate it with *Pépite* assessment for grade 8th students in France (13-14 years). We specify both the didactical model and the computer model that automatically generates generic tasks, analyses students' work and provides descriptions of students' profiles. Finally, we discuss the potentialities

[1] Pépite tools are available on LaboMep platform (developed by Sésamath, a French maths' teachers association): http://www.labomep.net/ and on WIMS environment (an educational online learning platform spanning learning from primary school to the university in many disciplines).

of *Pépite* online assessment to propose suitable courses adapted to students' learning needs for the domain of equations for grade 8th.

The theoretical and methodological framework

In the educational system, assessment is a complex issue. Usually, assessment results are generated from standardized and psychometric models. Studies highlight the strengths and limitations of such approaches to make instructional decisions (Kettelin-Geller & Yovanoff, 2009). To identify the features of appropriate online assessment for learning, we have chosen both a cognitive and epistemological approach and also an anthropological approach, the potentialities of which are described in Grugeon-Allys & al. (2012).

Epistemological and cognitive approach

Designing a test requires the selection of a set of tasks that enables the assessment to be realized. We agree with Vergnaud who stated, "Studying learning of an isolated concept, or an isolated technique, has no sense" (Vergnaud, 1986, p. 28). Furthermore, Vergnaud introduced a strong assumption: dialectics between genesis of a student's knowledge and mathematical knowledge structure. Beyond a quantitative analysis of responses, we have to define a qualitative didactical analysis (based on a collection of students' responses to the tasks) to identify the type of procedures and knowledge used by students in solving the tasks. To provide descriptions of a student's consistent reasoning, it is necessary to define a reference for modelling the mathematical competence, in a mathematical domain, at a particular school grade.

Anthropological approach

The epistemological approach is not sufficient so as to take into account the impact of the institutional context on students' learning. According to the anthropological approach, mathematical knowledge is strongly connected to the institutions where it has to live, to be learnt and to be taught; it is strongly connected to mathematical practices (curricula, etc.). Chevallard (1999) analyses knowledge in terms of praxeology, that is to say in terms of type of tasks, techniques used to solve these tasks, technological discourses developed in order to produce, explain and justify techniques, and last, theory that justifies technological discourses.

A reference epistemological praxeology for algebraic knowledge

For a given mathematical domain, we defined a reference epistemological praxeology (Garcia, Gascon, Higueras & Bosch, 2006) that makes it possible to create an *a priori* design that describes features of an appropriate assessment. For algebraic knowledge, such reference is based on results from didactics of algebra (Chevallard, 1989; Artigue & al., 2001; Kieran, 2007). In its *tool* dimension (Douady, 1985), there are tasks for *generalizing, modelling, substituting, proving.* In its *object* dimension, there are tasks focused on calculus with algebraic expressions (*calculating, substituting* a number for a letter, *developing, factorizing)* or equations (*solving*). This reference makes it possible to define appropriate technology for an intelligent and controlled algebraic calculus, based on equivalence of algebraic expressions and the dialectic between numeric and algebraic treatment modes.

The domain of equations for grade 8th students

The three following types of tasks are specifically related to equations (we will give some precise examples later about our experimentations):

- Modelling and putting a problem into equation (tool dimension). These tasks motivate the production of an equation in order to solve modelling problems and require semiotic conversions (Duval, 1993).
- Solving an equation using an algebraic technique; proving that two equations are equivalent (object dimension). These tasks use the concept of equivalence and require transformational activity (Kieran, 2007).
- Testing if a number is a solution of an equation; identifying the degree of an equation (object dimension). These tasks are based on substitution and polynomial properties.

Features of *Pépite* online assessment

The *Pépite* online diagnostic assessment is based on a reference epistemological praxeology of the algebraic domain, both for designing tasks and for analyzing responses to the test. We will base this on the *Pépite* test for grade 8th students.

The didactical model

Pépite test

The test (targeting 13-14 years old students) is composed of 10 diagnostic tasks and 22 individual items covering the types of tasks defined below (Table 1). The tasks may be multiple-choice or open-ended items (Figure 1).

Types of tasks	Number of items	Test items
Calculus	4 / 22	7.1 / 7.2 / 8.1 / 8.2
Producing numerical expressions	1 / 22	5
Producing algebraic expressions	2 / 22	3.1 / 6
Translation or recognition	14 / 22	1.1 / 1.2 / 1.3 / 2.1 / 2.2 / 2.3 / 3.2 / 4.1 / 4.2 / 9.1 / 9.2 / 9.3 / 9.4 / 10
Problem solving in different mathematics frameworks	1 / 22	6

Table 1: Organization of the 8th grade level test in terms of types of tasks

Exercise 6: Proof and calculation program

A student says to another student: “You will always find the same result if you take a number, you add 6 to that number, you multiply the result by 3, you subtract triple the initial number”.
Is this statement true for any number? Justify your answer.
Justification

Result
The statement is true for any given number: true or false?

Figure 1: Example of generalization task

Responses analysis: the multidimensional model of algebraic assessment

At the local assessment level (for each task), students' responses are not only evaluated as correct/incorrect, but also according to their technological discourse, that justifies the techniques. The analysis concerns validity of response (V) and seven dimensions: meaning of the equal sign (E), algebraic writings produced during symbolic transformations (EA), numerical writings produced during symbolic transformations (EN), use of letters as variables (L), algebraic rationality (J), connections between a semiotic register to another (T) and skills with negative and decimal numbers (N) (Table 2) (Grugeon-Allys, 2015). We code the responses with assessment criteria, which depend on knowledge and reasoning involved in the techniques[2].

Assessment dimensions	**Assessment criteria**
Validity of response	V0: No answer V1: Valid and optimal answer V2: Valid but non optimal answer V3: Invalid answer Vx: Unidentified answer
Algebraic writings produced during symbolic transformations	EA41: Incorrect rules make linear expressions a^2->2a EA42: Incorrect rules gather terms …
Connections between a semiotic register to another	T1: Correct translation T2: Correct but non optimal translation T3: Incorrect translation taking into account the relationships T4: Incorrect translation without taking into account the relationships Tx: No interpretation

Table 2: The multidimensional model of algebraic assessment (partial view)

We illustrate the multidimensional model of algebraic assessment on the task "Proof and calculation program" (Figure 1). In order to solve this task, two *a priori* strategies are possible: an arithmetic strategy using a number or an algebraic strategy mobilizing a variable. Several incorrect techniques can illustrate an arithmetic strategy (Table 3) according to the rules used to translate or transform numeric expressions. Algebraic strategy may be incorrect (J3) if the conversion rules (T3 or T4) or algebraic transformation rules (EA3 or EA4) are inadequate (Table 4).

Solutions	**Reasoning and technological discourse**	**Coding**
For number 5 $(5 + 6) \times 3 – 3 \times 5 = 18$	Correct arithmetic strategy with global expression that uses parenthesis	V3, L5, EA1, J2, T1
For number 5 $5 + 6 = 11$; $11 \times 3 = 33$; $3 \times 5 = 15$; $33 – 15 = 18$	Correct arithmetic strategy with partial expressions	V3, L5, EA1, J2, T2
For number 5 $5 + 6 \times 3 - 3 \times 5 = 8$	Erroneous arithmetic strategy with global expression that uses no parenthesis	V3, L5, EA3, J2, T3
For number 5 $5 + 6 = 11 \times 3 = 33 – 3 = 30 \times 5 = 150$	Erroneous arithmetic strategy with calculus by step (procedural aspect)	V3, L5, EA3, J2, T4

Table 3: *A priori* analysis for arithmetic strategies

[2] Contrary to usual practices in assessment, we do not attribute a code by technique for each task. This would lead to a multiplicity of codes on various tasks and would be unusable for a cross analysis on all the tasks of the test.

Solutions	Reasoning and technological discourse	Coding
$(x + 6) \times 3 – 3 \times x$ $= 3x + 18 – 3x$ $= 18$	Correct algebraic strategy with global expression that uses parenthesis	V1, L1, EA1, J1, T1
$(x+6) \times 3 = 3x + 18$; $(3x + 18) - 3x = 18;$	Correct algebraic strategy with calculus by step (procedural aspect)	V2, L1, EA1, J1, T2
$x + 6 \times 3 – 3 x$ $= x + 18 - 3 x$ $= - 2x + 18$	Erroneous algebraic strategy with global expression that uses no parenthesis	V3, L3, EA32, J3, T3
$(x + 6) \times 3 = 3x + 18 = 21x$; $21x - 3x = 18x;$	Erroneous algebraic strategy with calculus by step (procedural aspect)	V3, L3, EA42, J3, T4

Table 4: *A priori* analysis for algebraic strategies

Student's profile, groups and differentiated strategies

The *Pépite* diagnostic assessment proposes both individual and collective assessment. The individual assessment, at the global assessment level (on a set of tasks), builds the student's profile which aims to identify features of algebraic knowledge and skills for the seven dimensions. The collective assessment locates a student on a scale with four components: skill in Algebraic Calculus (CA), skill in Numerical Calculations (coded CN), Use of Algebra for solving tasks (UA) and flexibility in Translating a semiotic register to another (TA). For each of those four components, different technological levels and appropriate benchmarks have been identified (Grugeon-Allys & al., 2012). Regarding to a class, students are divided into three groups according to their skill in Algebraic Calculus: CA1 (group A) - reasoned and controlled calculation preserving the equivalence of expressions -, CA2 (group B) - calculation based on syntactic rules often in blind, not always preserving the equivalence of expressions - and CA3 (group C) - meaningless and non-operative calculation. Therefore, for a given learning objective, it is possible to assign tasks to each group, depending to didactical variables related to the associated technological levels (Delozanne & al., 2010, Pilet & al., 2013).

The computer model

An iterative process between educational researchers, computer scientists and teachers was used to design and test different *Pépite* prototypes in order to improve the didactical model. We defined the conceptual IT model of classes of tasks, which allows the characterizing of equivalent tasks (Delozanne & al, 2008). The software *PépiGen* (Delozanne & al., 2008) automatically generates the tasks and their analyses, at different grade levels. It uses *Pépinière*, a computer algebra system, to generate anticipated student correct or incorrect answers (according to the *a priori* analysis). *Pépite* automatically calculates a student's profile as well as profiles for groups of students. According to a learning objective defined by the teacher, *Pépite* generates tasks adapted to the related technological levels (Grugeon-Allys & al., 2012).

Results and discussion

The information provided by *Pépite* diagnostic assessment allows the teacher to identify students with close profiles in algebra. Then, *Pépite* automatically generates differentiated routes corresponding to these algebraic profiles. As mentioned above, these routes were designed on the basis of a reference epistemological praxeology.

Differentiated routes for learning equations

Three differentiated routes were created concerning equations. The first route "Motivating the production of an equation and solving it with an equation solver" motivates the production of an equation. It includes tasks like "*equalizing two calculation programs*" (see the example below). Students have to solve them using an equation solver. The second route "Algebraic resolution of an equation" requires technologies for solving equations by algebraic methods (by using the concept of equivalence of equations). In the last route "Algebraic resolution of problems that lead to an equation", tasks that require a problem to be expressed as an equation and then solved, such as, "*equalizing two perimeters of dynamic figures*", are proposed.

We give now two examples of tasks for the first route. The first one aims to introduce equations and to highlight the inadequacy of arithmetic techniques to solve problems of first degree. As we can see, thanks to a thoughtful choice of the didactic variables, this task prevents arithmetic strategies – because of the presence of the unknown in both calculation programs – or "trial and errors" methods – because the solution of this problem which is $\frac{7}{3}$ cannot be easily obtained by successive trials. Algebraic techniques are necessary.

<table>
<tr><td colspan="3">For groups A, B and C</td></tr>
<tr><td>Program A</td><td>Program B</td><td rowspan="2">Alex and Brenda choose the same start number.
Alex tests the calculation program A and Brenda tests the program B.
Then, Alex and Brenda find the same final result.
Which start number did they choose?</td></tr>
<tr><td>Choose a start number
Multiply it by 3
Add 5 to the result</td><td>Choose a start number
Multiply it by 6
Subtract 2 to the result</td></tr>
</table>

Table 5: Task for motivating the production of and equation and solving it with an equation solver

The second task is differentiated (Table 6) to take into account students' algebraic activity and makes the students work on semiotic conversions (from the representation register of algebraic writing to the representation register of calculation program). Differentiation relies on didactical variables: the left member of the equation for group A is a product and solving the equation needs to use the distributive property, while the equation for groups B and C do not require it to be solved. Moreover, the multiplication sign is used for groups B and C to suggest that one or more multiplications are expected in the expression.

For group A	For groups B and C
Write a problem with two calculation programs that correspond to the equation $2(x+7)=5-3x$.	Write a problem with two calculation programs that correspond to the equation $2\times x+7=5-3\times x$.

Table 6: Task for working on semiotic conversions solver

Experimentation in a grade 8th class

We now present the results of research carried out in 2016 with a mathematics teacher we will call M2. M2 has been working in a REP establishment (high-priority education network) for three years. We chose him because he is not an expert. After an observation phase (6 hours) of his teaching practices, we proposed a whole teaching sequence to him on equations that takes into account the main epistemological aspects of the reference epistemological praxeology. M2 was free to adapt this sequence to his practices; however, both teacher and researcher have worked together to plan the implementation in the class.

M2 is required to introduce equations in his grade 8th class. First, his 20 students (14 years old) completed the *Pépite* test. Then, they were been divided into three groups A, B and C. Only one student belonged to group A (reasoned and controlled calculation preserving the equivalence of expressions). The others students belonged to groups B (15 students who can calculate correctly expressions but without using semantic rules) and C (4 students who do not understand the calculus on algebraic expressions). M2 proposed to his students the three routes mentioned above, in the same order. Due to the fact that most of his students were in group B (15/20), M2 chose to give the same tasks to the whole class. After working on the three routes, the 20 students completed a written test on equations. We chose to focus on two tasks from this test to present our results. The first task was about solving three first-degree algebraic equations. Depending on the equation they solved, 7 to 11 students among the 20 students found the correct solutions. We particularly studied how many students used an algebraic technique. We observed that 17 out of 20 students solved the equations using the equivalence of equations. Even if they did not find the right solution, they had a strategy and transformed the equations in order to "eliminate" the unknown; they respected the concept of equivalence to do so. For the second task, equalizing two calculation programs (as presented above in table 5), 11 out of 20 students succeed for putting the problem into an equation.

Discussion

The *Pépite* assessment tool, based on an epistemological reference of the algebraic domain, allows the teacher to identify students' consistent reasoning and calculation in order to plan differentiated courses adapted to grade 8th students' learning needs for the domain of equations. The mathematics routes tested in our experimentation seemed to have effects on the students' technological level: most of them used algebraic techniques to put a problem into equation. But this experimentation only concerns one teacher. So, in the ERASMUS + project "Advise me" which has just started in September 2016, we aim to carry out a larger scale research study.

We intend to validate these results for the field of arithmetic of integers for grade 3-4 pupils. Grapin (2015) carried out a multidimensional model of assessment for this new domain in elementary school. She defined an epistemological reference of arithmetic of integers to design an assessment tool in order to define pupils' profiles and to highlight the epistemological aspects of arithmetic to work according to pupils' learning needs. She organized an experimentation to study the evolution of pupils' profiles according to differentiated routes adapted to students' learning needs for the domain of arithmetic of integers. Data analysis is underway.

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Analysing MOOCs in terms of their potential for teacher collaboration: The Italian experience

Eugenia Taranto[1], Ferdinando Arzarello[1], Ornella Robutti[1],

Virginia Alberti[2], Sara Labasin[3], Sara Gaido[1]

[1]University of Turin, Italy; eugenia.taranto@unito.it; ferdinando.arzarello@unito.it; ornella.robutti@unito.it; sara.gaido92@gmail.com

[2]I.I.S. "B. Castelli", Brescia, Italy; alberti.virginia@gmail.com,

[3]L.S. "P. Gobetti", Turin, Italy; saralabasin@yahoo.it

The main aim of this paper is to analyze the experience of a MOOC for mathematics teacher training implemented in Italy, in parallel with a similar experience in France[1]. The study focuses on teacher collaboration within such an online learning environment, in terms of co-working and co-learning. The Italian and the French teams outline a common starting point and set of concerns for the research (the two papers have a similar Introduction for this reason). Each team then reformulated the research questions and tried to answer them through specific theoretical lenses. The Italian team used a fresh theoretical framework called MOOC-MDT. We concentrate on practices implemented by teachers who attend the MOOC, in particular on their contributions to communication boards and the consequent conception and growth of their particular community. In the conclusions, we contrast our results with those of the French experience.

Keywords: MOOC, teachers' professional development, meta-didactical transposition, community, collaboration

Introduction

Internet communication tools provide the opportunity to develop new types of teaching methods that combine online courses, resources and discussions. MOOCs (Massive Open Online Courses) were born in 2008 at the initiative of prestigious American Universities (MIT, Harvard, Stanford, ...) that sought to enlarge the courses they offered to a wide audience. Since then, the MOOC phenomenon has been regularly growing and the worldwide number of MOOCs has doubled from 2014 to 2015 (Shah, 2015). Although there is a wide choice of many different topics, when looking specifically for a MOOC aimed at teacher training, the range is limited, especially in mathematics. Nevertheless, there is a growing interest in MOOCs involving mathematics teachers as participants, as shown by TSG44 work during the 13th ICME[2]. In particular, from our experiences, there is a need for designing and implementing a MOOC for teacher training in mathematics education with a focus on the development of communities of practice (Wenger, 1998) and the collaborative work among teachers as a basis for their professional development. Indeed, when people co-work (work together collaboratively) they can also co-learn (learn together collaboratively), as highlighted in the ICME

[1] See Panero et al. (2017). Since the Italian MOOC and the French MOOC were delivered at the same period of time, even if the contents were designed independently, our teams had the opportunity to exchange and to discuss about them.

[2] For more information, see http://www.icme13.org/files/tsg/TSG_44.pdf

survey of Robutti et al. (2016). The authors found that teachers can learn through discussion, conversation and reflection on their own teaching, on student learning and on the teaching of others. The methodology of the Italian MOOC Geometria aims to create collaborative contexts for teachers' work, where they can learn from these kinds of practices. Taking into account this necessity for teachers to be supported in exploiting affordances of technology affordances, the objectives of both the French and Italian MOOCs are shared, namely: accompanying teachers in the production of teaching resources, by examples of activities and reflection on their ongoing resource design; fostering a reasoned use of technology, encouraging teachers to choose appropriate digital tools for the classroom. Such aims are related to the interest in the design and the implementation of teacher professional development programmes to include the role of teachers working and learning in communities (Wenger, 1998; Jaworski & Goodchild, 2006). The originality of our research based on the data collected from two MOOCs (in Italy and in France), that share similar aims and objectives, is twofold.

First, our new framework (MOOC-MDT: see below) facilitates the study of the specific dynamics of the interactions among trainees and between trainees and trainers, which occur online and in totally virtual environments. It is topical and urgent to analyze these interactions in the context of such distance learning due to the increased interest in this approach in recent years. Consequently, we reviewed and revised an existing framework that had been used to describe face-to-face meetings for teacher professional development, namely the Meta-Didactical Transposition (see below).

Second, our new framwork analyzes suech dynamics according to the cultural constraints that shape the MOOCs' design and development. The French and Italian school environments have some remarkable differences and one of the most palpable is a wider freedom that institutional curriculum regulations traditionally give to the Italian teachers, compared with the major institutional constraints met by the French teachers. The Italian *Indicazioni*[3] (guidelines) highlight for each discipline the fundamental learning goals that students have to achieve at the end of each cycle of instruction (two or three scholastic grades). These guidelines have the character of general didactic guidelines and defer to teachers the responsibility of choosing and linking the specific mathematical contents to be developed in the classroom in order to reach the established learning goals. The French *Programmes*[4] (syllabi) are also based on competences for a given cycle of instruction, but they appear to be more detailed and normative: for each mathematical content, they provide some examples of activities. Moreover, they are accompanied by several additional resources intended to support for the curriculum implementation in the classroom.

In this paper, and in parallel with the French one, we draw on the common theoretical element of the MDT to highlight how the concepts of community and of collaborative work evolve to new and different forms, and the impacts on teachers' professional learning. As members of the Italian team we worked alongside members of the French team to compare the data from the two MOOCs, so in the conclusion we will discuss the relevance of cultural and institutional aspects in the specific dynamics of the two experiences.

[3] http://www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf

[4] Links to the French curriculum and supporting material are available at http://eduscol.education.fr/

The description of MOOC Geometria

The "MOOC Geometria" is the result of a long development process over many years by the researchers of the Mathematics Department of Turin University, and characterized by many previous experiences of teacher education projects in which the team has been involved (e.g. the M@t.abel project https://goo.gl/Q30Dn0) alongside years of research into teacher education. The MOOC was delivered on a Moodle platform (http://difima.i-learn.unito.it/) between October 2015 to January 2016 (6 modules in 8 weeks), and the 424 participants, all teachers in secondary school, were from all over Italy. 36% of the teachers completed all of the MOOC activities, which compares with reported average completion rate of about 5% (Bayne & Ross, 2013).

The MOOC team comprised 13 people (university researchers and expert in-service teachers). The MOOC had two main teacher education aims: professional training and raising awareness of the possibilities for technology use in schools. Every week the trainees worked individually to become familiar with different approaches. These activities included: watching a video where an expert introduced the conceptual knot of the week; watching a "cartoon video" with some guidelines to carry out the units; reading the geometry activities based on a mathematics laboratory (and the option to experiment with these in their classroom). Trainees were invited to share thoughts and comments about the activities and their contextualization within their personal experience, using specific communication message boards (forum, padlet, and tricider: see Table 1 for an outline description of each). The team of trainer chose to limit their own interventions in these message boards to a minimum in order to support the birth of a trainees-only community. The trainers were more active within the webinars: educational online event for trainees.

TOOL	AFFORDANCES	REASON OF THE CHOICE	IN WHICH MODULE WAS IT THERE?
Forum (web 1.0 tool)	For public discussion, where everyone can read and answer to messages, using nested replies.	To give teachers a friendly and known tool for discussion.	1, 2, 3
Padlet (https://it.padlet.com/) (web 2.0 tool)	Board of collaboration/sharing material (images, videos, documents, text) on common tasks.	To give a communication mode different from the forum, for supporting teachers in participatory methods.	1, 2, 3, 4, 5, 6
Tricider (https://www.tricider.com/) (web 2.0 tool)	For easy brainstorming and voting. For decision making, crowdsourcing and idea generation.	To facilitate decision making after any discussion by the request of a vote.	2, 3

Table 1: Collaborative a-synchronous tools for interaction

Theoretical framework

As previously mentioned, we developed the MOOC-MDT framework to suitably describe and analyze the MOOC's dynamics (presented by Taranto in TSG 44 of ICME 13). It integrates three

theoretical frameworks in a new form: the Meta-Didactical Transposition[5] (MDT: Arzarello et al., 2014), Connectivism (Siemens, 2004; Downes, 2012), and the Instrumental Approach (Verillon & Rabardel, 1995). In what follows we give a synthetic idea of this framework.

First, a MOOC can be considered as an *artifact*, that is a static set of materials. Connectivism allows us to picture the *MOOC-artifact* with its own network-based knowledge: its nodes are the content, the ideas, the images and videos used; the connections are the links between their node pairs. When a MOOC module is activated, it dynamically generates a complex structure (Siemens, *ibid*; Downes, *ibid*) that we call ***ecosystem***: "all the relations (exchange of materials, experiences and personal ideas/points of view) put in place by participants of an online community thanks to the technological tools through which they interact with each other, establishing connections within the given context". The network-knowledge of the *MOOC-ecosystem* is dynamic: it evolves as the MOOC-artifact's network, thanks to the participants' contribution. Also, the network-knowledge of individuals evolves as a personal self-organization (Siemens, *ibid*, p. 4) of the ecosystem. The process of transformation from artifact to instrument (Verillon & Rabardel, *ibid*) is here replaced by the evolution artifact-ecosystem-instrument.

In a MOOC there are two communities, a community of inquiry (the researchers and designers of MOOC) and one of practice (in the sense of Wenger, 1998), that is teachers as trainees in the MOOC[6]. The trainers evolve from their meta-didactical praxeologies (*m-dp*), to new ones, to deal with the MOOC's training in Geometry. These new *m-dp* are based on a double awareness. One is that learning within the MOOC is connectivist: each trainee is part of a community, with the opportunity to share her/his own views, self-organizing information, with which (s)he comes into contact, for creating new connections, and questioning the existing ones. The second is that what is shown in the MOOC should encourage experimentation. The trainers' *m-dp* constitute the network of the MOOC-artifact. During the implementation of the MOOC-artifact's network-knowledge, in fact, trainers foster its nature of ecosystem, sharing tools and posing appropriate key questions. Moreover, the tasks designed by trainers suggest to trainees, in a more or less explicit way, to use the proposed material in their classes. In such a way, the MOOC is enriched with reports about trainees' teaching experiences: this process increases a virtuous circle that encourages other trainees to experience the same materials. For this, the trainers' *m-dp* evolve themselves, because trainers analyze, observe and monitor the MOOC activities as an ecosystem, to understand what did work or not.

The community of trainees is not a unitary subject of learning: the MOOC-ecosystem is an *instrument* that belongs to each single trainee. The trainees have to solve multi-tasks, through multi-techniques, properly justified. In fact, they must look at the proposed material, share their thoughts through sharing tools, and experience their activities. These tasks are not predetermined, depending on the

[5] MDT is a model that describes the process of teachers' professional development with the aim of grasping its complexity. It is a tool to analyse the dynamic aspects of this process, namely the evolution of teachers and researchers' activity over time. This activity is described through teachers' and researchers' meta-didactical praxeologies (Arzarello et al., 2014, pp. 353-355), which consist on the task in which they are engaged in the educational programme, with the techniques used to solve it, along with its theoretical justification.

[6] In the following we use ***trainers*** to indicate both researchers and designers and ***trainees*** for participants of the MOOC.

time, approach and depth with which trainees address them. The multi-techniques are the ways in which the trainees extend and modify their network-knowledge, drawing on that of the ecosystem, and influencing it in turn (thus affecting all other trainees). The *m-dp* of each trainee trigger a "***double learning process***": firstly the MOOC-ecosystem is a specific learning tool for the individual, and secondly the use of MOOC-instrument by the individual generates learning for the whole ecosystem. The dynamic process has the following components, intertwined and self-feeding each other:

i. **Instrumentation/Self-organization** (from the ecosystem to the individual): process by which the network of MOOC-ecosystem expands the individual's network-knowledge. In particular, the **instrumentation** (Verillon & Rabardel, *ibid*) is the process by which the chaos (Siemens, *ibid*) of the ecosystem network reaches the individual. The many novelties of views and experiences make sure that the individual compares himself with new usage schemes. A phase of **self-organization** (Siemens, *ibid*) of the MOOC's information follows this process: when the individual selects which usage schemes proposed by the MOOC are valuable and which are not.

ii. **Instrumentalization/Sharing** (from the individual to the ecosystem): process by which the individual's network-knowledge expands the network of MOOC-ecosystem. The **instrumentalization** (Verillon & Rabardel, *ibid*) is the process by which the individual, with her/his renewed network-knowledge independently builds new connections. The individual is stimulated by a task requested by MOOC and (s)he caters to the ecosystem to turn it according to her own (new) usage schemes. (S)He wants to integrate it with her/his own cognitive structures. **Sharing** is the process by which the MOOC welcomes the contribution of the individual and makes it available to all: information goes towards all members.

Within this complex, iterative learning process lies the inherent difference between the frame of the MDT and the MOOC-MDT. In fact, in the MDT, the trainers shape their proposal according to the practices they think appropriate, and so they can realize how much the trainees learn such proposal. On the contrary, inside the MOOC-MDT the process appears to be more difficult to control. The trainers do not know "what" the user has really looked at among the presented materials, nor they can know how (s)he interpreted them. At the same time, the trainees benefit from material provided not only by trainers, but also by other trainees that share some of their own materials and ideas using the communication boards. The process evolves stochastically: a determining role is played by the individual trainees, and by their feeling as a community with whom to collaborate, to inspire and to share results. Basing on such a theoretical framework, it is now possible to suitably formulate a specific research question as follows: *How effective and in which form is the collaboration between involved teachers (trainers and trainees), and how does it develop because of the support of tools designed by the trainers?*

Data analysis

The accesses of the trainees in the MOOC (distinct from watching videos, reading materials and interventions in communication boards) have been in the order of tens of thousands. Accessing the MOOC, each trainee enters into an ecosystem, living in it through the use of free collaboratively a-synchronous tools (as shown in Table 1), through which (s)he interacts with a community. Each of these interactive tools has been carefully monitored by the trainers' team during the weeks of the MOOC delivery. The trainers' team met regularly and, at the end of each module, they shared what

they had observed during that specific module. In particular, the most significant trainees' interventions or sharing actions were discussed. After the first few weeks we realized that we were dealing with a unique community of trainees, which we will expand on in the last section. We explain below how the trainees have used these interactive tools, showing some examples (in italics).

The forum played a predominant role with respect to the other tools. Despite being an almost out-dated mode (based on web 1.0), the trainees were very fond of it and used it to share their experiences of learning or of working. There was no moderator in the discussions: each trainee had the opportunity to read a diversity of opinions and experiences, and when (s)he understood how it worked, then (s)he introduced her/himself, became an author of posts, influenced other colleagues, or appreciated the idea expounded by a colleague. For example, in the second module of the MOOC, the geometrical topic was the widespread (at least in Italy) misconception that students have between angle and arc. Several activities have suggested to teachers to tackling this problem and a forum was inserted in this module. It collected 31 discussions, each of them with from 1 to 21 response replicas. In the following, just an extract: *"The proposed activities have made me think about* (a) *how the conceptual articulation "Angle vs. arc" is delicate. When the guys study trigonometry at high secondary school* (b), *they know the Radian that […]allows you to no longer distinguish between (width of) angle and (length of) arc. I would like to know your thoughts* (c), *especially those who teach at lower secondary school*". In (a) there is an evident phase of Instrumentalization: the trainee is creating new connections between his network-knowledge and that of the ecosystem. He was stimulated by the activities that he saw in this module and he is connecting this thinking to his classroom (b). In particular, he invites another person to share their thoughts about this topic (c).

If the forum was the right place for the trainers to talk about themselves, including their strengths and weaknesses, the Padlet was the place where the trainees began to share photos, videos and, spontaneously, their own materials. It is clear that the Padlet did not help to structure the exchange, but many trainees obtained inspiration from the exchange of materials in this place. For example, it was re-used and proposed by a participant as a tool to track her training programme with the construction of a Learning Diary: "*I am reviewing all of the course materials ... Because of my age, I can hardly remember the various proposals, ideas offered in this course surely professionally enriching and among the best I've attended to! So I thought to produce a Learning Diary with Padlet. Step by step it will enrich it, even with external links, with the materials I have looked for during this course or suggested by colleagues in the forums. Can it be useful to anyone?"*.

The Tricider had the goal of triggering simple threads, most of all confined to the approval or not of ideas, by voting through "likes". However, the participants used it more for collecting ideas and comparing their didactical experiences – as a forum – rather than for the expected use. Practically, the trainees realized a *catachresis* (Verillon & Rabardel, 1995): an artifact is used to do something it was not conceived for. Due to the fact that they explored the tool for the first time, and also because they usually need to explain and to go in depth when they express an idea, so the simple vote would not have let them satisfied. The posts written in Tricider are rich of ideas for both trainees and trainers. The trainees were introduced to a new tool for them. The trainers acquired awareness about the necessity to be clear in writing the tasks, in exemplifying the use of the tools and in providing tutorials on their affordances.

Beyond some trainers' interventions in the forums, or email communications with administrative aims, the actual contact between trainees and trainers was realized through three online webinars (using the chamber BigBlueButton of Moodle): they supported the community with synchronous interaction. While the trainers in the webinar could use video and chat, the trainees could use only the text chat. The trainers (in this case, only the academic professors) presented themes linked to the didactics of geometry and from mathematics education research. In all the three webinars there was a high participation (from 90 participants in the first one to 50 in the last one) of trainees, who posed questions and doubts.

Discussion

The complex ecosystem structure developed as soon as the trainees had begun to access the MOOC. They are asked to enter into what, at first glance, may look like *chaos*, because of the multitude of materials and available technological resources. In fact, initially the trainees may not have enough self-confidence with the situation (*instrumentation*). Gradually they implemented the *self-organization* phase: appropriating the use of the MOOC's usage schemes, they began to use resources and materials (*instrumentalization*) and also to contribute comments to the communication boards (*sharing*). A community, in the sense of Wenger (1998), began to take shape. It is a community comprising individuals who are both looking for answers and helping others, by sharing their practices - a community that seeks to grow collaboratively. The will to establish the threads often leaks out, though it is very difficult that they take shape in a broad and articulated manner. In fact, the threads tend to split into different groups, which are formed and split locally and for a certain period of time, depending on the needs felt by the individual, but generally they contribute to give to all trainees the sense of a common participation in one unitary event, precisely the MOOC. Using a term from neuroscience, we call this property *plasticity*, which makes it possible to adapt to various situations in different groups and times. It is true that situations and times change, but within a community that preserves its global unity. This unity consists in the collaborative sharing of what happens, even if the active participation converges on more than one local theme. The sharing processes (of materials, thoughts, ideas, experiences) in fact gives life to the ecosystem, enhancing the materials and expanding the individual's network-knowledge. Even the "contact points" with trainers via webinars contribute to this purpose. Through sharing processes the ecosystem becomes more and more structured; fragments from the history of web communication (from web 1.0 on) coexist and complement each other, and are used by the trainees. This aspect is interesting and little pointed out in the literature. It is something similar to the multimodal interactions that take place in the classroom thanks to the activation of different registers: we call it *technological multimodality*.

Plasticity and *technological multimodality* are the two main properties distinguishing the evolution of a community in a MOOC from that in a traditional training course. It is primarily for this reason that we needed to change the framework of the MDT elaborating the lens of MOOC-MDT: it allowed us to give a first answer to our research question.

Comparison with the French experience and conclusion

The Italian team worked to observe a general community, studying the MOOC phenomenon at a macro level and they did not intervene in the interactions between trainees. By contrast, the French team (Panero et al., 2017) observed local communities of practice. They studied the phenomenon at

a micro level, intervening in the groups' discussions to support and encourage the development of the collaborative work. During MOOC Geometria local groups are generated "emerging from chaos" (Siemens, 2004), namely they are subject to a spontaneous generation. During MOOC eFAN Maths (the French MOOC) trainers induce the generation of local groups and regulate peer relationships. Despite the fact that the cultural aspects affect these differences for sure (as we underline in our similar introduction), for both MOOCs there is an affinity that relies on the fact that trainees' learning is often generated by self-feeding discussions and instrumentalization processes.

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Coaching and engaging. Developing teaching with CAS in High School

Henrik P. Bang[1], Niels Grønbæk[2] and Claus R. Larsen[3]

[1]Christianshavns Gymnasium, Denmark; hb@cg-gym.dk, [2]Dept. of Mathematical Sciences, University of Copenhagen, Denmark; gronbaek@mathematics.ku.dk, [3]Christianshavns Gymnasium, Denmark; cl@cg-gym.dk

Keywords: CAS, mathematics teaching, coaching

The extensive use of CAS at upper secondary school in Denmark provides a laboratory for research on the development of standards for CAS teaching. The poster focus on action research into teachers' development of lessons and student activities in an ongoing collaboration between university and high schools on use of CAS in mathematics teaching. Coaches mediate design processes, reflection and documentation, and enable sharing. We discuss coaching as a valuable part of action research, and how to draw findings from such collaboration.

Danish CAS context

Starting with handheld devices in the 90's, the use of CAS in upper secondary mathematics education has accelerated. The reform in 2005-07 of Danish high schools opened the door to extensive use of computers, resulting in a move to PC based programs like Maple or TI-Nspire, used in the classroom as well as for homework. Powerful CAS tools such as Maple drastically change the teaching environment, but have only led to minimum adjustments in the final examinations since the earlier CAS-days. Early adopting teachers *do* give access to their own material, but there is no systematic sharing of experiences or standards for use of CAS within Danish mathematics education.

CMU's[1] Agenda

Many teachers experience instances where CAS provides new insight or furnishes new possibilities to handle examples that are more interesting or more realistic. CAS can also provide possibilities for extensive drill and practice of taught methods. However, and especially when use is allowed in the final exams, CAS can turn mathematics into merely an instrumental enterprise and thus trivialize mathematics education. Moreover, this trivialization is hard to see, outside looking in. From a policy maker's, school leaders, parents or even students point of view, you hardly know what is missing. Understanding that CAS *can* work in ways where skilled students learn less because tasks are too easy, while at the same time the less able students are performing poorer, because they try to rely on a tool they do not know how to use, demands insight. Addressing these issues, lead to establishing CMU.

Principles for coached teacher training

CMU collaborates with teachers interested in developing and sharing their experience with the use of CAS as an instrument for learning. We used a bottom-up approach, drawing upon models for action research (Asiale et al., 1996, Borba & Skovsmose, 2004), to designed a project management model (Figure 1). In Denmark, teachers have wide latitude to organize their teaching, but a limited tradition

[1] CMU, Center for Computer Based Mathematics Education, Department of Mathematical Sciences, University of Copenhagen, Denmark, founded in 2013.

for addressing teaching and learning in didactical terms. Through individual or group coaching[2] we support teachers to develop their own ideas about mathematics with CAS. The coaches play an important role to promote teachers' reflections before, during and after teaching - our goal being twofold; to draw on teachers' experience and to promote teachers' professionalism (Dale 2003). We have designed a project report template to capture teachers' reflections alongside the teaching material, and made the projects available on our website (http://cmu.math.ku.dk/projekter/). The coaches also assist in this documentation process.

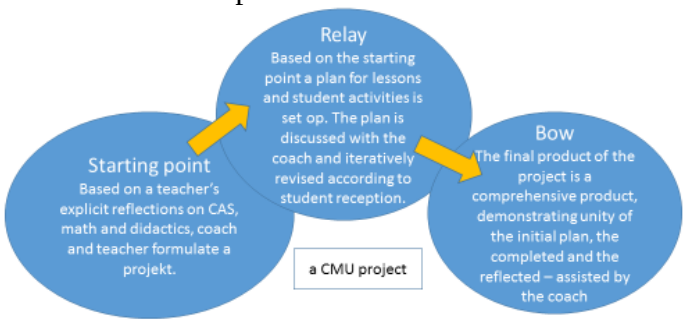

Figure 1: Systems model for a CMU project

Participants present their projects, goals and standards on an annual basis. It is essential to develop a common language and understanding of how mathematical content, and student activity changes in a CAS environment. From the discussions at our seminars, we can point to themes such as:

- which non-CAS activities should be introduced when working with CAS?
- what is the value of students mastering (details of) the CAS program or put in another way – how well should students know the CAS program in order to make real investigations?
- How can you work in ways that students both acquire useful skills and concept knowledge?

CMU has a double agenda of promoting sound use of CAS and of in service teacher training, so we pose our own research questions: 1. How to draw *general* conclusions about CAS standards based on individual projects? 2. Which teacher competences - CAS, didactical and mathematical - should coaching promote?

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[2] Coaches are high school teachers skilled in mathematics and CAS, with some didactical knowledge along with social skills.

A study of the development of mathematics student teachers' knowledge on the implementation of CAS in the teaching of mathematics -a praxeological analysis.

Louise Meier Carlsen

Department of Education, University of the Faroe Islands, The Faroe Islands,
Department of Science Education, University of Copenhagen, Denmark; LouiseC@setur.fo

Keywords: CAS, student teacher knowledge, praxeology, lesson study.

Introduction

The implementation of CAS in the teaching of mathematics introduces new challenges that concern: content, technology and didactics. There is a change in the mathematical focus from pragmatic to more epistemic, since the routine work is taken over by digital tools (Artigue, 2002). Furthermore the didactic changes, not only by the change in the mathematical knowledge, but also due to the need for the teacher to orchestrate a new component in the classroom (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). In the course "Numbers, Arithmetic and Algebra" for mathematic student teachers for lower secondary school on the Faroe Islands we implement the format of lesson study as a tool for the student teachers to develop their knowledge on teaching with CAS. This is a preliminary study of the transcripts from a reflection meeting, which will later form the basis for a more detailed study as part of my work towards the PhD.

Theoretical framework and research question

The anthropological theory of the didactic (ATD) suggests considering human activity as an amalgam of praxis and knowledge: praxeology. The praxis, also named the praxis block of the praxeology, consists of the constituting task and the corresponding technique. A technique can consist of several actions. The task and technique are in one-to-one correspondence. The knowledge, also called the logos block, is the discourse of the techniques and the theory that explains and verify the discourse. For more details see (Bosch & Gascón, 2014). This study considers the didactical praxeology and the mathematical praxeology of mathematics student teachers when implementing CAS in research lessons. Within ATD the task of solving an equation with CAS is categorized as mathematical praxeology, while the task how to teach the students to solve an equation using CAS is categorized as a didactic praxeology. The research question is "What didactical logos developed during the reflective meeting of a lesson study cycle?"

Context and study

The study is situated within a course focused on numbers, arithmetic and algebra. As part of the course, groups of three to four students participate in four lesson studies where CAS has to be implemented in the teaching of algebra in grade 7 and 8 (14 - 15 year). Each lesson study cycle consists of a planning phase resulting in a lesson plan, a research lesson, a reflection meeting and a new lesson plan. In the lesson plans, the student teachers will not only describe the intended lesson but also have to justify the instrumental orchestration, part of the didactical technique, in relation to the mathematical praxeology of the lesson. Reflection meetings in lesson studies are a rich environment for the development of knowledge for teaching (Miyakawa & Winsløw, 2013;

Rasmussen, 2015). As part of the protocol for these meetings, the student teachers reflect on the relationships between the instrumental orchestrations used during the research lesson, the development of the mathematical praxeology of the students and the mathematical praxeology of the teacher.

Conclusion

As an example, the mathematical praxeology of making two integer sliders in GeoGebra and the related didactical praxeology of how to teach students to make two integer sliders in GeoGebra is considered. As a didactical technique, the student teachers chose to hand out a booklet with step-by-step instructions in order to guide the students through the lengthy technique. Focusing on the didactical logos related to the technique of handing out a booklet during the reflection meeting the teacher students concluded that if the student did not get the exact same picture as in the booklet such as slider *b* above slider *a* instead of slider *a* above slider *b*, they would consider it an error and not usable. Additionally, the students were discouraged by having to read text in addition to carrying out the GeoGebra technique. It was agreed upon by the student teachers that a booklet is still a good didactical technique but has to be complemented or preceded by a board-demonstration for the students.

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A study investigating how mathematics teachers view and use tablets as a teaching tool

Kinga Petrovai

University of Oxford, Faculty of Education, United Kingdom; kinga.petrovai@education.ox.ac.uk

Mobile digital technologies, such as tablets, hold great potential for teaching and learning, but they are being introduced into schools with little evidence to guide how they are implemented. With little impact on learning outcomes there is increasing attention on the need for greater focus on the role teachers play in the use of digital technology in the classroom. This study investigates the way teachers at an English school, which has one tablet per student, view and use tablets in teaching mathematics. The results of my study form a framework on how tablets are used in teaching.

Keywords: Teaching, mobile technology, mathematics.

Introduction

"Despite considerable investments in computers, Internet connections and software for education use, there is little solid evidence that greater computer use among students leads to better scores in mathematics and reading" (OECD, 2015, pp. 145). Mobile technologies are increasingly being introduced in schools with little evidence to guide how they are implemented (Kiger et al., 2012). With the technology available today, some argue that there is a need for renewal of pedagogy (Jouneau-Sion & Sanchez, 2013), and greater understanding of the teachers' perspectives and practice regarding the integration of technologies in schools (Ertmer, 2005). Using a paper poster, I outline how my study addresses this by investigating how a group of lower secondary school mathematics teachers in the south of England – who meet regularly as a group to reflect on and develop their practice – view, use, and develop their use of tablets in their teaching.

Theoretical Lens

Within the second facet of Ruthven's (2008) examination of the incorporation of new technology into educational practice - the process of integrating a tool at the level of a community - Laborde (2001) identifies four stages of increasing degrees of mathematical/pedagogical innovation. This is the base of my analysis.

Methods

My study ran in two phases over the course of one year. Phase one established the context of the study. In phase two data was gathered by (1) group meetings in which teachers reflected on, and developed, their practice; (2) classroom observations; (3) post observation interviews with teachers.

Findings

I developed a framework of how mathematics teachers use tablets in their teaching. I adapted Laborde's (2001) framework of *instrumental evolution*, to which I developed new categories that are organized in the two distinct groups of *efficiency* and *engagement*. Efficiency includes tasks that help to organize the class structure to give more time to focus on the mathematics. An example includes the use of quick response (QR) codes, which speeds up the distribution of online material to students.

Engagement includes tasks that help to capture the attention of students so that they focus more intently on the mathematics. An example includes using virtual games to practice numeracy skills.

My study addresses the criticisms of education research that it does not investigate questions that are important to teachers, thus impacting on the lack of disruptive change in schools (Pring, 2002). Other studies have investigated the use of technology in mathematics education by (Ruthven et al. 2009; Galligan et al. 2010). However, the combination of a natural school setting, regular collaboration among teachers, observation of use over a longer period of time, focusing on the use of tablets, and focusing on multiple lessons per teacher, makes my study unique. This research can help guide future implementation of new technologies in schools and the associated teacher professional development.

Acknowledgment

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The role of automated reasoning of geometry statements in mathematics instruction

Francisco Botana[1], Tomás Recio[2] and M. Pilar Vélez[3]

[1]Universidade de Vigo, Spain; fbotana@uvigo.es. [2]Universidad de Cantabria, Spain; tomas.recio@unican.es. [3]Universidad Antonio de Nebrija, Spain; pvelez@nebrija.es

Keywords: Mathematics instruction, elementary geometry, automatic reasoning with GeoGebra.

The tools …

By "automated proving of geometry statements" we refer to tools that mechanically output a mathematically rigorous (e.g. not based upon a probabilistic proof) yes/no answer to the conjectured truth of a given geometric statement. By "automated derivation of geometry statements" we refer to tools that, rigorously, output some/all geometric relations verified by a collection of selected elements within a geometric construction. Finally, by "automated discovery of geometry statements" we refer to tools that systematically find complementary, necessary, hypotheses for a conjectured geometric statement to become true.

The community of mathematicians and computer scientist has been working on these goals for the past 50 years, with a variety of approaches, outcomes and popularization results (cf. the pioneer work of Gelertner (1959) in the Artifical Intelligence context, or the algebraic geometry framework to automated reasoning in geometry disseminated by the book of Chou (1988)). On the other hand, although we can mention the development of some intelligent tutorial systems designed to assist students to construct proofs in Geometry such as GRAMY (Matsuda and Vanlehn, 2004) or GeoGebraTutor (Tessier-Baillargeon, Richard, Leduc and Gagnon, 2014), it is fair to say that, up to now, the dissemination, use and impact of these findings in the educational context is very limited. Thus, the very recent survey by Sinclair et al. (2016), on geometry in education, although it includes a full section on the role of technologies and another one on "Advances in the understanding of the teaching and learning of the proving process", does no refer at all about automated reasoning tools.

…and the issues

Hence, we consider it quite relevant to address, in our poster, two issues: one, to announce the very recent implementation (2016) of tools for the automatic proving and discovery of geometric theorems over a free dynamic geometry software, with tens of millions of users worldwide, and a great impact in mathematics education. See Abánades, Botana, Kovács, Recio and Sôlyom-Gecse (2016) and Hohenwarter, Kovács and Recio (2016).

Then, recalling that the program where we have implemented our automatic reasoning tools (ART) is available over computers, tablets, smartphones, with and without internet connection, the second issue we would like to pose here is the consideration of the following questions: what could be the role, in mathematics instruction, of the ample availability of such tools? It was already 30 years ago (cf. the visionary ICMI Study "School Mathematics in the 1990's" (Howson and Wilson, 1986) or the inspiring paper by Davis (1995), with a section that refers to the "transfiguration" power of

computer-based proofs of geometry statements) when educators started reflecting about the potential role in education of software programs dealing with automatic theorem proving (automatic discovery and derivation were inexistent at that time). But these reflections were formulated rather as considerations about the future than as proposals for the present time of their authors…

Thus, in view of the current implementation of ART in well spread, dynamic geometry programs, our final goal is to make an open call to the community of math teachers and math education researchers, in order to join us preparing a research project to address the following questions: Are ART in geometry education good for anything? If yes, what are they good for? What should be the necessary changes and requirements in the educational context, if ART are to be considered good for anything?

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A study of the Cornerstone Maths project teachers' classroom practices: Geometric similarity using the Cornerstone Maths software

Ali Simsek[1] and Nicola Bretscher[2]

[1]UCL Institute of Education, London, UK; british.simsek.15@ucl.ac.uk

[2]UCL Institute of Education, London, UK; n.bretscher@ucl.ac.uk

The rationale of the study

More recently, a need for further studies examining the process of integrating digital technology into classroom practice in order to support the development of a better, more comprehensive understanding of classroom practice with digital technologies, and to refine evolving frameworks has been emphasised by a number of researchers (Artigue, 2010; Hoyles, Noss, & Kent, 2004). In this context, the present research is useful in various different areas. First, the study is concerned with the concept of geometric similarity in geometry teaching using Dynamic Geometry Software (DGS). This particular topic makes our study beneficial, specifically because geometric similarity has been overlooked in recent studies. Given the significance of geometric similarity to benefit students' spatial and geometric reasoning skills (Watson, Jones, & Pratt, 2013), the fact that this key area of mathematics has not received sufficient attention seems surprising. Secondly, this study uses and adapts a contemporary theoretical framework (the Structuring Features of Classroom Practice (SFCP) framework) (Ruthven, 2009) that aims to assist researchers in the identification and analysis of classroom practice using digital technology. Our study, therefore, helps identify how the SFCP framework supports and/or hinders the researcher in the process of such identification and analysis. Furthermore, an exploration of teachers' classroom practice using the new technological tool provides deeper insight into the issue of digital technology integration through the detailed analysis of two case studies.

Research design

The purpose of this research is to develop a holistic understanding of how the Cornerstone Maths (CM) software is integrated by teachers into their classroom practice when teaching geometric similarity. This entailed conducting a qualitative research study in order to gain a detailed in-depth understanding of teachers' use of the CM tool in the classroom. We adopted the case study approach where data is collected through multiple sources of information; i.e. in this study, observation followed by semi-structured post-lesson teacher interview based on the observations made during the lessons. While the more experienced teacher's classroom practice was observed in two lessons, the less experienced teacher's classroom practice was observed in one lesson. With each teacher, one follow-up teacher interview took place.

In addition, a multiple-case study design was used, so that a better holistic understanding of teachers' technology integration into classroom practice could be accomplished by comparing the teachers' two cases. Between the teachers who participated in the CM project professional development programmes, the participants were chosen based on their different experiences of teaching using digital technology because this offers a productive comparison that will highlight the

variables in the teachers' integration of technology in the classroom (see Bozkurt & Ruthven, 2015). The data analysis was made according to the five factors in the SFCP framework, namely: working environment, resource system, activity structure, curriculum script, and time economy.

Findings

Data analysis suggests several key findings related to five structuring features of classroom practice using digital technology. For instance, the varying levels of teachers' experience in using digital technology to teach have a considerable influence on the degree and type of their technology use during classroom activities. In addition, teachers think that pre-designed technological resources with good ideas support them to better exploit the didactic potential of digital technologies in classroom practice because they do not have enough time to prepare such resources. The evidence shows that teachers' preparation for integrating technology into classrooms leads them to not use their time productively enough on the mathematical content to be taught in the course of planning their lessons. Lastly, despite some technical difficulties appears during classroom practice, the use of digital technologies facilitates and accelerates students' learning of mathematical ideas.

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Resource systems of English teachers as they appropriate mastery approaches

Michael Achile Umameh
University of Leeds, UK; edmau@leeds.ac.uk

Keywords: Mathematics teaching, resources, digital tools, appropriation.

Introduction

I report on the resource systems (Gueudet & Trouche, 2009) of a sample of mathematics teachers' in England. An influence on these teachers' appropriation of resources for learning and teaching is current mathematics education reform that is focused on how mathematics performance in England compares to the highest attaining systems internationally. The OECD's Program for International Student Assessment (PISA, 2012) ranked England in the 25th position in mathematics achievement, with the assessment table headed by a clutch of south-east Asian jurisdictions. PISA results also showed that England's performance in mathematics has stagnated over the years. There is a government-backed mandate to explore, adapt and embed the Singapore/Shanghai model of mastery teaching and assessment for learning approaches across England (Hodgen, et al., 2014; NCETM, 2014). "Mastery learning can be described as a set of group-based, individualized, teaching and learning strategies based on the premise that virtually all students can and will, in time, learn what the school has to teach" (Anderson, 1975 p.4). Although mastery learning has morphed into various adaptations, essential elements remain the feedback, corrective and enrichment (Drury, 2014). Extant research points to the potential of well-implemented mastery teaching as enabling higher levels of achievement, deep understanding and confidence (Drury, 2014).

This study combines an activity theoretic approach with the more recent 'documentational approach' (Gueudet and Trouche, 2009) from French didactics as theoretical tools for developing an understanding of the teachers' appropriation of digital resources and building up a coherent explanation for its impacts on classroom practices. In this investigation, I document the context for the current motivation for adopting mastery teaching and examines the emergence of the resource systems of seven English mathematics teachers' as they 'resource for mastery', and the potential impact of this on classroom practices. The major aims of this study are to:

- Analyze how teachers' appropriate digital resources for classroom practices.
- Explore teachers' resource systems for mastery teaching.
- Contribute to the discourse on teachers' appropriation of digital resources.

Research design, methodology, data collection and analysis

A qualitative case study approach (Creswell, 2013) was adopted. Purposive sampling was used to select seven teachers from three schools based on the use of digital resources, access, proximity and the opportunity to observe rich and real life-context of teacher practice with digital resources. Data collection was undertaken during the 2015-2016 school year through periodic whole day school visits. Data were collected through: audio-recorded semi-structured interviews; classroom observations using an adapted *systematic classroom analysis notation* for mathematics lessons (SCAN, Beeby, Burkhardt & Fraser, 1979); screen capture software; and collation of documents.

Data analysis is ongoing at the time of writing and includes: coding and analyzing transcribed interviews using thematic analysis and taking into account key concepts from the literature and information emerging from the data alongside the SCAN analysis of classroom observations. Data is organized by teacher and by data type. The thematic mappings will be constantly grouped and regrouped into categories and themes for discussion.

Emergent results and implications

Preliminary results from ongoing data analysis includes:

1. Teachers' appropriation of digital tools for formative assessment (student seat-work is e-analyzed during lessons which allows the teacher to enact changes in the tasks).
2. The reality of 'emergent (in lesson) task design' afforded by access to multiple resources.
3. The emergence of Twitter as a key platform for 'massive live staffrooms' (teachers are constantly collaborating, developing task, and sharing expertise and resources on mastery approach).

I believe that this research will contribute to the ongoing discourse on issues and challenges of the integration of digital resources in classrooms and offer 'working hypotheses' for future research.

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TWG16: Learning Mathematics with Technology and Other Resources

Introduction to the papers of TWG16: Learning Mathematics with Technology and Other Resources

Paul Drijvers[1], Eleonora Faggiano[2], Eirini Geraniou[3], Hans-Georg Weigand[4]

[1]Utrecht University, Netherlands; p.drijvers@uu.nl

[2]Università di Bari Aldo Moro, Italy; eleonora.faggiano@uniba.it

[3]University College London Institute of Education, UK; e.geraniou@ucl.ac.uk

[4] Universität Würzburg, Germany; weigand@mathematik.uni-wuerzburg.de

The use of technology and other resources for mathematical learning is a current issue in the field of mathematics education and lags behind the rapid advances in Information and Communication Technology. Technological developments offer opportunities, which are not straightforward to exploit in regular teaching. In CERME10 TWG16, the recent research findings, issues and future questions have been explored and discussed in detail. In this introductory chapter, we will outline the scope and focus of the work, describe the results with respect to existing questions, and identify upcoming topics as well as missing topics that might set the agenda for future work in this domain.

Keywords: Digital resources, mathematical learning, educational technologies.

Scope and focus of the Working Group

In recent years, discussions within the CERME-technology-group have confirmed the relevance of Information and Communication Technology (ICT) for the learning of mathematics. ICT provides a range of resources, such as software, handheld devices and online classroom activities. This range of resources has been compared to non-digital resources, such as textbooks, worksheets and other types of tools and manipulatives. The impact of both digital and non-digital resources on mathematical learning has been of great interest to our working group. The scope of this working group was to explore and discuss opportunities and possibilities, as well as challenges and limitations, of technological resources for student learning. We wanted to establish an overview of the current state of the art in the use of technology in mathematics education, including both practice-oriented experiences and research-based evidence, as seen from an international perspective and with a focus on student learning, as well as to suggest important trends for technology-rich mathematics education in the future, including a research agenda. TWG 15 is closely related to this theme, but focuses on the teachers' roles and practices.

In the pre-conference call for papers and poster proposals, theoretical, methodological, empirical or developmental contributions were particularly welcomed on the following topics:

- Analyses of the impact of using digital and non-digital technology on students' learning;
- New forms of digital resources, including mobile devices and dynamic e-textbooks;
- Digital assessment of and for learning;
- E-learning, blended mathematics education and (Massive Open) Online Courses for mathematics;
- Influence and use of social media in students' perception of learning mathematics;
- Promoting communication and collaborative work between students through ICT;
- Using ICT for out-of-school informal mathematics learning;
- Examples of the use of technologies devoted to the support of students with disabilities.

This introduction provides an overview of the 24 presented papers and 6 posters and the discussions in TWG16 building up on theories and past research on digital technologies and other resources for mathematical learning. We especially refer to the CERME history of this technology group and consider the results of the 2017 conference as a continuation of the background, aims and scope of the conferences since 1999 (Trgalova, Clark-Wilson & Weigand, to appear). To do so, we will first address "old" questions, then describe upcoming topics, and close off with topics we missed.

Taking up "old" questions

Some contributions to TWG16 continued the discussion on topics that had been addressed in the past, such as the potential of digital tools to evoke the dynamical aspect of manipulating objects within a digital tool, functional thinking, and the use of e-books.

Interactivity, dynamics and multiple representations

Since the early years of using digital technologies in mathematics education in the 1970s and 1980s, interactivity, dynamic and multiple representations played an important role in developing new strategies for understanding mathematical concepts. Dynamic manipulations were prominently present in dragging opportunities in Dynamic Geometry Systems (e.g. Leung, 2008). Digital technologies created easy access to multiple representations and interactions between the user and the software (e.g. Noss & Hoyles, 1996; Moreno-Armella, Hegedus & Kaput, 2008). On a more elaborated level, the interactions between the *knowledge*, the *tool* and the *learner* built three main aspects of digital technologies and were also strongly represented in TWG16 of CERME 10.

Dynamic digital tools can promote conceptual understanding (e.g. Drijvers, 2015) and potentially support low-achieving students. An example is the interactive environment presented by Swidan, Daher and Darawsha, to support the learning of the concept of equivalent equations. An applet gives the possibility to work with numerical, algebraic and/or graphical representations. Moreover, a pan balance represents enactive experiments with "weights" and a slider allows to dynamically change the x-values. The idea is to represent enactive actions and to allow students to work with a visual mediator while changing mathematical objects. The difficulties, limits and obstacles of working with multiple representations are also highlighted. Low-achieving students, for example, can become overwhelmed when faced with a large number of representations, which may prevent their progress. The consequence is *not* to avoid working with multiple representations, but to create didactical reflected learning environments with a successive introduction of multiple representations and reciprocal interpretation of the transition between these representations.

Functional thinking

Another "old" question concerns the prototypical dynamic view of functions while filling bowls with water and asking for the height of water in a bowl as a function of the volume of water in the bowl (Carlson et al., 2002). Lisarelli's contribution to TWG16 involved the outcomes of investigating different dragging modalities in the frame of the above-mentioned problem, as shown in Figure 1. Users had to be familiar with different kinds of dragging possibilities: (quasi) continuous dragging, discrete dragging (e.g. if only natural numbers are allowed), or impossible dragging, (i.e. where the user tries to drag a dependent point). She argued for the importance of recognizing the aim for a specific type of dragging and considering whether it is a random movement, a movement for testing

possibilities or a guided dragging to reach a special configuration. Such a classification of dragging modalities gives the possibility to observe, describe and analyze students' processes involved in the exploration and solution of dynamic problem solving activities. This example shows clearly a digital tool as a medium, which is – or mediates – between the user and the mathematical concepts.

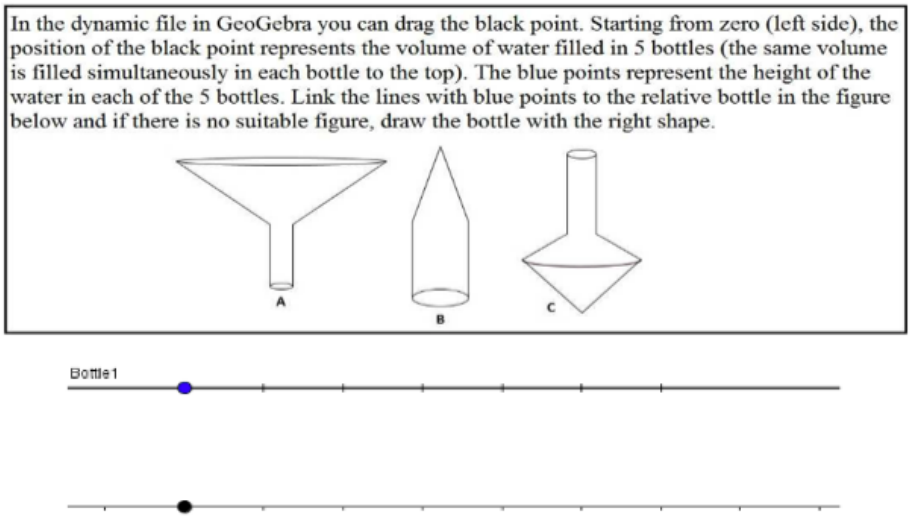

Figure 1. The Bottle Problem task and its dynamic representation

The interactive worksheets presented by Lindenbauer and Lavicza focus on functional thinking through a situational model (the area of a triangle) and a related graphical representation. The explanation and interpretation of the graphical representation is – especially for lower achieving students – challenging and as the author stated, the help of the teacher may be crucial. These graphical representations allow students to reflect on what the impact of moving the point on the x-axis is by showing the small or big changes to the area of a triangle. Such an approach provides students with an intuitive access to the concept of rate of change.

E-books

A great variety of digital books or e-books for classroom use exists. Such books may be more or less extended versions of the traditional schoolbooks, including dynamic activities and in-built assessments (Gueudet et al., 2017). The “Creative Electronic Book on Reflection” presented by Geraniou and Mavrikis allows students to explore mathematics situations individually and interactively, and it also encourages them to reflect on their actions while they are exploring and solving mathematical tasks. A key role in the students’ reflection is played by the so-called “bridging activities” which emphasize the mathematics integrated into the book. As claimed by the authors, the design and evaluation of such interactive learning environments, learning paths or trajectories and the promotion of their wider use in classrooms is a new challenge.

Theories

The discussion on theoretical approaches regarding digital technologies for mathematical learning is also an on-going one within CERME (Trgalova, Clark-Wilson & Weigand 2017). There are some well-developed and experimentally confirmed theories like semiotic mediation (Bartolini Bussi & Mariotti, 2008), instrumental genesis (Trouche, 2004) or the documentational approach (Gueudet & Trouche, 2009), which are also used in many papers and discussions in TWG16. Murphy and Calder, for example, applied a framework including social semiotics and multimodality to interpret screen

casts of students working in a problem-solving application on an ipad, to understand the learning that took place.

In spite of theoretical developments in the field (e.g., see Monaghan, Trouche & Borwein, 2017), Schacht's was the only contribution to TWG16 that paid attention to a new theoretical field. Taking an inferential perspective, he investigated the relationships between mathematical and tool language while working with digital technologies and the transition – or non-transition – from one to the other. He showed how the way in which this transition can be accomplished can have implications on the individual concept formation processes. He especially emphasized the meaning – but also the obstacles – of the transition in the language use (by students) from a tool-oriented language to a mathematical-oriented language. The philosophical discourse about the concept of "digital" (see Galloway, 2014) - "Any discourse that produces or maintains differences between two or more elements can be labelled digital" (Schacht) - might give orientation also in the evaluation of the language transfer in mathematics education.

Upcoming topics

The continuous development of technological tools, which are used both in and out of school, requires us to address old questions under a new perspective. On one hand, this new perspective has to consider new developments in hardware (tablets, smartphones) but also in software (social media, cloud computing). On the other hand, we have to consider new developments in society, science and (mathematics) education, for example with respect of online communication without any limitations in time and space. Goals in education have to be continually rethought and evaluated.

3D-geometry

Regarding the future development and progress of our working group, there are different topics for which we see the potential for further investigations. Kynigos and Zantzos presented a study, during which students were asked to construct the shortest path between two points on a cylindrical surface. To solve the problem, they had to see the relationship between 3D- and 2D-geometry and activated the "old idea" of a turtle geometry which allowed access to difficult concepts like the curvature of a special surface.

MOOCs and new kinds of e-learning

A second aspect is the meaning and the impact on mathematical learning of free available massive open online courses (MOOCs). Khan Academy[1] offers a free tool that allows teachers to monitor students' activity and provide them with feedback and guidance. Vančura used this tool at a Czech high school to provide feedback for students' homework. The investigation showed that weak knowledge of the English language might not be a barrier for students. Vančura also sees the danger of using such courses just for the training of algorithms without developing knowledge of underlying mathematical concepts.

Gray, Lindstrøm and Vestli also used the Khan Academy (KA) tool for pre-service teachers in mathematics who were allowed to substitute their compulsory mathematics assignment with exercises

[1] www.khanacademy.org/ (06.04.2017)

in KA. They compared their results with those of a control group, learning in the traditional way. At the end, there was no statistically significant difference in the performance of the two groups.

It is an open question whether MOOCs or SPOCs[2] will have an influence on the teaching and learning at schools and universities. Nevertheless, identifying good ways of e-learning will remain important, whether open resources on the internet or special courses integrated in learning management systems are used.

Tablets

Since the very first CERME conference, an important question has always been what kind of interactions take place between the tool and the learner. The goal has always been to bring the individual into the centre of learning. Digital technologies can mediate between mathematics and understanding. Nowadays, the relatively straightforward and intuitive use of digital technologies in the form of laptops and smartphones gives users the chance to not put too much emphasis on the technical aspects of the tool, but to concentrate on the learning. Palha and Koopman created the tablet-driven project Interactive Virtual Math: a tool to support self-construction of graphs through dynamical relations. The aim of the project is to develop a visualization tool that supports students' learning and relational understanding of graphical situations. The medium – here a tablet – allows the students to "draw" graphs using a finger, a digital pen or a mouse, to ask for help and to compare their own solution to the expected solution. According to the authors, this tool has the potential to help students understand functional relationships, but more importantly, allows the students to work on their own, experiment, create self-productions and reflect on them. Until now the authors only evaluated their tool in a small qualitative study.

Tablets will be important tools in the years to come. With multi-touch technologies, gestures have become an essential feature of user interface. The relation between touching and meaning-making might become more important. De Freitas and Sinclair used multi-touch technology and tangible gestures with young children to promote counting on and with fingers. These children used their fingers – one after another – while counting sequentially, they used their fingers simultaneously to represent numbers and they left a trace on the screen with one or more fingers. With the touchscreen interface, and particularly the multi-touch actions, they see the hand involved in a process of communicating and a process of inventing and interacting. "We interpret these speculative comments as an indication that the future of the gesturing hand in relation to new media may involve all sorts of surprises, and that perhaps even pre-school children may count 'on their hands' to 100 as they engage with these media" (De Freitas and Sinclair).

Smartphones

Nur Cahyono and Ludwig used smartphones to help students engage in meaningful mathematical activities. A math trail is a walk in which mathematics is explored in the environment by following a planned route and solving outdoor mathematical tasks related to what is encountered along the path. In the MathCityMap-Project students are confronted with special situations and questions along the path, supported by a GPS-enabled mobile phone app. Students were intrinsically and extrinsically

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motivated and engaged in this project. Moreover, they got to know more about their environment and model problems related to it.

Digital games

Computer game characteristics could also be exploited for the purpose of mathematical learning. As an example, Gjovik and Kohanova developed a mobile app on the topic of linear functions. The mobile phone is a tool we can expect to see more in mathematics education as learning becomes further individualized and online. In the "Lucky Hockey" game students have to strike a hockey puck along a straight line by entering a linear expression. Prior knowledge concerning the properties of linear functions is required when playing this game and in order to identify the path of the puck so that it hits the coins. The results of this project especially concerning the long term effect have not been satisfactory. The authors conclude that it might be difficult to make applications that facilitate exploration and discovery while doing mobile learning. It might be more effective if quite narrow mathematical topics are used. The concept of linear function might already be a too elaborate topic.

There are many questions around the use of games in mathematics classrooms which still need to be examined. How do we integrate games into the curriculum? When do students play these games? Is the motivation to play these games just an initial effect? What is the impact on students' learning and understanding? How sustainable is that knowledge over time?

Computational thinking

Robots are starting to play a more important role in our daily life. Robot competitions are quite popular in schools, but these activities usually take place outside regular lessons. The control of the robots, e.g. while walking through a labyrinth, needs algorithmic thinking similar to the turtle geometry of the 1980s. Seymour Papert (1980) originally created the label "computational thinking", but nowadays this concept has a much wider scope: it includes collecting, analysing and visualizing data, programming, creating computational models, and understanding relationships in systems. Broley, Buteau and Muller presented a model of computational thinking practices based on Weintrop et al.'s (2016) taxonomy for computational thinking in mathematics and science practices. The authors ask for further clarification of this concept and ways to integrate it into mathematics lessons.

Missing topics

If we compare the TWG16 call for proposals with the actual contributions made by the participants, we see some interesting gaps. Firstly, no attention was paid to digital assessment of and for the learning of mathematics. There are on one hand questions concerning written (final) examinations: Which technologies are allowed? Which tools are needed (Drijvers et al., 2016)? Which tasks are appropriate? How do students report their thinking? On the other hand, the question of how formative assessment might be a means to develop student competences is also of interest (Beck, 2017; Black & Wiliam, 2009). These topics have been addressed in some aspects in TWG 15 and in more detail in TWG21 on assessment.

Moreover, the topics of e-learning, blended mathematics education and (Massive Open) Online Courses for mathematics may set the agenda for CERME11. This includes issues such as personalized and adaptive learning, and the design of online feedback for students. The opportunities and constraints of using social media in students' perception of mathematics and their learning have

also been absent, as was the case for the intriguing topic of virtual and augmented reality. Examples of the use of technologies devoted to the support of students with disabilities have not been addressed either.

With respect to the methodologies in the reported studies, the focus was on small-scale qualitative studies, whereas large-scale experimental studies were not presented. Even if the latter may have pitfalls, the field might benefit from an integration of both qualitative and quantitative approaches, so as to gain sustainability and applicable knowledge on how mathematical learning can benefit from the interaction with digital resources.

Concluding remarks

Digital technologies are now an element across all CERME groups (e.g., see Ferrara & Ferrari, TWG24; Hogstad, Norbert Isabwe & Vos, TWG14; Montone, Faggiano & Mariotti, TWG4). This indicates how digital tools permeate the mathematics education research landscape and have gained legitimacy across the field. In today's mathematics classrooms, different types of digital technologies are integrated in daily practice: interactive whiteboards, tablets, notebooks, graphing calculators with and without CAS. We have noticed a significant gap between research findings and mathematics teaching and learning practices in the regular classroom. The overall impression is that we cannot yet speak of a sustainable change through the use of digital technology, scaled up beyond the incidental level. We should acknowledge that integrating digital tools in a way that is beneficial to student learning is not as straightforward as we might have thought some decades ago. Thus, a specific working group on digital tools in mathematics education is appropriate within the frame of CERME, even if the impact of technological developments is hard to isolate from its context and from the topics central to other CERME working groups. A TWG dedicated to this issue could make a distinct contribution to important questions on the future of mathematics education.

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A computer-based collaboration script to mediate verbal argumentation in mathematics

Giovannina Albano[1], Umberto Dello Iacono[1] and Maria Alessandra Mariotti[2]

[1]University of Salerno, Italy, {galbano,udelloiacono}@unisa.it

[2]University of Siena, Italy, mariotti21@unisi.it

This paper reports on a pilot study concerning a first implementation of a collaboration script, aiming at developing students' argumentative competences in mathematics, as part of an interactive digital storytelling. We discuss the outcomes of the transcripts' analysis, which seem to show that the collaboration script fosters the introduction of the student to the construction of arguments as cohesive texts, independently on the student's skill in mathematics, and that the success of the script depends on the learners' engagement in the story and on the team mood.

Keywords: Collaboration script, linguistic, argumentation, mathematics education, e-learning.

Introduction

This paper focuses on a part a wider research (Dello Iacono, 2015; Albano, Dello Iacono, Mariotti, 2016; Albano, Dello Iacono, Fiorentino, 2016), aimed to investigate the effectiveness of the design and implementation of a Digital Interactive Storytelling in Mathematics (DIST-M), that is a platform model organizing mathematical learning activities based on social virtual interactions. The DIST-M consists in collaboration scripts implementing a work methodology for the students that, according to Vygotskian perspective, is expected to mediate specific mathematical competences. A collaboration script is a scheme that regulates and structures roles and interaction in a collaborative setting (King, 2007). The choice of the use of storytelling is related not only to motivational aspects and cognitive effectiveness, but also to the possibility of integrating narrative and logical-scientific thought (Zan, 2011). In our DIST-M the student does not create the story, but she interacts with it. In "Programma Discovery" the student assumes the role of a scientist at NASA, member of a team led by Professor Garcia (head of story and voice platform). The goal of the team is to analyse the data coming from a probe launched on a new planet, trying to figure out if this can accommodate life. During the fruition of the story, the student will face problems, whose solution is needed to continue the work of the team. This paper reports on a pilot study concerning an implementation of the DIST-M focusing on the development of students' communication competences in expressing argumentative mathematical sentences, as they can be considered as critical to the advance of mathematical thinking (Ferrari, 2004). The goal is to introduce the student to the construction of arguments written according to a register shared in the mathematical scientific community. We analyse, from a qualitative point of view, the arguments produced by the students under a linguistic perspective, focusing on the organization of the verbal texts, as cohesive texts, which means words and sentences perceived as a whole entity. We also look at the functioning of the collaboration script in terms of the team work and its impact on the success of the activity. We expect that the collaboration script in organizing the roles and actions within the team fosters the production of arguments and counter-arguments, allowing each member to interiorize the so-born practice.

Theoretical framework

Collaboration script

In cognitive psychology, the internal memory structure corresponding to a sequence of actions that define a well-known situation is named script (Schank and Abeson, 1977). Here each actor has specified roles and actions to take. The script is activated every time the individual is in the same situation. In educational context such constructs differ mainly because of its external definition and they aim to regulate roles and actions of students in collaborative/cooperative learning in order to succeed in learning (King, 2007). The use of external scripts has been incremented in computer-supported environment, where the need of pre-structuring and regulating the social and cognitive processes is much more evident. Concerning argumentation, it is well known that the simple request of collaborating does not guarantee the development of argumentative competences. This can be fostered by means of computer-based scripted collaboration (Weinberger et al., 2007), taking advantages on the use of text-based interfaces that allow the students to have more time to read their written argumentations and their peers' ones and to come back to their writings every time they want. According to Vygotsky (1930), "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (inter-psychological) and then inside the child (intra-psychological)". Although the scripts are externally designed and imposed to the learners, the goal is that they are internalised along the time through the social practice. Only when the external script is interiorised, then it is successful; otherwise we have once again a repetition of actions externally imposed and learned by heart.

Language and cohesion

As the defined script aims to the construction of arguments, we are interested in how the student can propose verbal arguments to support the solution to a given question, apart from the correctness of the solution. Although some theoretical models regard arguments with no reference to language, as a matter of fact, a written argument is, first of all, a written text, and for the student, the tasks of producing a correct text and an acceptable explanation are closely intertwined. This is why we will use a linguistic perspective with related specific tools such as cohesion. This latter allows creating the texture, which is the quality of being a text instead of a disorganised set of words and sentences (Halliday and Hasan, 1976). Though related, cohesion is different from coherence. The first one refers to the linguistic devices needed to realize the second one, which is instead a mental process, proper of the individuals involved in the discourse. The production of an acceptable argument can be hindered by the lack of either mathematical or linguistic competence. Often students produce written explanations that are plain descriptions of the procedures they have carried out, by means of a set of more or less disconnected clauses where cohesion is marked just by the fact that the text they produce is semantically congruent to the actions they have performed. In other words, their cohesion markers are extra-linguistic, and we cannot tell whether or not the writer is aware of the semantical links among the clauses. We believe that the construction of cohesive texts is the first step towards the development of logically acceptable arguments. The script has been designed with the aim of fostering the student to construct cohesive arguments, which can be interpreted by their pairs, independently on their mathematical abilities.

DIST-M script

The DIST-M script presented here aims to allow the student to grasp a method of construction of mathematical arguments expressed verbally. The student is involved in tasks alternately individual and social. Social tasks are realized by means of chat and group forum (Figure 1).

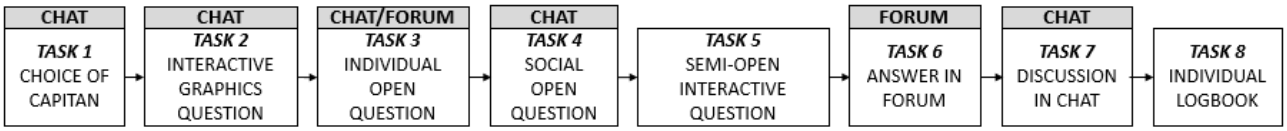

Figure 1: Design of the script

The chat supports the explicit comparison and it mediates (Bartolini Bussi & Mariotti, 2008) a modality of communal acting (to get an answer, an argument that supports its correctness, a reply adapting to the possible contradictory) that from social activity becomes an own way of working of the student. The forum, through its rules of use, supports the sharing and discussion, and in this way, it mediates the interaction inducing everyone to give their own contribution and to listen the one of others. In the forum, each student writes a description of his/her solution, reads / interprets the writings of others and can / must compare his/her texts with those of others. All this requires significant semiotic processes that besides being expected to foster the development of mathematical meanings, are expected to promote social argumentation experiences that might be internalized and become own internal process of each student. Thus, according to Vygotsky (1930), there is a development of "higher mental functions". In our case we refer to experiences of argued debate on manner of thinking / solving / answering the question, thus with higher mental functions we refer to argumentative skills, concerning the need to support the correctness of their answers with relevant topics, socially and mathematically acceptable. The functioning of the DIST-M requires different types of interactions: interaction with the script and interaction between the members of the team. The goal is to give a shared solution for the task, but the main achievement for the single student is to formulate his/her own argument (as a text) supporting the correctness of such a solution. In the script specific constrains have been designed to induce the production of personal arguments, their comparison and eventually the elaboration of a final answer, mediating the moving from an informal to a formal expression of the final individual answer.

In the following, we give a brief description of the various tasks constituting the script (Figure 1). At first the group chooses its own Captain talking to group friends in the group chat (Task 1). He is in charge of engaging all the team members in following the tasks of the script. The next task is the interaction with a GeoGebra interactive construction (Task 2). The aim is to investigate and solve a problem posed by the story. After a more purely experiential phase and subsequent guided reflection, the student answers on the forum to an open question aimed to generalize the experience and the results to which the student has come (Task 3). When all students have submitted the response to the Forum, the discussion continues chatting with the aim of achieving a common response that the captain reports on the chat (Task 4). In the next task, the student responds individually to a semi-open interactive question (Task 5). The interaction consists in manipulating the words-blocks available to build the response and motivation to the previous individual and group question, and in receiving a feedback on the correctness. The words-blocks have been constructed using some answers collected in a pilot. In order to highlight the causal structure of an

argument, the causal conjunctions, which are responsible of the cohesion, constitute separate words-blocks from the other ones. Then the student is required to report on the forum the phrase built by words-blocks with the received feedback and he can see the ones by his peers (Task 6). It follows a chat group discussion to reflect on the words-blocks sentences proposed by all the members with the aim of clarifying the correct answer and argument (Task 7). Finally, the student writes in the personal Log Book all information considered useful for the mission, the impressions on the activity, the difficulties encountered and how they were overcome (Task 8).

Experimentation

The prototype used for the experimentation has been realized by means of open-source or free tools, that have allowed to create new interactive graphical applications and semi-open interactive applications (Dello Iacono, 2015). The pilot study has involved 23 10th grade students of Liceo Scientifico in Pompei (NA, Italy). The students have been split into 6 teams, each of them constituted of 4 students, except one constituted of 3. The teams have been randomly assembled, so that each student at the beginning did not know his/her team mates. Students belonging to the same team could communicate only through the forum and the chat. In the following, we analyse and discuss the experiment with respect to the following key points: (i) the production of verbal arguments for supporting of the solution to a question; (ii) evidence of the different functioning of the script (that is the implementation of the designed learning activity) according to the student's engagement with respect to the story and the team work.

Concerning the first point, we analyse, from a qualitative point of view, the arguments produced during the individual open question and the answer in forum (Task 3 and 6 in Figure 1), that is before and after the semi-open interactive question, in order to investigate the effectiveness of the script. The students are required to answer if and why, fixed a sector in an aerogram, the angle varies according to the radius variation. As we will see in the following, the comparison between the nature of the individual arguments produced during the two tasks shows evidence of an improvement in the cohesion of the explanations constructed. In order to verify the cohesion, we look for the following cohesion markers in the texts produced by the students: lexical repetition (consisting in repetition of words), grammatical repetition (reference, that indicates something already appeared in the text, and ellipsis, that consists in the deliberate omission of words that are required to make up the sense), conjunction which allows to link two parts of a discourse (external, when it refers to a fact, internal, when it refers exclusively to the organization of the text).

Let us consider the team 2. At Task 3 only 1 student provides an argument explaining his/her answer, and he/she is the one who draws team's attention on this request, actually, replying to a mate enquiring of the platform's feedback on his/her answer, he/she says in chat:

1 S7 me too, but we are required to justify our answer

So next his/her answer in the forum is the following:

2	The quantity to be represented is equivalent to 20%.	A first reformulation of the data of the problem
3 4	360°:100%=x:20% x=(360x20):100=72°	S7 carries out a calculation

5	Although the radius changes, the size of the angle does not change	A conclusion is drawn on the previous calculation.

What is posted in the forum is mostly like a report of his reasoning (thinking aloud) without any cohesion marker. It can be seen as a report the mental process in the mind of the writer S7 and in his view it is coherent. This may not be the case for a reader, as it was for another student who asked for clarification. So S7, in order to explain to him, transformed such a personal reasoning in a new text.

6	The angle of the coloured part does not change varying the radius	The conclusion becomes the first statement expressing the answer to give.
7	Because in a circle the angle is always 360° and then 20% is always 72°	The previous calculation has been interpreted to become an explanation of answer.

The new text is cohesive. As a matter of fact we note lexical repetition (angle), external conjunctions (because, then), ellipsis (20% refers to 360°). The difference between the two texts consists in the fact that the cohesion of the text can help the reader to grasp its coherence, which may remain inaccessible for the first text. An effective use of cohesion promoted the shift from a personal report of reasoning to an argument: the sequence statement – calculations – conclusion became a statement plus an explaining argument. Such a cohesive text was generated for communication goals: the request of sharing his/her personal answer seems to have induced the student to better articulate the solution process transforming the calculations into a verbal text providing the reason of such calculation. So the collaborative script has promoted the construction of cohesive argumentations, because of the need of improving communication within the group.

In Task 6, we note a clear improvement: 4 students (that is all members of the team) produce an answer that includes an argumentation. In particular, 2 students (S5 and S8) who did not justify in Task 3,when reporting the answer made of words-blocks, not only produce a justification, but both of them go further the request and rephrase with their own words the arguments.

At Task 3, S8 writes the non-cohesive sentence (there is only an internal conjunction “anyway”):

8 S8 Varying the radius anyway the angle does not change

Then, at Task 6, he writes:

9	The angle does not vary because it is always equal to 20% of the circle angle.	External conjunction (“because”), reference (“it”).
10	The other scientists completely agree with me as varying the radius there is only an extension of it and the angle remains unchanged.	External conjunction (“as”), reference (“it”), lexical repetition (“angle”).

The first sentence is the one constructed by the words-blocks, as required in the Task. Then the student get back in touch with the story and he/she seems engaged and making reference to the scientists, he/she explains in his/her words why all the scientists agree and produces his/her own arguments for supporting the given answer. The second sentence is cohesive. Also in this case the script, requiring reporting the answer constructed by words-blocks and the scientists’ feedback, seems to promote the construction of arguments in terms of cohesive texts.

A similar evolution is shown by student S5. At Task 3, she produces a non cohesive text with no markers of cohesion:

11 S5 The angle does not change, only the radii vary

In following tasks, he/she writes:

12	The angle does not change because it is always equal to 20% of the circle angle.	External conjunction ("because"), reference ("it").
13	All the members of my team has the same idea. I have the angle is always the same because in a circumference the angle is always 360°, then 20% of 360° is always the same	Two lexical repetitions ("angle", "360°"), two external conjunctions ("because" and "then")

Also the student S5 at beginning reports the answer made by the words-blocks, but later he/she refers to the story and he/she seems so engaged to say "my team", the team of the scientists to which he/she belongs in the story, and when he/she refers to the story, he/she rephrases with his/her own words the answer and its motivation. The sentence constructed by S5 is cohesive. Thus, it seems that the functioning of the script, based on sharing the answers and impelling to find an agreement might lead to appropriate the meaning of argument as explaining and supporting the correctness of the solution by means of cohesive texts.

Let us consider a case where, even if the answer given by the student is not correct, we can anyway observe a shift towards the production of an argument. The student S22 at Task 3 writes:

14	Greater is the radius as much as the angle decreases.	This text is not cohesive (there is no markers).

At Task 6 he/she writes:

15	The angle decreases because it is inversely proportional to the radius but the other scientists do not agree	Two external conjunction ("because" and "but"), ellipsis ("the angle" is omitted in the secondary sentence)

Here we have a cohesive text constructed by means of the words-blocks. Even if the answer is not correct, there has been the production of arguments. So, the script seems to work according our goal (to foster verbal argumentation) independently on the correctness of the mathematical content.

Similar behaviour can be observed in the other teams: we have only 8 students among all teams that produced argumentation at beginning, whilst at the end all of the 23 students do it. In particular, the request of sharing on the chat seems to have a mediating function leading to transform a personal reasoning into a public argument. As shown above, the students do not limit themselves to report the sentences constructed with the words-blocks, but they also reproduce arguments with their own words assembled in a structure similar to the ones suggested by the script.

Concerning the second point, the transcripts show evidence that the effectiveness of the activity is strongly influenced by the students' engagement with respect to the story and to working in group. The effective work of Team 2 seems to occur because all the members were engaged in the story and shared a good mood allowing collaboration. However, in some other cases, we can see that the

activity fails if this does not occur. Let us consider for instance, team 5. The Task 3 seems to work well, the students are engaged in the activity and produce quite different argumentations:

16	S17	360:100
17		3.6x20=72
18		varying the radius the angle does not change
19		because the percentage is always the same
20	S20	varying the radius the angle indicating the percentage of the considered
21		stone does not change because 20% of 360° is always 72°, 360.1/5=72

Arriving at the Task 6, we find that they seem to have lost interest in the activity. Looking at the chat transcripts, there is evidence of a change of the team's mood. Actually, students start to become nervous around the end of Task 4:

22	S20	but we have not yet given the first agreed answer beep
23	S18	WE ARE DISCUSSING NOW TO HAVE AN AGREED ANSWER
24		20, you are a genius of evil, connect you brain

Team mood in chat get worse until the beginning of Task 6:

25	S18	HAVE YOU UNDERSTOOD?!
26		DO IT ALL OF YOU 4 IF YOU DID NOT DO IT YET!
27		I said to you in the session share not in the notes, I was sure that you did it
28		Hurry up to write in the sharing session
29		THERE ARE TWO SESSION OF beep. S20 who are you?

At the Task 6 the students have lost their initial engagement and do not satisfy completely the requests. It seems evident that the mode of operation has been strongly affected by the negative engagement, in particular of the Captain, and by the impossibility of collaborate.

Conclusions and future directions

In this paper we have reported on a pilot study concerning a computer-supported collaboration script, aiming at developing students' verbal argumentative competence. The analysis of the students' transcripts seems to show that the collaboration script fosters the introduction of the student to the construction of arguments as cohesive texts, independently on the student's skill in mathematics. Some students shift from producing computations to constructing cohesive argumentations that make use of the previous calculations; some other students, although they do not get the correct answer from the mathematics point of view, also produce cohesive texts as expression of their reasoning. We are now working on a quantitative data analysis, by means of statistical test, coding cohesion of the written students' productions to compare Task 3 and Task 6, in order to confirm the effectiveness of the script as shown by the transcripts in the previous section. Moreover, in order to check that students really interiorized the script and not only repeat what they did before, as well as the use of the cohesion, we are implementing a new script, as continuation of the story, without the word-blocks. There is also evidence that a negative mood in the work team can compromise the success of the learning activity. To this aim, we are implementing a new version foreseeing the introduction of a role for each member of the team, behind the Captain, avoiding that somebody in the group gives away the responsibility. We are going to prove that the

designed script, suitably modified in order to create a positive mood in the work team, promotes effectively the construction of cohesive texts and the fact that there is a strict interconnection between them and logically acceptable arguments.

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(Legitimate peripheral) computational thinking in mathematics

Laura Broley[1], Chantal Buteau[2] and Eric Muller[2]

[1]Concordia University, Canada; l_brole@live.concordia.ca

[2]Brock University, Canada; cbuteau@brocku.ca; emuller@brocku.ca

"Computational thinking" is a hot topic in math education, among teachers whose curricula now include the term, and researchers who wish to pinpoint what it means and how it could be promoted in classrooms. A recent study resulted in a theoretical model of the computational practices of professional mathematicians and scientists, with the aim of offering teachers a set of competencies around which to build activities for their students. Nonetheless, concrete examples that validate the model and exemplify its use in math classrooms have yet to be discussed. We wish to open up this discussion, which we see as crucial to understanding how to empower students to participate in the computational thinking that has become integral to the mathematics community and beyond.

Keywords: Computational thinking, mathematicians' practices, legitimate peripheral participation.

Introduction

Parallel to the invention of the personal computer, Papert (1980) envisioned a world where children fluently use the tool as young mathematicians. Some thirty years later, we've witnessed a widespread resurgence of interest in that vision, taking shape in educational reforms (e.g., in Europe; Bocconi, Chioccariello, Dettori, Ferrari, & Engelhardt, 2016) and research regimes (cf., www.ctmath.ca) in the name of *computational thinking*, deemed a 21st century skill. Yet, there is little consensus on what this "new" term encompasses or how/if it should be conceptualized within subject areas beyond computer science (Grover & Pea, 2013). In response to these issues, Weintrop et al. (2016) developed a taxonomy of computational thinking practices geared towards science and mathematics. They based their work on a literature review, an analysis of learning activities, and interviews with "biochemists, physicists, material engineers, astrophysicists, computer scientists, and biomedical engineers" (p. 134). They also show, through concrete examples, how the practices might be promoted in physics, biology, and chemistry classrooms. To build on this work, we could ask: What might the computational thinking practices look like in *mathematics* classrooms? Moreover, are they representative of professional *mathematicians*' practices?

In this contribution, we attempt to provide some answers to these questions by drawing on two resources: 1) fifteen years of experience in a sequence of three undergraduate Mathematics Integrated with Computers and Applications (MICA) courses at Brock University, where students create and use computer environments to explore mathematics concepts or real-world situations; and 2) reflections of mathematicians whose research falls within an area recognized by the European Mathematical Society in 2011: "Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of […] modeling, simulation, optimization, and visualization" (p. 2). The next section outlines the perspective underlining our work and our approach in preparing this paper. We then present our results, i.e., we exemplify (empowering legitimate peripheral) computational thinking practices in mathematics.

Theory and methods

The way we interpret "Learning mathematics with technology" in the context of this paper is well explained by Lave and Wenger's (1991) concept of "legitimate peripheral participation", whereby students are invited to become mathematicians through engaging in their shared practices. "Mathematics", then, is not seen as a body of knowledge to be acquired by the student, but rather as a social community to which the student gradually gains membership. Hence, we do not discuss computer "technology" from a cognitive point of view, for example, as a helpful tool in illustrating concepts. We focus, instead, on how mathematicians, the old-timers of their discipline, and students, the new-comers, create and use computer tools to engage in practices considered to be integral to the mathematical community.

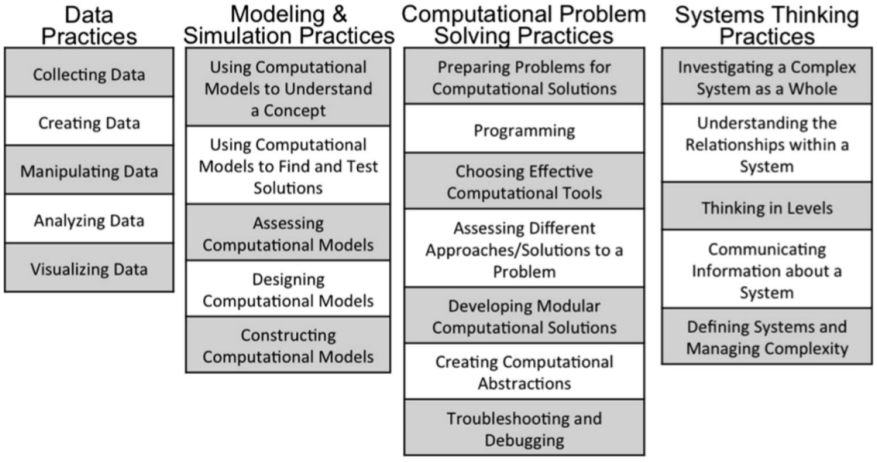

Figure 1: Computational Thinking Practices, taken from Weintrop et al., 2016, p. 135

In recent work, Weintrop et al. (2016) outline what they believe to be these integral practices (Figure 1). Their framework provides a detailed description, specific to STEM (i.e., science, technology, engineering, and mathematics), of one of three dimensions introduced by Brennan and Resnick (2012) to characterize "computational thinking": namely, computational concepts, practices, and perspectives. Such frameworks seek to elaborate on the general definitions on which they are grounded; for instance, that of Cuny, Snyder, and Wing, who describe computational thinking as "[t]he thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent" (2010). Computer programming plays a particularly important role, as it "is not only a fundamental skill of [computer science] and a key tool for supporting the cognitive tasks involved in [computational thinking] but a demonstration of computational competencies as well" (Grover & Pea, 2013, p. 40). Such ideas invoked in us vivid images of our own experiences with/as mathematicians and students creating and using computer tools to do mathematics. And this inspired us to address an apparent void: that is, a comparative picture that highlights the powerful computational thinking employed by professional mathematicians, on the one hand, and, on the other, the potential of students to participate in the same kind of thinking.

To prepare this picture, we re-examined Broley's (2015) research, which explored the use of programming by 14 mathematicians in their research and teaching. In an interview, each participant described research where they developed and used computer tools. Unbeknownst to us at the time, this provided examples of how some full-membership mathematicians engage in computational

thinking. Amongst them, we chose four that align with the groupings in Weintrop et al.'s (2016) taxonomy. We then reconsidered the data from another study (Buteau, Muller, Marshall, Sacristán, & Mgombelo, 2016) – the 14 MICA projects completed by one student, Ramona – as a source of four examples of peripheral computational thinking. MICA's goal of responding to society's need for professionals proficient in programmable technology made it a natural database for comparison.

Results

This section provides examples of computational thinking as it might be experienced by full and peripheral participants in the mathematics community. For each category identified by Weintrop et al. (2016), we describe a mathematician's project that we feel effectively exemplifies it. This is contrasted to a MICA project that we see as providing access to the same kind of practices.

Data practices

Adèle uses her expertise in mathematics and computing to solve problems in financial engineering. In one project, she developed a model that enables investors to judge the investment potential of various market entities. In particular, the model calculates the risk that an investor will lose money because the investee is unable to pay back what they owe. The tricky part is that most investees have never had such financial problems (e.g., with bankruptcy). To assess a given company or individual, Adèle considers their portfolio: She *collects* their history of actions (e.g., investments, bonds, shares) on the financial market. The basic idea is that as others agreed to invest a certain amount of money in the company or individual, they implicitly demanded to be compensated for the risks they were taking, thereby predicting the probability that the investment would be a good one. Mathematically-speaking, the problem is of an extremely high dimension: Adèle's model contains over 20 parameters that must be estimated for each portfolio by *manipulating* the corresponding data with optimization techniques. Intensive numerical methods are then applied to the specified model to *generate* the data necessary for evaluating the risk of the investee(s) being considered. The result is not a simple measure of the average risk. Adèle must perform a nuanced *analysis* to meet her clients' needs, calculating and *visualizing* probability distributions in order to portray the best and worst case scenarios. About the place of computation in her work, Adèle was blunt: She said there would be no project without it. In fact, when tasked with assessing the risk associated to hundreds of companies at once, she must use computer clusters to get the job done.

In their second of three MICA courses, students in Ramona's cohort were assigned a project similar to Adèle's. During lectures, they were introduced to mathematical ideas related to the stock market. In regards to programming, they also learned how to read data from files. Up to this point, they had worked with data they *created* through simulation; but during this project, they had to use data from Stock Market sources. During two (two-hour) lab sessions, the students initiated their individual work by *collecting* the S&P index, a measure of market conditions, from 1950 to 2002, and writing a program to *manipulate*, *visualize,* and *analyze the data* using standard statistical techniques. Students were also required to select ten stocks and, like Adèle, make recommendations to a fictive client based on their own *analysis*. In her report, Ramona grounded her recommendation on the mean and average yearly percentage of her stock selection. Then, as requested, she conducted a regression analysis of a stock over a decade and described how *visualizing the data* as a cloud of points confirmed her interpretation of the coefficient as representing a weak correlation.

Modelling and simulation practices

Alice's projects are often inspired by a collaborator in need of her modelling and simulation skills. She spoke, for example, of a kinesiologist who initiated a multi-year project about muscles. Alice began by learning about the application, which she knew little about. She could then *design* a system of equations that would allow her to study the features of interest, i.e., tensions, bulges, and fibers; but only once the model was implemented on a computer. During her interview, Alice joked that computation was essential because, unfortunately, the solution to a real-world mathematical model never simplifies to the quadratic formula. While some researchers use existing simulators to gain access to their models' solutions, Alice prefers to have the control of *constructing* her own. This comes at a price: Even if her team starts with an existing code, they still have to think very hard about how they implement their equations, import data, generate meshes, and so on. But all this hard work apparently paid off in this project: Alice described the resulting computational model as "the most complex simulator of its kind", and was hesitant to share its massive code during her interview. This tool was used systematically to investigate issues the researchers initially sought out to *understand*. But by varying parameters in an exploratory mode, they also *found and tested solutions* to an unreported problem: the forming of well-defined fiber structures. During her interview, it was clear that Alice was excited by this discovery, for her collaborator had observed the formation of the exact same fiber structures, but in a real human! In the end, the data collected during this ultrasound experiment of a person on a bicycle *assessed* Alice's model, confirming that it represented "the real thing" in more ways than expected.

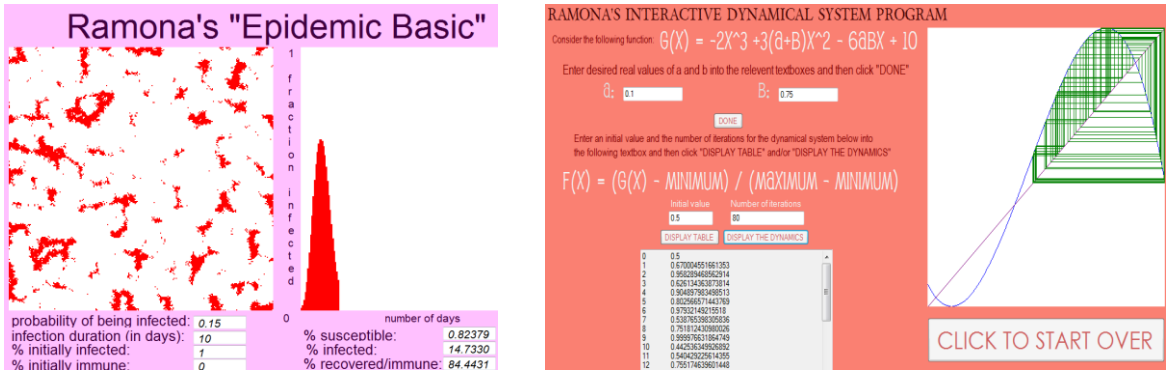

Figure 2: Ramona's epidemic simulator (left) and discrete dynamical system program (right)

Modelling and simulation practices that resemble Alice's are central to the MICA courses. At the end of the third course, students in Ramona's cohort were asked to use the theory of cellular automata to model and simulate the spread of an epidemic. Students worked individually to *construct* the computational model (i.e., to implement it in VB.Net), complete with a dynamic visualization of the cellular automatum and a complementary graph (Figure 2, left). Using this model, students were invited to observe real-time simulations of certain scenarios with the goal of coming *to understand* the effects of vaccination on the proliferation and diminution of epidemics. They were then told how to extend their models to include the cost of immunization and medical treatment, so to *find* (estimate) *the solution* of a minimal medical cost problem. In her report, Ramona went beyond finding the solution; as required, she also *assessed* the ability of her extended computational model to provide an accurate estimate, finishing with suggestions for improvement.

Computational problem solving practices

To understand Norman's pure mathematics research, some *preparation* is in order. In his work, a permutation of length n is just a string, $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, where each σ_i is a unique element from the set {1, 2, …, n}; for example, $\alpha = 624531$ is a permutation of length 6. Given another permutation, e.g. $\beta = 231$, we say that $\propto$ contains the pattern β if we can find in $\propto$ a subsequence (not necessarily consecutive) whose numbers have the same relative order as 231. The fact that $\propto$ contains the subsequence 451 – 1 is the smallest number, 5 is the highest, and 4 is in between – means that it contains β (we could have equally used subsequences 241, 251, 231, or 453). If a permutation does not contain a pattern, it is said to avoid it; for instance, α avoids 1234. An interesting problem for mathematicians is to determine the number of permutations p_n of length n that avoid a given pattern. It is known that p_n grows almost exponentially with n. The growth rate, however, is still unknown for many patterns. In search of one such rate, Norman's team had to build a complex computer tool. The *programming* was delegated to a student, whose life was simplified by the *development of a modular solution* based on an existing subroutine for another pattern. The creation of the entire algorithm, nonetheless, was a team effort, for it involved the careful *assessment of different approaches and solutions*. One option was to calculate the exact value of p_n for as many n as possible and then extrapolate the growth rate. But according to Norman, this approach was inefficient: At the time of his project, they could calculate the exact values only for $n \leq 25$, which was not enough to provide an acceptable solution. The mathematicians hence chose a probabilistic approach that uses estimates for p_n rather than exact values. This enabled them to calculate more data points; but their program was still slow. Seeking to *troubleshoot and debug* the problem, Norman suggested that his team try to visualize the permutations. Their decision to represent a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ as a function that sends i to σ_i led to the discovery of an unexpectedly striking structure (Figure 3, left). Norman insisted on the importance of *creating this particular computational abstraction*: The pattern would not have been observable, for example, had they produced only a list of matrix entries. And then Norman might have missed out on a novel research direction that occupied him for many years.

Since all MICA projects involve *programming*, computational problem solving practices like Norman's always form a major part of their completion. Starting in the first MICA course, students discuss what makes a math problem amenable to exploration through programming. Since this is new to most of them, they are also led to develop their computational skills through a carefully selected progression of projects, which increase in complexity in terms of both the mathematical content and the programming requirements. For example, Ramona and her peers learned about discrete dynamical systems alongside techniques of displaying graphics in VB.Net, which they applied by creating a program to numerically and graphically explore the logistic map. In a later project, the students were asked to build on this work (i.e., borrow *modular computational solutions* from it) and *program* a tool to explore the system of a two-parameter cubic (Figure 2, right). This new problem required more serious *preparation for a computational solution*, as the domain of the cubic called for the consideration of different cases. Inherent to the programming process was also *troubleshooting and debugging*, *creating computational abstractions*, and *assessing different kinds of solutions*, which may have contributed to Ramona's conclusion in her written report that "creating and working with this program has assisted me to fully grasp the way a dynamical system works by observing the table, the graphs, and the cobweb with countless test values."

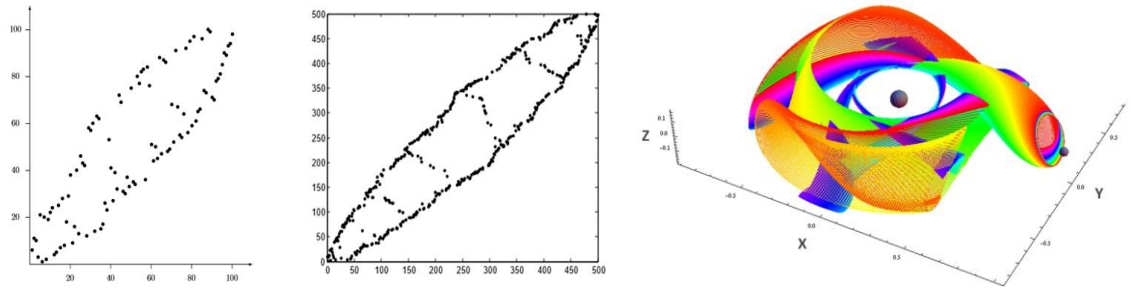

Figure 3: Norman's discovery of structure in pattern-avoiding permutations (left) and Albert's computation of trajectories resulting from a perturbation off an orbit close to the moon (right)

Systems thinking practices

Albert has studied many complex systems, including those *defined* in celestial mechanics. The three-body problem, for example, seeks to describe the motion of a spaceship in the presence of two bodies, like the Earth and its moon. The complexity of the system is *managed* by ignoring the presence of other bodies, taking the spaceship to have negligible mass, and assuming that the massive bodies move in circular orbits. These explicit boundaries do not render the system useless. In fact, the model has provided initial approximations for real space missions. Moreover, it serves as a rich source of problems that allow Albert to show off the mathematically and numerically sophisticated software he has developed, software that according to him can compute "amazing things" that are simply "not computable" by traditional methods. Albert's team has computed the uncomputable at different *levels.* Macroscopically-speaking, they have *investigated the three-body system as a whole* by finding and classifying an infinity of its periodic solutions (i.e., closed trajectories where a spaceship could remain in orbit). On a microscopic level, they have explored these orbits in family groups and individually. This latter consideration also helps them *understand some relationships between elements within the system*: For a given orbit, the researchers can determine the set of trajectories that a spaceship could follow after experiencing a slight perturbation. The resulting tube-like structures are like highways that enable space travel to far-away places with minimal effort (and money). One image (Figure 3, right) is enough to convey the importance of visualization in *communicating* Albert's results.

In each MICA course, the last two weeks are dedicated to challenging original projects wherein students select topics of interest to them. Ramona's terminal (14th) project is an example of how students might engage in practices similar to Albert's and, as the MICA course creators aimed, "develop their own strategies for handling complex real world problems" (Buteau et al., 2016, p. 144). With two of her colleagues, Ramona *investigated, as a whole, the complex system* associated to the water level changes in Lake Erie (Canada). In particular, they were interested in explaining how and why the level changes over time (i.e., in *understanding the relationships within the system*). They described their initial research in existing literature as "a crucial starting point in [their] project, allowing [them] to obtain an understanding for the changes in the water supply of Lake Erie." Based on the information gathered, they designed and programmed stochastic and deterministic models of the phenomenon. They then performed an analysis, through simulation, of six case studies, representing the system in various ways on *a different, more microscopic, level.* They used their initial research to justify the assumptions they made, the parameters they chose, and

the case studies they considered in order to *manage the complexity* of the system. This explanation was part of the 26-page report where Ramona's group *communicated their results*.

Discussion and conclusions

The four pairs of examples provided above aim to render Weintrop et al.'s (2016) framework more concrete, validate its correspondence with a diverse set of authentic professional practices, and provide some insight as to how students might be invited to gain access to them, all within the context of mathematics. Ramona's work differed from the mathematicians' in its magnitude: Her projects were more restricted in scope and length, her computer programs were more naïve, and her findings had less immediate value for the community at large. This is not surprising since Ramona was in a peripheral phase of participating in the mathematics community, where she was simultaneously negotiating entrance into a community of students at a particular university, with its own norms limiting engagement in full-membership mathematical activity. Nonetheless, in exposing Ramona to the computational practices of mathematicians, programs like MICA support a nuanced discussion of what it means to integrate digital tools in students' learning of mathematics.

Many scholars have reported on the ways in which building and/or interacting with digital tools might assist students in meaningfully acquiring mathematical ideas or ways of thinking that are embedded in current curricula. The collection of papers presented in the working group on learning mathematics with technology at this year's CERME conference provides numerous examples. In fact, the main framework used in this paper was built on the premise that learning activities involving computational thinking practices can enrich students' understanding of mathematics and science (Weintrop et al., 2016). This said, the framework was equally inspired by the ever-increasing computational nature of STEM-related disciplines. As evidenced by our examples, and much work that precedes us, the power of the computer has had a major impact on the way that STEM professionals (can) do their work. And so, the *computational thinking* trend presents an opportunity (or perhaps a necessity) for mathematics educators at all levels to reconsider not just the "how" of mathematics teaching, but also the "what", i.e., the knowledge and skills to be taught. After all, students' participation in the computational thinking practices of mathematicians might not just prepare them for a computational future in general; it may also widen their perspectives of the nature of mathematics and who is capable of learning (and doing) it.

Both research and experience tell us that reflecting on the above issues, developing curricula to address them, and enacting that curricula in classrooms are quite different feats. Detailed and extensive frameworks like the one developed by Weintrop et al. (2016) can certainly help support researchers, curricula developers, and teachers. But there is still a need to examine more closely and completely the experiences of students who are peripheral participants in computational communities of practice: What skilled knowledge (i.e., practices) do they *actually* develop? Moreover, how do they identify with communities they are both entering and (eventually) influencing? Given our analysis in this paper, the MICA program provides a rich context within which to study such questions. The answers could lead to an enlightening discussion about challenges and opportunities in bringing about a nuanced technology-rich mathematics education.

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Examining motivation in mobile app-supported math trail environments

Adi Nur Cahyono[1,2] and Matthias Ludwig[1]

[1]Institute of Mathematics and Computer Science Education, Goethe-University Frankfurt, Germany; ludwig@math.uni-frankfurt.de

[2]Mathematics Department, Semarang State University, Indonesia; cahyono@math.uni-frankfurt.de

The MathCityMap-Project was developed by combining the concept of a math trail program with advanced mobile technology. It aims at providing a new approach to promote student motivation to engage in meaningful mathematical activity. An explorative study was conducted as a pilot using nine secondary schools in the city of Semarang, in Indonesia, and 272 students and nine teachers were included. Using self-determination theory as a framework, we explored the motivation of students to engage in mobile app-supported math trail activity. Data collection procedures comprised observation, interviews, questionnaires, and student work analyses. Findings indicate that intrinsic motivation and identified regulation established an essential part of students' motivation to engage in this activity. The design of the learning environment, the use of mobile app, and the value of the mathematical task have contributed to this result.

Keywords: MathCityMap, mobile app, math trail, student motivation.

Introduction

A math trail, a path for discovering mathematics, was created as a medium for experiencing mathematics in all its characteristics, namely, communication, connections, reasoning, and problem solving (Shoaf, Pollak, & Schneider, 2004). In such a trail, students can simultaneously solve mathematical problems encountered along the path, make connections, communicate ideas and discuss them with their teammates, and use their reasoning and problem-solving skills. Although the math trail project is not new, supporting this outdoor education with mobile technology is an innovative approach to the program. This idea appears together with the fact that, in recent years, mobile technology has significantly improved and mobile phone use has significantly increased (Lankshear & Knobel, 2006). These advancements have been followed by the creation of many mobile phone applications (apps), including those intended for use in outdoor activities. In learning activities, Wijers, Jonker, and Drijvers (2010) suggested that mobile devices could be employed to facilitate learning outside the classroom. They also suggested that mobile technology could be exploited to support the outdoor educational program. Integrating advanced technology with the math trail program is the basis for the development of our project, called the MathCityMap-Project, in which math trails are facilitated by the use of GPS-enabled mobile phone technology. This project has been developed and implemented in Indonesia since 2013 and has been tailored to this country's situation. The main focus of this paper is to explore the motivation of students to engage in a math trail program supported by the use of a mobile app.

Theoretical background

The MathCityMap-Project is a project of the math trail program, which is supported by the use of a GPS-enabled mobile phone app and uses specialized mathematical outdoor tasks (Jesberg & Ludwig, 2012). This project was not conceived merely to design and/or use the math trails. Instead, it includes the entire process: preparation (how to design it), implementation (how it runs), and evaluation (how

it impacts student motivation). The mobile phone app, as a supporting tool, was also created and used during this project. Therefore, the theoretical framework for the MathCityMap-Project study is underpinned by the concept of the math trail program, the use of mobile technology in mathematics education, and student motivation in mathematics.

A math trail is a walk in which mathematics is explored in the environment by following a planned route and solving mathematical outdoor tasks related to what is encountered along the path (English, Humble, & Barnes, 2010). In math trail activities, "children use mathematics concepts they learned in the classroom and discover the varied uses of mathematics in everyday life" (Richardson, 2004, p. 8). They discover real problems related to mathematics in the environment and also gain experience connecting mathematics with other subjects. Among the many benefits of a math trail (Richardson, 2004) is the creation of an atmosphere of adventure and exploration resulting from the fact that it is located outside the classroom. A math trail guide, such as math trail map or human guide, must be prepared to inform walkers about the math trail task stops and to show the problems that exist at each location. It also tells about the tools needed to solve the problems, so that the walkers are prepared before starting to walk on a trail. With the rapid development of technology nowadays, it is possible to collect the tasks and design a math trail guide based on a digital map and database.

In recent years, rapid developments in technology have occurred in the scope, uses, and convergence of mobile devices (Lankshear & Knobel, 2006). These devices are used for computing, communications, and information. Mobile devices are portable and, usually, easily connected to the Internet from almost anywhere. This makes them ideal for storing reference materials and learning experiences, and they can be general-use tools for fieldwork (Tuomi & Multisilta, 2010). Their portability and wireless nature allow them to extend the learning environment beyond the classroom into authentic and appropriate contexts (Naismith, Lonsdale, Vavoula, & Sharples, 2004). Wireless technology provides the opportunity for expansion beyond the classroom and extends the duration of the school day so that teachers can gain flexibility in how they use precious classroom activities. The use of mobile devices can also promote positive emotions for students toward learning mathematics (Daher, 2011). These advantages have been exploited through the MathCityMap-Project. Math trail blazers can create and upload math trails into a database through a web portal, then the math trail walkers can access them and complete the math trail with the help of a GPS-enabled mobile app (Cahyono & Ludwig, 2014; Jesberg & Ludwig, 2012).

In this paper, we focus on the exploration of the factors that motivate students to engage in mobile app-supported math trail activity. The academic literature distinguishes between two motivational concepts namely: extrinsic motivation and intrinsic motivation. An influential theory that explicates intrinsic–extrinsic motivation in depth is self-determination theory (SDT, Deci & Ryan, 1985). The SDT model conceptualizes a range of regulation from intrinsic motivation to amotivation. Between these, there exist identified regulation and external regulation. Intrinsic motivation exists when a student is engaged in an activity for his/her own sake/pleasure/satisfaction. Identified regulation refers to engagement that is valued as being chosen by oneself. External regulation is the type of motivation when engagement is regulated by rewards or as a way to avoid negative consequences. Lastly, amotivation is associated with engagement that is neither intrinsically nor extrinsically motivated (Guay, Vallerand, & Blanchard, 2000).

Having outlined the theoretical background for this study, we can clarify the research question: *what is the nature of student motivation to engage in a math trail program supported by the use of a mobile app?*

Methods

An explorative study was conducted in the city of Semarang in Indonesia involving nine secondary schools. The participating schools represent three levels (high, medium, and low) and two location types (downtown and suburban). This study is a part of development research on the MathCityMap-Project for Indonesia. There were two main phases in this research, namely the design phase and the field experimentation phase. Here, we focused on studying student motivation to engage in the activity that was conducted in the second phase. This phase consisted of an introduction session, a treatment (math trails guided by the app), and a debriefing session. Student motivation was measured using the self-reported Situational Motivation Scale (SIMS) developed and validated by Guay, Vallerand, and Blanchard (2000) based on self-determination theory (SDT). The results of their study exposed that the SIMS represents a brief and adaptable self-report measure of situational intrinsic motivation, identified regulation, external regulation, and amotivation. 'Situational motivation' refers to the motivation individuals experience when they are currently engaging in an activity (Guay, Vallerand, and Blanchard, 2000). Therefore, this questionnaire is appropriate to be employed in this project to explore motivation of student to engage in the activity.

The SIMS is a 16-item questionnaire consisting of 4 subscales, intrinsic motivation (IM), identified regulation (IR), external regulation (ER), and amotivation (AM). In the first part of the instrument, the questionnaire asks, 'Why are you currently engaged in this activity?'. Respondents are to rate a number of answers using a 7-point Likert scale from 1 (*not at all in agreement*) to 7 (*completely in agreement*) for each item. The items are 'because I think that this activity is interesting' (IM), 'I am doing it for my own good' (IR), 'because I am supposed to do it' (ER), 'there may be good reasons to do this activity, but personally, I don't see any' (AM), 'because I think this activity is pleasant' (IM), 'because I think this activity is good for me' (IR), 'because it is something that I have to do' (ER), 'I do this activity, but I am not sure if it is worth it' (AM), 'because this activity is fun' (IM),'it was my personal decision' (IR), 'because I don't have any choice' (ER), 'I don't know; I don't see what this activity brings me' (AM), 'because I feel good when doing this activity' (IM), 'because I believe that this activity is important for me' (IR), 'because I feel I have to do it' (ER), and 'I do this activity, but I am not sure it is a good thing to pursuit it' (AM).

The four subscale scores are then used to calculate a single motivation score called the Self-Determination Index (SDI) for each student using the following formula: SDI = (2 x IM) + IR – ER – (2 x AM) (Sinelnikov, Hastie, & Prusak, 2007). The SDI score ranges between (2 x 1) + 1 – 7 - (2 x 7) = -18 and (2 x 7) + 7 – 1 - (2 x 1) = 18. A higher SDI score indicates the student is more self-determined and more intrinsically motivated to engage in the activity. A positive SDI score indicates that, overall, more self-determined forms of motivational type (IM & IR) are predominant (Vallerand & Ratelle, 2002). Then, open-ended follow-up questions were given to students to deepen the information about deciding factors affecting student engagement in this mathematical activity. Data were analyzed using qualitative methods to discover whether and what kind of motivations influenced these students. Quantitative data were also collected and analyzed. Non-parametric statistical

calculations were performed because the data consisted of ordinal scores, and normality could not be assumed.

Results and discussion

In the first phase of the MathCityMap-Project study in Indonesia, technical implementation of the project was formulated, and a mobile app was also created to support the program (Cahyono & Ludwig, 2014). Thirteen math trails containing 87 mathematical outdoor tasks were also designed around the city of Semarang (Cahyono, Ludwig, & Marée, 2015). Task authors found mathematical problems that involved objects or situations at particular places around the city. They then created tasks related to the problems and uploaded them to a portal (www.mathcitymap.eu). In this portal, the tasks were also pinned on a digital map and were saved in the database. Each task contained a question, brief information about the object, the tools needed to solve the problem, hint(s) if needed, and feedback on answers given. Math trail routes can be designed by connecting a few tasks (6-8) in consideration of the topic, level, or location. In designing the trails, it is also necessary to consider several factors such as: safety, comfort, duration, distance, and accessibility for teachers who would observe and supervise all student activity.

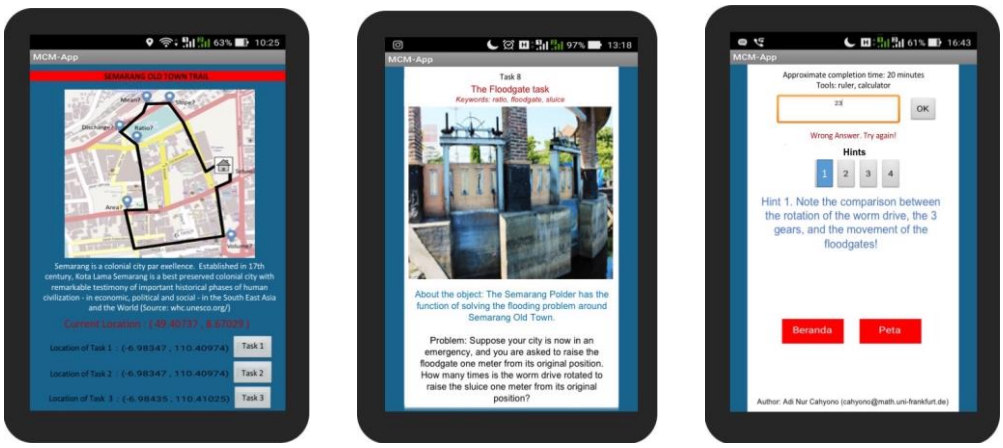

Figure 1: App interfaces (Map: ©OpenStreetMap contributors)

Figure 1 shows examples of the app's interfaces including an example route, task, feedback, and hint. Math trail routes can be accessed by students via the mobile app, a native app that was created by the research team as part of this project. Installation of a file in *.apk format was uploaded to the portal as well as the Google PlayStore™. From there, students could download and install the app, which works offline and runs on the Android mobile phone platform. Students can then carry out math trail activities with the help of the app. They follow a planned route, discover task locations, and answer task questions related to their encounters at site, then move on to subsequent tasks. The app informs them of the tools needed to solve the problems, the approximate length of the trail, and the estimated duration of the journey. On the trail, the app, supported by GPS coordinates, aids the users in finding the locations. Once on site, users can access the task, enter the answer, get the feedback, and ask for hints if needed.

In the second phase, field experiments were conducted with 272 students and nine mathematics teachers. Each school was represented by a class consisting of an average of 30 students. They were divided into groups of five to six members. Four schools carried out activities outside the school while five schools conducted activities in the school area. These choices were made because of

conditions and situations (such as: safety and availability of teaching and learning time) unique to each school. The activities were conducted during normal school hours over two 45-minute periods beginning with the teachers giving a brief explanation of the learning activities and goals. Groups then began their journeys, each from a task location that was different from the others (Group I started at task I, Group II from task II, and so on). As the groups trekked the trail, teachers observed and supervised student activities but were not expected to provide assistance because all the necessary information was to be provided by the app. Once the activity was completed or maximum time allowed for the activity had passed, the students moved to the next task. After completing the trail, each group returned to class, then had a discussion with the teacher about the task solutions and what they learned along the trail. At this time the questionnaire was also completed by the students. All 272 students' SIMS responses and SDI scores are summarized in Table 1.

	IM	IR	ER	AM	SDI
N	272	272	272	272	272
MIN	3.50	2.25	1.00	1.00	2.00
MAX	7.00	7.00	6.25	5.50	16.25
AVERAGE	5.9770	4.7215	3.4651	2.9779	7.3180
SD	0.82939	1.23314	0.93080	0.91471	2.92629

Table 1: Students' SIMS and SDI scores

Averages SIMS scores for the four subscales varied considerably, ranging from 1.00 to 7.00. The standard deviations indicated adequate variability in all subscales. It is apparent that the nature of these students' motivations to engage in the activity is diverse. However, all had positive SDI scores (ranging from MIN = 2.00 to MAX = 16.25 and average ± SD = 7.3180 ± 2.92629). This result indicates that overall, their motivation was more self-determined. Positive scores in this case indicate that internalized forms of motivation, namely intrinsic motivation and identified regulation, were predominant. Students perceived the activity to be interesting or enjoyable (an indicator of IM) and meaningful or valuable (an indicator of IR). They were engaged in the activities for their own sakes/pleasure/satisfaction, and their engagement was considered to be a self-choice. This finding is supported by the Independent-Samples Kruskal-Wallis Test (Table 2), which shows that there was a statistically significant difference in score between the SIMS subscales, $\chi^2(2) = 623.583$, $p = 0.000$, with a mean rank score of 886.37 for IM, 637.52 for IR, 379.97 for ER and 274.15 for AM.

Compared with others subscales scores (Table 1), the amotivation subscale had low average SIMS scores ($AM_{average} = 2.9779 \pm 0.91471$), which were contributing factors to the positive SDI scores. These low scores indicate that students enjoyed the activity and found value in it, which was reflected in the intrinsic motivation scores ($IM_{average} = 5.9770 \pm 0.82939$) and identified regulation scores ($IR_{average} = 4.7215 \pm 1.23314$). Students also reported being motivated by and were reacting to external demand, an indicator of extrinsic regulation. However, the ER scores ($ER_{average} = 3.4651 \pm 0.93080$) show they tended to be neutral on this subscale.

	Subscale	N	Mean Rank
Score	IM	272	886.37
	IR	272	637.52
	ER	272	379.97
	AM	272	274.15

	Score
Chi-Square	623.583
df	3
Asymp. Sig.	.000

Table 2: Kruskal-Wallis Test using subscale (IM, IR, ER, AM) as the grouping variable.

The open-ended questions asked in the next step focused on two types of motivation, namely intrinsic motivation and identified regulation. The first question was, '*Why did you enjoy the activity?*' We found that 30% of the students enjoyed learning outside the classroom, 23% were excited to use the advanced technology, 18% were satisfied with applying mathematics, 16% liked collaborating, 11% reported different reasons, 2% mentioned negative feelings, and 0% did not give a reason. The second question was '*What experiences influenced your consideration that this activity was valuable?*' (each student could mention more than one experience). Students mentioned application of mathematics in real life 158 times, outdoor mathematical activities 96 times, advanced technology for math 87 times, use of non-standard measuring tools 79 times, team work 72 times, activities in public places 65 times, and other 19 times. These answers showed that most students were delighted to engage in this activity because it was conducted outside the classroom, an unusual setting that offered comfortable conditions and it was a free and fun activity. The use of mobile devices for outdoor mathematics learning activities has become an attraction, encouraging students to engage in this activity. However, as a serious mathematical learning activity, it was not only enjoyable, but the students considered it a valuable activity. They reported that through this activity they learned how to apply mathematics in the real world, even where they had never thought about it in the past. The use of the latest technology in the learning process has also been reported as a valuable experience and new knowledge for them.

The self-reported data and answers to the follow-up questions were cross-checked with information obtained through field observations and the student works analyses. For example, here is one of the results of observations and analysis of student work on the Flood Gate Task, a task located on the Old Town Route. In this task, the problem statement is, *'Suppose your city is now in an emergency, and you are asked to raise the floodgate one meter from its original position. How many times must the worm drive be rotated to raise the sluice one meter from its original position?*'. This task is situated at one of the tourist attractions, an icon of the city, namely the Old Town area of Semarang City. All students agreed this was a pleasant place for learning math, and it was near the school where there were lots of trees, a pond, a garden, nice old buildings, and traffic was not too congested. These conditions made them feel joyful and comfortable in performing the activity there. Not only the location was exciting for them, the task was also considered by students to be a meaningful mathematical task because it was an important issue for them as citizens to know how this floodgate works. In this way, they could save their town if there were an impending disaster. Figure 2 shows an example of students working on this task.

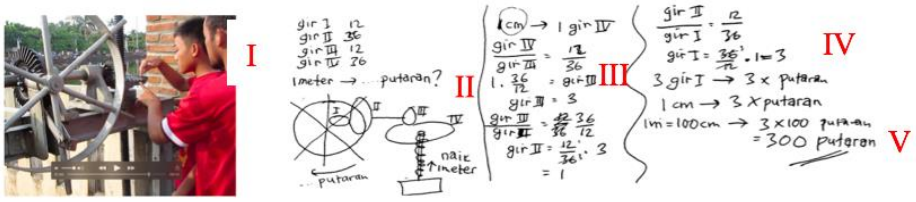

Figure 2: Students work on the flood gate task

It appears that the students had the opportunity to learn and practice ways of solving real problems by following the stages of mathematizing, namely, understanding a problem situated in reality (I), organizing the real-world problem according to mathematical concepts and identifying relevant mathematics (II), transforming the real-world problem into a mathematical problem that represents the problem situation (III), solving the mathematical problem (IV), and interpreting the mathematical solution in terms of the real situation (V). In addition, working in the environment to find the hidden task location was interesting and challenging for the students. They reported that the more hidden the task location, the more curious they were to find it. It was breathtaking for them when they had to match the coordinates of their current position and the coordinates of the task location. Here, students recognized the importance and attractiveness of utilizing a GPS-based mobile app as a navigation tool in the math trail activity. This is just one example task, and in general, the students' activities in this field experiment were similar. This explanation proves that the results of the student self-report instrument to determine their motivation to engage in activities coincided with actual conditions in the field.

Based on these findings, we conclude that the design and arrangement of the math trail and the mobile app as well as a combination of both have been successful in creating a pleasant situation and attractive environment offering valuable knowledge and experience in mathematics. They embody the aspects of enjoying or being interested in the activity and the use of advanced mobile technology for learning mathematics (an indicator of intrinsic motivation) and of value and meaningfulness of the mathematical tasks and the activity (an indicator of identified regulation), which were generated through the implementation of this project.

Conclusions

In conclusion, our findings indicate that student motivation to engage in the math trails program supported by the use of a mobile app was complex. Both intrinsic and extrinsic types of motivation, as well as amotivation, were found in the reasons for completing the activity. However, we also found that students reported and demonstrated more intrinsically motivating rather than extrinsically motivating and amotivating factors for engaging in the activity. While intrinsic motivation was an essential part, identified regulation was also important. The design of the project and its technical implementation contributed to these results, as reported by the student through the self-report instrument, and it was demonstrated through their activities and work. Therefore, in the implementation of the MathCityMap-Project, we must be aware of the important role of influencing student motivation when designing a mobile app-supported math trail activity. The relevance and value of the task must be clearly identified and linked to the objective of the project. Most importantly, students must enjoy and be attracted to the activity, both in completing the math trail task and in using the mobile app. These are the main factors that need to be considered when implementing the MathCityMap-Project.

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Mathematical gestures: Multitouch technology and the indexical trace

Elizabeth de Freitas[1] and Nathalie Sinclair[2]

[1] Manchester Metropolitan University, Manchester, UK; L.de-freitas@mmu.ac.uk

[2] Simon Fraser University, Canada; nathsinc@sfu.ca

This paper is about the threshold between gesture and touch in mathematical activity, focusing on the role of multitouch technology. Drawing on the work of gesture theorist Jürgen Streek, we propose and discuss the notion of the tangible gesture, in the context of mathematical explorations of young children with a novel, multitouch iPad environment called TouchCounts *designed to promote counting on and with the fingers.*

Keywords: Technology, multitouch, number and operations, gesture.

Introduction

With the advent of multitouch technologies, gestures have become an essential feature of user interface. Touch technologies break with previous computer-based norms, where the hand's actions were *indirectly* related to changes on the screen through mouse and keyboard manipulation. Touch technology invokes earlier drawing technologies in which the hand's actions had a more *direct* relation with a given surface. The digital nature of contemporary surfaces, however, significantly alters the relation between touching and meaning-making.

In this paper, we draw attention to the distinctive gestures new media elicit and produce and the way these new manual activities are changing the way we perform mathematics. We also interrogate the taken-for-granted distinction between the touchscreen gestures common in the technology world and the in-the-air gestures that have been the focus of study in mathematics education research. At first blush, they may seem quite distinct—albeit having the same word—but we follow Streeck (2009) in seeing them as being on a continuum, a perspective that enables us to better appreciate the role that touchscreen gestures might have in mathematics learning.

In the context of mathematics education research, many studies have focused on student and teacher use of gesture in classrooms, but this work tends to code and sort gestures insofar as they are *representations* of thinking. These studies tend to divorce the motoric hand from the feeling swipes and swishes of fingers on screens. Material semiotic approaches to the study of interaction, on the other hand, consider gesture less as representations and more in terms of the material effects they achieve (Roth, 2001; Radford, 2002; Nemirovsky, Kelton & Rhodehamel, 2012). Our goal in this paper is to unpack the implications for understanding mathematics learning in relation to new media. In order to illustrate how these new media gestures operate as expressions of numeracy, we draw on research involving a novel multitouch App in which fingers and gestures are used to count (Jackiw & Sinclair, 2014).

Gesture as movement

The predominant line of research in gesture studies focuses on movements of the body (especially the hand) and their interactions (i.e. correlations) with speech in communication (Kendon, 2000, 2004; Kita, 2003; McNeill, 1992; 2005). Researchers have identified different categories of gestures

(icon, metaphoric, deictic and beat) so as to distinguish different relations between gesture and speech. McNeill has drawn on Peirce's (1932) semiotics in which signs (icons, symbols, and indices) differ in terms of the nature of the relationships between the signifying sign and the signified. These categories have been used extensively in mathematics education research.

Research that codes gesture only in terms of linguistic potential tends to overlook the physicality of the hand movement, except insofar as such movement contributes to or obscures linguistic meaning (Rossini, 2012). As Streeck (2009) indicates, "it is common to treat gesture as a medium of expression, which meets both informational and pragmatic or social-interactional needs, but whose "manuality" is accidental and irrelevant" (p. 39). Streeck (2009) defines gesture:

> … not as a code or symbolic system or (part of) language, but as a constantly evolving set of largely improvised, heterogeneous, partly conventional, partly idiosyncratic, and partly culture-specific, partly universal practices of using the hands to produce situated understandings. (p.5)

Thus he studies gesture for how it is "communicative action of the hands" with emphasis on the term *action* (p.4). This focus on action allows Streeck to study gesture for how it couples with and intervenes in the material world in non-representational ways. Researchers often distinguish between hand movements *in the air* and hand movements *that make graphic marks*, where the former is deemed a gesture and the latter an act of inscription. However, such distinctions become fuzzy when we study the movement of the hand across and through media, where 'media' can be more or less receptive of trace or mark. In other words, all hand movements traverse and incorporate media. We see a trace in certain media, and not in others, but since the logic of new media is to break with current conventions of perception, this distinction is provisional. New media allow for new kinds of traces. This insight allows for new ways of studying numeracy and multitouch technologies. In the next section, we discuss a case study of children working with such technologies, showing how this material encounter entails a very different concept of number *precisely because of the indexical aspect of gesture*.

The question of trace and inscription returns us to how the indexical is different from the iconic. The contiguity aspect of indexicality (the smoke is materially caused and coupled with the fire) is aligned with touch and the way our body connects with another through touch. This approach to gesture supports the systems approach to bodies by which bodies become coupled with the environment they inhabit (Maturana & Varela, 1987). Streeck studies gestures as part of a nested scaling approach to this system, beginning in the world that the "hand knows best", and then examining how gestures operate at greater distance or remove from that world (p. 58). The touch or haptic factor of hand movement is precisely how the fragile interface of inscription or trace is currently produced. Again, we emphasize that this production of a trace is contingent on current configurations of sensory perception and material media.

Most of the hand's features (digits, degrees of freedom of movement, fatty palms, flexure lines, papillary ridges) evolved to facilitate grasping (prehension), and thus the hand "became a 'compromise organ'" in serving multiple purposes (Streeck, 2009). Prehensile "postures" are formed as the hand reaches its target (in our case this will be a screen), during which a pre-conscious calibration of speed and collective force determines the particular movement of the digits, hands and arm. As the hand moves towards the target, there is a strong reliance on peripheral vision

rather than vision directed at the target (Streeck, 2009, p.47). The speed of the gesture reduces as the hand reaches its target. But the moment of contact entails the forming of a new assemblage, when the entire body of the gesturer links up with that which it touches. Thus, we are focusing on how gesture is a hand action that does more than identify or code particular aspects of an object.

The video data discussed below is part of a larger project exploring the power of touchscreen technologies in teaching and learning mathematics in early childhood. Several research studies have already been carried out concerning the way that children learn various concepts using *TouchCounts*, including ordinality (Sinclair & Coles, 2015), place value (Coles & Sinclair, 2017) and finger gnosis (Sinclair & Pimm, 2015). The focus of this paper is less on the learning process using *TouchCounts* than on the various and distinctive forms of hand actions that are involved in creating and manipulating number in this environment.

What is distinctive about the index is that it is a sign that is materially linked or coupled to "its object". According to Peirce (1932), an index "refers to its object not so much because of any similarity or analogy with it, (…) as because it is in dynamical (including spatial) connection both with the individual object, on the one hand, and with the senses or memory of the person for whom it serves as a sign, on the other" (2.305). For instance, the chalk drawing of a parallelogram on a blackboard is often considered to be an iconic reference to a Platonic conception of parallelogram, but it is (also) an indexical sign that refers to the prior movement of the chalk. This latter indexical dimension is usually not emphasized in the semiotic study of mathematical meaning making, since we tend to focus on the completed trace and dislocate it from the labour that produced it. This focus on the completed sign neglects how the activity of the body and various other material encounters factor in mathematical activity.

TouchCounts: A multitouch early number App

In this paper, we discuss an application that author Sinclair has been involved in creating in which the digital gesture plays an even more central role in the mathematical activity. *TouchCounts* (Jackiw & Sinclair, 2014) is an application that permits young learners to coordinate simultaneously various forms of number: number names like 'three', number of taps on the screen, number of discs on the screen and number symbols like 3. It enacts a multimodal correspondence between finger touching, numeral seeing and number-word hearing (a one-to-one-to-one correspondence of touch, sight and sound). The App has two worlds: the Enumerating and the Operating worlds. In this paper, we focus on the Enumerating world, which is the one that children usually first experience.

In the Enumerating World, the screen starts almost blank, except for a horizontal bar called a shelf. In this world, a learner taps her fingers on the screen to summon numbered discs. The first tap produces a new yellow disc on which the numeral "1" appears. Subsequent taps produce sequentially higher numbered discs. As each tap summons a new numbered disc, *TouchCounts* audibly speaks the name of its numeral ("one," "two"). As long as the user's finger remains on the glass, it holds the numbered disc, but as soon as she "lets go" (by lifting her finger) virtual gravity makes the number object fall to and "off" the bottom of the screen. If the user releases her numbered disc above the shelf, or "flicks" it above the shelf on release, it falls only to the shelf, and comes to rest there, visibly and permanently on screen, rather than vanishing out of sight "below" (see https://www.youtube.com/watch?v=7xD-pqnsce0). Since each time a finger is placed on the

screen, a new numbered object is created, one cannot "catch" or reposition an existing numbered disc by retapping it. We note that, at least initially, the eye plays a prominent role in directing the finger above or below the shelf, but that if one does not care where the disc alights, the tapping of the finger needs little visual direction.

If we take the finger tap as a gesture involving the placement of a finger on the screen, and the subsequent production of an event featuring visual, mobile and aural aspects, then we might say that the gesture is iconic in its relation to the production of unitary quantities, or perhaps even metaphorical for the children for whom such unitary quantities are still "abstract". But what seems much more pertinent for the children as they engage with this application is the indexical nature of the gesture. The tap both points to the screen, designating one place of contact with it, but also creates a new numbered disc under the soft skin of the finger-pad, a disc which often falls with gravity-like weight. In addition, each tap produces a simultaneous sound. The children can also tap the Reset button, which makes all the numbered discs disappear and resets the count to 1.

While *TouchCounts* was designed to support the development of one-to-one correspondence between number and hand movement, by drawing on the tangible dimension of counting, its use by young children has prompted us to examine both the particular ways in which they use their hands and the implications of their hand actions on the meanings they make around counting, in particular the concepts of ordinality and cardinality (see Sinclair & Pimm, 2015).

A case study

This case study is drawn form a broader research project that was conducted in daycare and primary school settings over the course of three years. In the excerpt we present, co-author Sinclair was engaged in a clinical interview with a five-year-old kindergarten child named Katy, who is interacting with *TouchCounts* for the first time. (Indeed, it was the first time she was using a touchscreen tablet.) The interview occurs in June and therefore close to the end of the school year. We have chosen the excerpt because it illustrates a range of gestures that have been observed over the course of the research study, while also showing hand motions that have not been explicitly taught. In this case study, the hand actually operates very close to the surface of a screen: pointing to objects on the screen by tapping them; sliding objects along on the screen so as to leave visual and aural traces of the finger's path; pinching objects together in order to make new ones.

The room is quiet. Without prompting, Katy's hand approaches the screen, and her finger touches the top of it and slides down to the bottom. A yellow disc appears under her finger with the numeral '1' on it and the sound 'one' is made. The index finger moves back to the top of the screen, slowly swimming downwards. A chorus of 'two' comes both from her mouth and the iPad. This happens repeatedly, although sometimes only the iPad can be heard announcing the new numbered disc while Katy's lips move in synchrony (Figure 1a). The appearance of '10' on the tenth yellow disc attracts attention, perhaps because of its double digits, and Katy bends over to look closely.

Katy looks up again and her finger resumes touching the screen, but now only the iPad counts the numbers (Figure 1b): she no longer says them aloud herself. After 'seventeen', several fingers fall on the screen at once, and then 'twenty-one' is heard (since she has tapped the screen with several fingers, only the sound of the final number is said aloud, but the four discs all appear under where she has touched). This produces a pause in the action, and Katy's lips spread into a smile. All but

the index finger are tucked away, as the rhythmic tapping continues along with the chorus of named numbers. At 'twenty-seven' Katy looks up, no longer watching the screen (see Figure 1c), and she continues swiping and saying numbers. This continues until a finger accidentally land on Reset.

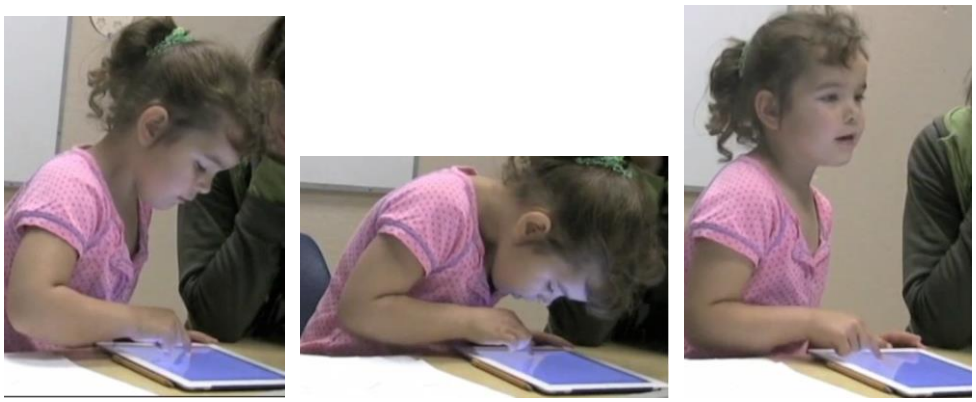

Figure 1(a). Katy swiping; (b) Following the yellow disc; (c) Tapping while looking up.

Katy's finger – as the main organ of touch in this encounter – takes on new capacities through the reset button. It is no longer the organ that can only move or drag the yellow circle. The power of the reset button to recalibrate the tempo and rhythm of this encounter, becomes part of the finger's potentiality, thereby redefining what is currently entailed in the sense of touch.

Discussion

Fingers can serve as both a physical extension of what Rotman (1987) calls the 'one-who-counts' (p. 27) (usually with an extended pointed finger reaching out to the world) as well as the thing-to-be-counted (in which the gaze is directed towards one's own fingers): fingers are thus simultaneously subject and object, both of the person and of the world (Alibali & diRosso, 1999). And this is what makes the finger actions of Katy so interesting; the mathematical act of counting with *TouchCounts* fuses this duality and in so doing changes the relationship between the hand and eye, as well as the ears.

Katy's hand actions change over the course of the episode, not only in the particular muscular form they take, first sliding down the screen as if lingering on the yellow discs to produce or partake in their falling off the screen, and then tapping impetuously so that each new touching of the screen follows the end of the sounds of the voiced numerical. The swiping gesture seems more exploratory while the tapping gesture seems to concatenate into a unit the touch-see-hear bundle of sensations involved in making a new disc-numeral-name. As Streeck writes, tapping is also "characteristic of ritualized behavior" (p. 76), which suggests that Katy has moved from exploration to practice. In both the swiping and the tapping, the finger can be seen as making an indexical gesture, with the trace being both visible and audible, not to mention tangible for Katy.

Although the initial movement and touch of her finger is what produces the disc, it is the disc that drives the swiping movement of her finger. Indeed, both her finger and her eyes *follow* the yellow disc as it heads down the screen. In shepherding the numbered disc off the screen, Katy sees when it's time to lift her finger and start making a new disc. But with the tapping, the eyes attend to the numerical sign on the disc—indeed, when 10 appears, Katy notices the change from the previous one-digit numerals. In this sense, the eye and the finger do very similar things in the swiping, the

visible trace is followed closely by Katy's eyes as the swiping takes place, so that the hand is subordinated to the watchful eye. With the tapping, the hand seems less subordinated, with the eye only interested when there's a novel situation. When Katy looks up, the hand is no longer subordinate. When Katy's eyes close, her fingers do the seeing and touching as they repetitively tap.

But of course, there is more than the eye and hand involved in this situation. The ear and voice feature importantly as well. Indeed, while the voice is subordinate to the touch (it only speaks while Katy taps), Katy's hand is also subordinate to the ear in that the ear judges the moment of the next tap. And the ear is disrupted by the hand, when several fingers touch the screen at once, causing the voice to jump from "seventeen" to "twenty-one." The eye, which was about to drift off, must return to survey the situation. And the hand returns to its single digit tapping. The importance of the aural and the vocal in this context is interesting in terms of the counting activity at play. Indeed, the ritual origins of counting are oral in nature, and counting with young children is often undertaken as the learning of a song that one memorises and chants. The involvement of the hand in this otherwise oral event provides a visual and tangible trace of the count, while also associating each counted number with a single swipe or a single tap.

One might question whether Katy's actions on the screen, which we might think of as touch-pointing, can really be thought of as gestures. Streeck argues that such touch-pointing gestures (and indeed all gestures) emerge from the touching and handling of things—the tracing (or other "data-gathering devices" such as caressing, probing, cupping) of objects that allows one to discover its texture and temperature (and, for young children, for instance, the difference between a cylinder and a pyramid). When the hand has done its exploring of the object, which fulfills an epistemic function in gathering information, it may then be lifted off the object and inclined to repeat the same movements 'in the air': "the hands' data-gathering methods are used as the basis for gestural communication" (p. 69). Streeck identifies such gestures as being communicative, which for him is the characteristic feature of a gesture. So perhaps Katy's touch-pointing becomes a gesture once she lifts her hand form the screen to do her tapping.

Distinguishing hand movements that explore from ones that communicate is problematic though. As Streeck writes, exploratory actions can become communicative when they are made visible to others, who may join in the action or infer tactile properties. If we look at Katy's swiping and tapping gestures, we might say that they are both exploratory, with the swiping gestures involving prolonged tactile contact that enables her to discover what would happen when her finger touches the glass—that a yellow disc would appear, with a numeral on it; that the disc would move down the screen; that the iPad would speak the numeral's name aloud, and that this could all be repeated as often as she wished. But Katy's swiping and, later, her tapping, are also communicative inasmuch as they tell *TouchCounts* what to do and say. The same might be said for clicks of the mouse or key presses of the keyboard, with the difference that the touchscreen is acted upon by direct hand motions. Instead of disentangling the tracing from the pointing (the exploration from the communication), we suggest that re-assembling them into an indexical enables us to see how Katy's hand movements can tap into the potentiality of the body by reconfiguring the relationships between sensations of touch, sight and sound that are at play. This potentiality mobilizes new mathematical meanings as Katy uses her fingers to count on, to count with and to count out one by one and indefinitely. Streeck recognises that hand-gestures "enable translations between the senses" (p. 70)

as tactile discoveries provide visual information for interlocutors. With Katy though, the tactile discoveries provide visual and auditory information to herself. She is her own interlocutor.

Conclusion

Streeck argues that hand-gestures cannot be taken only as components of a language system cast apart from the material world and used only to communicate *about* the world. Rather, they are *of* the world, and part of how we *feel* the world around us. This perspective requires us to see the moving hand as "environmentally coupled" (Goodwin, 2007), that is, as inextricable from the things it touches and engages with. But while Streeck implies a vector from the exploratory hand action to the communicative hand-gesture, our case studies reveal how the exploratory hand frees itself from the optic regime and invents meaning as much as it communicates it. This new kind of gesture is possible in large part because of the feedback mechanism of digital technologies, which can talk, push and show back. With the touchscreen interface, and particularly the multitouch actions, the hand is involved in a process of communicating that is also a process of inventing and interacting.

In the example we presented, we have shown that the gestures made by Katy in *TouchCounts* had an indexical nature both because they involved some kind of pointing (with one finger or more) and they left a trace that is both visible and audible. The trace is important in drawing attention to the material engagement of the gestures. The gestures arise out of movements of the hand, but they also result in material reconfigurations that can *give rise to* new movements of the hand. In discussing the effect of new digital technologies in mathematics, Rotman has written about the future cultural neoteny in which speech would "become reconfigured (as it was once before when transformed by alphabetic writing), re-mediated and transfigured into a more mobile, expressive, and affective apparatus by nascent gesturo-haptic recourses" (p. 49). We interpret these speculative comments as an indication that the future of the gesturing hand in relation to new media may involve all sorts of surprises, and that perhaps even pre-school children may count 'on their hands' to 100 as they engage with these media.

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Learning mathematics with technology. A review of recent CERME research

Ana Donevska-Todorova[1] and Jana Trgalová[2]

[1]Humboldt-University of Berlin, Germany; ana.todorova@hu-berlin.de

[2]University of Lyon, Claude Bernard University Lyon 1, S2HEP (EA4148), France; jana.trgalova@univ-lyon1.fr

This theoretical paper reports about our perception of the contributions that the working group TWG 16 about the learning of mathematics with digital media have made during the last CERME 9, taking into consideration the previous and the upcoming ERME conferences. Our analysis highlights the evolution of research questions, methodologies and theories through the lenses of the "didactical tetrahedron" metaphor and the networking strategies and methods. Finally, we point out themes that are, to our opinion, insufficiently addressed and need further discussions within the technology group.

Keywords: Mathematics, learning, technology, didactic tetrahedron, networking strategies.

Introduction and rationale

'State of the art' is a common expression used in surveys, review papers and up to date books reporting on the newest achievements in the research. This is also the ambition of the tenth Congress of the European Society for Research in Mathematics Education (CERME 10) TWG 16 leaders as they have announced:

> We want to establish an overview of the current state of the art in technology use in mathematics education, including both practice-oriented experiences and research-based evidence, as seen from an international perspective and with a focus on student learning [...] (Call for papers, CERME 10 TWG 16).

There are studies trying to establish such overviews (e.g., Drijvers et al., 2016), but also some claiming to report on the 'state of the art' research without sufficient argumentation and full justification of their statements throughout the text. The phrasal adjective 'state of the art' fits to advertise a 'product' but has our community become mature enough to respond to a challenge of offering 'state of the art' descriptions of complex phenomena like the use of technology in mathematics education which has a characteristic of enormous dynamism?

In this article, we do not claim that we have undertaken a meta-research beyond the scope of the CERME although we are aware of the variety of working groups on similar themes at other conferences as ICME, ICTMT, CADGME or ATCM and special issues of journals. Aiming to investigate how the CERME TWG 16 could capitalize knowledge of discussions regarding the learning with technologies, we have rather devoted ourselves to focus on two main issues: 1) *how have the research questions and methodologies about the learning of mathematics with technologies evolved* and 2) *is there a substantial progress regarding the use of the theories*. We begin discussing these two issues through the relations in a "didactic tetrahedron".

The "didactic tetrahedron" metaphor

The "didactic tetrahedron" metaphor (Fig. 1 right) was introduced by Tall (1986, p. 6) as an enlargement or adaptation of the "didactic triangle" (Fig. 1 left) commonly used before the advent of technology to analyze the teaching and learning of mathematical knowledge.

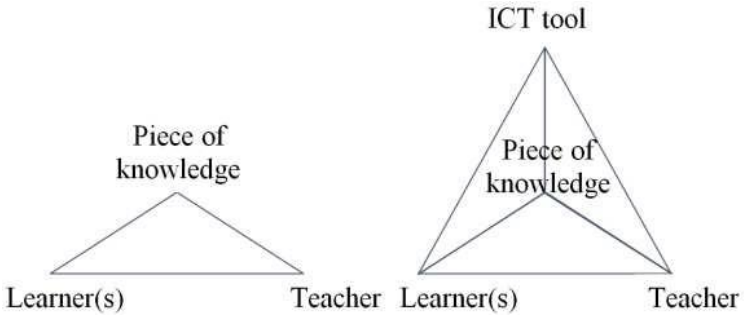

Figure 1: From a "didactic triangle" (left) to a "didactic tetrahedron" (right)

The integration of an artefact, e.g. an ICT tool, introduces a new component into the teaching/ learning system and creates new relationships between the components of the didactic triangle. Thus, for example the face ALK (A for Artefact or ICT, L for learner and K for Knowledge) represents phenomena related to learning mathematics with technology, such as students' conceptualizations of given mathematical concepts mediated by technology, or the edge AK highlights phenomena related to new approaches to given mathematical concepts offered by the affordances of a given digital artefact. The didactic tetrahedron has by now been used for analyzing mutual participation of artifacts and their users in a socio-cultural context (e.g. Rezat & Sträßer, 2012) or as a heuristic for studying the implementation of digital media in the teaching and learning praxis (Ruthven, 2012). In this paper, we use it to position the scopes of technology groups at the ERME conferences.

Until CERME 8, issues related to any vertex, edge or face fell within the range of a unique technology group, initially called "Tools and Technologies". The subsequent changes of the name into "Tools and technologies in mathematical didactics" from CERME 2 to 5 and "Technologies and resources in mathematics education" from CERME 6 to 8 indicate the appearance of enhanced specifications. The growing interest in the theme and the amount of research have led to splitting the technology group into two groups at CERME 9 which have progressed discussing topics focusing on edges and faces having "teacher" and "learner(s)" as a vertex, respectively.

Research method

In this paper we propose an analysis of the two issues 1) and 2) stated above based on the "didactic tetrahedron" through the: a) Calls for papers of the CERME 8-TWG 15, CERME 9-TWG 16 and CERME 10-TWG 16, b) Introductions to papers and posters of the groups published in the proceedings of the CERME 8 and 9, and c) Papers of these groups published in the proceedings of the CERME8 and 9. In this analysis we also refer to "networking strategies and methods" (Prediger et al., 2008, p. 170).

Findings and discussion

a) Evolutions tracked through the Calls for papers since the CERME8

The Call of the CERME 8-TWG 15 guided the discussions by posing three themes referring to design and uses of technologies, students' learning, and teacher professional development in presence of technologies. These three themes clearly refer to the three vertices of the triangular face "ALT (T for Teacher)" in the didactic tetrahedron (Figure 1). Although such structured shape for questioning the themes of interest may not appear straightforward by reading the text in the CERME 9 and 10 Calls, they are indeed meant to contribute to research related to the face "AKT" (TWG 15) and to the face "AKL" (TWG 16). Besides the split of the technology group in two groups, the relation between learning, teaching and digital tools is still present in the issues of the CERME 9-TWG 16 Call, as stated for example in the items "designs of teaching experiments with software and technologies concerning student learning" or "results of empirical studies and investigations especially concerning long-term learning with ICT, massive courses, national programmes of teachers' professional development". Thus, the face "ATL" remains relevant to both groups.

b) Evolutions tracked through the Introductions to the papers and posters of the CERME 8-TWG 15 and CERME 9-TWG 16

The Introduction of the CERME 8-TWG 15 corresponds to the Call and is structured according to the three themes (stated in a), i.e. the face "ATL". Moreover, it goes beyond the affirmed issues by raising a general one for "capitalization of research results" (Trgalová et al., 2013, p. 2500). This general issue has been addressed in an overview for mathematics, technology interventions and pedagogy based on systematic literature review by *Bray* (CERME 8, 2013) and in a survey reporting about undergraduate, master and doctoral studies for promoting the use of technologies in mathematics education by *Scheffer* (CERME 8, 2013). Further on, in this Introduction, it is claimed that a development of "specific methodologies enabling to assess the effectiveness of ICT in learning processes" (Trgalová et al., 2013, p. 2501) is required. The call for a "proper usage of research methods, which are informed by contemporary theories" (Lokar et al., 2015, p. 2438) is present in the Introduction of the CERME 9-TWG 16.

This paper builds on this claim and attempts to further investigate the usage of theories referring to the learning of mathematics in technology-rich environments in the next subsection.

c) Evolutions tracked through the Papers published in the proceedings for CERME 8-TWG 15 and CERME 9-TWG 16

Evolution of research questions (RQs) and methodologies

Unlike the frequent use of several methodologies and theories for exploring teaching (e.g. TPACK or instrumental approach), a large assortment of RQs and methodologies comes out from the papers regarding learning phenomena with technologies. We organize them in the following two categories:

- Category 1: RQs referring to at least two of the edges of the face "ALK"

While the most of the papers from this category discussed at CERME 8 focus on the impact of using technology on students' behavior, learning or performance, there is a greater variety of research issues addressed in papers at CERME 9. For example, the qualitative-empirical study by *Kaya, Akçakın, & Bulut* (CERME 8, 2013) related to the RQ: "does the use of Geogebra via interactive whiteboards as

an instructional tool affect students' academic achievement on transformational geometry?" (p. 2596) seems to meet all edges in this triangle. Likewise, a quasi-experimental study by *Kilic* (CERME 8, 2013) considers concepts in geometry (K), a development of geometric thinking and ability of proving in geometry (L) by using a Dynamic Geometry Software (A). Based on teaching experiments with high school students and prospective teachers, *Bairral and Arzarello* (CERME 9, 2015) have raised the RQ: "which domain (constructive or rational) of manipulation touch screen could be fruitful to improve student's strategies for justifying and proving?" (p. 2460). In this contribution, there is evidence not only of the three edges of the face "ALK" but also of the teaching component of the "didactic tetrahedron" by pointing out a lack of research about the teaching of mathematics with the use of touch screen devices besides task design concerns and cognitive implications (p. 2464-2465).

- Category 2: RQs referring to one of the edges of the face "ALK"

Exemplary studies addressing the edge "AL" are: a design based study by *Misfeldt* (CERME 8, 2013) about the students' instrumental genesis with GeoGebra board game, a study by *Persson* (CERME 8, 2013) grounded on students' interviews and teachers' questionnaires about instrumental and documentation genesis, or empirically based case study by *Storfossen* (CERME 8, 2013) about instrumented action of primary school students. It seems that the emphasis on RQs and methodologies studying instrumental genesis regarding the relation "AL" has slightly decreased from CERME 8 to CERME 9.

A paradigm which is noticeable in the CERME 9-TWG 16 papers and was not present before, except for one paper, is the online learning. Although the significant amount of RQs referring to learning through the Web (e.g., peer learning, collaborative learning, networking, flipped classroom) is visible (e.g., *Biton et al*., CERME 9, 2015; *Triantafyllou & Timcenko*, CERME 9, 2015), many specific questions related to the face "ALK" remain unanswered. For instance, what is the most relevant mathematical content available on the internet and how to locate it or what is a good quality of online teaching/ learning materials for mathematics and how to measure it. Another such question referring to the edge "AL", is about "students' perceptions if and how online resources contribute to mathematics learning and motivation" (*Triantafyllou & Timcenko*, ibid., p. 1573). The diverse nature and the complexity of these questions about online learning, in addition to the methodological approaches applied, mainly small scale studies or online surveys, do not allow generalizing conclusions about its truthful effects for the mathematics education.

Looking at the face "ALK" of the "didactic tetrahedron", an interesting question that could be worth exploring is whether a possession of a "(piece of) mathematical knowledge" leads to gaining an "other (piece of) knowledge" embedded in an ICT tool, e.g., knowledge in computer engineering. Except for one contribution by *Misfeldt & Ejsing-Duun* (CERME 9, 2015) about learning mathematics through programming and algorithms, we have not found others which would report on any kind of connections between learning mathematics and computer science or informatics. Neither have RQs about the learning of mathematics in relation to robotics, augmented reality and artificial intelligence been proposed in any of the calls, the introductions to papers or the papers in the technology group for the learning of mathematics at the CERME 8 and 9. This issue is neither mentioned in the CERME10-TWG16 Call, although we could expect that it may become an emerging one due to curricular changes in some European countries (e.g., France) highlighting algorithms in mathematics education.

Evolution of theoretical frameworks

Several observations can be drawn about theories and their networking in the papers.

First, the *instrumental approach* (Rabardel, 1995) appears as a widespread theoretical framework at CERME 8, while it is seldom mobilized at CERME 9. The hypothesis that may explain this fact is related to the shift in research questions reported above. However, in the terminology of "landscape of networking strategies and methods" (Prediger et al., 2008), it appears that the instrumental approach has been used for local organization and coordination, rarely combined with other theories. The heterogeneity of research questions at CEMRE 9 may be related to a greater diversity of ICT tools usage. Besides the commonly used technologies as dynamic geometry systems (DGS), computer algebra systems (CAS) or spreadsheets, innovative artefacts, such as multi-touch screen, Arbol software for developing combinatorics thinking or non-digital Fraction board, raise elderly and new concerns akin to those of tool affordances and multiple representations ("AK" edge of the didactic tetrahedron). Two main frameworks are called for exploring such questions: the t*heory of semiotic mediation* (Bartolini-Bussi & Mariotti, 2008) and the *approach of registers of semiotic representation* (Duval, 1993). These two theories seem to go along one with another and have a relatively high degree of integration founded on the strategies for understanding and making understandable, comparing and synthesizing (Prediger et al., 2008). Original digital devices, and possible novel teaching methods enabled by them (e.g., flipped classroom, learning on the Web) may lead to modifications of learners' perceptions of their efficiency or performance. These are explored through the Vygotskian perspective of object/meaning ratio.

Further observation leads to an assumption that there is a greater variety of theoretical frameworks used in CERME 9 compared to CERME 8 papers (Fig. 2). This seems to correspond to the previous argumentation. Besides the recognizable continuity of the usage of three theoretical frameworks, *instrumental approach*, *constructionism* and *learning by scientific abstraction*, there is a vivid occurrence of numerous others. Yet, "the multiplicity and isolated character of most theoretical frames used in technology enhanced learning in mathematics", brought to the fore by Artigue (2007) and considered by the author as "an obstacle to the exchange and mutualisation of knowledge" (p. 75), is still not overcome. The heterogeneity of the networking space may further be analyzed by using the flexible triple of principles, methodologies and paradigmatic questions (Radford, 2008).

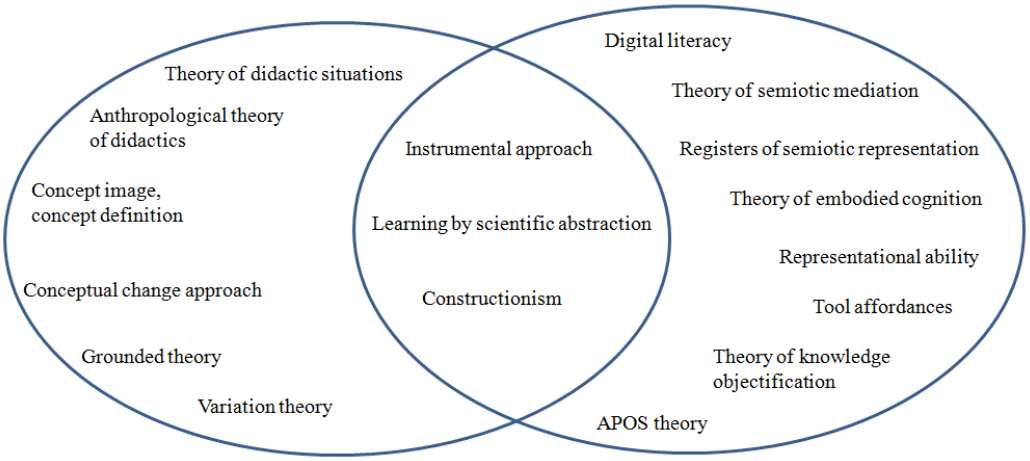

Figure 2: Theories used in paper at the CERME8-TWG15 (left) and CERME9-TWG16 (right)

It is worth noticing that most of the theoretical frameworks considered in the papers are not technology specific. In fact, the *instrumental approach*, *human-with-media concept* (Borba and Villareal, 2005) and the *theory of semiotic mediation* are rare frameworks addressing the interactions between learners and artifact(s), digital or not, besides those between learners and teachers. A widely used technology non-specific theoretical framework is the *theory of didactical situations* (Brousseau, 1997), which is occasionally combined or integrated locally with other theories.

Finally, we wish to draw attention to theoretical concepts that are not mentioned in the papers, although they are particularly relevant for addressing the relation "AK". Some of them, such as *computational transposition* (Balacheff, 1993) and *epistemological domain of validity* (Balacheff & Sutherland, 1994) are powerful means for ICT tool analysis in reference to a given field of knowledge and in terms of their possible contribution to the teaching and learning.

Conclusion

Looking through the lenses of the "didactic tetrahedron", the split of the CERME 8 technology group in two groups since the CERME 9 is not only a practical, organizational necessity due to the rapid growth of the number of scholars interested in the theme. It rather seems as a temporary solution to tackle and deeply investigate challenging questions about each of the faces of the tetrahedron before fabricating 'state of the art' reports.

Thinking about the capitalization of knowledge disseminated by the CERME 8-TWG 15 and the CERME 9-TWG 16 relating each of the two main issues in this survey paper, we may conclude the following.

1) *Evolution of RQs and methodologies*. Miscellaneous RQs are emerging rapidly, before the previous are being sufficiently explored. On the one hand, it seems that the trend of publishing findings about the influence of the World Wide Web including social networks and online educational platforms will continue in a relatively large amount despite an apparent lack of specific methodological and theoretical frameworks that could be commonly used to approach topical issues in the field of technology in mathematics education. Applied methodologies for approaching these questions belong within the frame of small scale qualitative empirical studies. On the other hand, research questions, appropriate methodologies and theories about attitudes, accomplishments and inclusion of specific groups of learners as low achieving, gifted and/or disabled students in technology supported learning environments remain urgent in the research agenda.

2) *Evolution of theories*. Is the use of current general theories like those referring to the "didactic triangle" sufficient or is there a need for a development of new ones, which would allow addressing issues specific to technology enhanced teaching and learning of mathematics? The latter seems to be more likely, as shown by a new item in the call for papers in the theory working group welcoming contributions on "theories for research in technology use in mathematics education" (CERME 10-TWG 17 Call for papers), which has not been part of the previous call of the group. Our analysis also shows that exploitation of the networking strategies and methods for understanding, comparing, contrasting, coordinating, combining, synthesizing and integrating theoretical frameworks (Prediger et al., 2008) may be beneficial for further truthful studies of the learning mathematics with technologies.

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Number, arithmetic, multiplicative thinking and coding

Krista Francis[1] and Brent Davis[2]

[1]University of Calgary, Werklund School of Education, Calgary, Canada; kfrancis@ucalgary.ca

[2]University of Calgary, Werklund School of Education, Calgary, Canada; brent.davis@ucalgary.ca

With frequent predictions of upcoming technological and economic difficulties triggered by an impending shortage of information and communications technologies (ICT) professionals, the calls are growing stronger to include coding as a core element of school curriculum. These calls are bolstered by the suggestion that coding supports the development of thinking skills – which echoes a longstanding argument for teaching mathematics. Motivated by the parallel, we attempted to investigate some of the common ground between learning to code and the development of core mathematical concepts. We photographed and video recorded children, aged 9–10, as they learned to build and program Lego Mindstorms™ EV3 robots over four days. Our findings suggest that programming supports children's understandings of decimal numbers and their transitions from additive to multiplicative thinking.

Keywords: Coding, robotics, arithmetic, number concepts, elementary education.

Introduction

In recent years there has been a growing recognition that information and communications technologies (ICT) are a major contributor to innovation and economic growth. For instance, the Organization for Economic Cooperation and Development (OECD, 2016) considers computer programming a necessity for a highly skilled labour force. Shortages are already felt across the world and demand for highly skilled ICT professionals is expected to rise. In our home country of Canada, for instance, there are predicted shortages of more than 150,000 skilled ICT workers in the next few years. This shortage is impacting IT innovations and revenues (see Arellano, 2015; Clendenin, 2014).

Canada is hardly unique on this count, as evidenced by major pushes around the world to include coding as a core part of school curriculum. In response, some educators and educational systems are shifting from teaching "how to use" software programs toward "how to code." Estonia and England, for example, have implemented a national curriculum that makes computer programming mandatory for all school-aged students across all grades, and other nations appear to be moving in this direction. For instance, it is currently a topic of political debate in Australia, where the opposition party is calling to have computer programming taught in every primary and secondary school in the country (Roumeliotis, 2015; Sterling, 2015).

In North America, national-level discussions and calls have yet to gather the same sort of momentum, but more and more initiatives are emerging at the local level. For example, the Chicago school district is adopting computer science as a core subject in all public high schools – prompted in large part by support from Google and Microsoft and through initiatives such as *Code.org* and *Hour of Code*, which are dedicated to expanding access to computer science for all U.S. students. Despite the absence of a national strategy in the U.S., messages on the importance of learning to code are frequent, with some emanating even from the President's office. In fact, coding skills have

been associated not only with empowering individuals and meeting employment needs, but with many aspects of the country's future and security (Pearce, 2013).

Trends toward including coding in school curriculum were preceded by a broadly effective worldwide push to get computers in schools. In 2011, most students (71%) in OECD countries reported having access to computers and the Internet at school. However, most students reported using the computers at school for email, browsing the Internet, word processing or doing individual homework. For the most part, such activities require low-level cognitive thinking and do not challenge students to develop more than basic user skills. Learning how to program a computer, it is typically argued, involves higher-level cognitive processes and provides opportunities for developing higher-level ICT skills.

These sorts of arguments for teaching computer coding parallel long-standing rationales for teaching mathematics. Similarly, many of the structures and strategies within coding bear strong resemblances to elements of mathematical concepts (Papert, 1980). We discuss a few of these resemblances in this paper, focusing on arithmetic.

Conceptual metaphors are one of the ways we understand mathematics (Núñez, 2000). With regard to the concept of number, Lakoff and Núñez (2000) describe "four fundamental metaphors of arithmetic": *arithmetic as object collection*, *arithmetic as object construction*, the *measuring stick metaphor*, and *arithmetic as object along a path*. The metaphor of *arithmetic as an object collection* is based on a one-one correspondence of numbers to physical objects. With this metaphor a greater size corresponds to a bigger number. For instance, 5 is *greater* than 2 because it forms a *bigger* collection. The metaphor of *arithmetic as object construction* is based on fitting objects/parts and arithmetic operations. For instance, 5 is greater than 2 because an object comprising 5 units is larger than one comprising two. The *measuring stick* metaphor maps numbers onto distances, whereby 5 is greater than 2 because it is *longer*. The metaphor of a*rithmetic as an object along the path* is based on arithmetic as motion, by which 5 is greater than 2 because it entails moving further from a common starting point (i.e., zero). Programing robots provides opportunities for illustrating and experiencing these arithmetic metaphors.

Context

In this interpretive study we asked what mathematics children learn by building and programming Lego Mindstorms^{TM} EV3. Interpretive research is about what meaning individuals construct in their lived experiences (Bhattacharya, 2008). We co-designed learning tasks with a graduate engineering student and co-taught the tasks with the classroom teachers over a course of four sequential days in three-hour daily sessions. The study's participants were 22 children, Grades 4-5 (aged 9–10), at Pakan School at Whitefish Lake 128 First Nation in rural Northern Alberta. Once the children knew the basic coding blocks for moving the robot, they were given Papert's (1980) task of programming a robot to follow a trace out of a triangle, square, pentagon or hexagon. On the third day, they were given the final challenge of building a robot that could find and douse a fire in any of four rooms in a building. Data included video-recordings, GoPro digital images, field notes, and artifacts including saved computer programs.

We video-recorded the four sessions to obtain rich contextual detail of children's mathematical interactions when programming the robots. Using interpretive video analysis (Knoblauch, 2013) we

selected videos and GoPro digital images that exemplified instances of children's mathematical thinking. Video data enables repeated viewing, slow motion, fast motion and frame-by-frame analysis. The selected videos formed the basis for emergent understandings of the children's experiences. The analysis developed through an iterative process of rereading the literature, reviewing the video and GoPro data, and rewriting. As is evident in our analysis, below, video data was vital. In particular, it permitted us to slow down the process and identify the integrated/nested processes of learning that occurred. The three instances that we use to focus our discussion were: (1) a trio of girls learning to program their robot for the final challenge to move a certain distance into the hallway to illustrate a developing understanding of number, (2) a boy tapping the vertices and sides of a triangle to count the number of programming steps necessary for the robot to move around the triangle as an example of additive thinking, and (3) a boy learning how the number of sides and angles of a polygon connects to the number of repeats in a loop, which illustrates a developing shift from additive to multiplicative thinking.

Findings

In the *numberline* video (see https://vimeo.com/144996708video), Krista was helping the pink team program their robot to move into the building. This action required manipulating one block of EV3 code to move the wheels a specified number of rotations. The team members started out with a guess of 0.4 rotations to move the robot into the first corridor of the building. After testing how far the robot moved and observing that the robot needed to move a considerably greater distance, Krista prompted the girls by asking what they should try next. Celina suggested they try 0.5. The small incremental change was still not enough, so Krista suggested they try 2. Two rotations moved the robot too far.

Krista: What is between 0.5 and 2?

Celina: 5.

Suspecting that Celina's response indicated that she and her teammates were unable to summon an appropriate interpretation of decimal numbers, Krista drew a simple number line on the whiteboard.

Krista: What is between 0.5 and 2?

Celina: Oh! 1.8.

The number Celina chose was close to the number of rotations actually required, which indicated she understood the meaning of 1.8. In the exchange above, we take Celina's immediate and satisfactory response to the repeated question as evidence that Krista was justified in her suspicion that the learners were lacking an appropriate interpretation for understanding decimal numbers – or, at least, were unable to extend whatever interpretations that had available to a situation in which distance was measured in wheel rotations. Coding the robot to move compelled the learners to elaborate their understandings. Invoking the number line appeared to provide an appropriate metaphor for helping Celina understand.

In the following sequence of images and descriptions, we summarize how the task of coding the robot to move into a room calls for all four of Lakoff and Núñez' (2000) representations of arithmetic. To begin, the metaphor of *arithmetic as an object collection* is used in most counting situations, whenever the forms being counting are perceived as discrete objects. It is by far the most

common interpretation of number through the task of assembling a robot, by simple virtue of the fact that the robots begin as large collections of separate items. Less obviously, it is also called for in coding moments as programmers translate complicated actions into discrete steps or instructions. And more obscurely too, such conceptual moves as the discretizing of wheel turns, so that they can be counted and thus used as a tool in programming, might be argued to rely on this metaphor. Figure 1 (left) presents an instance of this metaphor, showing that 2 turns is less (i.e., forms a smaller set than) 5 turns.

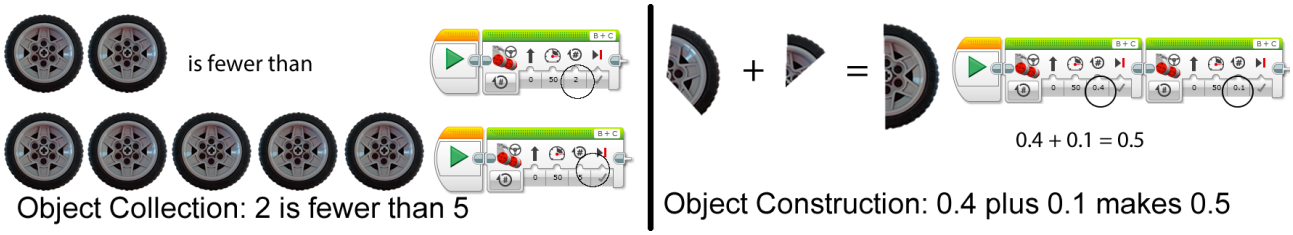

Figure 1: Arithmetic as object collection. Number of wheel rotations | Arithmetic as Object construction – combining portions of wheel rotations into single objects

Figure 1 (right) shows how the metaphor of *arithmetic as an object construction* might be encountered when programming a robot to move. Celina wanted a larger wheel rotation than 0.4, so she added an incremental amount of 0.1 wheel rotations to make 0.5 wheel rotations. Contrasted to the previous metaphor, in this instance, wheel turns are not perceived as discrete objects, but as parseable continuities. Those parsed elements can then be assembled into an appropriate object to move the robot a precise distance.

The *measuring stick* metaphor also featured prominently in the children's programming, and was particularly prominent in the frequent need to interpret wheel turns in terms of actual distances (e.g., when the phrase "1 wheel turn" was deployed not as a description of movement but was a reference to a distance of roughly 12 cm). Figure 2 (left) in reference to the instance in which the room of the hall was shorter than approximately 1.8 wheel turns. In this instance, programming the code block requires understanding measurement.

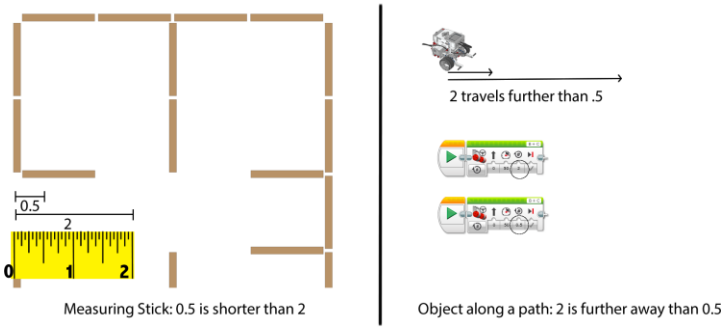

Figure 2: Measuring Stick: The length of hall | Arithmetic as an object along the path. The robot travels further with 2 than 0.5

Figure 2 (right) shows how programing the robot to move draws upon the metaphor of *arithmetic as on object along the path.* In this case, starting place becomes a critical element is that, for example, occurs when the robot enters the room and recurs in the opposite direction when the robot leaves.

To re-emphasize, we observed each of Lakoff and Núñez' four metaphors of arithmetic to be present in programming the robot to move a required distance in the room. The ability to identify to the particular metaphor(s) that a situation is calling for is a critically important teaching competence, as Krista demonstrated in the interaction with Celina. Re-interpreting that brief episode, Krista recognized that Celina was not interpreting number as a distance (i.e., she was not using a *measuring stick* metaphor), and thus reminded her of that metaphor by offering the image of a number line. No explanation other than an image of number that fitted the application at hand was required.

Arithmetic Topic 2 – Moving from "additive" thinking to "multiplicative" thinking.

The need for appropriate metaphors and images of number isn't sufficient for making sense of that entire episode, however, closer analysis reveals a further issue with the children's arithmetic, namely the tendency to default to additive actions rather when multiplicative actions would have been more suitable. That episode began with the group's realization that an entry of "0.4" moved the robot only a small portion of a desired distance. Asked what else they might try, they increased the distance only incrementally by 0.1 (to 0.5) rather than the necessary factor of (roughly) three.

This same tendency to default to additive actions when multiplicative action would have been more productive was witnessed many times across many groups over the four-day project. The *additive thinking* video (see https://vimeo.com/144820583) provides a window into any instance of the same phenomenon. In this case, Gene, who was on the floor in orange, is figuring out how many blocks of code were needed for the program. As he counted "one, two, three, four, five, six," he tapped each vertex and side of the yellow triangle, finally announcing that six steps are needed. Gene's step-by-step of the same two steps (straight, turn, straight, turn, straight, turn sequence) is an example of additive thinking – that is, of construing the situation in terms of a sequence of increments rather than a repetition.

Phrased in terms of coding, Gene opted to repeat the same line of instructions six times rather than employing a loop that ran six times. This happened in spite the fact that he and his group mates had learned how to use loops the day before when they programmed their robot to dance.

In fact, only one of the 8 groups in the class used a loop for the polygon task – suggesting that the move from additive/increment-based thinking to multiplicative/loop-based thinking is more conceptually demanding than is often assumed. The *additive to multiplicative thinking* video (see https://vimeo.com/144826969) further illustrates this point, as the classroom teacher along with Krista attempted to help Liam program with loops. Liam, on the left, identified that a pentagon has three sides. When asked to count the sides, he walked around the pentagon counting aloud and announced "5 times." Krista explains that 5 times is the number of times to repeat the two block codes (go straight and turn) in the loop. In response, Liam exclaimed excitedly, "Yes!"

In the same clip there are two boys who were fine-tuning their robot's program to follow a triangle. Their robot never stopped, which indicates that they are using an infinite loop – suggesting that they are making use of a concept of "repeating," but likely not a concept of multiplication. After three attempts at tracing out a triangle, they still hadn't crafted a program that would stop their robots.

Davis and Renert (2014) have identified a number of common instantiations for multiplication that are encountered in elementary school classrooms, including grouping, hopping, repeated sums,

stretching and compressing, array- and area-making, and making combinations. Looping, it seems, is another, distinct instantiation of multiplication that is particularly powerful in the activity of programming – in a manner, we suspect, that might be used reflexively to support mathematics learning. Figure 3 below, illustrates two programs for following a triangle. Additive thinking is found with the sequential accumulation of six programming blocks: move forward, turn, move forward, turn, move forward turn. Multiplicative thinking requires recognizing that the triangle can be traced by repeating the move forward and turn blocks three times in a loop. In the exchanges above, Liam appeared to be developing fluency with multiplicative thinking.

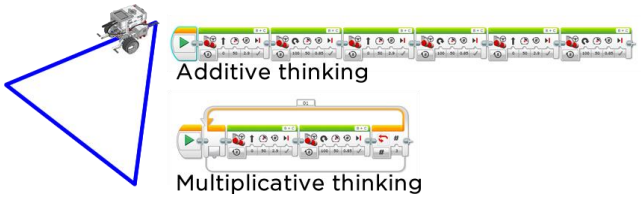

Figure 3: An additive and a multiplicative program for moving a robot in a triangle

Across the participants there was a pervasive tendency to program robots to trace out polygons as a sequence of same-steps rather than as a repetition of a set of steps (i.e., as enabled with a loop). This tendency was not easily interrupted through instruction, which provides evidence of the complexity of thinking multiplicatively. Even at the end of the four days, during the final challenge, only two of the teams had managed to appreciate the power of loops sufficiently to incorporate them into their programs. Not surprisingly, theirs were also the robots that performed the best. In one of these cases, the code for the winning robot (see https://vimeo.com/145404678) involved a loop determining if a fire is present, announcing "Yes" or "No" as appropriate, and activating an arm motion to dump retardant if "Yes."

Part of the reason that we dwell on this point is that the operation of multiplication is, arguably, the most important concept in grade-school mathematics. Multiplicative thinking is the cornerstone of proportional thinking, which is foundational to advanced mathematics for reasons that include the access it affords to an extended range of numbers (for example, larger whole numbers, decimals, common fractions, ratio and percent), its role in recognizing and solving a range of problems involving direct and indirect proportions, and the power it offers with its prominent place in school-based concepts and processes (Education and Training, 2013). In brief, multiplicative thinking is a key in the transition from early ideas to later ideas (see, e.g., ACME, 2011, p. 20).

Closing remarks

Our preliminary findings suggest programming robots can support learning mathematics. In the episodes reported, the tasks of programming robots required more than parsing complicated actions into singular direction; they entailed flexible engagement, Lakoff and Núñez' (2000) conceptual metaphors and mathematical models.

Computer programming aligns closely with concepts and structures in mathematics and we suspect that it might provide other powerful instantiations for mathematical concepts that have not yet been noticed. That suggestion is perhaps not surprising, given the mathematical roots of computer programming. However, to our reading, it is not an aspect of programming that has garnered much consideration in either mathematics education or the technology education literature. Considering

that mathematics literacy and competency with coding are of growing relevance, engagement with emergent technologies can complement and co-amplify mathematics learning, and contribute to evolving understandings of what "basic" mathematics might be for our era.

With regard to important complementarities between learning mathematics and learning to code, the Lego Mindstorms™ EV3 robots and the associated programming language provide a powerful instance of multiple solutions. They afford tremendous flexibility for accomplishing a range of tasks, from the trivial to the complex. None of the coding tasks set for the children in our study had pregiven or optimal "solutions." Despite that – or perhaps because of that – the children were able to engage in manners that they could recognize as successful, even when "complete" solutions were not reached. With incremental tasks and iterative refinements, children were able to learn more sophisticated and efficient methods for programming the robot. It is not difficult to imagine a mathematics class with similar standards of success.

That said, it is not a coincidence that the winning robot had the most efficient and sophisticated program of the group. Some answers are better than others, and those answers appear to reflect powerful mathematical thinking. Our future longitudinal research will investigate how children's understandings of mathematical concepts and programming robotics develop over several years.

We believe that the results of this study underscore the importance of developing and implementing a computer programming curriculum in schools. Coding is an emergent literacy that can amplify other critical literacies, while affording access to a diverse range of cultural capitals. The reasons to teach coding go beyond the technical and economic; for us, they are fundamentally ethical.

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Investigating the integration of a digital resource in the mathematics classroom: The case of a creative electronic book on reflection

Eirini Geraniou and Manolis Mavrikis

[1]UCL Institute of Education, University College London, UK; {e.geraniou, m.mavrikis}@ucl.ac.uk

This paper explores the potential impact of a full teacher-driven design and implementation cycle of an electronic book (c-book). We analyse data gathered from a school case study and identify the potential of the affordances of the c-book technology that allow the integration of various mathematical widgets and reflective activities. Our conjecture is that encouraging flexibility on playful tasks and reflection on 'bridging' activities early in the structure of the book prepared the students to more complex constructionist tasks around the concept of Reflection. Looking into the full cycle from design to evaluation this study demonstrates a successful integration of a digital resource in the mathematics classroom and highlights some of the successful components of the resource namely: playful activities for students, matched with carefully designed bridging activities, followed by constructionist activities that allow deeper exploration of the subject matter.

Keywords: Mathematical creativity, e-books, transformations, reflection, bridging activities.

Introduction

There is a lot of research and many projects that focus on developing digital resources for the teaching and learning of mathematics. Issues though regarding their successful use and integration in the mathematics classroom still remain (Clark-Wilson, Robutti & Sinclair, 2014; Geraniou & Mavrikis, 2015). One of the issues is whether and how students who may become experts in using a digital tool reflect and consolidate their mathematical knowledge (Geraniou & Mavrikis, 2015). Teachers then may not be convinced of the potential value of using digital tools in their mathematics lessons. In our view, a successful integration of such tools also involves the successful transition from interacting with a digital tool to a metacognitive understanding on behalf of the students that the interaction can support their knowledge 'outside' the tool.

Our work continues to focus on building 'bridges' to the maths involved (and may be 'hidden') in digital resources. We are looking into how we can encourage the consolidation of knowledge within digital tools and the 'transfer' of knowledge aiming at finding strategies to integrate them successfully into the classroom and the learning process. We define bridging activities as short tasks or questions that are used to intervene and encourage students to reflect upon mathematical concepts and problem-solving strategies they use throughout a sequence of activities (or simple interactions) with a digital tool. Such activities could take various arrangements from questions or prompts within the digital tool to paper-based worksheets or verbal teacher's interventions. In this paper, we focus on an electronic book resource and, particularly what the Mathematical Creativity (MC) Squared project (http://mc2-project.eu/) calls 'c-books', which are extended electronic 'creative' books that include widgets i.e. objects, other than text ranging from simple hyperlinks or videos to a broad range of interactive digital environments for mathematics such as GeoGebra and other microworlds (c.f. Kynigos, 2015). The project also includes an authorable intelligent support and data analytics engine that allows designers (e.g. teachers) to author the feedback that the system could provide to a student and the data they would like to see from their interaction (Karkalas & Mavrikis, 2016). The idea

behind the MC Squared project is to focus on social creativity in the design of digital media intended to enhance creativity in mathematical thinking (CMT). Researchers collaborating with math educators and teachers join Communities of Interest (COI) that work together to creatively think and design c-book resources reflecting 21st-century pedagogy for CMT.

The focus of the small study presented in this paper has been on designing a c-book including appropriate resources, such as bridging activities (Geraniou & Mavrikis, 2015) with the aim of enabling students to make connections to the mathematical concept the c-book is designed to teach them, in this case, Reflection[1]. We conjecture that designing resources that encourage flexibility on playful tasks and reflection early in the structure of the book prepared the students to more complex constructionist tasks. Looking into the full cycle from design to evaluation we demonstrate a successful integration of the c-book in a mathematics classroom and highlight some of its key components namely: playful activities for students, matched with carefully designed bridging activities, followed by constructionist activities that allow deeper exploration of the subject matter.

Theoretical framework

CMT has been given many definitions by various authors (e.g. El-Demerdash & Kortenkamp, 2009; Mann, 2005). In the MC Squared project, CMT has been drawn on Guilford's (1950) model of fluency (the ability to generate a number of solutions to a problem), flexibility (the ability to create different solutions), originality (the ability to generate new and unique solutions), and elaboration (the ability to redefine a problem). CMT has also been approached as a thinking 'process' that takes place in the context of a mathematical activity in order to produce a 'product' (e.g. a solution to a mathematical problem). As such the product and process are intertwined. For example, the construction of a geometric artefact is seen as a product that was started as a response to a task (problem), continued with the identification of a set of points, lines etc. that are underpinned by some properties that provide an answer to the task (product). Taking the above CMT's aspects as a starting point, we align our views to Papadopoulos et al.'s (2015; 2016) who consider CMT as the (i)'construction' of math ideas or objects, in accordance to constructionism that sees CMT being expressed through exploration, modification and creation of digital artefacts (Daskolia & Kynigos, 2012), (ii) Fluency (as many answers as possible) and Flexibility (different solutions/strategies for the same problem) and (iii) novelty/originality (new/unusual/unexpected ways of applying mathematical knowledge in posing and solving problems). Even though CMT seems to be at the core of mathematical thinking, its development through the use of exploratory and expressive digital media hasn't been thoroughly investigated (e.g. Healy & Kynigos, 2010) and the question about the best possible strategies for developing appropriate resources for integrating such digital media and promoting CMT inside and outside of the classroom remains.

[1] To distinguish between 'reflection' as a thought process and the mathematical concept 'Reflection', we will use capital letter 'R' for the mathematical concept.

Authoring c-books

As mentioned above, c-books are special electronic books that are designed within the Digital Mathematics Environment (DME)[2] which has been designed to allow teachers to create sequences of activities involving a number of widgets. It allows teachers to change the feedback messages students receive during their interactions with the c-book and stores all user interactions and scores. As part of a teacher training course, and based on our previous work, we encourage teachers to use DME's affordance to design bridging activities that promote students' reflective thinking on their interactions aligned with the various widgets. We expect these activities to 'bridge' the students' transition to the mathematical concepts, which the digital resource is designed to support (Geraniou & Mavrikis, 2015). These are questions presented and directly linked to the widget's tasks and can be viewed as interventions that encourage students' reflections on their interactions throughout a sequence of tasks, but also introduce and encourage the use of mathematical notation, not necessarily presented within the widgets. Authoring bridging activities within the digital medium of a c-book and recognising the potential value to students' learning progress and outcomes may encourage teachers to use such digital media more often.

The case of a c-book on reflection

The c-book on Reflection consists of a number of pages involving different tasks mostly in GeoGebra. This c-book (as opposed to others created in COI meetings, during which COI members brainstormed about ideas and activities that could be part of a c-book on a specific mathematical topic), was initially created by the class teacher in this study, who already had a number of prepared resources, which they put together using the affordances of the DME platform to form the c-book. These were resources like book chapters and GeoGebra worksheets. The c-book was also shared with the COI in an effort to gain constructive feedback and improve it.

The learning objective for the c-book was to remind students of the definition of Reflection, which had already been introduced about seven months before, define the Reflection ('mirror') line, consolidate students' prior knowledge and develop their understanding of the concept of Reflection. Even though the c-book technology allows a non-linear browsing of the c-book and students can work on any activity they want, this c-book was designed (and used) as a linear progression for constructing students' knowledge on Reflection by: (i) revising prior knowledge on Reflection through a series of multiple choice questions on certain reflected images where students had to decide which of the four images was the correct reflected image, (ii) revising and practicing on the GeoGebra widget (Figures 1A and 1B), (iii) challenging their understanding of Reflection through a competition task (Figures 1C and 1D) that promoted 'flexibility' in their solving approaches, (iv) challenging further their understanding of Reflection through a problem that challenged further their understanding and took them away from the standard style of questions such as 'Reflect this shape across the given Reflection line' (by not giving them the Reflection line, adding a constraint of the squared frame and giving them a story context to think about) (Figures 1E and 1F), and finally (v) a final assessment task mostly for those who finish faster aimed at recapping what students should know at the end of this c-book unit. We need to emphasize that all GeoGebra tasks were presented as bridging activities through the

[2] See http://ws.fisme.science.uu.nl/dwo/site/index_en.html and http://mc2-project.eu/

use of added text and reflective questions (see Figures 1A, 1C and 1E) on the side. These were designed as such to challenge students' thinking and understanding of Reflection and help them consider carefully their interactions rather than simply undertake the tasks. The feedback provided to students was of different types: (i) as a tick or cross for correct and incorrect responses, (ii) as a score for the GeoGebra competition task, which identified the number of correct Reflections students reached within the 5 minutes timeframe set by their teacher (Figure 1D) and (iii) as a written text to provoke their problem solving.

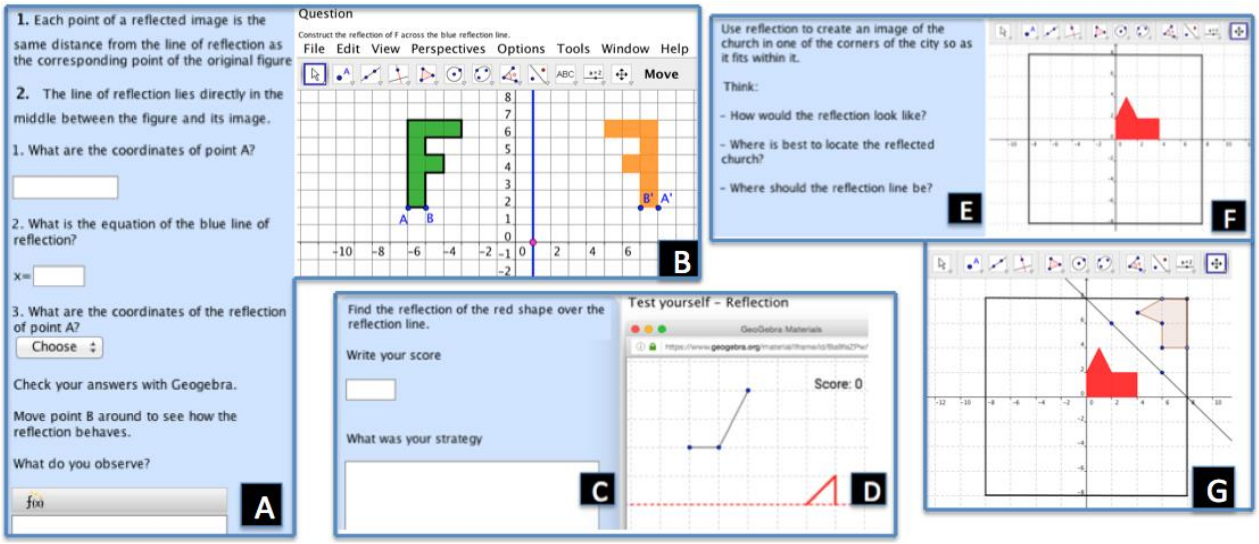

Figure 1: (A – F) Excerpts from the Reflection c-book and (G) a sample solution of (F)

Data collection

The aim of this case study was to explore the potential of both the Reflection digital book in the light of the affordances of the overall c-book technology i.e. beyond the ability to sequence activities, the potential for automated feedback and reflection that could be used to support bridging activities. The methodological tool used was that of a "design experiment" (Collins et al., 2004), that could act both as a way to 'engineer' and support the didactical situation and to systematically study it (Cobb et al., 2003). In this case, we, as a research team, collaborated with a teacher but left the decisions and responsibility of the classroom to the teacher.

Twenty-one 11-12 year old (Grade-7) students together with their class teacher and two researchers participated in the study, which was completed in two lessons in the school's computer lab. The students had been introduced to the concept of Reflection earlier in the year by working on some simple activities involving reflecting 2D shapes across the Reflection line. According to the teacher, the aim of these two sessions was to revise and consolidate their knowledge, but also to challenge their mathematical thinking against the concept of Reflection. The plan for the first lesson was (i) to remind them of what Reflection is and introduce the mathematical term of 'Reflection line' as opposed to 'mirror line' when they were first introduced to Reflection, (ii) introduce the c-book technology and (iii) allow students to familiarize themselves with GeoGebra through a challenging task, which acted as a bridging activity to recap prior knowledge. It involved working on some bridging activities, which included mathematical questions (such as 'find the coordinates') and reflective questions (such as 'what was your strategy?') within the platform. At the end of the first lesson, most students had reached the 'Church Challenge' task (see Figure 1E and 1F). During the

second lesson, students continued to work on the 'Church Challenge' and then answered a questionnaire to evaluate the c-book.

In addition, at the end of the second lesson, they were given a questionnaire to share their feedback on their learning experience with the Reflection c-book. The questionnaire was a Likert multiple-choice questionnaire consisting of questions such as: (1) How satisfied were you after completing the c-book activities?, (2) How easy to use do you think the c-book is?, (3) How free did you feel to experiment with the c-book and try out your ideas?, (4) I feel I understand Reflection now. Another two questions (5 and 6) gave them options to pick on their thoughts on the c-book and their preferred features. The questionnaire finished with three more questions to request suggestions from students (out of the scope of this paper).

Researchers took the role of 'participant observers' focusing on students' interactions with the digital medium and taking field notes. Besides working with the researchers and other COI members to design the Reflections c-book, the teachers' role was to offer assistance in technical issues when required during the two lessons and ensure that all students were on task and answered the bridging activities. Our data consists of the logged answers in DME and voice recordings as students elaborated on their interaction and answers. The data analysis was carried out by retrieving students' interactions with the c-book from the system and interpreting their responses against the CMT criteria presented earlier and by going through their answers on the questionnaire.

Results

The main outcome based on the data from the bridging activities, in particular, was that students were encouraged to reflect on the GeoGebra task from the start of their interactions. The teacher reminded students of the reflective questions (Figure 1A) and encouraged them to record their answers. The designed automated feedback supported all students to identify correctly the missing coordinates for the 'F' shape, its Reflected image and the equation of the Reflection line. In this first bridging activity, students were reminded of what Reflection is and the definition of the 'line of Reflection'. Both these terms were also introduced to the whole class and discussed with the class teacher at the start of the first lesson. But, we envisaged the repetition would give students a sense of familiarity and they would eventually start using mathematical terms in later tasks and would adopt mathematical ways of thinking. Fourteen (14) of the students provided sensible answers to the bridging question in relation to their strategy. Looking at students' responses to the bridging activity questions for the first couple of GeoGebra tasks, students were mostly using informal terminology:

Student: we have to flip the shape.

Student: count how many down from the mirror line.

But, in later bridging activities questions, students started to use mathematical terms, such as "the reflected church" or the "reflection line". For the question on what they notice when they move the 'F' shape, their responses were rather superficial:

Student: if you move the green shape, the orange shape moves with it.

They seemed to have noticed that the two shapes (green and orange 'F') are linked, but only 2 were able to articulate that they maintain the same distance from the Reflection line. Retrospectively, observing the students talking about their strategies, it might have been better to include some explicit

scaffolding questions here such as “What is the distance from the ‘F’ shape to the reflection line?”, “What do you notice?” etc. These could be followed up by the teacher to clarify what reflection is and how the reflected images are defined.

The bridging activities questions revealed students’ solving strategies and consequently their CMT. For the Competition task, students claimed to use three different strategies: (i) counting boxes across and down, (ii) tilt their head so that the reflection line becomes vertical and (iii) imagine using tracing paper on the screen. In this way, students demonstrated not only that they can come up with some original (for them) solutions but that they can also provide elaborate reflections on their strategies, which is linked to the originality/novelty CMT criteria described earlier. In retrospect, the c-book could have been designed to ask students for different strategies after they come up with one to challenge further their CMT in terms of the fluency and flexibility criteria.

Asking students about their strategy seems to promote reflection on their actions that helped them reach a solution. In particular, the Church Challenge (Figures 1E and 1F) posed a problem that ignited students’ thinking ‘process’ and resulted in a ‘product’, i.e. the reflected church image. In all the previous activities, students were given the Reflection line and their aim was to reflect a given shape. On the contrary with the Church challenge, students had to find the Reflection line and reflect the church image within the square town (see Figure 1G). By writing down their strategy, they recognised the solution ‘steps’ they took, questioned their actions and corrected them when needed. This open-ended problem allowed for exploration, construction of mathematical ideas and flexibility, which are all aspects we used to define CMT earlier (e.g. Papadopoulos et al., 2016).

Sixteen students (16/21 or 76%) managed to complete the task, whereas the rest ran out of time in the lesson. 10 of those got a correct answer. To reach the solution or the ‘product’, students produced creative solving strategies, which they were asked to justify. These strategies involved imagining a tracing paper used on the screen to reflect the church (14% or 3/21), which could be considered original in this context; trial and error technique by reflecting the church in all 4 quadrants and then thinking about reflecting each image within a quadrant to the corner of that quadrant to see which one fits within the square town (33% or 7/21); or another trial and error technique by constructing different Reflection lines and reflecting the church in one or more quadrants (52% or 11/21, see Figure 1G). These two latter strategies demonstrate students’ flexibility through the CMT criteria lens.

As far as the questionnaire is concerned, we are mostly interested in this paper on question #4 where most of the students (85% or 18/21) responded with an answer above 4 in the Likert scale. In relation to their thoughts on the c-books about 60% (13/21) answered that it helped them see the idea of reflection in different ways. This is really encouraging as one of our objectives was indeed to help students expand their understanding. About 43% (9/21) said that it included problems that they would not have tried to solve. This is also interesting as we want to encourage students to appreciate their mathematical abilities. In the open-ended questions, most students complimented the affordances of the c-book by commenting on enjoying the free explorations, testing of their ideas, experimenting, working on new questions and being challenged. While some students had comments for aesthetic improvements (fonts, games, colours etc.) three (3) students made comments that showed that they appreciate the advantages of digital technologies:

Student: the digital book help[s] because you could have actually test[ed] out your ideas and improve if it's wrong or not.

They recognised the dynamicity of such resources and how seeing the immediate feedback on their actions helps them validate their solution. At the end of the two lessons, the teacher also shared his reflections with the researchers and later with the COI. The teacher was impressed with how students were so engaged with the c-book, compared to past lessons without any digital resource and commented on the value of bridging activities and shared ideas on how to improve them.

Conclusion

This paper provides a good indication of the value of having a digital medium that combines free exploration, but encourages students to reflect upon their actions and make a link between their interaction in a digital environment and their mathematics through bridging activities. Such activities focus on mathematical terms, the definition of concepts, but also the justification for their solutions, throughout their work and 'bridge' the actions to solving a problem in the digital tool to the underlying mathematics (which could otherwise be 'lost').

Authoring activities using various widgets, designing Bridging Activities and in general, participating in the creation of the Reflection c-book re-enforced the teacher's keenness to continue to use digital technologies in their classroom. As a result of this study, the teacher and the COI revisited the c-book that led to further improvements in the book. The most notable of those was breaking down the bridging questions to smaller questions with guidance, and using the feedback affordances to encourage flexibility in terms of the strategies, as an aspect of CMT.

To conclude, this case study demonstrates how the c-book technology can be integrated into the mathematics classroom and promote a positive learning experience through the use of playful activities for students, matched with carefully designed bridging activities, followed by constructionist activities that allow deeper exploration of the subject matter.

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Preliminary testing of apps in mathematics, based on international collaborative work

Øistein Gjøvik[1] and Iveta Kohanová[2]

[1]NTNU, Faculty of Teacher and Interpreter Education, Trondheim, Norway;
oistein.gjovik@ntnu.no

[2]NTNU, Department of Teacher Education, Trondheim, Norway;
iveta.kohanova@ntnu.no

In this paper, we report on a project about developing mobile applications for learning mathematics through game playing. Several different types of applications were developed in a collaboration between universities in Norway and Slovakia, and between teacher education and information science. We give some preliminary results on how two of these applications were received and used by Slovakian pupils.

Keywords: Mobile learning, game based apps, mathematics learning, collaborative work.

Introduction

Mobile devices, such as smartphones, tablets and laptops, have become an integral part of our lives. Teachers and pupils use them daily for communication, searching for information or for entertainment. Pupils today, born from 1990 to 2010 and recognized as generation Z, are the most technologically advanced generation, often known as digital natives. They were born into the era of the Internet and Facebook; they always want to stay connected with their friends and to use high-speed digital devices (Baker & Evans, 2016). Hence, the wide spread of mobile devices causes a natural social pressure and challenge for educators to include these devices into education, to support learning. Computers, laptops, and netbooks have all been added to classroom settings with the hopes of revolutionizing education, promising vast improvements to pupil outcomes. These technologies, largely, have left education unchanged and in a continual state of need for improvement (McQuiggan, Kosturko, McQuiggan & Sabourin, 2015). Mobile learning offers a novel approach to reach current pupils. By the term *mobile learning* we follow McQuiggan, Kosturko, McQuiggan and Sabourin (2015, p. 31).

> It is anywhere, anytime learning enabled by instant, on-demand access to a personalized world filled with the tools and resources we prefer for creating our own knowledge, satisfying our curiosities, collaborating with others, and cultivating experiences otherwise unattainable. Mobile learning implies adapting and building upon the latest advances in mobile technology, redefining the responsibilities of teachers and students, and blurring the lines between formal and informal learning.

Mobile learning offers flexibility in when learning takes place, personalization of content, and gives pupils experience with contemporary technology and relevant skills for the future. So unsurprisingly, mobile learning has been considered as the future of learning or as an integral part of any other form of educational process in the future (Trifonova, 2003).

In June 2016, gaming apps were the most popular apps based on availability, as about 23 % of all apps available in the Apple App Store fit in this category. The second most popular category was

Business (10.22 %), closely followed by the Education category (9.21 %) ('Most popular', 2016). Shuler (2012) has analyzed the Education category from Apple App store. In 2011, more than three quarters (77 %) of the top selling apps targeted preschool or elementary aged children. Early learning was by far the most popular subject/skill-set (47 %), followed by mathematics (13 %). Drigas and Pappas (2015) have analyzed the most representative studies of recent years (2002 - 2013), involving online and mobile applications and tools for mathematics as well as their effect in the educational process. The results of the studies revealed that online and mobile learning applications motivated pupils, making mathematics instruction more enjoyable and interactive than ordinary teaching practices. The analyzed applications were targeted towards one specific area of mathematics, like graphs and functions, arithmetic, algebra, geometry, problem solving or mathematical programming and they were available only in English or Spanish. In light of this, we see it as an important contribution to ongoing research into mathematics education to engage in projects that examine the process of developing applications for mobile technologies as well as studying the effects they could have on learning. Also, providing tools readily available for school teachers was an important factor for running the Apps in Math project, as detailed in the next section.

Design and implementation of the Apps in Math applications

The main goal of the Apps in Math project (AiM) was to develop 25 applications in 15 months for supporting teaching and learning mathematics in lower and upper secondary schools in Slovakia and Norway. In Norway, pupils have relatively good access to technology, compared with European countries. Almost 90 % of pupils use Internet in schools but the most common use is probably the computer and not mobile platforms. After school hours, as much as 94 % of all children aged 9-16 have access to a mobile phone, and 83 % have a smartphone. (Medietilsynet & Trygg bruk, 2014) Several schools have a policy of buying one laptop for each child in school. Most publishing houses have their own apps and games connecting to their textbooks, and there are usually many choices teachers can do regarding software for their pupils. Much is not translated into Norwegian, but this is generally not seen as a big difficulty.

In Slovakia not all pupils have their own smartphone or tablet; the further east one goes, the less pupils have their own mobile device (Michálková, 2016). In the primary and secondary schools – the typicality is to have three computer rooms per school, in which Informatics is mainly taught, so there is rarely room for mathematics lessons in these specialized classrooms. Pupils usually do not have their own PC. During 2013-2015, thanks to national project supported by EU funding, 22 000 tablets were given to Slovak schools, which usually meant set of 30 tablets per school. Pupils in one school are sharing those tablets; teachers bring them for lesson, at the end of the lesson pupils have to return them, because they will be used in other classrooms. In Google Play or App Store there are very few mathematical apps in Slovak language that are intended to be used in mathematics classes at lower or upper secondary schools. So there is a need for applications, which teachers could use in math classes and for different levels of schooling.

The applications (modules) developed within Apps in Math project focus on various mathematical topics that are part of Slovak or Norwegian curriculum for pupils aged 9-19. The development of modules went in coherence with Design based research (Wang & Hannafin, 2005) and its iterative cycles. The mathematics teacher educators from Trondheim and Bratislava have cooperated with academics and bachelor students of applied informatics at the Comenius University in Bratislava.

Slovak bachelor students in Applied Informatics have programmed the modules based on the specifications from mathematics teacher educators and master and PhD students, as part of their bachelor thesis in informatics. The modules were tested extensively within the local participating groups in Slovakia and in Norway, as well as with pupils in Slovak and Norwegian schools. Reflective analysis of problems and obstacles was done and changes were implemented after each testing. All modules are part of one framework application called Apps in Math and they are divided into five main categories: Numbers, Functions, Geometry, Chance and Logic. Apps in Math is available for Android and iOS[1] platform and in Slovak, Norwegian and English language. Ebner (2015) has divided applications into four categories: stand-alone learning apps, game-based learning apps, collaborative apps and learning analytics apps. Apps in Math has the characteristics of being game-based learning application. Diah, Ehsan and Ismail (2010) have introduced the framework for mobile educational games consisting of four important segments: Learning Theories, Mobile Learning Approach, Games Development Approach and Learning and Education Medium. Most of the modules in Apps in Math apply the constructivism as the learning theory and for the mobile learning approach the games use activity-based themes for informal and lifelong learning.

Case studies

This section describes two case studies (Study 1 and Study 2) that were conducted to evaluate the effectiveness of mobile learning with Apps in Math application in real-world settings, with lower and upper secondary school pupils. We have chosen the SAMR-model for a quick categorization of the modules, where digital technologies can be placed on a scale from just replacing already existing practicing to facilitate types of tasks that could not have been done without the digital tools (Hudson, 2014). Limited resources and limited time made it necessary to choose for evaluation those modules that were closest to being finished. The module *Lucky Hockey* is based on the classic learning game *Green Globs* (Dugdale, 1982), and several versions of this game has been implemented over the years. The pupils who play the game are going to shoot a hockey puck across an ice hockey arena in order to collect as many coins as possible. The coins are shattered around the play field, sometimes in a random manner, sometimes to provoke a particular shot. The pupil shoots by entering a function expression, using the touch screen controls to alter the parameters of the function (Figure 2). By playing this game pupils should understand what impact the parameters of the function have on a graph. Using the SAMR-model we can say that Lucky Hockey acts as a direct tool substitute, but that the functional improvement allows for a more dynamic and dual view of the representations of a linear graph and the corresponding expression. Hence we can say this app is an augmentation of traditional instruction.

[1] http://www.project-aim.eu/eng/download

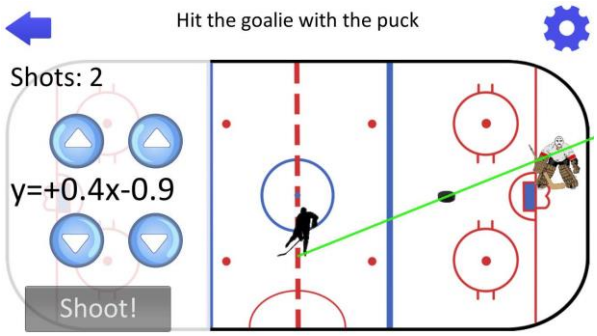

Figure 1: Learn mode of Lucky Hockey

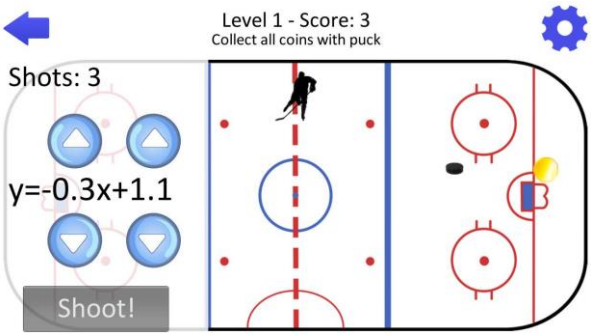

Figure 2: First level of Lucky Hockey

The module *House of cards* focus on arithmetic and geometric sequences in two separated submodules called *Arithmetricks* and *Geometricks*. By playing this game pupils should discover relations between the terms of the sequence and be able to write down basic formulae related to these relations. The number sequences are displayed on playing cards. Both submodules have a *Learn mode*, in which basic principles of the sequence are explained. The pupil has to determine the number, which is added/multiplied to/with each of the following sequence terms (Figure 3). The pupil has to answer five tasks correct within the time limit. After 3 incorrect attempts the correct answer is shown. In the next three levels the pupil should select the card, which belongs to the empty red spot in the given sequence within time limit (Figure 4). In the first level first 3 terms are given and the pupil should select the missing card for 4th and 5th term. Again, using the SAMR-model on Arithmetricks and Geometricks, we note that the effectiveness and readiness of the app makes work with sequences easier than in traditional teaching, or teaching done with real cards. Hence this app too provides an augmentation over traditional instruction.



Figure 3: Learn mode of Arithmetricks



Figure 4: Third level of Arithmetricks

The target group for the *Lucky Hockey* game study was Slovak pupils between the age 14 and 15 (grade 9), who had had no experience in linear functions yet. The goal of Study 1 was to determine which aspects of the linear function concept students seem to approach more effectively through the use of the Lucky Hockey game. Time limited gaming (25 minutes) was meant as an adidactical situation (Brousseau, 1997). The adidacticity was promoted by giving the students full responsibility for the technology-supported exploration of mathematical tasks by retroacting only with the milieu and not the teacher (Sollerval, de la Iglesia, 2015). All together 54 pupils from 2 different schools in Slovakia participated in Study 1 in November and December 2015.

The target group for the *House of Cards* game study was Slovak pupils between the age 16 and 17 (grade 11), who had not learned about sequences yet and had no previous knowledge about arithmetic

and geometric sequences. The goal of Study 2 was to determine which aspects of the arithmetic/geometric sequence concept students seem to approach more effectively through the use of the House of Cards game. All together 49 pupils from schools in Bratislava participated in Study 2 in March 2016. They first played the Arithmetricks game (starting with Learn mode and consequently going through all three levels) for 25 minutes. The next lesson (in the same day) they played the Geometricks game with the same conditions. During both Studies 1 and 2 all pupils used an iPad. No pre-test was conducted since pupils did not have any knowledge on these topics. The post-tests were used to determine the level of acquired knowledge. All pupils of Study 1 and 2 completed the post-tests as part of the evaluation, right after playing the game. The phase of institutionalization took place a few months after Studies 1 and 2, due to prescribed curriculum.

A preliminary study was conducted in September 2015 with 77 pupils of different age (7 - 16), in order to introduce them the early versions of five different games, including the Lucky Hockey game. At this stage, the game was more or less fully working, apart from minor graphical issues. Part of the group (about 20 pupils) tested the Lucky Hockey game. During the testing pupils thought (while playing the Learn mode – Figure 1), that the expression is always $y = 0x + b$, because they were able to hit the goalie only by changing parameter b. This was an obstacle in Level 1, so we had to refine Learn mode and control the possible movements of a shooting player. Most of the pupils liked the game and did grasp the notion of linear function. In the preliminary study we also asked all the pupils about their interest in using smartphones or tablets to learn mathematics in school. Figure 5 shows their answers. 92.3 % of pupils, who answered positively on this question, also said that they would like to play tested games at home. Out of them 46.7 % in the situation when they are bored, 28.3 % for practicing mathematics and 25 % when doing homework.

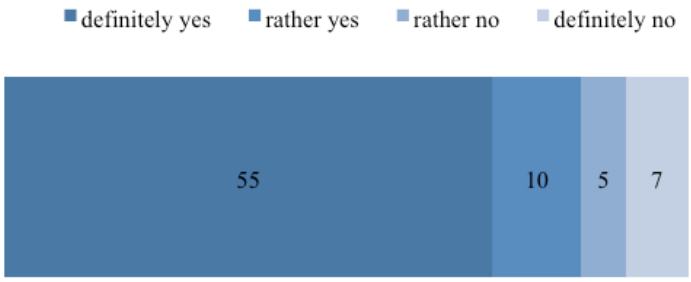

Figure 5: Interest of pupils to learn mathematics with mobile devices

Results

Figure 6 shows that pupils performed quite well in the post-test of Study 1. The average score was 5.11 and median score of 5. Pupils could obtain a maximum of 7 points, which were obtained by 13 pupils (24 %). Half of the pupils (50 %) scored 4-6.5 points, but there was also one pupil whose score was 0. The results indicate that most pupils learned the slope and intersection-aspects of the function concept on an acceptable level. The lowest score performance they had occurred in the last task, in which they were asked to explain what impact the parameters a and b in the expression $y = ax + b$ had on the corresponding linear graph. Only 46 % of pupils explained it correctly. Nevertheless, they performed better in tasks in which they were supposed to draw a line in correspondence with a given equation (76 %) or select the correct line/equation out of four possibilities that is corresponding to a given equation/line (86 %).

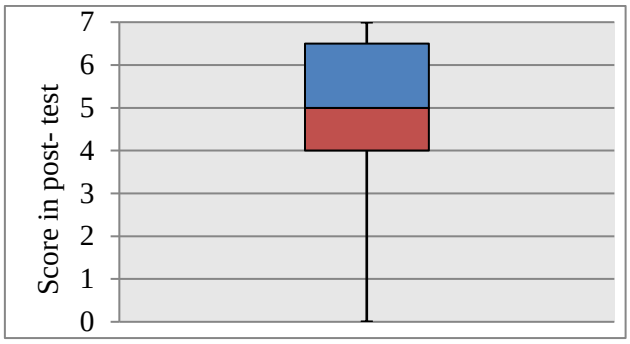

Figure 6: Box and Whisker Chart of Lucky Hockey Post-test Results

Figure 7 shows the results of pupils in the post-test of Study 2. It is clear that these pupils performed better in the Geometricks post-test. Here, 50 % of the pupils obtained 8-10 points, while 10 was the maximum. 18 pupils (36.7 %) obtained maximum score, and one pupil obtained the minimum score of 2. The second lowest score was 5, also obtained only by one pupil. The lowest performance was in the last task in which they had to write down the formula for how to find the 10th term, if they knew the quotient $\left(q = \frac{1}{3}\right)$ and the 1st term was given as a_1. Only 51 % of the pupils wrote the correct formula and explained their answer correctly. The most frequent error was made by 7 pupils (8.2 %), claiming that $a_{10} = a_1 . \frac{1}{3^{10}}$. In all the other tasks pupils were able to determine the unknown term, if they knew specified values of the 1st term and the quotient, or specified values of two consecutive or two nonconsecutive terms, with successfulness of 89 % - 100 %. The scores in the Arithmetricks post-test were slightly lower, with an average 6.82 and 6 as a median score. 8 pupils (16.3 %) obtained the maximum score and two pupils obtained the minimum score of 3. Distribution of scores within the box chart shows that approximately one quarter of the pupils obtained the same score, 6 points.

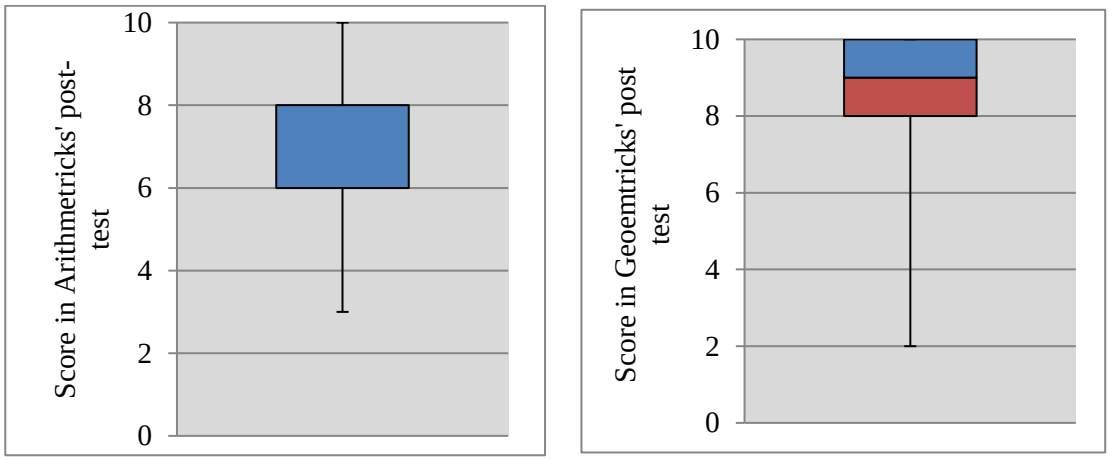

Figure 7: Box and Whisker Charts of Arithmetricks and Geometricks Post-tests Results

Discussion and conclusive remarks

We note from the results that most of the pupils did learn the important principles of linear functions or sequences at an acceptable level. However, only about 50 % of the pupils were able to answer the last questions correctly. This might have improved if the pupils were to play the games additional times. This hypothesis also arises from differences between how the pupils scored in Arithmetricks and Geometricks post-tests. The pupils did play the Geometricks game after playing the Arithmetricks game and since the principle is not very different, it could cause that they performed better in the

Geometricks post-test. As mentioned above, the phase of institutionalization took place 6 months after the Studies 1-2. While pupils who participated in Study 1 did not remember much about the linear function, it was different with pupils of Study 2. Pupils recalled the main principles of arithmetic/geometric sequence and told the teacher that it was not needed to explain it again: "It's like in that game we have played." From observation of the teacher we also note that traditional teaching of sequences went this time easier, probably also due to the mobile learning. The low performance on the last task of the Lucky Hockey post-test could be caused by the nature of the task. In all the previous questions we used numerals instead of parameters *a* and *b*, whereas on the last question some generalization and explanation were expected. While some pupils may have misunderstood the meaning of the parameters, some didn't give any reply at all or they only explained the role of one parameter. If the phase of institutionalization in form of, say, a discussion among pupils and a teacher took place right after the gaming activity, pupils' understanding of the parameters' role might have been better. The interest of pupils to learn mathematics with a mobile device was visible during testing both in Slovakia and in Norway. According to the results of the questionnaire it seems that most of the Slovak pupils would like to include mobile learning in their schooling. Testing of the other various applications from the project, not mentioned in this paper, also confirms that Slovak pupils and teachers consider mobile learning as a motivational way of learning and teaching mathematics (Michálková, 2016; Kapitulčinová, 2016). Mobile phone games in classroom is a novel idea and it might still cause the engagement of being a contemporary, "fresh" way of learning mathematics, which could be the reason of pupils' and teachers' enthusiasm.

The results of Study 1 and 2 suggest that mobile learning can be both motivational for pupils when learning mathematics, and helpful when acquiring new knowledge effectively. Gamification of education has also reached mathematics instruction but resources and research are just beginning to surface. Ideas from the project are being further developed at both participating institutions. Current issues can include utilizing the small touch screen sensibly and also collecting data from how and when pupils use the applications. The mobile phone is a tool we can expect to see more in mathematics education as learning becomes further individualized and online. One lesson learned from this project is the difficulty of communicating mathematical ideas from the idea stage to the actual implementation. This became quite apparent when collaborating with different countries, different levels of study and different study branches. Another lesson from the project is that it turned out to be much easier to develop ideas with a narrow mathematical theme, than to make applications that facilitates exploration and discovery.

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Khan Academy as a resource for pre-service teachers: A controlled study

James Gray[1], Christine Lindstrøm[1] and Kristian Vestli[1]

[1]Oslo and Akershus University College of Applied Sciences, Norway; James.Gray@hioa.no; Christine.Lindstrom@hioa.no; Kristian.Vestli@osloskolen.no

Khan Academy[1] (KA) is an online learning system of videos and exercises that is freely available and widely used. In this study, 131 students in a mathematics education class were split into two groups. Both groups followed normal instruction, but the treatment group was introduced to KA and given the opportunity to substitute their compulsory mathematics assignment with exercises in KA. This paper presents the results of students' performance on a mathematics pre- and post-test. The results show a statistically significant learning gain for both groups, but there were no statistically significant differences between the two groups on either test. This suggests that using the free and automated KA for self-study and assigned work was as *effective for students' learning as other standard resources. Student usage of KA beyond the compulsory exercises, however, did not correlate with results on the mathematics test, possibly due to the limited focus of the test.*

Keywords: Mathematics education, electronic learning, teacher education.

Introduction

Pre-service teachers in many countries struggle with mathematics. In Norway the TEDS-M study concluded that "a big problem in Norwegian teacher education is the poor academic skills of students in mathematics" (Grønmo & Onstad, 2012, p. 55, our translation). To address this challenge, the mathematics entry requirements for all Bachelor of Education students were increased from 2 to 3 (where 2 is the passing grade and 6 the highest grade) in 2005 (UFD, 2005), and increased again to 4 in 2016 (KD, 2014).

Fluency in school mathematics is essential for studying mathematics education. A consequence of pre-service teachers' weaknesses in mathematics is that class time has to be devoted to learning mathematics rather than mathematics education material. *Khan Academy* (*KA*) is one of many recent online resources offering structured sequences of videos and exercises. This paper reports on a first attempt to integrate KA as part of the mathematics instruction in a mathematics education class. More specifically, the research questions were: How do the learning gains of KA users compare to those in a control group? How much did the students use KA, and what were the associated learning gains?

Khan Academy

Khan Academy began as a collection of YouTube videos made by the founder Salman Khan to help his cousins with their schoolwork. These videos were later integrated into an online learning tool, which had 10 million unique users a month in 2014 (Murphy, Gallagher, Krumm, Mislevy, & Hafter, 2014). Beginning in 2010, the Bill and Melinda Gates Foundation and Google made a significant

[1] https://www.khanacademy.org/

investment in KA to develop new content and to translate it into other languages (Murphy et al., 2014).

One of the features of the tool is "missions", which are suggested sequences of videos, exercises and other materials. Learners can reach a level of "practiced" on an exercise by correctly answering 3–5 (depending on the exercise) questions correct in a row without using any hints. The level "mastered" is achieved by answering a mixed selection of questions a set time after the student has achieved the level "practiced". Gaming features, such as "badges" and "energy points", are designed to further incentivise completion of exercises and missions.

A KA user can also be a "coach" for other users, such as a class of students. A coach can see the time used by each learner, exercises practiced and mastered, and suggest other exercises, which then appear on the learners' KA home page.

Related research

There is a small but growing amount of research literature on use of videos for learning mathematics. These report that students see them as useful learning resources (Kay & Kletskin, 2012; Loch, Gill, & Croft, 2012; Loch, Jordan, Lowe, & Mestel, 2014; Wilson, 2013) and there is some indication that such videos can improve exam performance (Jordan, Loch, Lowe, Mestel, & Wilkins, 2012).

Wilson (2013) reports on the use of a flipped classroom approach with a university level statistics class, which resulted in increased student examination performance. KA was one of the resources used by Wilson to supply content to the students. A similar flipped classroom approach was employed by the second author in a physics course for pre-service science teachers (Lindstrøm, 2015). KA was found to have added value to the course based on the following: student compliance with using KA; positive student attitudes to KA; a learning gain measured using a pre-test–post-test design; and useful data in KA for the instructor to tailor teaching to the students' needs.

In California, (Murphy et al., 2014) conducted an implementation study using KA in nine schools. Schools were of varying type (public, charter and independent) and level (elementary, middle and high schools), and were located in areas with a spread of social-economic profiles. The amount of class time spent on KA varied, and KA was not used outside of school hours. The teachers who used KA reported positive outcomes for student engagement, and an increased capacity to meet the mathematical needs of all students. There was a positive relationship between KA use and test scores as well as students' attitudes towards mathematics.

In all of the studies mentioned above, the learning gains cannot be uniquely attributed to the online resources, because a control group was not used and there may have been other unreported factors that influenced the learning. This project is a first attempt at a controlled study of mathematics learning with KA.

Context

The requirements to qualify as a primary teacher in Norway are a four-year Bachelor of Primary Education or a relevant bachelor degree and a one-year Diploma of Education. The majority of primary teachers take the Bachelor of Primary Education. In this programme, students must take

courses in mathematics education equivalent to half a year of full time study, and have the option of taking additional courses in mathematics education to become a mathematics specialist teacher.

The students in this study were in their second year of the Bachelor of Primary Education. By the end this year, the students had completed the compulsory mathematics education requirement, which was spread evenly over the first two years. Teaching comprised of 22 sessions of 2 hours and 45 minutes over the course of the academic year with occasional breaks for study trips, thematic weeks and two placement periods (of two and four weeks duration). There were also four 2 hour and 45 minute plenary lectures for the whole year group.

Methodology

Four of the five parallel classes were included in the study, and two instructors each taught two classes. The first author held two of the four plenary lectures and taught the fifth class that was not included in the study, but was otherwise not involved with the instruction of the students. The other authors were not involved in the instruction of the students in any way.

One class from each of the two instructors was selected at random to be the KA group (the treatment group). There were 59 students in the KA group and 72 in the control group. In the third week of the first semester, the first author gave these two classes a short introduction to KA (10–15 minutes), which included showing how to set up an account, and an example of the videos and the exercises. The students were encouraged to get an account with the first author as coach. Only four students created an account in the first half of the semester, however, so the first author visited these classes a second time in the tenth week of the first semester to remind the students of how to set up an account. Throughout the first semester, the first author sent suggestions to the students for exercises related to the content in their mathematics education course both in KA and through the students' online learning management system, which was the main portal for communicating with the students. At the end of the semester there was still only four students with an account.

During the second semester, the students in the KA classes were given the option of completing their obligatory mathematics assignment in KA or as a written assignment. The KA assignment consisted of reaching the level "practiced" in the following KA exercises[2]: Recognizing fractions 2; Finding 1 on the number line; Equivalent fraction models; Naming the whole; Understanding multiplying fractions by fractions; Percentage word problems 1; Ordering fractions; Multiplying fractions by fractions word problems; and Converting multi-digit repeating decimals to fractions. The written assignment consisted of eight multi-part questions covering the same topics. For example two of the questions were:

> Write a number story for the following calculations and illustrate the last two:
> a) 13×0.8 b) $10{,}5 \div 0.3$ c) $\frac{1}{2} + \frac{1}{3}$ d) $\frac{1}{2} \times \frac{1}{3}$
>
> Convert to a fraction or a mixed number. Show your working.
> a) 0.375 b) $0.545454\ldots$ c) $1.88888\ldots$ d) $2.16666\ldots$ e) $0.461538\overline{461538}$

[2] KA is under constant development. These were the names of the exercises in the spring semester of 2015.

The assignment included instructions on how to set up an account (identical to that given in the first semester), and included the names of the KA exercises. The first author also sent the exercises as suggestions (three per week for three weeks corresponding to when the topics were covered in class). For the final submission, 42 students chose the KA assignment and 17 the written assignment. The control classes submitted the written assignment.

Progress was measured in all classes by a pre-test–post-test design using a 24-item mathematics test developed by the authors. The items were on mathematical topics associated with the second year mathematics education course, and all were within the scope of the grade 10 Norwegian mathematics curriculum (KD, 2013). The test contained: 11 items on fractions, decimals and percentages; 3 items on multiplication and measurement; 4 items on functions; and 6 items on algebra. There was an emphasis on fractions, decimals and percentages because that was the focus of the assignment. The authors wrote eight of the items and used published sources for the other items (Brekke, 1995; Brekke, Grønmo, & Rosén, 2000; Gjone, 1997; McIntosh, 2007; Utdanningsdirektoratet, 2011). Here are two examples of the questions on fractions:

Which of these fractions is half of the value of 3/8? A: 3/4 B: 6/4 C: 3/16 D: 6/16

Place in ascending order: 5/8 7/6 1/2 2/3 4/9

The pre-test was administered during the first teaching session of the first semester for each class. The same test was used for the post-test and was administered in the second semester during a session for the whole year group approximately one month before the final exam and after the compulsory mathematics assignment was submitted. The students had 30 minutes to complete the test on both occasions. On the cover page of the post-test, there were four brief questions about the students' use of KA, including an estimate of how many hours the student had used KA during the academic year. This information served as a check on the data collected from KA, and to see if anyone in the control group had used KA. The students filled out this information before the 30-minute testing period began. The first author marked the pre-test and post-test according to a marking key written by all the authors. Every item was allotted 2 points, so there was a maximum possible score of 48.

Matched pre- and post-test data were available for 51 students in the KA group and 58 students in the control group. Of the 51 students in the KA group, six students did not register any activity in KA or report using KA on the post-test cover sheet and were thus omitted from the analysis. Of the 58 students in the control group, two students reported on the post-test cover sheet that they *had* used KA during the trial period, and were also omitted.

The student data from KA on time usage was inconsistent (e.g. some students had completed many exercises but had a time usage of 0 minutes) and was thus discounted. On the post-test cover-sheet, not all of the students gave an estimate of their KA usage. In the KA group, those who did, reported an average of 4.4 hours total usage (SD = 3.6; N = 45). Historical self-reporting of work time is very unreliable (see e.g. Chambers (1992), so this estimate is only a very rough indication.

Results

The average score on the pre-test for the KA group was 24.1 (SD = 8.3; N = 45) and for the control group 25.9 (SD = 7.2; N = 56). This difference was not statistically significant ($t(99) = 1.17$; p = 0.246). The average score on the post-test for the KA group was 28.8 (SD = 8.1; N = 45) and for the

control group 31.4 (*SD* = 7.8; *N* = 56). Again, the difference was not statistically significant ($t(99) = 1.63$; $p = 0.107$). However, the gain for both groups *was* statistically significant: the gain for the KA group was 4.7 ($t(44) = 5.86$; $p = 0.000$) and for the control group was 5.5 ($t(55) = 7.74$; $p = 0.000$). Corresponding results were obtained when just the items on fractions, decimals and percentages were analyzed: there was a statistically significant improvement for both groups, but the difference between the groups was not statistically significant on either the pre-test or the post-test.

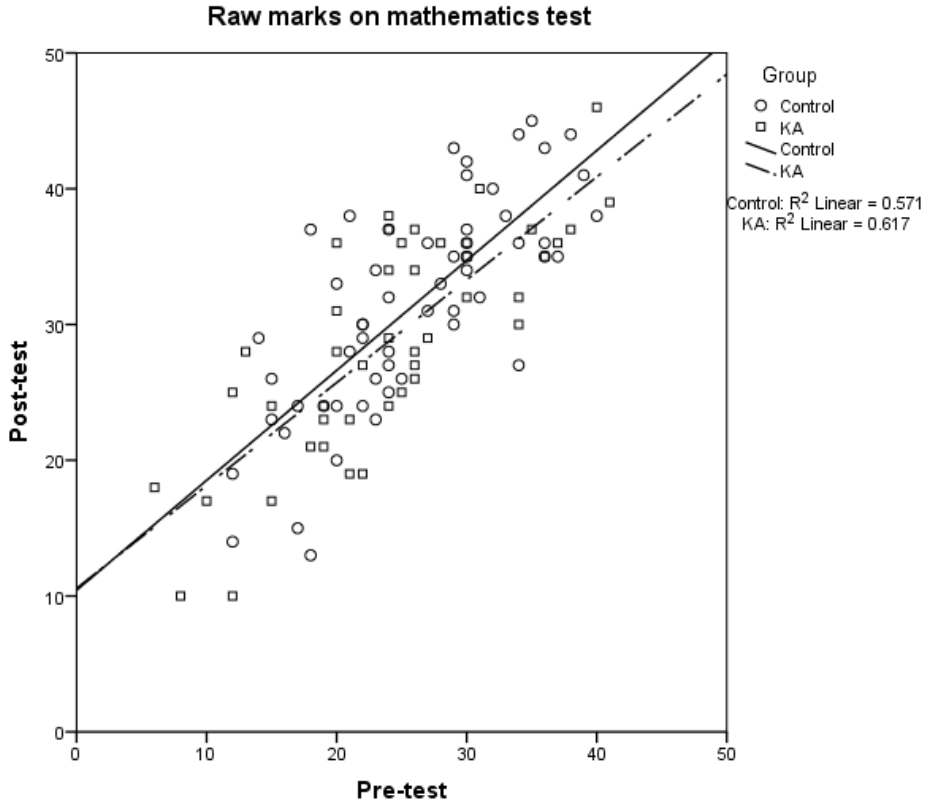

Figure 1: Post-test vs. pre-test results for KA and control groups

Analyzing the post-test versus pre-test scores, the linear regression lines for the two groups show similar trends (Figure 1). Again, corresponding results were obtained when performing the analysis using only the items on fractions, decimals and percentages.

"Improvement" refers to be the pointwise improvement on the mathematics test from pre to post. When compared with the pre-test results (Figure 2), there is no discernable difference between the two groups.

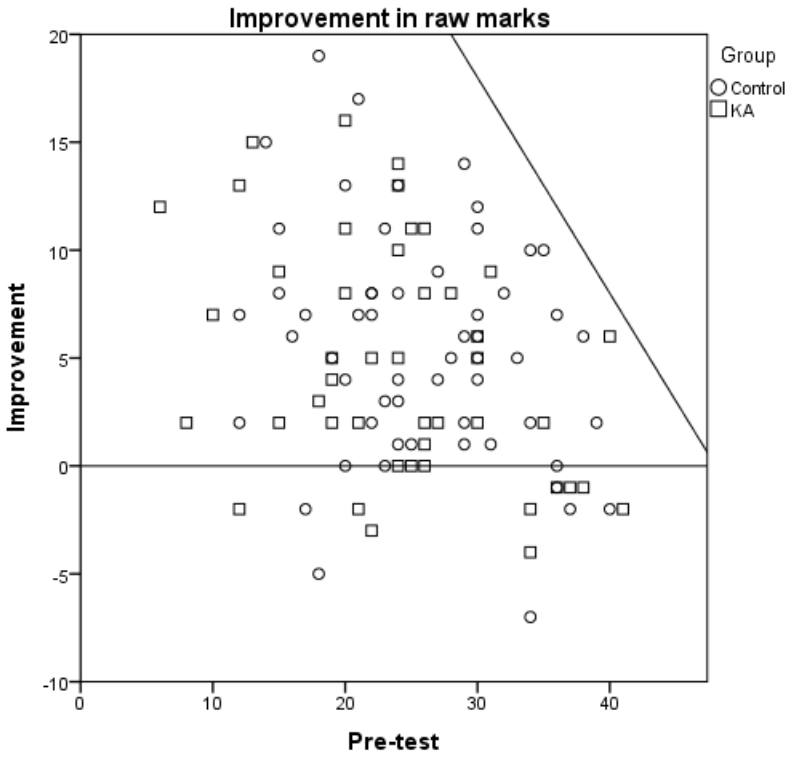

Figure 2: Improvement vs. pre-test results for KA and control groups. The diagonal line shows the ceiling for the scores, i.e. the total number of available marks minus the pre-test score.

Of the 59 students in the KA group, 49 set up a KA account with the first author as a coach by the end of the trial period, of which 44 registered activity by watching videos or doing exercises. Since the data on time usage was unreliable, "KA usage" refers to number of exercises in which the students achieved the level "practiced". The average KA usage was 44 exercises (SD = 21; N = 44).

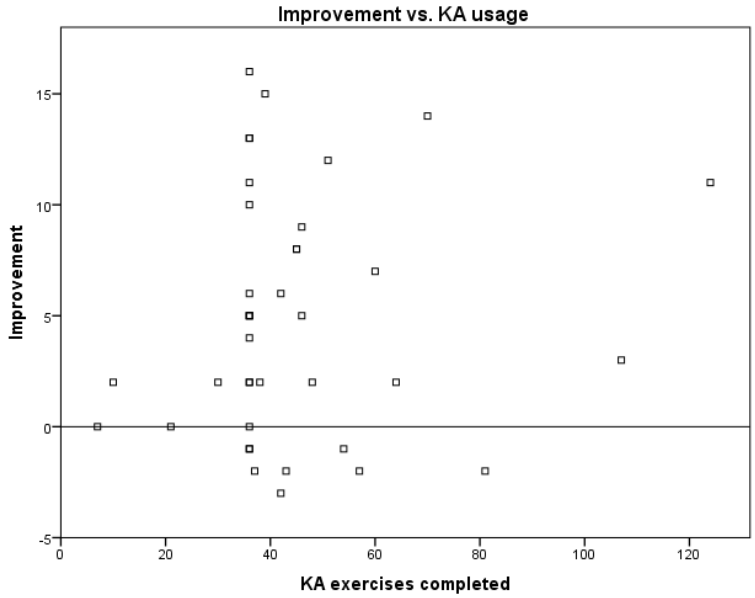

Figure 3: Improvement in raw marks on the mathematics test vs. KA usage, as measured by number of exercises completed.

Of the students who registered KA activity, there were 39 who submitted both the pre-test and the post-test. In Figure 3, KA usage is plotted against improvement for these students. The "vertical line" corresponding to 36 exercises represents completing the compulsory assignment. There were 21 students whose KA usage was greater than 36 exercises. It is clear from Figure 3 that there is no correlation between KA usage and improvement on the mathematics test. A similar analysis for the subset of items on fractions, decimals and percentages also showed no correlation.

Discussion

There were no statistically significant differences between the groups on either the pre-test or the post-test. Both groups had statistically significant gains over the trial period and they showed similar patterns in the scatter plots in Figures 1 and 2. Thus, in this study, KA was *equally beneficial* to the students as the other learning resources available to them. This non-significant result is of interest because KA has practical advantages over other learning materials (e.g., it is free and easily accessible) and the marking time saved by the instructor can be invested in other learning activities. In addition, Lindstrøm (2015) found KA to be beneficial for the instructor as a tool for formative assessment. KA may have yet other advantages for the learners, which could be investigated using qualitative methods. We are aware that the similar gains of the two groups may be due to the testing instrument being too coarse. However, addressing this is outside the scope of this study, and would require a qualitative analysis of students' learning processes with KA to develop a new testing instrument.

In the first semester, despite encouragement from the first author and messages with links to relevant topics, only four students set up an account. In the second semester, without any additional interventions, 45 students set up an account when the compulsory mathematics assignment could be completed using KA. This is consistent with the findings of Lindstrøm (2015) and Murphy et al. (2014) that high KA use is associated with a well planed integration into the course, including using it as part of the compulsory assigned work with consequences for non-compliance.

There were 21 students whose KA usage was greater than 36 exercises (which corresponded to the compulsory assignment), and some of these made extensive use of the tool (Figure 3). It may be surprising that there is no correlation between KA usage and improvement, with no indications of additional gains for the students who completed additional exercises. This may indicate a failure of the test to detect the learning gains; however, it may also be that the students worked on topics not covered by the test. Further qualitative research may be conducted to investigate what motivated these students and what possible learning gains resulted from the additional exercises completed.

Conclusions

A group of students who used KA showed similar learning gains to a control group that had no restriction on their learning resources but were not encouraged to use KA (and indeed did not, with two exceptions, use KA). As has been seen in earlier studies with KA, high use of KA was associated with a well-planed integration in the course. Some students made extensive use of KA, but there were no correlations between KA usage and measured learning gain. This raises the questions of what motivated the students to complete more exercises than required and whether there were other benefits not detected by our test.

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Constructing the shortest path on a cylindrical surface

Chronis Kynigos[1] and Ioannis Zantzos[2]

[1]Educational Technology Lab, School of Philosophy, University of Athens; kynigos@ppp.uoa.gr

[2]Educational Technology Lab, School of Philosophy, University of Athens; izantzos@ppp.uoa.gr

Digital media warrant a reappraisal of established conceptual fields and a search for new ones densely providing access to powerful mathematical ideas. This study reports secondary students' meaning making around the notion of intrinsically defined curvature in space by means of a tool integrating programming, dynamic manipulation of variable values and a simulation of 3D space. The study involved 15 ninth grade school students' attempt to design the shortest path between two points on a cylindrical surface are presented in this paper. Camera perusal and zoom allow for a change of viewpoints of the constructed figure. The findings yield meanings around concepts notoriously difficult even in undergraduate mathematics, such as differential stereometry, limits and curvature as systematic trihedron state change.

Keywords: Curvature, helix, stereometry, meaning-making, programmable media.

Introduction

Although curves appear in abundance in primary and secondary curricula, they are given the status of an auxiliary mathematical object to diverse structures either from geometry, e.g. circles-arcs, stereometry, e.g. cylinders-conic sections, or from algebra where the focus is of course on functions. Curvature is hardly discussed as a central notion, particularly in 3D space. Yet, in real physical space curves are truly abundant, in navigation they are key. Representations and notations from the pre-digital era are certainly one of the obstacles for students to access conceptual fields with curvature at their centre (we are intentionally using Vergnaud's construct, 1988). Here we use a digital medium integrating programmability with dynamic manipulation in simulated 3D space to get a sense of the meanings high school students may generate around differential curvature in space.

Curvature can be uniquely defined (apart from its position in space) by three elements of its arc, length, curvature and torsion (Lipschutz, 1969). The notion of curvature, the study of the properties of a curve and of the ways it can be approached consist one of the most important issues in tertiary education, as, for example, in differential geometry. The pre-digital formalism as well as the complicated formulas required consist a significant obstacle so that these notions and differential geometry in general can become approachable to many a student even at the tertiary level (Henderson, 1995; Kawski, 2003).

The encoding of the knowledge about curves, has historically gone through different stages. Euclid defines the curve as 'length without width' or 'end of a surface', without giving its definition in a general form restricted by general findings. But with the emergence of analytic geometry by Descartes, curves were defined as a mathematical sequence of points uniquely identified by two values. Later, the prevalence of the concept of function as a central concept in the curricula of secondary education, established functions as an umbrella under which large parts of mathematics

can be interpreted. As a consequence the only curves introduced in secondary education are graphical representations, namely curves which are represented only as secondary data representations or equations. The appearance of Turtle Geometry (Papert, 1980) constituted a first but most significant suggestion to consider restructuring knowledge (Wilensky, 2010) about curvature.

Papert proposed an intrinsic approach to geometry as a way to use digital media to provide kids with access to powerful ideas in environments rich in opportunity for meaning making (Kynigos, 1993). The intrinsic definition of curve on the plane was thus by means of the 'turtle', the cybernetic programmable unit vector (heading, position, zero length), making alternative state changes with a value approaching zero. So, this geometry addresses the problem of the local description of a curve using the kinematic picture of the curve as the line resulting from position changes (Loethe, 1992). But what about curvature in space? The first digital tool to simulate programmable turtle geometry in space appeared relatively early by Reggini in 1985, so it may be surprising that there was no further epistemological or pedagogical analysis regarding curvature represented with this medium. In space, the intrinsic description of a curve can be achieved by using a mobile system of perpendicular vectors describing the tangent vector and the osculating plane of a curve. The turtle moves only in the direction of the nose and 'sits' on the osculating plane (Loethe, 1992, p.72). Rotation of the trihedron as it moves is given by the curvature and torsion. Precisely, as the rate of change of the tangent is characterized by the curvature, thus the rate of change of the osculating plane is characterized by the torsion of the curve (Aleksandrov et al., 1969, p.75). Our research group has been interested in identifying meanings generated by students around the field of intrinsically defined curvature on the plane, using a tool we developed (we called it 'Turtleworlds') to integrate programmable turtle geometry with dynamic manipulation of variable procedure values (Kynigos & Psycharis, 2003). In this paper we address meanings of intrinsic curvature in space with a new version of the tool which we now call 'MaLT-Turtlesphere' (Kynigos & Latsi, 2007) and start from giving students the problem of the shortest path between two points on a cylinder.

The theoretical frame

Vergnaud (1988), introduced the notion of conceptual field as a set of situations the mastering of which requires mastery of several concepts of different nature. He claims that “a single concept does not refer to only one type of situation, and a single situation cannot be analyzed with only one concept” (p. 141), and he argues that teachers and researchers should study conceptual fields rather than isolated concepts. In our study we wanted to study meaning making on curvature by giving students the problem of finding the shortest path between two points on a cylinder. We thus perceived the problem as belonging to the conceptual field of ‘curvature in space’ as the notions, for example, of rate of change and arc length which are involved in the procedure of designing a curve based on the polygonal approximation, are directly related to the notions of curvature and torsion in space. With our basic aim being to examine the meanings the students develop (Noss and Hoyles, 1996) in relation with the notions of differential geometry we designed activities based on constructionism (Kafai and Resnick, 1996). Students would engage in meaning making through bricolage with digital artefacts. In recent years, we have developed a pedagogical design construct and method where we start students off by providing them with a 'half-baked microworld' (Kynigos, 2007). It is a specially designed digital artefact with one or more built in bugs resulting in some

faulty appearance and/or behavior when it is manipulated dynamically. It is designed to challenge students to decompose, change and debug the artefact and then construct something by using the correct version as a building block. Half-baked microworlds serve as starting points for the user to be acquainted with the ideas hidden behind the procedure of their construction.

The computational environment

The computational environment we used in our present research is MaLT-Turtlesphere (http://etl.uoa.gr/malt2) integrating Logo-based turtle geometry with dynamic manipulation of variable values resulting in DGS-like continuous change of the turtle figures at hand. This version of turtlesphere also afforded the insertion of stereometrical objects one of which was a cylinder, dynamically manipulable with respect to some key properties. The turtle movements are determined by following commands: fd(:n) and bk(:n) which command the turtle to take steps forwards or backwards, lt(:n) and rt(:n) move the turtle n degrees to the left or the right in its plane (osculating plane), and borrowed from Reggini's definition, dp(:n) and up(:n) turn the turtle upwards or downwards and rr(:n) , rl(:n) move the turtle around its axis. The basic tools of MaLT-Turtlesphere are the uni-dimensional variation tool (1DVT) which enables the user to dynamically manipulate the values of variables in a represented object and the 2d variation tool which is a two dimensional orthonormal system and is used to determine the co-variation of the values of two variables. An additional characteristic is its 3d Camera Controller which gives students the ability to dynamically manipulate the camera by means of the active vector and observes the object in the simulated 3d space from any side and direction he/she wishes. We should also point out the ability the user has got to insert ready-made 3d objects, such as a sphere or a cylinder, in a 3d virtual space and dynamically manipulate them.

The problem

The students were given the following problem: ‘Calculate and design the shortest path between two points on a cylindrical surface’. Our students were informed that this program would enable them to work out the way they could design such a path and that, at the end, they themselves could use it in order to construct their own models. The students were told that they were allowed to use any method and materials they liked (for example, paper and scissors) and the following half-baked microworld under the name the ‘shortest path’:

to shortestpath :n :s :dx :c

repeat :n [rl(:s) lt(:c) fd(:dx)] end ('end' is on a separate line, placed here to save space).

This microworld comprises a program with four variables each of which express the following: n expresses a number of repetitions, s expresses the turning of the turtle around the directions of its path, dx defining the length of the turtle step and c defining the turning of the turtle in its plane. The execution of the above code produces a polygonal line (either in space or in plane) or a straight line. But in the case when dx is very small, three kinds of curves can result from the aforementioned microworld, with the characteristic of stability of proportions ‘turning and twisting relative to traveled space’. If s=0, then we have a curve on a plane. For s=0 and c=0 line segments. For s=0 and $c \neq 0$ circle arcs. Solving the problem requires finding the shortest path between two points on a cylindrical surface, which means that the target is achieved when dx tends to zero. This leads to

limit procedures: $\lim_{dx \to 0} \frac{c}{dx} = k,$ which gives the curvature of the circle. If s is different from zero, helical lines are generated in space and similar conclusions are drawn (in case c = 0, a straight line arises). So, this code for creating a helix around the surface of a specific cylinder is half-baked in that it does not contain the property of each of the two turns being a function of the value of displacement (fd) and that the value of dx needs to tend to approach zero. If in the preceding code, we replace arguments with suitable functions and introduce a tail recursion, any line in space can occur. For example, if we replace the arguments of turning and twisting with trigonometric functions, a closed curve in space can occur.

The method

We adopted a design research method (Cobb et al., 2003). In this paper we discuss part of a broader research, which was developed in three phases: the first phase involving two students 3rd grade secondary school, in the second phase with the participation 15 students (a class 3rd grade secondary school) and which lasted 24 hours, and finally, in the third phase involving five higher education's students. These particular students of the second phase had already been familiarized with constructions in the logo programming language in the Turtleworlds environment. A sound and picture software (HyperCam 2) was used to record data and enabled the researcher to record the students' actions and the conversations amongst the participants. In order to analyze the students' mathematical thinking we were interested in the ways the students interacted with the available components of the software and in the ways they constructed mathematical meanings. We centrally used the notions of meaning making and situated abstractions, which enabled us to describe how the students construct mathematical meanings based on the functions of the particular software they were using and on the conversations between them (Noss & Hoyles, 1996). We also found the construct of 'instrumentalization' taken from the theory of instrumental genesis (Guin and Trouche, 1999, Kynigos & Psycharis, 2013) helpful in showing us was how the students were trying to change the functionalities of the 'faulty' microworld they were given aiming to produce a different artefact which automatically gives a circle and a helix with the shortest length.

Findings

The circle approach through limiting curvature

Even if the majority of students at first turned to the software they had been given in their effort to give an answer, they soon realized something else should be done first to make sense of the problem. They decided to use tangible objects first, paper, pen and the scissors, they had also been given. By selecting two points on the cylindrical surface and then rolling a piece of paper to form a cylinder and un-rolling it, they came to the conclusion that the shortest path could be a circle, a helical or a straight line. Upon un-rolling the cylinder they noticed that the line which was formed would be a straight line on the plane (geodesic in plane) but when they re–rolled up the cylinder, a helix or a circle was formed. Nevertheless, this conclusion, although it seemed to be the solution, did not seem to satisfy the students at all. Here is a typical answer from two students:

Student 1: If we could suppose that the cylinder opens, then okay it is a straight line
Student 2: But if the cylinder could not open? (Meaning: then how could we design the helix?)

They then started exploring the half-baked code firstly by dragging the variable values. All students decided to focus first on getting the code to create a circle around a fixed cylinder. Some kept the value of the rl command to zero, some decided to chuck it out of the procedure, starting to work on the formalism. The students at hand took the latter option and tried out dragging to understand the behavior of the turtle path (Brunström & Fahlgren, 2015). For this circle, a common technique was the winding of the polygonal line at a constant circle or at the bottom of an inserted cylinder from the software library. The completion of the first winding lead students to put values dx =1, c = 29. But when the researcher asked the question about the kind of path that was formed, students concluded by zooming that it was a polygonal line, and a further reduction of dx was needed. Students, with the help of changes decreased the value of dx from dx = 1 to dx = 0.1, and then did the same for dx = 0.01, while modifying the value of c as well, as the polygonal line continued to wind in a solid circle. Their attempts brought them to conclude that the turn value should be dependent on the displacement value if the turtle trace was to be a good fit to the base of the given cylinder. They then decided that the code should contain a proportional relation of the variables c and dx, and modified the half-baked microworld engaging in an intrumentalization activity. The result for these students and, as it turned out, for the majority of the participants, was a code like the following, with a differentiation in the arguments of the turtle turn:

```
to shortestpath  :n  :dx
repeat :n  [lt(29*:dx)  fd(:dx)] end
```

The dialogue continued yielding that the students considered the circle as a polygon with sides that are constantly decreasing in length:

Researcher: so, for which rates do you get the requested circle?

Student 1: for small dx, for example 0.1

Researcher: ie for dx = 0.1 will we have a circle?

Student 2: we will have a polygon

Researcher: and which may be the required rates?

Student 2: we can’t be exact because as we put smaller numbers, it will be approaching the solid circle (at the same time the student zoom and manipulate the slider of dx to continuously lower numbers to prove their claim)

The students' efforts show a change in the way they thought about curvature, starting from a static approach (with dx equaling a constant value of 1) to a dynamic (the more dx diminishes, the better approach to curvature). This was evident in their correction of the code initially achieved with dx to be small (usually teams chose for dx a tenth or hundredth approach). Thus, initially the circle formed by the mean curvature (c / dx = constant) determined the forward movement of the turtle in relation to the dx. This instrumentalization action resulted in a modification of the shortest path code and provided us researchers with a lens to students' development of a situated abstraction on the concept of curvature. The problem that was given to find the shortest path, thus led them at first to think of curvature as a limit and the circle as resulting from a limiting process and not simply by dx small. Although a strictly symbolic form of a limit was unknown to the students, the role of the limit process seemed to be played by the slider of dx.

From a static to a dynamic aspect of the helix

For the construction of the helix with Turtlesphere, students at first could not implement a technical approach as in the circle, since there was not a preplanned helix on the cylindrical surface. So they

resorted to properties discovered during the deformation process of the flat surface and the situated abstraction for the notion of helix which was delivered by them as follows: *'helix is a curve that is wrapped in a cylindrical surface and if it unfolds, a straight line emerges'.* The designing of such a curve though without the use of tangible materials, and the ability to generalize such a procedure demand the use of differential geometry notions which reflect the Frenet-Serret frame movement in space. The students appeared to realize the limitations of tangible materials, and the inability to generalize the procedure in situations when their use is impossible.

The students' speculation stimulated the researcher to turn their attention to the half–baked code they had already had at their disposal. The students chose again to insert a model cylinder with specific dimensions, and by dragging the variation tools they tried to achieve the construction of a helical line which twisted round the cylinder with its two ends being the ends of the generator of the cylinder with the above characteristics. Their initial suppositions referred to values which, although at first sight seemed to have achieved their goal (that is the helical line to twist round the cylinder), the use of the perusal camera proved wrong. Thus, from that time on each and every attempt of theirs initially comprised finding the values for n, c, s and dx with the simultaneous use of the camera and change of the values of the variables. A group of students, at their first correct attempt (with dx=1), came to the following values: n=14, c=25, s=5 and dx=1. Although they seemed to be satisfied with the result of their experimentations, they continued to experiment after the following questions on the researcher's part:

Researcher: Is this a helix? (They play with the camera, zooming in at the same time)
Student 1: They look like lots of straight lines (they are referring to the line segments which the helical line is composed of and with the execution of the half –baked microworld provides them with)
Researcher: What can you do so that you can turn it into a helix?
Student 1: Eliminate the angles
Researcher: How can you eliminate the angles?
Student 1: If we decrease dx, let's say to 0.1

Students' instrumentalization initially started by dragging the slider of dx and with the help of graphical feedback. Since dragging dx to dx = 1 gave a polygonal line and not the curve as the paper-folding approach, the students started to drag the slider of s. Their attention now concerned the discovery of the relationship between the three variables (c, dx, s) and the consequent changes to the half-baked code. The situated abstractions that this particular group of students seemed to have built arise from the need to prevent the distortion of the figure and satisfy the 'definition' that had created for the concept of helix based on the paper- folding approach. As dx took smaller and smaller values a line was given which looks like a helix with a length constantly decreasing and that the ratios c/dx and s/dx remain invariant and equal to 25 and 5 respectively. In fact, the rate of change of directions of the segments the turtle is moving on and its plane remain invariant. The replacement of the ratios they discovered in their initial code provided them with what they claimed was the 'correct' code and the solution in demand:

```
to shortestpath :n :dx
repeat :n [rl(5*:dx)  lt(25*:dx)  fd(:dx)]  end
```

Researcher: Which values provide us with the helix we are looking for?
Student 1: The smaller dx is the better.

The students seemed to realize that the solution they were looking for did not only consist of the specific values of the variables but it should also combine a limited procedure for dx.

Conclusions

The purpose of the present research was dual: Firstly, to study the degree to which this particular digital tool and microworld could form the basis for secondary level students to study notions in the conceptual field of curvature in space and secondly, to study the meanings developed by these particular students in their attempt to design the shortest path between two points on a cylindrical surface. The students expressed mathematical meanings for a number of notions of differential calculus (rate of change, limit) as well as of differential geometry (curvature, torsion and geodesic) which has been shown to be notions difficult to be approached by even math students. One of the major advantages of the method applied is the fact that, not only were students able to visualize the way a normal curve is constructed by the motion of a movable trihedron in space (the role of which was replaced by the 'turtle') but the students were also given the ability to study, explore and symbolically represent these movements via programming and dynamic manipulation. For example, the circle is constructed by the turtle avatar with the characteristic of working stability, and not just through the stability of the ratio c / dx, i.e. the curvature formula in Logo. The dynamic manipulation resulting in figural change helped the students focus on the limiting process of this ratio which reflects the notion of curvature. The students changed their conception of helix from a static approach to a dynamic aspect, i.e. as a line made from an avatar with the characteristic of stability in both its turning and rotating around the line of motion. Although the way they used to design the helix did not tally with the strict formalism of differential geometry, the answers the the meanings they generated are indicative of the fact that a restructuration of the notion of curve relying on concepts of curvature and torsion, and with the turtle replacing the role of the moving trihedron to create a curve in space, is feasible in secondary education.

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Using dynamic worksheets to support functional thinking in lower secondary school

Edith Lindenbauer[1] and Zsolt Lavicza[2]

[1]University College of Education Upper Austria, Linz, Austria; edith.lindenbauer@ph-ooe.at

[2]Johannes Kepler University Linz, Austria; lavicza@gmail.com

This paper reports on a current case study about the use of dynamic worksheets in a middle school in Austria. These worksheets were designed based on typical problems and misconceptions outlined in the literature concerning functional thinking, and they focus on the representational transfer between situational model and graphical representation. Grade 7 students were video recorded while working on these worksheets, pre- and post diagnostic tests and diagnostic interviews were conducted to examine their conceptions in relation to functions. This case study particularly pays attention to the intuitive conceptions of students, the influence of the dynamic worksheets on these conceptions, and whether or not dynamic worksheets are able to support students in developing appropriate mathematical conceptions. In this paper, some preliminary results are to be discussed.

Keywords: Functional thinking, technology, representational transfer, lower secondary school.

Introduction

Functional thinking is an important concept in mathematics education. For students, a variety of problems arise while working with functions and thus functional thinking has been widely investigated by numerous researchers. Considering the development of dynamic mathematics software, additional aspects of functional thinking appear. It needs to be examined whether or not dynamic mathematics tasks are able to support students in an early stage of learning functions in developing appropriate conceptions. Based on these issues, we developed dynamic worksheets visualizing the transfer between situational and graphical representations and integrated them into a qualitative case study to examine their influence on students' conceptions.

Theoretical background

Working with functions is a usual activity in mathematics lessons in school. Vollrath (1989) describes functional thinking as a typical way of thinking when dealing with function and he mentions different aspects of functional thinking. Malle (2000) refers to it and specifies the following aspects in a slightly altered version, which is better suited than Vollrath's (1989) description for the purposes of this research project: Relational aspect (each argument x is associated with exactly one function value f(x)) and co-variational aspect (if the argument x is changed, the function value f(x) will change in a specific way and vice versa). The relational aspect represents a static perspective of functional thinking whereas the co-variational aspect describes dynamic processes; particularly in this project, functional thinking comprises of both aspects.

In the context of functional thinking, various difficulties have been found and examined in the research literature. The graph-as-picture error occurs in various forms and means that students see function graphs as photographic images of a real situation (Clement, 1989; Schlöglhofer, 2000). Illusion of linearity means the preferable use of linear or direct proportional models for the description of relations, even if they are not appropriate (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

Difficulties arise also in the interpretation of slope and growth, for example, if the point of maximum growth is confused with the largest function value. This slope-height-confusion leads also to difficulties in the interpretation of path-time graphs (Clement, 1989). These problems can cause students' misinterpretations of functions and especially of graphs of functions. Vosniadou and Vamvakoussi (2006) suggest – to avoid that intuitive conceptions develop to misconceptions – considering the introduction of mathematical concepts at an earlier stage in mathematics education.

Vogel (2007) stresses that multiple representations of functions, such as graphs, situational representations, terms, and tables are able to represent aspects of functional thinking (relational as well as co-variational aspect) externally, and they have the potential to support students' ability to interpret functions. According to Duval (2006), only the flexible alternation between different representations allows a differentiated approach to mathematical content and forms the basis for sustainable acquisition of skills. But representations have to be considered critically as they influence ways of thinking, they may constrain students' thinking about the concepts involved and are interpreted by students according to their prior knowledge (Vosniadou & Vamvakoussi, 2006).

Dynamic mathematics software such as GeoGebra may support students' development of functional thinking, because it is suitable to emphasize different functional aspects through interactive representations (Barzel & Greefrath, 2015). Research findings in relation to the use of technology in teaching often only show small positive effects on students' learning achievements (Drijvers et al., 2016). Results concerning dynamic representations are more encouraging, because these representations can help students in understanding mathematical concepts (Hoyles, Noss, Vahey, & Roschelle, 2013). Thus, we need to examine in more detail the influence of technology on students' individual conceptions.

Conceptions in a dynamic mathematics environment

Based on problems and examples concerning misconceptions mentioned in literature several dynamic GeoGebra worksheets were designed reflecting multimedia design criteria (Clark & Mayer, 2011). We chose GeoGebra for this study, because it is the most widely employed mathematics software in Austrian schools. Due to the prior knowledge of selected students (experiences mainly with path-time diagrams, direct and inverse proportionality including their graphical representations, but none with the explicit function concept), these worksheets primarily address the representational transfer between situational model and graphical representation. Figure 1 displays a typical worksheet based on a task of Schlöglhofer (2000) addressing a graph-as-picture error.

This GeoGebra worksheet consists of a situational model, in particular an iconic representation of the situation, as well as a Cartesian coordinate system displaying the corresponding graph. In the situational representation on the left side a triangle is displayed. The shaded area left of the dotted line inside the triangle is treated as a function of x, which is the horizontal distance between the vertex A and the dotted line. Students can move the line and change the size of the coloured area. Afterwards they should formulate a hypothesis about the shape of the graph. In the diagram the coloured area is a function of the distance x. After clicking the checkbox, the size of the area is displayed. At the end, students should display the graph in order to examine their assumptions about the shape of the graph. In this research project, accompanying tasks assist the students in working with the representational transfer, which is considered particularly difficult conceptually.

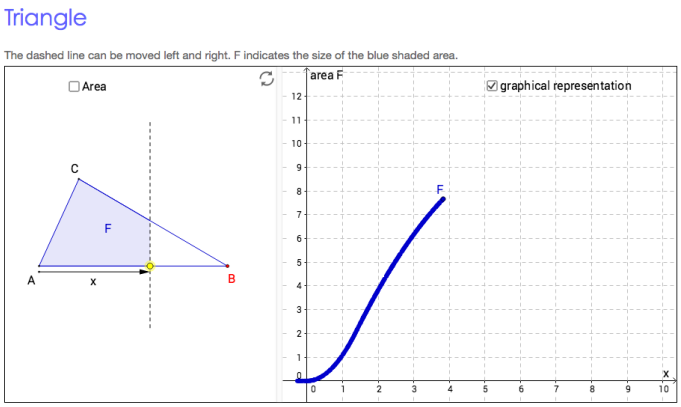


Figure 1: "Triangle", http://ggbm.at/GYeY4ayO

As the situational model shows an iconic representation, the corresponding task is likely to trigger a graph-as-picture error (Schlöglhofer, 2000). It is especially interesting if the dynamic worksheet has the potential to support students' ability to comprehend the graph.

Such problems concerning functional thinking and theoretical considerations have led to the research questions below. The first two offer a basis for research question three, as we believe this third question contributes the most to the field of inquiry about technology use related to misconceptions. Due to space restriction this paper focuses only on the first and third research question. Future papers will offer further details on research question two as well as more in-depth analyses.

1: What conceptions, with particular attention to pre- or intuitive conceptions, emerge concerning functional thinking of students in an early phase of learning functions (grade 7/8)?

2: How should dynamic materials addressing to this topic be designed to support students in developing appropriate mathematical conceptions?

3: What kinds of influence of these dynamic materials exist on conceptions and internal representations of students of lower secondary school concerning functional thinking?

Research design

To offer answers to the research questions, we selected a qualitative research approach. The overarching methodology for this research project is an exploratory and collective case study research, but integrating elements of Grounded Theory (Eisenhardt, 1989). The study was conducted in a 7th grade classroom of a rural middle school[1] in Austria with 28 students aged 12 to 13, who had some experience in working with graphs (mainly distance-time-graphs) but none with the function concept itself.

Figure 2 shows an overview of the research design. Piloting A was the first phase of the study aimed to evaluate the technical details of the recording procedure, to choose the tasks for the diagnostic tests and the worksheets for the intervention. The second phase – piloting B – consisted of one complete

[1] Rural middle schools in Austria usually have the most diverse student population concerning achievement levels. This is especially true for the selected school in this study, thus this choice offers us the possibility to examine as many different students and their conceptions as possible.

data collection process. After the data collection, the data was transcribed and analysed with qualitative methods described later in detail.

The data collection included five stages. First, all students participated in a diagnostic test based on ten different tasks from literature concerning conceptions (Schlöglhofer, 2000; De Bock et al., 2002) as well as a test instrument called CODI (Nitsch, 2015). Afterwards, eight students were chosen for diagnostic interviews (Hunting, 1997) depending on their test responses so that their – incorrect – results represent a wide range of different conceptions related to the various test tasks to obtain an in-depth view of their individual conceptions.

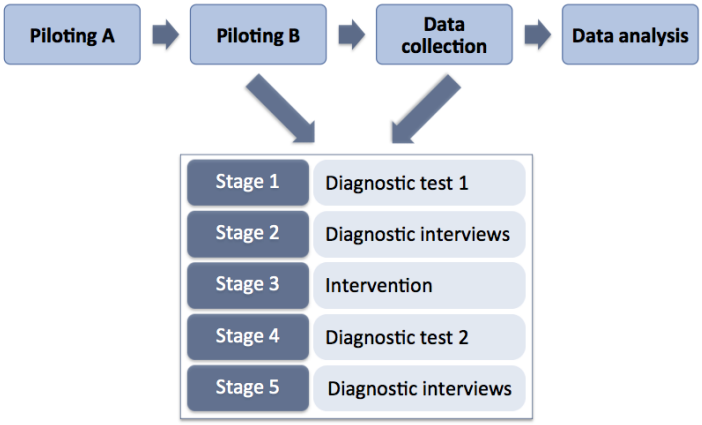


Figure 2: Research design overview

During the three-lesson-intervention, students worked in pairs with GeoGebra worksheets addressing different topics guided by accompanying tasks. While working, ten students were audio- and videotaped and the screens of their laptops were recorded. Also, students' paper worksheets were collected. After completing the intervention, another diagnostic test with slightly altered tasks was conducted. Based on the observational data and an analysis of the test results, eight students were selected for diagnostic interviews to investigate the influence of the worksheets on the students' conceptions.

Data analysis and preliminary results

The collected data is divided by the data source to address different research questions (the first test results and the corresponding interview data to approach the first research question, the recordings from the intervention and students' paper worksheets, the second test results and the corresponding interview data to focus on the second and the third research question).

Based on the research methodology, we conducted, for each student or pair of students, a within-case analysis using initial (or open) coding, then compared cases and searched for cross-case patterns using focused coding (Eisenhardt, 1989; Saldaña, 2013).

Further qualitative analysis of the observational data and the interview recordings will give an insight into the conceptions of the students concerning functional thinking as well as the influence of dynamic worksheets on these conceptions. In this section, preliminary results concerning the task "Area" from both diagnostic tests are to be presented, because both task and results exemplify the process of the research.

Diagnostic test 1

Figure 3 displays a task from the diagnostic test 1 that is similar to the GeoGebra worksheet in Figure 1 and was based on a standard test example concerning the graph-as-picture error from Nitsch (2015). The picture shows a trapezoid, and in the exercise students had to choose one diagram out of four that showed the grey marked area left of the dotted line as a function of the distance x, and to explain their decisions.

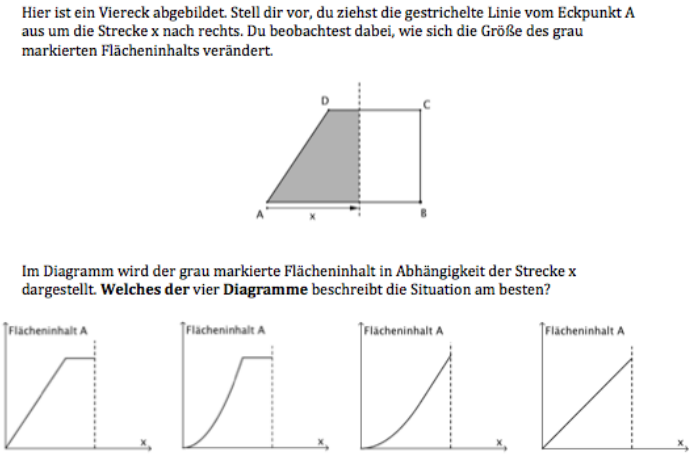


Figure 3: Screenshot task "Area" from diagnostic test 1

Students' explanations reveal different levels of conceptual understanding. A categorization of students' solutions and argumentations is visualized in Figure 4. The arrows represent the direction of the representational transfer from the situational model to the function graph, and the categories are arranged according to the correctness and elaborateness of students' understanding.

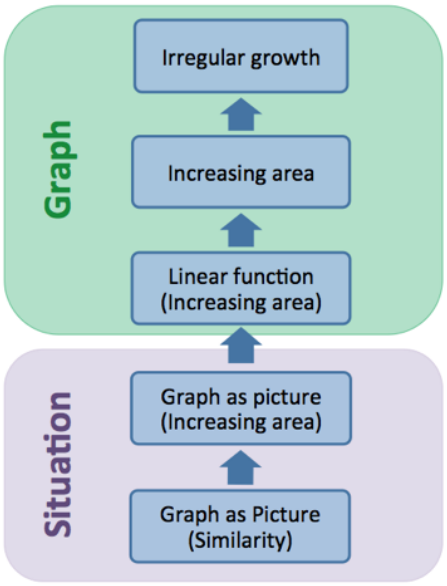


Figure 4: Categorized solutions task "Area"

The first two categories represent the choice of the first graph addressing the graph-as-picture error. Either the students marked the similarity between both representations, or they already recognized an increase of area but 'remained' at the shape of a trapezoid. These answers reveal reasoning from a situational perspective of students, who did not manage to transfer the situational model into a function graph.

Students with explanations of the next three categories achieved transfer to a graphical representation by recognizing an increasing function value, and these explanations were essentially correct. The third category of students, who selected the linear function, did not recognize the irregular change of the function value. Students who chose a correct graph form the last two categories. They either reasoned their choice with an increasing area or – the most elaborated explanation – with an irregular growth of area.

In the next section, we present three student answers to the corresponding task from the second diagnostic test. Students were chosen from the first category (Graph as Picture, Similarity) to

demonstrate the range of possible developments of their conceptions. The answers represent different extent of influence, also based on the achievement level of these students.

Diagnostic test 2

In the second diagnostic test after the intervention, the corresponding task was slightly altered – a trapezoid of another shape is displayed (see Figure 5).

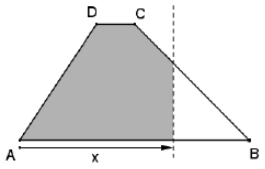


Figure 5: Trapezoid of task "Area" of diagnostic test 2

Corresponding to the four function graphs in diagnostic test 1, there are four possible choices for the solution. These function graphs represent the graph-as-picture error, the correct solution, a combination of graph-as-picture and correct graph, and a linear function.

Table 1 presents three students' solutions and explanations from the diagnostic test 1 category "Graph as Picture (Similarity)" exemplifying a possible diverse influence of the applets. These students were chosen because all three students changed their answers after working with the dynamic worksheets, and due to their different achievement levels they gave a wide range of changes in their results.

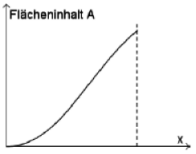
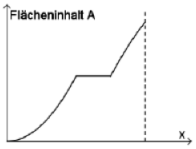
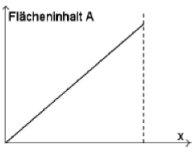
Achievement level	Solution	Explanation (translated from German)
High (student 1)		"In the beginning it [the area] increases strongly, then a bit more slowly, …"
Average (student 2)		"The area is always increasing, except in … the middle of the trapezoid, it [the area] remains the same."
Low (student 3)		"You have to consider the x-axis, and because the x-axis is straight, the last … ought to be correct."

Table 1: Students' answers concerning task "Area" of diagnostic test 2

Student 1 (high achiever) chose the correct solution, and the explanation reveals a correct understanding of irregular changes of the function value. Also student 2 (average achiever) described the change of area correctly, but he decided for the graph representing a combination of correct graph and graph-as-picture error.

Unlike in the diagnostic test, the GeoGebra worksheet displayed only a triangle and not a trapezoid. Student 2 managed to translate corresponding parts of the situation correctly to the graphical

representation, but was not able to transfer his knowledge to the part of the situational model where the dotted line is moved over the 'horizontal line' of the trapezoid.

The explanation of student 3 (low achiever) reveals a lack of understanding concerning the representational transfer and the meaning of Cartesian coordinates. It refers to the students' look for visual similarities, a solution strategy sometimes used by students during the intervention when confronted with a problem. Further results about students' test answers, their discussion during the work with the dynamic worksheet (Figure 1) as well as a detailed description of this worksheet and included instructions will be presented in upcoming research papers.

Discussion

For each task in the diagnostic test, several intuitive and incorrect conceptions appeared. For example, different levels of students' conceptual understanding have emerged during analysis of the diagnostic test answers. These levels are representing the translation process from situational to graphical representation. Concerning the graph-as-picture error, explanations also made visible that standardized multiple-choice test items were not always able to detect a corresponding incorrect conception. Other results reveal different tendencies of students to use relational or co-variational aspects of functions for explanations. Also the influence of everyday experience is visible in the data, and the influence of formal and informal language (e.g., when students interpret 'highest speed' as 'leading' or 'winning' in distance-time-diagrams) is especially interesting.

The dynamic worksheets have different purposes, for example, visualization, experimentation, and testing hypotheses. The visualization function is supporting students in translating the text into a situational model or into a correct identification of the interesting variable (e.g., the meaning of 'area' in the corresponding task in Figure 3). Also, the worksheets have an adaptational influence on students' conception (e.g., from linear to non-linear increase of function value). In other words they did not alter the conceptions but changed them partly to the direction of a correct conception.

Preliminary results seem to reveal that the extent of influence of these worksheets on students' conceptions depends on the intuitive conceptions of students and their achievement level. The interpretation as well as the perception of the GeoGebra worksheets is based on the prior knowledge of the students. The observational data repeatedly demonstrated that students tried to connect new content to their experience and knowledge. Considering that students worked without teacher instructions, for high achieving students the dynamic worksheets seem to be more appropriate, whereas lower achieving students would probably profit of teachers' assistance to reflect their perceptions and interpretations or to draw the attention to the important features of the worksheets.

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Students' use of movement in the exploration of dynamic functions

Giulia Lisarelli

University of Florence, Department of Mathematics, Italy; giulia.lisarelli@unifi.it

The concept of function has a central role both at school and in everyday situations. A dynamic algebra and geometry software program allows students to experience the dependence relation and to explore functions as covariation. In this paper we propose a description of different dragging madalities and the analysis of a protocol in which two students work together on a problem that involves coordinating two covarying quantities. The analysis has been carried out through this classification of dragging modalities, that can be efficiently used to observe, describe and analyze students' processes involved in the exploration and solution of dynamic activities.

Keywords: Function, variation, dynamic algebra and geometry software, dragging.

Introduction and conceptual framework

The concept of function is very important both in secondary school and university mathematics but it also has a central role in everyday situations. For a long time, this notion has been at the core of several studies in mathematics education, and a rich literature has revealed students' difficulties in understanding the concept in all its aspects (Vinner & Dreyfus, 1989; Tall, 1991; Dubinsky & Harel, 1992; Carlson & Oehrtman, 2005). Difficulties in interpreting the dependence relation as a dynamic relation between covarying quantities are widely reported (Goldenberg et al., 1992; Carlson et al., 2002). Falcade et al. (2007) suggest that the use of a dynamic algebra/geometry software, such as GeoGebra, allows students to experience functions as covariation, that is a crucial aspect of the idea of function (Confrey & Smith, 1995; Tall, 1996). According to these assumptions we are interested in studying students' cognitive processes involved in approaching functions represented in a specific dynamic environment.

Our study is an exploratory study aimed at analyzing students' use of movement in the exploration processes of the dynamic functions. We have adopted the idea of analyzing the movement because several studies have revealed that it can support a cognitive analysis of students' reasoning processes.

In order to analyze students' appropriation of movement Arzarello et al. (2002) identified different types of dragging which students use investigating a geometric problem, according to their different purposes. Antonini & Martignone (2009) proposed a similar classification in the case of physical artifacts. They introduced a classification of students' utilization schemes of pantographs, that are particular mathematical machines designed for geometrical transformations. Although the differences due to the different nature of the instruments these two studies concern, there are certain similarities. Especially the common purpose is to identify students' utilization schemes in order to analyze the cognitive processes involved in the investigation of geometric problems.

In this paper we shall present the first steps of our research: a classification of dragging modalities and the analysis of a protocol, that has been carried out through this classification. Some of our descriptions echo the classifications presented in the above-cited studies, but they have been completely transformed in order to suit the use of particular function representations in a specific dynamic environment. This is the original contribution of this paper. Indeed, while in literature we

can also find other studies on the use of dragging (Baccaglini & Mariotti, 2010; Robutti, 2013) they concern in particular the dynamic geometry.

Contextualization of the study

We analysed a sequence of classroom activities, 14 hours in total, implemented in an Italian high school for Math and Science, in which students explore the functional relationship in dynamic interactive files (in GeoGebra). The subjects of this investigation are 16 years old students who never met the concept of function before. The activities were led by the teacher and they have been video-recorded by three cameras present in the classroom simultaneously. The analysis is mainly based on the transcripts of the activities and it was led by paying a special attention to the use of dragging, the language employed and the gestures.

In this paper we present one of the activities carried out by two students. The activity selected concerns The Bottle Problem: an open problem about bottles filled with water (the task is reported in Figure 1), which involves coordinating the variations of two quantities. Students are asked to work in pairs so that they can form conjectures and explain their reasoning to each other. They have no time limits and are video-recorded through a camera behind them pointing at the computer screen. They are given the following task with an interactive dynamic file (in GeoGebra) for the explorations and some sheets of paper for the answer.

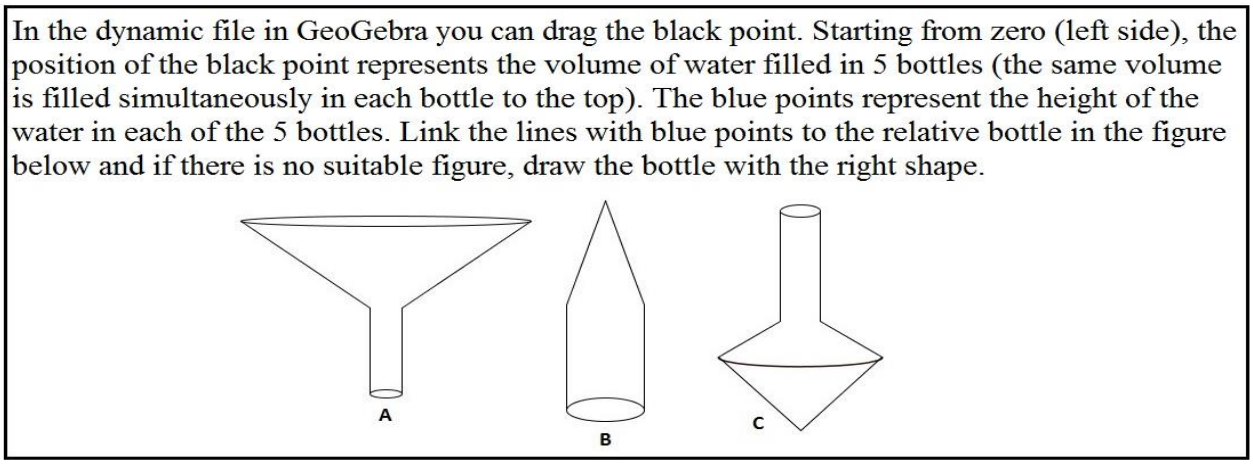

Figure 1: The task of the Bottle Problem

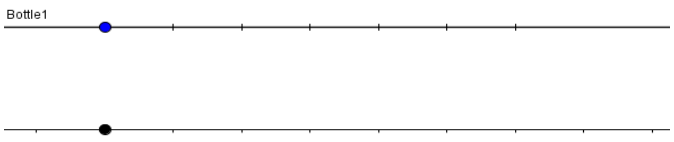

Figure 2: The dynamic file

Figure 2 shows a part of the GeoGebra file in which are presented the graphs of five functions representing the height with respect to the volume of water. They are not the "usual" graphs in the Cartesian plane: there is an unnamed horizontal line with a black point attached to it that represents the x-axis and five other horizontal lines, parallel to it and labelled "Bottle1, Bottle2, ..." with blue points moving on them. The motion of the blue points, bounded at the lines, is an indirect motion

because these points can not be dragged directly: they represent the dependent variables so their movement is determined by the dragging of the black point, that represents the independent variable. The height of each bottle is fixed equal to six, for this reason the blue points move in the interval [0, 6] and there are six notches on the lines which they move on. The black point can be dragged everywhere along the line without the magnetism, that is a property that GeoGebra allows to give to a point and makes it move on the real axis as if it has a magnet that attaches it to the whole numbers; and disabling this tool the dragging of the point is more uniform.

Dragging modalities

In this section we introduce a classification of dragging observed during students' exploration of dynamic interactive files. It can be efficiently used to observe, describe and analyze students' cognitive processes, involved in the exploration and solution of problems about functions represented in a specific dynamic environment.

The identified dragging modalities are divided into two families: the first one describes the quality of the movement, this type of dragging could be also recognized by a computer that captures how the mouse moves on the screen (Table 1) and the second one describes the use of dragging with regard to an aim, that is associated through the study of the language employed, the sight and the gestures (Table 2).

One of the potentialities of this classification is the fact that the two families of dragging modalities can be combined and, for example, keeping an element from the first one and an element from the second one allows a complete description of a students' process in solving problems.

First of all we observe that in our cases it is always a *bound dragging*, that according to Arzarello et al. (2002) consists of moving a semi–dragable point (a point which it is already linked to an object). Because the only point that students can move is bound to the x-axis, all the other points move depending on it.

	Description
Continuous dragging	Continuous movement
Discrete dragging	Movement with jumps, associated with counting
Impossible dragging	Trying to move a dependent[1] point that can not be dragged

Table 1: Types of dragging

	Description
Wandering dragging	Random movement, exploring the construction
Dragging test	Movement aimed at testing a possibly implicit conjecture

[1] We use this term to identify the point but we do not know if the students are aware of this dependence relation.

Handle dragging	Movement of the object as if it was a handle, in order to observe other objects' movements
Guided dragging	Movement aimed at reaching a particular configuration

Table 2: Dragging with an aim

A protocol

In this section we present an activity in which two students, Luca and Mara, work together at the bottle problem and we can identify some of the dragging modalities described before.

Their first approach to the problem involves dragging the black point, representing the volume of water filling the bottle, with a continuous movement (*continuous dragging*) and without apparently paying attention to the dragging of such point: it is used as a sort of handle that allows them to see the movement of the blue point, representing the height of the water in the bottle (*handle dragging*). Indeed, as we can see in Figure 3, during the dragging the arrow representing the mouse does not overlap the black point in every moment, suggesting a weak haptic control because the students' attention seems not on the dragged point.

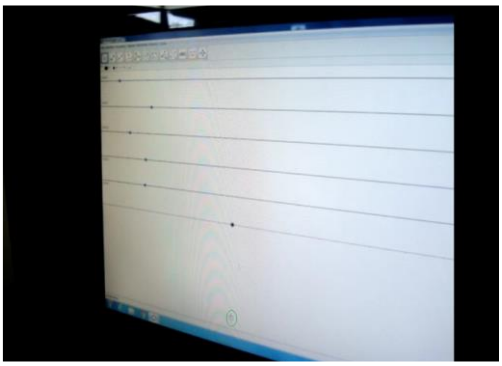

Figure 3: An example of handle dragging

The students do not express, through their words and gestures, awareness that as one variable changes, the other variable changes; they seem more concentrated on the differences between the movements of the blue points than on the relation that links the movement of a blue point to the movement of the black point. For example, they look for which one is the fastest in order to associate it to the tightest bottle, because the speed of blue points represents the speed at which the height increases if the water is poured in at a constant volume per time, and the tighter the bottle is, the faster the height increases; in the same way the slowest blue point will be associated to the widest bottle.

For example, the following dialogue takes place while students explore the file, dragging the black point very slowly and trying to keep a constant speed (*continuous dragging*):

Luca: The bottle three is the steepest in the lower part.

Mara: The bottle one goes very slow, also the bottle two.

Luca: Also the fifth, the bottle two is the slowest respect all the others.

Mara: No the five, the five does not move!

Luca: Yes and then it is steeper at the end.

What we can infer from this excerpt is that the students' attention is on the blue points and the independent black point is used as a handle (*handle dragging*), they compare the speed of these points, observing for example that the second is the slowest, or probably the fifth. Luca, in the last sentence, says "is steeper" instead of "goes faster" and this suggests that he mixes up the trend of the height of water in the bottle with the shape of the bottle.

Their initial approach changes: when they have to decide which one of the blue points represents the bottle B, shown in Figure 1 (that in the lower part has a cylindrical shape). They search for a point that has a constant speed and, in doing this, they compare the movements of the black and the blue points. So, first of all, they look at the picture of the bottle on the sheet of paper and imagine how the height of the water in the bottle should evolve, then they drag the point representing the volume of water in order to see whether there is a point, representing the height, with the needed properties (*guided dragging*).

In particular, they count how many notches of volume are necessary to let the blue point reach the first notch of height and then to let it reach the second and finally the third and finally they compare these numbers: if they are equal to each other they conclude that the bottle has a cylindrical shape. It is an example in which the two quantities that are varying are coordinated in order to establish the average speed of the blue point. This seems an attempt to make a continuous situation discrete and it is also suggested from their use of dragging: they drag the black point with jumps, while counting the notches (*discrete dragging*).

The following excerpt shows this combination of *discrete* and *guided dragging.* Luca summarizes their idea about how the blue point representing the bottle B should behave and searches for it:

Luca: So we have to find a point that is constant till the third notch and then it goes faster. I would see the bottle one, look: first, second, third more or less goes in the same way.

He drags the black point counting 1,2,3 and stops, 1,2,3 and stops, finally 1,2,3 and stops and during this process the mouse makes some jumps (*discrete dragging*).

Luca: We could say that it is constant till the third notch and then...

He drags the black point again, this time with a continuous movement and an almost constant speed (*continuous dragging*).

Luca: Then it goes faster!

The last part of the analysis reports students' explorations and conjectures when they have to draw the bottle looking at the movements of the points: the black point seems no longer to be only a handle for them. Indeed, as the next excerpts show, the students relate the changing values of height and volume in order to find whether the speed of the blue point is constant; their question is: how many notches of volume are necessary to have one notch of height? They fix the amount of change of the height (uniform increments) and find out the relative rate of change of the volume. In doing this, they consider the average rate of change locally, for a specific interval of the domain of the function.

It is not so clear how they conclude that "it is constant till the first notch" and this could be considered as an advanced statement because it requires an awareness that the instantaneous rate of change results

from smaller and smaller refinements of the average rate of change. From what they say it seems that at the beginning they observe a constant speed from the zero to the second notch:

Luca: Slowly at the beginning, it is wide, then it seems a constant velocity, then it is tighter and then wider again: this is a clepsydra. But a clepsydra that in the upper part is wider than in the tighter part. Wait, go back for a moment (she goes back with the black point onto the zero again: *dragging test*). How many notches of volume do you have to do, to have one notch of height?

Mara drags the black point very slowly and they count how many notches it crosses till the blue point reaches the first notch.

Luca: Five and a half, say five. Are these (notches) five again to reach the second notch?

The black point is dragged slowly again and they count how many notches it crosses till the blue point reaches the second notch. They count five notches, more or less. Therefore they conclude:

Luca: Yes, at the beginning it has a constant velocity.

Then by a similar process (*discrete dragging*) they observe that from the first to the second notch the average speed of the blue point is greater than from the zero to the first notch; so they decide that the bottle has to shrink at the first notch of height and before that point it has a cylindrical shape.

Luca: Now, count how many notches of volume: one, two, three, four, five let's round off (for a moment he stops dragging the black point, the blue point is on the first notch). Then from the first notch: one, two, three, four so it is tighter (for a moment he stops dragging the black point, the blue point is on the second notch), I mean the lower part is bigger than...

Finally, they check what they found out and they start drawing the bottle on the sheet of paper:

Luca: Therefore, constant till the first notch, then it is tighter and the third notch is the point in which it is the tightest. So: the first notch constant like this (he draws a vertical segment) then it starts to be tighter (he draws an oblique segment) up here. This is the tightest point (indicating the third notch on the sheet of paper).

Mara drags the black point (*continuous dragging*) without apparently paying attention to its movement, indeed the arrow representing the mouse is far from the point (*handle dragging*). She probably wants to find out where the blue point moves faster, because she puts the point on the second notch and explores a neighborhood of the third notch, that is the point suggested by Luca (*dragging test*).

Mara: It is in this passage that it is steeper (she stops dragging and indicates with the arrow of the mouse an interval between the third and the fourth notch).

Therefore they agree that the bottle has a choke point at half height. They conclude that in the upper part the bottle widens and it is wider than in the lower part because the height increases ever slower.

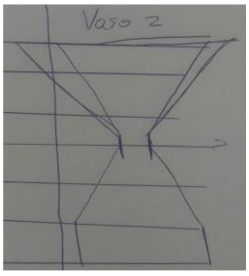

Figure 4: Luca's drawing of the bottle

Discussion

The studies on the interaction between a subject and a software have to take into account a variety of aspects because several components are involved. In this paper we have presented a study to better understand the explorations of functional dependence in a dynamic algebra and geometry environment: in particular, we have identified different dragging modalities and we have shown an analysis carried out through this classification. The analysis highlights how the proposed description of dragging modalities allows an insight into students' problem solving processes.

We noticed that the *handle dragging* is often recognizable through the observation of the mouse's position: if the attention is placed on an object that is not dragable, it is possible that the arrow representing the mouse does not overlap the dragged point in every moment, suggesting a weak haptic control of the solver. But this is not a generalization, because there could be some cases of *handle dragging,* recognizable for example from student's words, in which the student seems to reveal a good haptic control. We observe that there are two types of *continuous dragging,* in some cases it reveals a movement of the object trying to keep a constant velocity, in other cases the object is dragged with a continuous movement, without jumps, but with a variable velocity, for example a point that is dragged back and forth on a line. In the selected protocol there are no examples of *impossible dragging*, probably because the task says explicitly that the only dragable point is the black one; but we identified various examples of this type of dragging in other activities that we analyzed.

One of the potentialities of this classification is that, in order to better describe students' problem solving processes, it is possible to combine two dragging modalities, one indicating the quality of the movement and the other associated with an aim. It could be interesting to develop this study to investigate how a description of students' use of movement in a dynamic algebra and geometry environment is intertwined with the processes involved in conceptualization of functions, that could give an insight into covariation in the concept of function.

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Representing the one left over: A social semiotic perspective of students' use of screen casting

Carol Murphy[1] and Nigel Calder[2]

[1]University of Waikato, Tauranga, New Zealand; carolmm@waikato.ac.nz

[2]University of Waikato, Tauranga, New Zealand; ncalder@waikato.ac.nz

This paper examines the potential of using screen casting with an iPad to enhance learning in mathematics. Data are presented from two seven-year-old students as they use the Explain Everything app to solve a division with remainder problem (DWR). A social semiotic perspective was used to interpret students' use of multiple modes as they represented the mathematical ideas within the context of the problem. We consider how a social semiotic perspective has the potential to draw attention to the students' interests and emerging expressions in representing mathematical relationships. We further consider how the use of representations in the app might relate to student learning.

Keywords: Mobile technologies, multimodality, primary mathematics, representations, social semiotics.

Introduction

Several decades ago, Kaput (1987) predicted that the opportunities afforded by new digital technologies would mean "students of the near future … will be choosing how to represent given relationships" (p.21), and that students' choice in building and interpreting their own representations would be seen as important as the calculations themselves. With the recent introduction of mobile devices into mathematics classrooms, student choice in creating, selecting, and using representations has continued to widen and such new media has been seen to have the potential to "augment and enhance" student learning (Clark & Luckin, 2013, p. 2). In this paper we present data from part of a larger project that examined teacher and student use of iPad apps in primary mathematics classrooms in New Zealand. In particular, we focus on *Explain Everything*, a screen-casting app, with two students (aged seven years old) as they represented their solutions to a problem involving division with remainder (DWR).

Screen casting involves the use of a digital white board screen, which the user can write or draw on. The user can also add images and text. The digital board can then be recorded to capture the images, static or dynamic, along with a vocalisation of the user's thoughts. As such, in mathematics, students can create and present their solutions in real time and in a multi-modal format using text and images along with voice recording. Such apps are generally used as a tool for students to show their explanations in solving problems (Soto, 2015) as they have the appeal of exposing the students' thinking.

Screen casting enables multiple modes of communication, and can provide teachers with further insight into students' thinking and identification of misconceptions (Soto & Ambrose, 2015). Hence, their use as a tool for formative assessment. But might the creation of a screen cast go further than providing insight into thinking? Students can select from a range of modes, including writing, drawings, downloaded images, mathematical symbols, spoken and written language, so there is the

potential for choosing, creating and interpreting different representations for a given relationship (as predicted by Kaput). Furthermore, the use of the screen interface on iPads means that the students can manipulate representations by touch and hand actions (Sinclair & de Freitas, 2014). If the students are choosing to build and create their own representations along with hand actions, can such use go beyond the reporting of solution strategies? We also query whether screen casting, as an example of new media, has the potential to augment and enhance learning.

Theoretical framework: Social semiotics and multimodality

In order to understand the potential for learning with this new media we require a way of understanding how representations are selected and used by students in creating their screen casts. Whilst previous representational theories in mathematics education have been based on an epistemological view of learning as a constructive activity (e.g. Janvier, 1987), further theorising on representations in mathematics has focused on semiotics as intrinsic to mathematical thinking (Duval, 2008; Ernest, 2006). Ernest proposed that a study of mathematics teaching and learning from a semiotic perspective follows sociocultural Vygotskian theories in studying the appropriation of cultural signs and the underlying meaning structures that embody the relationships between signs.

In mathematics, signs are related to mathematical relationships and can only be understood as part of a complex system; there is a "pull towards abstraction" (Ernest, 2006, p.71). If mathematical signs become isolated as purely structural systems they lose meaning. A fundamental view of semiotics refers to representations, as sign production in a broader sense, standing for something else in order to make meaning. Ernest referred to such sign production as "primarily an agentic act" that "often has a creative aspect" (p.69). The students' use of representations in a screen cast may indicate this agentic, creative act, where the sign relates to a form that "strongly suggests the meaning [we] want to communicate." (Kress, 2010, p. 64). Rather than using a sign that pulls to abstraction, the student may choose a representation that indicates what he or she sees as critical in regard to their 'bit of the world' and the mathematical relationship in the context of a problem. As such, we can determine the interest and agency of the sign-maker, and what they attended to, in order to make meaning.

Drawing on both Ernests' theorisation in relation to semiotics in the teaching and learning of mathematics, and to broader theorists, such as Kress and social semiotics, students' choices of representations (text, image, verbal explanations, and hand actions) could be interpreted as sign-making with the potential to make meanings of mathematical relationships within their view of their world. These new meanings may then have the potential to change their understanding of mathematical relationships within a given problem. If we see learning from a social semiotic perspective as generating meaning through sign making (Kress, 2010) then screen casting may have the potential for students' representations to have a role as social and material resources "in and through which meaning is made and by which learning therefore takes place" (Kress, 2010, p.178).

Furthermore, direct interaction with the screen of an iPad allows students not just to choose representations, but to manipulate them through hand actions. The screen cast app also enables students to record verbal explanations. As such, the use of the app allows for students to be agentic in creating signs across a multiplicity of modes. In this paper we consider how a multimodal social semiotic theoretical perspective (Jewitt, 2013) can inform the interpretation of students' choices and dynamic use of symbols, and images along with their use of language. Social semiotics has been used

as a theoretical tool to explain phenomena by revealing things, which might not be evident otherwise (Jewitt & Oyama, 2001). In this paper, the intention is to examine the students' choices of representations, how they manipulate them, and to consider what they see as critical between their world and the mathematical relationship in the context of the problem.

In following a social semiotic theoretical perspective, the intention was to interpret the students' syntactic positioning of images as a source for representational meaning as well as temporal components (Jewitt & Omaya, 2001). That is, how the students placed images on the screen. For example, how the centrality of their placements and connections of objects showed some elements as held together, in contrast to more marginal or disconnected elements. In addition, the intention was to interpret the students' narrative and hand actions as syntactical temporal components. For example how the students' verbal explanations related to how they moved images or drew on the screen.

The study

Two seven-year-old students' use of the *Explain Everything* app are presented in this paper. These data come from a larger research project investigating how iPads apps were used in primary mathematics classrooms. The project involved researcher observation and the collection of video data over one year with three teachers experienced in using iPads in their mathematics classrooms. Further data was collected through student and teacher interview to investigate their views of using the apps. The research team met with the three teachers throughout the year for collaborative analysis and critical reflection of classroom practice and student learning. The use of screen-casting apps such as *Explain Everything* featured several times in the teachers' classrooms and in comments made by students and teachers as they were seen as beneficial for reporting solution strategies.

The data presented here come from one class of seven-year-old children. The problem was set by the class teacher and regarded sixteen dog biscuits shared equally among three dog bowls. The students were given five options, as shown in Figure 1. They were asked to determine which option gave the correct solution, and to explain their reasons using the *Explain Everything* app. The teacher projected the problem onto the screen in the classroom. The students took a photo of the problem to insert into a screen on their iPad, so that they could refer back to the five options.

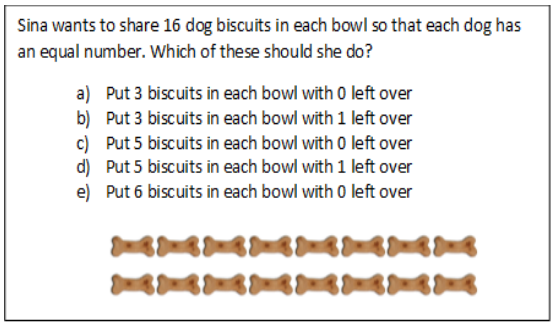

Figure 1: The division with remainder problem

Students worked individually on the problem with the intention to create a screen cast of their solution process for the teacher for her assessment. As they worked in the classroom, six students were selected at random by the researchers to explain more fully their solution strategies in relation to the representations on the screen cast they were developing. As Soto and Ambrose (2016) suggested, the completed screen casts of students may not "capture all the intricacies of students' explanations"

(p.282). As the research team was interested in gaining as much insight as possible, the researchers asked the students to elaborate on their thinking in representing their solutions in the screen cast. These elaborated explanations were videoed to show the iPad screen and students' hand actions, and to capture the students' explanations and responses to the researchers' questions. In this short paper, data from two of the students are presented. These two students are presented here because they showed contrasting approaches in relation to their mathematical solution using partitive and quotitive models. In the partitive or sharing model, the divisor indicates the number of groups and the quotient indicates the number of objects in each group. In the quotitive or grouping model, the divisor indicates the number of objects in each group and the quotient indicates the number of groups (Roche & Clarke, 2009).

Student 1: Fred

Fred downloaded images of dog bowls and biscuits from the internet and positioned five dog biscuits onto each bowl, see Figure 2.



Figure 2: Fred's screen with his solution (a sketch is also provided as the iPad screen is not clear)

Fred: This shows that the answer is (d) because five and five and five is fifteen with one more it's sixteen. So this is the one up here left over. (Fred circled the biscuit in the top right hand of the screen.) So they each get five. (Fred circled the five written above each dog bowl). So that makes it fair and there's one left over for nobody, so nobody has that because they're all full.

Researcher: Did you try any other questions using the bowls? Did you try (a) with the bowls?

Fred: No, I basically knew it was (d) from the start because there were three bowls and you have sixteen biscuits and you have to have one left over.

Fred chose to use realistic images. The dog biscuits were piled onto the dog bowls in a realistic fashion. Fred had also given different names to the dogs. Fred wrote the numeral five above each dog bowl as if in a 'bubble,' and placed the left over biscuit in the top right hand corner of the screen. As Fred said, the dog bowls were "full and fair" and the remaining biscuit was for "nobody." When talking to the researcher Fred used dynamic recordings and hand actions in circling the five numerals and the one biscuit left over in the top right hand corner.

Student 2: Jan

Jan had drawn three circles at the top of the screen. She downloaded images of dog biscuits from the internet and grouped them at the bottom of the screen. Then Jan moved each biscuit one by one to line up underneath each circle (see Figure 3).

Jan: I'm doing five and then I've got one left over. (Jan moved the left over biscuit around the screen with her finger.)

Researcher: Why do you think that is?

Jan: Ummm, I don't know. (Jan scanned back to the screen with the original problem and the options). Because (a) and (b) are not going to be right, but I haven't tried six (referred to the last option). So if I put six…

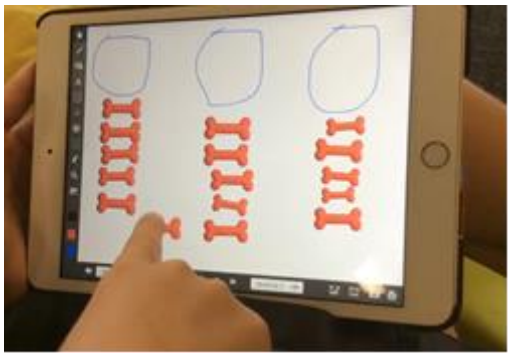

Figure 3: Jan's screen with her solution

Jan placed six biscuits under two bowls but then moved one biscuit from the middle line to the line of four to make five in two of the lines. She then counted the third line as six and moved the sixth biscuit away. Jan then moved the left over biscuit around the screen (Figure 3).

Researcher: What could you do with the spare one? What would you do if they were your dogs?

Jan: Ummm… I'd probably cut it in half so they'd have equal numbers.

Researcher: If you cut it in half how many pieces would you have?

Jan: (Jan used her finger to draw two lines on the left over biscuit) I'd have three halves. One for that one, one for that one, and one for that one (Jan indicated with her finger to the three lines of biscuits).

Jan used realistic images of the dog biscuits but drew circles for the bowls, and placed the dog biscuits in a vertical line underneath each bowl. Jan did not use any numerals, but she referred to the numbers in her oral explanation. Jan seemed in a quandary about the one left over, to the extent that she tried six biscuits, only to find she needed to redistribute them. Jan also moved the left over biscuit around the screen. She then marked the biscuit into three "halves" in order to share the remainder, pointing to each line as she did so. Whilst she used the term 'halves' incorrectly she was attempting to further divide the left over biscuit between the three dogs.

Discussion

In relation to the students' use of models of division, Fred used repeated addition to explain his solution; "five and five and five is fifteen with one more it's sixteen." Fred's solution demonstrated a quotitive model, in that he focused on the quotient as the size of the subset from one of the solutions in the options (i.e. five in each bowl). Jan, on the other hand, used a partitive strategy to share out the dog biscuits. Jan focused on the divisor as the number of subsets, that is the three dog bowls, and so she shared out each of the dog biscuits by counting. Jan then moved to the use of rational numbers by including fractions in further dividing the left over biscuit, although maybe she was influenced by

the reviewers' question. It is noted that neither of the students wrote their solution using mathematical symbols formally, such as 16 ÷ 3 = 5 remainder 1, and this may have been due to the way the problem was set where the options were stated verbally.

In relation to the use of representations, Fred used realistic images and features, along with the mathematical symbols. Fred's 'bubbles' over the dog bowls with the number five suggested a close connection between the number symbol and the quantity of dog biscuits in each bowl. Furthermore, he centralized the dog bowls, piled the dog biscuits onto the bowls and then positioned the left over dog biscuit in the corner of the screen, stating it was for nobody. Interpreting the positioning of the representations from a spatial syntax perspective, it could be said that Fred marginalized the left over dog biscuit both in positioning it on the screen and in verbally stating it was for no one and so indicating his own perspective of the remainder in the context of this problem. Interpreting the temporal syntax, Fred's hand actions in circling each of the five numerals and the left over biscuit, along with his explanation, suggested an emphasis on key features, and mirrored a formal recording of the solution.

Jan also used realistic images for the dog biscuits, but used drawn circles for the dog bowls. These circles represented a container in a more general sense, focusing on the shape but not the features. Jan did not include any number symbols, although she referred to the numbers in explaining her solution. Jan also centralized the circles and dog biscuit images as key features of the problem but she placed the circles at the top of the screen and aligned the biscuits under each bowl. This positioning was not as realistic as Fred's as he piled the biscuits onto the bowls. Interpreting the temporal syntax, Jan's movement of the biscuit around the screen suggested a dynamic visual 'doodle' as she thought about the remainder. Her uncertainty in where to position the dog biscuit was reflected in her comment "Ummm I don't know." Unlike Fred she did not seem satisfied that the left over biscuit should be for no one. In the end, Jan solved this problem in a realistic context that made sense to her, and used hand actions in drawing lines to show how the biscuit could be cut into three pieces.

In interpreting the students' use of representations in creating the screen cast, the intention was to see further into the students' placing of different semiotic modes (symbols, images and drawings) alongside temporal narrative and dynamic movements. As the students chose to use mathematical symbols and 'made up' the signs, they were being critical in relating the mathematics with their 'bit of the world', in order to make meaning. Fred already knew the solution and selected realistic representations to show this solution, tying the key mathematical signs, the chosen images and the quotient closely together. The remainder was redundant and hence placed marginally representing his understanding of the relationships in regard to his bit of the world. Jan chose a less real life representation of the problem but appeared to explore the solution with these representations. Her exploration then led her to the use of fractions in relation to sharing as her bit of the world.

Concluding remarks

The interpretation of the students' use of representations in relation to spatial and temporal syntax may provide further insight into what students attended to in order to make meaning of the mathematical relationships. In this regard, this paper has, arguably, presented an illustration of Kaput's prediction that students will choose to build and interpret their own representations, and that

their choice of representations will be seen as important as the calculation. However, how these choices relate to or augment learning is less clear.

It has been possible to consider how Jan was 'settling' an understanding of the mathematical ideas in solving a problem, maybe by virtual 'doodling' with the remainder. Her use of the representations was agentic and indicative of how she related to the problem, but they also appeared to change her understanding of the mathematical relationships in the problem. For Fred the representations were used to explain thinking that was already formed. He knew the solution. It is not clear that the use of these representations, whilst agentic and indicative of his bit of the world within the context of the problem, changed his understanding of the mathematical relationships. Although, they may have helped him explain or report his thinking.

In these examples it would seem that for Fred, as an example of a student who appeared to understand the mathematical relationships within the problem, the meaning making of the representations in the screen casting referred to an explanation or reporting of a solution strategy, and that this would relate to studies by Soto (2015) and Soto and Ambrose (2016). However for Jan, as an example of a student less certain of the mathematical relationships within the problem, the meaning making of the representations in the screen casting may also have changed her understanding and hence may have augmented her learning about the mathematical relationships in the given problem.

The intention of this paper was to consider whether screen casting, as a way of agentic sign making across multiple modes, has the potential for students' representations to make meaning and hence augment learning. Only two examples are presented here, and whilst a social semiotic approach may shed light on what the students attended to, the use of the screen casting app as new media to augment learning needs further investigation.

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Interactive Virtual Math: A tool to support self-construction graphs by dynamic events

Sonia Palha[1] and Stephan Koopman[1]

[1]University of Applied Sciences of Amsterdam, The Netherlands; s.abrantes.garcez.palha@hva.nl; s.a.h.koopman@hva.nl

An essential condition to use mathematics to solve problems is the ability to recognize, imagine and represent relations between quantities. In particular, covariational reasoning has been shown to be very challenging for students at all levels. The aim of the project Interactive Virtual Math (IVM) is to develop a visualization tool that supports students' learning of covariation graphs. In this paper we present the initial development of the tool and we discuss its main features based on the results of one preliminary study and one exploratory study. The results suggest that the tool has potential to help students to engage in covariational reasoning by affording construction and explanation of different representations and comparison, relation and generalization of these ones. The results also point to the importance of developing tools that elicit and build upon students' self-productions.

Keywords: Visualization, virtual reality, interactive tool, mathematical modeling, reasoning.

Introduction

Students' difficulties with constructing graphs that model dynamic events are well documented in literature (e.g. Thompson, 2011; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Oehrtman, & Engelke, 2010). When modeling a dynamic situation into a graph (e.g. the speed variating with time or the height of water in a bottle variating with volume), it has to be conceptualized as a covariation relation, that is a relationship between two variables that vary simultaneously (Thompson, 2011; Carlson et al., 2002). However, students have a tendency to view functions in terms of symbolic manipulations and procedures rather than as relationships of dependency between two variables. These students might encounter difficulties imagining how the output values of a function are changing while imagining changes in function input values. And therefore they might fail in successfully construct a graph of a function modeling a dynamic situation.

Research has revealed that traditional approaches have not been successful in overcoming the above described difficulties. Technological tools can however afford alternative approaches to the subject. Also, most of the research that provides insight in students difficulties with understanding graphical situations is done in clinical environments. We need to develop a better understanding of students learning in classroom settings.

In our research we developed a tool that intends to provide an alternative way to approach the learning of graphs by dynamic events and an opportunity for examining its learning in the classroom. The tool Interactive Virtual Math (IVM), which can be found at **https://virtualmath.hva.nl**, is designed to support 14-17 years old students at secondary school to understand the graphical representation of relations between variables in dynamic situations. IVM supports this process by addressing the visualization of these relationships. The aims of this paper are to introduce a prototype of the tool, its

main features and design and, to discuss its added value for students' learning based on the results of one preliminary study and one exploratory study.

Theoretical framework

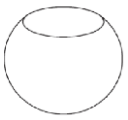
Covariational reasoning

An example of a mathematical task that requires understanding of covariational reasoning is Task A from Figure 1. The task is about a dynamic situation involving the height of water in a bowl and the volume and, it was taken from Carlson et al. (2010), who used it to diagnose students' understanding of graphs of this type of events.

Figure 1: tasks used in preliminary study

Task A

Imagine this bowl is steadily being filled with water.



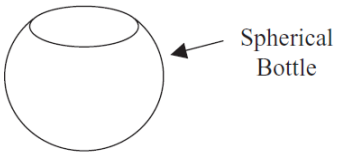
Sketch a graph of the water height in the bowl as a function of the amount of water in the bowl.
Explain the thinking you used to construct your graph.

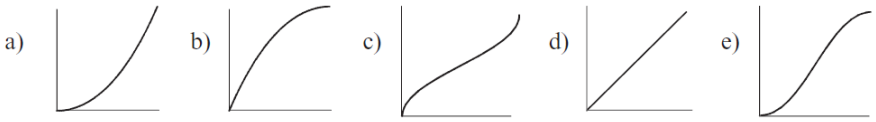
Task B

Assume that water is poured into a spherical bowl at a constant rate.

a) Which of the following graphs best represents the height of water in the bowl as a function of the amount of water in the bowl?

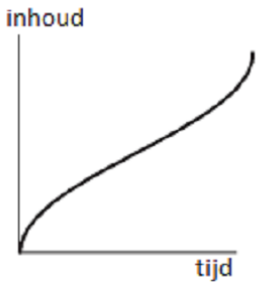
b) Explain the thinking you used to make your choice.





Task C

Assume that water is poured into a bowl at a constant rate. The graph in the figure represents the height of water in the bowl as a function of the amount of water in the bowl. Describe the filling in of the bowl in words,



a) Explain the thinking you used to make the description.
b) Draw a possible bowl

To solve task A, students will need to consider how the dependent variable (height) changes while imagining changes in the independent variable (volume). The coordination of such changes requires the ability to represent and interpret relevant features in the shape of the graph (Carlson et al., 2010).

Carlson et al., (2002, 2010), developed a framework that allows to investigate students' covariational reasoning abilities when responding to dynamic function tasks. The framework describes covariational reasoning as entailing five mental actions, which are successively more complex: (M1) coordinating the value of one quantity with changes in the other; (M2) coordinating the direction of the change; (M3) coordinating the amount of change of one quantity while imagining successive changes in the other quantity; (M4) coordinating the average rate of change of the function with uniform increments of change in the input variable; (M5) coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function. We used this framework to evaluate the quality of students' graphs and explanations in our study.

Guiding principles and main features of the tool

There are many technological tools available for learning graphs from dynamic events, but very few request students' own productions. They are often simulation-tools, which involve whole figures or part of figures that have to be moved, changed or dragged. When students are asked to construct a graph with these kind of tools, construction actually means using representations that are already given or can be synthesized by putting parts together. In this case there is not a true visualization of students' concept image (Vinner, 1983), since part of the representation is already given. A distinguishing feature of the IVM is that it builds solely on students' graphical productions.

The tool Interactive Virtual Math allows students to draw, analyze and compare graphs for themselves and improve the graphs if they conclude this improvement is needed. At CERME 10 we presented a second prototype version of the tool in which the students work on an assignment involving a single graphic situation: the dynamic event described in task A (Figure 1). In later versions we expect it to be possible to use more contexts and varied assignments so that all students can practice at their own level. In Table 1 we present a short description of the main features of the tool: *Self-construction, Contrast, Help 1 and Help 2, Reward* and *flow*. These features are based on general learning principles that include building on students' previous knowledge, interaction and feedback. We expect that the use of the tool will challenge students to create their own graphs and explanations, to make assumptions, conjectures and to reflect upon these (feature *Flow*).

The tool was also built according to topic specific learning principles. Thompson (2011) states that it is critical for students to first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs. Therefore, the graphs in the tool must be drawn by the student themselves and the tool elicit students to imagine relationships from scratch, without presenting any (partial) graphical representation that has not been drawn by the student themselves (feature *Self-construction*).

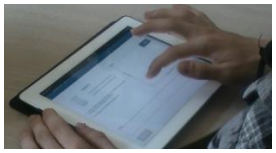
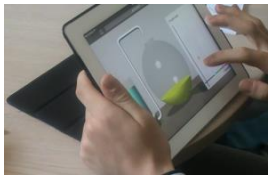
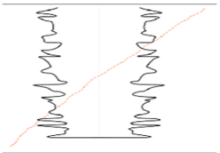
A second guiding idea behind the tool-design is the focus on visualizing quantities. Results from Ellis (2007) indicate that instruction encouraging a focus on quantities can support generalizations about relationships, connections between situations, and dynamic phenomena. To help students to focus on the relation between the height of the water and the volume we provide two kinds of help with the tool: the features *Help* and *Help* 2. In *Help 1* the student visualizes the increasing height of the water in the bowl and he can start and stop the water falling in bowl. In *Help 2* students must assume the height of the water in the bowl and represent it in the graph with dots. We expect that the students,

while guessing where to put the dot for the height, will notice that the difference in height between consecutive dots (values of the height) decreases in certain situations and increases in others.

Another guiding principle was to provide constructive feedback to the students' final graph and to give them a way to evaluate their production. The students get to see, after submitting their graph, the corresponding bowl-figure to the graph they draw (feature *Reward*).

Finally, the tool also includes the use of Virtual Reality (VR), which is still limited to Help 1. Here the use of VR (sound, movement, interaction) is expected to improve the experience of the graphic situation.

Table 1: main features of Interactive Virtual Math

Feature	Description
	***Self-construction*** The student is given two assignments. The first assignment is task A from Fig.1 and the second assignment is a variation of the same task with a cylinder instead of a bowl. In both assignments they are requested to draw a graph that describes the relationship between two variables in the corresponding dynamic situation. The student constructs the graph with a finger, a digital pen or a mouse.
	***Contrast*** The student compares her/his own graph and explanation of the two situations, referred to as *a* and *b*. The student can then submit the graphs or improve them.
	***Help 1*** The student visualizes the increasing height of the water in the bowl. He listens to the water he moves the platform with the ball and he can start and stop the water falling. Using a mobile device and a cardboard, Help 1 can be experienced as Virtual Reality
	***Help 2*** The student connects the graphical representation to the context representation. A Cartesian coordinate system in the plane and the bowl appear next to each other. The student must construct a dot graph that represents the height of the water in the Cartesian graph. He does this by dragging and dropping dots into the graph.
	***Reward*** The student gets the corresponding form of the bowl.

Methodology

Preliminary study

Previous to the development of the first version of the IVM tool, we conducted a preliminary study to explore students' knowledge, skills and difficulties with constructing covariation graphs. The study (February-March 2016) involved *N*=98 students from 4 classes age 15-17 years old and we used three versions of the same task with different questioning (Figure 1). The students in each of the four classes were divided into three groups and each group was presented with one of the three versions.

Analyses of students' written answers showed that the majority of the students (64%) failed to successfully solve task A (see also Table 2). Nineteen of them presented an increasing but incorrect graph, suggesting that they understand that the water increases or that the height increases with the amount of water but they don't have a consistent concept image of this process. Most of these students (13 out 19) produced one straight line (9 students) or a combination of two/three straight lines (4 students). These findings point that the majority of students that solved the self-construction tasks (tasks A and C) could not construct for themselves an acceptable representation. These results motivated the importance of engaging students in self-construction assignments and the development of the IVM-tool.

Table 2: results of preliminary study

	Task A (self-construction graph)	Task B (multiple choice)	Task C (self-construction bowl)
Acceptable	12 (36%)	**25 (66%)**	3 (11%)
Incorrect	**19 (58%)**	11 (29%)	**22 (79%)**
No answer	2 (6%)	2 (5%)	1 (4%)

Exploratory study about the first version of the tool

The first version of the tool was developed in February –April 2016 by a team composed by one researcher-math educator (first author), a high school teacher (second author) and ICT -designers. We decided to use task A (Fig.1) that we considered suitable to explore students' understanding of covariation and within a broad age group. To explore its learning potential and usability we investigated through a small qualitative study the learning of four students age 14-15 years old (two boys and two girls) with different school performance for mathematics. Kevin[1] has high grades for mathematics, Lisa and Anton have average grades and Wilma has low grades. We observed and interviewed the students while working with the tool. The aims of the exploratory study were: (i) to understand how the students construct a graphical representation with IVM; (ii) to identify features of the tool that support or constrain students' successful construction; (iii) to get a better understanding about how the guiding principles work and can be used to develop later versions of the tool. The collected data consisted of video records and students' written work and it was collected at two different moments in April 2016. In both situations the students were asked to go first through the whole application on their own. Lisa was the first student to be interviewed; she used the application on a computer. The other three students Kevin, Wilma and Anton were interviewed together at their

[1] The real names of the students were modified

school. Wilma and Anton use a tablet and Kevin a mobile device. The data was first organized chronologically with relation to each student's attempt to construct the graph and use of the tool. Secondly, a global description of how each student attempted to construct and transform the graph was made and how they used the main features of the tool. We used the covariational framework (Carlson et al., 2002) to get insight in students' covariational reasoning abilities. A summary of the results are presented in Table 3. These results and the data were shared and discussed with the ICT-team and used to evaluate the tool and to make decisions for the development of a next version.

Results and discussion

As we can see in Table 3, all four students improved their graphs on basis of the tool. Kevin produced in the first trial an incorrect graph with three straight lines and he improved it in second trial after comparing the form of the bowl he got in the *Reward* with the bowl in the bowl-assignment. Wilma produced in the bowl-assignment, in the first trial two incorrect graphs: a straight line and afterwards a raising curve. She 'improved' the graph after seeing the cylinder- assignment (*Contrast*). Through consulting *Help 1* and *Help 2* she constructed in a second trial a final acceptable graph. Anton produced in the bowl-assignment several incorrect graphs. His final graph in the first trial is a curve raising slowly. He consulted *Help 1* several times and, based on that, he produced a graph with three straight lines and adapted the length of the line segments. Anton's improvement did not lead to a final acceptable solution and the student remained in doubt whether the pieces of the graph should be curved or not.

Table 3: students' use of the features of the tool during the exploratory study

Features	Kevin	Wilma	Anton	Lisa
Construction (round bowl)	Acceptable final graph after two trials	Acceptable final graph after two trials	Incorrect final graph after two trials	Acceptable final graph after two trials
Construction (cylinder bowl)	All students have produced an acceptable graph at first trial (straight line)			
Contrast	First, all students draw a straight line at assignment one but improve their drawing after constructing the graph of assignment two.			
Help 1: Bowl is being filled up	Doesn't consult help 1 in first trial	changes a straight line into a rising curve	changes the middle line of the graph,	Consults but doesn't improve the graph
Help 2: relation figure - graph	Doesn't consult Help 2 in first trial	changes a rising curve in an acceptable curve	Consults Help 2	Does not understand how it works
Reward	Improves straight line to a curve.	Not observed	Not observed	Does not understand the reward
Flow	Constructs graphs without consulting Help 1 and 2.	Consults Help 1 and Help 2	Consults Help 1 and Help 2 several times	Consults Help 1 and Help 2
VR (Help 1 with cardboard)	Not used	Not used	Not used	rich experience

Based on the analyses of students reasoning while constructing and explaining their graphs, we identified a number of aspects through which students could be brought to a better understanding of

graphical situations, while working with the tool. One aspect is students engagement in covariational reasoning and their progression through the mental actions (Carlson et al., 2002). For instance, Wilma identifies and represents the two quantities changing together (M1). She draws initially a straight line which suggests that she attends only to the direction in which the height changed while imagining increases in the amount of water (M2). *After consulting help 1* she changes her straight line into a rising curve and then into a curve-down followed by a curve- up graph and she is able to explain how changes in the amount of water were related to changes in the height of the water at various locations in the bottle (M3).

Another aspect is students' involvement in actions that underpin mathematical reasoning such as the construction and explanation of different representations and, comparing, relating and generalizing these ones. Examples that we observed include students *comparing* their own graph and bowl filling up with water, which was the case of Wilma when she used Help 1 or Anton switching from Help 1 to his own graph several times; students *evaluating the relation* between the reward and initial graph. Visualizing the bowl of the reward made Kevin to think about the relation between the form of the bowl and the form of the graph. He used the reward to improve the smoothness of the graph curve; students *contrast* the relation between graphical situations of assignment one and two. For instance, Anton switches between one and two and adapt the graph one after seeing assignment two.

As Table 3 shows, different students used different features to improve their graph, which suggests that tool with possibilities to choose to view additional help or not and to be able to switch between the graphical situations, allows for diversity. Furthermore, all students had difficulty with constructing a graph, even with the tool support. This result suggests that self-construction tasks are needed to reveal these difficulties, which can remain unnoticed when using simulation-tools or tools in which the representations are already given.

A final aspect concerns the usability of the tool. Students valued the opportunity of choice and the interactivity of Help 2 (one can drag and decide where to put the point). And, one student (Lisa) who view Help 1 in VR with the cardboard valued this experience as a more enriching one.

There are also some critical issues with regard to the methodology of the study and the tool design. The small amount of students involved in the use of the IVM tool allowed for a fairly detailed study of their interaction with the tool. But, we should carefully interpret our findings since they regard only 4 students. We need to experiment more with the tool in classrooms, in combination with other tasks and forms of interaction and teacher support to better understand its potential and to what extend these findings can be generalized. With regard to the tool design, a number of aspects should be improved in follow up versions. One challenge concerns the self construction- and reward-features. It is left to the tool to decide what is an acceptable representation and how accurate it can be. We programmed the tool to accept any sketch of concave up followed by a concave down graphs starting at the origin. And, for the graph to be considered accurate, the line must be smoothly drawn. Sometimes the tool rejects answers that are accepted by the researchers and teachers. Another concern is the amount of variables involved in the assignments (height, accumulated volume, time, volume per unit of time, shape of the bottle). It is reasonable that the students should focus on one or two variables but not so many that are changing simultaneously. At the CERME conference we also received useful suggestions to improve the tool. For instance the time-counter in Help 1 can be replaced by a volume-counter and, students could fill the bowl by adding themselves cups of water.

This could help students to focus on the relation between height and volume rather than height and time. Another suggestion was allowing students to change the shape of the bottle as this might afford students' awareness of the phenomenon.

Concluding, this paper reports on the experiences of students learning graphical representations by dynamic events with the aid of a new learning technology (IVM); a topic which many students struggle to understand. We have learned that the prototype-tool has potential to engage students in covariational reasoning and we identified a number of aspects that could bring the students, while working with the tool, to a better understanding of graphical situations. Namely, the tool affords construction and explanation of different representations and, comparison, relation and generalization of these ones. The results also point to the importance of elicit and build upon students self-productions.

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Meaning-generation through an interplay between problem solving and constructionism in the C-book technology environment

Ioannis Papadopoulos[1], Dimitris Diamantidis[2], and Chronis Kynigos[3]

[1]Aristotle University of Thessaloniki and CTI & Press Diophantus, Greece, ypapadop@eled.auth.gr

[2] University of Athens and CTI & Press Diophantus, Greece, dimitrd@ppp.uoa.gr

[3]University of Athens and CTI & Press Diophantus, kynigos@ppp.uoa.gr

Starting from Silver's (1997) approach for the importance of the interplay between problem solving and posing in the agenda of creativity, a new kind of e-book, aiming to promote creative mathematical thinking to students, in which its designers enriched the posing element with a constructionist approach, is used and examined in a real classroom. The paper follows a pair of Grade-8 students while they are working on this book. The contribution of this new interplay in the meaning-generation process around the concept of covariation is examined and the change in the creativity landscape by the analogy between problem posing and constructionism is discussed.

Keywords: C-book technology, meaning making process, constructionism.

Introduction

According to Silver (1997) deep flexible knowledge is closely related to creativity and emerges during the interplay between problem solving and problem posing. On the other hand, new exploratory and expressive digital media provide users with access to and potential for engagement with creative mathematical thinking and meaning-generation activities (Hoyles & Noss, 2003). However, education systems fail to rise this challenge due to restrictions stemming from the emphasis given on conformity and standardization in testing (Chevallard, 2012). So, new designs are needed to support students' engagement with dynamic digital media that aim to foster creative mathematical thinking. In this spirit, a pair of students are working collaboratively to solve a problem using a new digital medium, we call 'c-book', ('c' for creativity), a new genre of authorable e-book, extending e-book technologies to include diverse dynamic widgets, interoperability and collective design. In this paper we draw on end-users' interactions examining their interplay between problem solving and constructionism as enabler of meaning-making and try to analyze the effect of this affordance of the medium to the meaning making process of the pair as well as to comment on this new role of Constructionism as facilitator/substitute of problem posing.

Theoretical framework

The close connection of mathematical knowledge with the interplay between problem solving and problem posing flows from the fact that most of the mathematicians do their research, mainly by formulating their own questions and problems and then trying to solve them, rather than solving problems posed for them by others (Borwein, Liljedahl, & Zhai, 2014). In this sense, the generation of new mathematical meanings for students, as an action, may be related with this kind of interplay. Cai and Cifarelli (2005) further refined this link between problem solving and problem posing, considering the posing and solving process to be mathematical exploration structured by this

recursive process. In the context of technology Abramovich and Cho (2015) found that a technological environment facilitates problem posing and turns it into discovery experience. And it is possible for meaning-generation processes to take place during such experience. Obviously, the affordances of the digital environment determine so much the kind of interplay that takes place as well as the meaning-generation process. Papadopoulos, Diamantidis and Kynigos (2016) describe how specific affordances of an expressive digital medium (c-book) led students to meaning-generation process around the concept of angle. However, in their study a possible relation between constructionism and posing is hardly examined. Constructionism is a theory that examines design and learning processes focusing on the ways in which these are part of individual or collective construction of digital artefacts. It illuminates how the representations, the affordances, the rules behind the behaviour of digital objects and the fields in which they reside and the ways in which these representations can be manipulated can all constitute representational registers around which meanings are generated, shared and developed (Kynigos & Psycharis, 2003). It thus provides an analytical lens to study the design and construction process in close interaction with the changes made to the artefact in question and the meanings those changes carry (Papert & Harel, 1991). In the case of a jointly constructed artifact by a group of students, the changes made to the artefact constitute externalization of the group's knowledge. Microworlds are such environments, allowing at the same time personal construction of objects and new meaning. C-books exploit half-baked microworlds which are incomplete by design, challenging students to fix them fostering thus learning through tinkering (Healy & Kynigos, 2010). Students have to solve problems that they encounter, in between and may come up, as a result of students' efforts to make new constructions, in order to fix the initial bug of the microworld. So, the question now is: How the affordance of the c-book technology to support the interplay between problem solving and problem posing/'Constructionism' might contribute to a process of meaning-generation?

The digital medium and the Don Quixote c-book unit

C-book is a new expressive medium that affords the design of modules named c-book units. Each c-book-unit is based on a storyline, and includes diverse 'widgets' between the lines of the narrative. The term 'widgets' is used for objects, such as hyperlinks, videos and mostly instances, or activities, from a range of educational digital tools such as Geogebra and MaLT2, a web-based Turtle Geometry environment that affords Logo-mathematics symbolic notation and dynamic manipulation of 3D geometrical objects, using sliders as variation tools. Most of the widgets refer to mathematical inquiries, constructions and problems. Students can navigate through the pages of the c-book unit and be involved in the included tasks through experimentation, reconstruction and problem solving.

The c-book unit used in this study presents a different twist of Don Quixote's story. It begins with Don-Quixote confronting 30-40 windmills he mistakenly considers giant enemies (first pages of the c-book unit). But, after being close to them he realizes that they are damaged windmills and he wants to repair them. Half-baked logo codes in MaLT2 represent the windmills' fans and sails in various geometrical figures and Don Quixote has to modify the codes so as to repair and reconstruct the fans and the sails.

The study

This study presents an educational intervention designed and implemented in a classroom. Adopting the methodology of "design experiments" (Collins et al., 2004) the focus was on seeking relationships between the learning process and the use of digital media used by the students during the implementation phase. Twenty-four students (18 from Grade-8 and 6 from Grade-9) from a public Experimental School in Athens participated in the study which took place in the pc-lab of the school during after-class mathematics courses for totally eight teaching hours within four weeks. The students were divided into pairs. Most of them were familiar with the usage of 2D E-slate Turtleworlds. Two teachers served as facilitators for technical issues, when necessary, whereas two researchers undertook the role of observers recording instances of the students' interactions with the digital medium. Voice recorders and a screen-capture software (HyperCam2) were used to record students' interactions with the c-book unit tools and their discussions, since both of them constituted our data. The students' interactions were transcribed and the protocols were parsed into episodes with emphasis on the transitions between episodes since these were the points at which the change from solving problems to creating new ones used to happen (Schoenfeld, 1985).

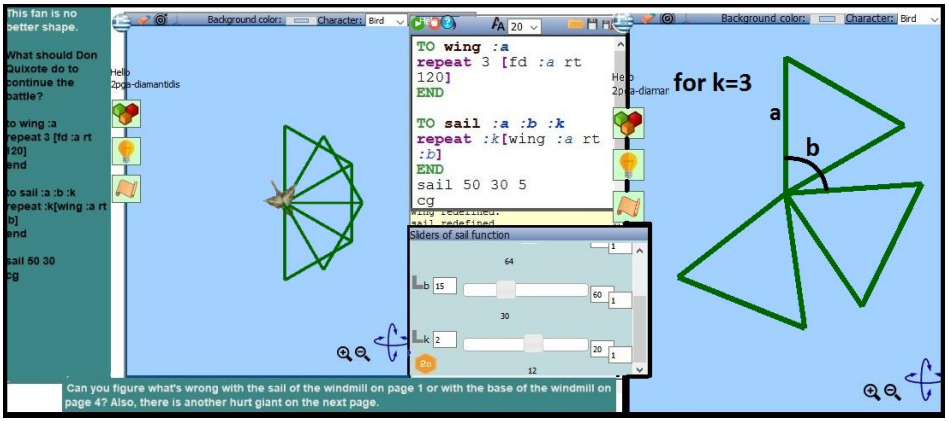

Figure 1: The 'buggy windmill's fan' task in the c-book unit environment

In this study, we follow two students as they are coping with a task asking them to fix a broken windmill. A Logo program was already developed producing a buggy and half-complete fan of windmill (Figure 1, left). It was needed to make changes in the Logo program, to fix the bug and shape up the fan.

Results

The students initially had to fix the bug on the windmill (Figure1, left). The fan was ill-constructed since its wings had not been joined in a proper way. So, they started using the variation tool to observe changes and identify the role/function of each slider/variable. The initial Logo-code construction contained three variables a, b and k, for the ray of the fan, the angle between two consecutive wings of the fan, and the total number of wings, respectively (Figure 1, right).

There are two procedures in this code. The "wing" which uses variable "a" to make an equilateral triangle with side length "a", and the "sail" (main procedure) which constructs the whole fan using "wing" as sub-procedure. Fixing the bug, is an open-ended problem with a variety of solutions (for example, for a polygon-shaped fan a feasible solution would be to replace b with 360/k).

After some back-and-forth of changing dynamically the values of all variables in the code, and examining the results of their actions on the screen the students found a pair of values that made the figure to look like a windmill's fan:

S1-23: We managed to make it well shaped, but only for a certain pair of values; 12 for k and 30 for b. We must put certain values instead of variables.

S2-24: It is not a fair solution; we should find a way to keep the fan well shaped, for any set of values. Is there a possibility that a, b and k vary analogous to each other?

S1-25: What do you mean by "analogous"?

S2-26: I mean that the change of only one value through the variation tool, results to changes for all of them, without our intervention.

S1-27: Let's see [*she changes dynamically the value of* a]. It is not worth dealing with a. It only changes the length. We should find a relation between k and b.

In the extract above, it seems that according to Student-2 the specific pair of the variable values cannot be considered as a proper solution. The references to "analogous" and "without our intervention" are indicative of the student's confidence that a more generic solution such as a relation between the variables, is needed. Therefore, they started actually talking about covariation.

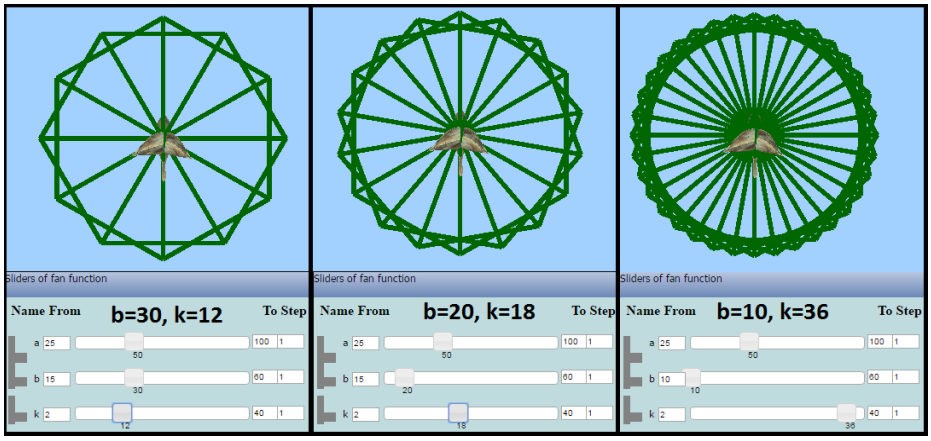

Figure 2: Three pairs of values that make the shape look like a fan.

Thus, in order to find the relation between "b" and "k", students went on with their investigation through dynamic manipulation, identifying pairs of values for b and k that made their construction to look like a proper fan (Figure 2). Their investigation resulted to the conclusion that "b" and "k" might be inversely proportional. Although they reached a conclusion about the kind of relation between "b" and "k", they did not take the next step to express this finding as a formula, so as to use it for fixing the bug, reducing thus the number of the necessary variables. On the contrary, they decided to go on with their investigation, adding a new variable in the sub-procedure "wing":

S1-33: We found the solution. But I think that we must go further. You see, in the program "wing", there is a right turn by 120 degrees, which means that our solution works only for this amount of turning.

S2-34: Yes, we should put a variable instead of 120, let's use the letter "k" again, in order to find out what is going on, and solve the problem for every case of turning right.

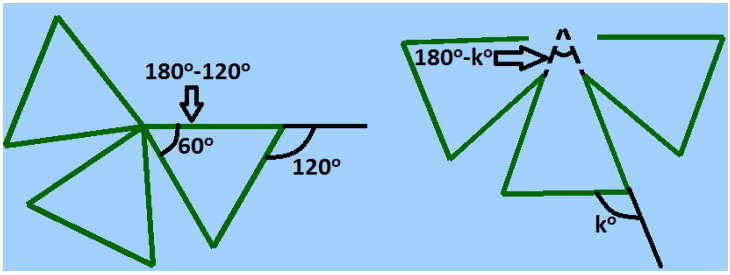

Figure 3: The role of right turn by 120° (left) and by k° (right)

Variable "k" now refers to the right turn for each new wing (instead for the total number of wings). Students made their own construction, by adding a variable in the 'wing' sub-procedure, which actually made the problem more complicated. The choice of a constant right turn by 120° is crucial for having the wings evenly delivered across the fan, since right turn by 120 degrees means that the sails will be equilateral triangles. Substituting the constant right turn by the new variable "k", has an impact on the angles of the triangular wing (Figure 3). Technically, this choice results to a fan even buggier than the original one. However, students did not see it as an obstacle. On the contrary, they accepted the challenge to solve a new problem that seemed to be more challenging to them:

S1-46: Let's use the same variable k, for both, the number of wings in 'sail' and the amount of right turn in 'wing'. [*They ran the program and moved hastily the slider that stands for k. This action changed not only the number of wings, but the shape of each wing of the fan as well.*] What a strange shape!

S2-47: Are b and k still inversely-proportional or proportional amounts? [*They moved the sliders, in order to find pairs of values, as they had done before* (Figure 4).]

S1-48: Variables b and k do not seem to be proportional.

S2-49: Nor inversely proportional. This is not fair!

S1-50: Is it possible that there is no connection, no relation between b and k?

S2-51: What other kind of relation other than proportional and inversely proportional may exist between them?

This question became the starting point for the students to be engaged in a new inquiry, about a new meaning that seemed to emerge. They started speaking about the notion of covariation in a more abstract sense than before (S1-25, S2-26). The spirit of this negotiation is mirrored in the final remark they made in order to solve the problem: "We think that there must be a relationship between b and the new k. We found that for b=30, if k equals to 120 or 240 or 480 or 960 the sail stays well-shaped, so there is a relation like $k=120\cdot 2^x$. We also discovered a pattern for the values of b that is much more complicated." They refer to their observation that if for example b=45 then the most 'acceptable' shapes are the ones with k multiple of 5 (Figure 4).

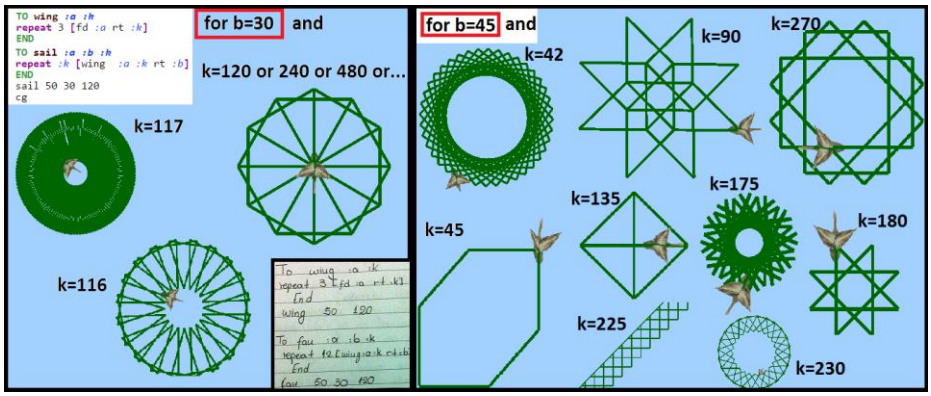

Figure 4: Snapshots for pairs of values of b and k

Discussion

The Don Quixote c-book unit is designed in alignment with the view that creativity lies in the interplay between problem-solving and problem-posing, an idea which is very much in accordance with Silver's approach (1997), arguing that it is in the interplay of formulating, attempting to solve, reformulating, and eventually solving a problem where creative activity may lie in. Indeed, as Papadopoulos et al. (2016) describe, students who used this c-book were able to show creative mathematical thinking that did not emerge instantly but was the result of the above mentioned continuous interplay combined with the provided affordances. In this paper we focus on the meaning-generation processes that took place before the creative moment, relating them with the entrance of constructionism in the agenda of creativity. The members of the designing team of the c-book unit showed an inclination to connect creativity with constructionist activities due to their background and familiarization with this theoretical tradition (Papadopoulos et al., 2015). This resulted to a fostering of the problem-posing element by a constructionism view. So, in this c-book unit the students were working in a context that enabled a continuous and more distinct interplay between problem solving and posing/constructionism and this interplay is examined in relation to whether it operates as enabler of meaning-making in mathematics. As we presented above, the students had initially to explore the problem of an ill-structured windmill and find the part of the problem that is ill-defined. The problem seemed to be solved for a specific set of values (S1-23, S2-26) but this does not ensure the generality of the solution. So, the new problem was to keep the fan well-shaped for any set of values. To solve this task, it was deemed necessary to identify the role of each variable in the solution of the problem which resulted to the knowledge that variable 'a' is not related to the bug (S1-27). This actually transformed the last problem to a new one, asking for the relation between variables 'b' and 'k'. This new problem contributed to the shift of the focus towards the notion of covariation. In order to find the relationship between the variables, a series of new constructions took place. They resulted to a collection of pairs of values for 'b' and 'k' that made the fan look like a proper one. This made the students think that the two variables were inversely proportional. However, the formula was still missing and this became their next problem. Therefore, a new variable was added to the code (S1-33, S2-34). A new, more complex construction took place. The feedback on their screen from their constructions made them doubt their claim for the proportionality of the variables (S1-48, S2-49) and the phrase "This is not fair!" (S2-49) opened the discussion about possibly another kind of relationship (S1-50, S2-51). That was the new

problem which resulted to the more focused discussion on covariation and the possible formula that might fix the bug.

It seems that students made a step further. They tried to answer two interim questions they themselves posed and which came up as they tried to fix the buggy windmill, reconstructing it in a way consistent and meaningful for them. So, they reformulated the initial problem, starting from their reconstructing efforts and came back to solve it anew in a process where new and perhaps more creative aspects of mathematical knowledge were expected to emerge. Thus, the interplay between problem solving and constructionism was apparent, while the interplay between problem solving and posing was not direct. The formulation of new inquiries by students indicates that constructionism facilitated problem posing. Actually, this view of constructionism is close to what Brown and Walter (1990) argue about the problem-posing process: the solver first makes a list of all attributes included in the statement of the original problem, and then, he proceeds in negating each of them formulating thus an alternative proposal, a new problem. However, negating an attribute makes the original problem ill-defined (or 'half-constructed'), and so the solver is challenged to proceed to the 'construction' of a 'new' problem. It is in this sense, we argue, that an interesting connection between problem-posing and the Constructionism perspective arises within the context of the c-book technology and c-book units, never having been identified in the literature so far. At the same time there is evidence that during this interplay a meaning-generation process takes place. The problem of finding the bug of an ill-structured windmill's fan, which seemed to be mostly related to spatial observation and Geometry, turned to be investigated by the students through algebraic procedures, use of symbols and looking for relationships between variables. Students while trying to understand what was going on with the shape of the fan by reconstructing it, actually moved back and forth between processes of problem solving and construction. It was during this interplay that students started moving from the specific notions of proportional and inversely proportional variables to the more abstract notion of covariance.

Conclusions

The problem solving and posing approach in creativity (Silver, 1997) attributes creative moments in the interplay between them. The entrance of Constructionism in the agenda of creativity seems to have an impact in the creativity landscape and perhaps opens new research challenges. The new c-book units that aim to foster creative mathematical thinking in students are based on a design principle that is characterized by 'Constructionism fostering/substituting problem posing'. Then the whole story is evolved around the continuous interplay between problem solving and posing/constructionism. So, on the one hand some new research questions arise about the role of Constructionism in fostering creative mathematical thinking. On the other hand, there is evidence that this interplay between problem solving and posing/constructionism in the path towards creative moments facilitates meaning-generation processes by the students.

Acknowledgment

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Three ways that use ICT to enlarge students' roles in learning

Jean-Jacques Salone

Centre Universitaire de Formation et de Recherche de Mayotte, France

Institut Montpelliérain Alexander Grothendieck, CNRS, Université de Montpellier, France
jean-jacques.salone@univ-mayotte.fr

How can the academic success of students be better ensured? Many math teachers ask this question. Educational researchers have proposed multiple solutions. In our own works we have considered three of them: diversifying the taught knowledge's sources of references and re-contextualize it, involving students in their learning process by giving them various responsibilities, enriching the class's didactical "milieu" with resources and digital tools. In this report we will focus on the second and third propositions with one main question: how can information and communication technologies help increase students' responsibilities in learning? We will expose three examples of how this aim could be achieved.

Keywords: Cooperative learning, teaching methods, computer assisted instruction, students' topos, anthropological theory of didactics.

Focus and rationale

Giving responsibility to students for their learning is a concern that educational researchers have taken for many years. For example Barnes (1977) or Lee and Smith (1996) show that achievement gains are significant when teachers enhance collective responsibilities, Scardamalia (2002) explores some possibilities of computer-supported environments and Coffman (2003) proposes strategies to increase students' roles. Theories also exist that give a frame to this issue, such as the Joint Action Theory in Didactics (Sensevy, 2010) or the Cooperative Learning theory (Slavin, 1995). Our purpose in this paper is to expose three examples of how web resources and digital intelligent systems allow math teachers to involve their students in cooperative activities where they are authors of the lesson tracks, where peer learning is promoted and where curricula are individualized. The intelligent system that will be used in the classroom is the web platform LABOMEP (http://www.labomep.net/). We will show that it is a tool likely to foster student-to-student monitoring, autonomous training and self-evaluation.

Theoretical framework

We will use in this paper concepts from the Anthropological Theory of Didactics (ATD) (Chevallard, 2006; Wozniak et al., 2008; Winslow, 2011), In ATD, learning and teaching are interpreted as ordinary human activities that can be described and analysed through the general concept praxeologies: "A *praxeology* is, in some way, the basic unit into which one can analyse human action at large." (Chevallard, 2006). At first a praxeology is built around a *type of task* which is usually expressed by a verb and a precise object. For example, "to climb a staircase" is a type of task, but to climb, short, is not one" (Chevallard, 1998, our translation). Secondly a praxeology precise a *technique*, a way to realize the type of task, a know-how. This technique is then often justified and lightened by a *technology*, a reasoned discourse which states that the technique is suitable for the type of task and explain how to perform it. "At his turn, the technological discourse contains some

statements, more or less explicit, for which one can ask the reason. We then reach a higher level of justification-explanation-production, the theoretical one" (Ibid.).

Another theoretical concept on which we rely in this paper is that of *topos*:

> In some contexts, didactic tasks actually are *cooperative*, meaning that they must be performed *together* by *several* persons $x_1,\ldots,x_n$, the *actors* in the task. It will be said that each of the actors x_i must in this case perform certain *gestures*, the whole of which constitutes its *role* in the fulfillment of the cooperative task *t*, these gestures being both differentiated (according to the actors) and coordinated by the collectively implemented technique τ. Some of these gestures will be seen as separate tasks, *t'*, in the accomplishment of which x_i will act (momentarily) in a *relative autonomy* compared to the other actors in the task. The set of all these tasks, which is a subset of the *role* of x_i when *t* is performed according to τ, is then called the *topos* of x_i in *t*. (Chevallard, 1998, p. 108, our translation)

A student's *topos* is thus the set of all of the gestures he will have to accomplish in didactic autonomy. In his dictionary of didactic, Chevallard (1996) describes at least three types of student's or teacher's *topos*: 1/ the math disciple/pupil who just listens and observes what is done by the master/teacher; 2/ the math practitioner who masters some techniques in order to realize some tasks and is guided by the animator/teacher; 3/ the math student/researcher who masters the theoretical and technological parts of the praxeologies and has a relative didactic autonomy when studying research question under the direction of its director/teacher. A way to look at the students' *topos* is to focus on what happens with their public speeches or texts. Most of time, these discourses are just communicated and appear in the milieu (Brousseau, 1997), but they are not included in the shared praxeologies which constitute the lesson and that is here termed the *class's praxeological equipment* (Salone, 2015b). Writing the *class's praxeological equipment* is usually a type of task reserved to the teacher; it is an element of his *topos*. The *topos* of the students relatively to the *class's praxeological equipment* is then just to copy and memorize it. But, in some contexts, it may be a cooperative work, so we proposes a four levels scale to analyse how students' public discourses evolve in a classroom: 1/ they are communicated; 2/ they are discussed; 3/ they are included in the *class's praxeological equipment*; 4/ they program the study. At first level, students' public discourses exist in the class' milieu. At second level, they become a local reference: students and teachers refer to them when debating. At third level, excerpts of the students' public discourses constitute the *class's praxeological equipment* and excerpts of them are directly inserted, with no rewording by the teacher; at fourth level, their function is to organize the study

In order to give the teachers some tools to go through these for levels, we develop some *didactic plans*. A *didactic plan* is a teaching technology, a way to conduct the study in a classroom. Chevallard (2006) proposes some examples: a lecture course is "teaching by giving a discourse on some subject", a seminar is "a small group of advanced students […] engaged in original research or intensive study under the guidance of a professor […]". Thus a *didactic plan* aims to shape the didactic relation between the teacher's *topos* and the students's *topos*; in this respect it contributes to the evolution of the didactical contract (Brousseau, 1997).

In this paper we describe *didactic plans* where students are involved in cooperative tasks with a relative autonomy, where they have a math practitioner *topos* and where their public discourses are at second and third levels (see above).

Methodology

Our research was conducted from 2010 to 2016 in math classes ranging from primary school to high school levels. It began with a team of three teachers, including myself, and twelve classes in middle school (students aged from 11 to 15 years), with two classes per grade (from grade 5 to grade 9). Later the team was joined by three more teachers from middle school (four classes per teacher), two teachers from high school (grade 10 to 12, three classes per teacher) and five teachers from primary schools (grade 4 and 5, one class per teacher). In addition two teacher's trainers joined the team. All the teachers involved in the research project agreed to implement *study and research activities* on specific topics and various *didactic plans* designed by an upstream engineering in order to diversify knowledge's sources of reference and to open classes on their surrounding world (Salone, 2015a). Teachers remained free to adapt and insert these activities and plans into their own mathematical progressions. For the research needs, they collected data in their classes: lectures, students' documents, teacher's online textbooks[1], students' notebooks. Twice or three times a year, we visited one of these teachers (that means we observed their classes without interacting) in order to make audio recordings of sessions, to take photographs of the classrooms and to interview some students that were chosen randomly. We did informal interviews with open questions on how the students appreciated the course and where notes were taken. From 2014 to 2016, the whole team also met twice a year in order to share teaching experiences. This was an opportunity to improve the *didactic plans* and to realize informal interviews of the teachers or to refine some of our *a posteriori* analysis.

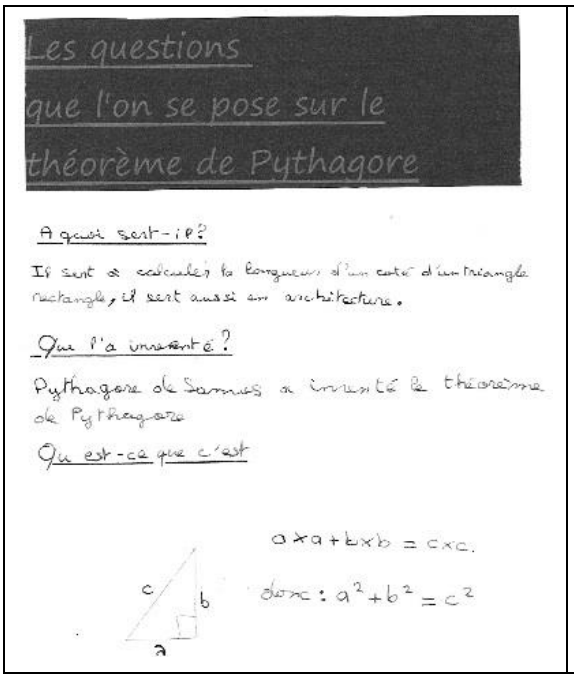
Learning the Pythagoras' theorem

In France, the Pythagoras'theorem is studied in grade 8. The Ministère de l'Éducation Nationale (2008) imposes two abilities: 1/ to characterize the right-angled triangle with the Pythagorean equality; 2/ to calculate the length of a side of a right-angled triangle from the lengths of the two others. It states also that the direct theorem must not be distinguished from its reciprocal (nor from its contraposed form). The case we report here concerns a class at third level of the middle school, with pupils aged 13-14 years (grade 8). The objective was the study of the Pythagoras' theorem. The teacher's online textbook shows his progression: 1/ a survey, at home, of the Pythagoras' theorem; 2/ group works to product synthesis on what is the Pythagorean theorem and its uses; 3/ a tutored training with Labomep; 4/ a selection of exercises' models

Exploration of the theorem and of its uses

As already said, the study began with an exploratory survey conducted at home, on the web and by asking the close family. In the first session students had realized written presentations on Pythagoras and his theorem (Salone, 2015a, p. 323):

[1] In France, teachers are required to write each day a summary of what they have taught in an online textbook. This textbook can be consulted by the students and their parents.



The questions we ask about Pythagoras' theorem

What are its uses?

It is used to calculate the length of a right-angled triangle. It is also used in architecture.

Who invented it?

Pythagoras from Samos invented the Pythagoras' theorem

What is it?

Figure 1: Excerpt of a presentation on Pythagoras (left) and our translation (right)

Four of these presentations were exposed on the blackboard and orally presented by their authors (10 minutes). The teacher then asked some questions: "Does someone have found some more information about Pythagoras?", "Do you agree with these statements of the theorem?", "What the Pythagoras' equality allows us to calculate or to do?" Then he invited the students to freely constitute six peer groups (4 to 6 students per group) to answer these questions and to produce a shared synthesis. In the groups, the students collected and compared their presentations. Their works lead to the emergence of shared statements of the theorem, some uses of it and some problems in line with the official programs. After 30 minutes, the teacher ordered each group to copy one single statement on the notebooks. He had a glance to these statements but, since they all were right, he did not reword them. His first teaching objective was thus reached. In addition, he exposed five of the students' synthesis on the classroom's walls. In this session, the students' *topos* was thus quite unusual; indeed they were first responsible at home of their own first encounter with the theorem (Chevallard, 1998); second they produced a synthesis in peer groups, by reviewing collaboratively one another's works, while the teacher facilitated their work; third they were the authors of the theoretical part of the *class's praxeological equipment* (third level on the *students' public discourse* scale). In this *didactic plan*, the ICT were a tool to access web resources. In the interviews, some students reported being pleasantly surprised by all the uses of the theorem.

Tutored training with a digital media

During a second session, the teacher animated a computer training shaped by a *didactic plan* we call a "tutored training" (Salone, 2015a). It's a moment where students perform training exercises and where they help each other and self-evaluate. In this *didactic plan* a digital media, here the web platform for math teachers Labomep (http://www.labomep.net), provides series of type of tasks. The teacher has to subscribe and then he is allowed to access and deposit resources to organize his courses. Many exercises are thus available, sorted by school grades, chapters and themes. Students may access Labomep freely, without subscription. But the teachers of our team preferred to enrol their students

so that they could control their works (see further). At first the teacher video-projected one problem from the series (Figure 2, left). Each student then individually sought an answer for it. Then the first students who had one consulted with the teacher who evaluated them. After a few minutes, some of the students who had correct responses were invited to help others. At this moment, these students had a *topos* enlarged with teaching task: they gave technological-theoretical explanations and methodological advices, they realized assessments. Meanwhile the teacher too had a specific *topos*: he regulated the activity, reminding some rules, giving some advices. When everyone had come to an answer techniques were finally discussed by the whole classroom and a common solution was chosen and copied in the notebooks (Figure 2, right). The process could then start again with a new exercise from the same set or from another one. In a third session, not observed, the students had also to gather in a file the problems along with their solutions (one problem from each Labomep series). Thus in these sessions several types of mathematical tasks associated with the Pythagoras' theorem appeared through problems and techniques gradually emerged. The students' *topos* was enlarged with monitoring tasks usually reserved for teachers and with writing tasks in order to constitute the *class's praxeological equipment*. ICT were at the heart of this *didactic plan* as they provided sequences of problems and allowed the existence of a joint action. In interviews, students often reflected the feeling they had that tutored trainings, with peer to peer exchanges, improve their understanding of mathematics. Teachers also highlighted that a long-term regular use of such a *didactic plan* enables students with learning difficulties to keep up with their classmates.

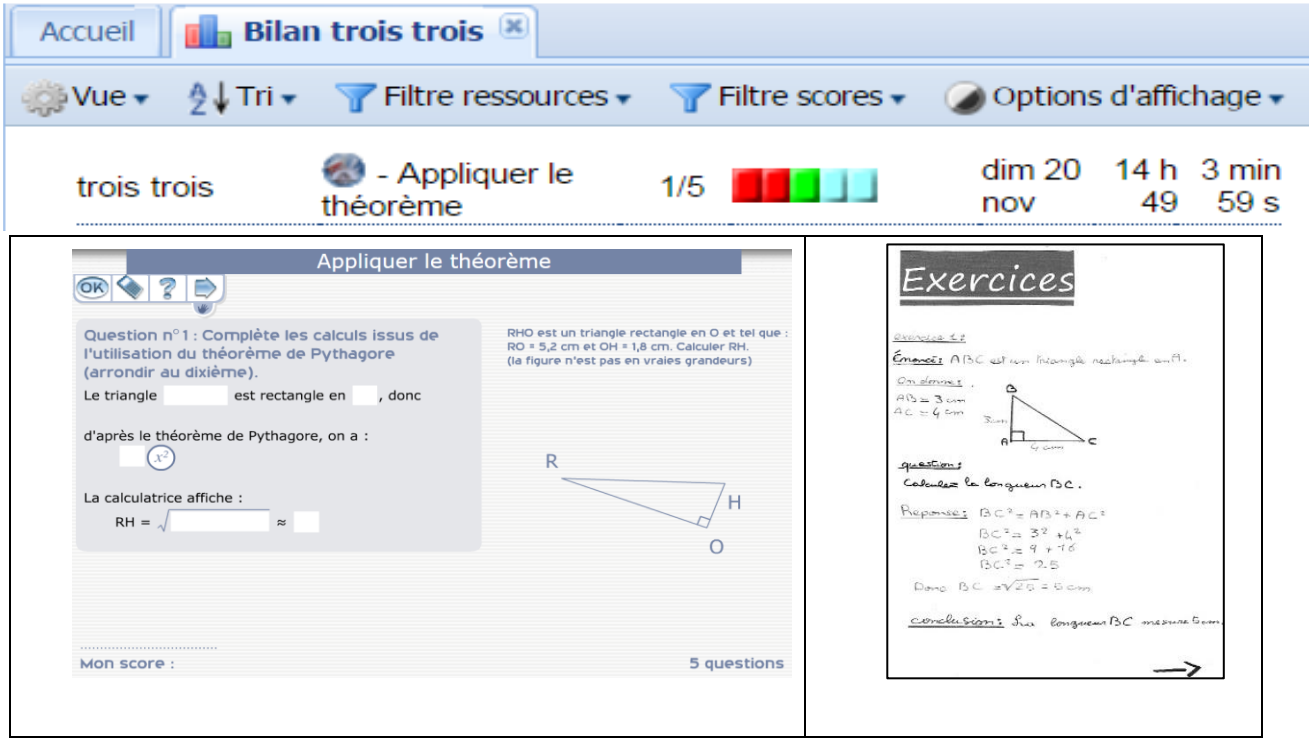


Figure 2: An exercise from Labomep (left) and a shared one (right)

Self-training and assessments

Websites as Labomep are not only resources for interactive exercises. They are also intelligent systems that assess the performance of individual students. In several of the classes involved in our research, teachers took advantage of this potential to develop training sessions in relative autonomy.

Each student had a personal account on Labomep and trained alone or with a classmate. The sets of exercises are either freely decided or defined in advance by the teacher. At the end of a series, Labomep assigns a score and suggests trying again if needed. Video animations reminiscent of technological-theoretical elements are also directly accessible or proposed. The greatest advantage of this *didactic plan* is that it can be continued outside the class. Indeed each student can extend the studies conducted in classroom by training, revision or exploratory sessions at home. Figure 3 shows an example of individual assessment which is made by Labomep and which the teacher can view. The first column is the name and first name of the student (here a generic one), the second column contains the title of the series, the third one is a score, the forth and the fifth ones are day and time. In the third column, the score is at first a mark (1 over 5 here) and the five rectangles corresponding to the five exercises of the series are coloured: when the colour is red, that means the student didn't succeed at all (he had two attempts to succeed), when it is light green he succeeded at the second try, when it is green he succeeded at the first try, and when it is blue he didn't answer the exercise.

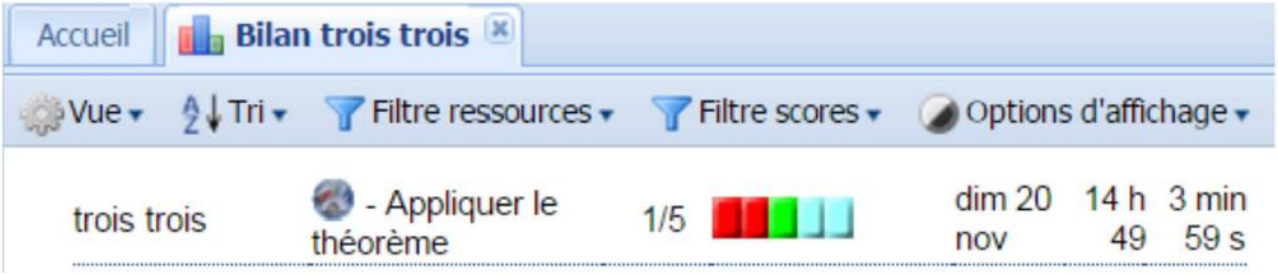


Figure 3: An individual assessment with Labomep

To go back to Pythagoras' theorem, Figure 4 shows the activity of two students on it and on the Pythagorean triples. This is an extract from a page with global statistics generated by Labomep that informs us about the different issues they addressed, adding scores or achieved grades, and the dates, times and durations of sessions. The two students, which we will call here Ali and Ame, had different profiles: Ali was ranked among the top students in his class, whereas Ame was facing some learning difficulties. Data on dates and hours show that both have used Labomep 3 times: twice during classroom sessions, on 24/09/2012 and 01/10/2012, and once outside the classroom on 03/10/2012. In class, within an hour and forty minutes of activity (rows 1 to 7), Ali mastered the first two types of tasks (applying the theorem and showing that a triangle is not right-angled). For the first type of task (rows 1 to 4), his score is three times 0/5 and then it becomes 5/5. For the second task (rows 5 and 6), his scores are 1/5 and 5/5. But he only achieved a score of 1/5 for the third type, at row 7 (use the Pythagorean triplets). Within the same time frame, Ame successfully completed the first two types of tasks, with a maximum score of two out of five for the first one (rows 13 to 18) and one out of five for the other (row 19).

ROW	eleveclasse	numnom ressource	score	sur	note	meilleur ?	date	heure	duree
1	ALI Ci4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-09-24	11:00:35	00:08:43
2	ALI Ci4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-09-24	11:09:18	00:06:57
3	ALI Ci4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-09-24	11:16:15	00:11:39
4	ALI Ci4ième Satie	1 Appliquer le théorème	5	5	20,00	o	2012-10-01	14:07:30	00:28:09
5	ALI Ci4ième Satie	2 Démontrer qu'un triangle n'est pas rectangle	1	5	4,00	n	2012-10-01	14:35:57	00:05:09
6	ALI Ci4ième Satie	2 Démontrer qu'un triangle n'est pas rectangle	5	5	20,00	o	2012-10-01	16:54:25	00:10:59
7	ALI Ci4ième Satie	3 Triplets pythagoriciens	1	5	4,00	n	2012-10-01	17:05:53	00:31:22
8	ALI Ci4ième Satie	3 Triplets pythagoriciens	1	5	4,00	n	2012-10-03	18:27:47	01:00:28
9	ALI Ci4ième Satie	3 Triplets pythagoriciens	2	5	8,00	o	2012-10-03	19:28:15	00:19:23
10	ALI Ci4ième Satie	3 Triplets pythagoriciens	2	5	8,00	n	2012-10-03	19:47:47	00:07:46
11	ALI Ci4ième Satie	4 En deux étapes	5	5	20,00	o	2012-10-03	19:55:41	00:53:14
12	ALI Ci4ième Satie	5 En deux étapes (bis)	1	5	4,00	o	2012-10-03	20:49:05	00:05:11
	AL ...Résultat				***7,66666667***				***04:09:00***
13	AMEF4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-09-24	10:57:19	00:19:55
14	AMEF4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-09-24	11:17:15	00:11:11
15	AMEF4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-10-01	14:12:32	00:09:23
16	AMEF4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-10-01	14:21:56	00:17:09
17	AMEF4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-10-01	14:39:05	00:01:58
18	AMEF4ième Satie	1 Appliquer le théorème	2	5	8,00	o	2012-10-01	16:57:29	00:23:50
19	AMEF4ième Satie	2 Démontrer qu'un triangle n'est pas rectangle	1	5	4,00	n	2012-10-01	17:21:31	00:14:19
20	AMEF4ième Satie	1 Appliquer le théorème	0	5	0,00	n	2012-10-02	15:10:30	00:03:03
21	AMEF4ième Satie	2 Démontrer qu'un triangle n'est pas rectangle	3	5	12,00	o	2012-10-02	15:13:40	00:11:48
	AM Résultat				***2,66666667***				***01:52:36***

Figure 4: Excerpt from statistical assessments of students in Labomep

Out of the classroom the path differences are even more marked. Ali returned to Labomep, two days later, more than two hours in the evening (rows 8 to 12); he trained himself to solve the third type and didn't succeed (his best score is 2/5). After that he went on working on two other types of more complex problems (rows 11 and 12). Ame just spent a quarter of an hour taking the first two types, in the afternoon one day after the second session (rows 20 and 21). He partially succeeded the second type of tasks, reaching a score of three out of five. Thus, with intelligent digital systems such as Labomep in such a *didactic plan*, courses and students' paths can be individualized. According to teachers, it is very beneficial for learning: it consolidates the skills of all students. Those who have difficulties have tools to progress at their own pace and perform better evaluations, those who already have a good level complement their knowledge. Some teachers have also chosen to look at these individual activities outside the classroom so that everyone's work is rewarded regardless of the initial or achieved levels in mathematics. Quarterly average scores are thereby increased, which greatly helps to maintain students' motivation.

Conclusion and perspectives

Through these examples we have therefore tried to identify some benefits on learning induced by the use of *didactic plans* including ICT and which enlarge students' *topos*. The first one concerns the *class's praxeological equipment*: students become authors of the lecture, of its content and its programming. The second benefit is related to the joint action: ICT facilitate peer exchanges in *didactic plans* where students endorse teaching tasks that are usually assigned to teachers. The third benefit is the differentiation of learning: intelligent tutoring systems such as Labomep allow tasks to be performed in individualized ways and to be continued at home. Can we conclude that students are more motivated when using ICT? And does this improve their learning of mathematics? The general consensus amongst the participating teachers and students was yes. But there are other factors that might explain this conclusion. First we worked with an extremely motivated team of teachers who were very dynamic and keen on interesting their classes. Second today's students easily understand and appreciate ICT related activities. So it is not sure that these methods would ensure success for all students. Our research objectives are now to study the conditions and constraints of implementing such *didactic plans* in regular classes.

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Nature and characteristics of digital discourse in mathematical construction tasks

Florian Schacht

Universität Duisburg-Essen, Didaktik der Mathematik, Germany; florian.schacht@uni-due.de

This paper introduces the concept of digital discourse in Mathematics using a philosophical framework by Alexander Galloway. The notion of the digital is discussed and the concept of digital discourse is elaborated on that basis. The empirical data shows its value by reporting on transitions in language when working with digital tools on geometrical tasks. Existing research findings show effects of DGS on language changes referring to geometrical objects and actions. The present study analyses qualitatively both students' language referring to mathematics as well as to the digital tool in the context of geometrical constructions. The empirical results give insights into processes and transitions in the language use (by students) from a tool-oriented language (e.g. referring to buttons) to a mathematical-oriented language (referring to mathematical concepts) and aim to explore the nature and the characteristics of digital discourse.

Keywords: Language, digital tools, geometry, discourse.

There are substantial research results concerning the changes in language used for describing mathematical actions and objects when working with DGS (Kaur, 2015; Sinclair & Yurita, 2008). This paper shows results focusing on empirical phenomena concerning a language that students use in order to describe actions and objects referring to the digital tool. The interplay between these two layers referring to the tool as well as to the mathematics is analyzed in detail. The analysis will give empirical insights into the transition-process between the language of mathematics and technology and, by doing so, will examine a central facet of digital discourses.

Etymological and philosophical aspects of digital discourse

What is digital discourse? Initially one might think of SMS-chats, of instant messaging via Skype or WhatsApp, of blogging or the like of tweets, of posting messages on social network sites, or of video-based online-discussions (e.g. Llinares & Valls, 2009). There is no doubt that all these forms of discursive practices have their roots in the way new media is used. But is it adequate to use the predicate *digital* for such discursive practices? For the examples above, this paper will rather use the term discourse "in the New Media" (Thrulow & Mroczek, 2010) to differentiate it from the term digital. But what is the *digital*? This paper wants to stress the notion of the digital in the mathematics classroom, especially the notion of what can be seen as digital discourse in mathematics. To do so, we will follow a philosophical path drawing on a work by the philosopher Alexander Galloway (2014), in which he gives an introduction to the work of the philosopher François Laruelle and—by doing so—tracing back the notion of the digital in philosophy to Plato and Sokrates and, especially, to Hegel's work. For Galloway, rather than distinguishing zeros and ones (the digital) compared to continuous variation (the analog), the "digital is the basic distinction that makes it possible to make any distinction at all. The digital is the capacity to divide things and make distinctions between them. Thus not so much zero and one, but *one and two*." (Galloway, 2014, p. xxix) In that sense, the digital is closely connected to the notion of difference. This philosophical perspective on the digital is used here to discuss its value for the mathematics

classroom and especially for discursive practices and processes of concept formation in it. By doing so, this approach does not claim to adopt Galloway's perspective on philosophy and his non-standard philosophical approach drawing on Laruelle (c.f. Laruelle 2010). It rather uses perspectives he offers to introduce the concept of digital discourse in mathematics and to better understand its nature and characteristics.

The understanding of discursive practices and the underlying norms have been a major subject of study in mathematics education. And yet there is a need for reactivating such analytical approaches in the light of the use of digital tools since digital tools "give rise to new ways of thinking that may conflict with the established discourse of formal mathematics" (Sinclair et al., 2016a). In their analysis, Sinclair and Yurita (2008) outline the way in which the use of dynamic geometry changes discourse, e.g. transitions from static to dynamic forms of discourse. Also, Schacht (2015a; 2015b) reports on shifts in language regarding student's documentations, in which the students use a language that clearly refers to the digital tool and not to mathematics. Both examples show—in different ways—that the digital tool affects the discursive practices in class. And still we know little about normative rules affecting the discursive practices: the conceptual (mathematical) norms involved, the social- and socio-mathematical norms, the norms established and the norms brought in by the technology in use. By introducing the concept of digital discourse, this paper approaches an understanding of discursive processes by using Galloway's perspective on the digital in order to distinguish different discourses referring to the tool or to mathematics with specific underlying norms. Therefore, we will first briefly highlight Galloway's (2014) notion of the digital, then introduce the definition of digital discourse in mathematics before applying it to the empirical data.

What is *the digital*?

Building on the broad definition of the theoretical concept of the digital that points at the notion of difference (see above), Galloway describes the operation of the digital as follows: "the making-discrete of the hitherto fluid, the hitherto whole, the hitherto integral. Such making-discrete can be effected via separation, individuation, exteriorization, extension, or alienation. Any process that produces or maintains identity differences between two or more elements can be labeled digital." (Galloway, 2014, p. 52) Although the digital can be seen as an archetype of philosophical thinking in general (tracing it back to Plato and Sokrates), Galloway (2014) describes the digital as fundamental to the dialectics of Hegel with its two moments: "the (digital, F.S.) moment of *analysis*, where the one divides in two (1→2, F.S.), and the (analog, F.S.) moment of *synthesis*, where the two combines as one (2→1, F.S.)" (Galloway, 2014, p. xxix). Galloway also attributes analog and digital to both moments (p. xxxi). Although Galloway (2014) does not focus on computers or new media in this work but rather on discussing the theoretical concept of the digital in general, the (digital) process of discretization and of computation however, of extending *the one* (or, in a mathematical context: the mathematical concept) "beyond its own bounds, thereby branching the one, splitting it" (Galloway, 2014, p. 52) is inherent to computers and hence to digital tools. In this sense, the dialectic idea is existentially present and closely related to the digital: "Hegel is dead, but he lives on inside the electric calculator." (Laruelle, Introduction aux sciences génériques, 28, cited in Galloway, 2014, p. xxxiv)

Digital discourse in mathematics

Following Galloway's discussion of the digital, the following definition is used here: Any discourse that produces or maintains differences between two or more elements can be labeled digital. This definition does not necessarily focus on technology. Also, the notion of digital discourse presented here certainly differs from studying discourses in the new media (Llinares & Valls, 2009). As a theoretical concept, this notion can be applied to any discursive practice. However, the analysis of the empirical data will demonstrate that the concept of digital discourses can be used to structure and describe conceptual processes and transitions underlying the work with digital tools because the (*digital*) distinction between expressive reference to the mathematics and to the digital tool and, in line with that, the corresponding underlying norms seem to play a central role in these processes. In this sense, the approach follows the "need to study the transition phases in the progress of geometrical concept formation" (Sinclair et al., 2016b, p. 696).

The term discourse is used here in a pragmatic (more precise: from an inferential) perspective (Schacht & Hußmann, 2015): Individual conceptual processes and mastering mathematical concepts is understood as being able to give reasons for the use of concepts within discursive practices (=master the inferential relations), similar to Wittgenstein's idea of mastering the rules of the language game. In this perspective, individual conceptual acting is highly normative since the individual acknowledges the reasons one has for applying a concept to be true or at least to be adequate in a certain situation. Hence, it is one of the tasks of this analytic approach to reconstruct the (normative) rules that the individuals (as concept-appliers) follow. Discursive practices, in this perspective, give access to individual conceptual processes and the underlying social and individual norms. Galloway's notion of the digital is used here for digital discourses to differentiate between different (normative and conceptual) discursive layers, and especially between the following two layers discussed in this paper concerning the mathematics and the tool. In this sense, it is the aim of this paper to introduce the concept of digital discourse and to better understand its characteristics and nature when working with digital tools exemplified by an example from geometry class.

Language, written documentations and digital tools in geometry

Before analyzing the digital discourse in the empirical example from a dynamic geometry environment (DGE), this paragraph will briefly highlight research findings on the changes in language when working with digital tools. Such tools affect the conceptual thinking and acting with the mathematical objects as well as the written solutions (Ball et al., 2005; Weigand, 2013). In geometrical contexts, DGS can support the grasping of both the geometrical objects as well as the actions conducted with these objects (Jore & Parzysz, 2005). Also, the use of DGS changes language and discourse in various ways. Hölzl reports that students tend to use active verb forms while working with the dragging mode (Hölzl, 1996) which mirrors the movement and the dynamic actions. Also young children (ages 7–8) tend to change forms of reasoning using action verbs influenced by dynamic and temporal elements of the DGS like the dragging tool (Kaur, 2015). In line with that, changes in discourse and vocabulary could be observed regarding the transition from static to dynamic forms of discourse about geometrical objects: "[S]hapes were discussed as if they were a multitude of objects, changing over time, rather than as a single object" (Sinclair & Yurita,

2008). Hence, not only the understanding of the objects and actions change, but also the language used in order to make actions and objects explicit.

Whereas most of these findings report on transitions of language regarding *mathematical actions and objects* when working with DGS, Schacht (2015) shows that students also use a rather naturalistic language to describe the *manual actions* precisely to get to their solution as well as *objects referring to the tool* like buttons or commands. This paper focuses on the way in which students describe actions and objects both referring to the mathematics and referring to the digital tool as well as on the transitions between these layers when working on geometrical constructions. These different referential units—the mathematics and the digital tool—each reflect certain conceptual—and, following the pragmatic approach, also normative—layers e.g. it makes a difference whether one describes the solution process in terms of the mathematical process or in terms of the manual actions conducted with the digital tool.

This paper studies the transitions of using *tool language* to using *technical language*. This paper reports on results of a qualitative study, in which processes like these were studied based on students' involvement on construction tasks using DGS (GeoGebra) with respect to the following underlying research question and the overarching interest to explore the nature and characteristics of digital discourses: *Which language transitions can be observed between a tool- and a maths-oriented language when working on construction tasks?*

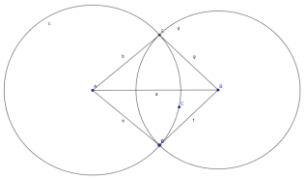
Methods and design

The empirical study was conducted with N=20 students (age 13–15) from different schools. All students worked in pairs with GeoGebra within clinical interviews. This paper uses some of the data of this bigger project in order to demonstrate the potentialities of digital discourses as a way to understand lexical transition processes better . A systematic analysis of such lexical processes is given in Schacht (in revision). All pairs that did the construction tasks were videotaped during the construction and afterwards. The interviews were analyzed qualitatively focusing on the oral and the written language by using lexical categories (Schacht, 2015). The written documents were also analyzed. Three to four geometrical tasks were given to each pair; the duration of working on these tasks ranged between 40 and 80 minutes. All students were introduced to the DGS first and they were encouraged to explore some main functionalities since most of the students had not have much experience working with GeoGebra. Most tasks had an explorative character, meaning for example that the students were asked to construct certain objects and try to formulate a description. Within a task aiming at exploring the concept of symmetry, the students were asked to describe and give reasons for their findings. In this sense, the information the experimenters were trying to obtain from the interviews focused on the way the students work on such geometrical tasks and the language they use when speaking and writing. A detailed description of the tasks relevant to this paper is given below.

Results and discussion: Transitions in language

The pairs of students were given a geometrical configuration and they were first asked to (reconstruct) the given Figure 1 with ruler and compass (on a sheet of paper) and formulate a written description of the given construction by using ruler and compass for themselves (line 1

below shows the description of student 1 (S1)). After that the students worked in pairs with a DGS. They were then asked to construct the same figure with GeoGebra and then formulate a description of the construction together (lines 2 & 3 shows the common description of S1 and S2 with the DGS).

Line 1	S1 (r&c): Draw a 5,7cm straight line (German: Linie) from A.	 Figure 1: The given geometrical configuration
Line 2	S1&S2 (DGS): Click on the button "segment" at the top.	
Line 3	S1&S2 (DGS): Segment from the center to each point of intersection.	

The analysis focuses on the language the students use to describe the objects they deal with. Using ruler and compass, S1 refers to a term from everyday language (line (*German: Linie*)) which is—in the German translation—not considered to be a proper mathematical term. When the students work with the DGS, they document their manual action (line 2) precisely by describing which button to click on. In this case, they choose the segment-button. In the version of GeoGebra they use, the name of the button "segment" is shown. Hence, the term *segment* refers to a *button* (c.f. Schacht 2015a) which is an object that refers to the digital tool. It is important to note that although *segment* can be considered as a mathematical object, the students describe a button, hence an object referring to the tool. On the other hand, in line 3 the students refer to the segment from the center to the intersection points. This description refers to the mathematical object of segment.

This analysis shows two transitions in the students' language within their documentations. First, there is a change from the description of a given object in everyday language (line 1) to the description of a certain button (line 2). For the students in line 2, *segment* is a signifier of the button that they use for their documentation. The second transition shows in which way this signifier is used as a mathematical term to refer to the mathematical object of the *segment* (line 3). The students do not refer to a button. They rather use the term segment within a mathematical description. Hence, the analysis gives insights into the process of the description of buttons (as objects referring to the tool) to the description of segments (as mathematical objects). This is a central characteristic of digital discourses: The distinction between the tool-oriented and the mathematical language offers a possibility for the students to use terms common in mathematical discursive practices. The example shows that digital discourses can follow a linear structure in which the difference between the tool oriented layer and the mathematical layer can even be bridged by the digital tool since it offers expressive resources that students adopt.

This first example has limitations though: The students can easily adopt the term *segment* because it connects to a colloquial use of the term. As the following contrasting example will show, obstacles may occur when students use terms that do not connect easily to a colloquial understanding. In this task, the students explored the phenomenon of symmetry. The students were given a GeoGebra file with two quadrilaterals, where one quadrilateral was the image of the other under a line reflection. The line itself was not depicted. By dragging the one quadrilateral, the other one moves according to the line reflection. The students were asked to construct a new geometrical figure by starting with the two rectangles (Figure 2) and the two students S3 and S4 formulated a description for a construction he made with the DGS afterwards.

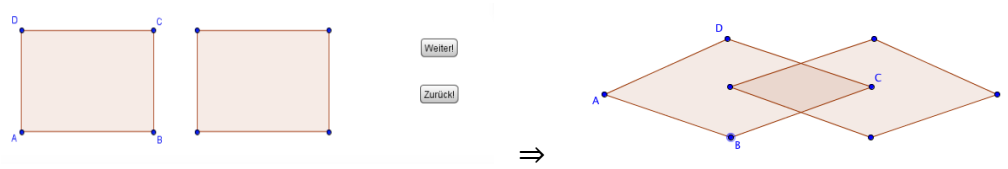
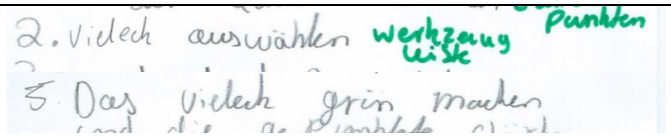


Figure 2: Exploring the properties of a line reflexion. By dragging ABCD (left) the image moves correspondingly

In the next step, only the written description was given to the other student who then had to (re-)construct the original figure with the DGS. In the following example, student S3 draws a triangle by using the *polygon-button* 

. He had seen the signifier of the button during his construction. In both descriptions below the student S3 refers to the triangle using the term *polygon*.



line 2	S3 (DGS): (toolbar)	2. Choose polygon
lines 6-8	S3 (DGS):	5. Colour the polygon green

Transcript of the oral description:

Turn 1 S3 (DGS): By pulling at the left polygon, here at the points, coordinates, in order to have three rectangles.

First, the student refers to the polygon-button (line 2) in order to describe a manual action precisely and the objects (referring to a *button*) needed in order to handle the DGS. Next, the student adopts the signifier in order to refer to the mathematical object (lines 6–8). In lines 6–8, the polygon is seen as a static object that has to be colored in green. Third, this mathematical object is used dynamically since the student describes how to pull at it (turn 1). Hence, this example shows a transition from the description of an object that refers to the tool (*polygon button*) to the description of first a static and then a dynamic mathematical object (*polygon*).

Although this transition seems to mirror the process that could be observed in the first example, there are significant differences. It is important to note that student S3 does not refer to the term *triangle* at all. Instead he sticks to the term polygon throughout the interview. When (his partner) S4 has difficulties with the description and asks S3 to describe precisely what he meant by *choose polygon,* S3 does not refer to the three points that his specific polygon (as a triangle) has (turn 1). Hence, this example shows that although S3 uses the signifier to deal with the mathematical object, he has difficulties with grasping the concept to distinguish between a polygon and a triangle since he has no conceptual understanding of this term. This analysis shows interesting obstacles. First, S3's description to *choose a polygon* is viable, but it is not precise enough for S4 to understand that he has to create a triangle. There are several other scenes in this interview, which lead to the conclusion that S3 does not grasp this concept as he cannot connect it to other mathematical concepts. Especially, it is not obvious for S3 that a triangle is a certain polygon. This example raises the question of how it is possible to support a *semantic* connection to the students' knowledge by using the digital tool and the given signifier. In terms of the characteristics of the digital discourse, the student does not manage to make the mathematical concept explicit by giving reasons for it in the course of the interview. He rather incorporates the expression *polygon* offered by the user interface. The second example gives insights into a divergent digital discourse because there is no mediation between the two layers in order to establish a commonly shared mathematical

understanding of the construction. The reason for that is the missing conceptual transition between the two layers mathematics and tool.

Final remarks

The two examples discussed in this paper show similar transitions in students' language when working with digital tools and give insights into digital discourses. They give insights into the process of the descriptions from objects related to the tool (e.g. buttons) to mathematical objects. Regarding the research question, the examples show different lexical phenomena when working on construction tasks. Example 1 reveals that digital tools can support lexical processes by adopting the signifiers' names in the description of the construction. The students manage the shift from a language that focuses on technology to a rather mathematical language. In other cases this can lead to obstacles which show the need for a conceptual connection to concepts the students already know (example 2). These different lexical transitions show that the use of a rather calculator-oriented language has potentialities as well as limitations. Although it is important for the mathematics classroom to support the use of both language referring to the tool (e.g. in exploration situations) as well as to the mathematics, it remains a challenge to develop means to bridge and support language processes in a way that students not only adopt signifiers but rather grasp a conceptual notion of the mathematical objects and actions. Hence, studying language processes like these shows the importance of a lexical consciousness of the different lexical norms of adequacy when language is used in specific situations (Schacht, 2015).

Studying such language transitions illustrates the role of the notion of difference in discursive practices when working with digital tools. The two examples in this paper show two different digital discourses, understood as discourses that produce or maintain differences between the mathematical and the technological discursive layer. On the one hand, a digital discourse can have a linear structure in which the difference between the mathematical and the technological discursive layer is bridged. On the other hand though, that difference between these two layers maintains throughout the discussion. Regarding the characteristics of such digital discourses, the examples show that it makes a difference if the students use expressions referring to the mathematics or to the digital tool. One the one hand, each layer implies specific underlying norms and specific conceptual aspects, which refer either to the mathematics or to the digital tool. On the other hand, the processes above show the dynamic of such discourses and the way in which these—linear or divergent—transition processes can have implications on the individual concept formation processes.

The results also show possibilities for further studies of the digital discourses in respect (i) to the underlying conceptual processes, (ii) to a deeper insight into the nature and characteristics of digital discourses and (iii) to the potential for classroom practices to foster digital discourses within conceptually supporting environments.

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Using dynamic and interactive technological tools to support conceptual learning of equations among low-achieving students

Osama Swidan[1,2] and Wajeeh Daher[2,3]and Nael Darawsha[2]

[1]Kaye Academic College, Beer Shiva, Israel; osamasw@gmail.com

[2] Al-Qasemi Academic College, Baka Al-Garbia, Israel; naal1975@gmail.com

[3] An-Najah National University, Nablus, Palestine; wajeehdaher@windowslive.com

Mathematics education researchers have been interested in students' understanding of the equality as equivalence relations. Doing so, they pointed out that the notion of equality is difficult for students to perceive. We provided one pair of 16-year-old low-achieving students with a productive environment (technological tool, supportive teacher and an authentic activity) to support their learning of equality sentences as equivalence relations. We examined the pair of students' routines in this environment. The research results indicated that the students followed a sequence of routines where the teacher and the technology had an effective role. Moreover, students' substantiation routines relied on empirical argument that utilized concrete realizations afforded by the applet.

Keywords: Commognition, dynamic technology, low-achieving students, equation, equivalence.

Introduction

The mathematics education of low-achieving students has attracted educators' attention for a long time. To support the mathematics learning of these students, one of the recommendations is to conduct a classroom environment that is conducive to learning (Leone, Wilson, & Mulcahy, 2010). This can be done, among other things, by giving students authentic tasks and dynamic tools (National Council for Curriculum and Assessment, 2003; National Council of Teachers of Mathematics, 2000), and, at the same time, by maintaining effective teaching (Ball, 2003), for example through questions. By authentic tasks we mean, tasks that are situated in meaningful contexts that reflect the way tasks might be found and approached in real life. In the present research, we tried to follow these principles by giving low-achieving students authentic activities related to equivalence relations. At the same time, the students worked with an applet suited for learning equations as equivalence relations; issues that have been indicated as critical to algebra (e.g., Stacey & Chick, 2004).

Students' understanding of the equivalence relations

Mathematics education researchers have been interested in students' understanding of the equivalence relations (e.g., Kieran, 1981, 1992; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008). Knuth et al. (2008) argue that the notion of equality is often complex, and thus difficult for students to perceive. Furthermore, Kieran (1992) considered the equivalence relations as a pre-requirement for understanding structural representations such as equations.

Knuth et al. (2008) examined middle school (grades 6-8) students' definition of the equal sign. They found that those students had three types of conceptions: a relational conception (when the student expressed the idea that the equal sign represented an equivalence relation between two quantities), operational conception (when the student expressed the idea that the equal sign meant "add the numbers" or "the answer"), and other conceptions; for example, when the student used the word

"equal" in the definition. Several researchers expressed the view that helping students acquire a relational conception of the equality sign would help them succeed in algebra and beyond (e.g., Hunter, 2007; Knuth et al., 2008). Generally, this concept, together with the related concepts, as equivalence and equation, are complex ones and difficult for students to understand (Hunter, 2007; Kieran, 1981).

A productive environment for students' learning of the equivalence relations

Students' difficulties in understanding the equivalence relation could be lessened in a learning environment that includes authentic tasks (Taylor-Cox, 2003), technology (Jones & Pratt, 2006) and teacher's guidance. As to the use of technology to assist the learning of the equivalence relations, Jones and Pratt (2006) report an experiment in which two students connected an onscreen '=' object with other arithmetic objects, which supported them in developing relational conceptions of the equal sign. As to the use of authentic activities to assist the learning of the equivalence relation, Taylor-Cox (2003) describes the Pan Balance scales as a tool to demonstrate equality, where students need to use and make scales. As to the teacher's guidance as means to facilitate students' learning of the equivalence relation, researchers have indicated the importance of the teachers' role and guidance in learning mathematics in general (NCTM, 2000), and learning the equivalence relation in particular (e.g., Taylor-Cox, 2003). Taylor-Cox (2003) describes the mathematics teacher's role in enhancing students' learning, for example by asking questions that promote mathematical dialogue and understanding. The mathematics teacher's actions are part of the classroom routines (using Sfard's terms) that assist the students in their mathematics learning.

We designed the learning environment taking into consideration the role of technology, the role of the teacher, and the type of the tasks. To better understand the students' learning in this environment, we analyzed this learning using Sfard's commognitive approach. Especially, we concentrated on the evolution of routines' use. In the following section, we briefly outline the commognitive approach.

Routines in the mathematics classroom

Sfard (2008) presents four components of the mathematical discourse that help analyze it: words, visual mediators, narratives and routines. *Mathematical words* are used by the participants in a mathematical discourse to express and communicate with the other participants about mathematical ideas. In this discourse, a student learns new uses of previously encountered mathematical words, but may also learn new mathematical words. *Visual mediators* are visual objects and means with which participants of mathematical discourses identify mathematical ideas. They include symbols such as numerals, algebraic letters, tables, graphs and diagrams. A *narrative* is a spoken or written text that describes objects, or relations between objects or activities with or by objects, and that could be accepted or rejected within the mathematical discourse. Mathematical examples of narratives could be theorems, definitions and equations.

Sfard (2008) defined *Routines* as "repetitive patterns characteristic of the given discourse" (p. 134). They characterize the use of mathematical words and visual mediators or the creation, substantiation or change of mathematical narratives. Examples on typical mathematical routines are methods of calculations and of proof (Sfard, 2008). She divides routines into explorations that aim to further discourse through producing or verifying endorsable narratives (as verifying a mathematical conjecture or proving a mathematical relation); deeds that aim to change the actual objects, physical

or discursive, not just the narratives; and rituals that aim to create and sustain social approval with other participants in the mathematical discourse. Furthermore, rituals could involve imitations of other participants' routines (Berger, 2013). Sfard further divided explorations into three types: construction that aims to create new endorsable narratives, substantiation that aims to decide whether to endorse previously created narratives, and recall that aim to summon narratives endorsed in the past.

Previous research has used the commognitive framework in different ways to examine the four components of the mathematical discourse, or just some of them (e.g., Berger, 2013; Viirman, 2012). Little research has been done on students' routines while learning the equality sentences as equivalence relations, where most of the research was done on students' word use or narratives related to these concepts. The present research intends to study the routines of low-achieving students while learning equations as an equivalence relations between quantities. The main research question is: what are the characteristics of low-achieving students' routines in the course of learning equations as an equivalence relations between quantities in a productive learning environment?

The design of the study

To answer the research question we analyzed approximately three hours of learning by Noha and Maha, one pair1 of 16-year-old low-achieving students in the math class taught by the third author.

The experiment took place in a school of low-achieving students who want to graduate as car mechanics or house/car electricians. The students volunteered to participate in four after-school meetings that aimed to teach the equations as an equivalence relations. In this study, we concentrated on the third meeting, which dealt with learning the equivalence between the two sides of an equation when performing arithmetic operation. The students who participated in this study had prior knowledge in operator precedence and the substitution of numeric values in algebraic expressions. They were not familiar with using technological software in learning mathematics. The two students shared a single computer, and the third author briefly introduced them to the functions of the software.

The students were video-recorded and their computer screens were captured. The video recording was performed with a computer program that captured the footage in two different windows; one for the computer screen and the other for the student's body. The third author conducted the learning activity. His main role was to ask clarification questions. The pair of students carried out four tasks presented in Figure 1.

Task 1

- Enter the expression 6x in the red pan and 18 in the blue. What happened to the pans? Why?
- Change the slider until the pans have equal values. Why do the pans have equal value?
- Add the value 2 to the red pan. What happed and why? What should you do now to make the pans balanced?
- Subtract the value 2 from the red pan. What happed and why? What should you do now to make the pans balanced?

[1] For reasons of space, we decided to perform the micro-analysis of the learning process with one pair of students from the three pairs participating in the research project.

Figure 1. Example of a task given to the students

The technological tool used in the experiment

The technological tool used in our study is the interactive applet Pan Balance Expressions (PBE; NCTM, 2015; Fig. 2). The interactive applet PBE allows numeric or algebraic expressions to be entered and compared. Students can "weigh" the expressions they want to compare by entering them on either side of the balance. Using this interactive applet, students can investigate the equivalence of equation. PBE consists of four main windows: a) the slider window, which allows the student to vary the x- values; b) the pans window, which contains symbolic expressions entered by the users; c) the keyboard window, which enables the student to enter and edit expressions in the pans; d) the graphic window, which represents the graphs of the expressions entered in the pans.

Data analysis

To analyze the data, we categorized the routines, as suggested in Sfard (2008). We considered a routine to be an exploration when the student's goal, from performing the routine, was to arrive at a narrative. More specifically, we considered a routine to be an exploration of the type 'construction', when its goal was to arrive at a mathematical relationship. Moreover, we considered a routine to be an exploration of the type 'construction', its goal was to verify a relationship that was arrived at or conjectured. Other categories that we found are: teacher's request (when the teacher requested the students to do an action), and students' actions with the applet (when the students worked with the applet for different reasons).

Results

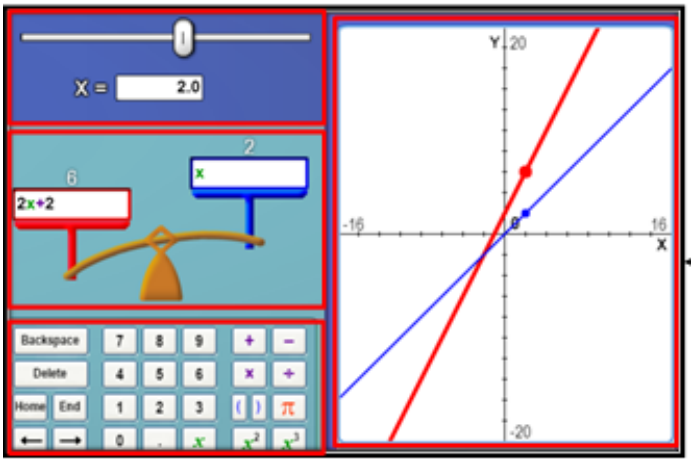

Figure. 2: The interface of the Pan Balance applet

The pair of low-achieving students worked with three groups of narratives; (a) Solving the equation Ax=B using the applet; (b) constructing the equivalence equations resulting from performing the same allowed operation on both sides of the equation; (c) solving linear equations using the equivalence principle. In the present paper, we will present students' routines related to constructing the equivalence concept resulting from performing the same allowed operation on both sides of the equation.

Transcript 1 describes the pair of students' work while adding the same number to both sides of an equation. At this phase, the expression 6x was in the red pan and 18 in the blue one. The slider was at x=3, which mean the pans were in balance.

25	T:	Add the number two to the blue pan
26	N:	*(she added two to the blue pan causing the red pan to rise)*
27	T:	What did you see?
28	N:	Eighteen plus two
29	T:	What happened?
30	M:	It rose.
31	N:	It is not equal; the red pan rose and the blue fell.
32	T:	Why did this occur?
33	M:	(*Looking at the Pan Balance*) Because we added the number two to the blue pan. They are not balanced; one pan is higher than the other.
34	T:	Could they balance now?
35	M:	*(adds 2 to the red pan)*
36	M:	Yes, if we added the number two to the red pan.
37	N:	Yes they are balanced now.
38	T:	Why are they balanced now?
39	N:	Previously there were 18 on both sides. Thereafter, we added two to the blue pan. It totaled 20. Now I added two to the 6x and it also totaled 20. It is now equal.

Transcript 1: Adding the same number to the two sides of an equation

This transcript illustrates the pair of students' routines, which led to the endorsement of the narrative "*Yes, if we added the number two to the red pan*" [36]. Students' routines started with a teacher's request [26] with an overall intention to allow the student to construct a narrative related to adding a number to an equation. The students got engaged in actions with the applet [26]. The teacher then started a construction routine, with the intention to make the students aware of the effect of adding a number on one pan [27-31]. Then the teacher started a routine of substantiation [32-33]. It can be seen that the students' exploration constituted of the following sequence of routines: teacher's request, students' actions with the applet, students' construction of a narrative, teacher's questioning, and students' substantiation of the narrative. The pair of students performed again the same sequence of routines to explore how to make the two pans equal: teacher's request [34], students' actions with the applet [35], students' construction of a narrative [36-37], teacher's question [38] and students' substantiation of the narrative [39].

In their exploration of the narrative related to subtracting a number from the two sides of an equation, the pair of students needed just one sequence of routines. Moreover, in their exploration of the narrative related to multiplying the two sides of an equation by the same number, the pair of students skipped performing actions with the applet to construct the narrative. However, and as transcript 2 shows, they performed these actions with the applet to substantiate the narrative about the equivalence of an equation under multiplication.

86	T:	What would happen if you multiplied the expressions in the pans by the same number?
87	N:	When multiplying, the balance of the two pans would remain unchanged.
88	N:	[*she inserted the expression 6x on one pan and 18 on the other; thereafter she fixed x=3 to balance the pans*].
89	N:	I will multiply both sides by 2.
90	N:	[*She multiplied both sides by 2*].
91	N:	I got it right.

Transcript 2: Multiplying the two sides of an equation by the same number

This transcript illustrates a modified sequence of routines: teacher's request [86], conjecture (as a part of a construction) [87], actions with the applet [88], substantiation [89-91].

The data analysis revealed some characteristics of students' routines. First, routines started with a teacher's request or questioning. It seems that one of the teacher's routines in the low-achieving classroom was to start the learning process by requesting the students to act or to answer. Second, the pair of students followed a sequence of routines to arrive at each of the narratives. This sequence consisted of teacher's request, students' actions with the applet, constructing a narrative, and substantiating it. This sequence of routines was not kept as is for every narrative, but a variation of it was followed. Third, students' actions with the applet, what we could call *deed routines*, supported the low- achieving students in their exploration routines, whether they were constructions or substantiations. Fourth, the data analysis revealed a pattern of evolution of the routines associated with the successive narratives, where the number of routines needed for the students to endorse narratives was decreased for each group of narratives.

Discussion

The goal of the present research was to examine the routines of a pair of low-achieving students, while learning the equivalence relations in a productive learning environment. The students worked with the Pan Balance, which illustrate the equation concept. Working with it, they actually worked with visual mediator which signifying the mathematical objects and relations (Sfard, 2008, p. 224). Moreover, the students' routines regard using the visual mediator were visual and dynamic, where they could scan the Pan Balance and manipulate it, and consequently watch the effects of this manipulation on the equivalence relations. It could be claimed that these visual and dynamic routines helped the low-achieving pair of students to signify the equivalence relation through construction and substantiation routines. Furthermore, the applet constituted for the pair of low-achieving students a prompt for construction and substantiation routines.

It was observed that the pair of low-achieving students used a sequence of routines: teacher's request, students' actions with the applet, students' construction of a narrative, teacher's questioning and students' substantiation of the narrative. Moreover, students' use of the sequence of routines satisfied the variability and flexibility principles (Felton & Nathan, 2009; Sfard, 2008, pp. 202-205), i.e. the students varied their use of the sequence to meet their needs. This happened for example, when they engaged with multiplying the two sides of an equation by the same number. Constructing the appropriate equivalence narrative, they skipped performing actions with the applet, but performed these actions to substantiate the narrative.

The sequence of routines described above shows the effect of the teacher's routines and of technology affordances on students' routines. It seems that the teacher's initiation of students' construction and substantiation routines was a prompt for them to follow routines that supported their successful construction of equivalence narratives. As for the technology affordances, the Pan Balance applet allowed the pair of low-achieving students to perform actions that supported them in their construction and substantiation of the equivalence narratives (e.g., scanning the equilibrium of the Pan Balance). Moreover, we argue regarding the pair of students' substantiation routines, that they relied on empirical argument that utilized the "concrete realizations of the focal signifiers and relies on their perceptually accessible features" (Sfard, 2008, p.233). This type of substantiation is probably expected of low-achieving students.

The present research reports the routines of one pair of low-achieving students. It shows that a productive learning environment that combines teacher's initiation and questioning, technology and authentic tasks will support these students' routines for arriving at mathematical narratives. Research that engages more low-achieving students' is needed to confirm this research findings regarding their routines in similar environments.

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Students' reasoning on linear transformations in a DGS: A semiotic perspective

Melih Turgut

Eskisehir Osmangazi University, Eskisehir, Turkey; mturgut@ogu.edu.tr

The aim of this paper is to analyze students' reasoning on linear transformations while using a Dynamic Geometry System (*DGS*) *from a semiotic mediation perspective. Considering design heuristics of Realistic Mathematics Education and the semiotic potential of certain tools and functions of DGS, I have developed a hypothetical learning trajectory and have designed a task for inventing fundamental properties of linear transformations. The task was field-tested in a case study with pair of undergraduate linear algebra students. An analysis of the task-based interviews, with a semiotic mediation lens, shows that the students managed to* (*re–*)*invent the fundamental properties of linear transformations.*

Keywords: Semiotic mediation, linear algebra, DGS.

Introduction

One major issue in the teaching and learning of linear algebra is providing students with ready–made mathematics using different representations and different contexts (Dorier, 1998) without considering the students' intellectual needs for learning (Harel, 1998). An example might be to introduce the notion of linear transformations with two fundamental properties as in numerous textbooks, where such introduction to the topic could trigger epistemological issues for students' conceptualization of non–linear transformations (Dreyfus, Hillel, & Sierpinska, 1998). In this paper, I acknowledge a contrary introduction to the topic and consider a research question: Is it possible for students to (re–)invent fundamental properties of linear transformations? To answer this question, I consider a dynamic geometry system (DGS), which invite students into a progressive process of epistemic exploring, conjecturing and generalizing (Leung, Baccaglini-Frank, & Mariotti, 2013). Consequently, I focus on specific tools and functions of GeoGebra, such as dragging and grid functions, ApplyMatrix command and slider tool of the DGS as a tool of semiotic mediation for students' reinvention of proposed mathematics.

Theoretical Perspectives

In this work, I consider two theoretical insights: (i) Realistic Mathematics Education and (ii) Theory of Semiotic Mediation, for designing instructional activity and analyzing the teaching–learning process.

Realistic Mathematics Education (RME)

RME is a domain–specific instructional theory developed by Dutch researchers (Van den Heuvel-Panhuizen & Drijvers, 2014). The word *realistic*, here, does not directly refer to real–world task situations, but to paradigmatic situations that invite the development of meaningful mathematics. The problem situations do not necessarily come from real life directly, they can be related to an imaginary world or to real mathematics that students experience as meaningful: task situations have to be *experientially real* (Gravemeijer, 1999) to students. In parallel to such views, RME offers three interacting design heuristics for curriculum developers and educational designers

(Freudenthal, 1983; Gravemeijer, 1999; Van den Heuvel-Panhuizen & Drijvers, 2014): guided reinvention, didactic phenomenology, and emergent modelling. Guided reinvention means providing students with an environment for their exploration, elaboration and inventing of mathematics. Didactic phenomenology refers to finding certain experientially real phenomena, which might form an environment where students create mathematics. The objective of emergent modelling is to enable students to shift from informal task situations to formal mathematics through support, enabling them to create their own informal mathematics.

Theory of Semiotic Mediation (TSM)

TSM was presented by Bartolini Bussi and Mariotti (2008) with the following main idea: to construct mathematical meanings, the teacher intentionally uses *artefacts* as a tool of semiotic mediation, which are used in carefully–designed tasks. The aim of the TSM is to transform students' personal meanings to mathematical meanings. The teacher exploits the *semiotic potential* of the artefact, in which he or she uses an epistemological and didactical analysis to picture out possible learning steps from personal meanings to shared conventional mathematical meanings. Here, taking into account the didactic goals, the teacher considers what students know, what their experience with the artefact is and how they will accomplish the task by using the artefact. As a next step, the teacher designs a *didactic cycle* for classroom interventions.

Students' interaction with the artefact produces a complex semiotic process. Artefact signs (aS) appear when students who use the artefact relate in some way to the activity; specifically, to the use of artefact. Mathematical signs (mS) appear, when the students make a definition, conjecture, generalization or proof corresponding to didactic goals. Pivot signs (pS) have an interpretative link between personal meanings and mathematical signs and can appear in the accomplishment of the task. In the application of the didactic cycles, the teacher's role is orchestrating students' learning.

Methodology

This paper, in which I focus and present the results of a single task, is part of an extensive Design–Based Research (project) (Bakker & van Eerde, 2015). Due to page limitation, I will present a case limited to a pair of students (A male, B female), who were sophomore level undergraduate linear algebra students, aged twenty. The students had experience solving linear equations, matrix algebra, (geometric) vector spaces and subspaces, and they had learned that every linear transformation can be represented through matrices. They also had experience in the use of GeoGebra's main functions, specifically forming a slider and a matrix, and applying the ApplyMatrix construction tool from previous task sequences, where they constructed meaning of a transformation and linear transformation. However, the students did not know the fundamental properties of linear transformations. Task–based interviews were video–recorded and lasted around half an hour. The data was analyzed through a semiotic lens using categories of signs (Bartolini Bussi & Mariotti, 2008): aS, mS and pS.

Mathematical context, semiotic potential of DGS and task design

A linear transformation is a specific transformation between V and W can be represented as $T:V\to W$ for vector spaces V and W, where T satisfies: (i) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all vectors $\mathbf{u},\mathbf{v}\in V$, and (ii) $T(k\mathbf{u})=kT(\mathbf{u})$ for all $\mathbf{u}\in V$ and all scalars $k\in\mathbf{R}$ (Lay, 2006). Here, I

took $V = W = \mathbf{R}^2$ because of DGS availability (for example as in GeoGebra) and considered the semiotic potential of the following tools and functions of DGS for students' (re–)invention of the fundamental properties above: (i) the *dragging* function allows the user to manipulate figures and explore independency–dependency of drawings and constructed objects, (ii) the *grid* function activates specific lines for integer values on the *x* and *y* axes and this function enables the user to observe variations of the coordinates of the objects in different windows, (iii) the *slider* tool offers a means to define a parameter and this may evoke meaning for dynamic (co)variation (Turgut & Drijvers, 2016), (iv) the *ApplyMatrix* tool works through an input line that enables the user to apply certain matrix transformations to geometric figures. I postulate that students' dragging sliders connected to a matrix and applying matrix transformations to arbitrary vectors could provide an understanding for a meaning: matrix (and therefore linear) transformations preserve vector addition and scalar multiplication.

The synergy between the definitions of guided reinvention and didactic phenomenology heuristics and the notion of semiotic potential in TSM implies the construction of a possible learning route, in other words, a Hypothetical Learning Trajectory (HLT) (Simon, 1995) which has to be elaborated on by the designer before the experiment by following four points (Bakker & van Eerde, 2015): (i) learning goals, (ii) students' pre-knowledge, (iii) assumptions for students' learning, and (iv) the teacher's role (also in our case, the role of artefacts). Therefore, in Table 1, I express (i), (ii) and (iii) points of a HLT for invention of fundamental properties of linear transformations in a DGS.

Associated Concepts	Expected Steps in the DGS	Exemplary Task	Epistemic Artefacts in DGS	Expected Mathematical Meanings
–Geometric vectors –Addition of vectors –Multiplication with scalars –Matrix transformations –Fundamental properties of linear transformations	–Exploring the effects of sliders on (arbitrary) linear transformations of arbitrary vectors –Comparing the initial and final versions of vectors while moving sliders or dragging the objects	–Form sliders –Construct 2×2 matrix –Form arbitrary vectors –Use Apply Matrix command –Move the sliders and drag the objects	–Dragging –Grid function –Apply Matrix construction tool –Slider tool	–Comprehending that the situation is independent from matrix entries or vectors –Formulating the first rule situation, $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ –Formulating the second $T(k\mathbf{u})=kT(\mathbf{u})$ –Proving such results in terms of matrix representations

Table 1: HLT for the inventing of fundamental rules for linear transformations

As aforementioned before, students worked on GeoGebra interface in the previous didactic cycles, which were about transformation of geometric vectors, figures, and constructing meaning for linear transformation. Consequently, the tools and functions of GeoGebra and proposed concepts were experientially real for them. Following Table 1 and considering guided reinvention heuristic, the task was formulated as follows (a possible interface for the task steps is presented in Figure 1), and also for students' making their own models (cf. emergent modeling).

Step 1: Open GeoGebra and activate grid function. Next, form two sliders *a* and *b* and, using *a* and *b*, form an arbitrary 2×2 matrix. *Step 2*: Form two arbitrary vectors $\mathbf{u}, \mathbf{v}$ and construct $\mathbf{u} + \mathbf{v}$

through an Input line. *Step 3*: Apply matrix transformation to $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$. Name these vectors, respectively: $\mathbf{u}'$, $\mathbf{v}'$ and $\mathbf{w}$ respectively, and then calculate $\mathbf{u}'+\mathbf{v}'$. Move the sliders and drag $\mathbf{u}$ and $\mathbf{v}$ in itself. Discuss with your pair and explain your observations. *Step 4*: Form a new slider k. Now, obtain matrix transformation of $k\mathbf{u}$ and also compute $k\mathbf{u}'$. Drag the vector $\mathbf{u}$ and explain your observations, and make conjectures. What happens when you move the sliders?

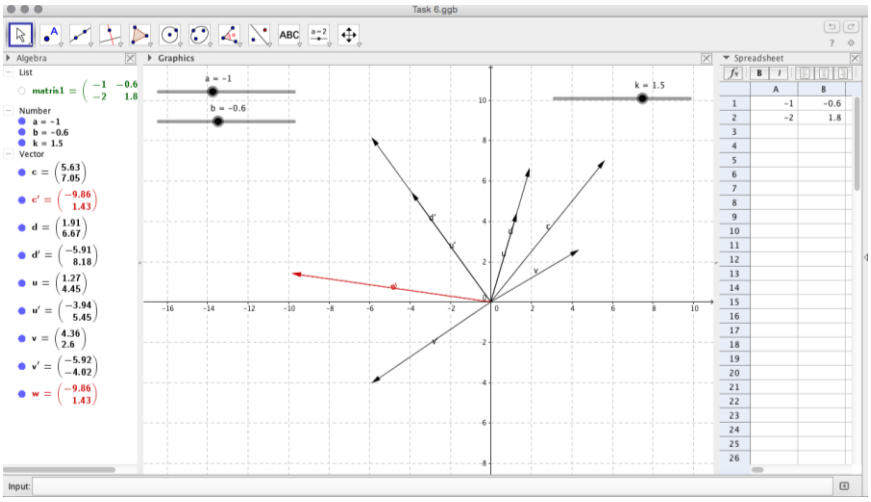


Figure 1: An expected DGS interface for the task

Teacher's (possible) underpinning questions in the interview are: *What is the role of sliders here*? *What is the role of the matrix*? *What are the relationships between initial vectors and transformations*? *How do you prove this?* [In case they make a generalization with matrix notation]. Within this task, I hypothesized that students would observe that the transformation of $\mathbf{u}+\mathbf{v}$, denoted by $T(\mathbf{u}+\mathbf{v})$, always overlaps on the $T(\mathbf{u})+T(\mathbf{v})$ vector and similarly, $T(k\mathbf{u})$ also always overlaps on $kT(\mathbf{u})$ vector, where situations were independent from the choice of matrix and/or choice of vector. This could be made possible through the semiotic potential of the aforementioned functions and tools of DGS and teacher's (T) guidance role for reinvention of the mathematics.

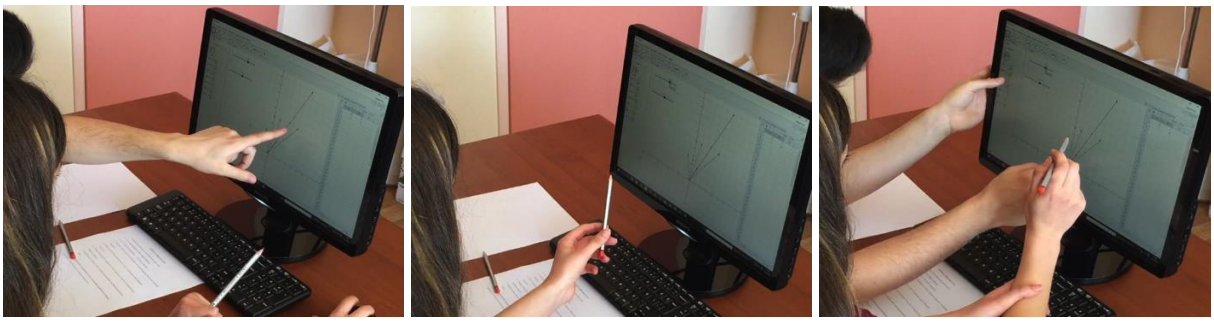
Analysis: Emergence of signs

Students followed the line of the task. First they constructed two sliders, *a* and *b*. Next, using such values in the spreadsheet window of Figure 1, they defined a 2×2 matrix as $\mathbf{A}=\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Through the Input line, they formed two vectors $\mathbf{u}=(1,2)$ and $\mathbf{v}=(-1,3)$. They first obtained the sum of the vectors and thereafter applied matrix transformation by the ApplyMatrix command. The software assigned $\mathbf{u}'$ for $T(\mathbf{u})$, similarly, $\mathbf{v}'$ for $T(\mathbf{v})$, and $\mathbf{u}+\mathbf{v}=\mathbf{w}$ and $\mathbf{u}'+\mathbf{v}'=\mathbf{d}$. For a while, the students discussed the steps of the task to determine which matrix application is the first, the second or the third, which seemed rather confusing for them. After they had completed the three steps, while dragging the sliders, they were surprised because a number of vectors and some transformation vectors overlapped. At this moment a few aS appeared (see 18–20):

18 A: … [*pointing on the grid* (see Figure 2a)] look, how this happened, these are overlapping…

19 B: No. I think, it is because of matrix, look, [*dragging sliders and pointing matrix entries with pencil* (see Figure 2b)] it is changing.

20 A: Let's analyze them, which is which and why overlaps… [*They are trying to separate the vectors* (see Figure 2c) *and taking notes*]



(a) (b) (c)

Figure 2: a, b, c Emergence of aS during analyzing the overlapping situation

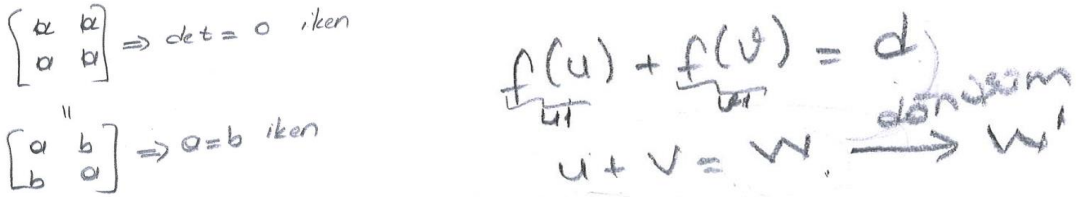
Next, the teacher intervened to make the students focus on the transformations of the vectors, because they had spent a lot of time dragging sliders, changing matrix entries (i.e., trying a unit or zero matrix and so on) to figure out why some overlapped (see Figure 3a). Then students re–checked the steps and wrote up the findings in their own way. Some pS appeared here, reflecting the students' new meanings through the semiotic potential of the artefact (37–38), and also appeared on the students' productions (Figure 3b).

26 T: … what about the transformations of vectors? What did you observe?

…

37 B: … I think we will find a relationship between these [*pointing on the notes* (see Figure 3b)]. Here, we have the sum vector's transformation and sum of each vector's transformation.

38 A: However, this could be dependent on the choice of matrix? What will happen for the matrices where their determinants are zero? …



(a) (b)

Figure 3: a, b Students' productions as pS

Interestingly, once more, they focused on the entries of the matrix, because in the previous step they had employed a unit or a zero matrix, and they began to check other possibilities for the cause of the overlapping situation. Consequently B figured 'they always overlap'. Here, aS 'overlaps' in the previous analyses, and can now be considered a pS (see 63, 86), because it is mediating the transformation of personal meanings to mathematical meanings.

63 B: It is clear that they always overlap … Why is this happening?

64 A: Exactly… but why?

…

86 B: $\mathbf{d}$ and $\mathbf{w}'$ always overlap and they are the same. I could not analyze the others.

87 A: … because of matrix transformation, I think.

As a next step, aS and pS interlaced with the students' personal meanings associated with matrix transformations. They re–analyzed their findings, and finally, mS corresponding to students' reinvention of the fundamental properties appeared (93-97).

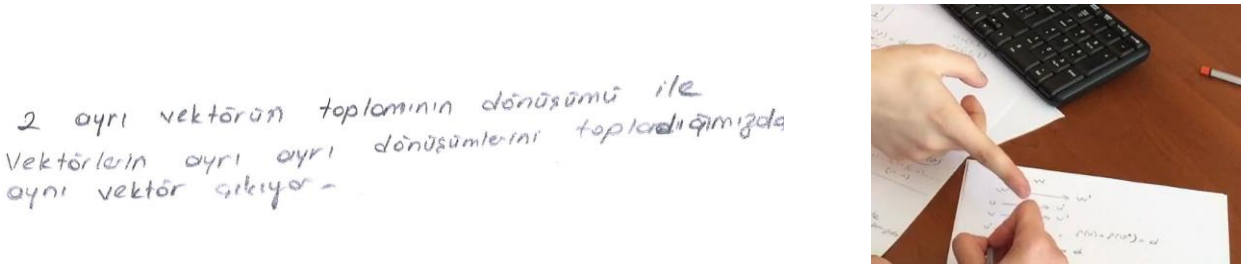
93 B: Just a second. What was the meaning of $\mathbf{w}'$? It was a transformation of $\mathbf{u}+\mathbf{v}$? … [*moving sliders and thinking*]…

94 A: We also applied matrix transformation to $\mathbf{u}+\mathbf{v}$?

95 B: Because, they are overlapping, this means, we have obtained the same vector. … Does transformation of the sum vector [*meaning* $\mathbf{u}+\mathbf{v}$] equals the sum of the separate transformations?

96 A: Absolutely, right. …

97 B: … [*First, she is writing her conclusion, but not mathematically* (see Figure 4a), *then she is trying to write it mathematically with her partner's help* (see Figure 4b)] …



(a) (b)

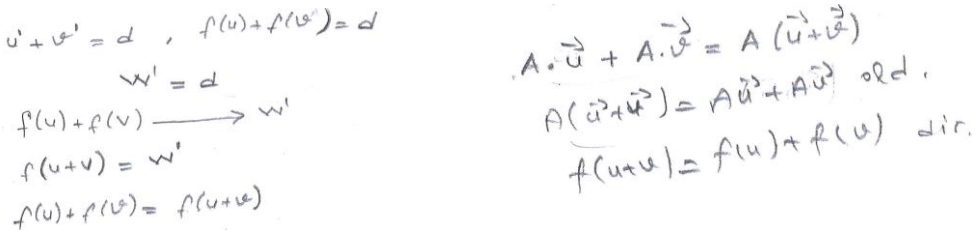
Figure 4: a, b Students' conclusions for the first fundamental property as mS

Next, while trying to express the situation mathematically, which was under the teacher's orchestration, the students reinvented the first fundamental property (see Figure 5a). However, the teacher was orienting students to prove their result considering their pre–knowledge on representing linear transformations with matrices. Student B immediately related the situation with her pre-knowledge and proved her conclusion (119 and Figure 5b).

114 T: Ok right. Please remember the matrix representations of linear transformations. Considering this, how do you prove your result?

…

119 B: … [*She is writing matrix representations* (see Figure 5b), *then explains*], yes … I now realize why this is happening. We can show every linear transformation with a matrix and matrix algebra has distribution property. Then I can do like this [*writing expressions* in Figure 5b]…



(a) (b)

Figure 5: a, b Emergence of mS in relation to the task's goal

As a final step, the teacher asked the students to consider Step 4. As expected they placed a slider for *k*, and applied a matrix transformation to $k\mathbf{u}$ and also computed vector $k\mathbf{u}'$. As soon as one student saw that the transformation of $k\mathbf{u}$ and $k\mathbf{u}'$ overlapped and were exactly the same, by the help of the first property she invented, B expressed her views. The final mS emerged in the discourse (129).

129 B: … Oh yes, I think this is obvious; this is also a result of property of matrix algebra. For a matrix, *k* can be multiplied with each entry of a matrix or it can be expressed a factor [writing $k\mathbf{u}' = f(k\mathbf{u})$]. Therefore, their [*meaning* $k\mathbf{u}$ and $k\mathbf{u}'$] transformations are the same.

Conclusions

In this paper, I consider the research question, 'Is it possible for students to (re–)invent fundamental properties of linear transformations?' Students work on the task formed through the design heuristics of RME and the semiotic potential of some tools and functions of GeoGebra provide an affirmative answer, but with some doubts and limitations. For instance, the students spent much time determining vectors when they overlapped. This issue to be considered is the students' frequent analysis of matrix entries, where they think that an overlapping situation depended on this. I think that such frequent analysis of matrix entries stems from previous experience, where the students were continually trying to find matrices of linear transformations. Interestingly, in the students' analyses for characterizing the matrix, different semiotic resources beyond aS, pS and mS appeared; for instance, gestures and mimics attached to students' analyses process. A multimodal perspective (Arzarello, 2006) could provide a detailed view for our case. However, in the present case within a TSM perspective, I observe a semiotic chain (Bartolini Bussi & Mariotti, 2008), which shows the connection between semiotic resources of students' learning, for inventing fundamental properties of linear transformations as follows (Figure 6).

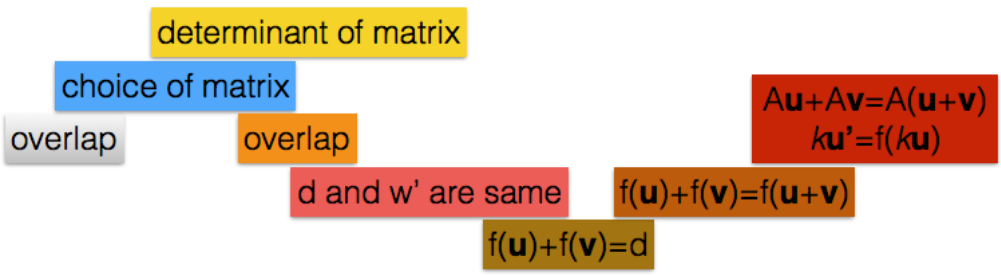


Figure 6: A semiotic chain for inventing fundamental properties

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Research on the language barriers of students who use Khan Academy as a mathematics homework platform

Jiří Vančura

Charles University, Faculty of Mathematics and Physics, Czech Republic; jivancura@gmail.com

Homework is a routine practice in maths classes, and research has shown that the immediate feedback and acknowledgement of effort is important for students. Unfortunately, the traditional classroom setting does not allow for this degree of feedback. Khan Academy offers a free tool that allows teachers to monitor students' activity and provide them with feedback and guidance. In this study, we investigated one Czech high school's use of Khan Academy as a homework platform, focusing specifically on language barriers and their impact on the ability of non-native English-speaking students to benefit from Khan Academy. We found that students who faced a lower language barrier were able to make better use of Khan Academy's educational resources. Surprisingly, we also found that a reported language barrier does not significantly correlate with a student's English grades.

Keywords: Homework, online assessment, language barrier, electronic resources.

Introduction

Every student does maths homework during his or her high school studies. I remember doing most of mine on the way to school or during the break before my maths classes. At the beginning of each maths class, my teacher would walk around the classroom, checking some notebooks randomly to see if there was anything that looked like homework. Since I really enjoyed maths, I did most of my maths homework by myself and then lent it to others to copy. We did not receive much feedback on our homework, so it was no wonder that many students were not very motivated to do their homework on their own. Unfortunately, maths teachers in large classes did not have much of a choice back then. Today, however, new online technologies, such as Khan Academy (KA), offer individualisation, guidance and immediate feedback for students, as well as a great amount of data about student activity for teachers (Khan Academy, 2016a).

The Khan Academy (KA) is a non-profit organisation that runs the website www.khanacademy.org. Since 2008, it has undergone a great deal of development. What started as a list of instructional maths videos has developed into a network of vast educational resources, including interactive exercises covering mathematics, the natural sciences and more, from the elementary to the undergraduate level. Thanks to generous donors, KA is able to provide all of its content for free, and it probably will not be cancelled or monetised any time soon. At the moment, KA's exercises and most of its educational videos are offered in English only, as is the case with many other online resources. We therefore decided to investigate the effects of language barriers on the preferences and attitudes towards KA of students who practise maths using KA exercises.

Theory

Homework and feedback

Homework assignments are routine in most mathematics classes, including those in high schools in the Czech Republic. There is a great deal of evidence suggesting that monitoring students' work and acknowledging their efforts is very important for students, as it increases the effort they put into their homework (Strandberg, 2013). When teachers do not grade a homework assignment and return it promptly, students report feeling like they have wasted their time on this activity (Strandberg, 2013; Wilson & Rhodes, 2010). Students need to believe that their homework is meaningful and that teachers value their efforts (Bempechat, Neier, Gillis & Holloway, 2011).

When it comes to feedback, there are still important gaps in our understanding (Shue, 2008). There is some evidence suggesting that when a task requires material or procedural understanding and analytical problem solving (e.g. mathematics), providing hints and allowing multiple attempts may lead to a greater increase in student performance than simply revealing the correct solutions (Clarina & Koul, 2003; Attali, 2015).

Students benefit greatly from timely and meaningful homework feedback. Unfortunately, it is often beyond a teacher's capacity to provide this to every student in a traditional classroom setting. However, technology might be able to help teachers with this. Moreover, technology can prevent misunderstandings between students, teachers and parents about the amount of time students spend on homework, as accurately estimating this can be difficult for teachers in the traditional classroom setting (Strandberg, 2013).

Language barriers

We did not find many recent studies investigating English language barriers in terms of learning mathematics or learning in general. There have been some studies that have investigated non-native English-speaking students in an English-speaking environment, both at the high school (Adams, Jessup, Criswell, & Weaver-High, 2015) and university (Variawa & McCahan, 2012) levels. However, these are not very relevant to our investigation, as they study foreign students in English-speaking communities and focus on different subjects (e.g. chemistry, engineering).

For the purposes of this study, we define a language barrier as English language difficulties, as perceived by students when interacting with the Khan Academy website.

Khan Academy

KA has been providing interactive exercises for only a few years, so it has not yet been heavily researched. However, videos have been used for educational purposes for decades. Two recent studies have investigated data from several Massive Open Online Courses to determine the attributes of the more engaging videos (Guo, Kim, & Rubin, 2014; Kim et al., 2014). In our previous study, we concluded that KA's videos align well with most of the aforementioned recommendations (Vančura, in press). Our investigation into the possible impact of KA as a homework platform on student attitudes towards mathematics demonstrated that negative impacts are very unlikely (ibid).

A large study was also conducted concerning the implementation of KA in U.S. classrooms (Murphy, Gallagher, Krumm, Mislevy & Hafter, 2014). The results revealed that only 45% of

American students reported being able to learn new skills using KA without teacher assistance. We found similar results (46%) in our previous study (Vančura, in press).

Context

In this study, student participants were assigned homework on a weekly basis in the form of KA's interactive exercises.

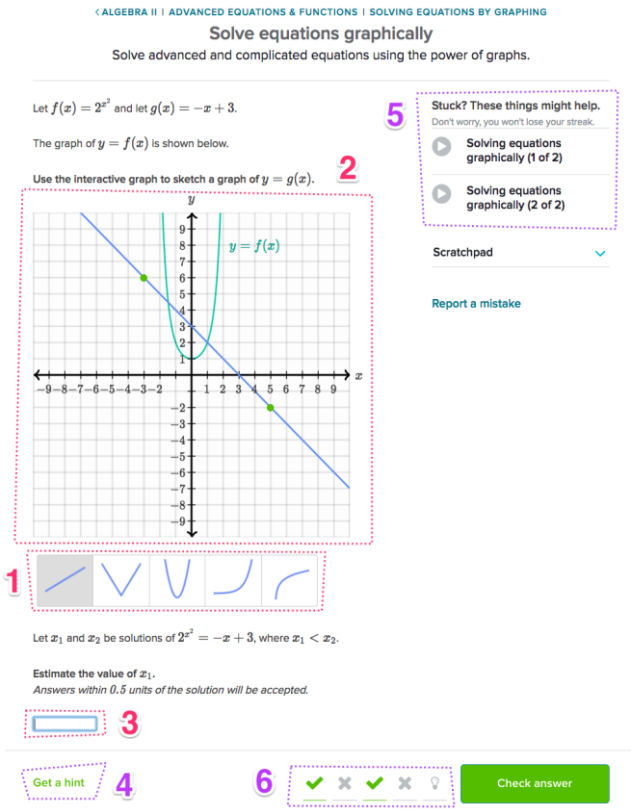

Figure 1: Interactive exercise (Khan Academy, 2016b)

Every exercise consists of a series of problems related to very specific topics. In the exercise shown in Figure 1, students are asked to practise estimating equation solutions using graphs. Specifically, students are required to select the shape of the graph for function *g* [1]; to graph the function *g* using the interactive graphing tool [2]; and to estimate the lower solution of equation $f(x) = g(x)$, where function *f* is given by the graph. Students cannot move on to the next problem until they solve the exercise correctly. If they cannot solve the problem, there are hints [4] that demonstrate step-by-step solutions. Even after the whole solution is revealed, students are still required to graph the function *g* and estimate the solution correctly. Only then can they continue on to the next problem. Students can also watch instructional videos that explain the solutions to a sample problem in detail [5]. Each student's progress is captured and displayed at the bottom of the screen [6]. Students receive a check mark for solving the problem correctly on the first try without any hints. They get an x mark for entering the wrong solution and a light bulb icon for solving the problem correctly on the first try with some hints.

In order for students to successfully finish an exercise, they must get five (or sometimes three) check marks in a row (i.e. solve five problems on the first try without any hints). This multiple-try

mechanism aligns well with the findings on feedback (Attali, 2015). However, this feedback does not tell students where they have made their mistakes, and it usually provides only one way to solve a problem. Some exercises consist of multiple-choice answers, and students might be tempted to guess the solutions—although the requirement of solving five problems in a row makes guessing time consuming. For example, even if students were able to narrow the choices down to two, they would still need to answer 62 questions on average in order to get 5 in a row correct. If students guessed blindly from 4 choices, they would need to answer 458 questions on average.

Another important tool that KA offers is the teacher dashboard, which allows teachers to monitor student activity. Teachers can see when students work on exercises, which exercises they work on and how well they solve the problems. Teachers can even see the amount of time students spend on each problem, as well as the total time they spend on KA. This data allows teachers to monitor, acknowledge and assess students' homework objectively and meaningfully. Moreover, in our study, these attributes allowed teachers to grade homework on a weekly basis.

Based on the results of one SRI study (Murphy et al., 2014), student participants were not required to learn new skills on KA; rather, they had to practise skills they had already acquired.

Methodology

Research questions

1. Does a student's language barrier influence whether he or she prefers KA homework over homework from traditional textbooks?
2. How does the language barrier influence students' attitudes towards KA and their ability to learn maths while using it?

As the research progressed, we saw that language barriers did play an important role, which made us add a third question of interest:

3. Can the language barriers of individual students be easily and reliably estimated (i.e. by asking the student's English teacher)?

Data collection

We developed two surveys based on the surveys used in the SRI study (SRI, 2015), although we added some questions about English language usage and omitted some questions that were irrelevant to our investigation. The first survey was administered in December 2015, and the second was administered in June 2016. Both surveys contained several pairs of verification questions to detect inconsistencies or carelessly filled-out surveys. To measure the language barrier, we used Likert-scale questions, such as, "My limited English knowledge prevents me from using Khan Academy effectively." To measure the preference for the KA homework platform, we used Likert-scale questions, such as, "I prefer to solve examples from common textbooks rather than from Khan Academy." Surveys were administered during an ordinary maths class so that the students had no reason to hurry. We also collected students' midterm and final grades in mathematics and English for the 2015–2016 school year.

At the beginning of September 2016, we asked the students' English teachers to estimate the reading and listening abilities of the participating students, as well as the effects of the students'

language barriers when using English mathematical software. The teachers were asked to use the Common European Framework of Reference for Languages (A1–C2) for their estimations (Council of Europe, 2016), which were then recoded on a scale of 1–6.

Participants and criteria of analysis

The first survey was administered to 141 non-native English-speaking students aged from 15 to 20 years old from 7 maths classes in 2 Prague high schools. For the second survey, the participants included 83 students from 5 out of the 7 classes that participated in the first survey. All of the students were learning English as their second or third language. A total of 64 students participated in both surveys. The students in our study were taught by six different English teachers, who were asked to estimate the students' language barriers. The author of this paper was the maths teacher for two of the seven classes. Therefore, we looked for relative patterns (i.e. connections between the students' language barriers and their learning independence) rather than for absolute results. When investigating absolute results, such as student preference for KA over traditional textbooks, we also considered the differences between the students who were taught by the researcher and those who were not.

To measure the language barriers, we required Cronbach's alpha to be greater than 0.7, which is generally considered to be an acceptable level of consistency. When it came to correlations and hypothesis testing, we used the 5% significance level.

Results

In the first survey, students reported a strong preference for KA over traditional textbooks (Vančura, in press). This preference decreased significantly in the second survey, although KA was still preferred. In both surveys, the students who were taught by the researcher did not report a stronger preference for KA than the other students. Students' preference for KA was significantly correlated with reported language barriers (see Figure 2). Students with greater language barriers tended to prefer KA less. Even so, students who reported significant difficulties with English preferred KA over traditional textbooks.

Both surveys revealed a significant connection between the students' reported language barriers and several other factors. In both surveys, students with lower language barriers

a. found KA videos and exercises to be more helpful for them (correlations 0.18–0.45);
b. reported higher autonomy when learning new skills using KA (0.17–0.28); and
c. reported a more adequate understanding of their skills while working in KA (0.22–0.29).

The reported levels of language barriers decreased slightly between the two surveys, but this decrease was not statistically significant. Surprisingly, the reported language barriers did not significantly correlate (-0.02, 0.14) with the English grades. We assumed that different teachers would have different grading strategies and standards, so we normalised the English grades within the groups of students taught by each teacher. The resulting correlation increased slightly to 0.16, which is still insignificant in our case. We also calculated the correlation between English grades and the decrease of language barriers between the two surveys; again, the correlation was insignificant (0.02).

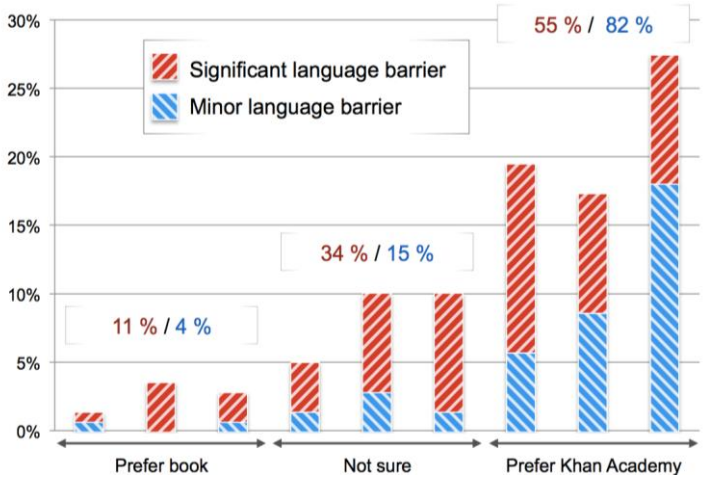

Figure 2: Student preference for the KA homework platform by language barrier in the first survey

Driven by these results, we asked the English teachers to evaluate the students' English listening and reading skills, which correlated moderately (0.45, 0.48) with the language barriers reported by the students.

Conclusions and discussion

Homework remains an important part of mathematics education in the Czech Republic. KA can provide students with guidance and immediate feedback, which we believe is the main factor that leads students to prefer KA over traditional textbooks. The decrease in KA preference over time may be attributed to the novelty of this new system wearing off. Still, it is worth noting that even six months (and many working hours) later, KA remained the preferred choice of the majority of the student participants.

We found that language barriers play an important role in both preference and reported utilisation of KA. Students with greater English-language capabilities reported a higher ability to use KA learning resources, which we believe to be a strong factor behind their stronger preference for KA over traditional textbooks. We can assume that similar patterns would appear with other online educational resources—the number and quality of which continues to grow rapidly, and which would take a great deal of time to translate into Czech. Notably, KA is one of the most-translated educational resources in the Czech Republic. However, despite great effort from the non-profit organisation Khanova Škola (Khanova škola, o.s., 2016), only about 35% of KA's videos have subtitles, while fewer than 1% have Czech dubbing. If we want students to benefit from these growing resources, then we need to prepare our students for learning in English.

The last result of our study was that students' English grades did not significantly correlate with their reported language barriers (i.e. the 'best' English students did not typically feel better able to overcome their language barriers than the struggling students). This pattern held true even after six months of using KA. We assumed that the students would learn to overcome these barriers over time, as they used the English resources over the course of the study. Our study did indeed show that the reported language barriers decreased slightly, albeit insignificantly.

The teachers' evaluation of the students' English reading and listening skills correlated significantly (0.45, 0.48) with the language barriers reported by the students, so such evaluation could provide a

very rough estimate of the language barriers faced by a group of students. However, this correlation was not strong enough to provide a reliable estimate of the barriers faced by the individual students, as it only explained about 23% of it ($0.48^2 = 0.23$). Ultimately, we were unable to find a quick and reliable way to estimate the individual students' self-reported language barriers.

Limitations and future research

The small size of our sample made it impossible for us to find small correlations or inconspicuous patterns. Larger samples would have also allowed us to verify our results at a higher confidence level. The disconnection between the reported language barriers and the students' English grades could also be a local phenomenon, since every Czech school has its own curriculum.

In this study, we relied mostly on students' opinions, which might not have been completely accurate (i.e. even though students reported that they could make good use of KA resources, that does not necessarily mean that they did).

The results also showed that the English courses currently being offered to students might not be sufficient to prepare them to learn mathematics in English using tools like KA. Therefore, determining how to help students learn in English might be an interesting question for researchers and a challenge for both maths and English teachers.

While KA offers a great variety of exercises, its narrow focus (i.e. graphing quadratic functions in vertex forms) and repetitive nature might produce very formal knowledge that cannot be transferred. In future study, we would like to investigate what students actually learn using KA and how it might be affected by their language barriers.

As with every digital resource, KA sometimes experiences technical problems. Exercises can fail to load properly and data might not show up in the teacher dashboard. Therefore, it is a good idea to consider possible technical problems before judging students too quickly.

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When students use podcasts to share their own expertise in building proofs.

Martine De Vleeschouwer[1] and Marie-Ange Remiche[2]

[1]University of Namur, Faculty of Sciences, Mathematics Department, Namur, Belgium;
mdv@math.unamur.be

[2]University of Namur, Faculty of Computer Science, Namur, Belgium;

Marie-Ange.Remiche@unamur.be

Keywords: Podcast, didactic contract, complete praxeology.

In 2012, the University of Namur (Belgium) launched the PUNCH project ('PUNCH', 2012). Within this framework, many experiences to rethink our teaching practices were sponsored. Among many others, the POD-EN-MATH project aims to help students in computer science to complete praxeologies (Chevallard, 2006; Winslow, 2008) when learning mathematical concepts and doing this with the help of high-quality podcasts. Students will gain experience by analyzing step-by-step our own procedure to make the link between theory and technics. Doing so would permit us to make the didactic contract more explicit (Brousseau, 1984). Indeed this procedure is not documented but only transmitted orally. The next step of the project is that students themselves provide content to our podcast database. This experience put into evidence the difficulty we had to help students to communicate, to teach their own knowledge focusing on a more didactic point of view.

Podcast we proposed

At the end of their graduate program, Computer Science students should be able to model the customer's need, compute the complexity's program (in terms of number of needed operations to perform) and prove that proposed program architecture fulfills the customer's demand. Discrete mathematics is mandatory as a corner stone to reach these outcomes. However, our students have great difficulties in building up the connection between theory and practice due to, among others, our ex cathedra teaching habit, as mentioned by Winslow (2008). Our podcasts aimed at filling this gap. We proposed high-quality podcasts of five to ten minutes. Our objective is to explain step-by-step, the reasoning that permits us to obtain the solution of a problem. Indeed the difficulty they often mention is to translate the problem into a mathematical model that needs to be solved, and not the theory they should use once the mathematical model is obtained. As proposed in Houston (2009), we want them to build up their mathematical reasoning and one way to do it is by viewing our own podcasts. As future analysts, the mathematical reasoning is of crucial interest for our computer scientists. Indeed they will have first to analyze the customers' needs and next to rephrase them in terms of programming objects and methods to programmers they manage. Mathematical reasoning is using the same skill.

Podcast they had to build

Our final objective was that students should be able to build up their own podcast in the second level of discrete mathematics program. They were expected to build these podcasts, with the same level of quality and accordingly contribute to the podcast database. The problem they have to solve is on building up a podcast explaining their proof using the so-called recursive method. A specialist in

mathematics education was available to answer all their questions via personal meetings. Our students did present almost all the difficulties reported by Grenier (2012) as explained in the poster. In the light of the personal didactic supervising, some have been corrected and others have unfortunately not. The question then arises on how to improve our methodology to give them the ability to discuss about how to build up an inductive proof.

Return on experience

None of the submitted podcasts reached our didactic quality requirements and could not be shared between peers. However, students greatly enjoyed producing podcasts as well as the personal coaching. They mentioned their understanding of the recursive approach increased in quality. Grades obtained at the final examination confirmed their belief.

This experience has highlighted to us the difficulty students have when building up their own expertise independently, as well as the difficulty they have to explain to their peers how to build up their own mathematical reasoning. The question that remains is to decide what kind of approach should be used to let our students become more and more autonomous in their learning of mathematics.

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Identifying the reasons behind students' engagement patterns

Emma Howard[1], Maria Meehan[1] and Andrew Parnell[1]

[1]University College Dublin, Ireland; emma.howard@ucdconnect.ie

Keywords: Engagement patterns, blended mathematics education, college mathematics.

Introduction/Literature review

Previous studies of courses with both face-to-face lectures and online lectures/videos (Inglis, Palipana, Trenholm, & Ward, 2011) have identified clusters of students based on their resource engagement. They found that students who attended face-to-face lectures or the maths support centre achieved higher grades than students who predominantly used online lectures. Inglis et al. (2011, p. 490) furthermore discuss how "what remains poorly understood is the overall pattern of study choices made when students are presented with many options", and comments on how valuable research into examining student choices would be. Other studies have made suggestions as to why students might opt for a particular engagement pattern including: performance in course to date; proficiency of IT; convenience; and personality type (Bassili, 2006). Bassili found that both promotion and prevention factors influence students' engagement decisions. This study seeks to expand on the literature by explaining reasons behind students' choices. Subsequently, the research questions for this study are:

- Which resources do students engage with when studying the course content?
- Why do students choose to engage with these resources?

Method

This study took place in University College Dublin (UCD). Data was collected from a large first year undergraduate module, Maths for Business. This is not a traditional, blended or e-learning course. Maths for Business is a unique course in that students have a choice of whether to complete the course material through lectures, online videos or a combination of both. The e-learning segment of the course consists of 68 short videos with average length of 7.6 minutes. The module co-ordinator has chosen to offer online support for students in response to: the large class sizes; acknowledging differences in learning styles and abilities of students; and additional support needed by 'weaker' students. This form of online learning is particularly suited for procedural mathematics courses of large mixed ability cohorts. For our study, we combine quantitative and qualitative survey data to identify engagement clusters based on resource usage, and explain the reasons behind students' engagement clusters.

In order to develop a complete understanding of students' engagement, the data for this study broadly covers three areas: survey response data, background information of students, and engagement data. Students' data was linked together from each of the sections. The first stage was cluster analysis. Rather than cluster students under total videos and lectures, we decided to cluster students based on what resources they engaged with for the lecture material they covered. We developed three variables to describe this; lecture usage, video usage and overlap of resources. Subsequently, cluster analysis was performed on the three variables; lecture usage, video usage and overlap of resources using model-based clustering. Qualitative data analysis is currently being

performed under the Braun and Clarke (2006) framework. Themes are considered to be semantic as students' responses are direct.

Initial results

Cluster analysis has identified four distinct clusters: high lecture usage cluster; high video usage cluster; a cluster with high lecture, high video usage and high overlap between resources; and a cluster which features both lecture usage and video usage but with little overlap. Initial qualitative analysis has suggested the high lecture usage cluster is formed by students who perceive videos as a secondary tool; they find the lecture content has more depth, and enjoy the interactive lecture environment. In comparison, the high video usage cluster is formed by students who have issues with the lecture environment, and find little if any benefit from lectures. Videos offer these students an efficient and flexible method to study. The third cluster, the cluster with high overlap, has occurred owing to weak students accessing all available resources and needing extra support. The final cluster with little overlap of resources is formed by students who have switched from lectures to videos during the semester or are avoiding a specific lecture every week owing to the inconvenience of the timetable.

Mathematical procedural courses differ in their nature and design from other disciplines. Maths for Business use of e-learning allows 'stronger' students to progress at a fast, flexible pace while supporting the 'weaker' students through providing access to multiple resources which can be repeatedly used. Students can choose their resources to suit their learning. Overall there is an opinion from students that "[online learning] works very well for maths however [students] don't know if it would work well for other modules". Understanding students' reasons for choosing their engagement pattern may help in the future design of resources, and identifying whether online resources are particularly beneficial for large mathematics classes of mixed abilities.

The poster will expand on the initial qualitative analysis of the survey responses by explaining in detail the reasons behind each engagement cluster. We would like to thank UCD IT services for providing the Virtual Learning Environment data.

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Using tangible technology to multimodally support algebra learning: The MAL project

Thomas Janßen, David Reid, Angelika Bikner-Ahsbahs, Anke Reinschlüssel, Tanja Döring, Dmitry Alexandrovsky, Rainer Malaka and Estela Vallejo

University of Bremen, Germany; janssent@uni-bremen.de

Keywords: Algebra, learning modalities, smart objects.

Scope and objective of the presented project

Based on the assumption that activity (as opposed to the passive consumption of verbal instruction) supports learning, and accompanying the rise of learning theories that take into account the bodily grounding of the brain and its development, the usage of *manipulatives* – tangible learning materials – enjoys widespread trust among algebra teachers. Over the last two decades a variety of virtual adaptations of such manipulatives on personal computers and touch devices has been developed. The potential of such programs lies in the possibility to discover a range of configurations that would not be feasible in the real world, and to get automatic feedback about the correctness of the chosen manipulations. However, these qualities come at the price of abandoning the materiality of the manipulatives. The miniaturization of existing technologies and the creation of new input and output channels raise the question if and how manipulatives in mathematics education could be conceptualized as *smart objects* or *tangibles* – manipulatives that have the ability to interact with its users. Such objects have already been designed for learning purposes in other domains (Marshall, 2007). For the field of school algebra, the possibilities and challenges of their implementation is to be investigated in the MAL (*Multimodal Algebra Learning*) project, a collaboration between mathematics education and digital media researchers, experts in data collection and evaluation, and commercial enterprises. This contribution focuses on the didactical conceptualization that underlies the work of the consortium.

Theoretical framework

Many theoretical approaches assume bodily action to be beneficial or even defining for learning processes. Bruner's well-known model proposing the distinction between enactive, iconic and symbolic action offers a starting point. Following Nakahara (2008), a distinction can be made at the enactive level (between hands-on manipulation and other real(istic) settings) and at the symbolic level (between natural and mathematical symbolic language). Although neither Bruner's nor Nakahara's terminology implies a strict proceeding order, the progression from concrete action with the smart objects to symbolic algebraic language follows from the pedagogical setting at hand.

Design ideas and questions for investigation

Because research regarding traditional manipulatives suggests that only prolonged engagement with such objects reliably supports learning (Sowell, 1989), the MAL project wants to design *one* set of smart objects that can be used for a whole range of topics (e.g., for the generation of algebraic expression from pattern sequences, for transforming expressions, building relations, and for solving

equations). To achieve this, we turn to *algebra tiles*, which are already integrated in some North American algebra textbooks (e.g., Dietiker, Kysh, Sallee, & Hoey, 2010).

Marshall (2007) offers a framework that allows for the systematic design analysis of tangibles in learning environments. For example, he points to possible learning benefits: the playfulness that can be designed, the potential novelty of links, enhanced accessibility, chances of collaboration, and the learning benefits that may arise from physicality itself. Furthermore, he proposes a distinction between *expressive* and *exploratory* activities, which reminds of the more algebra-related work by Drijvers, Boon, and van Reeuwijk (2011), who distinguish between (technological) *tools for doing algebra* and *tools for learning algebra*, with the latter being subdivided into *practicing skills* and *developing concepts* (p. 185). It seems that the existing implementations of algebra tiles are more in the expressive realm of doing algebra, as they always start with a given definition of variables as unknowns, opposed to the possible exploration of their potential in describing change. It is a central goal of the MAL project to bring these two together. In the process, the following questions can be addressed: How do the physical and technical features of the designed objects interact with the didactical goals? To what extend is the integration of many algebraic concepts into one system of smart objects helpful in creating a multi-faceted image of the role of variables in algebra? What are limitations of the smart algebra tiles that could be resolved by either returning to traditional manipulatives or going virtual?

Acknowledgments

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Self-assessment and achievement in mathematics education: The influence of immediate feedback included in a computer-based learning environment.

Concept of a research study

Elena Jedtke and Gilbert Greefrath

University of Münster, Germany; e.jedtke@uni-muenster.de

Theoretical background

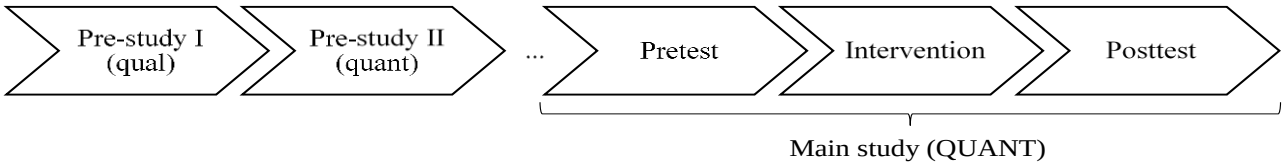
Roth (2015) defines computer-based learning environments based on mediawiki software as a structured pathway with a well-matched sequence of tasks. Central contents are interactive materials like for instance GeoGebra-applets. Roth adds that learners are encouraged to work self-regulated, self-reliant and activity-oriented within these pathways. In Germany suchlike learning environments are for example available on the website of ZUM-Wiki (https://wiki.zum.de/wiki/Hauptseite), a web page that is called on to yield a well-kept surrounding in which everybody is invited to add something (Vollrath & Roth 2012). In our study, we will focus on quadratic functions. The area of functional relationships is a central theme during secondary education in Germany in which commonly occur some learning difficulties as for example on figuring out the meaning of parameters (Nitsch 2015). A computer-based preparation is called to promote the understanding of the parameters due to the option to easily include dynamic and interactive visualizations (Vollrath & Roth 2012). Furthermore, the importance of feedback for learning is mentioned in several publications (e.g. Black & Wiliam 1998; Vollrath & Roth 2012; Hattie & Gan 2011). Hattie and Gan (2011) for example identified feedback as one of "the top 10 influences on achievement" (Hattie & Gan 2011, p. 249). Thereby the dimension of this influence would vary depending on different kinds of feedback. Apart from this, we are interested in the influence of feedback on self-assessment as a specific component of metacognition (cf. Ibabe & Jauregizar 2010). More specific we will look at students' self-rating ability. In our study, we will focus on feedback, which is typically encountered in computer-based learning environments. Suchlike feedback concerns the right answer and, in some cases, comments on how to get to this conclusion. Beside it "help me"-buttons with hints about how to proceed are added. Both the right answer and the hints can be activated by mouse click.

Research questions

1. Is students' self-assessment better if they work with a computer-based learning environment, including immediate feedback, than it is when they get the feedback outside the learning environment (research based on quadratic functions)?

2. Does a computer-based learning environment, including immediate feedback, have greater benefit on students' math achievement in comparison to their achievement when they get the feedback outside the learning environment (research based on quadratic functions)?

Method



We are currently conducting some preliminary studies, including a qualitative (pre-study I) and a quantitative part (pre-study II). The quantitative pilot study includes testing the self-assessment scales and the achievement test. Within this study, self-assessment is used in a context of self-rating the expected performance in the upcoming achievement test. The achievement test includes items about functional thinking as anchor items and items about linear functions (pretest) alternatively quadratic functions (posttest). In the qualitative pre-study, we are proving the designed computer-based learning environment about quadratic functions. Our aim is to reveal problems and misunderstandings and thus to rework and improve the learning environment.

The main study will be a quantitative research study with a quasi-experimental between-subject design. Students will work with the computer-based learning environment. The experimental group receives the learning environment including immediate feedback and hints for the tasks. The control group gets the same learning environment with absent feedback. Instead, this group gets correct-answer-feedback in outlying paper-sheets. Students' self-assessment as well as math achievement will be measured in a pre- and posttest design. The achievement test will measure the students' prior knowledge about linear functions (pretest) and their concepts on functional thinking (pre- and posttest) as well as their increase of learning after the self-regulated learning with the different types of feedback (posttest). As further research interest, we are going to compare the self-assessment ability of both groups related to the passage of time.

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Technology enhanced Resources for Mathematics Education (TeRMEd)

Ann O'Shea[1], Sinéad Breen[2], Connor Brennan[2], Frank Doheny[3], Fiona Lawless[4], Christine Kelly[1], Ciarán Mac an Bhaird[1], Seamus McLoone[1], Eabhnat Ní Fhloinn[2], Caitríona Ní Shé[1,2], and Brien Nolan[2].

Maynooth University[1], Dublin City University[2], Athlone Institute of Technology[3], Dundalk Institute of Technology[4].

Email for correspondence: caitriona.nishe2@mail.dcu.ie

Keywords: Technology enhanced resource, formative assessment, first year mathematics, evaluation.

Introduction

The aims of this project are to design and develop formative assessment resources for first year undergraduate mathematics modules and to evaluate the impact of these resources. The types of resources that have been developed to date include: targeted Khan Academy playlists and mastery challenges, a smart phone based audience response system that allows mathematical input, Moodle lessons, student generated screencasts and interactive tasks using Geogebra. The mathematical topics which are the focus of these resources were chosen based on the results of surveys of staff and students (Ní Shé, Mac an Bhaird, Ní Fhloinn, & O'Shea, 2016). In this poster we will present a snapshot of the resources, the evaluation methods and initial results. Augmented reality software and/or QR Codes are used in the poster to demonstrate the resources.

Theoretical framework

The National Research Council (National Research Council, 2001, p. 116) defined mathematical proficiency as comprising of five interwoven and interdependent strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. This description of mathematical proficiency guided our design of resources. We used the Black and Wiliam (Black & Wiliam, 1998, pp. 7–8) definition of formative assessment: 'encompassing all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged' to advise our implementation of formative assessment techniques. We used technology to design and deliver the formative assessment in this project. This is not a new idea; according to the JISC report (JISC, 2010, p. 9) the benefits of using technology in assessment include allowing a greater range of types of assessments, a greater flexibility on timing and location of assessment; improved student engagement especially with interactive tasks which incorporate instant feedback, timely evidence on the effectiveness of course design and delivery.

Methodology

The resources were developed by researchers affiliated to five different higher education institutes in Ireland who lecture on first year undergraduate mathematics. The resources were trialed in the 2015/2016 academic year. According to McKnight et al. (Mc Knight, Magid, Murphy, & McKnight,

2000, p. 10) mathematics education research is ’inquiry by carefully developed research methods’ that provides evidence of the nature of teaching and learning. For the evaluation we chose to conduct student surveys, resources usage, student grades and think aloud interviews. The questionnaires, developed from similar questionnaires (MacGeorge et al., 2008; Zaharias & Poylymenakou, 2009) contained 4 dimensions; confidence when learning mathematics; impact on engagement, impact on learning and usability of the resource.

Results, analysis and conclusions

Students were generally positive about the use of the resources, though there were differences between students’ opinions on the different resources. For example students using the audience response system found that the resource encouraged them to engage more in class (over 80%) whereas only 32% of students using the interactive tasks reported accessing the extra resources when not assigned on homeworks. Based on the results of this analysis the resources are currently being modified for use in the next academic year, 2016/2017. The data from the evaluations will be further analysed to answer the research questions that we have: What are the benefits of using technology in formative assessment design? How effective are the resources in developing mathematical proficiency?

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Real materials or simulations?
Searching for a way to foster functional thinking

Michaela Scheuring and Jürgen Roth

University of Koblenz-Landau, Landau, Germany; scheuring@uni-landau.de

Keywords: Real materials, simulation, functions, functional thinking.

Relevance

Living in a world strongly influenced by intelligent technology, it is indispensable to know in which contexts this technology can be beneficial and in which contexts the 'real world' should be used for teaching mathematics. Considering the topic of functional relationships, the need to foster pupils' functional thinking (FT) from the very first beginning arises. Even though the topic is important for mathematics education in every grade, pupils show a lot of misconceptions (Leinhardt et al., 1990). Therefore and because of our multimedia life, we need to ask, if FT should be fostered with computer-simulations (GeoGebra) or real materials (like cubes, pencils…).

Theoretical background

FT consists of three fundamental aspects: mapping, covariation, and function as object (Vollrath, 1989). Previous research shows that the use of real materials as well as the application of computer-simulations can lead to a learning progress in this complex topic. On the one hand, real materials make it possible to experience functional relationships physically (Ludwig & Oldenburg, 2007). Learning in such a setting has long-lasting effects, i.e. pupils can recall results and working methods better (Vollrath, 1978). On the other hand, simulations enable pupils to explore functions in different ways. Pupils can vary variables systematically and use the multiple-representation system (Balacheff & Kaput, 1997). Thus, e.g. covariation gets perceptible. Summing up, simulations become a mediator between pupils and mathematical phenomena (Hoyles & Noss, 2003).

Methodology

After constructing a test to measure FT we derived topics that can be used to foster FT with real materials and simulations from theory: the relationship between volume and fill height of vessels, edge length and volume of a cube, diameter and circumference of discs, rotation-number while sharping a pencil and its remaining length.

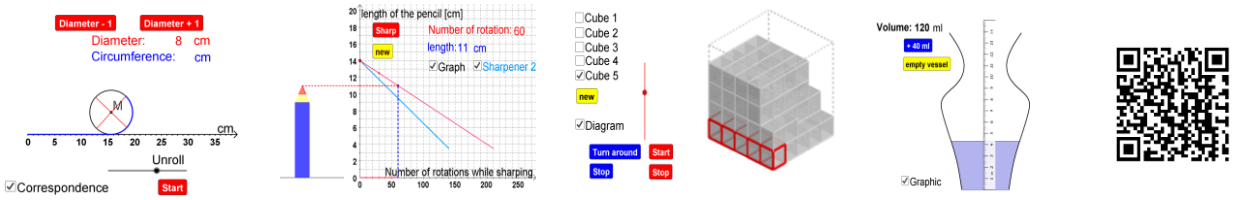

Figure 1 Interfaces of the used simulations done with GeoGebra

Then we designed an intervention-study (pre-post-control-group-design, randomized experimental-groups) that was implemented in grade 6 (age 11-12, N = 282). During the intervention (4 lessons) pupils had to work on learning-tasks individually to foster their FT. They were not instructed or

supported by a teacher. While part of them were using real materials (experimental-group 1, N = 111) the others worked with computer-simulations (experimental-group 2, N = 123). The learning-tasks in both groups were equivalent, only the medium differed. The control-group (N = 49) worked on pre- and posttest, only. Data analysis was done with item-response-theory (IRT). First, we estimated item difficulties by use of virtual persons. Then we did a 2-dim. Rasch-model (dim. 1: pretest, dim. 2: posttest) with fixed item difficulties to estimate the person ability FT. Finally, a mixed ANOVA (Field et al., 2013) using 10 sets of plausible values to compare pupils' FT in pre- and posttest was conducted.

Results

The mixed ANOVA (between: intervention, within: time) leads to the result, that there is a significant main effect of time ($F(1, 22.71) = 68.16$, $p < .001$, $\eta^2 = .089$) and also a significant interaction effect of time and intervention ($F(1, 23.69) = 7.65$, $p = .003$**, $\eta^2 = .044$) . A pairwise t-test showed that real materials as well as computer-simulations lead to a significant increase of FT (real materials: $p < .001$, Cohen's $d = .49$, computer-simulations: $p < .001$, Cohen's $d = .83$). In contrast, the control groups' FT increases not significantly ($p = 1$, Cohen's $d = .22$). Thus, it needs to be concluded that FT can be fostered by use of real materials as well as computer-simulations. Nevertheless, effect sizes show that computer-simulations should be the method of choice for fostering FT.

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The development of the mathematics curriculum in basic education with technology

Manuela Subtil [1] and António Domingos[2]

[1] *Agrupamento de Escolas Fragata do Tejo, Escola Básica 2º, 3º ciclos Fragata do Tejo, UIED, Moita, Portugal*; mm.pedro@campus.fct.unl.pt

[2] *Universidade Nova de Lisboa, Faculdade de Ciências e Tecnologia, UIED, Monte da Caparica, Portugal;* amdd@fct.unl.pt

Keywords: Technology, curriculum development, activity theory.

This poster refers to a project concerning an educational teaching experiment that focuses on the development of the curriculum on the 7th grade of basic education, by integrating technology. Using a learning environment with an exploratory nature, based on diversified tasks involving the use of graphing calculator, we aim to create an innovative curriculum in this level of education.

The teacher is the mediator of all the curricular decisions, having the responsibility to reorganize the main curriculum proposal, adapting, transforming, innovating and setting methodological teaching strategies that foster motivation and improve student outcomes (Pacheco, 2001).

According to several authors, the implementation of technology in the teaching of mathematics, namely computers and calculators (Domingos, 1994; Lee & McDougall, 2010; Tan & Tan, 2015) influences the way in which it is taught and enhances students' learning, for they build themselves their knowledge by premise creation. There are several benefits that emphasize the incorporation of technology in learning environments, namely, the increase of motivation, involvement, cooperation, hands-on learning opportunities, confidence and technological skills of students (Costley, 2014). In addition, those tasks are tools that generate activity in an interactive form, supporting the mathematical knowledge (Ainley et al., 2013).

This study is supported by the Activity Theory and seeks to understand how students, in solving specific tasks with the aid of the graphing calculator, builds their mathematical knowledge embedded in a learning community. Being the activity system within the classroom the unit of analysis, the third generation of the Activity Theory (Engeström, 2001) allows us to understand what happens when different activity systems interact. More specifically we seek to understand the instrumental genesis (Rabardel, 1995) and the semiotic potential played by technology in a student's activity system, developing the process of semiotic mediation (Bussi & Mariotti, 2008). In this sense, we intend to investigate how the graphing calculator influences, reinforces and facilitates the quality of teaching and learning and promotes the processes of instrumental genesis and semiotic mediation in the performance of tasks in curriculum development. We seek to get answers to the following research questions: Which are the instrumental action schemes created by students when they use a graphing calculator? How does the teacher promote the process of semiotic mediation? How does the graphing calculator act as semiotic mediation tool? How does the integration of technology in the curriculum influence the process of teaching and learning? What is the quality of the achieved learning?

Based on research of an interventionist nature, using innovative practices that aim to promote new ways of learning, enabling improvements on an educational level, a qualitative paradigm based on a Design Research process will be adopted. The techniques used to collect the data shall be based on the planning of the study units, the elaboration of reports by the students as a result of completing the tasks and reports resulting from the participant observation of the teacher as a mediator. We will

also consolidate the structured observation of lessons, using a logbook, photographs of the graphic representations in the calculator, videos and audio recordings, during the performance of the tasks (Cobb, Confrey, diSessa, Lehrer, & Schaube, 2003). Empirical data will be collected in the school year of 2016/17 in a public school in the district of Setúbal, in Portugal.

We therefore intend to present an innovative curriculum development project, that integrates a strong technological component in an educational level where traditionally this kind of approach is not commonly used. The use of tasks permeated by graphic technology seek to highlight the semiotic dimensions present in the instrumental genesis that are activated in different activity systems.

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TWG17: Theoretical perspectives and approaches in mathematics education research

Introduction to the papers of TWG17: Theoretical perspectives and approaches in mathematics education research

Angelika Bikner-Ahsbahs[1], Arthur Bakker[2], Mariam Haspekian[3], Mirko Maracci[4]

[1]University of Bremen, Faculty of Mathematics, Bremen, Germany; bikner@math.uni-bremen.de

[2]Utrecht University, Freudenthal Institute, Utrecht, the Netherlands, A.Bakker4@uu.nl

[3]University or Paris Descartes Sorbonne Paris Cité, France, mariam.haspekian@parisdescartes.fr

[4]University of Pavia, Department of Mathematics, Pavia, Italy, mirko.maracci@unipv.it

Keywords: Networking of theories, design research, home-grown and borrowed theories, educational theory, theory practice interplay.

Line of thought of the previous working groups in CERME5 to CERME9

Since CERME4 in 2005, theoretical approaches and perspectives have been the topic of an ongoing CERME working group. The idea of the "networking of theories" emerged at CERME 4 and was explored in the subsequent conferences. At CERME 5 (Arzarello et al., 2007), the diversity of theories in the field of mathematics education was regarded as a source of richness, and the networking of theories as a multi-theoretical approach which preserves theoretical identity but also while allowing to bridge the boundaries of theories for a better understanding of teaching and learning. Thereby, the effort was made to make hidden assumptions and relationships of theoretical approaches visible. Principles and heuristics of handling the diversity of theories in empirical research were explored as a new possibility to better grasp the complexity of empirical situations of teaching and learning mathematics, such as the interplay between the individual and the social. Typical heuristics to network theories were to relate different approaches based on research: bottom-up on the one hand and starting from theoretical views top-down on the other, but also mixed types were presented. One interesting result was that not only theoretical principles may be hidden in the use of theories but also the view on the nature of mathematics can be an implicit but relevant feature in the specific theoretical approach.

The central theme of TWG 9 of CERME 6 (Prediger et al., 2010) was investigating how the use of networking strategies may lead to a more comprehensive understanding of the empirical world, what kind of limits have to be faced, and what kind of difficulties have to be considered. In this respect, the questions of commensurability and complementarity of theories came into play. Radford's conceptualizing of theories (2008) as a triad of principles, methodologies and research questions built as a cultural entity of research practice in the semiosphere, a cultural-semiotic space of research as activities, was applied to structure the way networking strategies were used as guiding heuristics which link different aspects of theories. Examples showed that through the networking of theories new questions of 'balance' can be posed, concepts at the boundary of theories may become relevant to solve problems, and theoretically "zooming-in and zooming-out" can be a strategy when theories of different grain size are coordinated. The discussion in this TWG was captured by a dynamic view on theory "as a 'living entity' embedded in the researchers' social, cultural and institutional heritage" (Prediger et al., 2010, p. 1533).

Two years later, the TWG 16 at CERME 7 (Kidron et al., 2011) re-addressed meta-theoretical views on the networking of theories and recognized the need for a meta-theoretical frame for the networking of theories. Projects began to implement the networking of theories as a research practice following the new research aim of building a relationship between corresponding concepts of different theories. Besides the semiosphere (Radford, 2008) which was re-used in the paper presentations as a space where the networking of theories may be conducted, Artigue, Bosch and Gascón (2011) applied the Anthropological Theory of the Didactics to investigate the networking of theories as a research praxeology leading to specification of the relevance of problems and phenomena. Monaghan (2011) described Theoretical Genesis as an analogy to instrumental genesis on the part of the researcher adopting a theoretical view. It is a meta-view on the process of theorizing through the practice of "writing, learning, engagement with research and other voices" (ibid., p. 2498). The interesting point was the insight that also meta-theoretical views on theories largely depend on the cultural-semiotic way (meta-)theories are considered in the specific community. The contributions and discussion of this TWG substantiated the view that the networking of theories may advance the quality of research and lead to more linked and comprehensive results of research.

Whereas teaching and learning mathematics has been the main focus before, in TWG 16 of CERME 8, teacher education provided new directions of considering theories that involve new ways of theorizing on new research objects. This new topic also renewed the understanding of theories as epistemological tools: "the theoretical approaches need to be considered by what they enable researchers and practitioners to do, the questions raised, the regularities identified and described, that is, in a sense the results obtained" (Kidron et al., 2013, p. 2788). Besides addressing goals and practices of networking theories in research, the aspect of time as an additional dimension was emphasized; for example, because the networking strategy of coordinating theories may be executed as an intermediate step in time when theoretical elaborations have reached a particular status, or because coordinating theories may be executed fruitfully in a sequential way.

Although the networking of theories has also been an ongoing topic in TWG 17 of CERME 9, the main focus of this working group was on the notion of theories (Bosch et al., 2015). The discussions once more emphasized that theories are living entities that develop through processes of theorizing in research, beginning with local models, and developing towards more global entities dependent on the requirements of research. These processes result from exploring specific research questions, which may or may not broaden the theories' scope in mathematics education and beyond.

The Thematic Working Group 17 of CERME 10

The TWG 17 of CERME 10 continued the discussion on multi-theoretical approaches (the use of more than one theory in research, see the paper of Chan and Clark in the proceedings), particularly on the networking of theories, but also shifted its attention towards multi-theoretical approaches in design research and the problem of transfer inherent in the tension between home-grown and borrowed theories. The latter aspects have been an ongoing theme, already addressed by Steiner in the conferences on Theory of Mathematics Education of 1985 (Steiner, 1985). The description of the call of TWG 17 of CERME10 showed what kind of contribution was expected:

> This networking of theories approach is also addressed in the TWG 17 of CERME 10. With this working group we want to build on previous work of the group but this time we also want to address more specific topics: theories as prerequisite and result of design research, theorizing in research which involves technology, theories involved in interdisciplinary research with mathematics education. We want to explore how theories are used and built to better understand their role in and beyond mathematics education and the use of theories to inform practice.

Twelve papers and five posters were presented in the TWG 17. All but one poster abstract are published in the proceedings. They are grouped in three topics, the essentials of which will be extracted in the subsequent summary.

Networking of theories approaches

As in previous CERMEs, the discussion within this TWG has addressed the question of how to deal and work with theories, particularly concerning multi-theoretical approaches, which respect the theories' identities and at the same time are able to connect them fruitfully to solve problems in the field and understand the complexity of teaching and learning mathematics better. In terms of the networking of theories approach, the subsequent contributions witness a growing methodological maturity of handling the diversity of theories in research. This maturity is strongly related to deepening and broadening insight about the complex nature of the teaching and learning settings on two intertwined levels, the level of data analysis as well as the level of methodological and theoretical considerations and decisions, both being intertwined.

For example, Tabach, Rasmussen and Dreyfus conduct research to understand how learning in inquiry-based classrooms takes place individually and collectively and how these two learning planes are linked. They *coordinate* two theories, namely Abstraction in Context and Documenting Collective Activity, in a way that represents an innovative methodological step of research within the networking of theories strand that allows to identify how specific ways of coordinating may lead to in-depth insights into the functioning of inquiry-based learning, individually and collectively.

The effect of using networking strategies is directly investigated by Shinno, who has undertaken two case studies following two consecutive networking strategies; namely, coordinating and locally integrating. His research reveals: While coordinating preserves the meanings of the concepts involved as parts of theories, locally integrating theory elements changes the meanings of concepts. The reason for this seems to be that the concepts were *integrated* into a new theoretical framework, with new kinds of issues, questions, and aims. This result substantiates the fact that the meaning of a concept is deeply determined by the theory to which it belongs.

The mathematical workspace (MWS) presented by Nachache and Kuzniak even *requires* to be networked with further theory elements. The MWS originates in practical work with teachers, preserving its pragmatic character in linking semiotic, epistemic and cognitive genesis. Kuzniak et al. illustrate the plasticity of the model by connecting it to several theories or models for teaching mathematics. The reason why this connection is possible is the empirical load: The mathematical workspace model is empirically empty, and therefore allows models with high empirical load to complement it according to the three components offering ways to navigate through them.

Chan and Clarke's purpose in using a multi-theoretical approach is to explore the notions of complementarity and commensurability in an empirical way, a theme that has repeatedly been

addressed in previous working groups. They have instantiated a research project allowing to clarify the concepts of complementarity and commensurability based on analysis of the same data sets of problem solving activities from three different theoretical perspectives. Thus, the common data sets function as boundary objects (Star, 2010), objects that can function in different practices for different purposes, even without the need for consensus (Star, 2010).

In the fifth example, Behrens and Bikner-Ahsbahs add the perspective of the indexicality of signs to their theoretical framework for analyzing gestures related to speech, representations, and a technological tool. This choice is driven by the need to better understand the development of gestures from hand movements on the iPad's digital place value chart towards epistemic gestures, contributing to build knowledge. They show that the process of conceptualizing decimal fractions proceeds as an epistemic shift from *gesture-of* towards *gesture-for*, thus justifying their choice by a methodological result: The indexical nature of signs is a fruitful theoretical perspective for the analysis of epistemic processes as it allows tracing these processes back to their origins.

(Multi-)theoretical approaches in design research

The call explicitly asked for examples of theory use; specifically, in design research. This is particularly challenging because the purpose of theory use in this area is different from that in studies considered before in the networking of theories cases in the previous CERME-TWGs and the previous section. What is special about design research is that the justification of an educational goal requires normative theories, and the ways in which means are implemented to reach the goal – for example in design principles – require prescriptive theories. Finally, there is also a need for theoretical tools to analyze the empirical data of the implementation of the design, using descriptive or explanatory theories (Prediger, 2015). The normative and prescriptive theories developed, for example in the form of design principles, conjecture maps or hypothetical learning trajectories, raise the issue of methodology in relation to theory (cf. Radford, 2008). Kelly (2004) challenged design researchers to come up with what he calls an argumentative grammar – the reasoning from methods via analysis to warranted conclusions, which in the case of randomized controlled trials largely relies on the structure of argumentation.

Bakker takes up the challenge and argues that in design research, as in many other qualitative and mixed methods approaches, scholars cannot rely purely on the structure of argumentation. They need to account for the content too (content of the learning goals, content of core concepts used, context etc.). Bakker argues that design research may need several argumentative grammars and proposes elements of an argumentative grammar that he proposed to experts in design research during an interview study.

Given the multi-theoretical focus of the TWG, it was interesting to see how theories could play different roles in the design of curriculum or learning activities. For example, Johnson and colleagues used different theories for the design of their learning activities and for analyzing the resulting learning processes. The authors show how making theories of different grain sizes — grand theories (Piagetian theory), intermediate theories (Marton's variation theory), and domain specific theories (Thompson's theory of quantitative reasoning) — interact with each other allows designing effective dynamic computer environments and tasks to promote students' learning. Kouropatov and Dreyfus, on the other hand, used two theories for the design of a task-based

curriculum and for analyzing resulting learning processes to feed back into improved design. The two theories – Abstraction in Context and Proceptual Thinking – were of different grain size and had different focuses, which made them complementary. The authors argue that in the process of designing learning units, the different theories have been interwoven while keeping different roles from stage to stage.

Simon and his colleagues, building on constructivist theory and their empirical research, developed an elaboration of Piaget's construct of reflective abstraction for the purpose of undergirding an instructional design theory for promoting mathematical concepts. In conjunction with this elaborated construct, they have articulated an instructional design approach that fosters reflective abstraction of particular concepts. In doing so, they have afforded a change in design research (i.e., teaching experiment methodology) from a focus on students' mathematical reasoning and operations to a focus on the conceptual learning process and designs for promoting that process.

Transfer of theory elements: The tension between home-grown and borrowed theories

The tension between home-grown and borrowed theories was one question previous working groups have dealt with by several contributions. In this working group, the discussion focused on two main interrelated issues. On the one hand, home-grown concepts may bear with them meanings specific to the social and cultural context or the field in which they have been elaborated, and that raises the question how to transfer them into a foreign context or field. Research is needed to address the viability of adapted concepts. On the other hand, theoretical tools or perspectives which are borrowed from other fields must either be adapted to mathematics education or particularized or complemented with content-related theoretical tools in order to be fruitfully put to work. Some of the contributions presented in the working group faced one of these two issues.

For example, Roos' contribution shows how a home-grown concept which emerged in specific cultural context may be difficult to transfer or translate into a different one. More specifically, Roos presents an overview about the concept of *Grundvorstellungen*. This concept emerged in German-speaking countries as a practical tool for teaching. It is impossible to translate and even difficult to explain in English, even if one can recognize the existing links with the notion of *concept image* or with Vergnaud's *theory of conceptual fields*. This difficulty raises the question of how ideas, or even entire theories, which emerge within a specific cultural context and therefore bear cultural-historical meanings, can be communicated on an international scale.

In other cases, home-grown concepts seem to have the potential to be more easily tranferred and adopted in foreign contexts. This is shown in Liljekvist's contribution. She uses the concept of *prosumer*, which stems from sociological research, to understand mathematics teachers re-sourcing and using social media in a Web 2.0 world, linking the two activities of consuming and producing. Even if the concept of *prosumer* bears meanings and values from its native context and has to be futher developed for mathematics education purposes, it has the potential to be easily translated and spread outside as it carries its 'origins' in the term itself.

Adapting borrowed concepts and theories from other fields is not only a question of translation. Mathematics and mathematics education have their own specificities, which must be taken into account when borrowing theoretical tools and concepts from other fields. How that can be done is at the core of the tension between home-grown and borrowed theories. For example, Haspekian and

Roditi faced the issue of adopting general concepts from the field of assessment research in education to a specific research study in mathematics education. The authors developed a methodological tool for analyzing teacher-student interactions in mathematics classes as an adaptive dynamic process. The discussion on their uses of concepts from the assessment field illustrates a way of locally connecting research areas via a shared methodological tool.

Similarly, Georget and Sabra draw on general sociocultural theories to investigate the professional development process occurring in mathematics teachers' communities; however, their study emphasises the need to resort to a complementary theoretical focus addressing specifically the place of mathematics in such communities in order to account effectively for the dynamics which take place in the community. The same tension is also present in the contribution of Zerafa. Zerafa developed an intervention programme addressed to children experiencing mathematics learning difficulties. The design of this programme relies upon the adoption of a large number of borrowed theoretical tools. That raises the question of how to complement borrowed theoretical framework in order to take into account the specific mathematical content at stake.

Researchers can also meet problems concerning the adaptability of theories in context or for purposes different from those in which and for which theories have been developed, even within a given specific field. For example, Benedicto, Gutiérrez and Jaime faced such a problem when applying an existing model, developed to analyse cognitively demanding tasks in the areas of arithmetic and algebra, with the aim to analyse tasks in different mathematical topics. In fact, the original model revealed not to fit adequately their research needs. Their contribution illustrates the processes of analysing the model and the difficulties emerged. Thus, they adapted the model to the new needs and obtained an improved model that did not lose its core meaning while being more widely applicable.

Issues to be considered in future meetings

For future development, several participants expressed the wish to make progress on broader themes that superseded individual presentations. One way to do so is by proposing themes that participants commit themselves to for the next TWG, as was done in previous groups. This would allow the working group to continue working by themes, and discuss the studies in relation to a central theme (say one per day). This can make each author's contribution a case of a more general issue, and allows us to do cross-case analyses in the working group. However, the challenge of this theory group is to balance concreteness and generality in the discussions to make the huge number of theories (48 theories in TWG17) being presented in this working group accessible to all the participants even if they are familiar with some of them. Suggestions for central themes are:

1. **Progress** and **quality**: On what grounds can we decide whether the networking of theories is a contribution to the field? Concepts often used are: complementarity, commensurability, consistency, usability, and fit to purpose. What **methodologies** for research with networking theories are suitable? What **criteria** are suitable for selection and adaptation when networking theories? Criteria may be different for researchers who have a fundamental interest than for educators who work with models that teachers need to work with. Theories can be placed in a framework of different levels (diSessa & Cobb, 2004); are there any heuristics we can derive on good practices of how to coordinate multiple theories; and what

do disciplines outside mathematics education have to offer us in this respect (history and philosophy of science?)

2. What **work** do you do with theories to be able to use them for your purposes? In what respect do you have to adapt a theory or combine it with others? What is the nature of theories used: Describing and explaining learning processes versus offering design heuristics or guidelines? The incompleteness of theoretical models (discussed by Kuzniak et al.) can be an advantage because generality or emptiness can make a model or construct easy to transport (transportable). However, there are also risks when there is a lack of specificity. An issue raised was how theories are insensitive to differences that may matter.
3. What are appropriate **argumentative grammars** for types of research that explicitly have a normative and/or prescriptive element, such as design research? How do we ensure that design embodies theoretical ideas, and how to study the resulting learning as a consequence of the design?
4. How can we deal with concepts that are hard or even impossible to **translate** into English (*milieu*, *Grundvorstellung*, *Stoffdidaktik*, types of participation in Asian countries, …)? The Lexicon Project will have a lot to offer in this respect.

To deepen the understanding of theory and methodology in European research, this thematic working group of CERME should in the future address the issues of quality of the networking of theories in research practice, of the specificities of theories, of identifying different argumentation grammars and scientific ways of communicating culturally bound concepts and theories on the international plane.

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The Thematic Working Group 17 of CERME 10: Grouped topics of TW17 of CERME10

Networking of theories approaches

Abstraction in Context and Documenting Collective Activity

Michal Tabach[1], Chris Rasmussen[2], Tommy Dreyfus[3] and Rina Hershkowitz

Meta-theoretical aspects of the two case studies of networking theoretical perspectives: Focusing on the treatments of theoretical terms in different networking strategies

Yusuke Shinno

On dialectic and dynamic links between the Mathematical Working Space model and practice in the teaching and learning of mathematics

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The Mathematical Working Space model: An open and adaptable theoretical framework?

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Learning research in a laboratory classroom: Complementarity and commensurability in juxtaposing multiple interpretive accounts

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The perspective of indexicality: How tool-based actions and gestures contribute to concept-building

Daniela Behrens and Angelika Bikner-Ahsbahs

Abstraction in Context and Documenting Collective Activity

Michal Tabach[1], Chris Rasmussen[2], Tommy Dreyfus[3] and Rina Hershkowitz[4]

[1]Tel-Aviv University, School of Education, Israel; Tabach.family@gmail.com

[2]San Diego State University, Department of Mathematics, USA; crasmussen@mail.sdsu.edu

[3]Tel-Aviv University, School of Education, Israel; TommyD@post.tau.ac.il

[4]Weizman Institute, Science Teaching Department, Israel; rina.hershkowitz@weizmann.ac.il

In this report we advance the methodological and theoretical networking for documenting individual and collective mathematical progress. In particular, we draw together two approaches, Abstraction in Context (AiC) and Documenting Collective Activity (DCA). The coordination of these two approaches builds on prior analysis of grade 8 students working on probability problems to highlight the compatibility among the epistemic actions that ground each approach and drive the respective methodologies. The significance of this work lies in its contribution to coordinating what might otherwise be viewed as separate and distinct methodologies.

Keywords: Methodology, theories coordination, individual cognition, collective meaning making.

Background

In this report we advance the methodological and theoretical networking (Bikner-Ahsbahs, & Prediger, 2014) of two different approaches, the Abstraction in Context (AiC) approach with the RBC+C model commonly used for the analysis of knowledge construction by individuals or small groups; and the Documenting Collective Activity (DCA) approach with its methodology commonly used for establishing normative ways of reasoning in classrooms. In previous work related to this goal (Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014) we demonstrated how this coordination can illuminate the processes by which ideas shift from individuals and small group to the classroom community as a whole or vice versa. This combination revealed that some students functioned as "knowledge agents," meaning that they were active in shifts of knowledge among individuals in a small group, or from one group to another, or from their group to the whole class or within the whole class.

We take the coordination between AiC and DCA a step further by explicating theoretical and methodological commonalities between the two approaches. These commonalities, which we first pointed to at CERME9 (Tabach, Rasmussen, Hershkowitz, & Dreyfus, 2015), drives further the integration of the two approaches, including what we refer to as environmental, underlying, and internal commonalities. The analysis in the present case led us to enhance the theoretical commonalities with data driven ones. We explicate these commonalities to set the stage for the analysis of students' work, but first begin with a brief summary of the AiC and DCA approaches**.**

Abstraction in Context and the RBC+C model

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating abstract mathematical knowledge (Hershkowitz, Schwarz, & Dreyfus, 2001). Abstraction is defined as an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of

mathematical thinking so as to lead to a construct that is new to the learner. According to AiC, the genesis of an abstraction passes through three stages (ibid): (i) the arising of the need for a new construct, (ii) the emergence of the new construct, and (iii) the consolidation of that construct. AiC includes a theoretical/methodological model, according to which the description and analysis of the emergence of a new construct and its consolidation relies on a limited number of epistemic actions: Recognizing, Building-with, Constructing, and Consolidating (RBC+C).

These epistemic actions are often observable as they are expressed by learners verbally, graphically, or otherwise. Recognizing takes place when the learner recognizes a specific previous knowledge construct as relevant to the current problem. Building-with is an action comprising the combination of recognized constructs in order to achieve a localized goal, such as the solution of a problem or the justification of a claim. The model suggests Constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and interweaving previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner. Recognizing actions are nested within building-with actions, and recognizing and building-with actions are nested within constructing actions. Therefore, the model is called the nested epistemic actions model of abstraction in context, or simply the RBC+C model. The second "C" stands for Consolidation. The consolidation of a new construct is evidenced by students' ability to progressively recognize its relevance more readily and to use it more flexibly in further activity.

Documenting Collective Activity

The methodological approach of documenting collective activity (DCA) is theoretically grounded in the emergent perspective (Cobb & Yackel, 1996), a basic premise of which is that mathematical progress is both an individual constructive process and a process of enculturation into the emerging norms and practices of the local classroom community. That is, the personal and collective mathematical progress can be seen as two sides of the same coin. Collective activity of a class refers to the normative ways of reasoning that develop as students work together to solve problems, explain their thinking, represent their ideas, etc. These normative ways of reasoning can be used to describe the mathematical activity of a group and may or may not be appropriate descriptions of the characteristics of each individual student in the group. A mathematical idea or way of reasoning becomes normative when there is empirical evidence that it functions in the classroom as if it is shared. The empirical approach makes use of Toulmin's model of argumentation (1958), the core of which consists of Data, Claim, and Warrant. Typically, the data consist of facts or procedures that lead to the conclusion that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant. It is not uncommon, however, for Rebuttals or Qualifiers to arise once a claim, data, and warrant have been presented. Backing provides further support for the core of the argument.

The following three criteria are used to determine when a way of reasoning becomes normative: 1) When the backing and/or warrants for particular claim are initially present but then drop off, 2) When certain parts of an argument (the warrant, claim, data, or backing) shift position within subsequent arguments, or 3) When a particular idea is repeatedly used as either data or warrant for different claims across multiple days (Cole et al., 2012; Rasmussen & Stephan, 2008).

Environmental commonalities

The use of both methodologies, RBC+C and DCA, requires quite specific classroom social norms (Yackel & Cobb, 1996). First, they require classrooms in which students routinely explain their thinking, listen to and indicate agreement or disagreement with each other's reasoning, etc. If such norms are not in place, then evidence is unlikely to be found of challenges, rebuttals, and negotiations that lead to ideas where knowledge is constructed and starts functioning as if shared by the whole class. We call such classrooms "inquiry-oriented classrooms" (Rasmussen & Kwon, 2007). Second, these classrooms require the intentional use of tasks designed to offer students opportunities for constructing new knowledge by engaging them in problem solving and reflective activities allowing for vertical mathematization. Both methodologies focus on the ways in which mathematical progress is achieved and spreads in the classroom. RBC+C focuses on individuals or small groups working in the classroom and DCA focuses on group or whole class discussions. In this sense, the two methodologies complement each other in analyzing a sequence of lessons including individual and group work and learning in whole class discussion and in tracing how knowledge is constructed and becomes normative.

Underlying commonalities

Other characteristics of a classroom culture in which DCA and RBC+C methodologies might be enacted together are that the tasks are designed to afford inquiry and the emergence of new constructs from previous constructs by vertical mathematization (Treffers, & Goffree, 1985); such learning materials allow for interweaving collaborative work in both small-group work and whole-class discussions, where the teacher adopts a role that encourages inquiry in the above sense. Another underlying characteristic relates to the centrality of the shared knowledge. RBC+C characterizes shared knowledge as a common basis of knowledge which allows the students to make further progress. We find its counterpart in sociological terms, in the phrase "function as if shared" used by the DCA approach. What is common between the two constructs is the point that each operationalizes when particular ideas or ways of reasoning are, from a researcher's viewpoint, beyond justification for participants. At the collective level, ideas or ways of reasoning that function as if shared have the status of accepted mathematical truths for the group. At the individual level, consolidation results in individuals accepting something as a mathematical truth.

Internal commonalities

DCA analysis helps illuminate what is happening on the social plane, while RBC+C analysis helps illuminate what is happening on the cognitive side. To elaborate, we highlight relationship between constructs suggested by the cognitive RBC+C analysis and their sociological counterparts suggested by the DCA analysis. We do that from a theoretical perspective and from an empirical perspective. To achieve this goal we begin with the following excerpt 1, used also in Hershkowitz et al. (2014) but for different purposes. It is a discussion between Noa and Gil, two eighth grade students working on a probability problem (see turn 95) during a group work period taken from the third lesson on this topic, and a bit of whole class discussion. This excerpt includes a DCA analysis, in particular classification of the marked parts of students talk (shaded) according to Toulmin's model as data [D], claim [C], warrant [W], backing [B], rebuttal [R] or qualifier [Q]. In addition, RBC+C actions were identified in students' talk (*italic*), and marked as recognizing (R), building-with (B),

end of the constructing action (C) or consolidating (CC) with respect to two knowledge elements: E_{xp} - experiment is needed in order to determine the chances and E_{xd} – experiment detailed.

No.		Utterance [DCA analysis]	RBC +C
95	Noa	(reads) 'Is it possible to determine without experimenting what the chances are that we will take out a defective match from a matchbox? If yes, what is it?' *You can't know!* [D1] *Unless … you have to experiment* [C1]! *You can't know!* You need to experiment! I'm writing "You need to experiment!"	RB
96	Gil	*You don't have to*! [C2, counterclaim]	B
97	Noa	Of course you do!	
98	Gil	"What the chances are of taking out a defective match from a matchbox?" *It's 1 out of the number of matches in the box*. [D2]	R
100	Noa	*Right, so you take many boxes, how many, if, in the box* [W1]…	B
101	Gil	*Noa, it depends on how long you have been using the box, if you used it once then maybe it will be less …* [Q1]	B
102	Noa	*No! If it's defective! You have to take many boxes* [D1] *and see in each one if there is … if there are let's say 50 matches in each box and 1 is defective so it says on the box 1 out of 50* [W1], *so you have to experiment!* [C1, referring back to turn 95]	B
103	Gil	*So it's 1 per the number of matches in each box* [W2].	R
104	Noa	*Not 1, there may be 2 defective matches in the box* [R2].	B
105	Gil	But what are the chances?	
106	Noa	But with 2 defective ones?	
107	Gil	But Noa, you are speculating … *you can say 50 out of 50* [R to 104].	B
108	Noa	*But you can't say 1 out of 50! Out of … whatever!* [W to 104] What is the probability? It's not correct what you are saying!	B
109	Gil	What isn't correct?	
110	Noa	*Because just like you can't say 2 out of the matches because you don't know that it's 2 or that it's 1* [W1 = R2].	B
111	Gil	(writes) "*can't determine without experimenting.*" [C3]	$\underline{C_{xp}}$
112	Noa	*We can, if we experiment*. [C, slightly new claim of how to do the experiment]	CC_{xp}
113	Noa	*Ok, so what is the probability? It's, we have to write that we won't know* [D1] *until we experiment* [C1].	RB
114	Noa	Let's write at the bottom, that *we need a few boxes* [D4], *suggest an experiment (dictates: "we need to take a few boxes of matches and see out of them* [D4]…" [Dictate together].)	B
115	Noa	No, wait! *How many matches does the box contain, and see how many defective matches are in it* [D4]… [Dictate together].	B

116	Gil	(continues to dictate) "*then, check how many defective matches are in the box* [W4]" [Dictate together].	B
117	Noa	Then we will write "*the probability is the number of defective matches in the …* [C4, together with turn 123]" [Dictate together].	$\underline{C}_{xd}$
…			
122	Gil	Noa, *each box will come out differently*.	R
123	Noa	*So it's average* [C, note Data is previous argument], *not precise* [Q4]!	CC_{xd}
		Back to whole class discussion	
135	Noa	In my opinion you need to experiment [C10]!	
136	T	Why?	
137	Noa	I don't know. I can suggest an experiment [Q10]	
138	T	Friends, listen, you need to express your opinion on what they said	
139	Gil	[addressing Noa] Why can't you say why you need an experiment, *you can't know how many matches there are in the box* [D10].	B[1]
140	T	Let's say I can reveal to you that there are 45 matches in the box.	
141	Gil	*And inside you have to [check]*.	B
142	Noa	[you need to take some] matchboxes [D11], you need to see how many matches are in each box, and how many of them are defective [W11].	
143	T	Let's say we know that information, what do I do with it?	
144	Noa	So …	
145	Gil	So *I do the average* [C11, with 147]	B
146	T	What average?	
147	Gil	*Of the defective matches in each box* [C11, with 145]	$\underline{C}_{xd}$
148	T	And how is that going to help us know what the probability is that we take out a defective match?	
149	Noa	Let's say we have 2 defective matches in a box with 50 matches, so it's 2 divided by 50.	
150	T	2 to 50, what do you think?	
151	Gil	We are saying that you can't do it without an experiment [C10]. *You can't know how many defective matches there are because we don't know how many matches are in the box and we don't know either … We can't speculate how many defective matches there are* [W10]. *We wrote that we need to take a number of matchboxes and see how many matches they contain, then count how many out of them are defective* and *do an average of how many defective matches in each box* [C11]. If we got 3 then it's 3 divided by 50.	CC_{xd}

Table 1: Excerpt 1, Transcript from the class

[1] From this point on it is Gil who does the B and C actions

RBC+C and DCA analysis

We begin by relating elements of the RBC+C and DCA analyses to each other, and then we relate the three criteria of the DCA approach to consolidation.

Relationship between Recognizing and Data. Theoretically, we argue that Recognizing actions are largely associated with Data. One uses some piece of information as Data because that piece of information makes sense to him/her. That is, this piece of information is relevant for the person; it is what the person selects for use (as Data). In the above excerpt, we see that Recognizing actions are primarily associated with Data. In some cases (e.g., turn 103), Recognizing actions can be associated with Warrants. In carrying out the DCA analysis, disentangling Data and Warrant is at times non-trivial, in which cases Recognizing actions can be sensibly associated with Warrants.

Relationship between Building-with and Warrants. Theoretically, Warrants establish a connection between data and claim; in order to establish such a connection, one needs to build-with what one has recognized. In the above excerpt this commonality is largely the case with some exceptions that need clarification. In turns 95/96 we had claims associated with building-with. These are the first building-with actions of this excerpt and thus the first ones of the part where the students are working on the present task. As a consequence, the building-with actions are somewhat shallow and make only claims without really warranting them. As such, this example does not pose a substantive threat to the theoretical conjecture. Similarly, turns 114 - 116 and 139 do not pose a substantive threat to the conjecture. These are the final utterances belonging to a constructing action; as such, they complete the constructing by explicitly stating the claim that was constructed. As we noted above, at times data and warrant are difficult to disentangle with certainty, hence building-with can be associated with data. Empirically, this is the case for turns 114 and 115.

The relationship between Constructing and Arguments as a whole. Constructing requires vertical mathematization. Constructing actions are usually much more extended than Recognizing or Building-with actions; they incorporate sequences of interweaving Recognizing and Building-with actions (plus the 'glue' between them). Similarly, arguments interweave data-claim-warrants and backings as a whole. Hence, in a line by line coding it is not feasible to indicate the holistic nature of an argument and it is typically indicated after a line by line coding (see for example Tabach et al., 2014). Moreover, arguments are usually co-constructed by several participants over several turns. Such interaction is also frequent in constructing actions.

Consolidating and the three criteria for identifying function-as-if-shared ideas. In consolidating actions as well as across the three DCA criteria for identifying when an idea functions as if shared, repetition, reuse, revisiting, or repurposing of earlier ideas frequently occurs. To clarify, in Criterion 1 there is a repetition, but the repetition is partial in the sense that some parts of the argument (Data, Warrants) cease to be explicitly stated. In Criterion 2 there is repurposing of previous part of an argument (e.g., Claim) as either Data or Warrant. In this sense there is a repeating and reusing, but for a different purpose. In Criterion 3 there is a revisiting of either Data or Warrants to establish new Claims. In consolidation, previous constructs are recognized as relevant (i.e., revisited), and then built-with, which means they are reused, often for a new purpose such as a new constructing action. Hence there are strong parallels between consolidation and the three DCA criteria. For example, in 151, DCA analysis shows that W10 (the warrant for Claim 10) turns into D11 (i.e. the

data for Claim 11); hence Criterion 2 is satisfied: the same part of the argument is reused with a different function. RBC+C analysis shows that knowledge construct xd is consolidated by being used again, and at the same time elaborated.

Further commonalities between consolidating and the DCA criteria can be seen by considering the characteristics of consolidation: awareness, self-evidence, flexibility, immediacy, and confidence (Dreyfus & Tsamir, 2004). Self-evidence links to Criterion 1 because the evidence is the Data, which drops off in subsequent arguments. The subsequent argument also then relates to immediacy and confidence in the validity of the idea. Flexibility links to Criterion 2 because components of an argument are being reused and repurposed (as sign of flexibility) in subsequent arguments. Similarly, Criterion 3 relates to flexibility, but in a different way. Here the flexibility lies in the fact that one is able to use an idea (e.g, Build-with) as Data or Warrant for a variety of different Claims. Hence close relationships exist between the criteria and characteristics of consolidation.

We conclude this report by returning to vertical mathematization, which was highlighted as an Environmental commonality. We also see vertical mathematization as an Internal commonality. Both methodologies work from the premise that vertical mathematization is core to mathematical progress. In the RBC+C approach, consolidation is vertical mathematization and, as we argued above, the consolidation is closely linked to the three criteria.

Conclusion

We now turn to discussing some implications for research. In addition to offering a theoretically and empirically grounded approach for coordinating methodologies for individual and collective mathematical progress, there exist specific ways in which this coordination can play out. For example, one could choose an individual student within the classroom community and trace their constructing actions for the ways in which they contributed to the emergence of various normative ways of reasoning. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin analysis used to document normative ways of reasoning. How do those contributions coordinate with individual student constructions? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that they are authoring, and in what ways may that capture or be distinct from other students' individual mathematical meanings? Future research could take up more directly the role of the teacher in relation to individual and collective level mathematical progress. More generally, however, this report contributes to an emerging discourse on theories and ways in which different theoretical approaches can be profitably networked (e.g. Bikner-Ahsbahs & Prediger, 2014).

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Meta-theoretical aspects of the two case studies of networking theoretical perspectives: Focusing on the treatments of theoretical terms in different networking strategies

Yusuke Shinno

Osaka Kyoiku University, Faculty of Education, Osaka, Japan; shinno@cc.osaka-kyoiku.ac.jp

Keywords: Networking strategies, combining and coordinating, integrating, theoretical terms.

Networking strategies and the present study

In the past decade, strategies for connecting theories have been intensively discussed (e.g., Prediger et al., 2008). According to Prediger et al. (2008), the networking strategies are structured as follows: *understanding* and *making understandable*; *comparing* and *contrasting*; *combining* and *coordinating*; and *integrating locally* and *synthesizing*. The purpose of this study is to show my/our two case studies related to the networking theories in order to consider some meta-theoretical aspects of these studies. I will briefly introduce the two case studies, which have been developed by myself (Shinno, 2016) and by a Japanese research group (Shinno et al., 2015). The first study is concerned with combining and coordinating, and the second study is concerned with locally integrating.

By reflecting the researchers' practice on the two case studies at meta-theoretical aspects, I attempt to reconsider different treatments of theoretical terms in different strategies. This may allow us to analyse transition between pre/post statuses of the networking, although different strategies can be utilized for different purposes. Comparing the two studies, by focusing on the treatments of terms, I discuss how the elaboration of the original terms may influence the degree of integration.

The two case studies: Overview

Case study 1: *Combining* and *coordinating*

Shinno (2016) aims to characterize the development of mathematical discourses in a series of lessons in terms of the model of semiotic chaining by Norma Presmeg and the *commognitive* framework (Sfard, 2008). One of the research questions of this study is as follows: In what ways can the model of semiotic chaining be combined with the *commognitive* framework in the analysis of reification in the learning of square roots? For gaining multi-faceted insight of the reification phenomenon, Shinno (2016) attempts to coordinate the *commognitive* terms (such as *keywords*, *visual mediators*, *endorsed narratives*, and *routines*) with the semiotic terms (such as *signifier*, *signified*, and *interpretant*). By doing so, Shinno (2016) intends that the implicit meta-discursive rule (*routine*) may become explicitly identified as the semiotic component (*interpretant*).

Case study 2: *Integrating locally*

Shinno et al. (2015) aims to construct a theoretical framework for curriculum development for teaching proof by means of integrating different theoretical constructs related to proof. In developing a framework, the notion of Mathematical Theorem comprised of the three elements – *statement*, *proof*, and *theory* – is used as the foreground of the framework. Some other theoretical constructs, such as *mathematical proof* by Nicolas Balacheff and *local organization* by Hans Freudenthal, are locally integrated into the framework in order to consider the wide range of contents and levels of

statement, *proof*, and *theory* in curriculum. As a result, Shinno et al. (2015) elaborate some additional categories by introducing new terms, for example, *real world logic*, *local theory*, and *quasi-axiomatic theory*, which are included in a category of 'nature of system' based on the concepts of *local organization*.

Discussion: Meta-theoretical aspects

In the first study, when coordinating the semiotic and discursive terms, it seems that a theoretical term is interchangeable with another term (e.g., "an interpretant" with "a routine"). Therefore, even after the coordinating, it seems that the treatments of the terms can be preserved in both theoretical contexts. In other words, the results of the empirical studies can feedback to the original theories. In the case of Shinno (2016), it allows to analyse the reification phenomenon from the two different perspectives and to gain a deeper understanding of the phenomenon. In the second study, it seems that the basic constructs to be networked are 'concepts' rather than 'theories'. Some original concepts such as *local organization* can be elaborated and integrated locally into new terminologies or categorizations in the constructed theoretical framework. Since elaborated terms have a consistency within the new framework and these can create new meanings, such terms cannot preserve their original senses. Thus, this networking strategy may contribute to establish a new theoretical discourse rather than to understand a certain empirical phenomenon. It seems that this strategy also can be utilized for developing or elaborating a new theoretical model.

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Note: The digital poster can be available from the following link: http://www.osaka-kyoiku.ac.jp/~shinno/shinno/poster.html

On dialectic and dynamic links between the Mathematical Working Space model and practice in the teaching and learning of mathematics

Alain Kuzniak[1] and Assia Nechache[2]

[1]Université Paris-Diderot, LDAR (EA 4434), Paris, France; alain.kuzniak@univ-paris-diderot.fr

[2]Université d'Orleans, LDAR (EA 4434), Paris, France; assia.nechache@hotmail.fr

In this communication, we address the specific relationships between the Mathematical Working Space model (MWS model) and practice in the teaching and learning of mathematics. The strong and positive interactions existing between these two aspects are illustrated with two examples from geometry and probability teaching. They show how some theoretical constructs as MWS diagram can enlighten practice and, conversely, how studies on practice nourish the model with new tools such as "comics", "complete mathematical work" or "emblematic tasks".

Keywords: Mathematical work, MWS model, emblematic task, mathematics teaching.

A decade ago, the Mathematical Working Space (MWS) model has been introduced as a theoretical and methodological framework dedicated to identify and shape the mathematical work in schooling. Developed by researchers working collaboratively in various countries with, sometimes, very different educational approaches in Europe (France, Spain, Cyprus), Latin America (Chile, Mexico…) and North America (Canada), the model is deeply rooted in the teaching of mathematics in real classrooms. This communication aims at showing dynamics and dialectics between the MWS model and relevant questions to education practice. After a short presentation of the MWS model, we show first how it can be used to deal with the question of planning series of tasks in the teaching and learning of geometry. That leads us to introduce two new constructs: the methodological tool of "comics" used to describe the evolution and circulation of the mathematical work and the more theoretical idea of "complete mathematical work" which allows qualifying the final nature of this circulation within the diagram. Then, these two new tools are used for the study of the teaching of probability and statistics. In France, this teaching is relatively new and it is now initiated on modeling tasks with use of technological tools. This leads us to check the relevance of the above constructs and identify two kinds of incompleteness of mathematical work and, in addition, to draw out certain specific tasks, named "emblematic tasks". Designed on these former results, our present research aims at tracking transformations made by teachers when they adapt these "emblematic tasks" to their classrooms.

A short insight in the MWS model

Extending the research work developed by Houdement and Kuzniak (2006) in didactics of geometry, the MWS model emerged during the last decade. The model, especially in geometry, had already been presented during former CERME meetings (Kuzniak and Nechache, 2015). A recent issue of ZDM-Mathematics Education (48-6, 2016) is devoted to this model and we refer the reader to this issue for further details and discussions about the MWS model. Some elements of the introduction (Kuzniak, Tanguay and Elia, 2016) to this issue are used to present the framework.

The theoretical model of Mathematical Working Space (MWS) provides a tool for the specific study of the mathematical work in which students and teachers are effectively engaged during mathematics sessions. The abstract space thus conceived refers to a structure organized in a way that allows the mathematical activity of individuals who are facing mathematical problems. It establishes the reference to the complex setting in which the problem solver acts. In this approach, the crucial function of educational institutions and teachers is to develop a rich environment which enables students to properly solve mathematics problems. To describe the specific activity of students solving problems in mathematics, the idea of organizing the MWS into two articulated planes is retained: one of an epistemological nature in close relation to the mathematical content in the field being studied; the other of a cognitive nature, related to the thinking of the individual solving problems. Three components in interaction are characterized for the purpose of describing the work in its epistemological dimension, organized according to purely mathematical criteria: a set of concrete and tangible objects, the term *sign* or *representamen*[1] is used to summarize this component; a set of *artifacts* such as drawing instruments or software; a *theoretical system of reference* based on definitions, properties and theorems.

The second level of the MWS model is centered on the subject, considered as a cognitive subject. In close relation to the components of the epistemological level, three cognitive components are introduced as follows: *visualization* related to deciphering and interpreting signs, and to internally building (psychological) representation of the involved objects and relations; *construction* depending on the used artifacts and the associated techniques; *proving* conveyed through processes producing validations, and based on the theoretical frame of reference. Furthermore, the development by communities or an individual, whether generic or not, of appropriate mathematical work is a gradual process by which a suitable MWS is settled through a progressive approach and fine tuning. Therefore, analyzing mathematical work through the lens of MWSs allows tracking down how meaning is progressively constructed, as a process of bridging the epistemological plane and the cognitive plane, in accordance with different specific yet intertwined genetic developments, each being identified as a genesis related to a specific dimension in the model: semiotic, instrumental and discursive geneses. This set of relationships can be described proceeding from the elements of the following diagram (Figure 1) which, in addition, shows the interactions between the two levels with three different dimensions or geneses: semiotic, instrumental, and discursive:

- The Semiotic genesis is the process associated with representamen (or signifiers), and accounts for the dialectical relationship between the syntactic and the semantic perspectives on mathematical objects, displayed and organized through semiotic systems of representation.
- The Instrumental genesis enables making artifacts operational in the construction processes contributing to the achievement of mathematical work.
- The Discursive genesis of proof is the process by which the properties and results organized in the theoretical reference system are being actuated in order to be available for mathematical reasoning and discursive validations.

The epistemological and cognitive planes structure the MWS into two levels and help to understand the circulation of knowledge within mathematical work. How then, proceeding from here, may one

articulate efficiently the epistemological and cognitive levels in order to make possible the expected mathematical work? How may one organize and describe interrelationships existing between our former three geneses? In order to understand this complex process, the interactions that are specific to the execution of given mathematical tasks will be associated to the three vertical planes, naturally occurring in the diagram of Figure 1: the [Sem-Dis] plane, conjoining the semiotic genesis and the discursive genesis of proof, the [Ins-Dis] plane, conjoining the instrumental genesis and the discursive genesis of proof, the [Sem-Ins] plane, conjoining the semiotic genesis and the instrumental genesis (Figure 2). The three planes are valuable tools for describing the interrelationships between the different geneses, for identifying and characterizing phases in the solving processes, for analyzing the shifts occurring in the course of these processes when specific aspects are, unexpectedly or gradually, either left aside or given more prominence.

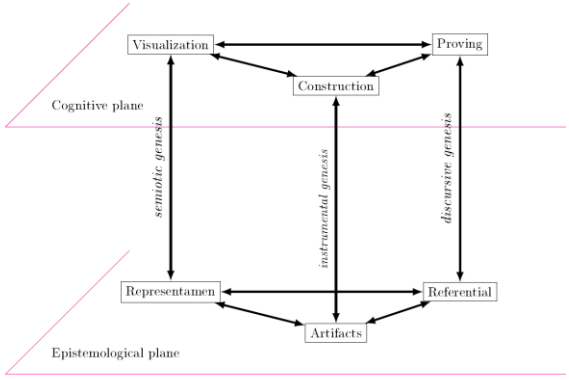


Figure 1: The Mathematical Working Space Diagram

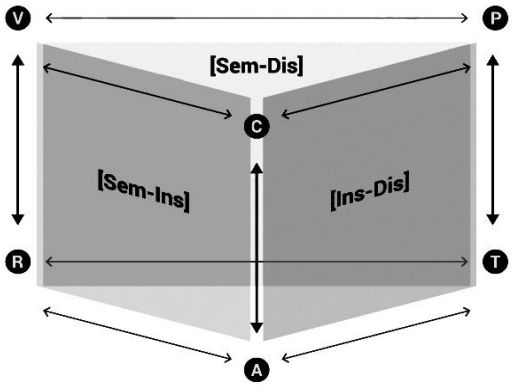


Figure 2: The three vertical planes in the MWS

The exact definition and precise description of the nature and dynamics between these planes during the solving of a series of mathematical problems remains a central concern for a deeper understanding of the MWS model. They vary with the mathematical field at issue, with the tasks, with the schooling level, with the type of work promoted or expected, etc.

Planning of a teaching sequence in geometry at primary school

In France, at primary school level, numerous and interesting tasks in geometry are available and relatively easy to access. By contrast, few resources are available to help teachers to plan a series of geometrical tasks and activities for elementary schools students. To move forward on this issue, the MWS model (Kuzniak & Nechache, 2015) was used to identify some key points in organizing a long teaching sequence on a specific topic. Designed by two well-known French researchers in the domain (Fenichel & Taveau, 2009), the selected sequence “Le cercle sans tourner en rond” is dedicated to Grade 4-6 students. The sequence includes eight sessions from half an hour to one hour. Its main objectives are the introduction of the global notion of circle as the set of all points equidistant from a given point, named the center; to use this property for solving distance problems and make constructions with compass used also to transfer distances. The MWS diagram was used to analyse each of the sessions and to observe various circulations of the geometrical work through the different planes of the MWS diagram (Figure 2). For example, in session 1, the objective is to identify the circle as the set of all points equidistant from a given point, the centre. Students are

asked to place a point A on a white sheet and then a point B (semiotic dimension). After that, they have to place 15 points "situated at a distance from A which is the same as the distance of B from A" (semiotic dimension). They may use various artifacts: blank and tracing paper, twine, square set, compass (instrumental dimension). The geometric work starts in the [Sem-Ins] plane. Then, during a formulation phase, some students' productions are displayed on the blackboard and discussed. The strategies used by the students to carry out the task are clarified and formulated. The notion of equidistance from a given point is expected to emerge. Some geometric terms are institutionalized and the characteristic property of the circle is given by the teacher and enriches the theoretical referential (discursive dimension) in the MWS. In summary, the geometric work starts in the [Sem-Ins] plane and is concluded in the discursive dimension (Dis). The same analysis has been made on five sessions and allows describing the dynamic evolution of geometric work. This evolution is visualised with the following "comics" which highlight the key-points of the sequence.

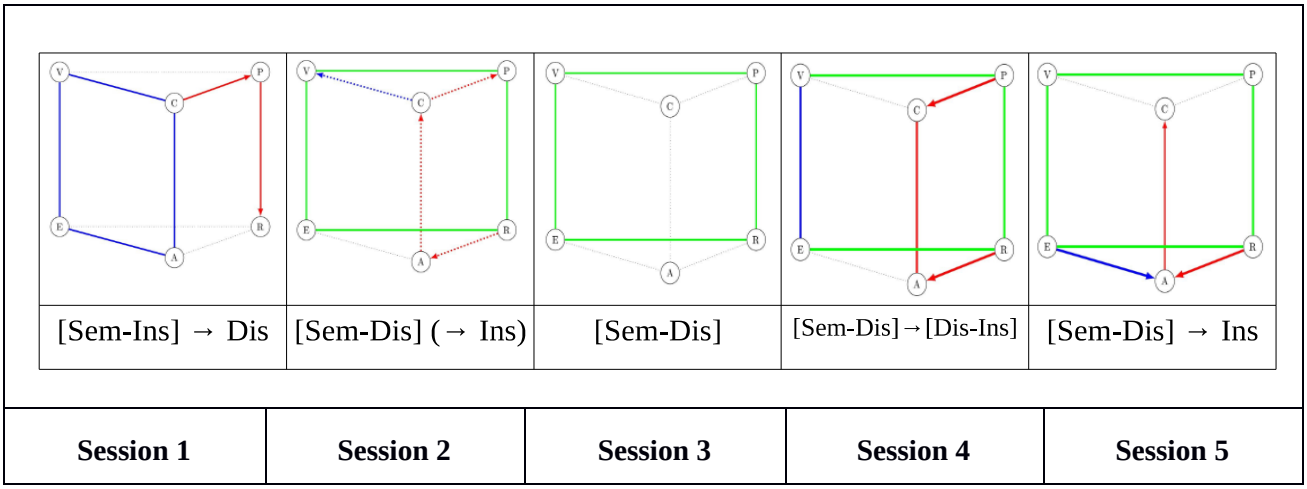


Figure 3: The dynamic evolution of the mathematical work during the five sessions

The analysis, supported on "comics", demonstrates a comprehensive circulation through the three vertical planes of the MWS model (figure 2) leading to what we identify as a "complete geometric work". More generally, a mathematical work is considered "complete" when both conditions (A) and (B) are satisfied:

(A) A genuine relationship between the epistemological and cognitive planes. This aspect means that students, be they generic or not, are able to select the useful tools to deal with a problem and then to use them appropriately as instruments to solve the given task.

(B) An articulation of a rich diversity between the different geneses and vertical planes of the model. This aspect means that various dimensions of the work related to tools, techniques and properties are taken into account.

Identifying blockages and misunderstandings and checking if the mathematical work is complete and coherent

Identifying blockages and misunderstandings requires observing how teachers implement tasks in their classroom. That allows us to describe what we call *suitable* MWS which depends on the institution involved, and is defined according to the way the knowledge must be taught, in relation to its specific place and function within the institutional curriculum.

Identifying blockages and misunderstandings through the study of circulation within the MWS diagram

Our analysis is based on a classroom session at Grade 10 (age 15) (Kuzniak, Nechache & Drouhard, 2016) in which a task is given to the students with two questions on the probability values of an event. The statement of the exercise is written as follows in the textbook used by the teacher:

> Two identical wallets are at disposal. The first contains 3 banknotes of 10 euro and 5 banknotes of 20 euro. The second contains 2 banknotes of 10 euro and 4 banknotes of 20 euro. One wallet is chosen randomly and a banknote is drawn "blindly" from this wallet. What is the probability of choosing one banknote of 10 euro? One banknote of 20 euro?

The underlying probabilistic model is that of equal probability. This model is not explicit, but the text makes reference to it with the following terms: identical, randomly, blindly. Moreover, this exercise involves a random experiment with two successive and not independent draws. The use of a weighted tree to solve the problem would be the most effective way to solve the problem. But, this particular type of tree only appears officially in Grade 12, the introduction of this kind of tree is something that is left for teachers to do. In the textbook, weighted trees are introduced before the exercise which is not the case in the observed class.

After some time left to search for a solution, a student is invited by the teacher to write his answer on the blackboard. He draws a non-weighted tree semiotic dimension to represent the situation and then gives his answer in the form of a fraction (Figure 4). The student gives numerical results without any justification and the tree is not only used for representing the situation but also as an implicit support for calculation instrumental dimension. His mathematical work starts in the semiotic dimension, which allows him to convert the problem into the form of a tree, the latter being then used to get the solution of the given problem. The student has performed his work in the [Sem-Ins] plane.

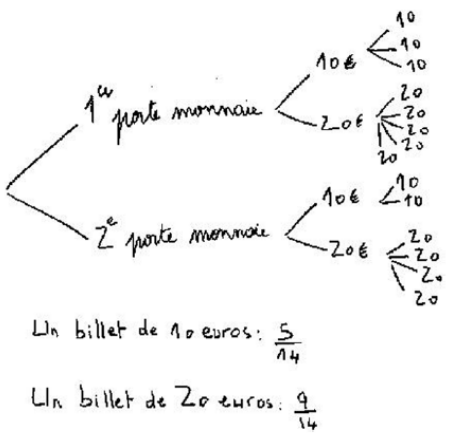


Figure 4: The student's tree on blackboard

Unsatisfied with the student's solution, the teacher asks him to explain his answer, and in particular, to explain the two results written on the blackboard (namely 5/14 and 9/14). Arguments given by the student are uniquely grounded on the semiotic dimension and the teacher is expecting one based on the discursive dimension, using properties. Then, asking various questions to the whole classroom, he attempts to shift the mathematical work to discursive dimension in order to develop a discursive proof of the results. The teacher emphasises strongly the importance of justification based on tools coming from the theoretical system of reference and this focus prevents him to notice the non-validity of the results provided by the student (the right results are 17/48 and 31/48). In fact, the mistake is linked to the student's insufficient knowledge about the nature and use of the tool "tree". The student draws a choice tree which allows counting the outcomes, but which is not a

weighted tree. At this Grade, the teacher avoids the use of probability trees which are spontaneously used by his students. The mathematical work done by students remains in the [Sem-Ins] plane while the teacher confines it in the discursive dimension to promote a discursive proof. Thus, this leads to a misunderstanding and blockages among some students which can be related to the two different forms of mathematical work expected to solve the task.

Mathematical work: Completeness and mathematical coherency

The following example is based on the analysis of a class session at Grade 9 (Kuzniak, Nechache & Drouhard, 2016) in which students are asked to solve the following task taken from Education Ministry resources:

> On a segment S, two points A and B are taken randomly. The following outcome is considered "The length of segment [AB] is strictly superior to half the length of segment S". What is the probability of this event?

The event "The length of segment [AB] is strictly superior to half the length of segment S" is labelled D. The solution suggested into the resource document is divided in two parts. In the first part, the reasoning work starts with an visual exploration on the segment (semiotic dimension) which is closely related to the use of an artefact (here a spreadsheet) for calculating numbers randomly with the *random* function (instrumental dimension). So, the mathematical work begins in the plane [Sem-Ins]. Then, based on the results given by the artefacts, an estimated value, closed to 0.25, is given and the estimation process is justified with the law of large numbers. The work done in this phase ends in the plane [Ins-Dis].

In the second part, the exact value (0.25) is justified with a discursive proof. It is first suggested to find all the couples $(X ; Y)$ such that $|X-Y| > 1/2$, where X and Y are two random variables with a continuous uniform distribution on the interval [0 ;1] (use of the theoretical referential). The inequation is solved graphically (Figure 5) on the square [0 ;1]×[0 ;1] (semiotic use of the square). Thus, the suitable couples $(X ; Y)$ belong to the gray zone (Figure 5), hence the probability of the event D is equal to ¼ (based on visualisation). The mathematical work, really implemented, is placed in the [Sem-Dis] plane.

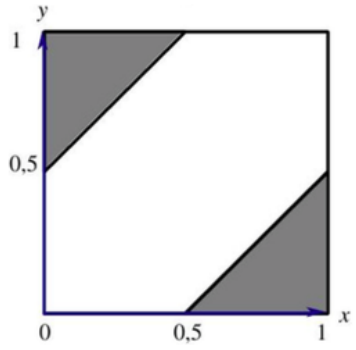
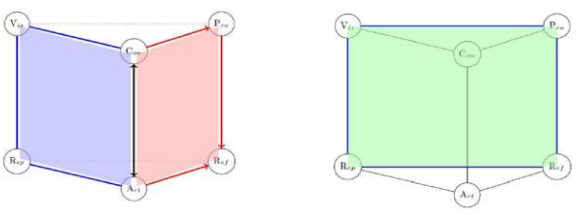


Figure 5: Geometric solution

In summary, the analysis, with the MWS model of the solution given by authors, of the resource document, serves to identify a circulation of the mathematical work through the three vertical planes of the diagram (Figure 2). Thus, a priory, the mathematical work can be regarded as potentially complete and mathematically coherent.



[Sem-Ins]→[Ins-Dis] [Sem-Dis]

Figure 6: The evolution of the mathematical work

In the session we observed, the *suitable* MWS implemented by the teacher, and, thus, the resulting mathematical work, is really different from the potential one described above. The teacher asks the students to realize the random experiment first. They have to draw a segment with a given length, place two points randomly on this segment and, measure the distance between this two points and, compare the measure to half the length of segment S. Then, the teacher engages students to use a discrete model of the experience with throws of two six-sided dice to get an experimental value of the probability of D and, they get 0,3. Finally, the teacher gives the students a table (6×6) with 36 cells to complete and asks them to calculate the probability of D, which is equal to 1/3.

In figurative terms, we can say that each phase favors one of the MWS vertical planes (Figure 2) moving from the [Sem-Ins] plane to help the understanding of the random experiment, to the [Ins-Dis] plane to obtain an experimental value of the probability, and finally to the [Sem-Dis] plane to give a theoretical validation based on counting numbers. In summary, the mathematical work proposed by the teacher provides an articulation between the various working contexts and can be considered complete. But, the probability of D in the model chosen by the teacher is 1/3 and is different from that expected in the official resource, which is 1/4. This difference is due to the fact that the teacher wants to adapt the task to his classroom and changes the initial task by using a discrete model instead of a continuous model. This difference highlights the contradiction between the reference MWS expected by the authors of the resource document and the suitable MWS developed by the teacher. The consequence is that the mathematical work is not mathematically coherent according the expectations of the reference MWS, at this level, even if the mathematical work can be considered complete.

On mutual influence of theory and practice on the MWS development

In this paper, we intend to show how analysis of tasks and teaching-learning sessions can benefit from and participate in the development of the MWS model. Is it possible to generalize our results to other theoretical approaches? We cannot assert, because the MWS model is still an emerging and growing model that is difficult to compare with mature theories. As Artigue (2016) underlines, one of the current characteristics of the model is precisely its plasticity and adaptability that, according her, big and mature French theories do not have. Moreover, conceived to describe and ensure the dynamics of mathematical work, the MWS model cannot be improved without a close and dialectic link with researches on tasks and activities favoring the tuning of the mathematical work.

Research perspective: Teaching trajectory and mathematical work

In the previous section, we have shown how, in some cases, teachers have transformed tasks in such a way that students have been blocked or engaged in mathematical work far from the intended one. In our present research and using the MWS model, we address the following questions: When do some blockages arise in the mathematical work? How can they be characterized? What is their origin? Which kind of teachers' adaptations and changes allows keeping (or not) a complete and mathematically coherent mathematical work? The research objective is to identify misunderstandings or resistance points or, instead, favorable rebounds which allow that an activity goes on nicely in the classroom. It is also possible to focus on tasks transformations leading to

denaturing when the intended mathematical objective is lost and questions of reproducibility and didactic obsolescence can be addressed.

To do this, some specific tasks, named “emblematic tasks” and verifying several conditions, are chosen. They must benefit first from an institutional recognition which ensures their compatibility with the intended mathematical work. Then, they are already provided by textbooks and, above all, implemented in some regular classrooms. Lastly, they may convey a complete mathematical work as defined above. We make the assumption that adequate and solid learning can result from the implementation of these tasks in classrooms if they are not too distorted through the teaching process. To study this assumption, these emblematic tasks are first implemented in pre-service teachers training by experienced teachers trainers and their transformations by the pre-service teachers are studied. The teachers training framework helps us to monitor the development and implementation of tasks in classrooms and makes easier the study of teaching trajectories according our research objectives. Moreover, two other specific objectives related to teacher training can be added to our research program: the use of “emblematic tasks” may initiate students to new and interesting forms of mathematical work for those who are not familiar with; the assessment of the impact of this approach on students' belief by analyzing the different transformations and adaptations of the tasks. In a way, emblematic tasks can help to understand the link between teaching and learning.

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The Mathematical Working Space model: An open and adaptable theoretical framework?

Charlotte Derouet, Alain Kuzniak, Assia Nechache, Bernard Parzysz, Laurent Vivier

Université Paris-Diderot, LDAR (EA 4434), Paris, France; alain.kuzniak@univ-paris-diderot.fr

Keywords: *Activity theory, cognitive approaches, MSW model, networking theories*

For more than a decade, a theoretical approach focusing on mathematical work in schooling has been developed by an international community of researchers. Grounded on geometry education research, the Mathematical Working Space (MWS) model emerged from this collaborative work and has been developed during symposia, the fifth of which being held in Florina in July 2016. Recent publications in English, French and Spanish (Relime 17(4), 2014; Bolema 30(24), 2016) and in English (ZDM-Mathematics Education 48(6), 2016) may be helpful for discovering this model and its current state.

This poster aims at illustrating and discussing one of the specificities of the model, which means that it was conceived to interact with other approaches. As Artigue (2016, p. 938) underlies:

> But the MWS construction is an object of a very different nature, at least in its current state. Its logic seems more that of an assembly that would incorporate, possibly with adaptation, a diversity of constructs and perspectives developed in the field, without privileging any of them. This gives the MWS structure a plasticity that big theories (...) do not have, and certainly contributes to its accessibility and attractivity.

Conversely, this plasticity and attractiveness pose the challenging question of the real nature of its relationships with other theoretical approaches, which may be grounded on very different epistemological and methodological principles. In this poster our purpose is to address this question through some examples. For that reason, some key-points of the model will be presented and, in particular, how the study of mathematical work in schooling is framed. Then, some examples will be given to illustrate possible interactions with other theoretical and exogenous frameworks. All the examples come from special issues on MWS model and MWS symposia. The list of examples is not complete and other frameworks have been used, although they do not appear in the poster (Didactical Situation Theory, Anthropological Didactical Theory, Semiotic registers, etc.). Naturally, all the examples cannot be considered in detail but the fact that the model is supported on a diagram assists to illustrate interactions. The poster is organized around diagrams showing the findings of the different papers and questioning the openness and adaptability of the MWS model.

Combining the model with Drouhard's epistemography. Drouhard's epistemography use has changed the view on tool and instrument in the MWS model (Kuzniak, Nechache & Drouhard, 2016). Depending on their nature and on the way they are being exploited to solve the problem, tools may be situated in any of the three poles of the epistemological plane. In the cognitive plane, one speaks of an instrument whenever a subject interacts with a tool in order to tackle a task effectively. Thus, a tool is associated with a corresponding instrument in the cognitive plane.

Interactions with Activity Theory. Hitt, Saboya and Cortés (2016), investigate the articulation between arithmetic thinking and early algebraic thinking, through the analysis of an experiment

which focuses on secondary school students' spontaneous productions. The experiment is conducted within a research methodology based on Activity theory. The MWS model, used as a framework, is here adapted into an 'Arithmetic-Algebraic Working Space' (A-AWS), whose cognitive plane displays an articulation between arithmetic and algebraic thinking.

Completing APOS theory with the MWS model. Camacho Espinoza and Oktaç (2016) provide a study, using APOS Theory, on an University teacher in Mexico solving a task in linear Algebra. APOS helps to understand the work at a micro level using the mental mechanism of desencapsulation of an Object into a Process, and the authors use the MWS model to understand the global logic of this work at a macro level.

Integrating the MWS model in cognition and affect studies. In a technological (with Dynamic Geometry Software) collaborative setting, Gómez-Chacón, Romero Albaladejo and García López (2016) study the interplay between cognition and affect in geometrical reasoning. Their study integrates the MWS frame to enable a detailed exploration of the transitions from instrumental to discursive geneses of reasoning, within teacher-student and student-student interactions, and also of the cognition-affect dynamics in this process, with a focus on mathematical attitudes.

Coordinating the MWS and MTSK model to understand teachers' knowledge and the role of the teacher in the classroom. The MWS model describes the mathematical work development by teachers through the teaching implemented. Carrillo et al. (2016) suggest an articulation between the MWS model and the MTSK theoretical model (Mathematics Teacher´s Specialised Knowledge) to emphasize the specific role of Teacher's Knowledge in this learning process.

See the poster: www.irem.univ-paris-iderot.fr/~kuzniak/publi/ETM_EN/2017_Cerme10_poster.pdf

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Learning research in a laboratory classroom: Complementarity and commensurability in juxtaposing multiple interpretive accounts

Man Ching Esther Chan[1] and David Clarke[2]

The University of Melbourne, Melbourne Graduate School of Education, Australia;

[1] mc.chan@unimelb.edu.au, [2] d.clarke@unimelb.edu.au

The utilisation of multiple theories in a single research study requires careful consideration with respect to the complementarity of the theories and the commensurability of the associated research accounts in relation to the specific setting or research site. This paper proposes that commensurability is constructed to facilitate the comparison that researchers are trying to make. The Social Unit of Learning project is conducted in a laboratory classroom facility equipped with 10 built-in cameras and up to 32 audio channels allowing structured, rigorous, fine-grained investigation of the social aspects of classroom practice. The rich and detailed data generated allows parallel analyses predicated on different theories. Complementarity of theories is distinguished from commensurability of research accounts, which requires the identification of operationalised constructs (e.g., categories or measures) common to the accounts generated.

Keywords: Classroom research, video technology, research methodology, research design.

Comparability as a challenge in learning research

With the abundance of theories and perspectives that have been generated through research over the years, a continuing challenge that researchers face relates to the difficulty of navigating the multitude of theories available (Bikner-Ahsbahs & Prediger, 2014; Cobb, 2007). In this paper, we consider the conditions under which multiple theories might be deployed for the simultaneous, parallel analysis of a single social setting, with a specific focus on the roles of complementarity and commensurability in undertaking comparison of either the theories or the analytical accounts arising from any such multi-theoretic research design.

Clarke and his colleagues (e.g., Clarke, Emanuelsson, Jablonka, & Mok, 2006) have advocated "complementarity" as central to the contemporary conceptual management of theory and methodology, particularly in their use of "complementary accounts" (Clarke, 1997). In the same way that two research accounts of a social situation may be different but equally legitimate and informative, so two theories may be complementary in their foregrounding of different constructs. Like the accounts, both may be simultaneously "true" within their own coherent conceptual framework so that they are disjoint but separately coherent. Tensions between theories emerge when we juxtapose the analytical accounts derived from two different theories in relation to the same research setting and the coherent body of practice that occurs there. A specific difficulty with juxtaposing and connecting existing theories and their associated constructs is the possible incommensurability of the accounts generated by their application, particularly because theories arise historically from observations based on different research designs, settings and participants.

Direct comparison of analyses employing different theories, without considering the contexts or settings in which the theories are being applied and the intended purpose of their application, empirically undermines the integrity of the comparison and the legitimacy of the conclusions drawn

from the comparison. Consideration of "the right to compare" (cf Stengers, 2011; Clarke, 2013) must take into account differences between findings or interpretive accounts that relate to different physical spaces, different times, and/or involve different actors, activities, or cultural contexts. Clarke (2013) expressed an analogous concern regarding international comparative research, where a single theory is applied across multiple culturally distinct settings for the purposes of comparison with respect to a specific construct (e.g., student achievement). In such studies, researchers can risk compromising the validity of the comparison made in their study by misrepresenting the valued performances, school knowledge, classroom practice, etc. that are differently conceived by the communities being compared. A construct such as "student participation" can be conceived so differently (both theoretically and in practice) in different cultural settings that it cannot be employed as a "boundary object" (Akkerman & Bakker, 2011), that is, as a point of connection by which classroom practice in the different settings might be compared. In the application of a single theory across different cultural settings, it is the questionable validity of application of the same construct in the compared settings that renders the accounts incommensurable.

We are employing commensurability in the sense of the construction of common points of distinction[1], which can be seen as related to the notion of boundary objects. *Boundary objects* have been described as "artifacts that live in different practices, but can be used in different ways" (Bakker, 2016, p. 272). Consistent with Akkerman and Bakker (2011), we caution against the identification of a boundary object simply on the basis of similarity of name without an empirical grounding suggesting functional equivalence. In the case of "participation," such assumed functional equivalence can conceal profound differences in nature. To illustrate the extent of such differences, the significance attached to student talk as facilitative of learning in Western theories (and practice) is contested by theorists writing from a different cultural and theoretical position (e.g., Kim & Markus, 2004), leading to entirely different theorisations of the constitution of student participation. Attempts to connect theories in which definitions of student participation were predicated on such different epistemologies would lead to accounts that not only lacked comparability, but were in fact incommensurable, since the connecting construct "student participation" would admit no common points of distinction in the application of the two theories to any setting. In this sense, one might describe the two *theories* as being incommensurable *in their application*, but we suggest that the theories are better thought of as complementary (disjoint, but separately coherent), and it is the *accounts* arising from their application that are incommensurable. Accounts, like theories, can be complementary (disjoint but separately coherent), but commensurability is an attribute of accounts alone, implying consideration of context and purpose.

We argue that the commensurability of two theories cannot be meaningfully examined except in so far as they are "put to work" in the analysis of data. The interpretive accounts generated by such analyses (whether qualitative or quantitative) can then be compared and assessment made of the points of correspondence or dislocation in the accounts (e.g., through the identification of common points of distinction). Such points of correspondence take the form of operationalised constructs

[1] Our usage of "points of distinction" draws on the comments of John Mason during a conversation on 10 February 2017.

having similar meaning within both theories and which therefore serve to align the interpretive accounts for the purposes of comparison and connection. Such operationalised constructs may be thought of as boundary objects (Akkerman & Bakker, 2011). A "boundary object" in this discussion is an operationalised construct that has conceptual legitimacy and similar meaning in both theories being applied, connected, and compared. Where no such constructs exist, the theories are disjoint and each may be applied independently of the other to investigate the same or different settings (Clarke et al., 2012). In such a case, the disjoint theories are complementary, although incommensurable with respect to any setting to which they might be applied analytically, sharing no constructs by which comparison of the resultant accounts might be undertaken.

How might different theoretical perspectives be juxtaposed and connected in a way that allows the commensurability of the analytical accounts to be examined? This paper proposes as one solution the construction of research designs that involve the generation of data, which are complex and rich in detail, while sufficiently structured to allow systematic investigation of both the research setting and the multiple theoretical perspectives employed. The affordances of such research designs are illustrated with examples from the Social Unit of Learning Project, which utilises the newly established laboratory classroom facility to generate data amenable to multi-theoretical analysis.

The Social Unit of Learning Project

The recent development of a laboratory classroom at the University of Melbourne (see https://pursuit.unimelb.edu.au/articles/high-tech-classroom-sheds-light-on-how-students-learn) has made possible research designs that combine better approximation to natural social settings, with the retention of some degree of control over the research setting, task characteristics, and possible forms of social interaction. Such designs allow conclusions about connections between interactive patterns of social negotiation and knowledge products (learning) to be made with greater confidence. The Social Unit of Learning Project used the Science of Learning Research Classroom (SLRC) at the University of Melbourne to examine individual, dyadic, small group (four to six students) and whole class problem solving in mathematics and the associated/consequent learning. The project aims to distinguish the social aspects of learning and, particularly, those for which "the social" represents the most fundamental and useful level of explanation, modelling and instructional intervention. The project conforms to an experimental rather than a naturalistic paradigm. The caveats for the experimental approach are discussed in greater depth elsewhere (Chan & Clarke, in press). The SLRC has the capability to capture classroom social interactions with a rich amount of detail using advanced video technology. The facility was purposefully designed to allow simultaneous and continuous documentation of classroom interactions using multiple cameras and microphones. The project collected multiple forms of data for analysis including student written products and high definition video and audio recordings of every student and the teacher in the classroom. This allows the examination of data from multiple perspectives by multiple researchers as well as the reciprocal interrogation of the different theoretical perspectives through answering research questions such as the following:

1. What commonalities and differences in process and product are evident during problem solving activities undertaken by learners as members of different social units (individual, pairs, small groups and whole class groupings)?

2. Which existing theories best accommodate the documented similarities and differences in process and product and in what ways do the accounts generated by parallel analyses predicated on different theories lead to differences in instructional advocacy?

The following presents work currently being carried out to lay the foundation for considerations of complementarity and comparability in a multi-theoretic research project.

Data generation

The SLRC is equipped with 10 built-in video cameras and up to 32 audio channels. Intact Year 7 classes were recruited with their usual teacher in order to exploit existing student-student and teacher-student interactive norms. Two classes of Year 7 students (12 to 13 years old; 50 students in total) provide the focus for this report. Each of the classes participated in a 60-minute session in the laboratory classroom involving three separate problem solving tasks that required them to produce written solutions. The students attempted the first task individually (10 minutes), the second task in pairs (15 minutes), and the third task in groups of four to six students (20 minutes).

The problem solving tasks used in the project were drawn from previous research (e.g., Sullivan & Clarke, 1991). All three tasks had multiple possible solutions, included symbolic or graphical elements, and afforded connection to contexts outside the classroom. These features can make the thinking and/or social processes of the problem solving activity more visible, as the students can express their thinking through multiple modes (e.g., verbal, graphical, textual, etc.) and approach the task using different strategies or prioritise different forms of knowledge or experience. Nonetheless, despite sharing some similar features, the content foci of the three tasks were deliberately disconnected to avoid carry-over effects between tasks.

Task 1 provided students with a graph with no labels or descriptions with the following instructions: "What might this be a graph of? Label your graph appropriately. What information is contained in your graph? Write a paragraph to describe your graph." Task 2 was specified as follows: "The average age of five people living in a house is 25. One of the five people is a Year 7 student. What are the ages of the other four people and how are the five people in the house related? Write a paragraph explaining your answer." Task 3 stated that "Fred's apartment has five rooms. The total area is 60 square metres. Draw a plan of Fred's apartment. Label each room, and show the dimensions (length and width) of all rooms."

The resulting data collected in the project include: all written material produced by the students; instructional material used by the teacher; video footage of all of the students during the session; video footage of the teacher tracked throughout the session; transcripts of teacher and student speech; and pre- and post-lesson teacher interviews.

Parallel data analyses

As an entry point for analysing the project data, the written solutions, transcripts, and video record are used to understand the social process that took place to produce the written solution. The instructional material and teacher pre- and post-lesson interviews provide insights about the class capabilities and social relationships that the researchers would not otherwise be able to access.

Several parallel analyses are currently being undertaken drawing on the established research expertise of classroom research communities in three countries. For example, in Australia, Clarke

and Chan are conducting an analysis which identifies the negotiative foci of the students' social interactions during collaborative problem solving taking the social negotiation of meaning as a key constitutive element of learning (e.g., Clarke, 1997); in Spain, Díez-Palomar is conducting an analysis of the dialogic character (Mercer & Howe, 2012) of the spoken interactions of students working in collaborative groups; and in Finland, Tuohilampi is carrying out an investigation of the affective enablers and disablers of student participation in collaborative group work that uses Goldin's motivating desires (Goldin, Epstein, Schorr, & Warner, 2011) to explore the extent to which a group of students established a productive affective micro-culture. A theory is recruited to this study for its capacity to address constructs, artefacts or situations distinct from those addressed in other theories being employed – that is for its capacity to complement those already selected.

Connection of these three analyses is made possible by their application to a common set of social events occurring in the same research setting. The validity of any connections between the parallel analyses is heightened by their grounding in data from the same source and their application to a common interactive sequence. For example, consider the following excerpt when Anna and Pandit were writing up their response to Task 2 (pair task):

Anna:	Okay. So let's explain it here.
Pandit:	So - so 7 ... //One kid...
Anna:	//Because we have to write it in words. (Note. // indicates overlapping speech.)
Pandit:	So one kid has to be four... 17.
Anna:	No, no, no. So ...
Pandit:	(Laughs)
Anna:	I'm going to write it.
Pandit:	One kid has to be 17.
Anna:	So ...
Pandit:	So wait, no, no, no, no.
Anna:	... because ...
Pandit:	Oh a seven - a Year 7 is 13.
Anna:	I'm ignoring you.
Pandit:	You can't - So - So sad. I'll draw.

From the excerpt, we can examine the focus of the students' negotiation on the task requirements or sociomathematical focus (Anna: "Because we have to write it in words."), the coordination of the mathematical components of the task or mathematical focus (Pandit: "One kid has to be 17."), and the social obligations of the participants or social focus (Anna: "I am ignoring you"; Pandit: "You can't.").

At the same time, the transcript allows the investigation of the dialogic character (García-Carrión & Díez-Palomar, 2015) of the participants, where the excerpt began with Pandit offering information to Anna for her writing up of the results and ended with Anna rejecting Pandit's contribution. The conversation shifted from the dialogic interaction initiated by Anna ("So let's explain it here … because we have to write it in words.") to non-dialogic or authoritarian talk (Anna: "I'm ignoring you."; Pandit: "You can't.").

From an affective perspective, Anna and Pandit both appeared to share the same motivating desire to "Get the Job Done" (Goldin et al., 2011, p. 553). However, Pandit appeared to also appeal to the motivating desire of "Let Me Teach You" (p. 554) by dictating the information to be written down by Anna ("So one kid has to be four... 17 … One kid has to be 17. … Oh a seven - a Year 7 is 13."). Her attempt to take on the higher epistemic role did not appear to be well received by Anna. Upon being rejected by Anna, Pandit's desire quickly changed to "Don't Disrespect Me" (p. 553) by being disengaged from the task and switched to off-task drawing.

Although all three analyses focus on the same interactive episode during collaborative problem solving, each analysis highlights different aspects of the social interaction. The multitheoretic research design of the project provides the context for the consideration of how commensurability may be conceptualised in relation to the parallel analyses.

Discussion and conclusion

This paper presented three analyses that are currently being applied to the data that have been generated from the laboratory classroom concerning the same interactive episode of collaborative problem solving. The approach allows direct comparisons to be made between the applications of the three analyses (negotiative foci; dialogic theory; and motivating desires) in terms of what constitutes evidence within the realm of each analytical framework, the unit of analysis, and the conclusions that can be drawn from the analyses, all of which could form the basis for the evaluation of the commensurability of the separate analyses. In the case of the project, commensurability can be evaluated in relation to a common construct with respect to which each of the analyses might be employed to make comparative distinctions (either descriptive or evaluative).

For example, for the purpose of distinguishing between different interactive episodes with respect to the construct of "student engagement", the analytical accounts derived from dialogic theory and the theory of motivating desires can be seen as commensurable, whereas it is more difficult for an analysis with respect to negotiative focus to make useful distinctions with respect to engagement. The analyses based on dialogic talk (in terms of the ways in which students put forward their arguments) and on motiving desires (in terms of the fulfilment of goals or beliefs through social interactions) can each be seen as potentially capable of distinguishing between interactive episodes in terms of some conception of the quality of "student engagement" during collaborative problem solving, even though the premises on which the two analyses might make such evaluative distinctions would be different. On the other hand, the consideration of the negotiative foci of particular interactive episodes distinguishes between types of "student engagement" in a descriptive but non-evaluative way. In this sense, the account provided by the analysis of negotiative focus does not suggest any points of evaluative distinction in terms of student engagement, in the way that is possible with the accounts provided by the other two analyses. This renders it incommensurable with the other two analyses with respect to the construct "student engagement".

We want to emphasise that commensurability between theoretically-grounded analytical accounts should not be seen as "an ideal state" but as a reference point for examining the connections between theories. Stengers (2011) makes the essential point: "Commensurability is created and it is never neutral, always relative to an aim" (p. 55). In the case of multi-theoretic research designs, researchers are obliged to construct commensurability to facilitate the comparison that they are

trying to make between theoretically-grounded analytical accounts. The utilisation of multiple theories is enhanced through the identification of shared operationalised constructs (such as "student engagement"), intrinsic to or derivable from the interpretive accounts in question, the existence of which is prerequisite for their commensurability. Complementarity between the theories discussed can be accommodated independently of arguments concerning commensurability. The emphasis on complementarity removes the obligation that interpretive accounts should converge to a single truth. We posit that theories can be complementary in their conceptual totality (having different epistemological and ontological bases) but nonetheless invoke operationalised versions of specific constructs common to both theories which could be used to interrogate the setting, and form the basis for interpretive accounts which can then be juxtaposed with respect to their implications for practice. The viability of multi-theoretic designs does not demand that all accounts be commensurable. Some accounts may be simultaneously coherent and consistent with the data, but disjoint, in that they employ different operationalised constructs.

In conclusion, this paper argues for the importance of considering the roles of complementarity and commensurability in multi-theoretic research designs. We suggest that the consideration of complementarity resides between theories while commensurability can only be examined in relation to the interpretive accounts arising from the application of the theories. By juxtaposing theories applied analytically to data generated from the same setting, the research design of the Social Unit of Learning Project accommodates the complementarity of theories and affords an informed judgement of the commensurability of the parallel interpretive accounts. The consideration of commensurability obliges researchers to articulate the nature of the comparability between theoretically-grounded interpretive accounts when juxtaposing theories. We feel that the explication of complementarity and commensurability in this paper should contribute to the further theorisation of multi-theoretical research approaches.

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The perspective of indexicality: How tool-based actions and gestures contribute to concept-building

Daniela Behrens[1] and Angelika Bikner-Ahsbahs[2]

[1]University of Bremen, Germany; daniela.behrens@uni-bremen.de

[2]University of Bremen, Germany; bikner@math.uni-bremen.de

In this paper we apply the theoretical perspective of indexicality to gesture use on a digital place value chart on the iPad and show that this perspective allows for explaining how mathematical meaning is accumulated linking specific gestures to the actions performed on the digital devise. Thus, practical dragging leads to structural dragging via operational dragging, resulting in a shift of the representational function of gesture (gesture-of) to the epistemic function of gesture (gesture-for).

Keywords: Digital tool, indexicality of gesture, gesture-of and gesture-for, modes of dragging.

Introduction

De Freitas and Sinclair (2013) have proposed to adopt a new materialistic view in research on learning mathematics, specifically for technology use. In this respect, the indexicality of signs is relevant (Sinclair and de Freitas, 2014). This paper investigates the added value of the theoretical perspective of indexicality while adopting it to the analysis of technology-based epistemic processes in the design project DeciPlace, to understand more deeply how acting based on a multi-touch surface contributes to building mathematical knowledge.

DeciPlace is a design-based research project. Its main goal is to develop a task sequence for conceptualizing decimal fractions as structures in small groups of students by the use of a digital place value chart (DPC) on the iPad (Behrens, 2016; Behrens & Bikner-Ahsbahs, 2016). The core approach is to act tool-based with the DPC. Thereby, conceptualizing is not considered as a pure individual cognitive process but as a collective communicative process of constructing mathematical structures in an embodied and multimodal way (see Krause, 2016). For this kind of learning, the instrumental approach (see Drijvers, Kieran & Mariotti, 2010), which is often used for research on technology learning settings, fails to attain insight into the epistemic process in all its aspects. Our focus is on tool-based acting and interacting. However, the way this contributes to knowledge construction is not yet understood in-depth. In this paper, we will first present the tool, then outline what we mean by indexicality and finally apply this perspective in the analysis of some episodes from the DeciPlace data corpus to show how the adoption of this perspective deepens insight into tool-based collective learning.

The digital place value chart (DPC)

The digital place value chart on the iPad (designed by Ladel & Kortenkamp, 2013; itunes-App: "Place Value Chart") can represent a number on three different levels in parallel: In the bottom row of the chart tokens can be displayed by tapping on the screen. In the upper row the name of the place value as well as the number of tokens in each column of the place value chart is indicated. Additionally the standard notation of the represented number can be displayed above the chart. When the user drags a token to the next (after next) column to the right, the token is de-bundled

automatically into ten (hundred) tokens and so on (see Figure 1). The other way around, by dragging a token to the next (after next) column to the left, either nine (99) tokens move along with the dragged token bundling together to one token in the new column or – if there are not enough tokens to come along with – the dragged token slides back to its origin. Hence, in contrast to traditional place value charts (paper and pencil or material tokens) this digital version de-bundles and bundles (if possible) automatically, while the represented value is kept constant, and it gives feedback if the intended manipulation is impossible.

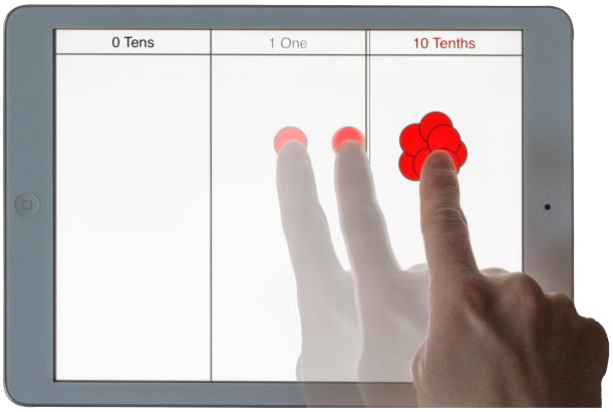

Figure 1: De-bundling a token from ones to tenths in the digital place value chart

Recent results: Three modes of dragging

In the analysis of activities on the DPC by a pair of grade 6 students who were introduced to the extension of the place value system from natural to decimal numbers, we noticed a shift in the students' activities from dragging-actions on the iPad to dragging-gestures (incorporated by the characteristic movement to the right or left), becoming more and more independent from the representation on the screen during the course of interaction with the DPC (see Behrens & Bikner-Ahsbahs, 2016).

Using an epistemic analysis based on ideas developed by Krause (2016) we were able to distinguish three modes of dragging:

- *Practical dragging* comprises actions of dragging tokens performed directly on the digital place value chart, when students use the function of bundling or de-bundling by dragging with a practical aim without scrutinizing the underlying principle.
- *Operational dragging* can be observed when students are able to foresee the result of bundling or de-bundling or when they want to test something by dragging a token intentionally in the chart, so that they use both transformations to fulfil a particular goal. This mode of dragging can be manifested either as a direct act of dragging in the digital place value chart or as a dragging-gesture referring directly to the chart.
- *Structural dragging* becomes apparent, when a particular mathematical structure is being described generally and the movement of dragging is represented in a gesture from left to right or vice versa performed independently from any concrete representation.

Aim of this paper

Using these three modes of dragging, we were able to describe the epistemic role of gestures in processes of building the decimal fractions' concept, which is mainly based on the principle of bun-

dling and debundling (see Behrens & Bikner-Ahsbahs, 2016). However, we were not able to understand in detail how these three modes of dragging contributed to processes of learning based on the digital artifact. In this paper, we will address this topic by answering the following research question:

How do actions and gestures "regarded as indices" contribute to conceptualizing the decimal fractions' concept based on the digital place value chart on the iPad?

Describing the theoretical approach: Actions and gestures regarded as indices

In this paper, we focus on the connection between the digital place value chart and a pair of individuals interacting with each other with regard to the device. To examine this interaction we focus on signs which are produced in the setting, such as gestures, inscriptions, tokens on the display, the artifact itself and so on.

The students' collective epistemic process is manifested in their actions (based on the tool), their verbal utterances and other semiotic resources. These actions can be analyzed in a multimodal way based on the concept of the semiotic bundle (Arzarello, 2006), which consists particularly of gestures, speech, inscriptions and relations among each other. To emphasize the influence of tool-based actions on speech, gesture and inscriptions, we adapt the perspective of the indexicality of actions on multi-touch devices described by Sinclair and de Freitas (2014). This perspective draws on Peirce's notion of semiotics, "in which signs (icons, indices and symbols) differ in terms of the nature of the relationships between the signifying sign and the signified" (Sinclair & de Freitas, 2014, p. 355). According to Peirce, a sign is defined by a triadic relation between sign, object and interpretant:

> A sign, or *representamen*, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. It stands for that object, not in all respects, but in reference to a sort of idea, which I have sometimes called the *ground* of the representamen. (Peirce, 1932, 2.228, emphasis in the original)[1].

As a consequence, a sign comes into being when there is an individual who produces an interpretant according to the relation between the sign and the object. This relation distinguishes a sign to be an icon, an index or a symbol (Peirce, 1994, p. 239).

While an icon is characterized by producing the idea of resemblance of sign and object in individuals and symbols are defined to be conventionalized signs, an index

> refers to its object not so much because of any similarity or analogy with it, […] as because it is in dynamical (including spatial) connection both with the individual object, on the one hand, and with the senses or memory of the person for whom it serves as a sign, on the other (Peirce, 1932, 2.305, cited in Sinclair & de Freitas, 2014, p. 355f.).

[1] This refers to Peirce in terms of '(section/page)', where 2.228 stands for 'volume 2, paragraph 228'.

By this, indexical signs "show something about things, on account of their being physically connected with them" (Peirce, [1894] 1998, p.5; cited in Sinclair & de Freitas, 2014, p. 355).

Sinclair and de Freitas (2014) emphasize that also the action that resulted in the emergence of another sign may be included in the concept of indexical signs:

> For instance, the chalk drawing of a parallelogram on a blackboard is often considered to be an iconic reference to a Platonic conception of parallelogram, but it is also an indexical sign that refers to the prior movement of the chalk. This latter indexical dimension is usually not emphasized in the semiotic study of mathematical meaning making, since we tend to focus on the completed trace and dislocate it from the labour that produced it. (Sinclair & de Freitas, 2014, p. 356)

Taking this assumption into account, we can further assume that every process of producing a sign is an indexical sign referring to the sign and the sign itself refers indexically back to the effort which produced it.

They conclude that "indexation becomes part of an experience that exceeds itself, and is thus self-generative" (p. 359). Thus an action on multi-touch screens leaves traces, hence, these traces as well as a hand gesture may refer to the original action when this gesture is produced in a similar way to the action on the device. Taking up this theoretical perspective we want to examine the above described modes of dragging with respect to "how they function as indexical, material actions" (p. 360) trying to explain how dragging movements contribute to build the concept of decimal fractions.

Applying the theoretical approach: Indexicality in dragging modes

We are re-analyzing episodes of our design-study DeciPlace (Behrens & Bikner-Ahsbahs, 2016) in order to answer the above posed research question. This way we will investigate the added value of this indexicality perspective.

As described above, practical dragging takes place when tokens are dragged directly in the digital place value chart from one column to the other either to the right or to the left without observable intention. The digital place value chart reacts to this practical dragging of one token to the next column on the right by de-bundling this token into ten tokens. Likewise, the DPC can bundle ten tokens to one in the next column, when one token is dragged into the next column on the left. Bundling and de-bundling by dragging keeps the value of the decimal number the same. This way, dragging can be linked to bundling or de-bundling in a material way being performed as an action of dragging from right to left or vice versa on the DPC. According to Sinclair and de Freitas (2014) an action of dragging leaves traces – e.g. the new arrangement of tokens within the chart – which refer back to the original action of dragging and vice versa.

In the following scene, two students are asked to find different representations for the number 101 in the place value chart:

1	Bella:	I'll just try (*drags a token from hundreds to tens within the digital place value chart on the iPad (in the tens' column ten tokens emerge), see Figure 2*) Woah
2	Hanna:	Ten and One.



Figure 2: De-bundling as dragging from hundreds to tens

In this scene of practical dragging the newly emerged bunch of ten tokens in the tens' column can be seen as an iconic sign representing ten tens and therefore the number 100. Additionally the bunch of ten tokens refers back to the action of dragging a token from the hundreds' column to the tens' column on the DPC and at the same time to the DPC's reaction by letting the token explode into ten tokens without changing the represented number (concept of de-bundling). Thus, a new sign emerged by indexation linking the action of dragging on the DPC with a visual representation of de-bundling. This in turn produced a reaction of astonishment expressed by Bella ("Woah"), again referring back to the DPC's reaction on her dragging.

When the students get more and more familiar with the DPC in phases of practical dragging, they may apply movements of dragging intentionally for a specific purpose, for example while making a conjecture or predicting what will happen in the case of dragging directly in the DPC. Operational dragging can take place as an action of dragging *on the surface of the device* and also as a gesture *directly above the surface* referring to the DPC but without touching it (the so-called potential level, see Krause, 2016, p. 134–139). Because of the material link between dragging actions and dragging gestures both being performed by a similar movement from left to right or vice versa, dragging gestures may refer to previous actions of dragging and what is already linked with them. In this respect, operational dragging being conducted as a gesture on the potential level of reference is materially linked to the performance of dragging in the DPC and at the same time linked to the traces which are potentially and materially produced by that.

In the following situation the place value "tenth" is just introduced by the interviewer adding the new column named "tenth". The students are asked to find further representations for the number 4. At first, the students tap in four tokens into the ones' column of the DPC.

3	Bella:	Can I drag over one (*moves her right hand at the bottom of the iPad from the ones' column to the tenths' column; see Figure 3*) and see what gets out?
4	Interviewer:	Try.
5	Bella:	(*drags a token from the ones' to the tenths' column, where ten tokens emerge; see Figure 4*)
6	Hanna:	Ohh.
7	Bella:	So ten are (1 sec.) one one (*pointing to the ones' column*) are ten tenths (*moves her hand to the right flexing and extending her index finger pointing to the tenths' column*).

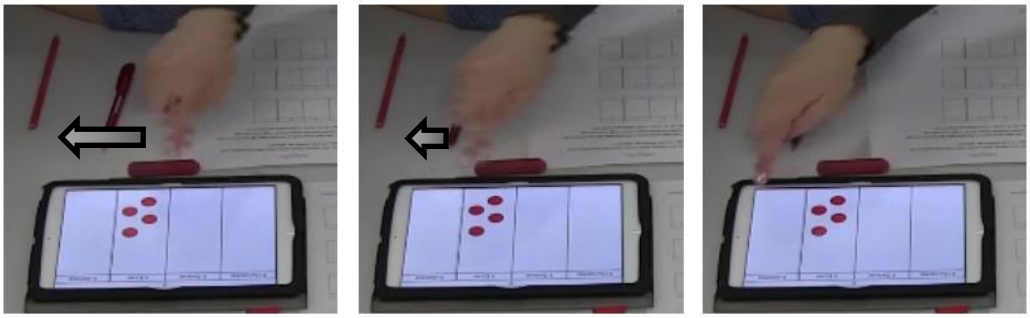

Figure 3: Dragging-gesture to the right from ones to tenths above the chart



Figure 4: De-bundling ones to tenths by dragging

In this scene Bella intentionally exploits the DPC's function of de-bundling to get insights into the relation between ones and tenths (lines 3 & 7). Thus, the two dragging-gestures (lines 3 & 7) as well as the dragging-action (line 5) represent operational dragging. From the perspective of indexicality, both dragging-gestures from left to right (acc. to the view of the students) can be assumed to be linked to previous actions of dragging from left to right on the DPC and their traces, because of the close resemblance between gesture and action. By this, dragging-gestures become indices of dragging-actions including the experiences and assumptions that have been made by dragging tokens on the DPC from left to right, e.g.: "when I drag a token to the right, the number of tokens changes / increases" or "when I drag a token to the right, the represented number remains the same".

Although both dragging-gestures are executed more or less equally, in relation to speech they function differently. The first dragging-gesture seems to focus on the movement of dragging to the right from ones to tenths specifying "what" and "where" (Krause, 2016, p. 125) to "drag over" (a wording frequently used by the students). In contrast, the second dragging-gesture adds the way by which the insight that "one one are ten tenths" (line 7) was gained, referring back again to the experiences and assumptions made by the action of dragging rightwards on the DPC just before (line 5). Similar to the notion of model-of and model-for (van den Heuvel-Panhuizen, 2003, p. 14) we have identified the development from a gesture-of (representing the action of dragging) to a gesture-for (representing the procedure of arriving at this particular conclusion) mediated by operational dragging.

Structural dragging is done when a dragging-gesture is conducted in the gesture space without visible references to any concrete representation of a place value chart. This was used particularly when describing the concept of bundling within the digital place value chart, which is a main step of conceptualizing the decimals' structure. Assuming that the characteristic movement from left to right or vice versa indicates the material link between dragging-actions and dragging-gestures, we can con-

sider structural dragging to be an indexical sign on the traces left by practical and operational dragging including all experiences, intentions, and conjectures made before.

During the whole design experiment, the two students established a shared context where they observed the other student dragging and negotiated shared answers to the tasks. In the first situation here, Bella performs the dragging (line 1), while Hanna sums up the emerged result (line 2). In the second scene Hanna reacts to Bella's dragging on the chart (line 5) and the chart's reaction by astonishment (line 6) and is therefore likewise involved in conceptualizing de-bundling. Thus, a dragging-gesture of one student can also be taken as an index to previous dragging-gestures by the other student. This way, both students and the device constitute an ecology of tool-based interaction to build the concept of decimal fractions.

Discussion: Reflections and consequences

Applying the theoretical perspective of indexicality we have reconstructed how the action and the gesture of dragging can accumulate more and more aspects about bundling and de-bundling. This process of mutual reference between indexical actions and gestures brought forward the conceptualization of bundling and de-bundling as the basis for the concept of decimal fractions. At the same time the dragging-gestures detached more and more from the concrete dragging-actions on the DPC and became shared signs by enriching indexical references.

Similar to the shift from model-of to model-for (van den Heuvel-Panhuizen, 2003, p. 14), a shift from gesture-of to gesture-for was observed, that is: A dragging gesture first represents the action of dragging (gesture-of), later the dragging-gesture is used as an epistemic means to structure the description of the base-ten structure (gesture-for). Operational dragging can be considered as an intermediate state. It produces a change of view from dragging tokens to bundling and de-bundling as the underlying concept. Hence, adding indexicality to gesture analysis may improve our understanding in how epistemic processes progress.

Whether or not this theoretical perspective keeps being fruitful for tool-based learning in general can only be answered by further empirical research. The main issue will be how this perspective can be fruitfully linked with local theories and models for learning specific contents, such as expanding natural numbers to decimals fractions. For that, we will apply the indexicality perspective to additional data from design experiments with another 15 student pairs in the DeciPlace project, attempting to prove our results and gain further insight into the role of tool-based dragging-actions and -gestures for contributing to conceptualize the decimal fraction's structure.

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The Thematic Working Group 17 of CERME 10: Grouped topics of TW17 of CERME10

(Multi-)theoretical approaches in design research

Towards argumentative grammars of design research

Arthur Bakker

When a critical aspect is a conception: Using multiple theories to design dynamic computer environments and tasks to promote secondary students' discernment of covariation

Heather Lynn Johnson, Evan McClintock, Peter Hornbein, Amber Gardner and Daniel Grieser

Combining two theories in the design for learning about integrals

Anatoli Kouropatov and Tommy Dreyfus

Learning Through Activity: A developing integrated theory of mathematics learning and teaching

Martin A. Simon

Towards argumentative grammars of design research

Arthur Bakker

Bremen University, Germany; Utrecht University, the Netherlands; a.bakker4@uu.nl

Design research is considered a valuable but demanding methodological framework that continues to generate theoretical and methodological reflection. An important topic to address is that of argumentative grammar, the logic guiding a method and supporting warranted claims, because critics consider it a weak spot of design research. With reference to the history of logic, I challenge these critics' demand of an argumentative grammar that relies solely on structure rather than also on content. The purpose of this paper is to think through what argumentative grammars of design research could look like. Because the literature is so limited on this topic, I draw on interviews with experts in design research to evaluate and discuss my own attempt to formulate an argumentative grammar in relation to possible research questions. One conclusion is that design research requires multiple argumentative grammars depending on the design and the research focus.

Keywords: Design-based research, expert interview study, methodology.

The need for an argumentative grammar of design research

In a special issue on design research, Kelly (2004) argues that design research (DR) is a valuable emerging set of methods in education, but he has methodological concerns. In his view, "[t]he next task is to establish the logos of design research so that we can argue, methodologically, for the scientific warrants for its claims." (p. 105) For design research to become a methodology (method + logos), he proposes, we need an "argumentative grammar," which he defines as "the logic that guides the use of a method and that supports reasoning about its data" (p. 118).

A methodology that already has a clear argumentative grammar is that of randomized field trials as introduced by Sir Ronald Fisher in the 1920s for agriculture. In such trials, also called randomized controlled trials, researchers randomly attribute objects or subjects to an experimental or control condition so that they can assume that these two groups are equal on average except for receiving the treatment or not. Any differences between these two groups as measured by means of pre- and posttests can therefore be attributed to the difference in treatment. One advantage of this methodology is that its argumentative grammar is a structure that can be

> described *separately* from its instantiation in any given study so that the logic of a proposed study and its later claims can be criticized. Thus, many reviewers reject studies not on the choice of method (procedure), but on their violation of the underlying logos that one expects to see with that choice of method. (Kelly, 2004, p. 118, emphasis in the original).

Kelly and other critical friends such as Shavelson et al. (2003) thus push design researchers to make warranted claims and go beyond purely narrative accounts. With Cobb et al. (2014), I think design researchers should indeed work towards an argumentative grammar (or grammars) to increase DR's methodological quality. However, drawing on the history of logic, I problematize the preference for a *separate* logical structure for design research (DR) that is irrespective of content. The purpose of this paper is then to think through what an argumentative grammar for design research could look like instead. Because the literature is so scant on this topic, I decided to interview experts in DR on a provisional grammar that I formulated myself.

Problematizing the need of a separate argumentative structure

This section first addresses randomized field trials (RFTs) as the best known example of a methodology with an argumentative grammar that relies on the structure of argumentation. Next I use the history of logic to argue that logic that is based only on the structure and not on its content is of limited scientific value. I argue that DR requires argumentative grammars that acknowledge content as part of their logic, where content can refer to many things including key concepts used in the research, information on local circumstances (context), and the content of what is learned or aimed to achieve.

Randomized field trial

With respect to RFTs, I highlight three themes to prepare the discussion of an argumentative grammar of DR. Theme 1: In an RFT the design of an intervention and the evaluative research are separate. From some theoretical perspective or hypothesis, educational design with particular characteristics is developed and then evaluated. When the effects of its implementation are positive, these are attributed to characteristics of the design, operationalized in particular variables. Where does this leave the design researcher who typically intertwines intervention and evaluation?

Theme 2: Typical research questions reveal the type of knowledge that RFT are often after: What are the effects of intervention I on D? Is intervention I1 better than I2? The design researcher typically asks how particular learning can be supported or how some problem can be resolved (of course RFTs can also focus on mechanisms rather than just effects).

Theme 3: RFTs have a clear argumentative structure that is separate from the content of what is researched. This allows the audience, even those who may not have the expertise to engage with the content of the studies (e.g., key concepts, learning content, situation, mechanisms) to judge the structure of the procedure and the scientific reasoning. What could an argumentative grammar of DR look like if it, as I argue later, cannot depend on structure alone?

Despite their power, RFTs also have their limitations (MRC, 2000). To know "what works" in general is of little value if it is unknown "how and under what conditions things work" or what the mechanisms or "active ingredients" are that make an intervention work (Biesta, 2007). It is further acknowledged that valid measurement is difficult and that RFTs typically have good internal but not necessarily good external validity (cf. Shavelson, 2008).

Logic: Content matters as much as structure

I now use examples from logic to clarify that Kelly's (2004) and others' focus on the *structure* of argumentation may hold back educational research. The discipline of logic started in Aristotle's *Prior Analytics* with syllogisms such as "All men are mortal; Socrates is a man; therefore Socrates is mortal." This logic purely depends on the structure of the inference: The non-logical terms such as "mortal," "Socrates," and "man" can all be replaced by other terms without loss of validity. The interpreter does not even need to know the meaning of these terms to judge the validity of the reasoning. This reasoning is thus rigorous but irrelevant in scientific reasoning:

> This kind of logic based on syllogisms came into disrepute in the seventeenth century when science was born. Scientists like Descartes found that all interesting propositions, all interesting inferences are in fact nonsyllogistic. (Lakatos, 1999, p. 39)

Logic has developed in multiple ways. One nonsyllogistic type of reasoning relevant to science is what Brandom calls non-monotonic. This means that new conditions can turn a valid inference into an invalid one. Brandom (2000, p. 88) gives an example from physics:

1. If I strike this dry, well-made match, then it will ignite. ($p \rightarrow q$)
2. If p and the match is inside a very strong electromagnetic field, then it will *not* ignite." ($p \& r \rightarrow \neg q$)
3. If p and r, but the match is in a Faraday cage, then it will light. ($p \& r \& s \rightarrow q$)
4. If p and r and s and the room is evacuated of oxygen, then it will not light. ($p \& r \& s \& t \rightarrow \neg q$)

Scientific reasoning in educational research is clearly non-monotonic: There are overwhelming numbers of factors that can influence learning. Any relevant positive factor can probably be counteracted by a negative one. Given the pragmatic nature of education, it is also worth mentioning progress on pragmatic reasoning: Walton et al. (2008), for example, identified 96 argumentation schemes that people use in reasoning. It has also become evident that valid argumentation does not depend purely on structure but also on content (and context). So-called material inferences even purely depend on content rather than on their structure. Brandom (2000, p. 85) uses the inference from “Pittsburgh is to the west of Philadelphia” to “Philadelphia is to the east of Pittsburgh,” as an example of an inference that is materially valid because it depends only on the content of the concepts of east and west.

These brief observations from logic suggest that scientific progress relies not only on the structure of argumentation but also on content. Why then should research methodology in education be judged by the separate structure of its argumentation? But what would an alternative look like? Design researchers are faced with the challenge to come up with an alternative grammar or, more likely, grammars. One attempt is that by Cobb et al. (2014):

1. Demonstrating that the students would not have developed particular forms of mathematical reasoning but for their participation in the design study.
2. Documenting how each successive form of reasoning emerged as a reorganization of prior forms of reasoning.
3. Identifying the specific aspects of the classroom learning environment that were necessary rather than contingent in supporting the emergence of these successive forms of reasoning. (p. 490)

The function of such grammars is that they “link research questions to data, data to analysis, and analysis to final claims and assertions” (p. 489). Given that little has been written about this, I formulated an argumentative structure myself based on discussions with Karel Stokking and my own experience with doing and supervising DR. The most efficient and sensible way to gauge its quality seemed to be an interview study with expert design researchers. In this way I could explore what they thought about the need for an argumentative grammar of DR, what they thought of my attempt, and how it could be improved.

Method: Interview study with experts

I interviewed eighteen well-known international design researchers on argumentative grammars of DR and related themes for about 60-90 minutes. These experts represent a variety of different

disciplines and traditions in DR (seven were mathematics educators). Before presenting my own grammar proposal, I asked them about issues that might elicit their view on the logic accompanying DR and the type of claims it renders. First, I asked about the intertwinement of design and research because it can make particular claims difficult: In line with the argumentative structure of RFTs and thinking in terms of variables, many researchers prefer to keep design of an intervention and the (evaluative) research separate. Second, I asked experts' views on Kelly's (2004) claim that DR has no clear argumentative grammar. Third, I asked if they had a preference for types of research questions (what- vs. how-questions). An example of a what-question I showed to the interviewees is: "What are characteristics of a valid and effective teaching and learning strategy to teach students about correlation and regression in such a way that they experience coherence between mathematics and the natural sciences?" (Dierdorp, 2013). A how-question I presented is: "How can students be fostered in their connecting of gene as a molecular-level concept to phenomena at higher levels of biological organization?" (reformulation of Van Mil's, 2013, question). Fourth, I asked experts about the argumentative grammar I propose in the next paragraphs.

The focus on *how* to support learning in DR implies that in my view at least four things need to be captured in an argumentative grammar of a DR project. First, learning goals need to be underpinned (or a problem or needs analysis should be done). A design criterion could be relevance and a research criterion content validity (Plomp & Nieveen, 2013). Several existing methods (review study, expert interviews, Delphi study) can be used to this end. Second, a design (e.g., tool, teaching-learning strategy, or program) could be described in relation to theoretical and empirical considerations. Criteria here can be "empirically and theoretically underpinned" and "innovativeness," but some may want to emphasize "feasibility or practicability." Third, only if intentions are realized, particular intended phenomena can be studied (e.g., whole-class scaffolding; Bakker & Smit, 2017). In RFTs, the criterion would be formulated as "implementation fidelity," necessary to check if any effects can be attributed to the intervention having particular characteristics (cf. Sandoval, 2014). Fourth, information about to what extent learning goals are achieved, or a problem solved, needs to be given in order to answer the main question. The main criterion here is effectiveness.

The structure of a DR project presented to all interviewees for their feedback was the following:

> *How can goal X be achieved for a particular group of learners* (in particular conditions or under particular constraints)? To answer this main question, a sensible list of research questions could be:
>
> 1. What is an appropriate learning goal for….?
> 2. What is a design that would help students/teachers to achieve this goal?
> 3. How well was this strategy/trajectory implemented?
> 4. What were the effects of this intervention?

In discussing this structure, several topics arose that are related to aspects of argumentative grammar such as links between different parts of research (data, claims), in particular in contrast to RFTs. I summarize the experts' responses in three themes.

Theme 1: Intermingling design and research

A key feature of DR is that design and research progress hand in hand. In response to this issue, the interviewees noted the following points. First, any natural scientist knows that scientific practice, in particular the context of discovery, is much messier than presented in textbooks or reports of experiments. Of course, there is a place for experiments, but a large part of science—even in physics—is trial and error with set-ups, designing new arrangements, philosophizing, thought experiments et cetera. In certain disciplines, take astronomy, experiments are even impossible. Serendipity (e.g., the discovery of penicillin) also points to the importance of the context of discovery. The relative importance of RFT as a methodology rests on the side of justification. Several experts said that RFT-type research often produces "false security" or that it struggles with similar issues as other types of research, but somehow it has become common practice to ignore particular problems or trust researchers on doing it well (e.g., validity of measurement, identifying relevant variables). However, many noted there is also a place for RFT as it helps for example policymakers to decide between various well-established options to be implemented.

Second, two interviewees emphasized that DR is about how education *could* be. Where much research is about current educational practice, and some about its past, DR is about its future. Design researchers may argue that educational goals should be different from current educational practice, and design for these new goals. Such DR is thus after proofs of principle, not proof of doing better than current practice which may have very different goals. Comparison with a control group that worked towards different learning goals would be unfair. The argumentative grammar of this type of "proof of principle" DR is thus clearly different from DR that aims for causal claims about effectiveness of particular means of support. This points to the need for multiple grammars.

Third, several interviewees noted that DR conceptualizes learning environments as ecologies rather than systems that can be captured with a few manipulable and unmanipulable variables. Attributing an effect to particular variables then becomes challenging. Rather the focus should in the experts' view be on design principles, hypothetical learning trajectories, or mechanisms of learning, in line with DR's intention to produce knowledge about *how* things work (cf. Sandoval, 2014).

Theme 2: Research questions

Most interviewees considered the examples of what- and how-questions presented to them as too broad. Some did not have a strong preference for either formulation: The researcher wants to know similar things in both cases. However, most experts preferred the how-questions because these emphasize the process of achieving particular learning goals or solving a particular problem. In terms of Cobb et al. (2003), DR typically aims to provide insight into how particular means can support particular learning. This hints at the type of knowledge claims that DR purports to deliver.

A view, expressed by Abrahamson and diSessa, was that DR is a methodological framework (not a method or a strategy) that provides a generative context (about how education could be). Because new types of learning are promoted, new phenomena may emerge and thus in turn become objects of investigation. This view fits with the image of DR as a context of discovery for researchers. Once such phenomena are implicated and objectified, they can be studied as interesting in their own right, with little or no reference to the broader design research context (e.g., Abrahamson et al., 2016). In

line with the generativity of DR, many interviewees emphasized that interesting research questions often emerge rather late in the research process. They are hard to formulate in advance.

Theme 3: Argumentative grammar

The interviewees were overall positive about the proposed grammar. The elements of learning/educational goals, design, implementation, and effects are key to DR, and can be studied empirically, perhaps even in separate publications. One interviewee expressed some resistance to categorizations and structures in research because each project is unique and requires flexibility and creativity. Yet structures could be useful to early career researchers as a starting point.

The experts' further comments were matters of detail. With regard to the learning goals, diSessa noted that he sometimes preferred learning goals that colleagues thought were impossible to achieve with certain age groups (e.g., comprehending velocity and acceleration as vectors in Grade 6). McKenney pointed out that design researchers often encounter obstacles that can become the topic of research. She tends to do a lot of "front-end" work in the early phases of DR in areas where too little is known to arrive at effective designs.

Judging the quality of implementation was considered a good idea, although several experts noted that the implementation process could be interesting to study even without judging its quality. diSessa remarked that failure can be interesting from a design perspective. In his experience, many colleagues respond with surprise when he reports failure, but as long as important lessons can be learned, contributions to the knowledge base can be made.

The terms "interventions" and "effects" elicited some resistance due to connotations with the RFT paradigm of thinking in terms of variables. Several experts preferred to talk in terms of learning ecologies instead. However, some found it important to measure what was achieved and thought that design researchers had measured too little in the past. Many noted that there is certainly a place for RFTs, as well as for quantitative measurement, in DR. Some indicated that RFT ideally gives insight into mechanism too, and can be part of DR.

Ruthven suggested a fifth element, namely an improved re-design, which is indeed in line with DR's emphasis on the hypothetical status of any claims. Citing Cronbach, Plomp emphasized this holds for any type of research: "When we give proper weight to local conditions, any generalization is a working hypothesis, not a conclusion" (Cronbach, 1975, p. 125).

Plomp noted that although he did not write about argumentative grammar, his approach with Nieveen (Plomp & Nieveen, 2013) has such a function. For each phase of a DR project one criterion was central: relevance for the exploratory phase (e.g., problem analysis), consistency (of the design), practicability (of using the design), and last effectiveness.

An issue raised was whether different criteria were needed for DR than for some other research approaches. Because many readers and reviewers are used to different commissive spaces, experts such as Cobb stressed that DR has to become clear on the criteria on which it wants to be judged. For example, we have to acknowledge that design researchers are part of the research, and that their qualities as designers and researchers matter. As Confrey noted in the interview: "You build a reputation for doing good work (…), but that's not great for newcomers because they don't have the track record yet." It certainly goes against the more conventional norm of reliability that research

should be independent of the researcher. Hence it seems necessary to think through the criteria by which design researchers want to be judged. However, McKenney preferred the research part of DR to be treated with the same criteria as other qualitative or mixed-methods approaches. Kelly suggested DR can learn from other research approaches such as single-subject and repeated-measures designs.

Not only the design researcher, but the audience has to make judgments as well. Where RFTs can yield results that sometimes seem to require little understanding of the topic at hand, DR asks for an audience that can appreciate the relevance of the educational goals chosen, the innovativeness of the design, and the learning processes reported. diSessa noted that the typical reasoning in DR is to show what types of reasoning can be promoted in a particular way, for instance by using particular software. Any well-informed domain-specific educational researcher with knowledge of the disciplinary (e.g., mathematical) content will know how rare or relevant such types of reasoning are for particular age groups, so will appreciate qualitative examples of even small samples.

Conclusion

In this paper I have argued that it is unreasonable to expect that educational research including DR should use an argumentative grammar that depends solely on structure rather than also content (key concepts, mathematical learning content, context etc.). Examples from logic illustrate the importance of types of reasoning that are also based on content. Argumentative grammars for DR should thus acknowledge content too. Cobb et al. (2014) offered an argumentative structure that can help convince readers about the development of students' mathematical reasoning and the aspects of the learning environment that supported them (see also Sandoval, 2014). My own proposal focused on the grammar of a DR project with the aim to contribute to knowledge about how particular educational goals could be achieved in general (or problems solved). Based on the interviews with experts, my proposal—after some modification—seems to make sense as a starting point for design researchers when they write a proposal or want to demarcate phases in their overall project (cf. McKenney & Reeves, 2012; Plomp & Nieveen, 2013) with criteria that are central in each phase. However, there is a need for more explicit argumentative grammars, for instance for "proof of principle"-type DR and for smaller-scale design studies that focus on interesting phenomena that are discovered during a larger DR project.

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When a critical aspect is a conception: Using multiple theories to design dynamic computer environments and tasks to promote secondary students' discernment of covariation

Heather Lynn Johnson, Evan McClintock, Peter Hornbein, Amber Gardner and Daniel Grieser

University of Colorado Denver, School of Education and Human Development, Denver, Colorado, USA; heather.johnson@ucdenver.edu; evan.mcclintock@ucdenver.edu; peter.hornbein@ucdenver.edu; amber.gardner@ucdenver.edu; daniel.grieser@ucdenver.edu

We explicate how we used different theories of learning to design dynamic computer environments and tasks to promote secondary students' discernment of covariation—central to students' study of fundamental mathematical ideas such as rate and function. Using Marton's variation theory, we designed task sequences to foster students' discernment of the critical aspect of covariation. Using Piaget's constructivist theory, we defined the critical aspect, covariation, in terms of students' conceptions of a relationship between attributes whose measures vary. Using Thompson's theory of quantitative reasoning, by quantities we mean attributes of objects that students can conceive of as being possible to measure. We provide data to demonstrate how a student's discernment of covariation advanced during her work on a task sequence. We discuss implications for the design of dynamic computer environments and tasks focused on the mathematics of change and variation.

Keywords: Secondary school mathematics, instructional design, computer simulation, learning theories.

By drawing on more than one theory of learning, researchers can combine tools and lenses to investigate complex phenomena (e.g., Cobb, 2007; Sfard, 1998; Simon, 2009). Cobb (2007) recommended that researchers "act as bricoleurs by adapting ideas from a range of theoretical sources" (p. 29). Sfard (1998) argued that researchers should not assume that theoretical "patches of coherence" somehow would combine to form a single, unifying theory of learning. Yet, using multiple theories can pose challenges, particularly if researchers view theories as competing, rather than complementary (Simon, 2009). To address challenges, it is useful for researchers to take into account the grain sizes of different theories (Kieran, Doorman, & Ohtani, 2015; Watson, 2016).

By distinguishing between the grain sizes of theories, researchers can more effectively interpret and use theory for task design purposes (Kieran et al., 2015; Watson, 2016). Broadly, grain sizes include grand theories (e.g., Piaget's constructivist theory), intermediate theories (e.g., Marton's variation theory), and domain specific/local theories (e.g., Thompson's theory of quantitative reasoning). Furthermore, it is useful for researchers to acknowledge interrelationships between theories of different grain sizes. Theories of smaller grain size depend upon or address particular aspects of theories of larger grain size (Watson, 2016). For example, Thompson's theory of quantitative reasoning depends upon Piaget's constructivist theory to define quantities in terms of students' conceptions. By drawing on theories of different grain sizes, researchers can adapt and interpret grand theories for task design and implementation (e.g., Cobb, 2007; Kieran et al., 2015; Thompson, 2002; Watson, 2016).

When researchers engage in task design, they should make explicit how their theory choice informs their task design (Watson, 2016). Using Marton's variation theory (2015), we designed task

sequences to engineer opportunities for students to discern critical aspects central to fundamental mathematical ideas, such as rate and function. We posit that one such critical aspect is covariation. Using Piaget's constructivist theory (1985), we define covariation in terms of individuals' conceptions. Using Thompson's theory of quantitative reasoning (1994, 2002), we articulate the conception; by covariation we mean a conception of a relationship between attributes whose measures vary. Students' conceptions of covariation impact their understanding and use of function (Thompson & Carlson, 2017).

We build on the work of researchers who have designed dynamic computer environments and tasks to foster students' study of the mathematics of change and variation (e.g., Kaput & Roschelle, 1999; Saldanha & Thompson, 1998; Thompson, 2002). In this paper, we explicate how we used theories of different grain sizes to design dynamic computer environments and tasks to promote secondary students' discernment of covariation. To avoid remaining only in the abstract, we provide data to demonstrate the utility of our approach in fostering students' discernment of covariation.

Theoretical and conceptual framework

We use Piaget's constructivist theory to orient our research. We foreground students' conceptions to explain how researchers might design dynamic computer environments and tasks to foster students' development of difficult to learn mathematical ideas such as function and rate. We focus on students' mental operations, which refer to actions that individuals can enact in thought or in the physical world (Piaget, 1985).

We use Marton's variation theory to guide the design of our dynamic computer environments and related task sequences. Broadly, Marton (2015) argued that instructional designers should develop task sequences that provide students opportunities to discern critical aspects. The task sequence should involve patterns of variation, then invariance in the critical aspects (Marton, 2015). We draw on Piaget's constructivist theory to orient our interpretation of the critical aspect that we intend for students to discern. By critical aspect, we mean a conception. The critical aspect—covariation—refers to a conception of a relationship between attributes whose measures vary.

When using variation theory it is important for instructional designers to determine if the critical aspect is comprised of a single aspect or of interrelated aspects (Marton, 2015). For example, suppose a designer intends for students to discern the color blue, which students could conceive of as being comprised of a single aspect. A task sequence should begin with variation in color and invariance in some unrelated feature (e.g., blue ball, green ball, red ball), then invariance in color and variation in the unrelated feature (e.g., blue ball, blue block, blue cone), and then variation in both. In contrast, if a designer intends for students to discern the *depth of blue color*, students would need to conceive of interrelated aspects (depth, color) that comprise the critical aspect. In this case, the task sequence should include variation and invariance in each interrelated aspect (e.g., different colors of the same depth, then different depths of the color blue), then move to variation in both aspects. A conception of covariation necessitates a conception of a relationship between interrelated aspects (attributes whose measures vary). For example, in a situation involving the varying height and distance of a car in a turning Ferris wheel, the height and distance are the interrelated aspects, and students' conceptions of a relationship between measures of height and distance (covariation) is the critical aspect.

We found Marton's variation theory and Piaget's constructivist theory to complement each other for the purposes of our task design. From a constructivist perspective, we do not assume that a relationship between attributes whose measures vary (covariation) is something that is "out there" for students to perceive. Marton (2015) argued that researchers should not assume that students already attend to the critical aspect prior to encountering a task sequence; therefore, task sequences should include variation in critical aspects (contrast) prior to variation in noncritical aspects. To foster students' discernment of covariation, we incorporated variation in the types of interrelated aspects (height, width, distance) prior to variation in the representation of those aspects.

We use Thompson's theory of quantitative reasoning to explain what we mean by the attributes whose measures vary—the interrelated aspects comprising the critical aspect of covariation. Drawing on Piaget's constructivist theory, Thompson (1994) defined quantities in terms of individuals' *conceptions* of attributes of objects. Therefore, quantities are not "things" that exist in the physical world. Following Thompson (1994), we claim that an individual conceives of some attribute as a *quantity*, if the individual can conceive of the possibility of measuring that attribute. For example, we would claim that a student conceived of "height" as a quantity if the student provided evidence of envisioning the possibility of measuring the height of some object.

We selected Thompson's theory of quantitative reasoning because we found it to be useful for interpreting Piaget's constructivist theory. Accordingly, the mental operations on which we focus are quantitative operations, which involve actions on attributes that students can conceive of as measurable, or in other words, actions on quantities (Thompson, 1994). Specifically, we focus on students' conceptions of covariation, which entail the quantitative operations involved in forming and interpreting relationships between attributes whose measures vary. For example, a student conceiving of covariation could form and interpret relationships between the varying measures of height and distance for a Ferris wheel car traveling around one revolution of a Ferris wheel.

The diagram in Figure 1 illustrates how we used different theories to inform our task design.

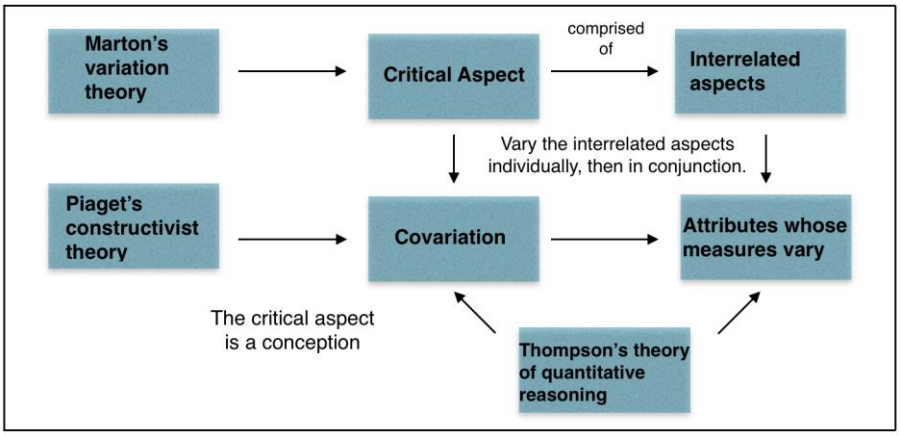

Figure 1: Relationships between the different theories we used to inform our task design

We placed Marton's variation theory at the top to foreground our intention to design a task sequence to foster students' discernment of a critical aspect comprised of interrelated aspects. We placed Piaget's constructivist theory in the center to communicate how we used this grand theory to define the critical aspect, covariation, in terms of individuals' conceptions. We placed Thompson's theory

of quantitative reasoning at the base to show how we used this local theory to explain what we mean by covariation—a conception of a relationship between attributes whose measures vary.

Ferris wheel dynamic computer environments

Using Geometer's Sketchpad software, Johnson developed two dynamic computer environments for use with the task sequence. The environments consisted of a Ferris wheel animation and linked graph, each of which students could control separately or in conjunction. The environments related either the *height* of a Ferris wheel car from the ground or the *width* of the car from the center to the *distance* traveled around one revolution of the wheel (Figure 2 shows height and distance). See Johnson (2015) for more details about the environments.

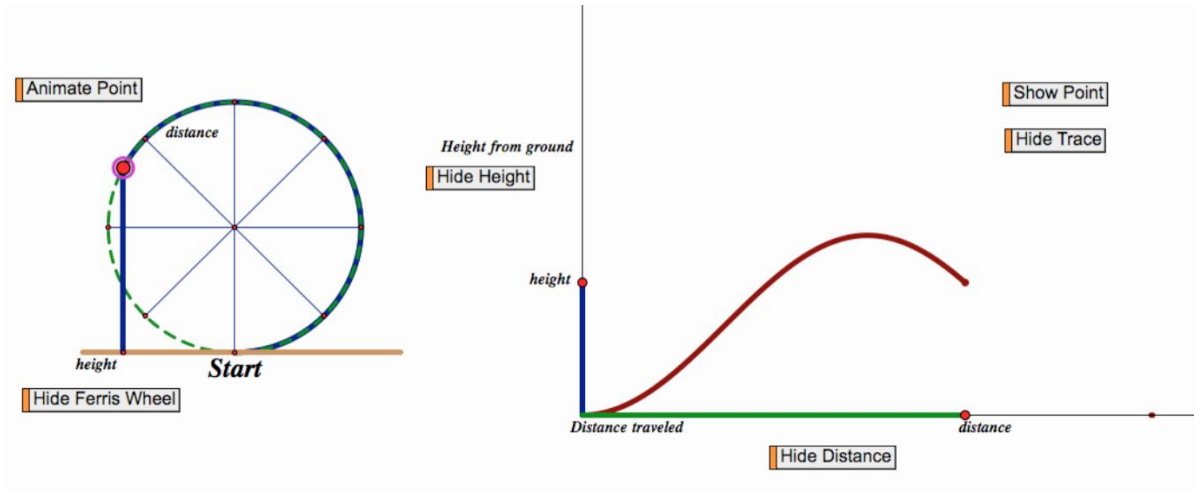

Figure 2: Ferris wheel dynamic computer environment, distance and height

The Ferris wheel environments contained three affordances particularly relevant to our use of variation theory. First, students could vary each of the interrelated aspects (e.g., height and distance) individually by dragging or animating the dynamic segments on the vertical and horizontal axes. Second, the environments included different interrelated aspects—height and distance (Figure 2), and width and distance (not shown). Third, the environments included variation in the axes used to represent the interrelated aspects on the Cartesian plane (e.g., height and distance represented on the vertical and horizontal axes [Figure 2], then horizontal and vertical axes, respectively [not shown]).

Our use of Piaget's constructivist learning theory and Thompson's theory of quantitative reasoning informed our choices about the types of quantities to include on each of the axes. Specifically, we included quantities measurable with linear units, because it is less difficult for students to conceive of using linear units to measure quantities (see also Piaget, 1970). Furthermore, Thompson (2002) recommended students use their fingers as tools to represent change in individual quantities. In the Ferris wheel environments, students could use either their fingers or the dynamic segments on the vertical and horizontal axes to represent change in individual quantities.

The Ferris wheel task sequence

Purpose and setting

We view tasks as problems designed for particular audiences and settings (see Sierpinska, 2004) Johnson designed the Ferris wheel task sequence to provide students opportunities to discern covariation. In a small neighborhood school in an industrial region of a large U.S. city, Johnson conducted a series of small group interviews with five ninth grade students (~15 years old), enrolled in an introductory algebra course. Interviews occurred approximately once per week. During the interviews, students completed the Ferris wheel task sequence (see Table 1). Johnson designed the task sequence for a small group interview setting; however, teacher/researchers could adapt the tasks for use in different settings (see Johnson, Hornbein, & Azeem, 2016).

Variation and invariance in the Ferris wheel task sequence

To foster students' discernment of critical aspects comprised of interrelated aspects, Marton (2015) recommended that instructional designers begin with task sequences containing variation in individual interrelated aspects, then variation in the both interrelated aspects, against a background of invariance. In the Ferris wheel task sequence, we intended for the situation of a turning Ferris wheel to provide a background of invariance. Furthermore, Marton (2015) recommended variation in features (or dimensions) of those interrelated aspects. We provided two types of variation in features: the type of interrelated aspects (width, height, or distance), and the representation of each aspect on the Cartesian plane (horizontal or vertical axis). Table 1 shows the Ferris wheel task sequence, including variation in interrelated aspects and representations on the Cartesian plane.

Task	Interrelated aspects	Representation on axes on Cartesian Plane
1	Height, Distance	Distance – horizontal, Height – vertical
2	Width, Distance	Distance – horizontal, Width– vertical
3	Height, Distance	Height – horizontal, Distance – vertical
4	Width, Distance	Width – horizontal, Distance – vertical

Table 1: Ferris wheel task sequence: Variation in interrelated aspects and representation

Covariation and quantity in the Ferris wheel task sequence

When the critical aspect is a mental operation, instructional designers should provide students opportunities to engage in activities in thought as well as action. Each task in the Ferris wheel task sequence contained five parts: (1) Explain what the interrelated aspects measure in the Ferris wheel situation; (2) Sketch a graph relating both aspects; (3) Use dynamic segments to represent change in individual aspects (e.g., height or distance); (4) Predict a car's location on the Ferris wheel given only dynamic segments representing the changing individual aspects; and (5) Compare the computer generated graph to the sketch in (2). Through each of the five part tasks, Johnson provided students multiple opportunities to discuss and represent their thinking about how the interrelated aspects (height and distance or width and distance) were changing individually and together. For example, students sketched a graph relating the interrelated aspects prior to viewing any facets of the dynamic graph. Furthermore, Johnson provided students opportunities to discuss and show the possibility of

measuring interrelated aspects of the Ferris wheel situation (e.g., "height" represents the vertical distance from the car to the base of the Ferris wheel, see Figure 2).

A case of a student's discernment of covariation

We use the work of one student—Ana—to demonstrate the promise of this design approach for fostering students' discernment of covariation. Ana's work demonstrates the range of reasoning of all five students who completed the Ferris wheel task sequence. Building from Ana's work, we present a case of a student's discernment of covariation.

We share data from Ana's work for Part 2 of Tasks 1 and 3: *Prior to viewing the dynamic graph, sketch a graph relating both aspects*. We selected data from Part 2 to illustrate how Ana's sketches changed prior to viewing aspects of the dynamic graph. We attribute the changes in her sketches to changes in her conceptions of a relationship between varying measures of height and distance.

Figure 3 shows Ana's written work for Part 2 of Tasks 1 and 3. For Part 2 of Task 1 (left), Ana drew the curved graph, labeling it "height," and then drew the line graph, labeling it "distance" (Figure 3, left). When asked what her labels meant, Ana stated: "This (points to the curved graph) would be the graph shape if we were dealing with the height, and this (points to the line graph) would be the shape if we were dealing with the distance." For Part 2 of Task 3, using one continual motion, Ana sketched a single graph (Figure 3, right). When asked to explain her thinking, she stated that the "distance keeps on going," but the height will reach "a certain amount," and then "it goes back down." To illustrate, she drew an arrow along the left of the distance axis. Next, she drew a small, darkened segment on the graph, and two arrows extending along the graph.

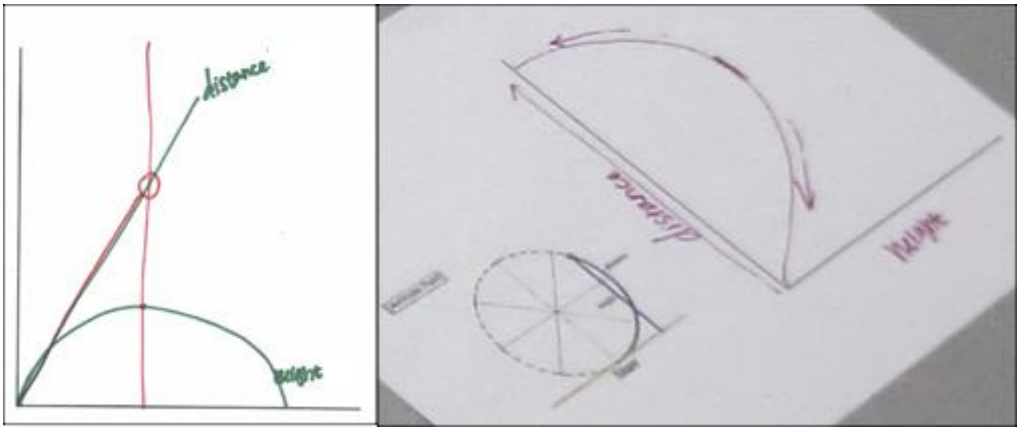

Figure 3: Ana's graphs in Part 2 of Task 1 (left) and Part 2 of Task 3 (right)

For Part 2 of Task 1, we interpret that Ana represented individual variation occurring in the measures of height and distance. Not only did she sketch two graphs, she labeled the actual sketches, rather than the axes. We use Ana's work for Part 2 of Task 1 to demonstrate that Ana did not enter the Ferris wheel task sequence already conceiving of a relationship between the varying measures of height and distance, or in other words, conceiving of covariation. Moving forward to Part 2 of Task 3, Ana used a single graph to represent a relationship between the varying measures of height and distance. Not only did she sketch a single graph, she annotated the graph to show how the single graph represented variation in the measures of both height and distance. Therefore, we claim that Ana demonstrated discernment of covariation during her work for Part 2 of Task 3

(conceived of a relationship between attributes whose measures vary). Furthermore, Ana's discernment of covariation was not limited to the aspects of height and distance. She also demonstrated discernment of covariation when working with width and distance in Tasks 2 and 4.

Discussion/Implications

When critical aspects involve interrelated aspects, Marton (2015) recommended that instructional designers develop task sequences that include different backgrounds. In our task sequence, we used only a Ferris wheel situation, and we recommend that researchers designing task sequences to foster students' discernment of covariation also include different situations. However, we provide our recommendation with a caveat—the different situations should include interrelated aspects measurable with linear units. For example, if we were to design a task sequence for a filling bottle situation, we might ask students to relate the height of the liquid in the bottle to the diameter of the liquid in the bottle. Our caveat stems from our use of Piaget's constructivist theoretical perspective. It is less difficult for students to conceive of the possibility of using linear units to measure attributes (e.g., Piaget, 1970). Therefore, we recommend that task sequences designed for students to discern covariation (a critical aspect involving interrelated aspects) should include interrelated aspects measurable with linear units. Researchers have shown that even successful university students have difficulty using graphs to represent relationships between height and volume in a filling bottle situation (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). If students have difficulty conceiving of the possibility of using a three-dimensional unit to measure volume, it may impact their discernment of covariation for situations involving such attributes.

By using theories of different grain sizes, we were able to employ multiple, compatible lenses to engage in task design that looked both *across* and *within* the sequence of tasks. By guiding our variance and invariance of interrelated aspects, Marton's variation theory informed design *across* the task sequence. By fostering our choices about the kinds of aspects to vary, Thompson's theory of quantitative reasoning informed our design *within* tasks in the sequence. The ability to view a task sequence from different perspectives—in our research, looking both across and within—is a productive result emerging from the use of multiple theories to do compatible explanatory work to augment the design of task sequences intended to foster students' discernment of critical aspects.

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Combining two theories in the design for learning about integrals

Anatoli Kouropatov[1] and Tommy Dreyfus[2]

[1]Levinsky College of Education, Israel; anatoliko@gmail.com

[2]Tel-Aviv University, Israel; tommyd@post.tau.ac.il

We show how a combination of two theories, Abstraction in Context and Proceptual Thinking, served as basis for design decisions in the framework of a research study about learning the integral concept in high school via constructing knowledge about accumulation.

Keywords: Design of instruction, Abstraction in Context, proceptual thinking, integration.

Introduction

Designing learning units involves decisions about the transition from syllabus to curriculum (Dreyfus, Hershkowitz, & Bruckheimer, 1987); Mathematics Education offers home-grown theories supporting this transition; curriculum designers use these theories when design is a main goal.

Here, we present a case study of using theories for the design of a didactical tool – a learning unit intended for students' construction of a specific mathematical notion. This study is part of a larger research project (Kouropatov, 2016) offering a didactic approach supporting high school students in acquiring a conceptual understanding of the integral. For this purpose, we asked ourselves: a) What does "a conceptual understanding of the integral" mean? and b) How can we support students in acquiring such understanding? Answering question (b) required design. The design was preceded by a thorough didactical-mathematical analysis of approaching integration via the idea of accumulation (Thompson & Silverman, 2008). A tight interrelationship developed between the design process and the relevant theories, Abstraction in Context and Proceptual Thinking. In this paper, we exhibit this relationship via the need for and effect of these theories in the design process and hence the contribution of these theories to a basis for the transition from syllabus to curriculum.

Theoretical frameworks and their influence on the design

We follow Tabach, Hershkowitz, Arcavi, and Dreyfus (2008) in distinguishing

- a pre-design stage involving considerations before starting the actual development and research work with students;
- an initial design-research-redesign stage of sporadic and isolated activities, and the observation, data collection, and analysis of their implementation with a few students;
- a final stage that comprises further redesign, for the creation of a coherent, complete task-based curriculum, and its implementation (limited, in our case, to four pairs of students).

These stages indicate what to do but not how to do it. Design decisions require theories. For our study, we have adopted Abstraction in Context (AiC) because its roots in constructivism and in activity theory make it suitable to design for and analyse the construction of abstract knowledge during the learning process (Hershkowitz, Schwarz, & Dreyfus, 2001). We have also adopted the theory of Proceptual Thinking (PT) because of the proceptual nature of the mathematical notions in focus, in particular accumulation (Gray & Tall, 1994). The theories are compatible but different: PT deals with how students see mathematics; AiC deals with how students acquire knowledge.

Theoretical considerations at the pre-design stage

AiC as basis for design

AiC takes abstraction to be a learner's activity of vertical (in the sense of Freudenthal) reorganization of previous mathematical constructs in order to arrive at a new (to the learner) construct. Abstraction leads from an initial, vague first form, which may lack consistency, to an elaborated form (Davydov, 1990). The activity is interpreted in terms of epistemic actions performed by a learner, or a group of learners, for a specific purpose, in a particular context. The context includes the social setting as well as the learners' personal background, in particular the previous mathematical constructs resulting from previous processes of abstraction. 'Reorganization' includes establishing new connections between previous constructs, making mathematical generalizations, and discovering new strategies for solving problems. 'Vertical' implies building a new level of abstraction on top of a previous level. For the researcher, the question arises how to support and unveil the processes by which the students' new constructs may emerge as a vertical reorganization of previous constructs in the current context. AiC argues that for this purpose, it is crucial to carry out an a priori analysis. This theoretical content analysis aims at identifying the elements of knowledge (mathematical facts, notions, claims, strategies, representations, etc.) that together constitute 'learning a concept' and have a didactical perspective, namely can be constructed by learners in a suitable context using appropriate didactical tools. The results of the a priori analysis are *descriptions* of the elements of knowledge that belong to the world of mathematics but may be linked to the current context, and operational *definitions* that constitute descriptions of observable student behaviour: utterances or actions that provide criteria for assessing whether a student's constructing action corresponding to the said knowledge element has occurred.

AiC has originally been proposed in the framework of a curriculum development project, in which abstraction was a central concern, and Hershkowitz et al. (2001) have already then expressed the hope that it will be useful not only for analysing students' processes of abstraction but also for designing sequences of activities supporting students in such processes. At the pre-design stage, AiC requires a sequence of activities, each intended toward the construction of an appropriate element of knowledge, while the sequence is hierarchically structured as imposed by verticality. In other words, AiC helps us find a structure of the subject knowledge that is appropriate for implementation in the design. For example, based on the a priori analysis of the mathematical content, we decided that the unit should be designed as a four level vertical structure of the following conceptual components: *Approximation* in the context of given geometrical objects; *Accumulation Value* (the definite integral) in the context of given analytical objects or situations; *Accumulation Function* (the definite integral with varying upper bound); and *Integration-Differentiation interplay*, mainly the Fundamental Theorem of Calculus (FTC). Further analysis of each level provided the vertical structure of the elements of knowledge intended to be constructed by the students. We present two of the four levels in some detail.

For didactical reasons and based on verticality, *Approximation* was interpreted as Geometrical Shapes Approximation (GSA) with the following three knowledge elements:

AP_G "General approximation": The size of a given object can be approximated by replacing the given object with known objects;

AP_R "Refined approximation": The approximation can be made more precise by decreasing the size of the replacing objects and increasing their number;

AP_L "Approximation limit": The size of a given object can be determined as precisely as one wants by continued refinement.

The corresponding operational definitions are that we will say students have constructed

AP_G if they explicitly (verbally and/or graphically) replace a given object with known objects;

AP_R if they explicitly (verbally and/or graphically) refine the approximation by decreasing the size of the replacing objects and increasing their number;

AP_L if they explicitly (verbally and/or graphically) identify a value as the exact size of a given object by continued refinement.

Knowledge elements for the other three conceptual components were similarly described and defined (Kouropatov, 2016; Kouropatov & Dreyfus, 2014). These descriptions and definitions constitute the framework for the design of activities in the learning unit. In other words, by means of the description of the elements of knowledge, AiC informs the decisions of what should be designed and in what hierarchy it should be organized. AiC does not inform how to design for each notion by means of micro-tasks. For that purpose we used PT.

PT as basis for content design

Gray and Tall (1994) defined the notion of procept as an aggregate of three things: process, concept (or object), and symbol. For example, the symbol $\int_0^x f(t)dt$ is meant to evoke both the process of accumulation (integration) and the concept of accumulation function (integral), with the cognitive combination of all three, process, concept, and symbol, being called a procept. This stance has crucial didactical implications: students might first meet a process; later, a symbol is introduced for that process and/or its product, and this symbol takes on the dual meaning of the process and the object created by the process. Proceptual Thinking is then defined as the ability to switch one's focus between these dual roles of the symbols as is useful and efficient in the current context, for example solving a problem. Someone who has the ability to think in this way may be described as versatile (Tall & Thomas, 1991). Versatility includes a global picture of a concept as well as the ability to break it down into a process, seeing each stage as part of the whole concept. According to Hong and Thomas (1998) versatility is critical for comprehension of the integral concept.

We see the integral as a multilevel, hierarchic procept, which is composed of (in the sense of AiC, and hence intended to be constructed by students from) other procepts including function, graph, approximation, sum, and accumulation; hence, we continue the pre-design stage by

- using the result of the above a priori analysis in order to identify and describe the main sub-procepts of the integral procept;
- identifying the hierarchic structure of the integral as an aggregate of procepts.

The main didactical flow of ideas was derived directly from the procept hierarchy of the mathematical notion of the concept of the integral. In particular, the didactical goals are: to create an opportunity for the learners *to carry out a process* that is meaningful for them (e.g., to

approximate an unknown area of some shape by accumulating the known areas of small parts of this shape); to give the learner the possibility *to internalize this process as a concept* (e.g., by quantifying the process, by discussing the characteristics of this process); *to introduce* the learner to the common mathematical *symbol* as encapsulation of the completed process and the internalized object; these considerations became the leading considerations of the initial stage of design. In other words, PT allows us to answer the question of how the learning activities should be designed.

We present two examples from the learning unit that show how we took into account the proceptual nature of the intended elements. The first one, is the initial activity for introducing *Approximation* via GSA. And the second one is from the middle of the unit, and is intended to lead students to constructing the concept of *Accumulation Function*. As mentioned above, these two concepts, together with the *Accumulation Value* and the *FTC,* are the four components of the vertical structure of the suggested design of the unit.

Regarding GSA, students *carried out the process* of approximating the length of a given (sketched) curve (interval, semicircle, non-standard curves) using a ruler, compass, protractor, square paper (with two different mesh sizes) and calculator. Then, students discussed the "quality" of the resulting approximation and were asked to refine it (for example by using more sophisticated measurements) with the intention to lead to *internalization of this process as a concept*. Finally, students were asked to *find the length* as precisely as possible (the existence of such a value was taken as intuitively obvious). This “process, concept, existence” triad constitutes the GSA procept according to the above analysis.

Regarding the notion of *accumulation function*, the activities offered students opportunities to *carry out the process* of co-variational change of the accumulation value according to the value of the right end-point of a certain sub-interval (using approximation or algebraic considerations); students dealt with a table and/or graph and/or verbal representation of this change with the intention to lead to *internalization of this process as a concept*; finally, *the symbol* $A(x)=\int_a^x f(t)dt$ was introduced. This “process, concept, symbol” triad constitutes the accumulation function procept.

Theoretical considerations at the initial stage of design

The influence of AiC and PT on the design of the unit could, in principle, best be demonstrated by the design of activities about the procept of accumulation, the central notion of the learning unit. Because of space limitations, we concentrate instead on a small part of this: When describing a process of accumulation, one should know "how to start accumulating" - in other words, how to calculate an initial quantity. Then, one should know how to calculate further pieces of the accumulating quantity. The general answer to this problem is *approximation*. Here, the proceptual nature of approximation is particularly important: We see approximation as a process, and the result of this process, of calculating (as accurately as required) some unknown value (length, area or volume) by using known values.

While approximation refers to many kinds of quantity, verticality suggests a sequence of activities that starts with concrete geometrical shapes (lines, 2-dimensional and 3-dimensional shapes) followed by geometrical shapes that are given analytically (using elementary functions) in a coordinate plane (space); only then, more general quantities, given analytically, are considered.

Such a sequence allows students to construct their knowledge, starting in a concrete context of geometrical drawings and bodies that is intuitively clear to them, and where all quantities (i.e. length, area, volume) have positive values. This context requires relatively little previous knowledge and allows for rather linear and smooth vertical reorganization. Next, follows a more formal context of analytically given objects or situations (all quantities still having positive values). And finally, students are asked to deal with general quantities. Practically, approximation may be made by measurement, by geometrical consideration (with known formulas), or by algebraic considerations (analysing some algebraic term). In light of these considerations we have designed a sequence of activities, which we now, following an a priori analysis, interpret as focusing on (i) GSA with its three elements of knowledge 'General Approximation' (replacing the given object by known objects), 'Refined Approximation', and 'Approximation Limit'; and (ii) parallel elements of knowledge for 'Analytical Shape Approximation'.

Similar considerations apply to the concept of *accumulation function*. We see the accumulation function as a process of change (e.g. the change of accumulating area beneath the graph of the function while "the upper bound is moving") and the product corresponding to this process (e.g., a graph of this process demonstrating the ability to characterize it). This approach to the accumulation function immediately leads to the following conclusions: for constructing the accumulation function element of knowledge, students should know (even if only intuitively) that if we change the upper end-point of some sub-interval of the function domain, the appropriate value of the given function and the accumulation value of the given function will also change; and they should know how to characterize the process of the changing of the accumulation value. In light of these considerations, we have designed a sequence of activities, focusing on the *Accumulation Function* element of knowledge via its component elements (not specified in this paper).

On the basis of the above we claim that the combination of AiC and PT allows us to make decisions regarding the design of activities for the learning unit.

Theoretical considerations at the final stage of design

As a result of the two previous stages (pre-design and initial design) we developed a sequence of activities that were organized according to the above four component vertical structure. Each of its four components constitutes a hierarchical procept that is vertically composed of sub-procepts. We interpret this whole structure as the procept of Integral.

At the final stage of the design we analysed the developed activities with the purpose of avoiding inconsistent usage of terms, symbols, and visual representations. Another important issue we took into account at this stage was adaptation of the unit to students' previous knowledge. Thus, for example, at the previous stages we had used the number e for some of tasks. We recognized that this notion is not familiar to the students, so at the final stage of the design certain activities have been changed (e.g., by using π instead of e).

The implementation of the unit was organized in the form of learning sessions of pairs of students with in the presence of a researcher. The time interval between the sessions was typically between one and two weeks. We considered that for every part of the unit, students need some introductory and some summarizing activities. These activities aimed to provide a smooth flow of the learning

process and were developed (in the form of a short discussion that was led by researcher) at the final stage of design.

As a result of the final stage, we created a task-based curriculum unit introducing the concept of integral via the idea of accumulation with a fair measure of internal coherence. This unit has been implemented with four pairs of students.

AiC as a tool for design evaluation

An essential component of AiC is the nested epistemic actions model for describing and analysing, at the micro-level, processes of abstraction by which learners construct new knowledge. The model uses the three epistemic actions of Recognizing (previous constructs as relevant in the present situation, R), Building-with (the recognized constructs to achieve a local goal, B), and Constructing (assemble and integrate previous constructs so that a new construct emerges by vertical mathematization, C). In processes of abstraction, R-actions are nested in B-actions, and R and B-actions are nested in C-actions.

Following Dreyfus, Hershkowitz and Schwarz (2015) the core of the method is the analysis, utterance by utterance, of transcripts to identify R, B and C epistemic actions as building blocks of abstraction. The RBC methodology helps making processes of knowledge constructing observable. This claim is based on empirical results regarding many content areas including integration (Kouropatov & Dreyfus, 2014).

RBC analysis of the learning sessions has been successfully used for evaluating the design of the activities by identifying problems with the implementation; this evaluation has uncovered instances where the design (or micro-design) of activities or their sequencing needed to be improved. We present two examples.

The first example concerns the concept of approximation limit (AP_L) referred to above. The RBC-analysis of the performance of one pair of students (A and B) supplied empirical evidence about students' constructing processes of AP_G and of AP_R but not of AP_L, which is a crucial component of approximation. Therefore, the design of the activity for the following groups of students has been refined in a way that supports the constructing process of AP_L. The elaboration consisted of adding questions leading the students to intuitively distinguish between overestimates (decreasing to the exact value) and underestimates (increasing to the exact value) of the approximated value. The revision was successful in the sense that all following student pairs succeeded in constructing AP_L.

The second example relates to the issue of lacking previous constructs assumed by the design. For example, when constructing the procept of approximation via GSA, students M and N demonstrated a lack of previous constructs such as identifying coordinates of points on a graph, or calculating lengths of segments. For example, in the activity of finding the length of a quarter-circle, the students quickly recognized the relevance of approximation. They replaced the curve with a set of chords but then got stuck because they didn't know how to calculate the chord-lengths. The idea of choosing the segment endpoints according to some division of the given interval was new them and outside their current grasp. The teacher's intervention was needed and was locally helpful. So, we can argue that there was a need for an additional element of knowledge that our design did not take into account: division of the given interval creating an appropriate division of the graph.

Conclusions and further questions

We presented a case study of using theories for design decisions; this case dealt with learning the integral concept in high school via constructing knowledge about accumulation. The theories were most significant but not the only resource for decision making. The decisions were inspired by theory (e.g., in the case of verticality of the structure of the elements of the intended knowledge), by practical experience (e.g., in the case of assumption regarding intuitive accessibility of some elements of knowledge for students), or by both (e.g., in the case of building the system of sub-procepts of the procept of the integral or in the case of designing the sequence of learning activities). However, we argue that in the process of designing the learning unit on integrals for high school students, the theories have been interwoven and have played crucial roles in the process of development and implementation of the unit at all stages: the pre-design, the initial, and the final stages, as well as for fine-tuning the design after its evaluation.

The theories that we adopted for the purpose of the design are Abstraction in Context (AiC) and Proceptual Thinking (PT). These theories were adopted on two levels: AiC on a cognitive-epistemological level with the purpose of coming to design decisions regarding the nature and the structure of knowledge be learned (at a macro-level, which seems to be efficient in a more general context); PT on a didactic-implementation level with the purpose of coming to design decisions regarding how to help learners to achieve this knowledge (at a micro-level, which seems to be efficient in the context of mathematics procepts). The role of the theories differed from stage to stage: AiC was more essential at the pre-design stage while PT was more fruitful at the initial stage. However, the synergy of the theories was more influential than their diversity: Our design aims at supporting students in constructing proceptual knowledge of the Integral that we interpret as a hierarchical procept that is vertically composed of sub-procepts. Our research allows us to follow how students acquire a proceptual view while they construct their knowledge. We speculate that students' behaviour that is coherent with the suggested operational definitions (in terms of AiC) can be interpreted as evidence for the acquisition of a proceptual view (in terms of PT).

Following the research, we find ourselves in a better position to pose two relevant yet unsolved problems, a practical and a theoretical one. The practical problem is how to optimally profit from theory when designing instruction. The findings of the research show that the AiC and PT frameworks can be used for development and evaluation of an important instructional instrument – a learning unit. What about other instructional instruments, such as homework assignments, tests, and so on? Could we also use these theories for the design of such instruments? Additional research and experiments are needed in order to suggest the adaptation of the discussed theories for these types of instrument. The theoretical problem concerns the consistency of theories: Could we have used other theoretical frameworks instead or in addition to AiC and PT, what consistency issues would have arisen, and how different a design would have resulted?

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Learning Through Activity: A developing integrated theory of mathematics learning and teaching

Martin A. Simon

New York University, New York, NY, USA; msimon@nyu.edu

Sociocultural theory (Vygotsky, 1978) has left the process of internalization relatively unexplored. In the Learning Through Activity (LTA) research program, we use basic constructs of constructivism to address this issue. The goal of our empirical and theoretical work has been to elaborate an integrated theory of mathematics learning and teaching. Towards this end and grounded in our empirical research, we have engaged in explicating reflective abstraction for mathematical concepts and developing a design approach for fostering reflective abstraction. The LTA approach is complimentary to a problem-solving approach; learning engendered by the LTA approach is not dependent on the uncertainty inherent in solving authentic problems.

Keywords: Constructivism, learning theories, task design, mathematical concepts, learning trajectories.

In Simon (2009), I argued that different theories of learning can be thought of as different tools for research affording different kinds of work. It is the job of mathematics education researchers/theorists to not only develop and articulate theories, but to specify the work for which they are designed. Such specification allows fruitful discussion about the relationship of different theories and the possibilities of particular multi-theoretical approaches. In this paper, I describe a developing theory and the work for which it is intended.

Background

In the Learning Through Activity (LTA) research program, we use multiple theories. In particular, we use sociocultural theories to think about cultural factors and the role of artifacts, social theories to think about the norms negotiated in situations of learning and teaching, and cognitive (constructivist) theories to think about the development of particular concepts. The latter has been our primary theoretical tool and the theoretical area to which we have been contributing.

Sociocultural theory views all knowledge as socially constructed. Knowledge development proceeds from a social level to an individual level through a process termed "internalization" (Vygotsky, 1978). Bereiter (1985) wrote, "How does internalization take place? It is evident from Luria's first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved (p. 206). My colleagues and I see constructivism as a theory that has the potential to explicate internalization.

Constructivism, particularly the work of Piaget (1985), is a major theory of learning and has been the basis of important research on the learning of mathematics (Steffe & Kieren, 1994). However, Piaget's work has not had a comparable effect on mathematics pedagogy.[1] DiSessa and Cobb (2004) observed, "Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to

[1] I use "pedagogy" to refer to all contributions to instruction, instructional design, instructional planning, and teaching.

do so deeply and effectively" (p. 81). We believe that it is Piaget's later work on reflective abstraction, rather than his earlier work on equilibration, that has the potential to be the basis for pedagogical theory development.

The goal of our empirical and theoretical work is to elaborate an integrated theory of mathematics learning and teaching. This involves articulating a theory of conceptual learning that is useful for orienting mathematics pedagogy and building on that theory to explicate mathematics instructional design and teaching. Towards this end and grounded in our empirical research, we have engaged in explicating reflective abstraction for mathematical concepts and developing a design approach for fostering reflective abstraction. This empirical and theoretical work has focused on the learning and teaching of mathematical concepts (as opposed to problem solving or other areas of mathematics learning), and on the generation of hypothetical learning trajectories (Simon, 1995), including the design of task sequences.

The term "mathematical concept" is an underspecified construct. Because it is central to our empirical and theoretical work, I have characterized the construct for research and development purposes (Simon, 2017). I refer to one aspect of that characterization here:

> A mathematical concept is a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship.

Elaborating reflective abstraction

Outline of the theory[2]

One challenge that we accepted in explaining the development of new mathematical concepts was that the explanation must account for building more advanced concepts from prior concepts. Thus, we endeavored to describe a recursive structure in which the result of conceptual development at one level serves as a building block of a concept at the next level. Piaget (1980, p. 90) described reflective abstraction as a "coordination of actions." We built on this idea in the following way.

1. We specify a concept as a complex of a goal and an action (represented as G_n-A_n) constructed through reflective abstraction. We represent the prior concepts of the learner as G_0-A_0 and the concept whose development we are attempting to explain as G_1-A_1.
2. The learning process begins with the learner setting a new goal (generally in response to a mathematical task) and calling on a sequence of *available* actions to achieve that goal. This sequence of available actions is what we call an activity, represented as ($A_{0a} \rightarrow A_{0b} \rightarrow A_{0c}$). An activity is the precursor to a new concept.
3. The actions that are part of the activity do not exist in isolation. Each is part of an existing concept (e.g., G_{0a}-A_{0a}), and each is called upon, because the goal of that existing concept (e.g., G_{0a}) is a subgoal of the activity used to solve the task. Thus, the set of actions that make up the activity are part of a set of concepts that are activated to achieve the goal (solution of the task). Thus, whereas Piaget defined reflective abstraction as a coordination

[2] This description has been abbreviated. In particular, there is no discussion of the stages of concept development. For a full treatment of reflective abstraction and discussion of the stages, see Simon, Placa and Avitzur (2016).

of actions, we assert that this coordination takes place in the context of a coordination of concepts. This point is important, because it allows us to explain how concepts build on concepts recursively.

4. The coordination of actions results in a new, higher-level action linked to a goal (G_1-A_1). Reflective abstraction results in a learned anticipation. That is, the learner can now solve the task without going through the sequence of actions that was originally necessary, but rather by enacting the higher-level action. (This will be demonstrated in the example.)

Example from data

The following example is taken from Simon, Placa and Avitzur (2016). The data derived from a single-subject teaching experiment focusing on learning fraction concepts. Kylie was 10 years old at the time of the study. "R" refers to the researcher, Simon, and "K" to Kylie.

> In this example, Kylie is developing an abstraction of recursive partitioning (i.e., a unit fraction of a unit fraction). Hackenberg and Tillema (2009) defined recursive partitioning as "partitioning a partition in service of a non-partitioning goal, such as determining the size of 1/3 of 1/5 of one yard in relation to the whole yard" (p. 2).
>
> • ***Task 4.1:* [*Using JavaBars, R draws a bar on the screen.*] *This is one third of a unit. Make a bar that is one sixth of a unit.*** Kylie made it clear that she did not know how to just "cut up" the bar on the screen. She made the whole by iterating the third three times and then cut the first third in half. She indicated immediately that one of the small pieces is one sixth.
>
> • ***Task 4.2: This is one fifth of a unit. Make one tenth of a unit.*** Kylie used the same process. She iterated the one fifth 5 times to make the whole and then partitioned the first fifth into two subparts. She reported, "Here, you have one tenth of a unit."
>
> • ***Task 4.3: This is one third of a unit. Make one ninth of a unit.*** This time Kylie immediately divided the one-third bar into three pieces (without iterating to make the whole).
>
> *K:* One of those is one ninth.
>
> *R:* How do you know?
>
> *K:* Because, um. How many times does three go into nine? . . . Three times. And it's one third! So. Three times three is nine, and one of—if you cut it up into thirds again. That is, um. . . . And you take one, it would be . . . one third. . . . But that's really one ninth of a unit.
>
> Kylie seemed to indicate that she thought about what number of parts would iterate three times to the whole. She therefore knew that one third of the one third would iterate nine times in the whole.
>
> • ***Task 4.4: This is one fifth of a unit. Make one twentieth of a unit.*** She immediately cut the fifth into four pieces. She went on to complete two more tasks in this way. (pp. 77-78)

In this example, Kylie developed an abstraction that taking 1/*m* of 1/*n* creates 1/*mn*, that is, a part that when iterated *mn* times recreates the unit. The example illustrates several aspects of the theory discussed above. Initially, Kylie had no way to think about making 1/*mn* by simply partitioning 1/*n*. However, she did have knowledge that allowed her to make 1/*mn*. That is, she had concepts that she

was able to call on producing a sequence of actions (an activity) to achieve her goal. She conceptualized 1/*n* of a unit as a part that can be iterated *n* times to make the unit. She also knew that she could partition the unit to make any unit fraction. In Task 4.1, she called on these two concepts. She sequentially iterated the original part, 1/3, three times to make the unit and then partitioned the unit to make 1/6 of a unit. However, because the unit bar that she created was already partitioned into three parts, she called on her concept of partitive division (6 divided by 3) to determine how many times to partition each of the three parts. Thus, Kylie created an activity made up of three actions involving three extant concepts: iterating the part to make the unit, using partitive division to determine the number of subparts per part, partitioning a subpart.

The activity Kylie employed for Tasks 4.1 and 4.2 led to the abstraction that was apparent in Task 4.3 and beyond. In Task 4.3, Kylie no longer needed to go through the sequence of actions used in the preceding tasks. The actions that made up the activity were now coordinated into a single higher-level action. She knew immediately in Task 4.4 that cutting 1/5 into 4 subparts allows a subpart to iterate 5x4 times to the whole.[3] That is, she had developed an anticipation that the denominator of the part has a multiplicative effect on the number of times the subpart iterates to the unit.

Building a pedagogical theory: the LTA instructional approach

As stated our goal was to generate a theory of mathematics concept learning that can serve as a basis for mathematics pedagogy. In this section, I describe how we have built an instructional design approach on the basis of the explication of reflective abstraction, discussed above.

The first two steps in our design of instructional sequences are part of various design approaches. In Step 1, we specify the prior knowledge needed to engage with the sequence. This is particularly important in our design approach, because it identifies the concepts that students can call upon as components of their activity. In Step 2, we identify specific learning goals for the students, that is, we articulate the particular abstractions we intend to promote.

Step 3 makes direct use of our explication of reflective abstraction. In this step, we specify an activity (sequence of concepts/actions) available to the (actual or hypothetical) students that could serve as the raw material for the intended abstraction. There are two requirements here. First, the students must be able to call on the activity. Second, the researchers/designers must be able to describe how the students could come to the abstraction as a result of engaging in the activity. In our example above, the activity would be iterating the part to make the unit and then partitioning the unit by subdividing each part – the number of partitions determined through partitive division.

Step 4 involves generating a sequence of tasks designed to elicit the activity specified in Step 3 (in our example Tasks 4.1-4.4) and promote reflective abstraction. Sometimes the tasks that promote the activity are sufficient as in the example presented (by the third task, Kylie had made the abstraction). In some cases, subsequent tasks are created that restrict the student's ability to carry out the sequence of actions in the activity – prompting the students to use developing anticipations

[3] Although Kylie's justification was given for Task 4.3, I refer to the numbers from Task 4.4, because in Task 4.3, the use of 1/3 as both the fraction of the unit and the fraction of the part makes articulation of the ideas confusing.

if available. For example, in our work on promoting a reinvention/abstraction of the multiplication of fractions algorithm, Kylie had developed a reasoned activity beginning with thinking through the effect of the denominator of the multiplier on the denominator of the multiplicand. Her reasoning then included the numerator of the multiplicand and finally the numerator of the multiplier. Each step was dependent on the prior one. To promote and elicit use of a developing anticipation, we gave her tasks with the denominators hidden and asked for the numerator of the product. She was only able to do these tasks when she had developed sufficient anticipation of the effect of the numerator of the multiplicand in the context of her activity. In other cases, particular tasks are sequenced to increase the probability that students will attend to the commonality in their activity. In Simon et al (2010), Erin was reinventing/abstracting a common-denominator algorithm for multiplication of fractions. She had developed diagram solutions to division tasks whose dividend and divisor had common denominators. She had also developed the ability to talk through a diagram solution (without drawing). For example, she was able to talk through 37/31 ÷ 17/31. However, she also made it clear that without talking through the solution, she could not come up with the quotient. At this point, I gave her consecutively two tasks with the same pair of numerators, but different common denominators (e.g., 7/167 ÷ 2/167 and 7/103 ÷ 2/103). Although she needed to talk through the first, she was able to give the answer immediately to the second. Not only that, she was able to elegantly explain the abstraction she had made and do subsequent tasks (involving common denominators) simply by dividing the numerators.

I have highlighted the first four steps. However, as in other approaches to instruction, these steps might be followed by symbolization, introduction of vocabulary, and institutionalization of ideas.[4]

A couple of clarifications are in order. First, when we refer to a *task*, it includes the resources available to the students for solving it. Second, the goal of our research is to specify a sequence of tasks that can promote a particular abstraction. Thus, the sequence should work without the instructor asking leading questions, telling or showing solutions, or giving hints or suggestions. Also, the sequence should allow students to make the abstraction without needing to hear the solutions of others. This does not mean that there is not a role for teachers. The teacher is important in developing norms for the mathematical work, promoting justification at appropriate times, introducing symbols and vocabulary, and leading discussions that institutionalize the learned abstraction. Also, teachers should be able to monitor student progress and modify task sequences in response to student progress.

Affordances of the LTA instructional approach

To provide a context for discussing the affordances of the LTA instructional approach, I first discuss a commonly used and important approach to instruction, a problem-solving approach. I will then highlight some of the contrasts and complementarities between the LTA approach and a problem-solving approach.[5]

[4] See also Simon (2016).

[5] Discussion of contrast with Harel's DNR can be found in Simon (2013).

Although there is no single problem-solving approach, I will discuss some typical features. One of the main strengths of a problem-solving approach is the engagement of students in the critical activity of mathematical problem solving, attacking a problem for which the student has no solution at the outset. I will not highlight here the abilities and dispositions that can be developed through regular engagement in problem solving; these have been well documented. Rather, I will focus on one feature of this approach that provides a contrast with the LTA approach. Problem solving is by definition uncertain. There is no assurance that those who engage in solving an authentic problem will succeed in solving it. When a diverse set of students in a mathematics class attempt to solve a problem, it is likely that only those who are the stronger problem solvers and who have the more powerful mathematical concepts will succeed in solving the problem. The other students must try to *follow* the reasoning (in small groups or whole class discussions) of their more able peers.

The LTA approach is intended to provide a complimentary approach, one in which learning of a concept is not dependent on the uncertainty inherent in attempting to solve authentic problems. If an LTA sequence is designed effectively, students should be successful in solving every task in the sequence.[6] In the LTA approach the learning (the new abstraction) is not the ability to solve the task. Rather it is the insight that is gleaned through the students' solutions to tasks using available activity. In our example, Kylie was successful in solving each of the four tasks. She was not trying to learn anything – just to solve the tasks presented. However, by the third task, she understood something that she had not understood at the beginning of the instructional sequence.

We conceive of the LTA instructional approach as a technology for engendering the construction of particular mathematical concepts on the basis of particular prior knowledge. I call attention to two potential contributions of this approach:

1. For concepts that tend to be difficult to teach and learn, the LTA approach provides a technology for building up those understandings (promoting particular abstractions).
2. For students who have previously been unsuccessful in learning mathematical concepts, it provides a specific methodology for building up their conceptual foundation.

Affordances of the LTA theory for research and development

The elaboration of reflective abstraction discussed above provides a lens for looking at conceptual learning in different situations, not just in situations designed using the LTA instructional approach. For example, the LTA elaboration of reflective abstraction could be used to understand conceptual learning in the context of a problem-solving approach to instruction. How do we explain the success or failure of a lesson for particular students? Of course, the students' prior knowledge and problem solving skills are important. But how can we consider the usefulness of the problem or problems? The LTA elaboration of reflective abstraction allows for analysis of the students' activity and its relationship to the abstraction that they make.

[6] Of course, there is no curriculum that works flawlessly for all students. The issue is not whether we can create a sequence in which every student can solve every task. Rather the issue is that we intentionally create tasks that we predict students will be able to solve. This is in contrast to putting them in a problem-solving situation.

In Simon (1995), I postulated the construct of a *hypothetical learning trajectory (HLT)*. An HLT can describe a hypothesized trajectory for a single lesson or for a sequence of concepts in a conceptual area (also referred to as a "learning progression"). Learning trajectories has become a hot area for mathematics education researchers. HLTs are not just a series of conceptual steps through which learners progress, they involve articulation of sequences of learning situations and hypotheses of how these situations will be used by the students to learn the target concepts.

The LTA integrated theory of teaching and learning can provide the framework for learning trajectories in various conceptual areas. As a framework, it provides a basis for selecting and sequencing tasks and for hypothesizing how the students will abstract from their activity in working with those tasks. In our current project, we designed, enacted, and modified in teaching experiments nine trajectories for different concepts involving fractions.[7] This work has been grounded in and contributed to the LTA theory of teaching and learning. Also essential to the design and modification of the trajectories has been (but beyond the scope of this short paper) our work on reversibility (Simon, Kara, Placa, & Sandir, 2016) and on the stages of concept development (Simon, Placa, & Avitzur, 2016). Both build on the theory described in this paper.

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[7] I am currently planning the subsequent project in which we will use these trajectories, which were developed primarily in single subject teaching experiments, as the bases for creating and studying curricular interventions for whole-class lessons.

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The Thematic Working Group 17 of CERME 10: Grouped topics of TW17 of CERME10

Transfer of theory elements: The tension between home-grown and borrowed theories

'Grundvorstellungen' and their application to the concept of extreme points

Anna-Katharina Roos

Mathematics teachers' re-sourcing and use of social media: can the 'prosumer' concept convey what's going on?

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Analyzing verbal interactions in mathematics classroom: Connecting two different research fields via a methodological tool

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An analytical tool for identifying what works with children with mathematics learning difficulties

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When the theoretical model does not fit our data: A process of adaptation of the Cognitive Demand Model

Clara Benedicto, Angel Gutiérrez and Adela Jaime

‘Grundvorstellungen’ and their application to the concept of extreme points

Anna-Katharina Roos

University of Würzburg, Germany; anna-katharina.roos@mathematik.uni-wuerzburg.de

In this article, we discuss the German construct of Grundvorstellungen and highlight the connection with mathematical aspects. After recalling the definition of these terms, we compare it with similar notions used in the literature such as “concept image/concept definition”, and “metaphors”, pointing out some common features and differences. Moreover, we use Grundvorstellungen and aspects for the analysis of the concept of extreme points. Finally, we briefly sketch how it may be used for the discovery and interpretation of students’ misconceptions.

Keywords: Grundvorstellung, calculus, extreme point, misconception.

Motivation

There exists a vast literature on problems, mistakes, and misconceptions of mathematical topics in school. In order to discuss these misconceptions in connection with “correct” ideas, several approaches have been proposed, such as the construct of so-called “Grundvorstellung” (or GV, for short), which is mainly used in German literature (e.g. vom Hofe, 1995; Greefrath et al., 2016b). In this article, we briefly recall this construct and compare it with similar constructs appearing in the literature. In particular, we show how the GV construct can be applied to extreme points, with an emphasis on the transition from school to university. To this end, we give a short overview over GVs for extreme points, which helps us to classify students’ problems concerning extreme points.

Theoretical background

Let us first explain the notion of *Grundvorstellungen.* Grundvorstellungen originate from the traditional German didactical approach of subject-matter didactics and have a long tradition in German mathematics education (see vom Hofe, 1995). Vom Hofe extended the subject matter didactics approach by his understanding of Grundvorstellungen: he suggested to additionally take into account the learners’ perspectives (see Straesser, 2014, p. 569). As a consequence, he distinguishes between GVs from a normative viewpoint, on the one hand, and GVs from a descriptive viewpoint, on the other hand. Vom Hofe (1992, p. 347) recalls the main characteristics of this construct: It gives meaning to a concept by connecting it with already known facts or experiences. It brings along the development of corresponding representations and the ability to use the concept. Scrutiny reveals that it is useful to introduce different types of GVs, which read as follows: A *primary GV* is a GV which gives meaning to a concept by connecting it with concrete experiences and objects from everyday life. A *secondary GV* is a GV which gives meaning to a concept by connecting it with ideas and representations of other mathematical concepts (see e.g. vom Hofe & Blum, 2016, p. 234).

To clarify the connection between GVs and the mathematical content of a concept, Greefrath et al. introduce a distinction between aspects and GVs of a mathematical concept:

“A Grundvorstellung of a mathematical concept is a conceptual interpretation that gives meaning to it.” (Greefrath et al., 2016a, p. 101). However, an *aspect* of a mathematical concept is a part of this

concept, which refers to its mathematical content, like its definition, theorems related to the concept, its properties, or connections to other concepts. It is a part of this concept, which can be used to characterize it (see Greefrath et al., 2016a, p. 101).

Another distinction is given by Greefrath et al. (2016b, pp. 18–19) who introduce the notions *universal* and *individual* GVs: A *universal GV* works as a normative guidance for teachers and answers the question what students should think of a concept, i.e. a universal GV refers to vom Hofe's normative viewpoint. Universal GVs refer to mathematical aspects. However, there is no one-to-one correspondence between GVs and aspects: One GV can refer to several aspects, and one aspect can be the basis for several GVs (see ibid., p. 17–18). On the other hand, an *individual GV* answers the question what a special student actually thinks of a concept, i.e. an individual GV is related to vom Hofe's descriptive viewpoint. Individual GVs can be detected by watching students solving problems or by analyzing their explanations with regard to certain questions. Therefore it is a GV that an individual person possesses regarding a certain concept.

For our purposes–to find reasons for students' misconceptions–it is necessary to introduce yet another kind of GV which we call *partial GV*. A *partial GV* is an idea that gives meaning to a concept in a limited context. It also goes back to mathematical aspects of a concept, and thus to a normative viewpoint. However, it loses its generality through restrictive premises. Depending on the goal, it could also serve as a (preliminary) educational objective. An example is the idea "multiplication makes bigger" which is correct only in certain cases, e.g. the natural numbers (larger than 1). To highlight the differences between these types of GVs, we want to add the following point to the description of a universal GV: "Universal" means that it works in every given context. Consequently, we categorize an individual GV either as a universal GV, as a partial GV, or as a part of them. The following graphic puts the characterization of the GVs in a nutshell:

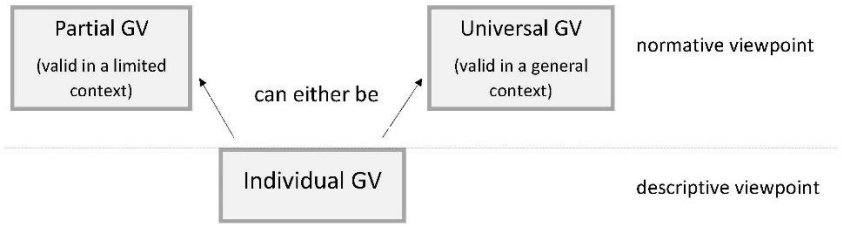


Figure 1: Connection of universal, partial and individual GVs

The notion Grundvorstellung could be translated, loosely speaking, as "basic idea", "basic notion" or "basic mental model" (see vom Hofe & Blum, 2016, p. 226; Greefrath et al. 2016a, p. 101). However, we decided not to translate the notion to not mix it up with other (mainly in Germany used) constructs like fundamental ideas, universal ideas, central ideas, big ideas, leading ideas (for further details see Vohns, 2016).

GVs are especially interesting regarding misconceptions. These "erroneous conceptions" have different names (preconceptions, alternative conceptions, misconceptions) and are defined in various ways in the literature (see e.g. Leinhardt et al., 1990; Hammer, 1996). We just recall the main facts: They are a repeatable, robust, "well-formulated system of ideas" (Leinhardt et al., 1990, p. 5) and they are cause for errors, although they do not need to constitute a complete theory (ibid., p. 5–6). If one takes a closer look at these, one can categorize some as so called epistemological

obstacles (see e.g. Brousseau, 1989), or overgeneralizations, like in our background as partial GVs used in the wrong context. Vom Hofe (1996, p. 259) describes the importance of GVs regarding misconceptions as follows: Firstly, GVs as a normative guidance should help during the learning process so that individual ideas become individual GVs and not misconceptions (see vom Hofe, 1996, p. 259). In this respect, GVs work as a *prevention* of misconceptions. Secondly, GVs should work as a plausible alternative to already existing misconceptions. If learners already have misconceptions they need not just be dissatisfied with the former idea, but also need to be given an intelligible, plausible and fruitful alternative by the teacher (see Posner et al., 1982, p. 214). In this respect, GVs work as a remedy for misconceptions.

The purpose of this article is to give a survey on universal and partial GVs at university level that refer to aspects of the concept of extreme points. To begin with, let us briefly recall the role of GVs in the existing literature.

GVs and concept image/concept definition

One of the most important and popular works on the properties of concept image, concept definition, and their distinction is due to Tall and Vinner (1981). A *concept definition* consists of the words used to specify a concept. The definition of a concept given by the mathematical community is called *formal concept definition*, whereas the definition given by an individual is called a *personal concept definition*. A *concept image* consists of all non-verbal connections to and connotations inferred by an individual regarding a certain concept. These connections and connotations may include all kinds of representations. There exists some literature describing similarities and differences between concept image/concept definition and GVs: vom Hofe and Blum (2016, p. 237) state that GVs refer to the concept image from both a normative and descriptive level. Greefrath et al. write:

“A concept image may contain several individual Grundvorstellungen that conceptualize different perspectives of that concept. Individual Grundvorstellungen are central components of a valid concept image[…].” (Greefrath et al., 2016a, p. 103)

However, for some learners, also misconceptions may be part of their concept image. The individual GVs can either be partial GVs or universal GVs. As Greefrath et al. write:

“These Grundvorstellungen give meaning to mathematical concepts that may be studied with respect to various aspects. Each of these aspects may be expressed with one of the various [formal] concept definitions that one reads in textbooks. Thus, a [formal] concept definition is a specific realization of an aspect.” (Greefrath et al., 2016a, p. 103)

The personal concept definition could either be similar to the formal concept definition or differ from it. It also has a relationship to the concept image.

GVs and metaphors

Apart from concept image/concept definition tools, metaphors have also been used to describe mathematical cognitive processes (see Lakoff & Núñez, 1997, 2000). First, we recall some facts about the concept of metaphors used by Lakoff and Núñez. They analyze the (conceptual) structure of mathematics and therefore use results of metaphor theory of cognitive linguistics. This theory is about how mathematics is constructed. To describe mathematics, they distinguish several kinds of

metaphors, the two crucial ones being *grounding metaphors* and *linking metaphors*, which are defined as follows:

"Grounding metaphors ground mathematical ideas in everyday experience." In addition, "[...] linking metaphors allow us to link one branch of mathematics to another. For example, when we metaphorically understand numbers as points on a line, we are linking arithmetic and geometry." (Lakoff & Núñez, 1997, p. 34)

Let us now point out some similarities and differences between the concepts of GV and metaphor. Both GVs and metaphors reveal several perspectives of one concept through a detailed analysis of it. GVs and metaphors suggest using this analysis for educational purposes. Grounding metaphors resemble primary GVs, since they connect everyday life experiences with mathematics. Linking metaphors resemble secondary GVs, because they connect different domains within mathematics. However, a fundamental difference is the aim of these theories: finding of GVs through a subject matter analysis aims at making the most important aspects of a concept comprehensible and giving them meaning, respectively. Referring to metaphors in the sense of Lakoff and Núñez (1997, p. 31), this educational aim is named "peripheral" for the theory of metaphors. The discussion about metaphors tends more toward revealing the structure of mathematics and unconscious connections to non-mathematical concepts as mentioned above. Metaphors are used to explain our linguistic vocabulary when talking about mathematical contents.

There are also connections with other theories like the theory of conceptual fields of Vergnaud (1996), which also works as a framework for organizing didactic situations. For lack of space we do not go into detail, but just remark that an exact analogue to the GV construct does not exist in the literature, but constructs which exhibit some similarities with GVs.

Aspects and GVs of extreme points

After defining the concept of GVs and comparing it with other concepts from the literature, we are now going to use it for the concept of extreme point, which is the major topic of this article. The following GVs were identified by both discussions with lecturers of analysis courses and a subject matter analysis of the concept of extreme point.

Aspects of extreme points

We will discuss three aspects of the concept of a (local) extreme point:

<u>The aspect of "largest/smallest value"</u>

Extreme points are the points (x,y) with the largest/smallest y-value with respect to a neighbourhood of x.

This aspect refers directly to the definition of an extreme point. It illustrates the connection of an extreme point (of a graph of a function) with the maximum/minimum of a (totally ordered) set: extreme points are the points, whose *y*-values coincide with the maximum/minimum of the range (on a given domain). Already children have some experience with this aspect.

Non-mathematical examples:

- Searching for the highest point (of a mountain)

- Searching for the best object of category (fastest car, most expensive house, highest building)

Mathematical examples:

- Interpretation and discussion of graphs: where is the highest/lowest point?

The aspect of "largest/smallest value" is the most fundamental one for the concept of extreme point.

The aspect of "derivative zero"

Extreme points are located at points where the derivative becomes zero.

In this connection, however, one has to be careful and take into account two important points. First, there are points, which are not extreme points, although the derivative becomes zero at these points. Second, since this aspect refers to the necessary condition for the existence of (local) extrema, one has to impose two additional premises: the point must be an interior point of the domain, and differentiability of the function is required. By dropping one of these premises, one can easily find examples of extreme points where the derivative does not become zero (e.g. boundary points of an interval).

Non-mathematical examples:

- From a physical viewpoint: searching for the points where the velocity becomes zero.

Mathematical examples:

- The algorithm for the identification of extreme points most frequently taught at school: find the zeros of the first derivative. Afterwards, check the sign of the second derivative at these zeros to detect maxima and minima.

The aspect of "change of monotonicity"

Extreme points are located at points where the sense of monotonicity of a function changes, i.e., from increasing to decreasing or vice versa.

It is, in fact, easy to see that a change of monotonicity always implies the existence of an extreme point. The converse, however, is false: not every extreme point induces a change of monotonicity (consider again extreme points at the boundary of an interval).

Non-mathematical examples:

Change of the direction of movement:

- Activities: mountain climbing (change of uphill to downhill), bike riding (change from pedaling to idling).

Mathematical examples:

- At school (premise: differentiable functions): a table of sign changes for the first derivative (distinguishing extreme points from saddle points).

GVs of extreme points

Let us now study three GVs of the concept of (local) extreme points of a real function (defined on a nondegenerate interval). The first GV is universal, the second and third are partial.

Universal GV

The GV of "largest/smallest value"

This GV refers directly to the aspect of largest/smallest value and may be interpreted in a concrete geometrical way: extreme points are the points with the largest/smallest *y*-value with respect to a certain neighbourhood. The students should:

- Identify extreme points through analyzing *y*-values of points of a function graph algebraically or graphically.
- Demonstrate the connection between a maximum/minimum of a (totally ordered) set and an extreme point of a function by projecting (pieces of) the graph onto the *y*-axis.

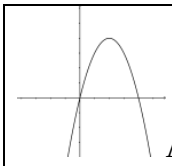
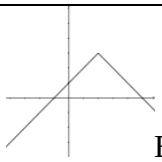
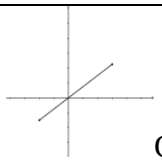
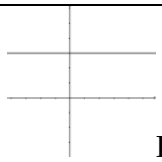
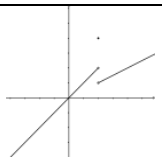
 A	 B	 C	 D	 E
Extrema as hill/valley (differentiable function)	Extrema as hill/valley (not differentiable function)	Extrema at the boundaries of an interval	Extrema of a constant function	Extrema at a discontinuity point

Table 1: Subcategories of extreme points

Once the students understand this GV correctly, they can specify different subcategories of extreme points, such as those sketched in the synoptic Table 1.

This GV is universal and should thus be emphasized when introducing the concept of extreme point. For applications, however, it is not that useful: it would take an infinite long time to analyze every point with regard to the value of its *y*-component, because the domain of definition is a continuum. However, there exist another two GVs, which are more helpful in applications.

Partial GVs

The GV of "slope zero"

This GV reflects the fact that the graph of a function has slope zero at an extreme point. Since this idea is intimately related to the aspect of "derivative zero", it should be assumed that the function is differentiable at the (interior) point in question. As this is usually the case for functions dealt with at school, it is mainly consolidated as a consequence of math classes. The GV of "slope zero" calls upon the GVs of the derivative of a function like the GVs "Tangent slope" or "Amplification factor" (see Greefrath et al., 2016a, pp. 106–113):

- Tangent slope: slope zero entails the existence of a horizontal tangent at the extreme point in question that approximates the graph locally. This means that the graph is also almost horizontal.
- Amplification factor: slope zero implies here that a change of the independent variable *x* leads to almost no change in the dependent variable *y*.

The reason, why this is just a partial GV is the fact, that neither the argument "extreme point implies slope zero" nor the converse argument "slope zero implies extreme point" is true.

The GV "change of monotonicity"

If you look at the graph of a function and see that it is strictly increasing up to a certain point x_0, and then strictly decreasing, you could intuitively argue that x_0 is a (local) maximal point. This idea exhibits an analogy to the curvature of a graph at inflection points; if a function changes at some point from left curvature to right curvature, or vice versa, this is an inflection point. As far as differentiable functions are concerned, passing from an increasing piece to a decreasing piece can be interpreted as the transition of the first derivative from the upper half-plane to the lower half-plane. The GV of "change of monotonicity" goes back to a specific "dynamical" experience of the students: if you regard the graph as trajectory of a moving particle, the movement changes in the *y*-direction precisely at minimal or maximal points. The students should:

- Partition the graph of a function into monotonically increasing and decreasing pieces and identify extreme points as boundary points of the partition intervals.

The reason, why this is just a partial GV is the fact that it works only "in one direction", because the logical converse is again false: not every extreme point leads to a change of monotonicity.

Discussion

These GVs play an important role when analyzing mistakes of students in their first semester at university concerning the concept of extreme point. The partial GVs especially can lead to mistakes when their restrictive premises are not considered. Our study aims, on the one hand, at finding mistakes and misconceptions of mathematics students after their first analysis course and, on the other hand, at giving insight into possible reasons for these mistakes. It turns out that the concept of GVs helps to find normative education objectives of extreme points, and it supplies–used in a descriptive way–an orientation of what ideas students have of extreme points.

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Mathematics teachers' re-sourcing and use of social media: Can the 'prosumer' concept convey what's going on?

Yvonne Liljekvist

Karlstad University, Department of Mathematics and Computer Science, Sweden;
yvonne.liljekvist@kau.se

Keywords: Mathematics teachers, prosumer, social media, theoretical discussion.

Poster summary

The interrelated roles of social media as both a platform and a phenomenon of interactivity, invite teachers to produce and consume knowledge of teaching and learning mathematics (e.g., Liljekvist, 2016; van Bommel & Liljekvist, 2015, 2016). The availability and user-friendliness of the social media platform alter the behaviour of the mathematics teacher, and 'stories' of the good mathematics teaching are made (e.g. kinds of curricular material, or kinds of questions raised, see e.g., Liljekvist (2016)). Analysing the affordances of this new environment is necessary to understand how the subject didactical discourse on learning and teaching is simultaneously constructed and consumed in mathematics teachers' digitalized every-day practice. It is a matter of probing the characteristics of the interaction, that is, the ways in which the activity on the Internet supports knowledge development and re-sourcing in mathematics teaching (cf. Liljekvist, 2016; Ruthven, 2016).

The primary aim of this poster is to initiate a discussion in the TWG17 that elaborates on theoretical constructs that may be fruitful in the research of mathematics teachers' digitally extended every-day practice and collaboration. This arena for teacher learning and collaboration is under-researched (see, e.g., Robutti, Cusi, Clark-Wilson, Jaworski, Chapman, Esteley, Gnoos, Isoda, & Joubert, 2016)).

The prosumer concept, that is, people as producers and consumers of products and services (cf. Beighton, 2016; Ritzer, Dean, & Jurgenson, 2012; Zajc, 2015) shows some possibilities to theoretically model mathematics teachers' re-sourcing on social media as it centres on the phenomenon per se (i.e., producing and consuming 'value' for the user). Thus it is closely tied to the raison d'être of social media (Zajc, 2015) and mathematics teachers' activities there (van Bommel & Liljekvist, 2015, 2016). However, the concept needs to be operationalized in an educational setting and in a mathematical discourse in order to have sufficient explanatory power for our purposes. Here are some examples; In business and sociology, the driving forces for investigating 'prosuming' is to understand consumers' behaviour (e.g., Ritzer et al. 2012; Zajc, 2015), but in educational research mathematics teachers' performance, for instance, as a learner and as a colleague is of interest (e.g., Liljekvist, 2016; Ruthven, 2016; van Bommel & Liljekvist, 2015, 2016). Further, Beighton (2016) discusses in his article how the prosumeristical behaviour can also work as a tool for control, where creativity and knowledge development, and professional learning may not be supported. This aspect of mathematics teachers' online activities is relevant, as it, for instance, may explain some of the quality problems in the curricular material shared (e.g., Liljekvist, 2016).

Mathematics teachers nowadays use social media to re-source and collaborate. This is an arena where the every-day practice of subject didactics is made (Liljekvist, van Bommel, & Olin-Scheller, 2017). Analysing this activity is to understand teachers as both producers and consumers of subject didactical contributions and of peer-learning. The core research question, then, is how we can theoretically describe mathematics teachers' simultaneous processes of producing and consuming subject didactical knowledge on social media.

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Access to the poster: www.kau.se/files/2017-04/CERME_YL.pdf

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Analyzing verbal interactions in mathematics classroom: Connecting two different research fields via a methodological tool

Mariam Haspekian[1], Eric Roditi[2]

University of Paris Descartes, Sorbonne Paris Cité, France,

[1]mariam.haspekian@parisdescartes.fr, [2]eric.roditi@parisdescartes.fr

This paper presents a small case of junction of two research fields that remained, until recently, relatively cut off from each other. Using concepts of the Assessment field, specified in a didactic approach, we develop and test a methodological tool on the analysis of students-teacher interactions in mathematics classroom. We discuss then its potential in the Assessment field. This illustrates a way of locally connect research areas, via a shared methodological tool.

Keywords: Mathematics teaching practices, evaluation, interactions, formative regulation.

Introduction

Initially inscribed in a particular research area of the didactics of mathematics field, the Double Approach (didactic and ergonomic) theory (Robert & Rogalski, 2005), our study aims at investigating the teacher's coherence in the processes of information-adaptation, when adjusting the situations, making decisions on the spot. Research on Assessment, a field traditionally cut from didactic ones, though with some exceptions (Vantourout and Goasdoué, 2014; Veldhuis, 2015), handles these questions with concepts as *formative* assessment or *regulations*. How can we describe these regulations and understand the way interactions guide them?

To study these questions, we thus went beyond the specific fields of didactics theories and carried out a literature review on formative interactions in Assessment, a research field in education, with its own models and conceptual frames[1]. Our review briefly summarized in section 1 shows needs for didactic references on learning and teacher activity, and led us to a theoretical and methodological development (reported in section 2) to support our analyses regarding the teacher-students interactions in mathematics as an adaptive dynamic process. The last part tests potentialities and limits of this tool on a short episode, before coming back to theoretical questions.

Literature in the field of assessment: Needs for a didactic approach

Various theoretical concepts relate to teacher-students interactions, for instance the "diagnosis and responsiveness" of "scaffolding", (Bakker et al., 2015). In Assessment research, they relate to three overlapping concepts we focus on: *feedback, formative assessment (*or *for learning*), and *regulation*.

Theoretical needs for a didactic approach

The teachers' feedback to learners' activities constitute an element of the process of formative evaluation, and research has been deepened to understand their different effects according to their nature. Yet, Crahay (2007) notes that research on this concept is full of opposed results. He proposes to focus on the processual aspect of the link between teachers' reactions and students' learning considered as an activity rather than a product, and taking into account the characteristics of the

[1]This includes research on measurement in education, psychometric and edumetric approaches.

tasks, the school disciplines or the taught objects.

As for the concept of formative assessment, initially restrained to dedicated moments, it has been extended to "*all those activities undertaken by teachers, and/*or *by their students, which provide information to be used as feedback to modify the teaching and learning activities*" (Black & Wiliam (1998, p.10). Research points out a new concept, that of *regulation*, which can be interactive, retro-active or proactive. In this widened sense, the interactive regulations that may happen in didactic situations make a junction between evaluation and teaching-learning situations. This also fits the idea of "whole-class scaffolding" (Smit et al.). Crahay's synthesis (2007) or that of Allal and Motier Lopez (2005) evoke a strong dependence between the formative evaluation and the very organization of teaching, encouraging research to integrate didactic issues (task, disciplinary contents…) in the analyses. More recently, Vantourout and Goasdoué (2014, p.142) claim the need for taking into account both didactical and psychological approaches by arguing that willing to foster learning requires understanding students' cognitive functioning with a task. Although some attempts exist in mathematics education, they stress the lack of research investing in this joint perspective.

The theoretical needs addressed in this review join our didactic concerns, i.e. to pay particular attention to specificities of the taught and evaluated contents by analyzing the teacher-students interactions. In this perspective, the concept of interactive regulation described in research on formative assessment seems an interesting object regarding our didactical concern.

Regulation in formative assessment

Formative assessment research has been driven by two underpinning teaching and learning theories: originally the neo-behaviorism, then the cognitivist one with its strong necessity to consider classroom interactions, particularly real-time regulations. In Allal and Mottier Lopez's synthesis (2005), the concept of regulation is a process referring to the features of the concept of formative evaluation (collect, interpret, decide). Our didactic concerns relate to one of the five distinguished conceptual forms of regulations they bring out from literature: the online regulations resulting from interactions. Their model of regulations includes the situation; the teachers' interventions; and the interactions between students based on a Vygotskian approach using the ZPD concept, which makes again a link with scaffolding. Aligned with this, recent research on assessment for learning (e.g., Pryor & Crossouard, 2008) is referring to Activity theory (Leontiev, 1975/84) in order to consider the formative assessment as a cultural historical activity shaped by teachers' and students' reciprocal acting, and co-determined by the subject and situation, which should not be considered separately.

The Double Approach also borrows from Activity theory, exploited in the field of ergonomics, but with a different unit of analysis, there the term "activity" focuses on that of an individual subject (Leplat, 1997). This didactic approach considers not only the mathematical knowledge to be taught, but also the procedures and the student's activity to solve problems. Also considered as cultural-historical processes, the activity of a subject is here again co-determined by the subject herself and the situation carried out, which includes socio-cultural interactions with other subjects (Robert & Rogalski, 2005). In these interactions, we are interested in how the teacher's activity depends on that of the students; particularly while adapting, adjusting teaching to optimize learning. Activity theory constitutes thus a theoretical link, enabling us to articulate teaching practices and students' learning. The next part clarifies these links and outlines our theoretical and methodological tool.

Connecting with a didactical approach of regulations of students learning

Towards tools for a didactic analysis of the formative regulations

We consider two "sub-activities" of one activity system, reciprocally influencing each other: that of teacher, whose result is a situation for the students, impacting them; that of the students, co-determined by the students and the situation produced by the teacher. The teachers' activity therefore involves two levels: the double regulation, resulting from the activity they address to themselves (to teach) and that resulting from the activity they address to the students (to make learn).

As for Crahay, in the Double Approach, teachers' feedbacks should not be studied independently of student's activity that creates them. From the didactic point of view, the analysis of the students' activity needs to consider the situation at stake, including the task and possible elements of the didactic contract; the students' knowledge and procedures (that are contextual elements), needed to solve the task; and the product (oral or written) of students' activity (answer and any element on how the task has been realized). Aligned with research on assessment, we consider that the teacher's activity in class interactions, is formative when he/she takes information on the students' activity in order to act on the learning. Our analyses of the teacher's regulations therefore need to specify the nature of information she collects and the actions (in Leontiev's sense, 1975) she subsequently carries out to reduce the gap she can observe with her learning objectives.

Our research is therefore guided by the principle of identifying what is the gathered information about: a product (answer, result), a procedure, or a piece of knowledge; and what the feedback directly aims at (again a result, procedure or knowledge). Even if we do not have hierarchical assumptions on these types of intervention as for learning, not all interventions are equivalent. In case of an error for example, some interventions aim at correcting by giving the expected result (this lets the student the responsibility to find the underlying mathematical concept). Other interventions address the underlying procedure or knowledge (conforms or not with what was aimed at). All constitute forms of accompaniment of the student's learning.

Results, procedures, knowledge

Activity theory (Leplat, 1997) distinguishes the result of the activity from the procedure implemented and from the state of knowledge of the subject. This framework adapted to teaching, leads to associate to the student (the subject) a state of knowledge, which allows him to analyze and redefine the task prescribed by the teacher, to implement a procedure leading to an answer, guided by some knowledge (explicit or implicit). This student's activity (generally carried out in thought, but possibly verbal or written) can thus be observable in what it produces: an answer/ a mathematical result... Among the possible feedback of the teacher, one can also distinguish those addressing the result only (indicating for example that the answer is false), or the procedure (indicating for example that the theorem used does not correspond to the hypotheses of the statement), or the student's state of knowledge (indicating for example confusion between a theorem and its reciprocal etc.). The choice of a type of feedback depends on various factors (time, prior explanations/examples, teacher's experience, etc.). In the same way, if the student says he ignores how to apply such theorem, the teacher perceives information on the procedure thought by the student. The possible feedback varies here again. It can remain on the procedure level, for example explain that the rule is not adapted; or change level by giving the solution (or begin the resolution and let the student finish), or it can

approach the student's difficulty by clarifying the mathematical knowledge.

This way of analyzing interactive regulations leads to identify in each student-teacher interaction a couple information-feedback where the information, as well as the feedback, could be associated to a *result* (R), *procedure* (P) or *knowledge* ("*connaissance*", C), leading to 9 possible types (table 1). A qualitative didactic analysis of each interaction in a class session allows reporting the dynamics at play in these interactions by associating each of them to one of the nine possible types of regulation.

Action / Information	Result	Procedure	Connaissance
Result	RR	RP	RC
Procedure	PR	PP	PC
Connaissance	CP	CP	CC

Table 1: Information-Action: 9 possible pairs

In some case, the "coding" could depend on what has been previously done, so that there is a need for interviews to support the coding.

The following section implements this tool to analyze an effective classroom episode.

A minute of verbal interactions in class of mathematics

We use a classroom video, collected from the French research project NEOPRAEVAL on evaluative practices in mathematics to analyze there the students-teacher interactions with the "RPC tool". The extract, situated at the beginning of a one-hour session with Grade 8 (14 y.o.), consists of "Flash", a series of calculations ritual in this class, intended to be treated quickly. A slide titled "Mental calculus" then five calculations are successively displayed 30sec each, during which the students carry out calculations. Their written answers/calculations are neither collected nor looked at by the teacher. After the fifth calculation, the teaching undertakes a collective correction, questioning students and writing the answers herself on the board. Our analysis concerns the collective 1min dialogue-correction of the very first calculation displayed: $\mathbf{3x10^{-2}}$. We aim at identifying verbal interactions that can be interpreted as moments of formative regulation, based on an *a priori* analysis of the task, of the knowledge concerned (presented in Appendix), of possible interventions; and an *a posteriori* analysis of the effective interactions (presented in the next section).

Analysis *a priori* of the teacher's interventions

In situations of interaction, the teacher faces various cases (answers are correct or not, wished or not, expected or surprising…), which could raise various types of regulation (explicitly agree or not, develop the procedure or not, indirectly disapprove by repeating an explanation/ not reacting/ questioning someone else…), depending, among others, on the objectives fixed by the teacher prior to the session and/ or on the spot. In all cases, the intervention is said formative if it enables the student to recognize whether a behavior, answer, is correct or not[2].

For the calculation $3x10^{-2}$, answers *a* or *b*, or even *f* (Annex) would be the acceptable correct ones (expected/not). Answer *a* could bring a simple approval or a development to clarify the procedure, for example by an oral explanation on the product by 10^{-2}, leading 3 at the hundredths digit by a

[2] The exactness of an answer does not prejudge the teacher's reaction. A correct and awaited answer could still draw to repeat the solution, to develop a procedure, take the opportunity to review some concepts, etc.

technique on the rows of the decimal writing, or by using one of the other possible answers *b*, *f*, *g*, *h*, *i*, *j* or *k* as intermediate calculus. As for the answer *b*, it contains already a calculative step of procedure, but here again; the teacher could accept it or add an explanation via the answer *c*, even *d* or *e*. These *a priori* reactions not only depend on the mathematics aimed at but are to be adjusted with the context where the task takes place. As already underlined, neither the task alone, nor the students' answers (correct or not) suffice to explain the feedbacks. Here, the mental calculations context, in the "flash" ritual, requires not to spend too much time on the task; therefore answers *c*, *d*, *e*, or *j*, *k* (fractional answers) are less likely to be acceptable (even if the teacher may want to expose them during the correction). We assume that these answers will lead the teacher to some "negative" feedback, i.e. indicating, by a way or another, that it's not the expected answer in such context.

Analysis of the 1 minute interactions

Table 2 transcribes the turns to speak that occurred during the minute of this correction. The two last columns indicate whether the information (I) brought by the student's relate to result (R), procedure to obtain it (P), or subjacent knowledge ("connaissance", C); the same for the resulting teacher's action (A). The coding actually limits the researchers' inferences by addressing the facts: the nature of the feedback (from a mathematical point of view: a result, a procedure or knowledge). This feedback can be done by addressing the result level, others the procedure, still others the knowledge. When the teacher does not react verbally to an answer[3], we consider this silence an information too about the validity (or not) of the student's answer.

	Turns of speak	**I**	**A**
01	P : so / Camélia ?		
02	Camelia : three times one over ten times ten/	P	
03	P : what is ten to the negative two ?		P
04	Camelia : one over ten times ten	P	
05	P : and we know how to calculate it ?		P
06	E2 : one over three hundreds	R	
07	E3 : what's that ?		
08	P : you don't know how to calculate one over hundred??		R
09	Camelia : Ah yes ///		
10	E2 : it makes one over three hundreds. **P: --**	R	R
11	E3 : it's not possible	R	
12	P : Orlane		R
13	Orlane : one over three hundreds. **P: --**	R	R
14	E5 : three hundreds… one over three hundreds	R	
15	P : there is indeed a negative here [showing the sign of the exponent – 2 on the statement]		P
16	E5 : bah / negative one over three hundreds. **P: --**	R	R
17	E6 : zero point zero three	R	
18	P : Yasmine		R
19	Yasmine : zero point zero three	R	
20	P : [writing on board 0,03 then $10^{-2} = \frac{1}{10 \times 10} = \frac{1}{100} = \frac{1}{10 \times 10} = \frac{1}{100} =$] So ten, negative two/ indeed Camélia / this is one / over ten times ten / so / one over hundred / one hundredth / you know how to write this?		P
21	Camélia : yes		
22	E8 : this is one percent !	R	
23	P : [she writes 0,01 after 1/100] It is 0,01 / so if one asks to make 3 times this/ 0,03		P

Table 2: A minute of verbal interactions

The pairs (I; A) then constitute basic units of formative evaluation and finally, the analysis of the interactions in this extract results in the following table:

[3] We consider only obvious cases (the student action is clearly audible and the teacher takes a time "not paying attention") and we count only utterances directly linked to the task as R, P or C, no other ones (as lines 7 or 21).

Information/Action	Result	Procedure	Connaissance (knowledge)
Result	6	3	0
Procedure	0	2	0
Connaissance	0	0	0

Table 3: Synthesis of the interactions

Several observations can be made: 1) a majority of pairs remain on the same level, with a majority of RR among these. Only 3 feedbacks are changing the student's level of information making it pass from result R to procedure P; 2) Very few student intervention are situated at the P level in this particular case; 3) No interactions from students, nor from teacher, directly address knowledge. Many of these remarks could reasonably be explained by the specific nature of this episode: mental calculation, intended to be already mastered by students, the correction of which should be easy and short. Yet, we can notice a swift along teacher's interactions. After a time respecting the students' "result" level, she switches feedbacks towards procedural level. Indeed, the rare RP events occur all at the end of the exchange (line 15/20/23). This can reflect teacher's up-taking of information from the students' speeches. Seeing numerous mistakes instead of the expected answer, she adapts her feedback by entering into a procedural level. She does not always verbally react when the answer is wrong (see E2; repeated twice; or E3; E5); she possibly prolongs the interaction when the answer is acceptable even if not expected (it is the case for Camelia's answer, but not for the answer "1%" of E8) and writes on the board what is correct and acceptable ("10^{-2}=0,01 and 0,03"). Thus, when Camelia starts an answer that could lead to *d* (fractional expression) or *g* or *j* (decimal expression), she interrupts her to orient the discourse towards the meaning of the powers of 10 (here 10^{-2}) that lead to decimal notations ("0,01"). Referring to our *a priori* analysis, we assume that the teacher thus expects the answer *b* (direct decimal expression) not *f*, nor *i* that have intermediate fractional steps. Yet, the student goes on with fractions. The expected answer turns to be long delayed. Students make the classical mistake which is not picked up by the teacher (E5), then an answer using percentage is expressed, not much taken into account by the teacher, who seems to get eager for the expected decimal. Observing the mistakes and students' difficulties, she reconsiders the change of the power of ten into decimal notation via fraction calculation, in accordance with the procedure suggested by Camelia, thus finally accepting to align with the cognitive path taken by her students.

Discussion and perspectives on didactic regulations

Aiming at analyzing, in a didactic approach, teachers' practices of formative regulation, we reviewed literature, which characterize these practices as both very few and little diversified. We thus looked for a theoretical and methodological tool that enables comparison between teachers and also various contents, in order to agglomerate results from various data and seek correlations. The analysis of the short episode above brings up many questions on the observed session as on the RPC. On the data, it suggests to test longer episodes, other contexts, other mathematical contents, and with other teachers. In the purpose of describing and understanding the teachers' adaptive activities to the students' one, it would be interesting to enrich analyses with various mathematical contents, and various didactical contexts of learning (application of former knowledge, problem solving, institutionalization…). The RPC tool makes possible the determination of trends in the regulating practices (between-variability of teachers or within-variability of teacher). On the theoretical tool, it results in prolonging the use of these RPC tables by devising tables being "average" of many same contex-

tual tables, to characterize the regulation types for a given teacher. One could indeed calculate a table of the variations to the average of the tables corresponding to several sessions for the same teacher (intra-variability, to characterize the regulation practices for a given teacher), or examine the variations to the average for several teachers on a given content (inter-variability to characterize the profession, or how regulating practices depends on the specific knowledge).

Similarly, the other few analyses we have carried out with this tool look quite promising for research on teaching regulations. They confirm the variability of the practices observed in former research. Yet, they also reveal some tendencies: 1) the couples information-feedback observed are not of the same type but distributed among five or six of the nine possible types; 2) these regulations were not numerous in our data; 3) the couples RR are dominant for all the teachers, a result converging with research on evaluative practices quoted in our literature review; 4) the distribution of interactive regulations, for a given type of information (R, P or C) is higher along the diagonal, i.e. pairs RR are more frequent than RP or RC; pairs PP are more frequent than PR or PC. This suggests that teachers generally produce feedbacks on the same level as the students' answers. They rarely turn a R-answer onto a procedure level; or help a student, who is indicating a procedure, to formulate the underlying knowledge C. Therefore, are these specific cases of *regulation* (when the teacher's feedback and received information are not on the same level) fostering the students' learning? And are the ascending ones (RP, RC, PC) more supportive for learning?

In conclusion, the joint perspective taken here led us to a tool to analyze the mechanisms of formative regulation in classroom interactions. If the tool appears fruitful here, could it be applied more largely to other forms of formative assessment in mathematics education? Indeed, "formative" evaluation requires means from the teacher to collect information on the students' learning. Identifying results, procedures or knowledge in interactions could *a priori* be means of taking into account the specificity of the *savoirs* in the forms of formative assessment set-up by the teacher (questioning, discussions, peer/ self-assessments…). Such considerations lead to reflections about the nature of the beginning networking case here. The connection between didactic fields of research and the assessment one is realized here by elaborating a possible common methodological tool that possibly informs both research fields. This case of networking is rather original, although it is justified in Radford's remark: Using the semiosphere's spatial metaphor, theories Ti and Tj can be visualized as being "closer" or "further" depending on their own (Pi, Mi, Qi) and (Pj, Mj, Qj) structures. The connection ck of Ti and Tj requires the identification of research questions Qij (tasks, problems, etc.) that guide the enterprise as well as the building of a new methodology Mij to answer the research questions under consideration. (Radford, 2014, p. 284).

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Appendix: Excerpt of the a priori analysis of the task

Possible answers and procedures: Various categories of correct procedures and answers can be discerned following the knowledge used. **I)** Use of the scientific notation: directly move from this notation to the decimal one: **a.** $3x10^{-2}$ =0,03 (note that the lack of explicit calculation can lead students to reject this procedure by didactic contract effect). **II)** Use of the decimal notation in calculation: **b.**3x0,01=0,03 **III)** Use of the fractional notation: different procedures according to the fractional form applied to ten powers: **c.** $3 \times \frac{1}{10^2} = 3 \times \frac{1}{100} = \frac{3}{100}$ **d.** $3 \times \frac{1}{10 \times 10} = 3 \times \frac{1}{100} = \frac{3}{100}$ **e.** $3 \times \frac{1}{10^2} = 3 \times \frac{1}{10 \times 10} = 3 \times \frac{1}{100} = \frac{3}{100}$. **IV)** Mix fractional/ notation: various procedures depending on the forms applied to ten powers: **f.** $3 \times \frac{1}{10^2} = 3 \times \frac{1}{100} = 3 \times 0{,}01 = 0{,}03$ **g.** $3 \times \frac{1}{10 \times 10} = 3 \times \frac{1}{100} = 3 \times 0{,}01 = 0{,}03$ **h.** $3 \times \frac{1}{10^2} = 3 \times \frac{1}{10 \times 10} = 3 \times \frac{1}{100} = 3 \times 0{,}01 = 0{,}03$ **i.** $3 \times \frac{1}{10^2} = 3 \times \frac{1}{100} = \frac{3}{100} = 0{,}03$ **j.** $3 \times \frac{1}{10 \times 10} = 3 \times \frac{1}{100} = \frac{3}{100} = 0{,}03$ **k.** $3 \times \frac{1}{10^2} = 3 \times \frac{1}{10 \times 10} = 3 \times \frac{1}{100} = \frac{3}{100} = 0{,}03$.

Mathematical knowledge at stake: The task could be legitimately interpreted in multiple ways. Yet, in the context here (a series of quick calculus), short procedures (and short corrections), mentally easy, might be awaited by the teacher, so *a, b, c, f* or *i*. There, the passage from 10 powers to decimal or fractional form, do not explicitly rely on the definition of the exponent. This is important to interpret her feedbacks taking into account professional constraints as the time factor.

Networked theories for a didactical study of communities of mathematics teachers

Jean-Philippe Georget[1] and Hussein Sabra[2]

[1]Normandie univ, Unicaen, Cirnef EA 7454, Caen, France; jean-philippe.georget@unicaen.fr

[2]Université de Reims Champagne Ardenne, Cérep EA 4692, Reims, France; hussein.sabra@univ-reims.fr

Existing research about communities of mathematics teachers mainly draws on sociocultural theories. In complementarity, our project consists in studying more deeply the place of mathematics within communities of teachers. We also define the notion of didactical study of communities of mathematics teachers to focus on a general research project on the place of mathematics in teacher community dynamics. We propose a theoretical model to analyze the problems of the community stemming from the network of several theoretical frameworks and approaches. The theoretical model is particularly based on arguments given by the members of the community and mathematics teaching resources. Two contrasted case studies show the relevance of our focus on mathematics.

Keywords: mathematics teaching, teacher communities, problem of a community, argumentation, mathematics teaching resources.

Introduction

This paper focuses on communities of mathematics teachers. We are particularly interested in the role of mathematics in community dynamics, its epistemological characteristics, its validation modes and its conditions of teaching. We wonder about the singularity of a community of mathematics teachers: in what way the study of a community of teachers in mathematics education is different from the same study in physics education, in geographic education and so on?

In this contribution, we discuss a general question which is the focus of our research project: "What network of theoretical approaches is needed to determine the singularity of communities of mathematics teachers?" We are in an exploratory stage in which we develop theoretical constructs. In the three last CERME workgroups about networks of theoretical approaches, two authors discussed the issue of communities of mathematics: Palmer (2013) and Castela (2013). Palmer notes that Wenger´s (1998) theory of communities of practice does not focus on mathematics education and/or teaching. Castela (2013) stresses the fact that the activity of communities of teachers and different modes of validation of mathematical knowledge in these communities are determined by the institutional conditions and constraints. For instance, teacher's communities do not have the same role as the researcher communities in the knowledge validation process. These researchers mainly draw on sociocultural theories in order to understand some phenomena. In complementarity to these research works, we aim to determine the singularity of communities of mathematics teachers. Our research also contributes to the French ReVEA[1] project, which concerns

[1] ReVEA is a project funded by the French National Agency for Research www.anr-revea.fr/

the interactions between teachers (individual and collective aspects of their work) and their resources in four disciplines, including mathematics.

Before presenting the kind of networking needed to study the communities of mathematics teachers, we shall define the way we use some words in the following. Many researchers use *communities of teachers* to talk about the forms of collaboration between teachers (Robutti et al., 2016). In the same line, we use the term *community*, loosely defined and not aligned with particular theoretical ideas. A community of teachers designates several teachers engaged together professionally to achieve a common project related to mathematics teaching. We propose also to characterize "*community dynamics*" as the role of the members and their individual projects, the verbal interactions, and the interactions with the resources for teaching.

Communities of mathematics teachers: Networking of theories needed

For a didactical study of communities of mathematics teachers

Several researchers on the practices of mathematics teachers have already studied the issue of communities of teachers (Graven, 2004; Gueudet, Pepin, Sabra, & Trouche, 2016; Jaworski, 2006; Krainer, 2003).

Some researchers explore the place of collective work in teacher training (Llinares & Krainer, 2006; Jaworski, 2006) and more recently its place in online teacher training (Borba & Llinaras, 2012). Furthermore, relying on the theory of CoP, Jaworski (2006) develops the *communities of inquiry* approach. She stresses that developing critical thinking on teaching practice takes place in long-term processes of collaboration between teachers, trainers and researchers. This approach takes the specifics of mathematical epistemology in teaching and learning practices into account. However, from the standpoint of research, the communities of inquiry approach only allows the study of communities determined by the particular operational process of this approach.

Other researchers in mathematics education characterized different forms of communities but the place of mathematics is not always considered as central in the theories mobilized and developed (for example, see Krainer, 2003).

In order to identify the singularity of communities of mathematics teachers, we define the notion of *didactical study of a community of mathematics teachers* as the study of the conditions and the constraints of learning and sharing knowledge related to mathematics education through the design of resources and interaction between members of this community. Let us specify each element of this definition:

- The conditions and constraints of learning and sharing knowledge are those that allow, encourage, restrict or inhibit learning in the community.
- Knowledge related to mathematics education consists of *subject matter knowledge* and *pedagogical content knowledge* (Ball, Thames, & Phelps, 2008) that affect the condition of mathematics diffusion.
- Resources are anything that is developed and used by teachers (and students) in their interaction with mathematics to teach or learn inside/outside the classroom (Pepin,

Gueudet & Trouche, 2013). This element is preponderant in the study of a community and plays the role of a mediation and communication tool.

- By "interaction between members", we do not only mean simple verbal, oral or written, exchanges but also sharing resources and experiences through online activities.

The next sections discuss the theories and approaches that help us to take into account each of the elements above.

Learning and sharing knowledge in CoP: through problems of the community?

Wenger (1998) proposes a general theory built in the context of knowledge management to study communities of practice (CoP) defined as apprenticeship communities. He emphasizes that learning takes place during the exchanges between members, especially those exchanges related to their joint (community) problems. The treatment of these problems leads to new knowledge emerging from the participation process.

Wenger's theory does not give specific tools to consider mathematics, though we rely on this general idea proposed by Wenger of considering the problems of the community. We distinguish between *problems of the community* – identified by the researcher – and *community problems* – identified by the members – even if both can sometimes be the same. In this contribution, according to our focus on mathematics, a *problem of the community* is a phenomena or event that prevents or hinders the achievement of the common project and that is linked with mathematics as a science or as a teaching field.

At this point of our contribution, we hypothesize that the study of the problems of a community permits to determine a part of the community dynamics. We propose to realize it by following up the resources and the argumentation process.

Resources for teaching mathematics

The documentational approach of didactics (Gueudet & Trouche, 2012) attempts to frame the process of the design of teaching from the point of view of the teachers' work. It is a socio-cultural approach that permits us to study the interleaving between the community resources and the teachers resources that are involved (Gueudet et al., 2016). The teacher interacts with resources, selects them, and works on them (adapting, reviewing, reorganizing, etc.) in some processes where *design* and *enacting* are intertwined. The intertwining of new resources and interactions with other actors in the educational system creates new teaching knowledge and carries teachers' professional development (Gueudet & Trouche, 2012).

In order to achieve common projects, the different individual documentational geneses interact (Gueudet et al., 2016). The consideration of resources in the community of teachers should take into account: 1) the gathering, creating and sharing of resources in order to achieve the mathematics teaching goals of the community; 2) the result of this process, the shared resources and shared associated knowledge (what teachers learn together from conceiving, implementing, and discussing resources).

In the documentational approach, we use the definition given by (Pepin, Gueudet, & Trouche, 2013): "all the resources which are developed and used by teachers (and pupils) in their interaction

with mathematics in/for teaching and learning, inside and outside the classroom". The advantage of this definition is that it determines the documentation work in terms of interaction with mathematics. In this perspective, the resources of the community could be considered like a highly structured system, where resources are linked according to the level of teaching, the mathematical topic, the teaching mode, and the evolution of the curricula.

The documentary approach is a framework designed to articulate with some other ones. To answer our question, we need a complementary framework to consider the epistemological characteristic of mathematics.

Interaction between members: the place of argumentation

Taking into account the interaction between the members of a community implies taking the oral or written verbal exchanges into account. We have chosen to identify the arguments given by the members during the exchanges and to interpret them with the help of the resources that were mobilized or designed. We define argument as the reason that one presents to defend a certain point of view (Plantin, 1990). It is always oriented toward the decision that speaker wants to take (Ibid.).

Pedemonte (2007) stresses, in the case of mathematics, what can be considered an argument. She notes that is necessary to look at the proposition and at the context, which allows us to remove misunderstanding. More specifically, she distinguishes between arguments to convince – based on rationality – and arguments to persuade – for example based on authority.

Plantin (1990) and Pedemonte (2007) considered argumentation in the learning processes. Which kind of transfer could we make in the case of teachers? Some aspects need to be developed and clarified in our future work: typology of arguments, impact on the community dynamics.

Emerging model for analyzing a problem of the community: criss-crossing of several theories and approaches

Problems of the community that we consider are linked to mathematics as subject matter knowledge and as pedagogical content knowledge (Ball et al., 2008). We hypothesize that the treatment of these problems is related to: (1) the role of members in the community of teachers; (2) the arguments given by members; and (3) the use of resources that the teachers mobilize or produce to support the process of argumentation. We aim to determine the articulation between these components that could emerge (or not) in the treatment of these problems (members, arguments, resources) and the decision-making. We consider two case studies.

Problem of the community treatment: two contrasted case studies of a didactical study of a community

We developed the model on two contrasting cases to: 1) highlight the importance of considering the problem of the community; 2) extend the validity of the model.
We conducted the first case study at high school level. The community is spontaneous. The mathematical knowledge considered is the notion of "function". The researcher is outside of the community.

We conducted the second case study at the primary level. The community is intentional. The mathematical knowledge considered are research situations and proof. The researcher is within the community.

In both cases, we present some elements of context, the arguments given by the members of the community and their role in the treatment process of the problem of the community. Finally, we present the interpretation of the arguments exchanged by specifying the determining factors in the decision-making. We provided the details of the analysis in another paper (Georget & Sabra, 2015).

First case study: grade 10 textbook project

Sesamath is a mathematics teachers association founded in 2001. It aims to design and disseminate free and open resources. Interactions between the Sesamath members take place on online platforms and by mailing lists. Sesamath is structured around several project communities. We consider one of these communities: the grade 10 textbook project.
The chosen corpus corresponds to a thread of discussion about the progression of the topic "functions" in the textbook it is intended to design. "Functions" is the main topic to introduce in mathematical analysis at secondary level in France. The curriculum introduces functions for the first time at the grade 10 (15-16 years old), which generates a problem of the community that led to a controversy that we formulate by the following words: "What progression of the topic 'functions' must be adopted?"

Three members of the community are mathematics teachers with more than ten years of experience each. They play a particular role in the discussions: Mr W as a designer and a commentator and reviewer of resources, Ms A as a designer and a reviewer and a commentator of resources, and Mr H as a coordinator for several Sesamath projects and coordinator between Sesamath and other external communities.

This first case study illustrates arguments given to treat a problem of the community: (1) mathematical arguments with a discussion on the mathematical contents (connectivity between some concepts); (2) epistemological arguments about "functions" with a level of abstraction, depersonalization, and decontextualization of associated concepts; (3) didactical arguments coming from a personal experience or by consideration of the topic "functions" in the different level of teaching; and (4) arguments mainly linked to the design of resources.

The interpretation of the arguments during the discussions cannot be made independently of the context of the problem (structuring of the "functions" topic in the textbook). The mathematical argument takes importance in the formulation of the decision. We consider some arguments made by Mr H (argument to persuade) as a strong determinant factor in the decision. Mathematical arguments launched by Mr W and confrontation of his proposal with existing textbooks are also a second determinant factor in the decision. The members formulated the decision in terms of mathematical arguments.

Second case study: CoP and situations of research and proof at the primary level

The second case study is about an experiment of an intentional CoP during three years. We implemented a CoP to develop the practice of situations of research and proof between peers in

some classes at the primary level. It was composed of teachers and the researcher-coordinator of the CoP. We proposed several "mathematical situations" on online resources to the teacher and the design of these resources has been a key point in this CoP.

The data analyzed are about an extract of transcription of a meeting at the end of the second year with three teachers and the researcher-coordinator. One of the teachers, Mr D, does not understand the proof of a mathematical problem. He brings the usability of the resources into question by pointing out that the resources do not take his lack of mathematical knowledge into account.

For Mr D, the introduction of the proposed situations in his practice is rather exceptional. Ms S is voluntary in her implication in the CoP and for practicing these situations with her pupils. Mr H is more experienced than Mr D and Ms S, who have also problems of understanding for several resources, especially Mr D about a certain resource which Ms S understands well. Some exchanges help Mr D to understand the proof of the mathematical problem after which Mr H closes the debate implicitly. Mr D and Ms S do not rekindle it. Following some controversies about the usability of several resources, there is therefore an implicit decision not to modify them.

During the interactions, both kinds of arguments are exchanged: (1) resource design arguments about the usability of the teaching resources given by Ms S and Mr D who launch the interactions; and (2) mathematical arguments given by Mr H and the researcher/coordinator that permit the understanding of the proof by Mr D.

At the end of the exchanges, the decision is implicitly validated by mathematical arguments: the proof presented by the resource is valid and understood by the majority of those present at the meeting, so it is not necessary to modify the resource once the explanations have been given.

Results and discussions

In both cases studied, community members use mathematical arguments to validate the final decision. Mathematics appears to have at least two different roles. In the first case, the validation by some mathematical arguments seems to be essential to legitimate the decision-making even if the determining factor concerns the design of teaching resources. In the second case, the decision-making is implicit and validated by mathematical arguments while the common project of the community concerns the design of "pertinent" teaching resources.

Thus, from a methodological point of view, it is relevant to analyze the specific role of mathematics in the exchanges about a problem of a community. From the study of both case studies, a model emerges (see figure 1) as a proposal for understanding the community dynamics. We base this model on three components: (1) The members of the community depending on their role and their implication in the treatment of the problem, (2) the arguments, and (3) the resources for mathematics teaching that support the argumentation. The model gives the opportunity to analyze the problem of all type of communities.

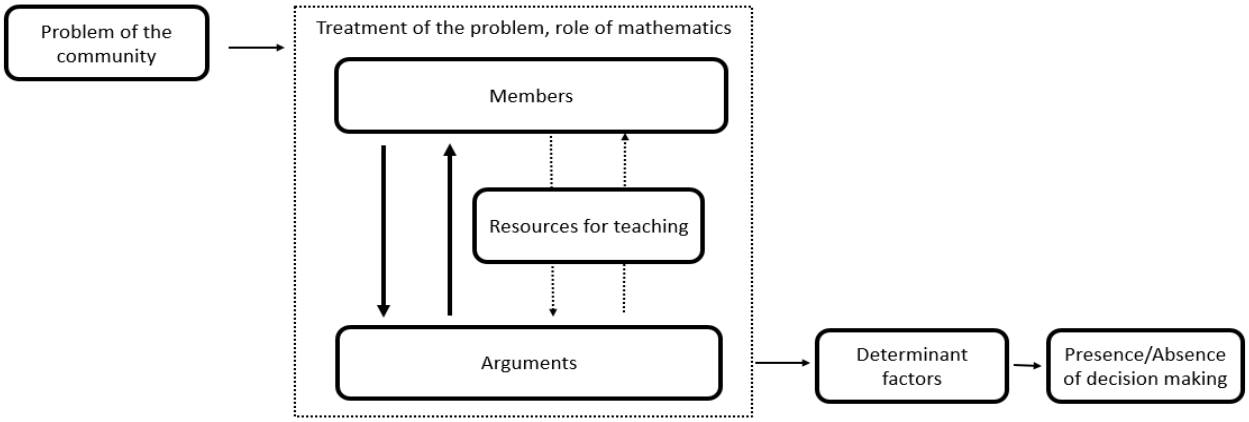

Figure 1: Model to analyze the role of mathematics (didactical, epistemological, etc.) in the treatment of a problem of the community

Conclusion

Existing research about mathematics teacher communities draw mainly on sociocultural theories. In complementarity, our project consists in studying more deeply the role of mathematics within communities of teachers. We propose a model of analysis resulting from articulation of several theories and approaches. We have also defined the notion of *didactical study of communities of mathematics teachers* to clarify the frame of this general research program. We have presented a model based on (1) the identification of a problem of the community, (2) the analysis of this problem in terms of community dynamics, (3) the characterization of the specific role of mathematics among arguments and resources for mathematics teaching. Both contrasted case studies contribute to show the relevance of our proposals. The model of analysis allows us to distinguish two roles of mathematics in the treatment of problems: 1) to explicitly formulate a decision already taken; 2) to implicitly validate a decision that might even be against the goal of the community. However, the model must now be tested on other data and on other kinds of communities. We also aim to test it in the perspective of identifying other roles of mathematics in community dynamics.

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An analytical tool for identifying what works with children with mathematics learning difficulties

Esmeralda Zerafa
University of Malta, Malta; esmeralda.zerafa@hotmail.com

Keywords: Analytical tool, Vygotsky's perspectives, mathematics learning difficulties.

As a theoretical framework I chose to use social constructivism on development and learning, mainly Vygotsky's perspectives. As highlighted by Ernest (1998), social constructivist theorists propose that knowledge is constructed through socially situated interactions or conversations. My study is underpinned by social constructivism because in this paradigm:

- The teacher has a fundamental role as the more knowledgeable other in guiding the learner through the learning process;
- Rich social interaction can enhance learning especially through the rich tool of social conversation and other forms of language; and
- Resources and language can be used to facilitate learning and thus to take the learner from the current point of learning to that which s/he has the potential to get to.

In Vygotsky's works the learner's developmental trajectory is often referred to as the Zone of Proximal Development (ZPD), "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (1978, p. 86). In this journey, the role of the More Knowledgeable Other (MKO) is thus crucial. Vygotsky frequently makes use of specific terms in his works. In the poster, I intend to provide a brief explanation for each term. Vygotsky's perspective accentuates that a child develops through a transformative collaborative practice which involves cultural tools, cultural influences, and other significant adults (Vianna and Stetsenko, 2006). This process, known as 'cultural mediation' underlies my use of an intervention programme to support learners with MLD. The programme sought to provide the participants with meaningful interactions that would help them to master the ten numeracy components (Catch Up, 2009). Vygotsky (1978) suggests that the analysis of the data collected by himself and his team "accords symbolic activity as a specific organizing function that penetrates the process of tool use and produces fundamentally new forms of behaviour" (p. 24). Therefore, whilst analysing the initial data I looked closely at situations in which the interactions provided would have altered the child's behaviour to such an extent that they now have mastered a specific concept, skill or knowledge vis-à-vis a specific numeracy component being focused on. Dunphy and Dunphy (2003) suggest that there are four stages within the ZPD, the first being that in which performance is assisted by the MKO, hence the choice of a teacher-led intervention programme. Based on Vygotsky's works, Tharp (1993) identifies seven means of supporting the learners to develop within their Zone of Proximal Development. These are: *modelling, feedback, contingency management, instructing, questioning, cognitive structuring "explanations"* and *task structuring* (Tharp, 1993, p. 271-272). Internalisation might take place through the use of what Vygotsky names as "cultural tools". Vygotsky (1978) distinguishes between technical tools and psychological tools. Technical tools "serve as the conductor of human influence on the object of activity; it is externally oriented" (p. 55). Such tools include the use of a ruler or protractor, for

example. Psychological tools are directed inward and gear the mind and the process of thinking such as language. In my programme I made use of both forms of tools.

Poster format chosen

The study focuses on exploring effective strategies for helping 9 – 10 year old Maltese children struggling with mathematics. Six participants were selected through standardised tests for numeracy and reading. Three children were identified as having only mathematics learning difficulties (MLD) and three having comorbid difficulties in mathematics and reading (MLDRD). In the poster, I present some crucial theoretical perspectives and the process of data collection. Its focus is to present the analytical tool developed to analyse the data collected through the multiple case studies carried out as part of an intervention programme conducted on a one-to-one basis with each of the participants.

Possible implications for existing research in the area

Research about what works with children having MLD is still very limited. Hence the importance of this study and the process of analysing the data gathered from the multiple case studies. Through the data analysed so far some important annotations have been made. These include the observation of two other modes of assistance that had not been mentioned by Tharp (1993). These I decided to label as 'recapturing' and 'role inversion'. Moreover it also seemed evident that interaction between the learner, the MKO and Cultural Tools was fundamental in providing the necessary assistance in guiding the learner within the ZPD to towards the potential zone of development. This means that this interplay was crucial in supporting the internalisation process. Therefore strategies that are effective with learners having MLDRD seem to be those that bring together all these aspects. This is represented in the analytical tool developed, which shows this symbiosis and its facilitation of internalisation.

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When the theoretical model does not fit our data: A process of adaptation of the Cognitive Demand model

Clara Benedicto, Angel Gutiérrez and Adela Jaime

Universidad de Valencia, Departamento de Didáctica de la Matemática, Valencia, Spain
clabebal@alumni.uv.es angel.gutierrez@uv.es adela.jaime@uv.es

Some theoretical models emerged from specific mathematical contents or educational levels, so they are well adapted to the context where they were created but, sometimes, they do not fit well in a different context. Related to the issue of the need to go beyond a specific theory when researching a phenomenon, to solve the tension between home-grown needs and borrowed theories, we present the reasons for and steps of an adaptation we have had to do of the Cognitive Demand model so that it fits the requirements of our research questions, methods and data. We systematized the definition of the model stated by its authors, and completed this definition with some necessary statements. Then, we re-stated some characteristics of the model to avoid inconsistencies when using it. Finally, we particularized the general model to several specific mathematical topics.

Keywords: Theoretical framework, adaptation, cognitive demand, mathematical problem solving.

Introduction

The choice of a specific theoretical framework is a key decision for researchers when they start working on a new research project. However, sometimes, researchers wish to use a theoretical model in a context that the model does not fit well, and they decide to modify the theoretical model to adapt it to the new requirements. We present our experience of fitting a theoretical model with flexibility so it can be adapted to contexts and used in ways the model had never been used.

We are developing a research project[1] aimed to better understand the cognitive processes of primary and lower secondary mathematically talented students (i.e., students showing mathematical abilities clearly over average students) when solving problems. A characteristic of those students is that they demand problems making them engage in high level of reasoning. A way for teachers to succeed in it is by posing them that kind of problems. Then, we had to find a way to determine the cognitive effort required by problems and done by students when solving them. The *Cognitive Demand model* (Smith & Stein, 1998) fitted our requirements, so we integrated it in our theoretical and methodological research frameworks. However, when we used this model to analyze our data, we found some difficulties in applying it due to ambiguity or inconsistency of results, so we decided to adapt it to our needs. We detail in next pages the steps in the process of adaptation, the reason for each step, and the resulting theoretical model. We present a reconstruction of the main parts of the real recurrent process, which has taken several years and still has not been finished.

The next section presents the characterization of the Cognitive Demand model as stated by its authors. Third to sixth sections present examples of the difficulties we found and the steps we followed to adapt the original model: organizing the original characterization, completing such charac-

[1] The results presented are part of the R&D&I research projects EDU2012-37259 (MINECO) and EDU2015-69731-R (MINECO/FEDER), funded by the Spanish Government and the European Fund for Regional Development.

terization with some new statements, re-wording some characteristics of the model to avoid inconsistencies when using it, and particularizing the new general characterization of the model to different specific mathematical topics. Due to space limitations we cannot show the details for all levels of cognitive demand, but only an example of each step for certain levels.

The starting point: The levels of Cognitive Demand

The Cognitive Demand model resulted after a process of characterization of mathematical tasks according to their "potential to engage students in high-level thinking" (Smith & Stein, 1998, p. 344). It includes four *levels of cognitive demand* that assess the cognitive effort required from students to solve a mathematical task. These levels are labelled (Smith & Stein, 1998) as *memorization*, *procedures without connections* to concepts or meaning, *procedures with connections* to concepts and meaning, and *doing mathematics*, when complex mathematical thinking is required. Each level is defined by a set of characteristics paying attention to different aspects of the solutions of problems. We present in Table 1 the characteristics of two levels that are the ground for the rest of the paper.

Procedures without connections
2.1. Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
2.2. Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
2.3. Have no connection to the concepts or meaning that underlie the procedure being used.
2.4. Are focused on producing correct answers instead of on developing mathematical understanding.
2.5. Require no explanations or explanations that focus solely on describing the procedure that was used.
Procedures with connections
3.1. Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
3.2. Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
3.3. Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
3.4. Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

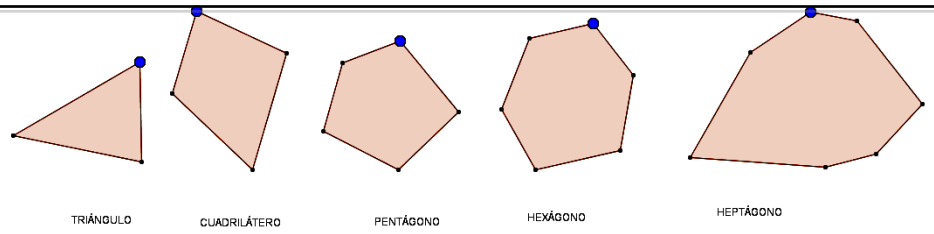
Table 1: Definition of the levels of cognitive demand of *procedures without connections* and *procedures with connections* (Smith & Stein, 1998, p. 348). Order numbers are added for easy reference

To attain our research objectives, we created *rich problems* intended to be posed to whole-class groups, consisting of several related questions of increasing complexity, in such a way that all students should be able to solve the first questions but only the more able students could solve all of them. The levels of cognitive demand allow us to classify the questions in a problem and decide whether each question is more appropriate for average students or for talented students.

The Cognitive Demand model was created after the analysis of problems that, most of them, were in the quite algorithmic areas of school arithmetic and algebra (Stein & Smith, 1998; Stein et al., 2009), but we are trying to use the model to analyze problems in mathematical topics very different from the previous ones, as are plane geometry, geometric patterns (pre-algebra) and visualization.

Identifying inconsistencies between the characteristics of the levels

After using the model to analyze different problems, a difficulty related to lack of consistency between some levels arose. We exemplify it by analyzing a problem consisting of several questions guiding students to discover and prove a formula to provide the number of diagonals of any polygon. The problem has two parts: the first one can be seen in Figure 1; the second part asks students to draw and count all the diagonals of the same polygons and to fill in a table, to calculate the number of diagonals of a 20-sided polygon, to generalize the procedure of calculation of the diagonals to any given polygon, and to prove this relationship. We present our analysis of question 1a (Figure 1) by considering typical average students' solutions that do not go further than what is asked to do.



1a) In each polygon, draw all the diagonals starting from the marked vertex. Change the shape of the polygons by dragging that vertex. Count the number of diagonals. Fill in the table below.

Polygon	Nº of sides	Nº of diagonals from one vertex
Triangle		
Quadrilateral		
Pentagon		
Hexagon		
Heptagon		

1b) What is the relationship between the number of sides of a polygon and the number of diagonals from one vertex? Why?

Figure 1: First part of a problem focused to discover the number of diagonals of any polygon

Regarding the level of *procedures without connections*, question 1a is algorithmic and the statement suggests the procedure to be used, namely to draw the diagonals from a given vertex and count them to fill in the table, so it fits characteristic 2.1 in Table 1. The polygons drawn in the statement guide students to draw the diagonals from a specific vertex, so there is little ambiguity about what they have to do and how to do it (it fits 2.2), and it does not require explanations (it fits 2.5). On the other hand, the procedure to be used has connections with the relationship between the number of sides and diagonals from a vertex of polygons (it does not fit 2.3), although question 1a is focused on producing correct answers, not on developing understanding of that relationship (it fits 2.4).

Regarding the level of *procedures with connections*, question 1a does not fit 3.1, since it is not focused to make students develop understanding of the underlying relationship. However, it fits 3.3,

since, to answer it, students use geometric and numeric representations of information about polygons and diagonals: the numeric representation shows the general relationship between number of sides and diagonals, while the geometric representation may help students understand why it is true. So the question has the potential to let students connect both representations, which would help them develop the meaning for the relationship. Question 1a fits 3.2, since it explicitly suggests a procedure that is closely connected to the underlying concepts, the number of sides and diagonals from a vertex of polygons. Question 1a does not fit 3.4 since its procedure may be followed without need of being mindful, and a correct solution to it does not require understanding the underlying relationship between number of sides and diagonals from a vertex of polygons.

The epistemological conception of the levels of cognitive demand is that they are mutually exclusive. We see that question 1a fits several characteristics of each level *procedures without connections* and *procedures with connections*, so it is unclear to which level of cognitive demand should it be assigned. This happens because some characteristics of these levels, as stated in Table 1, are not precise enough, which can lead to errors when trying to assign a level of cognitive demand to some problems. The most evident vagueness, or contradiction, happens with characteristics 2.1 and 3.2.

Organizing the characteristics of the levels and filling their gaps

After having identified the difficulty analysed above, we made a detailed comparison of the characteristics of each pair of consecutive levels, to identify possible weaknesses and modify their wording to correct them. We noted that the characteristics of levels (see examples in Table 1) focused on six domains of objectives of a problem or its process of solution. These six domains of characteristics are: *Procedure of solution*, *objective* of the problem, required student's *cognitive effort*, mathematical *contents implicit* in the problem, kind of *explanations* required, and types of *representations* used in the solution. The domains helped us arrange the characteristics of the levels of cognitive demand provided by Smith and Stein (1998) and identify some gaps in the definitions of the levels.

Levels of cogn. demand / Domains	Memorization	Procedures without connections	Procedures with connections	Doing mathematics
Procedure of solution	1.2	2.1	3.2	4.1, 4.5
Objective	1.1	2.4	3.1	4.2
Cognitive effort	1.3	2.2	3.4	4.3, 4.6
Implicit contents	1.4	2.3	3.4	4.4
Explanations	--	2.5	--	--
Representations	--	--	3.3	--

Table 2: Domains of the characteristics of the levels of cognitive demand in Smith and Stein (1998)

Table 2 shows the assignation of the characteristics of the levels to the domains. It also shows that two domains are considered only in the definitions of a level, and that 3.4 includes references to two domains, while several characteristics of the level *doing mathematics* refer to a same domain.

Next step to improve the usability of the original definition of the Cognitive Demand model was to complete the definitions of the levels, by including characteristics referring to the missed domains, taking care that each new characteristic is consistent with the corresponding characteristics of the other levels. Table 3 shows the new characteristics, to be added to those in Table 1 to make a more

complete description of the levels of cognitive demand.

Procedures without connections
2.6 (representations). One or more representations may be used (arithmetical, geometrical, visual diagrams, manipulatives, etc.). When several representations are used, students use them independently, i.e., without establishing connections neither between them nor with the underlying concepts and ideas.
Procedures with connections
3.5 (explanations). Require explanations that focus on the underlying relationships by using specific examples.

Table 3: Characteristics added to the levels of *procedures without* and *with connections*

Having completed the definitions of the levels by merging Table 1 and Table 3, we were ready to refine the characteristics that induced wrong or multiple identifications of the cognitive demand in some problems, like the problem analysed above (number of diagonals of polygons).

Refining the characteristics of each level

A necessary feature of any set of disjoint categories is that their definitions have to make it clear the border between adjacent categories. As we showed above, this is not the case for the levels of cognitive demand. To refine the definitions of the levels, we made a systematic comparison of the characteristics in the same domain and decided to do some changes in their wording to make them more explicit and to clearly raise the particularities of each level.

The key difference between the levels of *procedures without connections* and *procedures with connections* is that, in the lower level, students do not need to be aware of the mathematical relationships implicit in the problem to solve it correctly but, in the higher level, students need to use consciously such relationships to solve correctly the problem. Table 4 shows the result of the comparison between those levels, where we have italicised the new characteristics (see Table 3) and the characteristics in Smith and Stein (1998) that we re-worded. Characteristic 3.4 was split because it included parts corresponding to two domains. The new wording of the characteristics of the levels has highlighted this key difference and now the border between those levels is clear.

If we repeat now the analysis of the problem in Figure 1, question 1a fits new characteristics 2.1, 2.3 and 2.4, because it focus students' attention to draw the diagonals from a vertex of each polygon and count them, so it can be easily solved without being aware of the relationship between the number of sides and diagonals from a vertex.

	Levels of cognitive demand	
Domains	**Procedures without connections**	**Procedures with connections**
Procedure of solution	2.1. Are algorithmic. *The procedure to be used* either is specifically called for or is evident *from the context. It is a simple procedure that students can follow without the need to connect to underlying concepts and ideas.*	3.2. *Are algorithmic*. They suggest explicitly or implicitly pathways to follow, that are general procedures *that students can follow only if they have established a close connection to underlying concepts and ideas*.

Objective	2.4. *Focus students' attention* on producing correct answers. *Students can solve them correctly without the need to understand underlying concepts and ideas.*	3.1. Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of *underlying* concepts and ideas.
Cognitive effort	2.2. Require limited *cognitive effort* for successful completion. Little ambiguity exists about what needs to be done and how to do it.	3.4a. *Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly*.
Implicit contents	2.3. *There may be implicit connection between the algorithms used and underlying concepts or ideas. However, students do not need to be aware of it to solve the problem correctly*.	3.4b. *Students need to engage with concepts and ideas that underlie the procedures to complete the problem successfully and that develop understanding*.
Explanations	2.5. *Require explanations* that focus solely on describing the procedure that was used.	*3.5. Require explanations that focus on the underlying relationships by using specific examples.*
Representations	*2.6. One or more representations may be used (arithmetical, geometrical, visual diagrams, manipulatives, etc.). When several representations are used, students use them independently, i.e., without establishing connections neither between them nor with the underlying concepts and ideas.*	3.3. Usually are represented in multiple ways, *(arithmetical, geometrical, visual diagrams, manipulatives, etc.). To solve correctly the problem, students have to establish connections between different representations by using underlying concepts and ideas, which help them develop meaning.*

Table 4: Comparison between the characteristics of problems in two levels of cognitive demand

Question 1a also fits new characteristics 2.2 (since there is no ambiguity about how to solve it) and 2.6 (students will use geometrical and arithmetical representations but without needing to connect them), but it does not fit characteristic 2.5, since this question does not ask for an explanation (so, this analysis is also useful to uncover flaws in the statements of problems). On the other hand, question 1a does not fit new characteristics 3.1, 3.2, 3.3, 3.4a, 3.4b and 3.5. So, now it is clear that question 1a requires a cognitive demand in the level of algorithms without connections, which agrees with our experimental analysis of real students' answers.

Particularizing the new cognitive demand model to specific topics

As mentioned above, the Cognitive Demand model was generated after analysing problems that, in most cases, were related to school arithmetic or algebra. When we tried to use it to analyze problems in other areas of mathematics (plane geometry, geometric pattern problems and visualization), we found that the wording of quite characteristics of the levels were too generic and they did not help us to give meaning to the levels specific to those contexts. This forced us to re-word those characteristics of the levels to mention specific features of a given topic. We present here the particularization we have made to the context of geometric pattern problems.

Geometric pattern problems have proved to be a very fruitful way to introduce basic algebra to students (Amit & Neria, 2008; Rivera, 2013). A typical geometric pattern problem presents (Figure 2) a graphical representation of the first terms of a sequence of whole numbers, and asks students to calculate the value of certain terms of the sequence, to verbalize a general procedure to calculate the value of any given term, and to write an algebraic expression to calculate the value of any term.

You can see below a shape made with one dot, another shape made with three dots, and so on.



1. How many dots has the shape in the 4th position?
2. How many dots has the shape in the 6th position?
3. How many dots has the shape in the 20th position? How do you know it?
4. Is there some rule that could allow us calculate the number of dots of any given shape, for instance the one in the 100th position? Justify your answer.
5. Is there some rule that could allow us calculate the number of dots of the shape in the *n*th position? Justify your answer.

Figure 2: A typical statement of a geometric pattern problem

We are interested in analysing the relationships among the geometric patterns and the cognitive demand required by different kinds of students' answers. When we first used the definitions of the levels of cognitive demand (Tables 1 and 4) to classify answers to geometric pattern problems, we found that some characteristics were meaningless in this context, so we made a complete particularization of the characteristics of the levels to describe the answers to this specific type of problems.

Table 5 presents, as an example, the characteristics of the level of *procedures without connections* for the context of geometric pattern problems. It may be noted that most characteristics include reference to peculiar and unique aspects of those problems.

Procedures without connections (question 2)	
Procedure of solution	• Are algorithmic. The procedure consists in drawing a few terms by following the pattern of the terms in the statement, and counting the items. It can be followed without the need to connect to the arithmetic structure of the sequence.
Objective	• Focus students' attention on producing a correct answer, the number of items in an immediate or near term, but not on developing understanding of the structure of the sequence.
Cognitive effort	• Solving it correctly requires a limited cognitive effort. Little ambiguity exists about what has to be done and how to do it, because the statement clearly shows how to continue the sequence.
Implicit contents	• There is implicit connection between the underlying structure of the sequence and the procedure used. However, students do not need to be aware of it and they may answer the question by drawing terms and counting their items.
Explanations	• Require explanations that focus only on describing the procedure used. It is not necessary to identify the relationship between the answer and the term.
Representations	• A geometric representation is used to get the number of items and an arithme-

	tic one to write the result. Students use the representations without establishing connections neither between them nor with the structure of the sequence.

Table 5: Particularization of the Cognitive Demand model to the geometric pattern problems

This description of the levels of cognitive demand has proved to be very useful to analyze this kind of problems and students' answers to them.

Conclusions

We have presented a case of modification of a theoretical model to adapt it to the specific requirements of the analysis we had to do of our data. The Cognitive Demand model was a pertinent theoretical framework for our research project, with the potential to ground a deep analysis of our data, although the practice showed that the initial definition of this model, as formulated by its authors, did not fit well the requirements of our analysis. We have shown some difficulties that arose when we tried to apply the initial model. The way to overcome these difficulties was to analyze the theoretical model, to identify and understand the origin of and the reason for the difficulties, and to make adequate changes in the definition of the levels of cognitive demand to make it more accurate and useful. Finally, we had also to particularize the new definition of the levels to the specific context of geometric pattern problems. This general way of proceed may be applied, perhaps after an adequate adaptation, to modify other theoretical models not fitting adequately researchers' needs.

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TWG18: Mathematics teacher education and professional development

Introduction to the papers of TWG18: International research on mathematics teacher education and professional development

Stefan Zehetmeier[1], Laurinda Brown[2], Maria Mellone[3], Leonor Santos[4], Gulseren Karagoz Akar[5]

[1]University of Klagenfurt, Austria, stefan.zehetmeier@aau.at

[2]University of Bristol, United Kingdom, laurinda.brown@bristol.ac.uk

[3]University Federico II of Naples, Italy, maria.mellone@unina.it

[4]Instituto de Educação, Universidade de Lisboa, Portugal, mlsantos@ie.ul.pt

[5]Boğaziçi University, Turkeygulseren.akar@boun.edu.tr

Keywords: Professional development, pre-service teacher education, in-service teacher education, teachers, teacher educators.

Rationale

The study of mathematics teacher education and professional development has been a central focus of research during the last decades. Various research activities have focused on this topic. Within TWG 18, we focus on mathematics teacher education (pre-service and in-service), professional development and teachers' professional growth, teachers' professional development practices, collaboration and communities of practice, models and programmes of professional development (contents, methods and impacts) and the professional development of teacher educators and academic researchers. TWG 18 offers a communicative, collegial and critical forum for the discussion of these and other related issues, which allows diverse perspectives and theoretical approaches and which contributes to the development of our knowledge and understanding as researchers, educators and practitioners.

Participants

52 papers were originally submitted to TWG18. 22 of them were re-directed to other TWGs. Thus, 30 papers underwent a peer review process in TWG18: during this process, all papers were revised by authors, according to reviewers' remarks. 29 papers were accepted as paper presentations, one was re-submitted for a poster presentation. One of the accepted papers was withdrawn. Finally, 28 papers were presented during the TWG sessions.

Two posters were originally submitted and underwent a peer review process in TWG18: both authors revised their posters, according to the reviewers' remarks; both posters were accepted. Together with the re-submitted poster (see above), finally, 3 posters were presented during the conference poster session.

Organisation

TWG sessions comprised both plenary and sub-group working phases. During the plenary phases, two (or three) papers were presented for a maximum of five minutes each, in which the authors provided their paper's central message(s) and challenging questions for discussion. These plenaries were followed by parallel sub-groups, which were each managed by one of the presenting authors.

Participants were free to choose and join one sub-group, where they discussed the paper for 20-30 minutes. Afterwards, the TWG's participants met in plenary to hear reports of each sub-groups' central topics and to summarise emerging issues.

Topics

The presentations were categorised into four main topics:

- Noticing Students' Work
- Teacher Instructional Practice
- Impact of Professional Development
- Pre-Service Teachers.

Open questions and emerging issues

This section provides several questions and issues, which emerged during the sessions of TWG18:

Noticing Students' Work:

- **Open questions:**
  - How do we guide pre-service student teachers to notice particular things such as children's learning?
  - How do the different global country contexts and the constraints of each system locally, make a difference when you apply a learning trajectory?
  - Is there a connection between different teachers' views, what they see and their beliefs, based on noticing?

- **Emerging issues:**
  - Use of the language of the teacher educators in talking about errors e.g., concept image.
  - Different points of views about errors in different teacher education programmes and how we use/ understand errors in our teacher education programmes. Also differences in practices of 'noticing'.
  - Importance of context of the countries and coming to understand these to understand the organisation of the different teacher education programmes.

Teacher Instructional Practice:

- **Open questions:**
  - How to motivate teachers to document more of their work?
  - How do we change teachers feeling judged when getting feedback?
  - How do we understand the different notions of inquiry? What are the effective strategies to implement this approach in mathematics lessons?
  - How do we get to the mathematics? What mathematical knowledge do teachers need for inquiry approaches in mathematics lessons?
  - When we do our research, we use different tools such as philosophical perspectives and analytical frameworks. Is it the different tools that lead to different results? Is it the analytical framework that produces the results? What do we learn as teacher educators? Is that dependent on the frameworks used? Does working with multiple perspectives help us?

- **Emerging issues:**
  - Ethical and practical issues of using children to make interventions in the professional development of teachers e.g. use of video.
  - There might need to be different kinds of innovation dependent on the teachers.
  - Publishing negative cases as well as positive ones.

Impact of Professional Development:

- **Open questions:**
  - How could we ask questions to measure impact (e.g. changing of beliefs)? Are there other ways to what we do now?
  - What impact do 'we' want to sustain? Static image of change or dynamic change of teachers?
  - In discussing professional development, how do you track development/ learning, through the discourse, through anecdotes and, or?

- **Emerging issues:**
  - Comparing behaviour of teachers in lessons and in PD raises lots of ideas to consider, such as the perceived gap between what teachers do in classrooms and how they articulate their practice in PD.
  - The challenge is how to sustain the impact of professional development beyond its delivery.
  - There are differences in language and discourses between teachers, teacher educators and politicians. There is the need to define what is a good argument in the different contexts.
  - Describing frameworks when they are not familiar.

Pre-Service Teachers:

- **Open questions:**
  - Of the language use, such as theory, used in professional discussions at the university, what do pre-service teachers learn? Are they just naming names rather than gaining a deeper understanding of pedagogical concepts?
  - Can we measure the process of pre-service teachers' development and conclude something from it? Do we want to measure these kinds of dimensions?
  - What does it take for one to become explorative both in learning and in teaching?

- **Emerging issues:**
  - Using videos and not having access to videos in the language of the teachers e.g. any non-English language. Resource implication for creating a local bank of videos.
  - The relationship between the mathematics that a teacher has to teach and the mathematics learned by pre-service teachers in education courses offered by the teacher educator focused on conceptual development.

- Finding ways to disrupt previous experiences of teaching and learning mathematics that pre-service teachers bring with them.

Moreover, some further general questions and issues concerning mathematics teacher education and professional development were discussed during the TWG sessions:

- **Open questions:**
  - Can we observe the development of teachers in their pupils' learning?
  - Does research in mathematics teacher education support the work in schools? Are we critical about our own practices?
  - How do we recruit participants for professional teacher development? How do we promote PD effectively? Use of social media?

- **Emerging issues:**
  - Complexity of appropriation of frameworks, the language across cultures does not always transfer.
  - Writing scientific and professional journal papers with teachers that address teachers' communities.
  - There is work to be done to create dynamical relationships between theories.
  - In researching with teachers, we attend to different things, such as the teacher might be focusing on the learning of the children and the teacher educator the development of the teacher, and we need to find ways to understand each other.

Upper secondary teachers' *stages of concern* related to curricular innovations before and after a professional development course on teaching probability and statistics including the use of digital tools

Rolf Biehler and Ralf Nieszporek

Paderborn University, Germany; biehler@math.upb.de, ralf.nieszporek@math.upb.de

In 2013 the German Centre for Mathematics Teachers Education (DZLM) developed a professional development course called "stochastics compact". This course was held three times during the period from 2013 to 2015 and reached more than 270 teachers. One of their goals was to increase the upper secondary teachers' competence of teaching probability and statistics in combination with the use of graphic calculators (GC). A part of the research was to examine the stages of concern (SoC) linked to the implementation. Questionnaires were used as a survey method. In this paper, we present two selected and preliminary findings. At first we point out the development of SoC from 2013 to 2015. After that we will present the changes of SoC while participating in the course of 2015.

Keywords: Professional development, stages of concern, probability and statistic, graphic calculator.

Context of the professional development course

Our research is related to a four-day long (spread over several months) professional development course on teaching probability and statistics at upper secondary schools (grade 10-12) in the German federal state of North Rhine-Westphalia (NRW) from 2013 to 2015. About 270 teachers participated in this course. Due to new national standards (KMK, 2012) and subsequent new state curricula in NRW, probability and statistics became an obligatory part of the curriculum and the final examination (Abitur). Moreover, the use of graphic calculators became obligatory in the classroom and in the examinations. This was a challenge for many teachers and caused a high need for professional development. The German Center for Mathematics Teacher Education (DZLM) recognized this need and gathered a team of experienced school teachers and researchers to originate a professional development course called "stochastics compact" (Biehler, 2016). The design of this course was based on results from stochastics education (Biehler, Ben-Zvi, Bakker, & Makar, 2013; Burrill & Biehler, 2011; Oesterhaus & Biehler, 2014), their interpretation of the standards and the design principles of the DZLM (Barzel & Selter, 2015). The first implementation was done in 2013 and was followed by the second in 2014 and a third one in 2015. Accompanying research addressed the change of competences and beliefs of the participating teachers as well as teachers' feedback to the courses they attended. Moreover we were interested in the teachers' stages of concern related to the innovation. The main purpose of this article is to present results of the stages of concern questionnaire (SoC), which may be a relevant instrument for doing research on professional development courses and on teachers' attitudes to innovations.

Theoretical framework and related research on the Stages of Concern

The research of interests and concerns of teachers described in this article is based on the Concern-Based Adoption Model (CBAM). This model was developed by the Research and Development

Center for Teacher Education at the University of Texas in the early 1970s (Hall, Wallace, & Dosset, 1973). This model is partly based on Fuller's work on concerns of teachers (Fuller, 1969). One of the three diagnostic dimensions of CBAM is the Stages of Concern Model.[1] It is a framework which helps to understand the personal aspects of adopting an innovation and the connected change progress. The researchers of the University of Texas identified seven stages (see Table 1) which a person runs through while implementing an innovation. There are several techniques to monitor the SoC such as one-legged interviews, open-end concerns statements or the SoC questionnaires. For several reasons the most rigorous method for measuring SoC is the questionnaire (Hall & Hord, 2006). The team around Hall developed a SoC specific questionnaire with 35 questions, 5 per stage (Hall, George, & Rutherford, 1977). As Hall points out, changing anything besides the word "innovation"[2] will risk the reliability and validity of the items (see Schaafsma and Athanasou (1994) as a negative example).

Stages of Concern		Label	Typical items
Self	0	Unconcerned	I am more concerned about another innovation.
	I	Informational	I would like to know how this innovation is better than what we have now.
	II	Personal	I would like to know the effect of the innovation on my professional status.
Task	III	Management	I am concerned about time spent working with non-academic problems related to this innovation.
Impact	IV	Consequence	I am concerned about how the innovation affects students.
	V	Collaboration	I would like to develop working relationships with both our faculty and outside faculty using this innovation.
	VI	Refocusing	I now know of some other approaches that might work better.

Table 1: Typical items of the different Stages of Concern, based on George et al. (2008)

The course "stochastics compact" took place in Germany, so there was a need for a German translation of the English SoC Questionnaire. We used the translation that was used by Pant, Vock, Pöhlmann, and Köller (2008a)[3]. There are three international studies and three German studies which we compared our data to and whose results we will shortly summarize.

One of the first studies of SoC was executed by Hall et al. (1977). Hall's team identified several profiles and their characteristic graphical shape. These profiles were the basis for other SoC

[1] The development process of the SoC-Model is in greater detail described in George, Hall, Stiegelbauer, and Litke (2008).

[2] The innovation can be replaced by the name of the innovation or other phrases which respondents are more familiar with.

[3] We are grateful to Doreen Prasse for providing a copy of the German version of the questionnaire.

research like Liu and Huang (2005), who examined American teachers and their problems related to the integration of technology. They found out that the greatest concerns depend on the teachers' experience. Inexperienced teachers tend to have personal and informational concerns, while experienced teachers were mainly concerned about the consequences for their students, and renewing[4] educators placed their focus on collaboration and refocusing concerns. A second finding was that the strongest concerns were in the early stages of personal, informational and refocusing concerns. The three SoC profiles that were re-identified by Liu and Huang were first constructed by Hall et al. (1977). A second survey regarding concerns of 659 American pre K-12 teachers about the use of technology like computers in school was conducted by Casey and Rakes (2002). Peaks in the SoC profiles were found in informational, personal and collaboration concerns. The interpretation was that school teachers are still uncomfortable and in an initial stage of understanding the benefits of technology in school. Pant, Vock, Pöhlmann, and Köller (2008b) came to the conclusion that most German elementary and middle school teachers of their study have high self and impact concerns regarding the recently implemented national standards, thus they show a typical M-shaped profile (see Figure 1) of a *cooperator*. This profile was also found by other researchers like Bitan-Friedlander, Dreyfus, and Milgrom (2004) or Pöhlmann, Pant, Frenzel, Roppelt, and Köller (2014). Bitan-Friedlander et al. were able to identify five types of primary school science teachers which were confronted with the implementation of an innovation. Another result was that most of the participants were able to "adopt"[5] the innovation and developed a personal perception. Pöhlmann et al. (2014) chose a control group design to measure the efficiency of a new developed intervention to help teachers who are dealing with the new German national standards for the first time. The SoC Questionnaire shows that control and test groups were on a comparable level at the beginning and the participants show a high level of self-concerns. Impact concerns were secondary in both groups. After a year of training an increase in impact concerns as opposed to self-concerns was observed. This can be interpreted as consequences for teachers, for pupils, and for their mutual cooperation, which resulted in different foci. The control group also showed a different SoC profile, similar to one of an earlier test. There were no peaks recognizable, which might be due to a feeling of exaggerated demands.

Research question

We will address the following research questions in this paper:

1. How does the SoC differ when a PD course is backed up by an official obligatory innovation and not only by an innovation suggested by the PD course designers?
2. How are the SoC towards probability and statistics (including the use of the GC) change distributed before and after a professional development course on the topic?
3. Does a professional development course change the stages of concerns of the participants?

[4] Liu and Huang defined renewing teachers as persons who understand the innovation and are adopting or thinking about different kinds of use of the innovation based on their experience.

[5] For primary school teachers the meaning of adoption needed to be redefined, because they did not challenge the theoretical knowledge or ground or mentioned a personal opinion about the implementation of the innovation.

Those questions are just one facet of a wider research project. We used our access to the participants not only to determine their SoC levels but also to identify other important aspects which we plan to associate with the SoC-profiles in our future research.

From a methodological point of view we were interested how well the SoC scales can be used to identify important characteristics and sub-groups of teachers, which are important to take into account, when designing and evaluating PD courses.

Design of the intervention

There was a fundamental difference between the 2013 course and the 2014 and 2015 course. The 2013 course was run before the new curricula became obligatory and the GC was prescribed. The 2013 focused in day 3 and 4 on an approach to the teaching of hypothesis testing that was innovative for most German teachers, focusing on p-value hypothesis testing as a start, using authentic examples from real statistical studies instead of artificial problems, and discussing possible misinterpretations of hypothesis testing that are well known from studies in school and in statistical practice. We called our approach Best@Kontext (Oesterhaus & Biehler, 2014). The 2014 and 2015 courses (after the new state curricula) build on this approach but included more systematically the use of GC for interactive visualizations, simulations and calculations not only on day 3 and 4, where hypothesis testing remained the focus. Day 1 and day 2 was completely revised and restructured using simulation and the GC technology.

The SoC in 2013 was related to day 3 and 4 of our course and "our own" innovation Best@Kontext, the SoC in 2014/2015 was related to the whole course and to the state innovation "Teaching Probability and statistics with graphic calculators". We communicated to the teachers that our course is compatible with the new state innovations, but that our specific foci are based on research in probability and statistics education related to student difficulties, valuable teaching approaches but also on normative aspects concerning the fundamental ideas in probability and statistics that should structure the course.

Data collection and data analysis

In our study we collected 38 questionnaires in 2013 (post test), 55 in 2014 (post test) and 74 in 2015 (pre and post test) which were accepted for evaluation. The others had incomplete SoC Questionnaires or were not traceable in the pre and post test design of 2015.

We used the manual of Hall et al. (1977) as a guideline for the program SPSS 23 for analyzing our data. Therefore our statistical analysis is comparable to the above mentioned studies. For the determination of SoC subgroups, we used a cluster analysis of the individual subscale means. The ward method was chosen with the squared Euclidean distance as measure in every step of our analysis to divide the participants as recommended by Bortz and Schuster (2010). The clusters were created with the data of all four measurements so that we are able to analyze shifts in the distribution of participants into the identified clusters

Results

The reliability of the SoC subscales (see Table 2) can be compared to other studies like Pant et al. (2008b) or George et al. (2008).

Subscale	0 Unconcerned	I Informational	II Personal	III Management	IV Consequence	V Collaboration	VI Refocusing
Cronbach's α	0.717	0.560	0.729	0.727	0.782	0.836	0,765

Table 2: Cronbach's α of the seven SoC subscales in our sample

To answer research question 1, we constructed average SoC-profiles for the three years (including pre- and post test in 2015). Figure 1 shows that the subscale means of the 2013 questionnaire are considerably different to all other years. This observation is supported by *t*-test for every combination of subscales except for stage III of 2014 (p=.057) and 2015's post test stage 0 (p=.176). The graph shape of 2013 belongs to an *interested nonuser* (George et al., 2008).

The differences of stages I, II and IV in 2014 compared to the pre test 2015 and stage 0 of the post test 2015 are above the significance threshold of p=.05. In the comparison between the pre and post test of 2015 stages 0 and IV are the only stages without a significant difference (p>0.35). Therefore it is unsurprising that those three measurements' graph shapes only deviate slightly and can be interpreted as *cooperators* (Bitan-Friedlander et al., 2004). Attendants of those three years have got a split attention focus in self (peak at stage I informational) and impact concerns (peak at stage IV collaboration) regarding the implementation of statistics and the GC at school and show an M-shaped profile.

This very clear difference can be related to different kind of innovations (related to our project in 2013 – state based innovations in 2014/2015). We have to be aware that below the average there is a lot of variability in the individual SoCs. We will discuss this below.

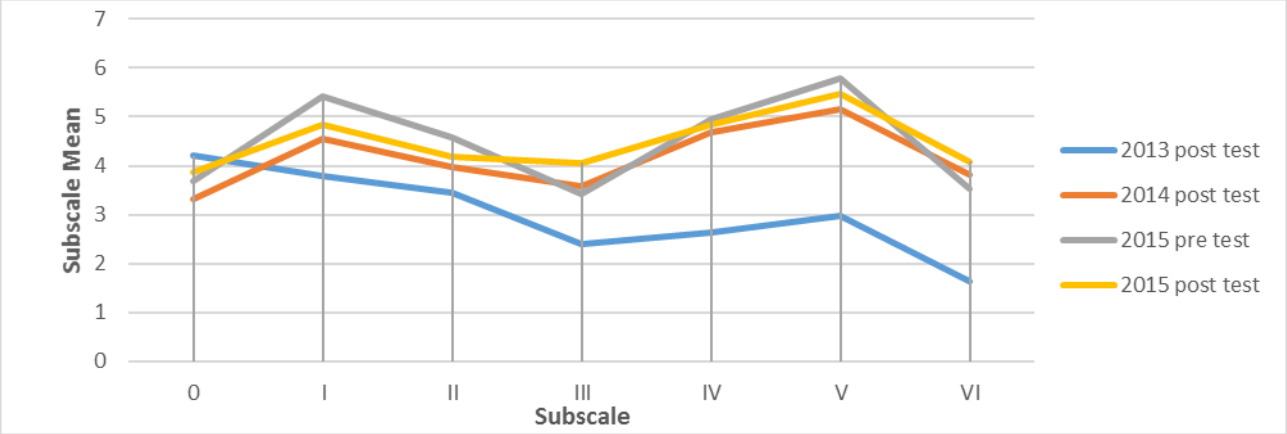

Figure 1: Subscale means divided by measurement

With regard to research question 2 and 3 we do not see a substantial difference between 2014 and 2015 (although one might have expected this because the teachers had been aware of the state innovations for one more year. On the level of the average profile we see a systematic difference, which however is not statistically significant.

Related to research questions 2 and 3 we did a cluster analysis to identify different types of participants and how often they occur in the various points of measurement. We decided to put all data (n = 167) together for identifying clusters. This is useful, when the distribution into the clusters is to be compared for the different measurement points. At first six clusters were identified by our

cluster method and upon closer inspection of these groups, two times we found two clusters that were identical in their graph shape and were shifted by one scale point. So we decided to combine those two similar clusters into a bigger group. Finally, we worked with four clusters, which had been identified in a very similar way before in other studies (George et al., 2008; Pant et al., 2008b) These clusters can be labeled as *unconcerned innovation user* (n=4), *typical nonusers* (n=29), *information seeking cooperator* (n=59) and *self-orientated cooperator* (n=93) after their characteristic graph shapes (see Figure 2).

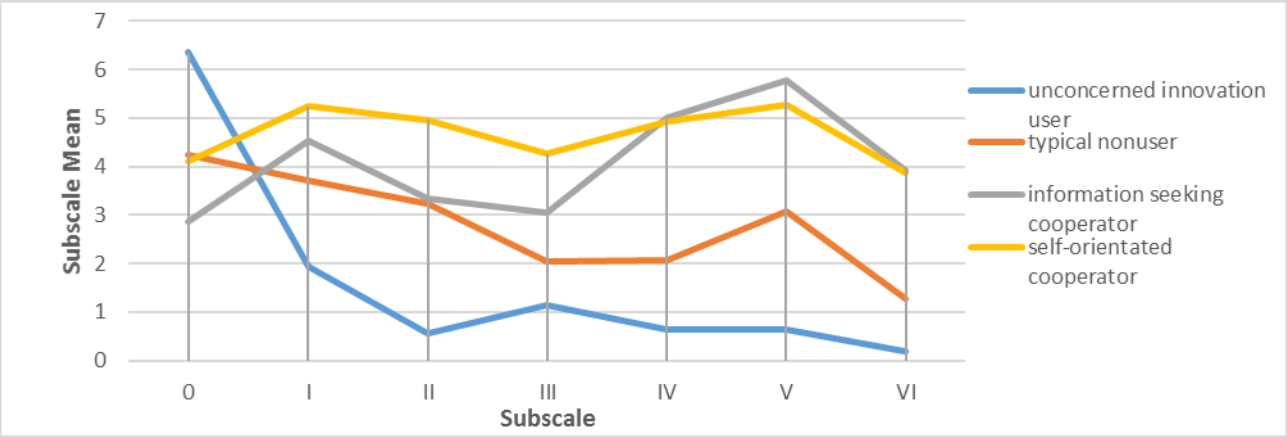

Figure 2: Subscale means profile by cluster

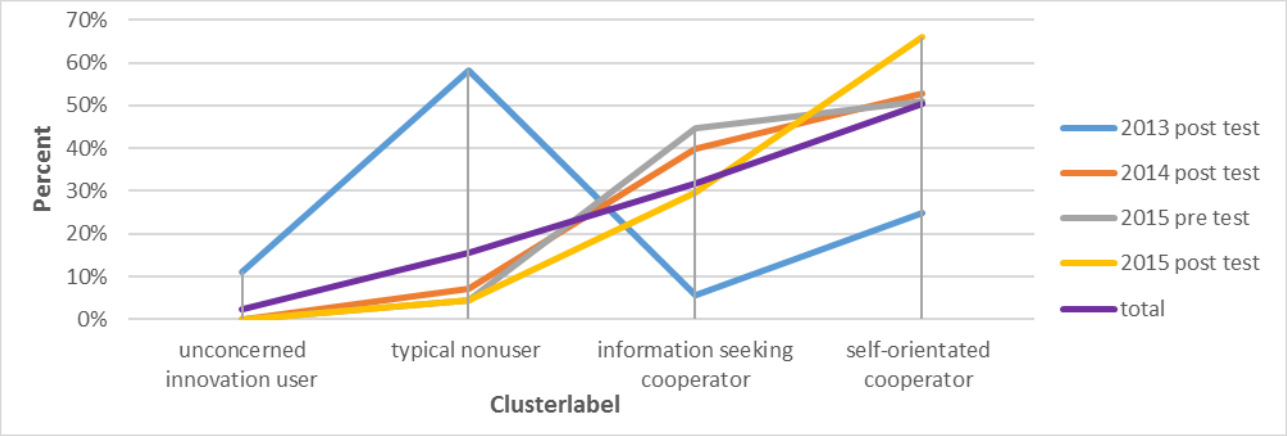

Figure 3: Distribution of persons of one measurement into clusters in percent.

It is noteworthy that all four people belonging to the *unconcerned innovation* user cluster were found in 2013, see Figure 3. Also the majority of the *typical nonuser* group attended the course in 2013. As mentioned before, the 2013 year differed from the other years. This impression continues for the distribution of persons into clusters. The differences to the others measurements are below the significance threshold ($p<0.001$). Participants in 2014 and 2015 are often assigned to one of the two *cooperator* clusters. The 2014 distribution shows an insignificant difference ($p=0.444$) to both tests from 2015. The pre and post test from 2015 show a slight deviation of the cluster distribution ($p=0.057$). In 2015 there is a migration of 14 persons into different clusters. 71.42% of those (n=10) shift from the *information seeking cooperator* cluster to the higher *self-orientated cooperator*

cluster. Another person switched from *typical nonuser* to *information seeking cooperator*. Only three swapped to a "lower" cluster.

Discussion and remarks

According to the study we can distinguish two main groups in the 2014 and 2015 course, the *information seeking cooperator* and the *self-orientated cooperator*. One of the "effects" of the course is the shift from the first to the second. We have to study in more detail in which respect these two groups differ and what factors influence to which cluster teachers belong. The design of the course can take this into account by addressing specific course elements to the two different groups. As mentioned before, our further goal is to combine the SoC profiles with other parts of our study. Doing so will allow us to validate our results, gain new insights and recognize a correlation between to aspects. In 2016/2017 we are implementing a fourth course. We also intend to expand our study by adding questions to the level of use (Hall, Loucks, Rutherford, & Newlove, 1975), interviewing participants and conducting another surveys six months after the course's end in order to measure the long term effects.

Acknowledgements

We are grateful to Janina Oesterhaus (Janina Niemann since 2014) for starting the research on SoC in 2013 and 2014 and for doing preliminary analyses of the 2013 SoC data. We are also grateful to Ruben Loest for supporting the SoC data collection in 2015.

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Characteristics of pre-service primary teachers' noticing of students' thinking related to fraction concept sub-constructs

Àngela Buforn[1], Ceneida Fernández[1], Salvador Llinares[1] and Gloria Sánchez-Matamoros[2]

[1]Universidad de Alicante, España; angela.buforn@ua.es; ceneida.fernandez@ua.es; sllinares@ua.es

[2]Universidad de Sevilla, España; gsanchezmatamoros@us.es

The aim of this research is to characterise how pre-service primary teachers notice students' reasoning related to the fraction concept sub-constructs: part-whole, measure, quotient, ratio, operator and reasoning up and down. 82 pre-service teachers analysed primary school students' answers to five fraction problems. Each student's answer shows different characteristics of students' reasoning in each sub-construct of the fraction concept. Five profiles of pre-service primary teachers have been identified according to how they used the mathematical elements to recognise students' reasoning.

Keywords: Fraction, students' reasoning, noticing.

Introduction and theoretical background

The study reported here is part of a larger study focused on how pre-service primary school teachers notice characteristics of students' proportional reasoning (Buforn, & Fernández, 2014). Several studies have indicated that the development of primary school students' fraction concept is important in order to develop relational thinking and proportional reasoning (Empson, & Levi, 2011; Lamon, 2007; Naik, & Subramaniam, 2008). However, the fraction concept is complex since it consists of multiple sub-constructs: part-whole, measure, quotient, ratio and operator (Behr, Harel, Post, & Lesh, 1992). In this paper, we are going to focus on how pre-service primary teachers notice students' reasoning related to the fraction concept sub-constructs. We also include the sub-construct *reasoning up and down* since it is an important component to develop proportional reasoning (Lamon, 2007; Pitta-Pantazi & Christou, 2011).

The skill of noticing students' mathematical reasoning

Recent research has shown that being able to identify relevant aspects of teaching and learning situations and interpret them to take instructional decisions (Mason, 2002) is an important teaching skill (professional noticing). Focusing on the skill of noticing students' mathematical thinking, Jacobs, Lamb and Philipp (2010) characterise this teaching competence as three interrelated skills: (1) attending to students' strategies that implies identifying important mathematical details in students' strategies; (2) interpreting students' mathematical reasoning taking into account the mathematical details previously identified; and (3) deciding how to respond on the basis of students' reasoning.

Studies, in this line of research, have indicated that identifying the relevant mathematical elements of the problem plays an important role to recognise characteristics of students' mathematical reasoning and also to take instructional decisions (Bartell, Webel, Bowen, & Dyson, 2013; Callejo, & Zapatera, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015). In the last years, researchers

have focused on different mathematical domains such as the derivative concept (Sánchez-Matamoros et al., 2015), classification of quadrilaterals (Llinares, Fernández, & Sánchez-Matamoros, 2016), algebra (Magiera, van den Kieboom, & Moyer, 2013) and ratio and proportion (Son, 2013) showing that the development of the noticing skill is not easy for pre-service teachers during teacher education programs.

Our study is embedded in this line of research and focuses on analysing how pre-service teachers interpret students' reasoning related to the fraction concept and how they use their interpretation of students' reasoning to propose new activities to help students progress in their reasoning.

Sub-constructs of the fraction concept

In our study, we consider the following sub-constructs of the fraction concept:

- Part-whole: it is defined as a situation in which a continuous quantity or a set of discrete objects is partitioned into parts of equal size (Lamon, 2005).
- Measure: it can be considered as a number which expresses the quantitative character of fractions, its size; or the measure assigned to some interval (Behr, Lesh, Post, & Silver, 1983; Pitta-Pantazi & Christou, 2011).
- Quotient: it can be seen as a result of a division situation (Pitta-Pantazi & Chrsitou, 2011) and interprets a rational number as an indicated quotient (it is exemplified by sharing contexts).
- Operator: it is seen as a function applied to a number, an object or a set (Berh et al., 1992).
- Reasoning up and down: it is a particular case of the part-whole sub-construct where the unit in a task is implicitly defined (Lamon, 2005) and students need to reason up from a rational number to the unit and then back down from the unit to another rational number.

Participants and the task

The participants in this study were 82 pre-service primary teachers (PTs) during their third year in an initial teacher education program at the University of Alicante (Spain). In previous years, pre-service teachers had attended a subject focused on numerical sense (first year) and a subject focused on geometrical sense (second year). In the third year, they were attending a subject related to the teaching and learning of mathematics in primary school. One of the units of this subject was about teaching and learning of the fraction concept and proportional reasoning. The aim of this unit is focusing pre-service teachers' attention on how primary school students learn the fraction concept including features of students' understanding of the different sub-constructs. Data were collected after this unit.

Pre-service teachers solved a professional task focused on interpreting three primary school students' answers to five primary school problems related to the five sub-constructs of the fraction concept (part-whole, measure, quotient, operator, and reasoning up and down) (Table 1).

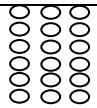
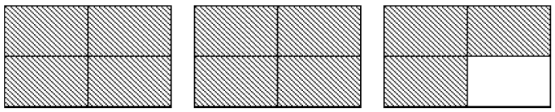
Problems	Characteristics
1. How many spots are in 2/3 of the set?  Explain your answer.	Part-whole. Partitioning the set in 3 equal groups and selecting 2.
2. Indicate which number is X in the following number line. Explain your answer. 	Measure. Identifying a unit fraction (for instance 1/10) and iterating it to find X.
3. Four people are going to share three identical pepperoni pizzas. How much pizza will each person get?	Quotient. Result of a division situation in which it is required the division of 3 pizzas between 4 people.
4. The teacher asked Nicolas to make some photocopies. Nicholas made a mistake and pressed the button that reduce the size of each copy by ¾. By how much should Nicholas increase each of the reduced copies to reproduce the original size?	Inverse operator. Inverse function has to be applied: ¾·x=1.
5. The shaded portion of this picture represents 3+2/3. How much do the 4 small rectangles represent? 	Reasoning up and down. Reasoning that implies identifying the unit "3 small rectangles" and then, representing a fraction.

Table 1: Problems related to the five sub-constructs of the fraction concept considered in the task

Each student's answer shows different characteristics of students' reasoning in each sub-construct of the fraction concept. In Figure 1, the three primary school students' answers to the part-whole problem presented to pre-service teachers are given. To interpret students' answers, pre-service teachers answered the following four questions (Table 2).

Questions	Aim
a) What mathematical concepts must a primary school student know to solve this problem? Explain your answer.	Identifying the learning objective of the primary school problem
b) What are the characteristics of students' mathematical reasoning involved in each student's answer? Explain your answer.	Recognising characteristics of students' mathematical reasoning
c) How would you change the problem to help students progress in their mathematical reasoning if they have had difficulties solving the problem? Explain your answer.	Responding on the basis of students' mathematical reasoning, supporting (question c) or extending (question d).
d) How would you change the problem to help students progress in their mathematical reasoning if they have not had difficulties solving the problem? Explain your answer.	

Table 2: Questions of the task

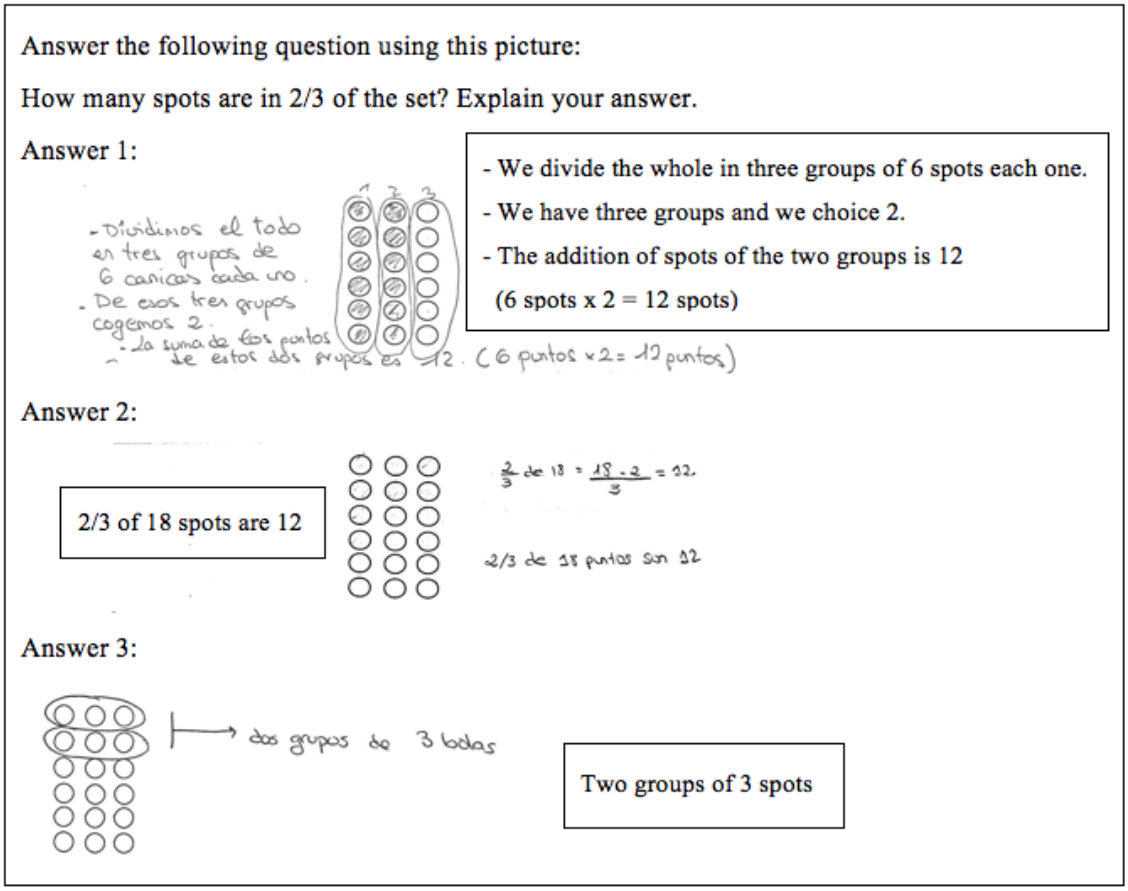


Figure 1: Primary students' answers to the part-whole problem

Analysis

Data of this study are pre-service teachers' answers to the first two questions (a and b) of the professional task (Table 2). Therefore, we focus on how pre-service teachers interpret students' reasoning related to the fraction concept in this paper. The answers to each question were analysed individually by three researchers and agreements and disagreements were discussed. We observed how pre-service teachers identified the mathematical elements involved in each problem and how they used them to recognise characteristics of students' mathematical reasoning.

From this analysis, we have identified six different profiles of pre-service teachers considering how they used the mathematical elements of the problem to recognise students' reasoning (Table 3).

Results

Results show that 41 out of 82 pre-service teachers had difficulties in recognizing characteristics of students' reasoning (Profiles 0 and 1). However, 19 out of these 41 pre-service teachers identified the mathematical elements involved in each problem. This data suggests that recognising the important mathematical elements of the problem is not enough to recognise characteristics of students' reasoning.

How pre-service teachers identified and used the mathematical elements of the problem to recognise students' reasoning	Number of PT's
Profile 0: They do not identify the mathematical elements and do not recognise characteristics of students' reasoning in any task	22
Profile 1: They identify the mathematical elements related to all sub-constructs of fraction concept but do not recognise characteristics of students' reasoning in any task	19
Profile 2: They identify the mathematical elements and recognise characteristics of students' reasoning related to part-whole, measure, quotient, and operator	8
Profile 3a: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to part-whole, measure, quotient, operator and reasoning up and down (but not related to the inverse operator)	25
Profile 3b: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to part-whole, measure, quotient, operator and inverse operator (but not related to reasoning up and down)	5
Profile 4: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to all sub-constructs of the fraction concept	3

Table 3: Profiles of pre-service teachers identified

Pre-service teachers of Profile 0 did not identify the mathematical elements and used general expressions such as "*fractions and operations with fractions*". Pre-service teachers of Profile 1 were more specific, identifying the mathematical elements implied in all the problems. For example, pre-service teachers of Profile 1 indicated: *"In problem 1, the mathematical element involved is part-whole. In problem 2, the idea of measure or number line. In problem 3, quotient. In problem 4, the idea of operator. In problem 5, part-whole and unit*". However, pre-service teachers in these both profiles did not recognise characteristics of students' reasoning. These pre-service teachers provided general comments based on the correctness of the answer: "*answer 1 is correct; answer 2 is correct; answer 3 is not correct, the student doesn't understand the concept*"; gave a description of the student answer "*the student 1 divides in 3 groups and choices 2 groups, student 2 makes a multiplication and then a division, and student 3 doesn't understand the problem*"; or interpreted incorrectly students' answers "*the three students solved the problem correctly but using different strategies*".

Pre-service teachers of profiles 2, 3a, and 3b identified the mathematical elements involved in each problem and recognised evidence of students' reasoning in some sub-constructs. Particularly, pre-service teachers of Profile 2 recognised characteristics of students' reasoning related to the sub-constructs part-whole, measure, quotient and operator. For instance, the next excerpt is a pre-service teacher's answer to the part-whole problem (problem 1): "*Answer 1: the student shows the understanding of the part-whole concept because identifies the whole and re-group the spots in equal groups (dividing the whole in equal parts). Answer 2: the student identifies the total of spots (whole) and selects 2/3. He interprets the fraction as an operator. Answer 3: He doesn't identify the whole and doesn't re-group in equal groups";* to the measure problem (problem 2): *"Answer 1: he*

solves the problem correctly because he identifies the unit fraction (1/5) in the number line. Answer 2: he solves the problem iterating 2/5 and then uses the idea of operator to obtain ½ of the interval. Answer 3: he doesn't identify the unit fraction and doesn't take into account what means 2/5 in the number line"; and to the quotient problem (problem 3): *"In answers 1 and 2, the student understands the fraction as a quotient because he divides the pizzas in equal parts. Answer 3: he doesn't understand the meaning of quotient because he divides the pizzas in different parts".*

Pre-service teachers of Profile 3a identified the mathematical elements and recognised characteristics of students' reasoning related to the sub-constructs part-whole, measure, quotient, operator and reasoning up and down (but not related to the inverse operator). The difference with pre-service teachers of Profile 2 is that pre-service teachers of Profile 3a recognised characteristics of students' reasoning related to the reasoning up and down sub-construct: "*In answer 1, the student doesn't identify the unit and the unit fraction. In answer 2, the student identifies the unit but doesn't identify the fraction that represents 4 small rectangles. In answer 3, the student identifies the unit and identifies correctly which fraction represents 4 small rectangles*"; and pre-service teachers of Profile 3b recognised characteristics of students' reasoning related to the inverse operator instead of the reasoning up and down sub-construct "*A1: he uses an additive wrong strategy. A2: he doesn't know how to make the reduction and the enlargement. A3: he knows how to obtain the original paper multiplying by the inverse fraction of 3/4".*

Finally, only 3 pre-service teachers (Profile 4) identified the mathematical elements and recognised characteristics of students' reasoning in all the sub-constructs of the fraction concept.

The different sub-constructs of the fraction concept were used by pre-service teachers to recognise characteristics of students' reasoning in different ways. The way in which pre-service teachers used the sub-constructs operator (and its inverse) and the reasoning up-and-down promoted the emergence of different pre-service teachers' profiles.

Conclusions

The five pre-service teachers' profiles show characteristics of the way in which pre-service teachers notice students' fractional reasoning. The difference between profile 0 and profile 1 is that pre-service teachers start to identify the mathematical elements of the problems but continue giving general comments based on the correctness of answers. The difference between profile 1 and 2 is that pre-service teachers of profile 2 are able to recognise characteristics of students' reasoning related to part-whole, measure, quotient, and operator sub-constructs. However, these pre-service teachers were not able to recognise characteristic of students' reasoning in problems where the unit was implicit (inverse operator and reasoning up and down). The difference between profile 2 and profile 4 is the fact that pre-service teachers of profile 4 recognise characteristics of students' reasoning in all the sub-constructs. However, there are two possible profiles between the profile 2 and profile 4 characterised by: recognising characteristics of students' reasoning related to the inverse operator (but not related to the reasoning up and down, Profile 3a), and recognising characteristics related to the reasoning up and down sub-construct (but not related to the inverse operator, Profile 3b).

These results provide information about different pre-service teachers' stages in the development of the skill of interpreting students' mathematical reasoning related to some sub-constructs of the

fraction concept. This information provides data to conjecture a pre-service primary teacher's hypothetical learning trajectory of noticing students' mathematical reasoning related to those sub-constructs (Figure 2). This hypothetical learning trajectory could inform us about the pre-service teachers' learning process of the skill of interpreting students' mathematical reasoning in the particular mathematical domain of the fraction concept.

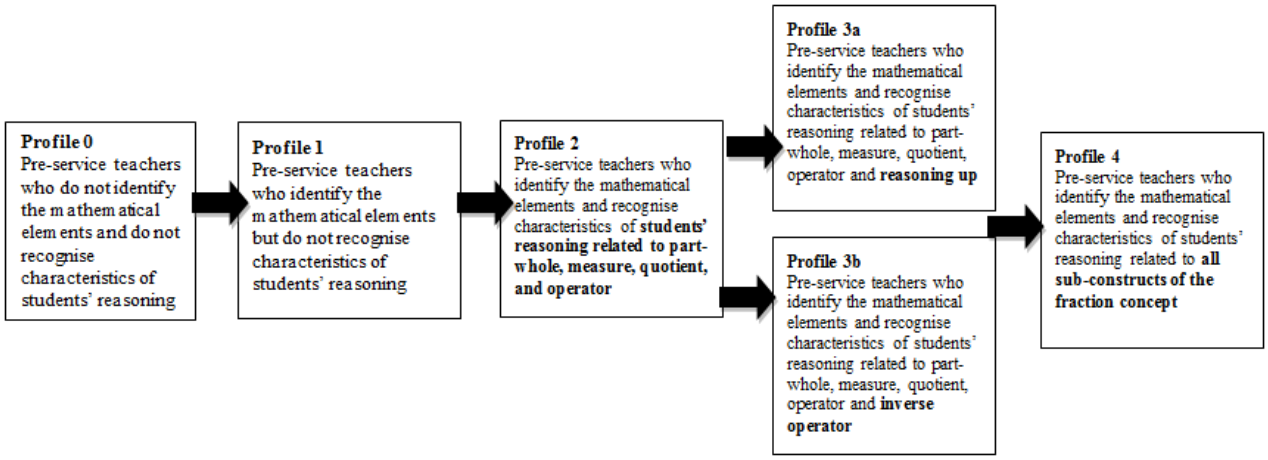


Figure 2: A pre-service primary teacher's hypothetical learning trajectory of noticing students' mathematical reasoning related to the fraction concept

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Analysing mathematics teacher learning in Lesson Study - a proposed theoretical framework

Stéphane Clivaz[1] and Aoibhinn Ni Shuilleabhain[2]

[1] Lausanne Laboratory Lesson Study, Lausanne University of Teacher Education, Switzerland, stephane.clivaz@hepl.ch

[2] School of Mathematics & Statistics, University College Dublin, Ireland, aoibhinn.nishuilleabhain@ucd.ie

The purpose of this paper is to analyse mathematics teacher knowledge incorporated during one cycle of lesson study. Analysis is undertaken utilising an extended framework which combines both the theoretical frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005). The proposed framework is situated as a tool to detail and analyse the use of mathematics teacher knowledge in planning, conducting, and reflecting on research lessons in a lesson study cycle in a primary-school case study in Switzerland.

Keywords: Teacher collaboration, Lesson Study, professional development, mathematical knowledge for teaching, levels of teacher activity.

Lesson Study and models of knowledge for teaching

Lesson study is a collaborative model of professional development which supports teacher learning (Huang & Shimizu (Eds.), 2016). Originating in Japan, this model has grown in international popularity over the past two decades, particularly in the field of mathematics education, and much research has detailed evidence of mathematics teacher learning through lesson study (e.g. Lewis et al., 2009; Murata et al., 2012; Ni Shuilleabhain, 2016).

Lesson study provides teachers with opportunity to contextualize representations of their classroom activities, while also making their implicit knowledge and practices explicit through their conversations within the group (Fujii, 2016). Each lesson study cycle consists of a number of steps where teachers begin by studying the curriculum and deciding on a research theme, planning a research lesson according to that theme, conducting and observing the live research lesson, and reflecting on student learning within the lesson (see Fig. 1) (Lewis 2016; Lewis et al., 2009)

With increased international educational research on lesson study, there have been calls to deepen the knowledge base of the development of teacher knowledge within this model in order to provide a solid theoretical foundation for its use in teacher education (Clivaz, 2015; Miyakawa & Winsløw, 2009). In this paper, we hope to contribute to the literature on professional development for mathematics teachers by analysing the mathematical knowledge utilized by teachers in their participation in lesson study, utilizing our proposed theoretical framework.

The two authors of this paper, in their analysis of teacher knowledge and learning in lesson study, seek to deliberately build on previous existing frameworks of teacher knowledge: Mathematical Knowledge for Teaching (Ball et al., 2008) and the Levels of Teacher Activity (Margolinas et al., 2005). Through analysis utilizing a combination of these frameworks (Prediger et al., 2008), we will detail features of the mathematical knowledge for teaching utilized by teachers in their participation in lesson study and will also track the movement of this knowledge.

MKT/ levels in LS: Towards a coordinated model

In this paper, we propose a framework which was developed based on data generated in two case study sites - with eight participating primary (grade 3-4) teachers in Switzerland and five lower secondary (middle school, grade 7) teachers in the Republic of Ireland. Analysis began by utilizing the Mathematical Knowledge for Teaching framework (Ball et al. 2008) to investigate the contributions by teachers in a lesson study cycle. However, we found that this model did not fully incorporate all the elements of teacher knowledge included in the lesson study cycle, particularly in capturing the educational values and conceptions of teaching and delineating between the layers of planning sequenced content of instruction, while also attending to students' thinking during the lesson. At this point in the analysis, Margolinas et al.'s (2005) Levels of Teacher Activity was identified as a framework which could encapsulate these elements of teachers' knowledge. Building on qualitative data generated through audio/video recordings of teacher conversations during lesson study meetings, teacher notes from lesson study meetings, researcher field notes, and selected samples of student work from research lessons, we present an extended model of the categorization of knowledge required for the teaching of mathematics. Our first example of analysis presented is from the Swiss case study and future work will present further analysis from the Irish case study data. In analysing and comparing these sets of local data, we attempt to demonstrate a more global sense of this proposed framework of mathematics teacher knowledge in lesson study.

Mathematical Knowledge for Teaching

In their ground-breaking work in 2008, Ball, Thames and Phelps addressed the concepts of content and pedagogical content knowledge in their model of Mathematical Knowledge for Teaching (MKT - see upper part of Figure 1). In this paper, they identified domains of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) used in teaching, which further defined the knowledge and skills required of mathematics teachers in relation to student learning and to mathematics content.

Research on these different categories of MKT has demonstrated direct links between teacher knowledge and high-level teaching practices (Clivaz, 2014; Hill, Ball, & Schilling, 2008) and with subsequent student learning outcomes (Hill, 2010).

Incorporating this model of MKT with teacher learning in lesson study, research has shown that Knowledge of Content and Students and Knowledge of Content and Teaching (features of PCK as defined by Ball et al. (2008)) are important elements of teacher knowledge utilized in lesson study cycles (Leavy, 2015; Ni Shuilleabhain, 2015b; Tepylo & Moss, 2011). However, considering the multitude of teacher knowledge and practices incorporated within each lesson study cycle in planning, conducting, and reflecting on a mathematics lesson, this model may not capture all the decisions, actions, practices, and skills required of mathematics teachers participating in lesson study.

Levels of teacher activity and MKT

To describe teacher activity, both in and outside of the classroom, Margolinas developed a model of the mathematics teacher's milieu based on Brousseau (1997). This model was designed to take into account the complexity of teachers' actions and to capture the broad range of activities contained in teaching and learning (Margolinas et al., 2005, p. 207).

+3	*Values and conceptions about learning and teaching*
+2	*The global didactic project*
+1	*The local didactic project*
0	*Didactic action*
- 1	*Observation of pupils' activity*

Table 1: Levels of a teacher's activity (Margolinas et al., 2005, p. 207)

At every level of environment (or milieu) the teacher must consider all that is occurring at the current level as well as those levels that are directly above and below. These multidimensional tensions relate to a non-linear and non-hierarchical interpretation of teacher's work (Margolinas et al., 2005, p. 208). In addition to commonplace professional opportunities where teachers speak about their beliefs and experiences on general educational concepts or about teaching and learning mathematics (level +3), about teaching and learning of a particular mathematical subject (level +2), or about the lesson they are preparing (level +1), during the phases of planning and reflection in LS teachers also have opportunity to discuss their classroom activities (level 0) or observations of student activity from a lesson (level -1).

This activity model was used by Clivaz (2014) and aligned with the MKT model in order to capture the movement of didactical situations, beyond the possible static characterisation which may be interpreted in the MKT model (Ball et al., 2008, p. 403). The combination of these frameworks allowed teacher knowledge to not only be analysed in terms of mathematical knowledge for teaching, but also mathematical knowledge in teaching (Rowland & Ruthven, 2011). Similarly, Ni Shuilleabhain (2015a) used the MKT model to analyse teacher learning in lesson study, but combined this with the idea of the 'student lens' (as suggested by Fernandez, Cannon, & Chokshi, 2003, p. 180), in proposing an additional layer of the model put forward by Ball et al. (2008). This concept of a 'student lens' incorporated the PCK a teacher utilises in seeing mathematics "through the eyes of their students" (Fernandez et al., 2003, p. 179).

When aligned with Margolinas et al.'s (2005) model, this layer of teacher knowledge relates partly to the -1 'Observation of pupils' activity' which can be anticipated and interpreted, but extends this observation to thinking of the mathematical content from the students' perspective. In our proposed framework, we therefore see this view of the mathematics through the eyes of the student as a layer below the observation of a students' work and include a new level of -2 level relevant to teacher knowledge titled the "student lens" (see Figure 1).

Proposed theoretical framework

Explicitly combining these two approaches to analyse the knowledge utilized by mathematics teachers during lesson study, the authors here present a new theoretical framework (see Figure 1). This framework attempts to capture the knowledge required of mathematics teachers, in the broad and complex range of teaching and learning activities, and represents teacher knowledge and activities incorporated during each phase of a lesson study cycle (see Lewis & Hurd, 2006, p. 4).

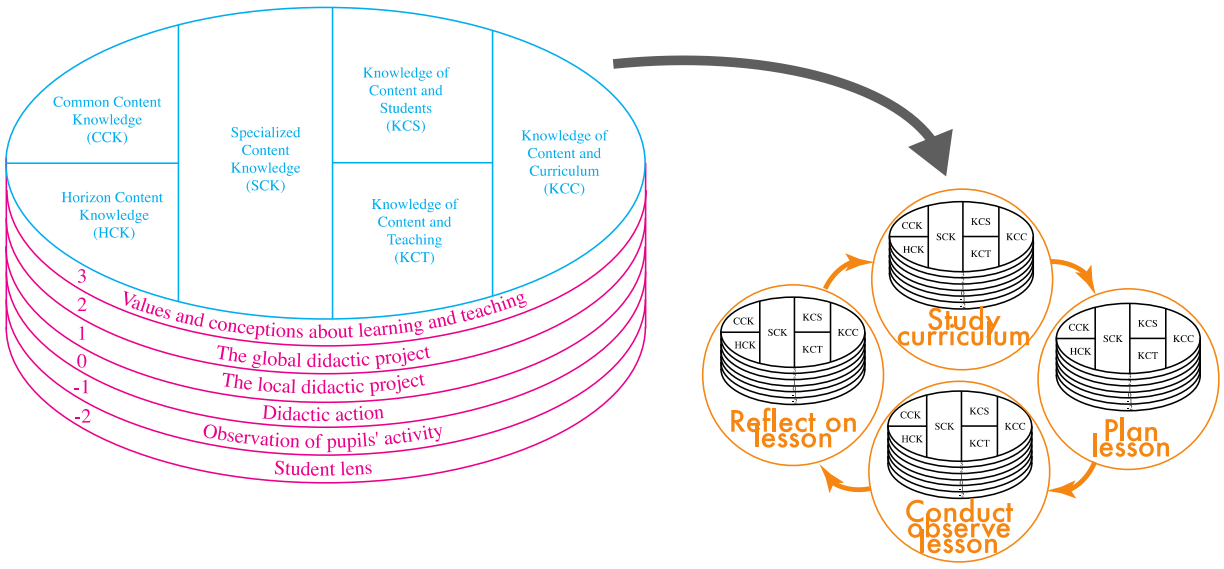

Figure 1: MKT and levels of teacher activity at lesson study phases

We first utilize the model to categorize the knowledge (MKT and levels of activity) appearing during the lesson study cycle. The knowledge about a particular mathematical topic will then be tracked over each phase of lesson study and the relations between the occurrence of this knowledge examined. At this stage, 'knowledge' is considered as collective (e.g. Ni Shuilleabhain, 2016).

Analysis

In this paper, data generated though video recordings of the Swiss case study are analysed utilising the proposed framework. Eight primary generalist teachers, new to lesson study, and two facilitators (one specialist in teaching and learning and the other a specialist in mathematics didactic (first author of this paper)) participated in the research which occurred over two academic years. Four cycles of lesson study were undertaken in this time, with a meeting held on average every two weeks during the school year (Clivaz, 2016). Each of these 37 meetings (about 90' each) were videotaped and transcribed and form the base of the analysis utilizing the framework outlined above (Figure 1) and incorporating defined features of KCS and KCT as utilized in lesson study (Ni Shuilleabhain, 2015b).

We present analysis of the first lesson study cycle where teachers chose to focus on the topic of integers and place value. The main reason for choosing this subject was the difficulty students had with whole numbers. In the first session, teachers discussed a particular difficulty their students had with counting through to new groups in base 10:

Océane: The counting through to the next ten.

Caroline: But each time they have to count through to (tens, hundreds, …)

Stéphane (facilitator): What's happening with counting through to the next ten?

Caroline: It's… that we have no more to write here! We have to use the digits which already exist. So, we count through to come back to one… In fact… Yes, it is the abacus, in fact, we need to move by one each time we arrive at a nine at the end. We need to move by one.

Océane: We exchange one packet of ten.

In this passage, during the *study curriculum* phase, teachers are at level of the *global didactic project* (+3) and this unpacking of mathematical knowledge is a Specialised Content Knowledge (SCK) i.e. the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students (Ball et al., 2008, p. 399). At this stage, the place aspect of number system was predominant in teachers' discourse and, when the value aspect appeared, it was linked with the value. To further address this knowledge, the facilitators suggested working on students' actual mistakes. Teachers and facilitators proposed mistakes like:

5 hundreds + 12 tens + 3 units = 515

This work prompted teachers to do the task as if they themselves were students. At some moments during the activity teachers even spoke like students - placing them at the level of *student lens* (-2). This allowed the teachers to go deeper into potential difficulties for students and by further studying curriculum materials (referred to as *kyozai kenkyu* by Takahashi & McDougal, 2016), teachers had opportunity to clarify this aspect for the research lesson. This passage is situated at the same phase, level and type of MKT as the previous excerpt above.

Anne (facilitator): […] It's a particular type of exchange since it's in the place value system. So, we can distinguish the two dimensions: the dimension of the place and the dimension of the decimal value which is revealed in the exchanges.

Stéphane: In fact, I prefer to talk about grouping/ungrouping instead of exchanging.

Océane: Oh, I see!

Following these two excerpts, we will briefly summarize the work undertaken by these teachers planning the second research lesson and focus on this phase for analysis. The group chose a task in the form of a board game involving the exchange of "1 hundred", "1 ten" and "1 unit" cards. Following a planning exploration of the task, this research lesson was taught by one of the group and, during the post lesson discussion, teachers agreed that the task should be modified to allow students practice the exchange of values and relate these to aspects of the number system. This revised lesson was taught by another member of the group to a different group of students.

At the beginning of the game a student, Julie, arrived on the square "give 35". She had three cards of "1 unit", three cards of "1 ten" and four cards of "1 hundred". In order to get three cards of "1 ten" and two cards of "1 unit", Julie wanted to exchange two "1 hundred" cards. The teacher, Edith, wanted to explain to Julie that two "1 hundred" cards were worth more than these three cards of "1 ten" and two cards of "1 unit".

Edith: So, two hundreds - that's how many?

Julie: Two hundred.

Edith: That's two hundreds. If you tell me: "I want three tens and two units." Three tens, how many is that?

Julie: Thirty.

Edith: You told me three tens makes thirty. And what about two units?

Julie: Two.

Edith: If you put the thirty and the two together? How many is that?

Julie: Thirty-two.

Edith: So you swap two-hundred for thirty-two! You're very generous!

In this passage situated during the *conduct lesson* phase, at level 0 (*didactic action*), the teacher converted all cards into numbers to compare them, instead of doing direct exchanges. Julie followed the teacher without expressing her own way of reasoning (which can be observed in another passage and demonstrates a 'direct exchange' way of thinking). In this case, we categorize the MKT in two ways. First as a KCS, where Edith did not notice or interpret Julie's mathematical thinking or strategies. Second as a SCK, related to the unpacking of mathematical knowledge, as detailed in the following excerpt.

During her dialogue with the class, Edith had to explain that one hundred is the same as ten tens. Here, again, her argument is to convert to units - which requires students to already understand place value. This argument can be summarized as follows:

1 hundred = 100 units

and 10 tens = 100 units

therefore, 1 hundred = 10 tens

In the final lesson plan, the group of teachers reflected on this strategy and argued against it:

> "Often exchanges are not really carried out and we go through the number. For example, when asked to exchange 12 hundreds into tens, many students (and adults) will go through the number 1200, namely 1200 units, to say that that 1200 is 120 tens, without being able to make a direct exchange from hundred to tens. Teachers also often explain this exchange in this way. In this case, we are in a type of vicious circle, since it means that it is necessary to have understood number system to understand the grouping/ungrouping in the place value system."

This episode appears in our data in the research lesson (*conduct and observe lesson* phase, level 0), in the notes of the observing teachers (*conduct and observe lesson* phase, level 0), in the post lesson discussion (*reflect on lesson* phase, level 0) and, in the above extract, in the lesson plan (*reflect on lesson* phase, level +2) where observations and analysis of the group were generalized and decontextualized from the particular lesson to the level of a global didactic project. In each case the knowledge represents a typical SCK.

The final example of this knowledge was found at the end of the *reflect on the lesson* phase. After discussing the lesson and the mathematical difficulty of directly converting hundreds into tens, Valentine (a teacher with over 30 years of teaching experience) realized she had observed a similar difficulty her own students in this topic, outside of the lesson study group. As a result of their collaborative reflection conversations, she began to realize that her students' errors were likely due to her use of only one strategy in teaching this topic:

Valentine: But, I've got a question. For example, in nine-hundred-sixty-three - how many tens are there? Ninety-six. But my students, they learned a trick - they write the number 963 and just go to the tens digit and write what is left: 96. I'm convinced they just use this trick. I probably didn't know how to explain that to them! Myself… I always convert in money! You will have nine hundred and sixty three one-franc coins. If you need to only have ten-francs notes… then you will have ninety-six ten-francs notes.

Although this observation was not directly related to observations during the research lesson, we still categorize it as level -1 since Valentine put herself in the position of observing her students converting 963 into tens. This conversation incorporates teacher KCS in interpreting students' responses and is situated at level -1 (*observation of pupils' activity*).

Utilising our proposed framework and building on our analysis of teachers' collective conversations, we can detail the types and levels of knowledge incorporated by mathematic teachers in their participation of lesson study. Utilising this framework provides us with opportunity to track the knowledge included in the planning and reflection of mathematics research lesson over various phases of lesson study.

Conclusion

This paper proposes an extended theoretical framework of mathematics teacher learning in lesson study combining the existing frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) with Levels of Teacher Activity (Margolinas et al., 2005). In this paper the proposed framework is situated as a tool used to detail and analyse the use and movement of mathematics teacher knowledge in planning, conducting, and reflecting on research lessons. Based on case study data generated through mathematics teachers' participation in lesson study, we have analysed teachers' qualitative conversations and considered the potential evolution of mathematics teacher knowledge over a cycle of lesson study. Analysis to date has demonstrated that in planning and reflecting on research lessons, teacher knowledge of various forms (e.g. SCK and KCS (Ball et al., 2008)) and across varying levels of activity (Margolinas et al., 2005) are incorporated in these separate phases of lesson study.

We hope this model will contribute to the literature on professional development of mathematics teachers and may serve to underpin further evidence of teacher learning in lesson study.

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Innovating teachers' practices: Potentiate the teaching of mathematics through experimental activities

Maria Cristina Costa[1] and António Domingos[2]

[1]Instituto Politécnico de Tomar, UIED[3], Portugal; ccosta@ipt.pt

[2]Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, UIED, Portugal; amdd@fct.unl.pt

This paper aims to potentiate the teaching of mathematics through hands-on experimental activities at the primary school level and by promoting teachers' professional development, using innovative practices. A Teacher Design Research cycle involving a group of primary school teachers during one school year was performed. This cycle consists in teacher training sessions, including an introduction to science and mathematics content, hands-on workshops for teachers and also classroom interventions in order to promote experimental activities and observe teachers in action. A particular case study of teacher Luísa will be presented. It was found that she gained motivation and self-confidence to innovate her practices, showing enhanced ability to perform experimental activities with her students.

Keywords: Teacher professional development, training practices, hands-on, teacher knowledge, primary school.

Introduction

This paper aims to potentiate the teaching of mathematics, through hands-on student-centered experimental activities at the primary school level and by promoting teachers' professional development using innovative practices.

Several studies show the importance of teaching mathematics and science through experimental activities in the early years of schooling to motivate the younger generations to scientific and technological areas, considered crucial for economic development and scientific literacy (Coll, Dahsah, & Faikhamta, 2010; Hallstrom, Hulten & Lovheim, 2014; Osborne, 2009; Perera, 2014).

Treacy and O'Donoghue (2014) also refer to scarce research about the integration of mathematics and science in classroom contexts as well as the lack of a widely-adopted teaching model. These authors advocate that "hands-on, practical, student-centered tasks should form a central element when designing an effective model for the integration of mathematics and science" (p. 1).

This study is part of a first round of a Teacher Design Research programme involving a group of primary school teachers in a cluster of schools of a region of central Portugal, held during the 2015/2016 school year. This experiment was carried out in the framework of a pedagogic intervention project that aims at introducing new methodologies to promote the learning of mathematics and science through hands-on experimentation and using the "inquiry" method.

To achieve this purpose, a lifelong training course was designed, with the collaboration of university researchers and a formation center, to include mathematics and science hands-on workshops, in order to help teachers develop their teaching skills and update their knowledge on these topics.

In this study, a particular case of a teacher, who used one of the proposed science experimental activities, to explore mathematical concepts and student-centered tasks, using the inquiry-based and problem-solving approach, will be presented.

Literature review

The great lack of professionals in the STEM (Science, Technology, Engineering and Mathematics) areas must be countered with an intervention at the level of the early years of schooling, being crucial to provide quality scientific practices (DeJarnette, 2012; Eshach & Fried; 2005; Johnston, 2005). The incorporation of hands-on experimental activities into the classroom, with scientifically well-prepared adults, lead to significant improvements in performance and produce positive attitudes towards science and learning (Mody, 2015; Myers Spencer & Huss, 2013).

The inquiry approach calls on the natural curiosity of children and develop their creativity and critical questioning at an age when they have the urge to discover the world around them (Alake-Tuenter et al., 2012; Krogh & Morehouse, 2014; Rocard et al., 2007).

Teachers are the cornerstone of any renewal of science education and being part of a network motivates them, contributes to improve the quality of teaching and promotes the sustainability of their professional development (Abell & Lederman, 2007; Rocard et al., 2007; Zehetmeier & Krainer, 2011). Martins (2006) claims to be a priority to strengthen investment in scientific research in the field of science education in the early years of schooling and continuing teacher training. Murphy, Varley and Veale (2012) recommend a professional development for teachers that will allow them to enhance their conceptual and pedagogic knowledge on the inquiry-based approach. Ball (2003) says that an intervention to combat failure in math's performance will only be effective if it is focused on teaching methods: "No curriculum teaches itself and standards do not operate independently of professionals' use of them" (Ball, 2003, p. 1).

Ball, Thames and Phelps (2008) investigated competences that are required to teach and developed an empirical approach to determine the content knowledge needed for teaching (figure 1).

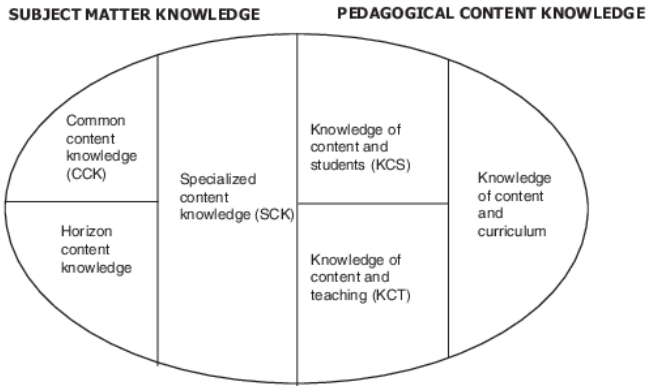


Figure 1: Mathematical knowledge needed for effective instruction (Ball, Thames, & Phelps, 2008)

Ball (2003) concludes that to improve children's mathematical learning, it is crucial to provide learning opportunities to teachers such as tailored courses, workshops and well-designed and taught materials. Afonso, Neves and Morais (2005) recommend that teachers should be given the opportunity to explore and experiment the contents to be developed in class in a reflective, collaborative environment where they feel supported. Also, Kuzle and Biehler (2015) sustain the

importance of "stimulate cooperation among the participants, and between the participants and the professional developer" (p. 2849).

Methodology

Teacher Design Research

Teacher Design Research (TDR) (Kelly, Lesh, & Baek, 2014) aims to promote the development of teachers as adaptive experts, using inquiry. This approach involves a collaborative work with research teams and teachers participating in the process, with the main objective to promote their professional development, leading them to innovate their practices and improve the whole teaching and learning process. The TDR premise is that the involvement of teachers in long-term (e.g. one year) design research periods can promote in-depth content learning and improve their ability to adapt to classroom environment and rethink their teaching beliefs and practices.

The pilot experiment: First cycle of the TDR

A pilot experiment with primary school teachers has been conducted in the academic year 2015/2016. This experiment included continuous training sessions where proposals of new content and experimental mathematics and science hands-on activities, to be used in the classroom, have been put forward. The teachers had the opportunity to explore the content and manipulate the materials to be able to apply them later with their students. In addition, the instructors also visited the trainees' own classroom to carry out experimental activities to exemplify them, and observe the teachers in action with their students. The teachers have also been encouraged to develop their autonomy by creating and implementing their own experimental activities.

Focus Group (FG) (Williams & Katz, 2001) was one of the working methodologies used throughout the training sessions to support the teachers and improve their practices. The last session was mainly focused on FG to promote reflection on the practices developed and make proposals for the following cycles of TDR. "Innovations need to be owned by the person implementing them on a personal level and transformed into their own practice in order to have practical effect" (Zehetmeier, Andreitz, Erlacher, & Rauch, 2015, p. 168).

This paper describes the case study of teacher Luísa (fictitious name) who participated in the study, proposed and carried out math-based activities using the inquiry method.

Participants

The participants in the pilot project comprised 14 teachers of 5 primary schools. These teachers participated in a first round of design research beginning in September 2015 and ending in July 2016. In this paper, we will study teacher Luísa who is 56 years old, has 37 years of service and is in charge of a third-year grade class with 25 students aged 8 and 9 years.

Data collection

Data collection consisted in observations (first author of the paper was a participant observer), semi-structured interviews, written records, and video (Cohen, Lawrence, & Keith, 2007). The action took place in two main moments: workshops with the teachers (to learn and practice what they are expected to implement) and at their classrooms (to support and observe them in action). At the end

of the training action, the participant teachers presented a written report, with a critical account on the pilot experiment and their proposals of innovative practices.

Data analysis and discussion

Teacher Luísa participated in the first TDR round. For about nine months, she attended the continuing training programme that consisted of seven workshops with, 3 to 4-hours duration, which introduced content, hands-on activities and methods of implementation in the classroom.

The first experimental activities carried out on the workshops was about electricity (http://www.academiacap.ipt.pt/pt/atividades/ciencia/fisica/77/). Before the intervention, Luísa completed a questionnaire that characterized her and her class. On the questionnaire, she refers that "the experimental aspect was not addressed in my graduation course" and "in the complementary training program (BA-equivalent degree) I attended it was dealt with only too briefly". Although she has attended several training sessions throughout her career, none of them was about electricity.

In the course of the training programme, Luísa was very participatory showing a great interest in the tasks performed. However, on several occasions she stated that: “I’m not comfortable to teach some of the content because I don't have full mastery of concepts and techniques and don’t know how to apply them”. She also admitted that: “I’m not able to handle some of the materials used in the experimental activities”. As early as in the first sessions she posed a series of questions such as: “What if the students ask me a question about this theme and I don't know the answer?” or “And what if an experiment does not yield the expected results?”

In addition to concerns about specialized content knowledge, the teacher also reveals concerns about pedagogical content knowledge. These insecurities have led us to rethink the approach to content and experimental activities, because we realized that it was very important to adapt training to the knowledge and the needs of the teacher, to make her feel secure and motivated to implement the tasks. We also realized that she gives great importance to specialized content knowledge and that she hardly will perform experiments that involve concepts she does not fully understand.

Given the great commitment of teacher Luísa to learn and her pedagogical concerns, she has been selected to receive the trainers in her own classroom to carry out experimental activities with her students, to exemplify the experiments, support the teacher and observe her in action.

During the intervention in the classroom, we have observed that teacher Luísa had a posture of inquiry, making questions to her students to guide them through the tasks, leading them to investigate, in order to find answers to the questions. It was interesting to observe Luísa making a reflection with her students, questioning them about the classroom hands-on activities, what they had learned, and what they would like to explore in the next experiments. Observations and interviews revealed she has a good knowledge of her students and knows how to introduce and adequate the content to each of them, according to their individual needs (KCS and KCT).

Two more sessions with teacher Luísa's students have been held which included tasks not covered by the training course with the other teachers. She again felt insecure and reported that she wouldn't be able to implement it without the support of the instructors. This shows the importance of the training workshops with the teachers before going to their classroom.

It has been suggested to her that she should propose activities involving mathematical content. Due to the difficulty shown by the teacher to achieve this objective, the researchers proposed a worksheet in which, based on experimental records like weigh, fruit diameter, potential difference measurements, it was possible to address the topic "organization and processing of data" that is part of the primary school syllabus. With this proposal, the teacher created some tasks (figures 2, 3 and 4). Figure 2 shows the method used by the teacher to propose mathematical problems, based on the classroom experiment performed by the team of instructors. In writing "electromagnet" she is showing that she knows the content acquired in training but she chooses to present mainly math specific content: problems involving operations, specially multiplication.

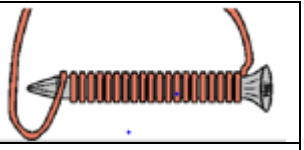
<table>
<tr><td colspan="2">When you used the nail in the electromagnet it was wrapped in copper wire.
I unwrapped one and measured the wire. It measured 40 cm.
</td></tr>
<tr><td>1. How many meters of wire were needed for the whole class?</td><td>2. One meter of wire costs 5 euros. How much was spent to wrap all the nails?</td></tr>
</table>

Figure 2: Math exercises suggested and implemented by teacher Luísa inspired by the activity performed in the classroom by the instructors

Based on the same experiment Luísa was invited to suggest tasks for data handling and processing, but she decided to collect data with her students (Figure 3) for further processing (Figure 4).

OCTOBER 2015		
NAME	**WEIGHT**	**HEIGTH**
Adriana	30,7 Kg	129 cm
André	31 Kg	139 cm
António	36,4 Kg	139 cm

APRIL 2016		
NAME	**WEIGHT**	**HEIGTH**
Adriana	32,2Kg	133 cm
André	31,9 Kg	142 cm
António	38,4 Kg	142 cm

Figure 3: Recording of heights and weights of the students in teacher Luísa's class

I grew _____kg I gained _____kg

In **October**, the **tallest** in my class was_____

In **May**, the **heaviest** in my class was_____

Which student **grew the most**? _____

Which student **gained more weight** between **October** and **May?** _____

Build a chart with the **weights** of the students in the class (**October-May**)

Figure 4: Math activity suggested and implemented by teacher Luísa

This attitude shows some autonomy on the part of the teacher to propose activities that are not provided for, in the school books. It also shows ability to adapt content to the specific needs of students. On the other hand, this could mean some resistance in using an experiment that was not familiar, preferring to use a context where she felt more comfortable, i.e. collecting data from the students. A possible explanation can be drawn from her report, where she shows lack of SCK:

> However, and given the nature of the subject matter addressed and the tools used, I do not feel comfortable to implement, in a natural/individual and consolidated process, many of the tasks proposed. (Teacher Luísa final report)

It also appears that she realized the importance of finding new ways of teaching math as she says in her report that math is part of day-to-day life: “The math activities performed in the class gained a new meaning as it was applied to practical real-life situations of individual students to complete the tasks proposed” (Teacher Luísa final report).

Luísa gained SCK and KCT, showing ability to do research, particularly on the internet, collecting information that she uses to make new approaches on teaching:

> (…) finding new ways of teaching math so that people understand that we think mathematically all the time and solve problems at several moments during the day (…) Math is thus part of our life and can be learned in a dynamic, challenging and funny way. (Teacher Luísa final report)

Finally, the teacher recognizes the impact of the project on her students:

> The class revealed very motivated when completing the tasks proposed by the instructors. The students adopted a cooperative, experimental attitude in which failure was regarded as a part of the scientific process. (Teacher Luísa final report)

In fact, student’s comments such as "this is the best experience of my life" "this is awesome", "you should come more often", among others, had impact on teacher Luísa motivation, contributing to make her recognize the importance of implementing hand-on experiments.

Final considerations

Visits to the classroom to support the teacher during the implementation of the experimental activities revealed very useful to improve teacher knowledge and confidence. Also, the enthusiasm, involvement and participation of the students in the classroom activities (mentioned in the teacher's final report) served to raise her awareness to the importance/relevance of these approaches. Such motivation has been observed in teacher Luísa who gained confidence to innovate her practices after receiving training and guidance. This teacher developed and implemented hands-on experimental activities with her students in classroom context, exploring their curiosity (using inquiry) and proposing problems requiring the use of math.

Although she created mathematics hands-on experiments, there still was, on the part of the teacher, some lack of confidence to innovate without the support of instructors. Like her, almost all teachers who participated in the continuous professional course suggested, during the final FG, that some of the experimental activities should be carried out by the instructors in their classrooms. All teachers were reluctant to propose innovative activities promoting by resorting to the inquiry method. It was noted that strong encouragement and responses on the part of the instructors were required to make the teachers change habitual teaching practices. However, throughout the sessions increased trust of teachers on their instructors and a better response to the tasks proposed has been observed.

It follows therefore that it is necessary to invest more in training and monitoring of teachers to further engage them in these approaches and improve their confidence and autonomy. Special mention should be made to the importance of getting the teachers to work out the activities before implementing them. Finally, it is concluded that this is a process that takes some time to be implemented and further work is needed to achieve the desired results.

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From emergency sirens to birdsong – Narratives of becoming a mathematics teacher

Elisabeta Eriksen[1], Bjørn Smestad[1], Annette Hessen Bjerke[1], Camilla Rodal[1], and Yvette Solomon[2]

[1]Oslo and Akershus University College of Applied Sciences, Norway; elisabeta.eriksen@hioa.no, bjorn.smestad@hioa.no, annette.hessen@hioa.no, camilla.rodal@hioa.no,

[2]Manchester Metropolitan University, UK; y.solomon@mmu.ac.uk

As part of a larger research project, we asked third-year PSTs to reflect on what they had learned about being mathematics teachers of the teacher education programme. The reflections were intended as presentations for first-year PSTs. In this article, we analyse the films by the third-year PSTs to understand the messages more experienced PSTs choose to communicate to novices. Concepts from Gert Biesta are the framework for the content analysis, and we find a complex picture of how qualification, socialization and subjectification interact in the narratives.

Keywords: Preservice teacher education, mathematics education, mentors.

Introduction

This article presents partial results from a research project examining pre-service teachers' (PSTs) developing identities as mathematics teachers, and, in particular, their experiences of mathematics in school placement. Previously published results from the project report how first-year PSTs value what they learn from mentors in practice more than the 'theoretical' input of the university based courses, not seeing the theoretical knowledge as transferable into teaching practices (Bjerke, Eriksen, Rodal, Smestad, & Solomon, 2013). In one intervention addressing the challenge, third-year PSTs presented to the first-year PSTs films describing experiences of becoming mathematics teachers in the course of the first three years of the programme. In this article, we analyze the third-year PSTs' presentations to understand how they view their own emerging professional identities.

Our research question is: What domains of their educational experiences do PSTs highlight in their presentations of their first three years in mathematics teacher education?

Research background and theoretical underpinnings

While learning to teach is about acquiring professional knowledge and skills, it is also about developing a teacher identity (Haniford, 2010). Adding to the identity work of experienced teachers, PSTs have the responsibility of successfully positioning themselves in relation to their teacher education programmes and cooperating teachers (Haniford, 2010). Identity formation is driven by the individual's goal state of what he/she wants to become (Smeby, 2007). Biesta (2012) discusses these processes under the headings of qualification, socialisation and subjectification, and we choose his concepts as the starting point of our analyses of PSTs' narratives.

For Biesta (2012), all education (including teacher education) is a question of judgement, because educators' decisions about the purpose of what they do occur within domains that may be in synergy with each other, but may also be in conflict. He notes three such domains, which interact and overlap: the domains of qualification, socialisation and subjectification. Qualification is about knowledge, skills and dispositions; socialisation and subjectification can be seen as opposites: while

socialisation applies to the induction of novices into existing practices, subjectification denotes how education contributes to a process of individuation, of becoming an independent subject. Educational judgements are underpinned by an understanding of the interdependencies between the three domains. In teacher education, the situation is more complex still: the purpose should not just be PSTs own qualification, socialisation and subjectification, but also to enable PSTs to become 'educationally wise', aspiring towards virtuosity in making educational judgements themselves. Such judgements are situated: they are made during the practice of teaching, and cannot be set out in advance, or in general - they are rooted in concrete situations and relate to the need to handle tensions and see possible synergies. To become 'educationally wise', one needs experience, together with opportunities to see more experienced others in action, and to discuss those actions in terms of the virtuosity of judgement which underpins them.

Method

The 32 PSTs in this study were enrolled in a four-year programme for primary school teachers (grades 1-7, ages 6-13) in Norway. They had chosen to continue with mathematics beyond the compulsory course spanning the two first years in teacher education, and were asked to look back on their mathematics education course and four school placements and to prepare group presentations describing their development as mathematics teachers in grades 1-7. They were asked to reflect on what they know now which would have been good to know in their first year; how they plan their mathematics teaching now compared to in their first year; what they have learned along the way; and what are the pitfalls and experiences to bear in mind for future mathematics teachers.

Six presentations were made - all short films (F1-F6) incorporating line drawings and sometimes photographs, with voice-over commentary and music. Five of the six films were organised as developmental stories from their novice anticipation and preparation of their first placement, to their reflective stance as third-year PSTs. The remaining film focused on four pitfalls for novice PSTs.

To analyse the data, we operationalised Biesta's (2012) concepts in terms of: descriptions of their knowledge, skills and dispositions; processes and accounts of making educational judgements and justification of judgement; and accounts of being and becoming a mathematics teacher, as exemplified below:

Qualification: References to knowledge, skills and dispositions, processes and practices from the teaching profession.

Socialisation: References to learning and to expectations in the school context.

Subjectification: References to inner feelings, identity and becoming a teacher and to perception of self (as a teacher).

The presentations were transcribed and analysed in several steps. First, they were coded according to Biesta's terms *qualification, socialisation* and *subjectification*, synergies and conflicts between these domains, and professional judgement. Then disagreements among the researchers were resolved, and all presentations were re-read and coded by another member of the group. A decision was made to organise the analysis in two parts - one around PSTs' novice anticipation, and one around their reflective stance in third year. Finally, we re-read each presentation to make sure that important longitudinal messages were not lost in our attempt to organise the analysis in two parts.

Looking back at early experiences in school placement - Emergency sirens

Qualification

In the films, there are a few examples of what PSTs learned in the first year, for instance pieces of 'theory', such as the importance of using multiple representations, giving feedback, and using the "didaktisk relasjonsmodell" (F1) (a hexagon connecting elements relevant for lesson planning: topic, learning objectives, the pupils in that class, etc.). The first-year PSTs are, however, uncertain about how to put the knowledge into practice. The uncertainty about effective use of manipulatives is most usual (F1, F4, F5), but other dilemmas are also identified:

> I also thought about all the theory I learned at university: Piaget's theory of stages of development, Bruner's representation theory and theory of scaffolding, Vygotsky's focus on cooperation, Bandura and his theories on motivation and self-efficacy. How should I use these theories to plan a lesson on fractions? [...] Which pupils should work together: the ones that are on the same level in terms of subject knowledge, or should the strong ones help the weak? (F1)

Several of the films show students meeting concepts as an overwhelming mass of words (Figure 1). At the same time, there are many statements about elements of qualification the PSTs perceive as lacking, both in terms of knowledge ("Do I know enough about this topic?" (F5)) and of processes:

> Just think if I have to explain several ways of doing something, to support conceptual understanding! It won't work. It just won't work. Manipulatives, manipulatives. (F5)

> It's only natural to carry on from where we left off [in the textbook]. It's not like I have other suggestions on what to do from here onwards. (F2)

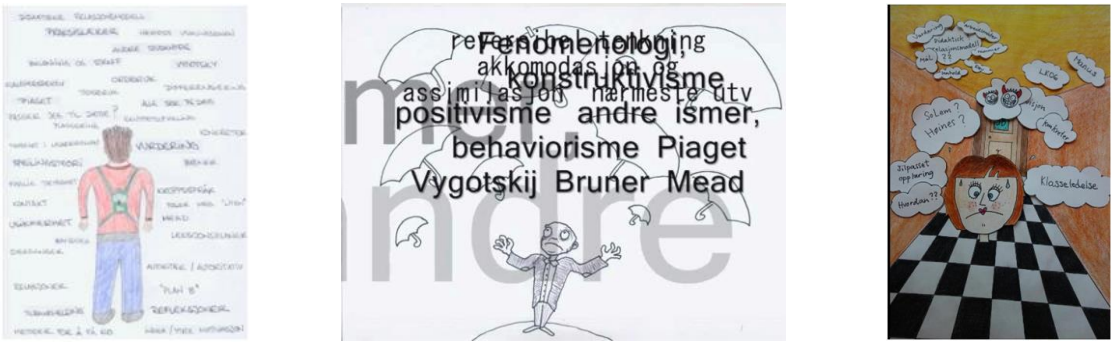

Figure 1: Knowledge in overwhelming amounts in the first year (F1, F4, F5)

Socialisation

The mentor is, naturally, the main role model for students in their first year of teacher education, and the mentor features in most of the (few) examples we find of socialisation when describing the first year. The PSTs are uncertain about what the mentor expects of them, other than using the "didaktisk relasjonsmodell" (mentioned above) which is common in Norway:

> As first-year students we used it [the didaktisk relasjonsmodell] slavishly (F6)

> ...I have to carry on from here [in the textbook], that must be what [the mentor] expects (F2)

The mentor can be viewed as an evaluator:

> Shit! This took the entire hour. The mentor glares at me. It didn't go as planned. (F3)

However, the anxiety of seeing the mentor write “like there was no tomorrow” is followed by “I had so many questions” - suggesting that the mentor is regarded as a person to ask for advice, as well.

Subjectification

A basis for subjectification is the development of a certain degree of self-confidence. In the description of the first year, we see little self-confidence - uncertainty and fear dominates:

> What if I don’t succeed? (F5)

> I went from being one of the best in mathematics to being perplexed when the pupils asked me questions about the subject. (F1)

Conflicts and synergies between domains

The three domains of education overlap. These non-empty intersections are implicitly present in the films. There are clear examples that a perceived lack of qualification (being overwhelmed by new concepts and by making sense of these in practice) leads to a lack of self-confidence - “I felt unsure and very, very small” (F1) - which we regard as part of subjectification. This can also work the other way: lack of confidence leads to lack in qualification:

> In the first school placement, I struggled a lot with getting the class to settle down. Later on I came to think it was because I did not feel like a confident and clear classroom manager. (F1)

There can also be a conflict between qualification and subjectification, in the sense that learning more makes you aware of your shortcomings:

> The more I learnt, the more I discovered what I didn’t know. [...] Based on Piaget’s theory I knew most of the pupils were at the concrete-operational stage. But which of Bruner’s representations should I use? [...] Or should I use the strange Cuisenaire rods that I still haven’t really gotten to grips with? (F1)

> Drowned in the curriculum he feels puzzled. What is most important? (F4)

Inside the domain of qualifications there are interactions between elements. In one case, the confidence in mathematics is shaken by the practice of teaching:

> I went from being one of the best in mathematics to being perplexed when pupils asked me questions. (F1)

At the same time, during the first year the process of lesson planning is weighed down by the awareness that there are many considerations to be taken. This is visible in form of the time that goes into writing a lesson plan (shown with clocks in the films), and the number of books that fill the desk in the process (Figure 2).

There can be a tension between socialisation and subjectification in meeting the mentor: in one example, the role model (supposed to provide socialisation) is so impressive that the PST’s self-confidence suffers:

> The meeting with the mentor was scary. I saw him as a Superman who really knew his work. He was confident, clear and, not the least, had strong subject knowledge. (F1)

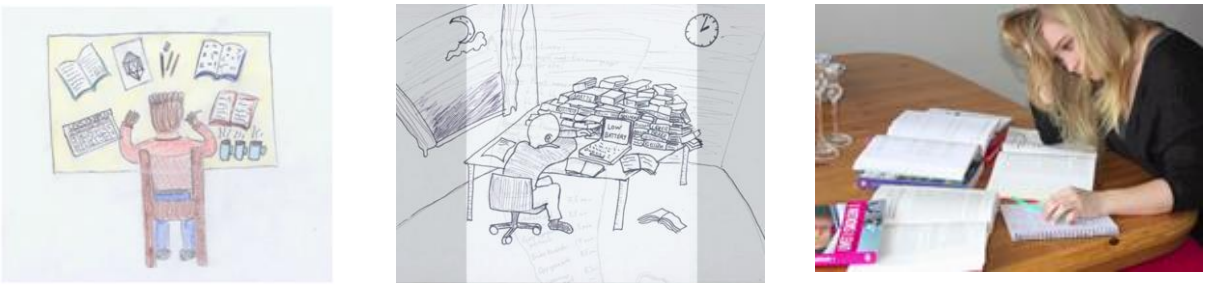

Figure 2: Lesson planning during first-year school placements (F1, F4, F6)

In another example, a PST's attempt at making a choice outside of the textbook is struck down by the mentor:

> Hmmm….I think maybe we should stick to the textbook. (F3)

With an emerging sense of agency, the PST questions the mentor's view and asks herself: "Should we always stick to the textbook?" (F3).

To conclude, there are synergies between (a lack of) qualification and (a lack of) subjectification, but also a conflict between qualification and subjectification, as well as between socialization and subjectification.

Practicing educational judgement

Judgement is difficult. A lack of self-confidence leads to a very detailed plan with little room for judgement on-the-fly.

> As a first-year PST the plan for the lesson was a long script. We had written down word for word what to say during the lesson. We were dependent on this script and could not improvise along the way. We even planned how to explain simple mathematical things that we actually knew well. This is also about lack of experience and confidence as a teacher. (F6)

At the same time, a lack of qualification translates into constraints on opportunities for judgement in the process of lesson planning:

> It's natural to continue from where we left off, it's not like I have other suggestions. (F2)

Looking at their recent experiences in school placement - Birdsong

Qualification

Changes from first to third year are visible in all aspects of qualification, from subject knowledge and knowledge of students and teaching, to the practices of teaching. In terms of knowledge, some films refer to knowing more mathematics, but, in terms of mathematics pedagogy, the films stress that the understanding is deeper, the knowledge can be operationalised to a greater extent.

The process of lesson planning during the first year involved long hours dedicated to the task (F1, F4, F6), and resulted in long scripts produced for each lesson (F4, F6). The films highlight, in comparison, how much quicker lesson planning goes (F1, F6), and how much shorter the scripts become (F1, F4) by third year, but the films give different suggestions on how to take advantage of the reduced burden, from watching TV and playing with the dog (F6) to investing time and energy on the 'frills' of differentiation and using a variety of teaching methods (F1).

Teaching practices out of reach during the first year are now on the agenda (F1): motivating pupils, providing them with opportunities to feel both confident and challenged, seeing the individuals as well as the class as a whole, giving more room to children's contributions, and encouraging enquiry.

Socialisation

The main presence that embodies the socialisation component is the teacher mentor, although some PSTs also mention peers and other colleagues playing a role. At this stage the mentor has transitioned from a feared judge to a colleague (F1), in some cases a role model (F2, F4), although disagreements between the views of PSTs and their mentor may occur, for example regarding the role of textbooks (F3). However, adopting established practices of the teaching community, such as body language (F1) or ways of saying or doing things in the classroom, seems to be perceived by PSTs as a sign of having become teacher-like:

> I've even put together extra handouts [for those who might need another type of challenge]. (F2)

Subjectification

Through the journey from first to third year, the PSTs have grown into teachers who are aware that teaching is not just about what you know, it is about making choices about complex situations. As there are no deterministic answers to these dilemmas, neither objectively speaking nor in terms of what is the established way of the teaching community, these choices come down to the individual, they are drawing on the domain of subjectification: "We're more aware that there should be a reason behind our choices" (F4), "I understand my own thoughts" (F5). In their third year, we hear the PSTs stress the importance of trusting their own choices (F1, F2, F4), and being yourself (F2).

Planning lessons is now an altogether more positive experience, described with attributes such as joy, and belief in oneself. Importantly, some of the PSTs realize that becoming a teacher is a continuous process, and experimenting is a part of it:

> Don't be afraid to try out new things. (F2)

> A lesson plan can never be too good. It's like a piece of silverware that you take out and polish from time to time. (F4)

Conflicts and synergies between domains

As PSTs become more comfortable as teachers (subjectification), some find reassurance in their theoretical knowledge (qualification) as well as their awareness of what is acceptable among teachers (socialisation):

> Not everything has to be perfect [...]. The theory I used to think about while planning lessons in my first year is now under my skin. (F1)

The routines of teachers (socialisation) also contribute to being more successful in the practices of teaching, such as lesson planning (qualification): "You don't have to reinvent the wheel (F1)". There is an aspect of growing confidence (subjectification) when the PSTs reuse lesson plans they have had positive experiences with (F1).

Unlike in their first year, lesson planning in the third year takes less time (F1, F6) and the scripts for the lesson are shorter (F1, F4) or even disappear altogether ("We've thrown out the script", F6). The

change is attributed in general to an increase in confidence (subjectification) but in some cases also to an aggregated influence of all three domains:

> … more confident in myself and the mathematics, I know more about the pupils' level in mathematics, I have become a clear leader, I dare to make mistakes, I am better at dealing with things as they happen. (F1)

In another film, the three domains come together in synergy to express the PSTs' development:

> By contrast with first year, when we used the the syllabus for the course a lot, we now have more knowledge of the subject and of pedagogy. We've become better at making use of our own knowledge, we cooperate more closely with colleagues. (F6)

The way these sentences are linked, makes it possible to interpret it as meaning that better qualification leads to better self-confidence (subjectification) which again leads to better cooperation with others (socialization).

Practicing educational judgement

Increased self-confidence by the third year is not synonymous with knowing just what to do:

> How can I connect algorithms and conceptual understanding? I need to be able to show them different strategies, to be sure as many as possible understand. How many strategies for division are there? Maybe they come with some I haven't thought of? Maybe some misconceptions will surface during the lesson? How can I then, in the best possible way, deal with this? (F4)

The difference from the first year is being able to deal with dilemmas, to practice professional judgement, guided by what they see as the goals of teaching:

> There's still a lot to think about, but I understand my own thoughts now, I know where I'm heading (F5)

In the third year practicing educational judgement features as a defining factor of the PST-mentor relationship at this stage: the detailed scripts for lesson plans that were in the first year in part written for the sake of the mentor (and in part to boost one's confidence, to feel prepared) are now shorter. A mentor's voice sets expectations:

> Just show me that you are aware of the choices you make, and that you can argue for them (F2)

During the third year, educational judgement is visible in reflections on one's own teaching:

> I've become better at assessing myself, and I can more readily explain what went well and what could have been done better in class (F1)

Teaching analysis draws on and at the same time feeds into the domain of qualification and perhaps also socialization. This way of assessing oneself - and the knowledge that you do it well - can be regarded as an engine for development also after graduation, it feeds into subjectification.

Concluding remarks

The titles of two subsections of the analysis reflect the soundtrack of a film where the experiences of first- and third-year school placements are introduced with emergency sirens and birdsong respectively. In terms of Biesta's (2012) framework, the overall picture the presentations paint is

that, looking back on their first-year school placement, PSTs remember a combination of a lack of qualifications, unclear expectations from mentors and low self-confidence. The fear many PSTs report on, seems rooted in their low self-confidence and the unclear expectations. Although first-year PSTs are allowed to try different approaches and to fail, the same combination of a lack of repertoire, uncertainty of the mentor's role and lack of self-confidence holds them back. By the third year their qualifications have increased, their role as PSTs is clearer and their self-confidence has grown. Because of this, they also find themselves practicing educational judgement more often.

Such narratives, perhaps in combination with logs from early placements, could be part of educational experiences, supporting PSTs' identity work. First-years might also benefit from watching the films, as not all challenges discussed can be dealt with by the teacher educators. While teacher education can and should make explicit what is expected of PSTs in their school placement, it cannot rush becoming educationally wise. However, we hypothesise that creating spaces where first- and third-year PSTs can discuss their experiences would contribute to the domain of socialisation and subjectification for both groups. Analyzing the students' contributions in terms of Biesta's concepts reveal the complex relationships between qualification, socialization and subjectification in teacher education. The three domains are interdependent, with conflicts and synergies which influences PSTs overall experience. More insight into these conflicts and synergies may contribute to better understanding of PSTs' experiences of their school placements.

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Core practices and mathematical tasks of teaching in teacher education: What can be learned from a challenging case?

Janne Fauskanger and Reidar Mosvold

University of Stavanger, Faculty of Arts and Education, Department of Education and Sports Science, Norway; janne.fauskanger@uis.no, reidar.mosvold@uis.no

In order to successfully carry out ambitious mathematics teaching, awareness about the underlying mathematical tasks of teaching involved is necessary. This paper presents the case of Martin, who started mathematics teacher education because he likes mathematics and feels that he knows the subject. We observe him in a period of field practice where he is supervised by an experienced mathematics mentor teacher. While planning, conducting and reflecting on a lesson on multiplication of fractions, neither Martin nor his mentor teacher focus on core tasks of teaching mathematics. We use this case as a starting point for discussing challenges and possibilities of increasing the emphasis on core practices and the embedded mathematical tasks of teaching mathematics in teacher education.

Keywords: Core practices, mathematical tasks of teaching, teacher education.

Introduction and theoretical background

Mathematics teaching is complex, and different attempts have been made to decompose it. Some have tried to identify the most critical practices involved in the work of teaching mathematics, and they describe them as "core practices" (e.g., McDonald, Kazemi, & Kavanagh, 2013) or "high-leverage practices" (Ball & Forzani, 2009; Forzani, 2014). These practices are fundamental for supporting students' learning. Others emphasize the mathematical tasks of teaching that are embedded in the professional work of teaching mathematics (e.g., Ball, Thames, & Phelps, 2008; Hoover, Mosvold, & Fauskanger, 2014). In this paper, we focus on skills necessary to carry out the work of teaching mathematics by emphasizing mathematical tasks of teaching that are embedded in core teaching practices in teacher education. High-leverage practices and the underlying mathematical tasks of teaching "are essential for skillful beginning teachers to understand, take responsibility for, and be prepared to carry out in order to enact their core instructional responsibilities" (Ball & Forzani, 2009, p. 504). It is thus important to develop these practices during teacher education (TeachingWorks, 2015). The mathematical tasks included in these practices are instructional tasks that are mathematical and not pedagogical. When referring to "mathematical tasks of teaching", we follow the conceptualization by Ball et al. (2008).

Researchers have made various attempts to categorize core practices (Forzani, 2014). In the TeachingWorks project (2015), a register of high-leverage practices is presented to serve as the basis for a core curriculum for the professional training of teachers. For instance, the first two of the nineteen high-leverage practices are: 1) "leading a group discussion", and 2) "explaining and modeling content, practices, and strategies." Both of these practices contain some mathematical tasks of teaching. For instance, the second practice obviously involves the task of "presenting mathematical ideas." It may also involve "finding an example to make a specific mathematical point" (Ball et al., 2008, p. 400), a mathematical task of teaching that has proven to be difficult for pre-service teachers (Zodik & Zaslavsky, 2008). This practice may also involve the tasks of "recognizing what is involved

in using a particular representation" and "linking representations to underlying ideas and to other representations" (Ball et al., 2008, p. 400). Other mathematical tasks of teaching—some are embedded in more than one high-leverage practice—include using language carefully, highlighting core mathematical ideas while sidelining potentially distracting ones, and make their own thinking visible while modeling and demonstrating. The third high-leverage practice presented by TeachingWorks (2015) is "eliciting and interpreting individual students' thinking." This practice may involve asking and responding to questions, or presenting the students with exercises that provoke or allow them to share their mathematical thinking in order to evaluate student understanding, guide instructional decisions, and surface ideas that will benefit other students. When engaging in these practices, the teacher is faced with several mathematical tasks of teaching. For instance, teachers are challenged to ask "productive mathematical questions" and to evaluate "the plausibility of students' claims" (Ball et al., 2008, p. 400). The mathematical tasks of teaching included in various core practices may vary depending on the context.

In Norway, the new national guidelines for primary and lower secondary teacher education use "core practices" as a term of reference (Ministry of Education and Research, 2016). Our study aims at discussing challenges and possibilities of implementing core practices and identifying underlying mathematical tasks of teaching mathematics in mathematics teacher education.

The Norwegian teacher education context

Norwegian teacher education is politically controlled (Hammerness, 2013), and national curriculum guidelines direct the focus and content of all teacher education programs. There are differentiated teacher education programs for primary (years 1–7) and lower secondary (years 5–10) levels; both are four-year bachelor programs. In the primary teacher education program, a mathematics course of 30 ECTS is compulsory for all pre-service teachers, whereas the lower secondary teacher education program requires 60 ECTS in mathematics for pre-service mathematics teachers.

Field practice is a compulsory part of teacher education, but studies indicate that pre-service teachers' opportunities to learn in this context are not sufficiently utilized (Hammerness, 2013). Pre-service teachers are required to complete 100 days of field practice at partner schools. The aim is that field practice should focus on the subject that pre-service teachers study on campus the current year. In field practice, pre-service teachers normally work in groups that are supervised by an experienced mentor teacher. Prospective mentor teachers are required to take a training course of 15 ECTS, and they are employed by the universities as teacher educators. From 2017, the Norwegian teacher education will be a five-year master program, and the field practice component will be extended to a minimum of 115 days spread across five years (Ministry of Education and Research, 2016).

Methods

In order to discuss challenges and possibilities of implementing high-leverage practices (e.g., TeachingWorks, 2015) and identify underlying mathematical tasks of teaching (e.g., Ball et al., 2008), we consider an empirical case from a cross-disciplinary project in Norwegian teacher education entitled Teachers as Students (TasS). The TasS project has a focus on pre-service teachers' learning in field practice.

Data collection includes video recordings of group interviews held with each group of pre-service teachers before and after their period of field practice. Based on analyses of these interviews, Martin

(pseudonym) stood out as a special case. He was one of only two pre-service teachers in the project who selected mathematics because they liked it and were good at it. A recent literature review suggests that being good at mathematics is important for pre-service teachers, for instance in perceiving and interpreting students' work (Stahnke, Schueler, & Roesken-Winter, 2016). In this case we only know that Martin sees himself as good, and this was a criterion for selecting him as a case. We observed Martin in a period of field practice in his fourth semester. In the previous semester, he had completed the 60 study points (ECTS) in mathematics/mathematics education that is required to teach mathematics in grades 5–10. Even though Martin enjoyed mathematics, his teaching practices cannot be seen as ambitious teaching practices (Lampert et al., 2010) fundamental for supporting students' learning (e.g., McDonald et al., 2013). In the following, we first show some glimpses from a lesson where Martin teaches multiplication of fractions in grade 7, followed by his discussions with the mentor teacher in the post-lesson mentoring session. The selected episode was typical for Martin's lessons in his three weeks of field practice. The lesson lasted for 38 minutes (we only focus on the whole class teaching in the selected episode), whereas the mentoring session lasted for 13 minutes.

The case of Martin

After a brief repetition from the previous lesson, Martin introduces multiplication of fractions as the focus of this lesson. He writes $\frac{1}{3}$ of $\frac{3}{4}$ = $\frac{1}{3} \times \frac{3}{4}$ on the blackboard and emphasizes that we say "one third of three quarters" when we write an expression like this. "What this means," he continues, "is that we first have a fraction of one third and split it into three by using two horizontal lines." He draws a quadrilateral on the blackboard, partitions it and shades the top third part. He draws another quadrilateral, partitions it vertically into fourths and shades three of these (see Figure 1). He then draws a third quadrilateral and says, "If we now want to take a third of this, we partition it into three [draws two horizontal lines in the figure]. How much is one third now?"

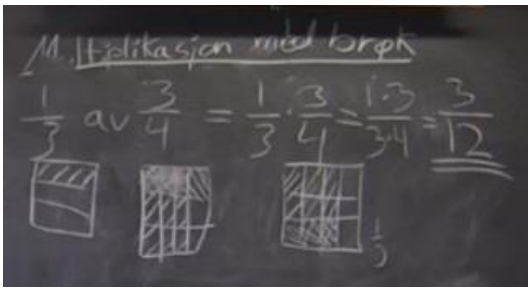

Figure 1. Martin's illustration of $\frac{1}{3} \times \frac{3}{4}$

When the students struggle to respond, Martin points at one of the parts in the figure. "It is one of these. And then we only have a third of what is shaded—look at this one! Did you understand that?" Some students say no, whereas others shake their heads. Martin tries again: "What we can also do, is to say that we put the two fractions on top of each other." He points at the first figure, pretends to move it over to the second figure and draws two horizontal lines in the middle figure. "What we focus on," he continues, "is that which has been shaded twice." When noticing that the students still do not seem to understand, he writes the expression. "It is as simple as taking this one [pointing at the first numerator] times this one [pointing at the second numerator], and then we take the first denominator times the second denominator." While saying this, he writes it out on the blackboard. "Do you see

that $\frac{3}{12}$ is the answer?" When the students are still hesitant, he quickly wipes everything out. "Let's take one more example," he continues.

In the next example, Martin writes $\frac{1}{2}$ of $\frac{1}{3}$ on the blackboard and draws two figures that he partitions and shades—this time by using colored crayons. The figures are of different sizes. "How much of this is both red and blue?" he asks. A girl presents $\frac{4}{6}$ as an answer, whereupon Martin repeats, "That is both blue AND red?" When a boy provides the answer, $\frac{1}{6}$, Martin confirms. "We have one part that is both blue and red, meaning both fractions. There are six parts altogether, and then we get one sixth," he continues. Martin tries to point the students' attention to the procedural approach. "If I want to solve this expression [pointing at $\frac{1}{2} \times \frac{1}{3}$ on the blackboard], the operation, how do I do it?" A boy mumbles that you are supposed to multiply, and Martin continues, "We are going to multiply denominator by denominator [pointing at the numerators (!) in the expression], and numerator by numerator [pointing at the denominators (!)]." He writes it out on the blackboard, seemingly ignorant about the fact that he has just mixed numerators with denominators. "Do you think you can make it if you try the tasks for yourselves now?" He then turns to the blackboard again and emphasizes how important it is to remember that although we write $\frac{1}{2} \times \frac{1}{3}$, we say one half of one third. "Important to remember," he says. "If we don't remember this, it will be very, very hard to solve the word problems!" Then he writes down $\frac{3}{7} \times \frac{10}{2}$ as another example. "How many of you know how to solve this one?" he asks. "The very operation," he continues, "forget about the figures!" After this introduction (14 minutes into the lesson) the students start working on similar tasks from their textbooks. When realizing that many students still have problems, Martin presents another example on the blackboard (after 18 minutes). Towards the end of the lesson, when summing up, he tells the students that he forgot to mention that the quadrilaterals (in Figure 1) are supposed to be equal in size.

Martin introduces the post-lesson mentoring session by saying that he can see what went wrong. When asked to elaborate, he points out that he should have used different colors from the beginning, and that he should have presented more examples before letting the students work individually on tasks. He continues to say that in his figures (e.g., Figure 1) he should have "mentioned that they [the quadrilaterals] were equally big". The mentor teacher supports this by saying: "When you don't show on the blackboard that they are equal, you cannot expect the students to understand it." In the next part of the mentoring session, they discuss what different students managed to do during the part of the lesson that involved individual work on textbook tasks. This part of the lesson is not the focus of attention in this paper. Towards the end of the mentoring session, the mentor teacher ends the discussion about the figures used to illustrate fraction multiplication by saying, "I think they [the quadrilaterals] would have worked out very well, as you are pointing out, if you had used colors. And if you had thought about making them equal in size, it would have worked out very well." Martin agrees, and adds that it is important to be careful about how you draw such figures.

Learning from the case of Martin

The case of Martin illustrates the core practice of "explaining and modeling content, practices, and strategies" (TeachingWorks, 2015). A simplified response to the presented episodes and vignettes could be that Martin does not carry out the core practice of explaining and modeling content, practices, and strategies well, and he needs more practice. Based on research indicating that pre-

service teachers do not necessarily learn from their field practice (Hammerness, 2013), we believe there is more to it than this. Although Martin and his mentor teacher discuss his explanations and modeling of the content, they do not appear to get to the heart of the issue. Ball and Forzani (2009) suggest that certain mathematical tasks of teaching are embedded in these core practices (e.g., Ball et al., 2008; Hoover et al., 2014). In our discussion of challenges and possibilities of implementing core practices, we identify four challenges and discuss the possibilities for highlighting some of the embedded mathematical tasks of teaching.

First, and most importantly, Martin is faced with the mathematical task of "recognizing what is involved in using a particular representation" (Ball et al., 2008, p. 400). In his attempt to use the area model to represent multiplication of fractions, Martin draws three quadrilaterals (Figure 1). The area model for multiplication of fractions requires use of one rectangle only as unit. Martin does, however, say that the third rectangle illustrates the two others "on top of each other", and he points at the first rectangle and pretends to drag it over the second rectangle. Still, this use of the model appears to confuse the students, and it would have been natural to focus on this mathematical task of teaching in a post-lesson mentoring session. In the given example one can draw three horizontal lines to show four equally large horizontal strips and then divide an area of three of these in three columns by drawing two vertical lines. Then there is still some work to do in order to understand that numerators can be multiplied as well as denominators. Martin tries to help the students develop this understanding by drawing two horizontal lines in the middle figure to illustrate how the four parts have now been divided into three and saying: "what we focus on, is that which has been shaded twice."

In the post-lesson mentoring session, Martin starts by stating that he can see what went wrong. He points out that he should have used different colors (instead of double shading) and that he should have presented more examples. He continues to say that he should have "mentioned that they [the rectangles] were equally big." The mentor teacher expresses his agreement. The area model requires use of one rectangle only as unit, but this is not discussed. "Recognizing what is involved in using a particular representation" (Ball et al., 2008, p. 400) is a mathematical task of teaching which might be fruitful in order to facilitate student teachers' learning to carry out the core practice of explaining and modeling content in teacher education. This mathematical task of teaching is, however, not discussed in the post-lesson mentoring session.

Second, and related to the first, Martin consistently reads the product as "$\frac{1}{3}$ *of* $\frac{3}{4}$" indicating another model for multiplication of fractions than the area model: the multiplicative comparison model. In this model, one of the fractions is an operator, one is represented as a portion of the area of one rectangle, and the result is represented as another portion of the area of the same rectangle. In this case the rectangles might well be drawn separately, but Martin's comment that the two first rectangles in Figure 1 should be placed on "top of each other" to make the third indicates that he does not have this model in mind. This is also not discussed in the post-lesson mentoring session.

Third, and related to the mathematical task of selecting appropriate examples "to make a specific mathematical point" (Ball et al., 2008, p. 400), Martin could have selected more appropriate examples when presenting the students with the area model for multiplication of fractions. In using this area model, it is necessary to not only use simple fractions as one-third and three-fourth. One can clarify much better that numerators can be multiplied as well as denominators with say three-fifth of four-

seventh. This illustrates an embedded mathematical task of teaching related to finding examples to make specific a certain mathematical point. In the post-lesson mentoring session, only the number of examples is discussed. For instance, they discuss that Martin should have presented more examples. Selection of examples, which has proven to be difficult for pre-service teachers (Zodik & Zaslavsky, 2008), is not discussed. One way to meet this challenge of randomly generated examples, when careful choices should be made, is to exemplify and discuss carefully selected examples in mathematics teacher education

Fourth, Martin is faced with the mathematical task of using correct mathematical language when presenting the mathematical idea of multiplication of fractions. On a couple of occasions, we observe that he mixes numerator with denominator. This might be regarded as a minor mistake of speaking mathematics, and we do not believe it is the most critical issue in the case of Martin. Correct use of mathematical language and notation is still important, however, and we suggest that this is a mathematical task of teaching that the mentor teacher could have discussed with Martin.

Martin is using a model for representing multiplication of fractions (Figure 1), but the students struggle to understand it. The high-leverage practice of eliciting and interpreting students' thinking (TeachingWorks, 2015) focuses on teachers' practice related to posing questions or tasks that provoke or allow students to share their mathematical thinking in order to evaluate student understanding and guide instructional decisions. To do this effectively, a teacher needs to draw out a student's thinking through carefully selected questions and tasks and to consider and check alternative interpretations of the student's ideas and methods. Although Martin knows the mathematical content himself, he struggles to understand the problems faced by the students. This illustrates the importance of pre-service teachers' mathematical knowledge in order to perceive and interpret students' work (Stahnke et al., 2016). In his communication with the students, Martin does not invite the students to engage in mathematical discussions and reasoning. Instead, the students are invited to give short and confirmative responses only. During the 14-minute whole-class introduction, as well as in the brief wrapping up of the lesson, the students were mostly invited to answer yes or no questions. Two examples of such questions are: "Did you understand that?" and "Do you think you can make it if you try the tasks for yourselves now?" Some questions are asked when the answer is already visible through the example presented on the board, like: "How much is one third now?" and "How much of this is both red and blue?" These questions invite students to answer by single words. Only once were the students invited to answer a how-question, and this question was related to how to carry out a routine procedure. Based on our analyses of how Martin invites the students to participate, we conclude that they are not invited to speak or reason mathematically. This kind of communication does not allow Martin to elicit students' thinking. We thus suggest that the core practice of eliciting and interpreting students' thinking is important to develop in teacher education.

Conclusion

Learning to successfully carry out high-leverage practices in mathematics teaching requires awareness about the underlying mathematical tasks of teaching involved (Ball et al., 2008), and mathematics teaching is a professional practice that requires training (Hoover et al., 2014). In this paper, we have used illustrative data from field practice in Norwegian mathematics teacher education as a starting point for our discussion. From this challenging case, we observe that neither the pre-service mathematics teacher (Martin) nor his mentor teacher appear to be conscious about the

mathematical tasks of teaching that are embedded in the high-leverage practices that have been analyzed and discussed in this paper—all of which are involved in the planning, conducting and reflecting on a lesson on multiplication of fractions.

What challenges and possibilities of implementing core practices and identifying underlying mathematical tasks of teaching mathematics in mathematics teacher education can be identified by analyzing Martin's lesson? In this lesson, the pre-service teacher was challenged to carry out the practice of "Explaining and modeling content, practices, and strategies" (TeachingWorks, 2015). Previous research indicates that strong teacher knowledge supports teachers in using representations to attach meaning to mathematical procedures (e.g., Charalambous, 2010). Although he has completed all of his coursework in mathematics, Martin does not appear to be prepared to carry out this core practice. Martin is also challenged to elicit students' thinking (TeachingWorks, 2015), but he does not seem prepared to carry out this core practice of teaching mathematics either, at least not in an ambitious way (Lampert et al., 2010). From these illustrative data, it appears that there is a lack of awareness about the underlying mathematical tasks of teaching, and the post-lesson mentoring session includes little discussion of these underlying mathematical tasks of teaching.

At least four lessons can be learned from the case of Martin. First, the task of recognizing what is involved in using a particular representation challenges teacher education to include detailed discussions of the area model for representing multiplication of fractions, highlighting the importance of using only one rectangle and how this model relates to numerators are multiplied as well as denominators. Second, selecting appropriate examples using carefully chosen numbers is important in order to clarify that numerators can be multiplied as well as denominators, and therefore important to discuss in teacher education. Third, Martin could have selected better examples when presenting the students with the area model for multiplication of fractions. Fourth, and finally, the case of Martin illustrates the mathematical task of presenting mathematical ideas using correct mathematical language. These mathematical tasks of teaching all seem related to the high-leverage practice of explaining and modeling content (TeachingWorks, 2015).

Mathematics teacher education needs to focus more on preparing prospective teachers to carry out high-leverage practices. Discussions of the embedded mathematical tasks of teaching are then necessary. The vignettes and episodes discussed in this paper indicate that the "unnatural" and complicated work of teaching needs to be explicitly taught in teacher education. One way of approaching this is to practice carrying out the mathematical tasks of teaching on campus as well as in field practice. The national guidelines for current primary and lower secondary teacher education that are now being developed focus on core or high-leverage practices (Ministry of Education and Research, 2016). More research and development efforts are needed to ensure a high-leverage implementation of these ideas in teacher education.

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Characteristics of a learning environment to support prospective secondary mathematics teachers' noticing of students' thinking related to the limit concept

Fernández, C.[1], Llinares, S.[1], Callejo, M.L.[1], Moreno, M.[1] and Sánchez-Matamoros, G.[2]

[1]Universidad de Alicante, Spain, ceneida.fernandez@ua.es; sllinares@ua.es; luz.callejo@ua.es; mmoreno@ua.es

[2]Universidad de Sevilla (Spain), gsanchezmatamoros@us.es

The aim of this study is to describe changes in the way that prospective secondary school teachers notice students' mathematical thinking related to the limit concept in a learning environment designed ad hoc. The learning environment progressively nests the skills of attending to, interpreting and deciding as three interrelated skills of professional noticing. Results show characteristics of how prospective teachers gained expertise in the three skills since four out of five groups of prospective teachers interpreted students' mathematical reasoning attending to the mathematical elements of the dynamic conception of limit. The links between attending to and interpreting helped prospective teachers justify the teaching activities proposed to support the progression of students' mathematical reasoning: from a mathematical point of view or considering mathematical cognitive processes involved.

Keywords: Noticing, prospective teachers' learning, learning environment.

Introduction and theoretical background

Research has shown that noticing is an important component of teaching expertise (Mason, 2002). Teachers need to attend to students' mathematical reasoning and make sense of it in order to teach in ways that build on students' thinking (Choy, 2016; Sherin, Jacobs, & Philipp, 2011). Noticing has been conceptualised from different perspectives. One of them consists of two main processes: attending to particular teaching events and making sense of these events (Sherin et al., 2011). Jacobs, Lamb, and Philipp (2010) particularise the notion of noticing to children's mathematical thinking, conceptualising this notion as a set of three interrelated skills: attending to children's strategies, interpreting children's mathematical thinking, and deciding how to respond on the basis of children's mathematical thinking.

Previous research has focused on pre-service teachers' ability to interpret students' mathematical thinking (Bartell, Webel, Bowen, & Dyson, 2013; Callejo, & Zapatera, 2016; Fernández, Llinares, & Valls, 2012; Llinares, Fernández, & Sánchez-Matamoros, 2016; Magiera, van den Kieboom, & Moyer, 2013; Sánchez-Matamoros, Fernández, & Llinares, 2015) showing that the identification of the mathematical elements involved in the problem (mathematical content knowledge) plays a significant role in interpreting students' mathematical reasoning. Furthermore, previous research has shown that some contexts can help pre-service or prospective teachers develop the noticing skill: watching video clips (Coles, 2012; van Es, & Sherin, 2002), participating in online debates (Fernández et al., 2012) or participating in learning environments (interventions) designed considering specific mathematical topics. For example, Schack et al. (2013) in the area of early

numeracy; Magiera et al. (2013) in algebra; Callejo and Zapatera (2016) in pattern generalization; Llinares et al. (2016) in classification of quadrilaterals; Sánchez-Matamoros et al. (2015) in the derivative concept; and Son (2013) in the concepts of ratio and proportion. These previous studies underline that the skill of deciding how to respond on the basis of children's mathematical thinking is the most difficult one to develop in teacher education programs. As Choy (2013) pointed *"the specificity of what teachers notice while necessary, is not sufficient for improved practices"* (p. 187). In other words, teachers can be very specific about what they notice without having a teaching decision in mind. So, the relation between how prospective teachers develop the skills of interpreting students' mathematical thinking and deciding how to respond on the basis of students' mathematical thinking deserves further research.

On the other hand, the concept of limit of a function is a difficult notion for high school students (16-18 years old) and is a key concept in the Spanish curriculum (Contreras, & García, 2011). Cottrill and colleagues (1996) indicated that the difficulty of students' understanding of the limit concept could be the result of a limited understanding of the dynamic conception. A way of overcoming this difficulty is by coordinating the processes of approaching in the domain and in the range in different modes of representation. Knowing these characteristics of students' understanding could provide prospective teachers with information to interpret students' mathematical thinking and to make instructional decisions based on students' reasoning.

Therefore, our study analyses changes in the way that prospective teachers notice students' mathematical thinking (attending to, interpreting and deciding) in relation to the limit concept when they participate in a learning environment designed ad hoc. The learning environment designed progressively nests the skills of attending to, interpreting and deciding and its relations. We hypothesise that the structure of the learning environment help prospective teachers to decide how to respond taking into account their previous interpretations of students' mathematical thinking.

Method

Participants and the learning environment

The participants were 25 prospective secondary school teachers (mathematics, physics and engineering) who were enrolled in an initial secondary mathematics teacher training program. One of the subjects of this program is focused on developing the skill of noticing students' mathematical thinking in different mathematical topics and on planning the instruction attending to students' mathematical thinking. One of the mathematical topics considered was the limit concept.

The learning environment consisted of 5 sessions of two hours each and was designed taking into account the nested nature of the skills of attending to, interpreting and deciding (Jacobs et al., 2010). Prospective teachers were divided into five groups of 5 persons to perform the tasks of the learning environment. Firstly, prospective teachers solved three problems related to the limit concept selected from high school textbooks (Figure 1) in order to unpack the important mathematical elements of the limit concept (session 1). Then, prospective teachers had to anticipate hypothetical students' answers to these problems reflecting different characteristics of conceptual development (session 2). That is, they had to anticipate what students are likely to do. Prospective teachers had a document with the definition of the dynamic conception of limit and its mathematical elements (Pons, 2014): (i) approaches from the right and from the left (in the domain and in the range), and

(ii) coordination of the processes of approaching in the domain and in the range considering different modes of representation (graphical, algebraic and numerical).

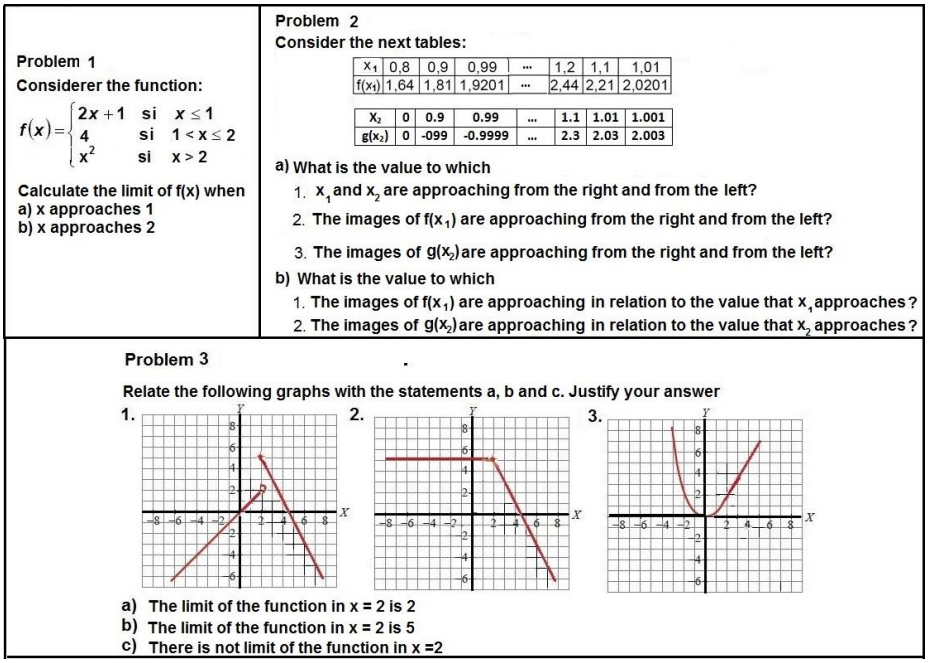


Figure 1: The three problems related to the limit concept

The aim of the tasks of identifying the mathematical elements in the resolution of the problems and anticipating hypothetical students' answers was to help prospective teachers focus their attention on the relationship between the specific mathematical content and students' mathematical thinking. We conjecture that focusing on this relationship is needed to develop the skill of noticing in a first step. Next, prospective teachers analysed a set of four high school students' answers (Pablo, Rebecca, Luiggi and Jorge) to the same problems. Prospective teachers had to attend to students' strategies, interpret students' mathematical thinking and propose new activities (or modify them) to help students progress in their conceptual reasoning (according to their previous interpretations of students' mathematical thinking) (session 3 and 4). The high school students' answers, provided to prospective teachers, reflected different levels of high school students' reasoning of the limit concept (Table 1; Pons, 2014). We also provided prospective teachers with theoretical information that summarise the characteristics of high school students' reasoning of the limit concept from previous research to solve the task (Cornu, 1991; Cottrill et al., 1996; Swinyard, & Larsen, 2012). In figure 2, the answers of Pablo to the three problems are given.

Prospective teachers had to answer the next three questions: (i) which mathematical elements has the student used in each problem? Indicate if he/she has had difficulties with them; (ii) identify some characteristics of how the student understands the limit of a function. Explain your answer using the mathematical elements identified before; (iii) considering the student reasoning, propose an activity that helps the student progress in their conceptual reasoning of the limit concept. Therefore, the objective of sessions 3 and 4 was that prospective teachers focus their attention on the relation between identifying-interpreting and between interpreting-deciding. We conjecture that these relationships are necessary to develop the skill of noticing. Finally, in the session 5, prospective teachers had to answer a similar task individually.

High school students	Level	Levels of students' reasoning about the dynamical conception of the limit concept
Pablo and Luiggi	High	Pablo and Luiggi coordinate the processes of approaching in the domain and in the range in the three modes of representation
Rebecca	Low	Rebecca coordinates the processes of approaching in the domain and in the range in the graphical mode of representation when limits coincide
Jorge	Intermediate	Jorge coordinates the processes of approaching in the domain and in the range in the algebraic and graphical mode of representation (when limits coincide in this last mode of representation)

Table 1: Characteristics of high school students' answers

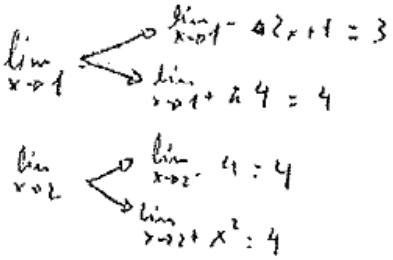
Answer to Problem 1	Answer to Problem 2	Answer to Problem 3
	a*1) x_1 is approaching to 1 from the left and from the right and x_2 is approaching to 1 from the left and from the right* *a2) The images of $f(x_1)$ are approaching to 2 from the left and from the right* *a3) The images of $g(x_2)$ are approaching to 2 from the right and is approaching to -1 from the left* *b1) when x_1 is approaching to 1, images of $f(x_1)$ tend to 2* *b2) when x_2 is approaching to 1, the images of $g(x_2)$ tend to -1 from the left and tend to 2 from the right.*	*a) Graph 3 because the limit of the function in x=2 from the right and from the left is 2.* *b) Graph 2 because the limit of the function in x=2 from the left and from the right is 5.* *c) Graph 1 because the limit of the function in x=2 is not the same from the right and from the left.*

Figure 2: Pablo's answers to the three problems

Data and analysis

Data of this study are prospective teachers' answers to the tasks of session 2 (anticipation) and sessions 3 and 4 (interpretation). Through an inductive analysis (Strauss & Corbin, 1994), we generated similarities and differences about how prospective teachers conceived high school students' reasoning of the limit concept and the type of activities they provided to help students progress in their conceptual reasoning. To carry out this analysis, five researchers analysed individually prospective teachers' answers to the anticipation and interpretation tasks and then, the agreements and disagreements were discussed to reach a consensus on these issues.

This analysis let us identify two ways of how prospective teachers conceived high school students' reasoning: as dichotomous (right or wrong) and as a progression (identifying different levels of students' reasoning). The type of activities that prospective teachers provided were categorised in three categories: general decisions, decision based on curricula contents and decisions based on cognitive processes. Examples of these categories are presented in the results section.

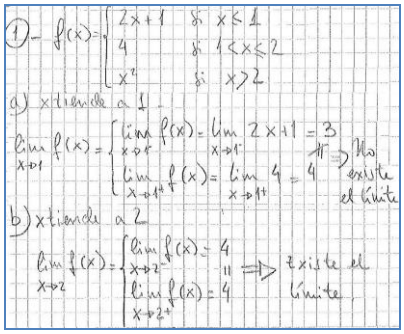
Finally, we compared categories obtained in the anticipation task with the categories obtained in the interpretation task to identify changes in the way of how prospective teachers conceived high school students' reasoning and proposed activities to help students progress in their conceptual reasoning.

Results

Our results show that prospective teachers changed the way that they conceived students' reasoning from a dichotomous to a progression way and this shift influenced the type of activities that they proposed to help students progress in their conceptual reasoning.

Changes in the way that prospective teachers conceived students' reasoning: From a dichotomous to a progression

In the anticipation task, three out of five groups conceived students' reasoning as dichotomous (right or wrong). For example, the group of prospective teachers G2 anticipated that a high school student with high level of reasoning of the limit concept (Maria) would coordinate in all modes of representation. For example, this group of prospective teachers anticipated the next answer for the algebraic representation:



Maria understands the limit concept. The idea of approximation in the domain corresponds to the fact that she properly selects the branch of the function and uses the notion of approximation in the range adequately. It is demonstrated when she replaces on the limit the approach of the independent variable. This student also coordinates the approximations to establish the value of the limit according to the branch.

Furthermore, these prospective teachers (G2) anticipated that a high school student with a not suitable level of reasoning of the limit concept (Pedro) would not coordinate in any mode of representation pointing out: *"Pedro only approximates (from the left or from the right) when the function is defined".*

Then (in the interpretation task), four out of five groups of prospective teachers were able to interpret students' mathematical reasoning. They linked students' reasoning with the mathematical elements of the dynamic conception of limit: the approaches from the right and from the left (in the domain and in the range), and the coordination of the approaches in the domain and in the range considering different modes of representation (graphical, algebraic and numerical). For instance, the group of prospective teachers (G2) interpreted the student's answer of problem 1 (Figure 2) as:

> The resolution of the student is correct (Pablo). We can notice that the student has identified the kind of function (piecewise function) since he (the student) has approximated in the range (he has calculated the approximation to x=1 from the left and from the right and the approximation to x=2 from the left and from the right) and in the domain (taking the correct definition of function in each interval). Furthermore, he has coordinated the processes of approaching in the domain and in the range since he has written, for example, that when x tends to 1 from the left, the image of the function tends to 3 (using the function 2x+1).

This group gave similar comments for the student's answers to the other two problems linking students' reasoning with the important mathematical elements in the other two modes of representations (problems 2 and 3). Afterwards, they wrote a summary about this student level of reasoning:

> This student understands the limit concept since he approaches from the right and from the left (in the domain and in the range), and coordinates the processes of approaching in the domain and in the range in the three modes of representation (graphical, algebraic and numerical). This student would be in the high level of reasoning.

These prospective teachers were able to identify different levels of students' reasoning. Therefore, they conceived students' reasoning as gradual.

Changes in the type of activities they proposed to help students progress in their conceptual reasoning

Prospective teachers who conceived students' reasoning as dichotomous did not propose specific activities to help students progress in their reasoning. These prospective teachers gave general comments about teaching as instructional actions. For instance, the group of prospective teachers G2 proposed to Maria (in the anticipation task) the representation of the graph of the function of problem 1. This decision was not based on the conceptual progression of the student.

When prospective teachers interpreted students' mathematical thinking identifying different levels of students' reasoning (linking students' mathematical reasoning with the important mathematical elements), they were able to provide specific activities to help students progress in their conceptual reasoning. For the students who only coordinate the approaches in the domain and in the range in one mode of representation, they proposed new activities to integrate these mathematical elements gradually in the different modes of representation. The proposed activities required a coordination of approaches in the domain and in the range in the different modes of representation. For the students who coordinate the approximations in the domain and in the rage in all modes of representation (such as the student of Figure 2), they also provided activities to help students progress in their reasoning.

We have identified two ways in which they justified their new activities: some justifications were based on the mathematical elements and others on the cognitive processes involved. In the first case, prospective teachers focused their attention on introducing a new mathematic content. In the following example, they introduced a new type of discontinuity – an avoidable discontinuity. The justification of this type of activities was based on the use of new mathematical elements (in this case, introducing other type of functions).

The activity: We would modify the function of problem 1 and we would use:

$$f(x) = \begin{cases} 2x & \text{si } x > 2 \\ 5 & \text{si } x = 2 \\ 2x & \text{si } x < 2 \end{cases}$$

Our justification: The student (Figure 2) seems to understand the limit concept in the three modes of representation, so we would provide him a more difficult function with an avoidable discontinuity.

In the second case, prospective teachers focused their attention on the cognitive processes involved to understand the limit concept. In the next example, prospective teachers focused on the reversal as a cognitive mechanism that leads students to a new reasoning level. That is to say, prospective teachers justified the proposed activity by the need of generating learning opportunities to develop the reverse mechanism that allows the construction of cognitive objects.

The activity: Represent a graph of a function which limit in x=-1 is 4 and that there is not limit in x=1.
Our justification: with this activity students need to do the inverse process that is, they need to use all the important mathematical elements to build that function.

Discussion and conclusions

Results show that after the participation in a learning environment that progressively nests the three interrelated skills of professional noticing (attending to, interpreting and deciding), prospective secondary school teachers gained expertise in noticing. Four out of five groups of prospective teachers were able to interpret students' mathematical thinking linking students' reasoning with the important mathematical elements of the dynamic conception of limit. These findings support the claim that some characteristics of the learning environment such as considering the nested nature of the skills help prospective teachers develop the skill of noticing (Sánchez-Matamoros et al, 2015; Schack et al., 2013).

Furthermore, prospective teachers were able to provide specific activities to help students progress in their conceptual reasoning. Therefore, the characteristics of the learning environment in which prospective teachers were engaged in the analysis of mathematical elements of limit problems, in the analysis of students' reasoning and in proposing new activities to support students' conceptual development enable them to gain more accurate understanding of the relation between the mathematical content and students' mathematical thinking. This new understanding provides prospective teachers with the needed knowledge to give their teaching decisions based on the progression of students' reasoning: from a mathematical point of view or considering the cognitive processes involved.

Acknowledgment

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Evaluation of a discovery, inquiry- and project-based learning program

Hanna Gärtner[1] and Matthias Ludwig[2]

Goethe-University, Frankfurt am Main, Germany;

[1]hgaertne@math.uni-frankfurt.de; [2]ludwig@math.uni-frankfurt.de

The program Math.Researchers ("Mathe.Forscher") funded by the German foundation "Stiftung Rechnen" has the goal to implement discovery, inquiry- and project-based learning in mathematic school lessons. The students are meant to explore and discover mathematics in their every-day-life with the help of their teachers. The teachers are well educated in supporting their students by in-service education organized by the program. In addition to finding out if the in-service education is well established and if any correlation between the students' or the teachers' view on mathematics and the positive identification with the program is detectable, an evaluation was planned. The goal is to investigate if while taking part in the program Math.Researchers teachers' beliefs concerning mathematics are influenced and if an effect on the students' view on mathematics is detectable.

Keywords: In-service education; secondary school mathematics; discovery, inquiry- and project-based learning; beliefs.

Theoretical framework

The program Math.Researchers ("Mathe.Forscher") funded in 2010 by the German foundation "Stiftung Rechnen" has the goal to implement discovery, inquiry- and project-based learning in mathematic school lessons. One motivation is that inquiry-based learning may increase the students' mathematical achievement (Hattie, 2015). Bruner considers the benefit of learning through discoveries that one makes for oneself in the 60's (Bruner, 1961). His discussed main benefits are similar to Winter's arguments consisting the thesis that active discoveries by students themselves are the more effective way of especially learning mathematics (Winter, 1989, p. 1–3). The students are meant to explore and discover mathematics in their every-day-life with the help of their teachers. The program Math.Researchers supports the teachers in several sessions, with training, booklets and material for best-practice-activities. In addition a team of scientific assistants and process assistants support the program members. One example for a Math.Researcher activity is "Mathematics at the Zoo" (Ludwig, Lutz-Westphal, 2016). The pupils have to leave their classroom and visit the zoo to develop their own research questions there. The teacher supports the students' ideas with mathematical knowhow. Back at school they work on their research questions and present their results. A reflection on the presented solutions completes this Math.Researcher activity.Although the teachers were supported in the terms of the program they were insecure whether their lessons were Math.Researchers-lessons. To improve transparency in this respect five special normative program-dimensions were developed by experts of the program in 2014 that can be used as a kind of checklist.

Math.Researcher-dimensions

Derived from the program goals, the following five Math.Researcher-dimensions were determined in 2014: using the Math.Researcher-principles (MRP), opening the classroom (OC), working with researchers' questions (RQ), acting as a learning guide (LG), and visualizing mathematics (VM).

Each dimension consists of three main elements (Table 1). The more elements considered, the more Math.Researcher-like is the planned unit. The table can be used as a checklist so that the teachers are able to find out whether their planned lessons are conforming to the program. An ideal Math.Researcher-activity contains a minimum one element of each dimension.

Dimension	Main Elements		
Using the Math. Researcher Principles (MRP)	Inquiry-based learning	discovery learning	project-based learning
Opening the Classroom (OC)	interdisciplinary instruction	inviting experts	outdoor lessons
Working with Researcher's Questions (RQ)	including the students living environment	exercising asking questions with the students	allowing multiple approaches
Acting as a Learning Guide (LG)	enable active students role	constructive handling with students ideas	working out milestones together
Visualizing and presenting Mathematics (VM)	documentation of mathematics	using mathematical language	talking about ideas

Table 1: The five dimensions and its main elements

Considering the above-mentioned Math.Researcher activity's example "Mathematics at the Zoo" elements of all dimensions can be found here. The activity is project-based (MRP) and includes outdoor lessons at the zoo (OC). The pupils develop their own mathematical questions in an environment they usually do not connect to an educational context (RQ). The teacher enables student activities and supports them in mathematizing their ideas (LG). The phase of presentation and reflection at the end of this project represents the fifth dimension (VM).

To establish the dimensions a special PD course was conducted (figure 1). At a kick-off-meeting in December 2014 with all participating teachers the dimensions were introduced. The following months the teachers were meant to implement the dimensions in their teaching to get more familiar with them. So called process-assistance visited the teachers at school, watched the lessons, gave feedback. At the Math.Researchers' camp in May 2015 concrete activities were planned. 17 teachers met with program's scientific and process assistance to develop units focusing the dimensions. These units were conducted in May, June and July 2015. Every unit should be documented by a uniform document tool provided on the internet platform. The assistants analyzed these documentations and gave individual feedback.

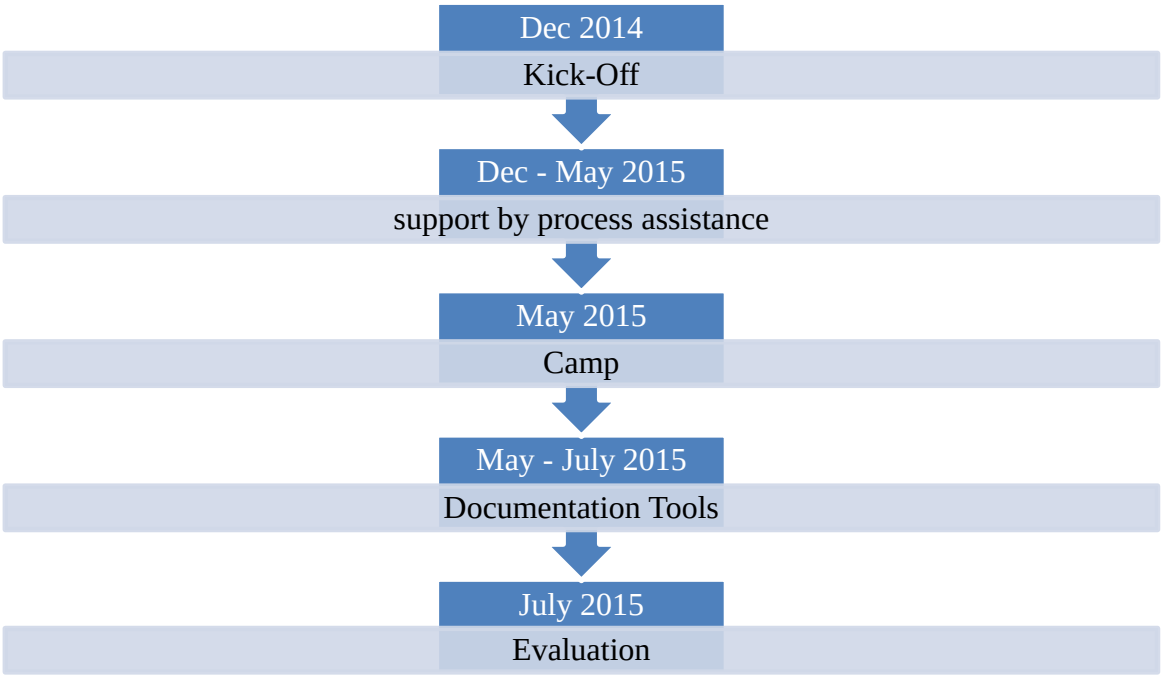

Figure 1: Timeline of treatments

While carrying out the Math.Researcher program with a special focus on the dimensions it was not sure whether and how the terms and ideas of the program were implemented in mathematical school lessons after May 2015. The dimensions can be seen as a kind of program characteristics. So a natural research question came up: Did the development and application of these dimensions succeed sustainable? That means on the one side that they are helpful for the teachers to plan their Math.Researcher lessons, and on the other side that the included main elements are recognizable by the students in the way the teachers planned it. Furthermore, we wanted to know if there is any effect on the view on mathematics of the teachers and also the students. By taking part in the program the students are meant to explore and discover mathematics in their every-day-life. Does this goal reach the students, or is the main opinion after they have taken part in the program that mathematics is still a school topic where you have to learn formulas and not necessary for every-day-life?

Presenting the design principles of in-service education by the German Centre for Mathematics Teacher Education (DZLM) Barzel and Selter (2015) reviewed the current state of research on in-service education with a special focus on mathematics. The effectiveness of in-service education is possible on different levels: Among other things there can be an impact on the acceptance of the training, an impact on the professional competencies and an impact on the classroom teaching (Barzel & Selter, 2015, p. 266). To improve transparency in the respect which impact can be found with the Math.Researcher-program an evaluation was planned with a focus on the following research questions:

- Are the dimensions well established, do the teachers identify with them and integrate belonging elements in their classroom teaching? **(1)**
- Which view on mathematics has been built up by the participants identifying with the Math.Researcher program? **(2)**

Therefore the dimensions were related to the mathematical beliefs. The mathematical world view of teachers (Törner, 1997) can be described by a belief system including four main aspects: the aspect of formalism (F), the schematic aspect (S), aspect of process (P) and the aspect of application (A). The aspect of formalism and the schematic aspect can be interpreted as a static view on mathematics education: mathematics as a system. The dynamic process character allows understanding facts, recognizing connections and building knowledge: mathematics is an activity. This belief system is also part of the competencies framework for in-service education of the DZLM (Barzel & Selter, 2015,p. 263).

The pilot study

A pilot study was conducted from May to July 2015 in the region Rhein-Neckar to find out whether the dimensions are well established and if any correlation between the students' or the teachers' view on mathematics and the positive identification with the dimensions are detectable. The program started in this region in 2012 in ten secondary schools. All respondents – students and teachers – of the pilot study have taken part in at least one Math.Researcher-activity within the last two years and especially in the PD course focusing the dimensions listed above.

Methods

The questionnaire contained belief-questions (taken from surveys by Grigutsch/Raatz/Törner, 1998), questions belonging to the program and its dimensions and some general questions (mostly developed by ourselves and taken from an earlier Math.Researcher evaluation in the region Nord conducted by the TU Berlin, Lubke et al., 2011). The teachers (n=20) answered an online-questionnaire containing 139 questions. The students (n=168) answered a paper print-version with 79 questions. The items were scaled with 1="totally don't agree" to 5= "totally agree". The higher the scale of questions belonging to the dimensions the higher is the identification with the program. All questions were formulated to be understandable for students not familiar with the dimensions. For example the students were asked whether they liked talking about their own mathematical ideas, which can be related to the dimension visualizing and presenting mathematics (VM). Or if they sometimes do outdoor mathematics, which is related to the dimension opening the classroom (OC). Compared to that the teachers were asked whether their students' have to present own results at the end of a project (VM). Or if they sometimes do outdoor mathematics with their students (OC). The items were related to the dimensions by an experts' rating.

Additionally, some teachers (n=14, 8 congruent with the respondents of the questionnaire) and some of the students (n=31, all of them also filled a questionnaire) were interviewed. The interviews only contained questions about the program. For example both groups were asked to characterize Math.Researcher lessons by identifying features or describing which special Math.Researcher-activity they did.

Results

The analysis of the data (factor analysis and experts' rating) showed that the identification of the separate Math.Researcher dimensions was not possible. So in the following results the dimensions are considered in total, the belief aspects are separate. The five resulting factors formalism (F), scheme (S), process (P), application (A) and Math.Researcher-Dimension (Dim) can be seen as reliable (Cronbachs Alpha ,728 to ,800, Table 2).

	F	S	P	A	Dim
Number of Items	11	8	10	7	25
Cronbachs α	,728	,764	,798	,800	,769

Table 2: Number of items and Cronbachs Alpha of each main factor

Figure 2 shows the means of the four belief-aspects and the mean of the Math.Researcher-dimensions of all teachers in total. Comparing them, application (A), process (P) and the Math.Researcher-Dimensions (Dim) are rated similarly positive.

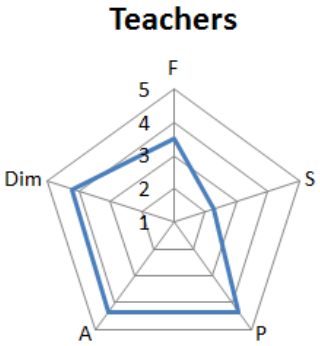

Figure 2: Means of the teachers' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

The exact scales can be seen in Table 2.The aspect of formalism (F) is rated not as positive as these, but considerably better than the scheme aspect (S). Scheme (S) is rated worse. (P) and (Dim) show a significant correlation of r= .59 so 35 % of the processes variance can be explained by the dimensions. The other beliefs cannot be explained significantly by the dimensions. Comparing the standard deviations of the beliefs and the dimensions in total shows that the teachers have rated the single items reasonably homogeneously (Table 2). The scheme aspect rated average worse is perceived the most ambivalent.

n=20	F	S	P	A	M.R-Dim
M	3,51	2,26	4,35	4,34	4,22
SD	0,44	0,56	0,42	0,51	0,35

Table 3: Mean and standard deviation of the teachers' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

Comparing the teachers' results with the students' a decline in the values of the scheme aspect is not recognizable (Figure 3). The five means do not differ that noticeably. Comparing the standard deviations (Table 4) shows that the students' answers differ a lot. The process aspect has the highest standard deviation of nearly 1.

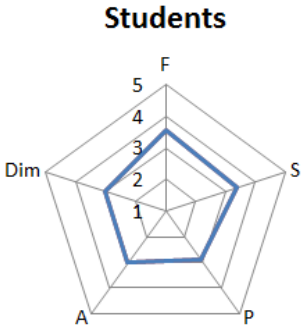

Figure 3: Means of the students' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

The process aspect has the highest standard deviation of nearly 1.The students´ values are much more ambivalent than the values of the teacher´s evaluation.

n=164	F	S	P	A	M.R-Dim
M	3,55	3,39	2,91	3,02	3,01
SD	0,69	0,68	0,91	0,85	0,47

Table 4: Mean and standard deviation of the students' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

The students were also asked whether they would recommend the Math.Researcher program. It is noticeable that the students who would recommend the program to friends and other students (values of 4 and 5) show a significantly higher identification with the Math.Researcher dimensions. The correlation of students' (P) and (Dim) is low but highly significant (r= .43).

In order to answer the question whether the dimensions are well established **(1)** the dimensions have also been considered separately. The analysis revealed that the dimension "working with researchers questions" (RQ) could not be identified separately. Items of this dimension (RQ) were assigned to the dimension "using the Math.Researcher principles" (MRP) or the dimension "acting as a learning guide" (LG). That's why in the following (RQ) is not included.

N=20	MRP	OC	LG	VM	RQ
M	4,26	3,56	3,99	4.43	-
SD	0,43	0,43	0,33	0,41	-

Table 5: Mean and standard deviation of the teachers' M.R dimensions separated (1= do not agree at all, 5= totally agree)

The dimensions "using the Math.Researcher principles" (MRP) and "visualizing mathematics" (VM) were high and homogeneously rated (Table 5). The identification with the dimension "opening the classroom" (OC) is the lowest. This assumption gets strengthened by the students' data. In interviews they were asked to characterize Math.Researcher lessons by conspicuous features. Compared to the teachers who are familiar with the dimensions, the students have not been educated in the Math.Researcher dimensions. Features belonging to the dimension (OC) with its main elements

"interdisciplinary instruction", "inviting experts" and "outdoor lessons" were only rated by 10 % of the interviewed students. Features belonging to the dimensions (MRP), (LG) or (VM) were more often mentioned noteworthy.

Discussion

In summarizing these results we can say that the dimensions are well established **(1)**. During the interviews the teachers often answered that the Math.Researcher program became more structured and concrete since the dimensions were introduced. They reported that the dimensions made them more certain about their planned lessons.

We can also say that a dynamic view on mathematics is more predominant if the participants identify with the Math.Researcher program **(2)**.

However, the pilot study gives no data about a sustainable impact of the Math.Researcher program. So the researched questions of the evaluation of the pilot study were reused and extended into a main study. The main study still wants to investigate whether the dimensions are well established (1) and which view on mathematics is connected to the participants' identification with the Math.Researcher program (2). Additionally, the main study wants to find out whether taking part in the program influences the students' or the teachers' view on mathematics (3). This third question is similar to previous research questions, for example of Cooper and Touitou (Cooper, Touitou, 2013). They wanted to find out, which beliefs can be found and how these identified beliefs change during a one-year PD course.

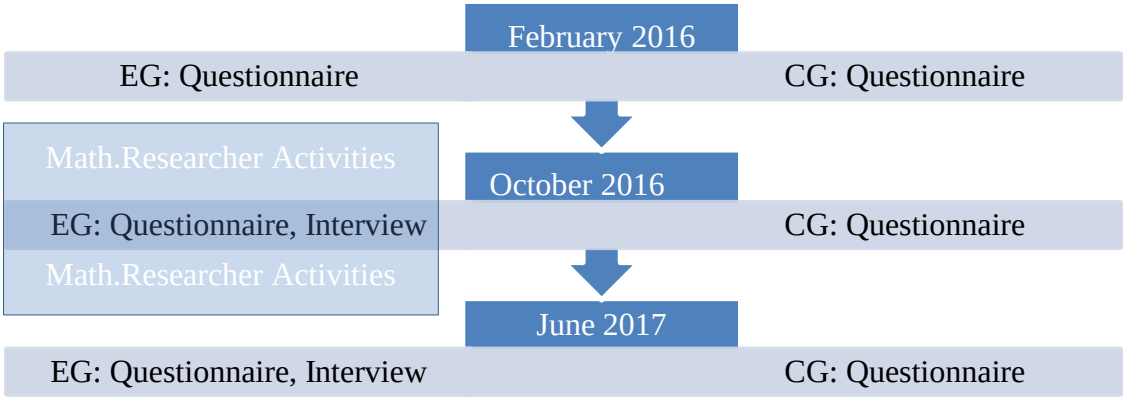

Figure 4: Three survey periods

The program-region Heilbronn-Franken joined in the Math.Researchers program in December 2015 and started activities in February and March 2016. Three survey periods from 02/2016 to 06/2017 are planned: one before, one during and one after taking part in the program (figure 4). The experimental group (EG) fills in a questionnaire in all of the three periods. In addition, teachers and students of the experimental group get interviewed in the second and third survey period. The control group (CG) gets the same questionnaire as the experimental group in all the three survey periods, but the control group will not be interviewed. If the control group and the experimental group show the same results at all of the three times of measurement, it is not possible to attribute any influence on the view on mathematics to the Math.Researcher program.

The questionnaires of the pilot study were shortened with the help of factor analysis and expert ratings. Items that were rated clearly by the experts, items with loadings exceeding ,39 and some

items of special interest stayed in the questionnaires. For example, the students' questionnaire had 79 content-related questions in the pilot study. Their questionnaire in the main study only has 29 content-related questions.

The next step is the second survey period in October 2016. First comparisons to the results of the first survey period and further details of the pilot study will be presented at CERME 10 in Dublin.

Acknowledgment

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Factors influencing the developing effectiveness of early career teachers' mathematics teaching – initial findings

Alison Godfrey

University of Leicester, United Kingdom; ag384@le.ac.uk

Early career primary school teachers in the UK enter a varied and complex situation in terms of professional development opportunities and support for their ongoing career progression as teachers of mathematics. Literature suggests that the professional development a teacher receives impacts on both their subject knowledge for teaching and their beliefs and attitudes to the subject. This multiple case study looks to fill gaps in the research undertaken in this area through detailed analysis of the personal perspectives of the participants over a two year period. This paper reports on the initial findings of a comparative analysis of the trajectory of two teachers to date. These teachers were in seemingly similar contexts for their first year of teaching, yet had very different experiences and held very different perspectives on their development.

Keywords: Teacher characteristics, professional development, elementary school mathematics.

Introduction

Students start a primary teacher education course in England with a range of qualifications in mathematics, a range of experiences in school mathematics and a range of attitudes and beliefs about the subject. During the course student teachers develop their subject and pedagogical knowledge and gain experience in mathematics teaching. They then enter a complex and changeable situation in schools in terms of provision for their ongoing development (Advisory Committee on Mathematics Education, 2013). Using a multiple case study approach, the aim of this study is to gain a deeper understanding of how the effectiveness of early career primary school teachers' mathematics teaching develops and what impacts on this, with a particular focus on each teacher's own perspective.

Theoretical background

The notion of effectiveness as applied to mathematics teaching is complex and ideas vary about what this looks like and even what its impact should be (Cai, 2007). Although generally agreed that teaching can only be considered effective if there is an impact on those being taught, i.e. effective teaching leads to effective learning and gains in understanding (Bryan, Wang, Perry, Wong, & Cai, 2007), the notion of 'understanding' is complex. Skemp (1976) made a clear distinction between instrumental understanding, simply knowing rules and procedures at a shallow level, and relational understanding which enables pupils to build conceptual schemas. Most mathematics educators agree that this second type of understanding is the most desirable and teachers internationally agree that an indicator of mathematical understanding is that pupils can use this understanding to problem solve flexibly in a range of situations (Byran et al., 2007). The significance of understanding the connected nature of mathematics is very apparent (e.g. Askew, Brown, Rhodes, Johnson, & Wiliam, 1997) and is consistent with the notion of relational understanding. The concept of effective learning in mathematics being based on this type of understanding within the context of appropriate intellectual challenge is endorsed by current policy makers and teacher inspection systems in the UK (Department for Education (DfE), 2013; Ofsted, 2012) and is the definition used in this study.

The literature suggests that teachers' effectiveness is impacted by their subject knowledge and by affective dispositions including beliefs, attitudes, motivations and emotions. Definition of these constructs are debated in the literature and indeed there are a range of ideas as to how they overlap and directions of influence (Lewis 2016). In agreement with Hannula (2011) I consider beliefs are both cognitively and affectively situated. As a starting point for discussions of attitude and emotion, I am adopting the three dimensional model of attitude from Di Martio and Zan (2010): emotional disposition towards mathematics, vision of mathematics (relational/instrumental) and perceived competence in mathematics.

Many studies, e.g. Ernest (1989) and Askew et al. (1997), have looked at teachers' beliefs about the nature of mathematics and the impact these beliefs have on their teaching. Ernest (1989), for example, argues the most effective teachers see mathematics as a dynamic, creative, problem solving subject and adopt a 'Facilitator' teaching approach.

Teachers' attitudes to the subject, including their emotional disposition, also impact on effectiveness. Particularly in primary teaching, teachers and student teachers find themselves teaching subjects that they have not necessarily enjoyed or been particularly successful at learning themselves. Many feel insecure with their subject knowledge in mathematics and recount their experience of mathematics at school as a subject that caused difficulties and even "real emotional turbulence" (Brown, 2005 p. 21). For many teachers therefore, mathematics is linked with negative emotions, particularly anxiety, and this can impact on their wider attitudes to it. It seems that teachers even protect pupils from mathematics and, in seeking to simplify it, emphasise the step by step procedures that are likely to lead pupils into developing instrumental understanding and potentially negative attitudes (Hodgen & Askew, 2007), reducing their effectiveness.

A further influence on the effectiveness of teaching of mathematics is subject knowledge. Indeed, evidence suggests that the impact of beliefs and attitudes and subject knowledge are interdependent (Askew et al., 1997). Most recent research on teachers' subject knowledge for mathematics uses Shulman's (1986; 1987) seminal papers as a starting point. Others have applied his ideas to mathematics teaching, debated which aspects are most essential and relevant, and sought to measure or evaluate them. Ball, Thames & Phelps (2008) argue that there is knowledge that is specific to teachers of mathematics and that might have an identifiable impact on the effectiveness of their teaching; Specialised Content Knowledge. This includes, for example, the understanding needed to be able to explain procedures, to analyse errors and strategies, and to consider appropriate examples. Baumert et al. (2010) conclude that pedagogical content knowledge (PCK) makes the greatest contribution to pupil progress, but weaknesses in mathematical content knowledge are not offset by greater PCK. Askew et al. (1997) found that it was not the formal qualifications or the amount of subject knowledge that the teachers had which was significant in the effectiveness of their teaching, but rather the connectedness of their subject knowledge "in terms of the depth and multi-faceted nature" of the meanings and uses of concepts in mathematics (p. 69). Other authors highlight the importance of pupil voice and the ability of the teacher to choreograph classroom discourse as key characteristics of effective teaching in mathematics (e.g. Schoenfeld, 2013), and Barwell's (2013) discursive psychology perspective emphasises that knowledge is contextual and can be changed or reconstructed accordingly. There is much critiquing of Shulman's and Ball's ideas but there seems

to be general agreement that the subject knowledge needed for effective teaching goes further than just having a strong conceptual knowledge of the subject being taught.

There is considerable evidence that a teacher's depth of reflection and their beliefs and attitudes towards mathematics are a crucial influence over their trajectory of development (Di Martino & Zan, 2010; Turner, 2008). Hodgen and Askew (2007) and Schoenfeld (2013) suggest that the development of some teachers is hindered by their linking of mathematics with emotion and that, for some teachers, professional development activities and goals should be much more about changes in their beliefs and attitudes than about improving their subject knowledge.

Teachers' work is within a social context, (Levine, 2010), and this can have a significant impact on the nature of a teacher's professional development. The community within the school might have a range of different foci and agendas and there seems to be a wide variation in practice between schools. The very structured collaborative approach to teacher development in China seems high effective although dependent on the considerable time given to teachers to discuss, prepare and analyse their work (Paine & Ma, 1993). In contrast Ball, Ben-Peretz &Cohen (2014) consider that in the US most teachers work in isolation and the potential benefits of sharing good practice are lost. In England, although the need for quality on-going professional development opportunities for primary teachers is recognised and highlighted by Ofsted (2012), the current context is of variable provision in formal professional development (ACME, 2013).

In summary, the literature suggests that a teacher's trajectory as a teacher of mathematics is influenced by the interaction of their beliefs and attitudes, their subject knowledge and the professional development they receive, through both formal education opportunities and personal reflection, and these factors therefore influence the effectiveness of their teaching and the effectiveness of their pupils' learning. My study sits within this theoretical framework and seeks to extend the existing literature particularly though highlighting teachers' own perspectives on this process.

Methodology

A multiple case study approach is being employed to follow the trajectories of a small sample of teachers as they progress into their first two years as a qualified teacher. Four participants were chosen for the pilot study, two with mathematics qualifications beyond the minimum required for primary school teachers in England. An initial interview at the end of their one year postgraduate teaching course focused on their relationship with, and attitude to, mathematics and their progress in teaching the subject as a student teacher. To facilitate discussion, Lewis's (2016) idea of a graphic display was adopted; participants were asked to draw and explain their relationship with mathematics over time. This gave insights into participants' attitudes to mathematics and their perspectives as learners of the subject as well as in their student teacher role. Participants were able to reflect on how their relationship with the subject influenced their current teaching approaches. Twice yearly interviews, including further graphing at the end of each year, and discussion of documentation related to their progress as early career teachers, such as formal observation feedback, provide evidence of their ongoing development as teachers of mathematics. Interview questions have been designed to probe about the participants' beliefs, attitudes and subject knowledge for teaching mathematics, what they perceive to be the characteristics of effective teachers of mathematics and their perspectives on their development as teachers of the subject. Within each interview they also describe two particular

lessons: their chosen ‘best’ and ‘most challenging’ lessons since the previous interview, providing insights into what they consider to be effective and ineffective teaching and their subject knowledge. In addition, they keep records of their professional development in teaching mathematics.

To begin to analyse the data, a mind map has been drawn of each data collection point for each participant. The mind map contains evidence of the participants’ changing context and development as a teacher of mathematics, including both factual and non-factual information, and also the participant’s interpretation and my interpretation of this evidence. The mind maps allow for a thematic approach, enabling identification of concepts and themes such as ‘awareness of self as a mathematician’ and ‘priorities in teaching mathematics’. In regard to their relationship with mathematics and their development as teachers of the subject, the participants have stories to tell and thus narrative analysis techniques are being used to support analysis of these stories. Whilst these techniques have enabled useful analysis of the early findings, the next stage is to further research and develop these analysis methods so that data can be analysed increasingly effectively and efficiently**.**

Results and discussion

In this paper the focus is on the early findings of a comparative analysis of the trajectory in the first year of the study of two of the teachers, Gina and Rama (pseudonyms), who in their first year as qualified teachers were in seemingly similar contexts: they both taught children aged 5-6 years in schools with a two class entry, working alongside more experienced colleagues and teaching mathematics to their own, mixed attainment, classes. Rama has a stronger mathematics background and also chose to study a mathematics specialism as part of the teaching course, but both students achieved the highest teaching course grade in all areas of the Teachers’ Standards (DfE, 2011). In their interviews, both were able to identify many ways in which they had evolved as teachers of mathematics to date, but they answered questions in ways that revealed very different perspectives on their development. In addition, their professional development records indicated that they received very different opportunities and approaches to their professional development. This raises questions about why these are so different and the extent to which this is school dependent or dependent on the approach and philosophy of the teachers themselves.

Descriptions of each participant’s relationship with mathematics and how this had evolved over time revealed interesting and complex relationships (Figure 1) and it was clear that past experiences impacted their current thinking. Whilst the graphs drawn are not directly comparable as the interpretation of the vertical scale was left open, the trajectories illustrated and discussion of these provide some scope for comparison. The two peaks in Gina’s relationship with mathematics indicate two different perceived aspects of success in mathematics. Firstly she recalls being successful in the subject in Year 9 (aged 13-14) when she felt she responded well to the high expectations her teachers had of her. The second period of success, during the teaching course, was due to her own development of the conceptual understanding that the literature suggests is essential for learners to gain for long term, secure understanding of mathematics. This enabled her to reflect on her previous period of ‘success’ which she then realised was based on superficial learning:

Gina: I could do the methods, but I didn’t understand them.

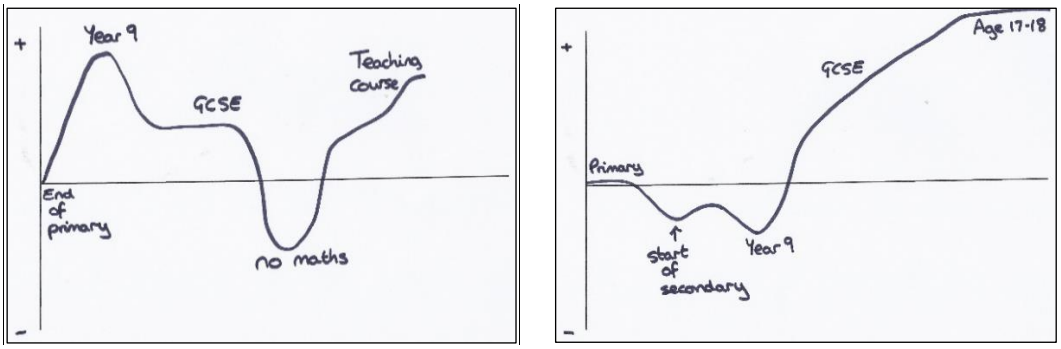

Figure 1: Gina and Rama's relationship with mathematics, drawn at end of teaching course

Gina's personal experience seems to have given her a particular focus in her own teaching of mathematics, particularly how strategies she is aiming to use in her teaching could prevent the children taking the path of superficial learning that she had followed. Discussing how she developed her teaching in her first few months as a qualified teacher, for example, she identified the use of representations as significant and something that she perceived as missing in her own learning:

Gina: And the pictorial things, definitely. I don't think I've ever done that as a maths learner myself and I think maybe if I had it would have been much more easy.

Rama's mathematical background also seems to have impacted on her priorities in teaching mathematics. Although she finished her studies with a degree where she used mathematics, she finished primary school not enjoying the subject and lacking in confidence. She seems to have been a shy and hesitant learner, fearful of getting the wrong answer in a subject she saw as right or wrong. At secondary school she was initially placed in lower sets but talked with great enthusiasm about her Year 11 class (aged 15/16), a top set, where the environment of the classroom was such that she felt she could make mistakes without fear. She described too a change at this point from memorising how to do certain methods to taking ownership of her learning and finding her own ways of solving problems. Priorities identified by Rama in her teaching include the importance of children enjoying mathematics alongside gaining conceptual understanding, both aspects that for her were missing until the later stages of her schooling.

It seems that Gina and Rama entered two very different teaching communities with different agendas which also impacted on the nature of their professional development (Levine, 2010), and graphs drawn at the end of their first year of teaching revealed very different perspectives on their year (see Figure 2). Rama's saw her journey as a teacher of mathematics as a smooth upward trend, with occasional dips when she taught poorer lessons. Whilst having few formal professional development opportunities, she indicated that she is confident in her own subject knowledge and in independently planning and teaching the subject. Gina described a much more structured and intensive programme with regards to the support she received. However, her graph illustrates her perspective that she has had a sometimes difficult year and narrative analysis revealed, interestingly, that the intense involvement of her school in her professional development seems to have led to a crisis in her confidence and a turbulent second half of her first year of teaching.

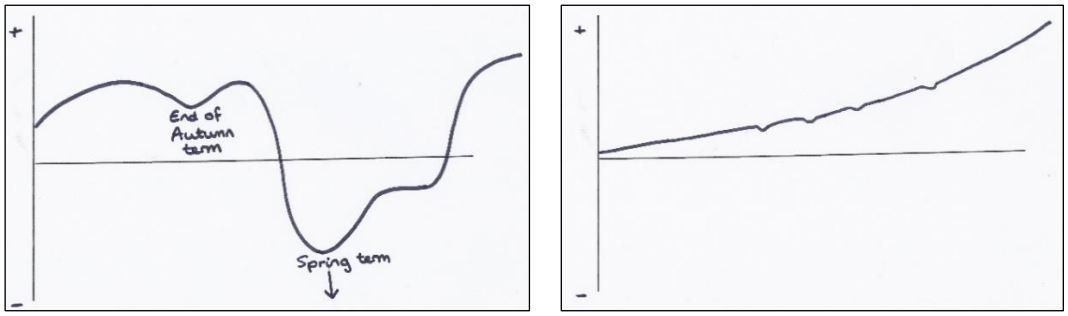

Figure 2: Gina and Rama's relationship with mathematics during their first year of teaching

This analysis highlights the emotional impact on Gina of the series of intervention events that followed on from a formal observation of her teaching that she labelled several times as a 'disaster', and then her subsequent recovery. In her story particular emphasis is evident of the impact on both the perceived competence and emotional disposition aspects of her attitude (Di Martino and Zan, 2010), with a consequent impact on her confidence in her ability to teach mathematics:

> Gina: It seemed no matter what I was doing to change, I was still getting negative feedback and it was disheartening. There was a big period of time when I literally dreaded every single lesson, because you just think "What could happen? I don't know what I'm doing"

The records of professional development kept by Gina and Rama over the year confirm that Rama's in school support has been much more informal and she places a high priority on targets she generally sets for herself, seeking to improve the quality of her teaching through a reflective approach (Turner, 2008). In contrast to Gina's record of events, her record is a series of targets. Contrasts in their priorities in planning and teaching of mathematics, and indications of their subject knowledge have been evident through different ideas about the characteristics of effective teachers and descriptions of their best and most challenging lessons. It seems that both teachers are seeking to teach with a relational approach, with Rama at this stage in a stronger position regarding subject knowledge. It is interesting that end of year assessment information indicates that children in both Rama and Gina's classes progressed well in their mathematical understanding over the year; this seems to have contributed to Gina finishing the year in a more confident frame of mind.

Conclusion

The findings of these case studies to date confirm my analysis of the literature in that there are two related and interwoven, but distinct, categories of factors that might influence an early career teacher's trajectory in relation to the effectiveness of their mathematics teaching - those that are related to the teacher themselves, in terms of beliefs, attitudes and subject knowledge and those that are related to their teaching context. Gina's experience during her first year of teaching provides a particularly interesting example of professional development that impacts in complex ways on beliefs and attitudes to the subject. Although it is too early to draw more than very tentative conclusions, it is interesting to ponder to what extent Gina and Rama's trajectories would have been different had they been in one another's school, and this leads to potential implications for initial teacher education.

These case studies align with current professional documentation (ACME, 2013) to suggest that there is a lack of uniformity in teachers' professional development experiences despite there being systems in place in the UK promoting uniformity. The existence of these issues has implications for the

preparation of student teachers in Initial Teacher Education (ITE) and suggest that ITE providers could seek to more explicitly discuss differences and similarities in primary school approaches to teacher development and further strengthen their policies of students gaining experience in a range of contrasting placements, within which they can not only participate in whole school professional development events, but also talk with teachers about their development experiences.

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A theoretical framework for analyzing training situations in mathematics teacher education

Claire Guille-Biel Winder[1] and Frédérick Tempier[2]

[1]COPIRELEM, CAFEP-ESPE de Nice Université de Nice, France; claire.winder@unice.fr

[2]COPIRELEM, LDAR Université de Cergy-Pontoise, France; frederick.tempier@u-cergy.fr

The thoughts on the primary schoolteacher training have led to the production of many resources for primary schoolteachers. Faces of the abundance of such documents, teacher educators need some tools to identify the knowledge potentially at stake in training situations and to allow them to implement such situations according to their own objectives and context. We present a five-level analysis framework that characterizes the training tasks, taking account of the activities induced by the task, according to the expected posture of the prospective teacher, to the type of the knowledge at stake and to possible degrees of decontextualization. We illustrate this analysis framework by presenting an example of a training scenario based on the principle of role-play.

Keywords: Teacher education, professional development, primary education, knowledge for teaching, analysis framework.

Introduction

The research about primary schoolteacher education in mathematics and professional development has led to the production of many resources for educators. In France, the COPIRELEM[1] group produced many of them. These resources provide "training situations" based on various training strategies (Houdement & Kuzniak, 1996), and are generally accompanied with information about their implementation (phases, steps, instructions, elements of institutionalisation) with regard to the stakes of the training. But their quality does not guarantee an accurate appropriation by teacher educators. Our questioning is: how is it possible to help teacher educators to exploit any "training situations" in a relevant way, according to their objectives?

The research literature usually provides studies about knowledge for teaching, teacher conceptions, and their evolution (Shulman, 1986; Houdement & Kuzniak, 1996; Ball, Thames, & Phelps, 2008). Other studies present one training situation, and generally focus on its effect on the prospective teachers. For example Horoks and Grugeon (2015) "analyse the contents and methods of an initiation course in research in mathematics education, and […] how it can influence the beginner teachers' practises" (p.2811). To our knowledge, no study focuses on the characteristics of training situations nor provides specific framework in order to analyse any training situation. This led us to develop an analysis framework for training situations. The paper presents this COPIRELEM's work in progress.

[1] The COPIRELEM is a commission dedicated to the education to the primary school. It is stemming from the network of IREM (French institute of research on mathematical education).

Presentation of the analysis framework

We define a *training situation* as a situation that involves prospective teachers (students, pre-service or in-service teachers) and educators within an institution of teacher education. It is composed of a set of tasks that could be conducted by a teacher educator. We take into account all the tasks proposed by the teacher educator. From a training situation, the educator may elaborate a training scenario that is to say a set of chronologically organized tasks chosen among all the tasks that constitute the training situation. We voluntarily distinguish situations from "scenarios" because we intend to underline the dynamic aspect of the scenario.

In response to each task of a training situation, prospective teachers develop an *activity* that corresponds to "what [they engage] in during the completion of the task" (Rogalski, 2013, p.4). We distinguish five different types of activity: "mathematical activity" (doing maths during the completion of a mathematical task), "mathematical analysis activity" (analysing the maths at stake in a mathematical task), "didactical and/or pedagogical activity" (highlighting didactical and/or pedagogical choices related to the mathematical task), "didactical and/or pedagogical analysis activity" (analysing didactical and/or pedagogical choices related to the mathematical task), "problematisation activity" (identifying and investigating professional issues by mobilizing mathematical, didactical and pedagogical concepts). For each type of activity we take into account three dimensions (Fig. 1): the type of knowledge at stake; the degree of decontextualization of this knowledge; the posture of the prospective teachers expected by the teacher educator. These dimensions are specified in the next sections.

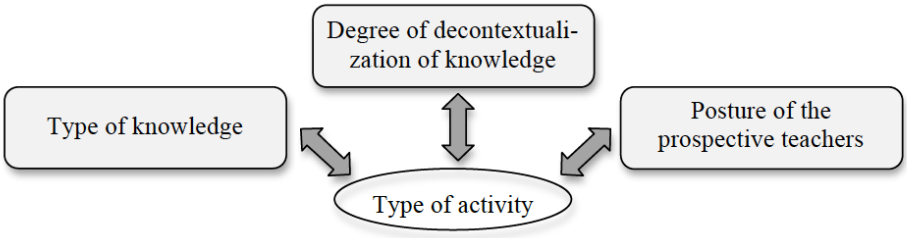

Figure 1: Three dimensions for characterizing a type of activity

Three types of knowledge

We rely on the three types of knowledge for teaching mathematics identified by Houdement and Kuzniak (1996): mathematical knowledge, pedagogical knowledge and didactical knowledge. "Mathematical knowledge corresponds to mathematics that a teacher needs to know in order to prepare, regulate and evaluate his lesson and his students" (Houdement, 2013, p.12). It "includes and specifies the content knowledge" identified by Shulman (1986). It roughly can be related to Subject Matter Knowledge (Ball and *al.*, 2008), and the specific didactical nature of mathematical knowledge can be identified to the Specialized Content Knowledge (SCK). According to (Houdement, 2013), didactical knowledge is linked to the mathematical content and fed by research in the field of mathematics didactics. It corresponds to analysis of teaching and learning phenomenon and to propositions of engineering. Therefore it can be associated with at least two categories (Ball and *al.*, 2008): Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT).

Pedagogical knowledge[2] is characterised as "knowledge of experience" (Portugais, 1995). It is related to teaching and learning conceptions and to the organisation and management of the class. It is less dependent of the mathematical content than other types of knowledge. It is important to take this knowledge into account because schoolteachers deal with various school subjects.

Three degrees of decontextualization

Brousseau (1997) and Douady (1985) identify three degrees of decontextualization of a mathematical knowledge: implicitly mobilized, explicitly mobilized in context or decontextualized (to become available in other contexts). We extend this notion to didactical and pedagogical knowledge. A mathematical knowledge is *(implicitly) mobilized in context* (in act) if it is used as tool (Douady, 1985) in a mathematical task. This task can be carried out: what is asked is effectively achieved (manipulation, elaboration and writing a solution for example). But the task can only be evocated: it is mentally achieved. A mathematical knowledge is *explicit in context* if its use (as tool) is identified and formulated. At least, a mathematical knowledge is *decontextualized* if it is identified as an object of learning: a status of object is given (by the educator) to the concept used previously as tool, usually during an institutionalisation phase[3] (Brousseau, 1997). The didactical/pedagogical knowledge is *mobilized in context* when the didactical/pedagogical choices are made for the considered mathematical task. It is *explicit in context* during the analysis about the consequences of these choices. At least, it is *decontextualized* when the underlying didactical/pedagogical concepts are highlighted.

Four postures of the prospective teachers

In conjunction with the teacher trainer's relationship to the prospective teachers identified by Sayac (2008), we define four specific postures of prospective teachers, which are expected by the educator during a training situation[4]. Prospective teachers are in a posture of *student* relatively to the mathematical knowledge when they have to perform mathematical activity or when they are concerned with the mathematical knowledge of this activity. They are in a *student/teacher* posture when they investigate mathematical tasks for students or students' works, or when they analyse the conditions of implementation of a task in the classroom. They are in a *teacher* posture when entering in a broader questioning on classroom practices and issues of mathematical learning. Finally, they are in a *practitioner/researcher* posture when they problematize a professional issue related to mathematical learning or teaching.

Five study levels

In order to analyse a training situation, we define five study levels. To each level corresponds a type of activity, that induces (implicitly or explicitly) a posture of the prospective teacher (expected by

[2] According to (Houdement, 2013), Ball's, Phelps' and Thames' typology doesn't seem to take into account this type of knowledge.

[3] In institutionalisation phase (Brousseau 1997), the teacher gives a cultural (mathematical) status to some knowledge emerging from students' actions during the situation.

[4] We notice that prospective teachers are not always aware of these postures.

the educator), and that involves different types of knowledge in a certain degree of decontextualization (see Fig. 2).

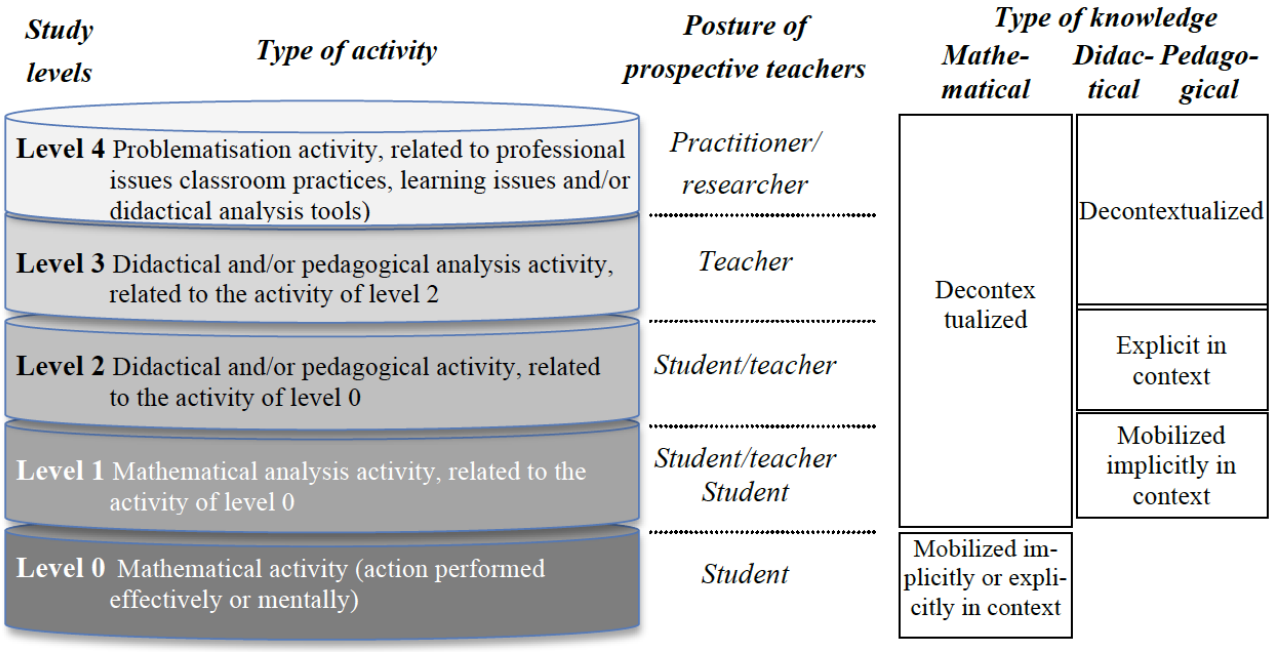

Figure 2: Characteristics of the five study levels

Level 0. A task may induce a mathematical activity. This activity can be performed or evocated (mentally performed). The mathematical knowledge is mobilized (implicitly or explicitly) in context. The prospective teachers are in a posture of student (relatively to the mathematical knowledge).

Level 1. A task may induce a mathematical analysis activity related to the activity of level 0 when it highlights decontextualized mathematical knowledge and the prospective teachers are in a posture of (learning mathematics) student. In this task, the didactical and/or pedagogical knowledge can be implicitly mobilized in context and then initiates the change toward a student/teacher posture of prospective teachers.

Level 2. A task may induce a didactical and/or pedagogical activity related to the activity of level 0 when it corresponds to the analysis of implementation conditions - actual or anticipated only of the mathematical task. The didactical and pedagogical knowledge is explicit in context. The prospective teachers are in a student/teacher posture.

Level 3. A task may induce a didactical and/or pedagogical analysis activity related to the activity of level 2 when it is for example a questioning on classroom practice (specific learning tasks, professional actions...) or on issues of mathematical learning for one or several contents (curriculum, progressions...), or even a highlighting of didactical analysis concepts (didactic situation phases, types of tasks...). This analysis leads to the decontextualization of didactical and/or pedagogical knowledge. The prospective teachers are in a posture of teacher.

Level 4. A task may induce a problematisation activity when it corresponds to the problematisation of professional issues related to classroom practices, learning issues and/or didactical analysis tools.

The prospective teachers are in a posture of practitioner/researcher, especially when it comes to developing an analysis methodology of this issue and to infer results.

Each study level is based on the study of the activity of previous levels and involves some mathematical, didactical and/or pedagogical knowledge. The change from study level *n* to study level *n + 1* is linked either to a change of the prospective teachers' posture or to a change of degree of decontextualization for at least one type of knowledge (from implicitly mobilized in context to explicit in context, from explicit in context to decontextualized). But the different activities induced by a training situation don't usually appear in a chronological order (from level 0 to level 4). For examples, see the analysis of various training situations developed in French context by the COPIRELEM group (Guille-Biel Winder, Petitfour, Masselot & Girmens, 2015; Bueno-Ravel and al., 2017). We think that the analysis could be extended to situations based on different training strategies. That is why we present here the analysis of a training scenario based on the principle of role-play developed in an international context (Lajoie and Pallascio, 2001; Lajoie and Maheux, 2013; Lajoie, accepted).

An example of use of the analysis framework

Definition of role-play

As Lajoie and Pallascio (2001) state "role-play involves staging a problematic situation with characters taking roles". It is used over many years in mathematics education course in UQAM (University of Québec in Montréal) and is organized as follows:

> First, the 'theme' on which students will need to role-play is introduced (introduction time). Second, students then have about 30 minutes to prepare in small groups (preparation time). Third comes the play itself (play time), where students chosen by the educator come in front of the classroom and improvise a teacher-student(s) interaction (sometimes, like in the case reported here, involving the whole class). Finally, we have a whole classroom discussion on the play (discussion time). (Lajoie, accepted)

We designed a role-play on the teaching of proportions based on a problem from a textbook. We use the analysis framework to illustrate an example of analysis aimed at highlight the potential of this situation.

An example of role-play

The role-play presented below is intended for pre-service schoolteacher education. We describe the different phases.

Introduction time. The educator distributes to prospective teachers an excerpt from a fifth grade (10-11 year old pupils) handbook presenting a problem of proportions (Fig. 3), and various productions of pupils. The teaching issue announced by the educator is the following: to manage a class discussion about the pupils' strategies and about their ideas and solutions, in order to share them in the class community and to determine their validity and efficiency.

Preparation time. The prospective teachers have to prepare the discussion class about the pupils' strategies.

Play time. At the end of the preparation time, the educator chooses prospective teachers to play the game: some of them play pupils, one of them plays the teacher, while the others are watching the discussion class and taking notes.
Discussion time. The debate intends to highlight and to analyse the choices of the « teacher » during the play game: what worked well during the implementation of the discussion class? What was difficult? What seemed to be important? What alternative implementations could be realized?
Institutionalisation time[5]. The educator institutionalizes the knowledge at stake: he generalizes some elements about how to manage a discussion class or about proportion problems solving.

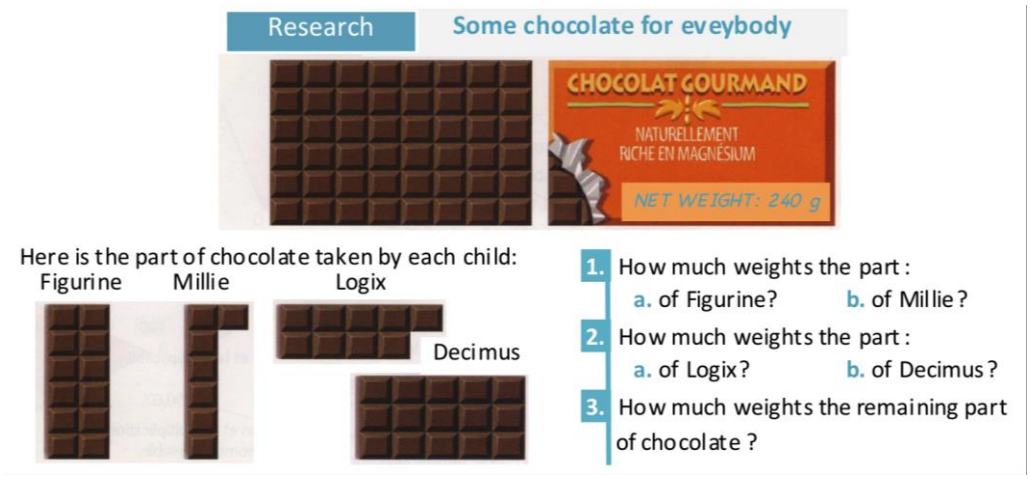

Figure 3: A proportion problem

Analysis of this role-play

The initiating task is a professional situation and corresponds to a level 2 activity: the prospective teachers are initially in a student/teacher posture. But they will need « to go down » to a student posture and « to go up » to a teacher posture during the phases of the scenario. The preparation time of the discussion class leads the activity of the prospective teachers "to go back and forth" to the study levels 0, 1 and 2. The problem solving corresponds to a level 0 activity and the mathematical analysis of the problem solving to a level 1 activity. Moreover there are various strategies to solve this proportion problem. Preparing the discussion class of the pupils' strategies (level 2 activity) hence needs to analyse and rank them (from the least to the most elaborate). This analysis corresponds to a level 2 activity. The prospective teachers don't have the same activity during the playtime. The study level is different according to the role to play: mostly levels 0 and 1 for the students' roles and level 2 for the teacher's role. The discussion time corresponds to a level 2 activity when the prospective teachers analyse how the discussion class has been managed. But it can also correspond to lower levels activities, when they discuss about pupils' strategies, difficulties, mistakes and their exploitation during the discussion class. Various institutionalisations can be considered, according to the knowledge that was developed at different study levels. The institutionalized elements will be more or less developed according to the teacher educator's objectives and progression, the prospective teachers' knowledge, etc. Here are some propositions organized in ascending order of study levels. The teacher educator can institutionalize some

[5] We add this new time to the four ones proposed by Lajoie and Pallascio (2001).

mathematical knowledge at stake (level 1) and related to the proportionality field: various methods to solve a proportion problem, mathematical justifications and mathematical theories they are relied on. He can situate the proportion problems in the more general category of multiplicative problems, or he can explicit some didactical variables usually at stake in proportion problems (level 3). He also can identify some difficulties or mistakes revealed by the pupils' productions as « usual » and highlight mistaken conceptions: identification of quantities, choice of an adapted strategy, persistence of an « additive model », etc. At least, in regard of the announced objective of the role-play, the teacher educator also can institutionalize some didactical knowledge, relatively to the organization of a discussion class (level 3): formulation and validation in mathematics; teacher's tasks before, during and after the discussion class…

Conclusion

The example of role-play situation shows how the analysis framework can be a tool for an *a priori* analysis. Moreover this example shows that the organization of the study levels is not a chronological but a hierarchical one: the initiating task can induce an activity of level 0, 1, 2, 3 or 4. But the transition to lower levels activity is often necessary. The conceptual maps of the knowledge for teaching developed by Houdement and Kuzniak (1996) or by Ball and *al.* (2008) have a descriptive, predictive and prescriptive dimension (Ball & *al.*, 2008, p.405). But beyond their interest, (Houdement, 2013, p. 21) stressed the importance of the knowledge reconfiguration in connection with the mathematical content. The analysis framework reports how, during a training situation, the types of knowledge for mathematics' teaching are dynamically hinged to one another in connection with the mathematical content. The analysis framework allows teacher educators to identify the potentialities of a full range of training situations. We intend to extend its use to study other types of training situations (for example e-learning situations). By clarifying the stakes of the various phases of the implementation, the analysis framework reveals various possible strategies for the teacher educator. Thereafter it could be a useful tool for elaborating different training scenarios. Hence, the teacher educator should be able to implement situations in a specific context according to his objectives and constraints (time and period of training, place in a progression which take into account the mathematical, didactical and pedagogical knowledge ever studied…). Besides it is possible to consider a sequence of successive scenarios. The analysis framework could also highlight various possible "training paths", which should reveal the educator's training strategy at a more global scale. A perspective is now to study how teacher educators appropriate this framework and how it supports their teaching practises.

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Changes in beliefs of teachers on a PD course after reflecting on students' learning results

Hahn, Thomas and Eichler, Andreas

University Kassel, Germany; thahn@mathematik.uni-kassel.de, eichler@mathematik.uni-kassel.de

In this paper, we discuss results of a qualitative part of a research project aiming to investigate the impact of a professional development course with a specific focus on reflecting of students' learning results on teachers' beliefs towards teaching and learning of mathematics. We refer briefly to some aspects of teachers' professional development. Afterwards, we discuss teachers' beliefs as the main theoretical construct for our research. Data for this paper about teachers' beliefs were collected by semi-structured interviews with three teachers after the professional development course. Results show that teachers report in interviews changes referring to teaching and learning of mathematics from a transmission-oriented view to a more constructivist-oriented view.

Keywords: Beliefs, belief change, professional development, reflecting students' learning results.

Introduction

There is a consensus that university studies and internships are not enough to prepare future teachers for all challenges in their professional career (Mayr & Neuweg, 2009). For this reason, professional development (PD) is understood as being a key factor for innovating and reforming mathematics teaching in school (Garet et al., 2001). However, every change seems to be dependent on specific characteristics of PD: Desimone (2009) summed up five key features of high quality teacher PD. She described that PD will be effective, if a PD course is content-focused, enables active learning, is coherent, has a critical duration and if teachers take part in a PD course collectively. In addition, Franke et al. (1998) reported that reflecting students' learning results makes teachers' learning sustainable. Reflecting students' learning results means, that teachers collect in distance phases of a PD course students' results when working in a specific learning environment or working on a specific task developed in the PD course. Afterwards these students' learning results were the basis of the next face-to-face meeting in the PD course.

Although research identified several features of PD, in most cases the effectiveness is not clear (Yoon et al., 2007). In addition, it is not clear how they influence teachers' learning. For example, Franke et al. (1998) could show that a mix of several characteristics including reflecting students' learning results is effective for teachers' learning, but they did not focus on reflecting students' learning results as a single variable. In a qualitative design, for example also Strahan (2003) found that teachers reflecting on students' learning results increased students' achievement on elementary school level. In addition, Schorr (2000) showed that students' achievement increases when teachers completed a PD including the analysis of students' problem solving processes. Like a conclusion, Little et al. (2003) state that reflecting students' learning results has the potential to bring students more explicitly into deliberations of teachers. However, more research is needed to understand how teachers learn from reflecting students' learning results and whether it impacts on teachers' professional competence including particularly teachers' beliefs.

To investigate the efficiency of reflecting students' learning results in PD courses on the teachers' knowledge, beliefs or motivation, the study as a whole considers two PD courses with a quasi-

experimental setting. Content, teacher trainer and learning time were mostly the same in both courses. However, we integrated reflecting students' learning results in the first PD course, but not in the second. In this paper we will not refer to quantitative results of our research that we reported elsewhere (e. g. Hahn, & Eichler, 2016). We further do not refer to differences between the effects of the two PD groups, but we emphasize results of a qualitative interview study including three teachers of the PD course with reflecting students' learning results. In these interviews, we primarily refer to the teachers' beliefs concerning the teaching and learning of mathematics.

Beliefs about teaching and learning of mathematics

Beliefs can be defined as "psychologically held understandings, premises, or propositions about the world that are thought to be true. […] Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action." (Philipp, 2007, p. 259).

For our research a crucial question is whether beliefs could be viewed to be stable or changeable. Partly, researchers use stability as a part of their definition of beliefs (Fives & Buehl, 2012; Liljedahl, Oesterle, & Bernèche, 2012). In contradiction to that, there are studies which demonstrate belief change by a special intervention. For example, Decker, Kunter, and Voss (2015) reported changes referring to teaching and learning of preservice teachers and teacher trainees. Liljedahl et al. (2012) analyzed related literature and conclude that there is a different meaning of stability in research studies. As a result, they propose to avoid stability in the definition of beliefs. For them, belief change is a natural process that requires a sufficient extent of time. For this reason, in this research study beliefs are described as changeable.

A further crucial question concerns the definition of the belief object that should be changed by PD. Beliefs can refer to a special subdomain of mathematics (Eichler & Erens, 2015), to mathematics itself, teaching and learning of mathematics or an overachieving orientation that is independent from a subdomain (Staub & Stern, 2002). Beliefs referring to teaching and learning of mathematics can be divided into two paradigms: the transmission-oriented and the constructivist-oriented view of learning (Fives, Lacatena, and Gerard, 2015). Transmission-oriented beliefs of teaching imply that knowledge is directly transmitted from teacher to the learners and learners absorb all information. In this case, learners are passive recipients. For this reason, teachers' role is to prepare all information for students to enable an effective storing and an optimal recall. In contradiction, the constructivist-oriented view of learning reflects the active role of learners as constructers of their own knowledge structures. In this case, students learn new information based on their existing knowledge and beliefs to enable an integration of information in their mental networks (Decker et al., 2015; Staub & Stern, 2002). For this reason, teachers take a role as constructor of learning environments that enable students to learn self-directed. Voss, Kleickmann, Kunter, and Hachfeld (2013) proposed evidence that these two dimensions are not the endpoints of a continuum. Instead of this, the authors proposed to understand these two dimensions as two distinct, negative correlated dimensions. These dimensions can be assessed on different scales. For this reason, it is possible that teachers have a high extent of both views on teaching and learning of mathematics.

Based on the definition of beliefs and the two main aspects of beliefs for our research, we primarily focus on the following research question:

Which influence show the PD course with reflecting on students' learning results on teachers' beliefs referring to teaching and learning of mathematics out of teachers' perspective?

Methods

We regard three groups of teachers in our study as a whole. Two groups of teachers were enrolled in a PD course that focused on problem solving and modelling in secondary school. An ongoing task for both PD courses was to develop tasks that meet different criteria of problem solving and modelling. The first group further was asked to give these tasks to their students and to collect the results of students' work that we call results of students' learning. These results of students' learning were the basis of the next face-to-face phase of the PD course. Teachers of second group were asked to improve problem solving tasks and a third group of teachers did not get any intervention.

We conducted pre- and posttests to measure the efficiency of the mentioned specific aspect of teachers' PD, i.e. reflecting students' learning results. Further we conducted interviews with the teachers. In this paper we regard three teachers of the first group who took part in the interview session voluntarily. The interviews took place about one month after the last meeting of the PD course. In semi-structured interviews the teachers were asked to report about their changes towards beliefs of teaching and learning of mathematics. All teachers are teachers of upper secondary schools and at the age of 40 to 50 and were women. These teachers could be representative for the whole group, because it consists of 21 teachers at the mentioned age. In addition, seventeen of these teachers were women.

We analyzed the interviews with a coding method including deductive and inductive codes (Mayring, 2015). The deductive codes were based on existing research referring to teaching and learning of mathematics considering the transmission-oriented and the constructivist-oriented beliefs. According to both types of beliefs, we created codes for teachers' answers. In this context, we distinguish between teachers' beliefs before and after the professional development course. The distinction is based on hints in teachers' statements which enables us to match beliefs to the appropriate point in time. In addition, we analyzed the role of "reflecting students' learning results" for teachers' learning during the PD course.

Results

The results section is structured into three parts. At first, we want to show how teachers' statements are coded according to beliefs about teaching and learning of mathematics. Second, we sum up beliefs of the three teachers. And third, the effects of "reflecting students' learning results" are considered.

In the interviews teachers were asked to report about their beliefs before and after taking part in the professional development course. In particular, they should consider changes in their statements. For example, Mrs. B states:

Mrs. B: […] it has changed that the tasks are different. Students should argue more and I do not have to work off stacks of tasks. I can work off all facets more determined and I do not have to say that I must work off a model and then practice, practice, practice ….

Mrs. B reported about changes in her beliefs about teaching and learning mathematics. We interpreted her statement as follows: At the end of her statement, she mentioned that she now does not have to introduce a mathematical model followed by many exercises in the lessons. Furthermore, she reported

on working off stacks of tasks before the PD course. Both parts of the statement are coded as transmission-oriented beliefs, because she emphasized practicing as repetition of information or procedures which is represented in stacks of tasks. In addition, the mentioned parts of the statement were coded as beliefs at the beginning of PD course, since she reported about changes during the PD (see table 1). In the first part of her statement, she considered a change in tasks and argumentation in her classroom. She also mentioned that she can work off the mathematical ideas more determined. These parts of the statement were coded as constructivist-oriented beliefs, because Mrs. B reported a student centered teaching style where students are asked to argue about mathematical concepts and to talk about mathematical problems with each other (see student-oriented perspective in teaching in table 2). In addition, these parts of the statement were coded as beliefs after taking part in the PD course, because in the whole statement Mrs. B reported about changes. In conclusion, the statement of Mrs. B shows a belief change from a transmission-oriented view to a constructivist-oriented view of teaching.

The three teachers showed similarities in reporting aspects of a transmission orientation and a constructivists orientation. A code that seemed to be crucial for all three teachers, but was not included in the quotation of Mrs. B, was expressed by Mrs. C:

Mrs. C: Students should do more and I have to restrain myself a little bit more.

We interpret the statement as following: Mrs. C also reported changes in her beliefs. The whole statement was coded as constructivist-oriented belief, because she wanted the students to be more active in her classroom when they are learning mathematics. In addition, Mrs. C reports on restraining herself in lessons. This is in line with the constructivist view, because on this perspective, teachers are creators of learning environments and students shall learn self-directed. For this reason, it is necessary that teachers shall restrain themselves in lessons. This statement was coded as beliefs after taking part in PD course, since “more” indicates that she had other beliefs at the beginning of PD. In particular, the beliefs at the beginning compared to those reported in the statement included that Mrs. C was more in the center of the lesson and students were more passive which are in line with transmission-oriented beliefs.

The results of the analysis of the three teachers’ beliefs showed that they were transmission-oriented before they took part in the PD course. However, they seem to change their beliefs towards a constructivist-orientation during the PD course. The teachers also reported existing constructivist-beliefs they had at the beginning of PD course, but they stated changes towards more constructivist beliefs while they reduced the strength of transmission-oriented beliefs. Table 1 shows the results of coding for beliefs before teachers take part in the PD course:

	Mrs. A	Mrs. B	Mrs. C
repetition of information (practicing)	X	X	X
frontal teaching	X	X	X
exact instruction	X	X	
teacher is in center of lesson	X	X	X

Table 1: Predominant beliefs before taking part in PD course (mentioned by teachers)

Teachers reported in the interviews that they changed their beliefs referring to teaching and learning from a teacher-centered perspective to a more constructivist-oriented teaching perspective. The following table shows statements of teachers referring to beliefs of teaching and learning of mathematics after taking part in the PD course.

	Mrs. A	Mrs. B	Mrs. C
active role of students	X	X	X
knowledge construction	X		
teacher withdrawing in lessons	X	X	X
students' discussions are important	X		X
students should analyze own mistakes	X		X
tasks with a meaningful context within real life			X
cooperative learning (group work, …)	X		X
student-oriented perspective in teaching		X	
pool of teaching methods		X	X

Table 2: Predominant beliefs after taking part in PD course (mentioned by teachers)

Both tables show different beliefs of teachers before and after the PD. These tables do not imply that all teachers had only teacher-centered beliefs at the beginning of the PD. They reported also that they had constructivist beliefs. For example, the statement of Mrs. C shows that she has more constructivist beliefs about teaching. This does not imply that she has not had constructivist-oriented beliefs at the beginning of the PD course. In her statement, Mrs. C only reported about more constructivist-oriented beliefs after taking part in PD. In addition, all the coded statements show that they also have transmission-oriented beliefs after PD course, but they report to think of teaching with more student-centered beliefs. Note, both tables only show predominant beliefs of teachers before and after PD. For this reason, peripheral beliefs about teaching and learning of mathematics are left out.

The results about beliefs show changes. For this reason, we want to know how reflection of students' learning results impact on teachers learning. In this context, Mrs. A and Mrs. C state:

Mrs. A: […] I have learnt a lot about my students and I have also learnt a lot about myself and for this reason I have tested some things.

Mrs. C: I think it was good. The students' learning results show how other teachers proceed in teaching, which approach they use and how they describe. Within this action, you could take new ideas that I found in students' solutions. […].

The statements of the three teachers show that "reflecting students learning results" were used to reflect the own practice of teaching. In this context, these teachers learnt about characteristics of their students and about themselves as teachers. As a consequence, Mrs. A tested some ideas contained in the PD course. In addition, the three teachers recognized other teaching styles when they reflected on students' learning results. For this reason, they also reflected about the practice of other teachers and their teaching approaches. In particular, they also reflected about their own practice compared to the

one of others to get new ideas (Mrs. C). These statements are typical for all three teachers. According to this, the reflection of teaching style could be understood as one factor that result in changes of beliefs about teaching and learning.

Discussion

The data from the interviews indicate that the three teachers changed their beliefs from a more transmission orientation to a more constructivist orientation of teaching and learning of mathematics. This is shown by statements of teachers after taking part in the PD course. The phrases in both tables consider the aspects of both beliefs about teaching and beliefs about learning. Based on the analysis in the results section, statements of table 1 refer to transmission-oriented beliefs and statements of table 2 to constructivist-oriented beliefs (e.g. Fives et al., 2015). In fact, teachers focused on constructivist-oriented beliefs, but they still expressed some transmission-oriented beliefs. This is in line with research findings of Voss et al. (2013), who supposed that both types of beliefs can co-exist. The statement of Mrs. C supports this assumption. Although she expressed transmission-oriented beliefs before the PD, she said that students should be more active, which indicates that she has had constructivist-oriented beliefs at the beginning of the PD.

Belief changes in teacher education are also shown in the study of Franke et al. (1998) for teachers on primary level who examined a student-centered framework in their PD. In this context, our results are similar to those of Franke et al. on secondary level, because we also used reflecting students' learning results to emphasize student-centered teaching. Furthermore, the teachers reported that they used students' solution to reflect their own teaching and to get to know information of other teachers' approaches and also about their students. This is in line with the results of Little et al. (2003), because teachers consider the ideas of their students in their deliberations more strongly after taking part in the PD course. In addition, belief change caused by reflection is reported by Decker et al. (2015) in the way that there is a relationship between the extent of reflection and teachers' belief change. For this reason, it is possible that all teachers of this study changed beliefs because they reflect intensively about their own practice. Skott (2015) stated that substantial new experiences are necessary to change beliefs. Concerning our research, we hypothesize that the intensive reflection of the own classroom practice includes the mentioned new experiences.

There are some limitations of this research. At first the interviews took place after the whole PD course. For this reason, teacher reports about the practice before the PD course can be influenced by the experiences of the PD. For a more precise research approach it would have been necessary to interview the teachers at the beginning of the PD. Considering the external circumstances in this research project, it was not possible to interview teachers, because they took part in PD voluntarily and the first meetings last about the whole day. In addition, these qualitative data provide no proof for "reflecting students learning results" as a feature of effective PD, but there are hints that beliefs can change and teacher reflect their own practice during this part of PD. Furthermore, the teachers of the second group were not interviewed. For this reason, it is not possible to indicate whether the teachers of the second group also changed their beliefs. In addition, we do not know whether second group teachers reflect on their own practice as deep as teachers of the first group.

Future research

This part of the whole research project considers qualitative data of the first PD course. Future research should use also quantitative data (measured by items of Staub & Stern (2002)) to support the results of teachers' belief change during PD. This could also show whether reflecting students learning results is one element that is connected with teachers' belief change empirically. In addition, it is necessary to link the qualitative data analyzed in this paper and the quantitative data which will we analyzed in the future. Combination of both resources can show the positive impact of reflecting students' learning results on teachers' knowledge, beliefs and motivation.

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Shifting frames - A teacher's change towards explorative instruction

Einat Heyd-Metzuyanim

Technion, Israel Institute of Technology, Israel; einat.metz@google.com

The case of one middle school teacher's change in practice is examined through the lens of "shifting frames" from ritual to explorative instruction. These frames are collection of coherent mathematical as well as subjectifying (people-related) meta-rules. The teacher, who participated in a year-long PD program, started out in a ritual frame and gradually shifted to a more explorative frame. The shift was not uniform, and could be seen first in the subjectifying meta-rules, and only later (and very partially) in mathematical meta-rules of exploring objects. In addition, newly learned practices were partially distorted through the old ritual frame.

Keywords: Mathematics instruction, professional development, frames, explorative practice.

Classrooms where students engage in explorative mathematical learning produce more robust, conceptual learning, and more positive mathematical identities (Schoenfeld, 2014). Despite these findings, teaching in the US and worldwide still provides mostly opportunities for ritual participation, where learning is made up of reciting procedures and facts (McCloskey, 2014; Resnick, 2015). Efforts at changing mathematical instruction, have shown ritual instruction is difficult to change (e.g. Santagata, Kersting, Givvin, & Stigler, 2011). However, the complex processes of change that do occur in teachers' practice, in the context of certain professional development (PD) programs, have largely remained obscure, perhaps because of the lack of theoretical frameworks for examining such complex processes. In this study, I offer the concept of "frames" for tracking subtle changes in one teacher's practice over a period of a school year.

Theoretical background

I take as a starting point Sfard's (2008) view, that the gist of mathematical activity, both in the classroom and over history, is the construction and exploration of "mathematical objects." These discursive objects, like numbers, triangles, and functions are not existing objects in the physical world, yet the "metaphor of object" is used by any skilled mathematician to talk about them as though they were indeed such physical objects. Mathematical objects can have many realizations, or physical symbolic representations. A function, for instance, can be represented as a graph, a table, an algebraic expression or a verbal statement. In the process of learning, according to Sfard, students come to see all these realizations as signifying one object. Once they do that, they start talking about it as an object existing of itself, a process termed "objectification."

Within this view, Sfard and Lavie (2005) theorize the process of mathematical learning as moving from a ritual, peripheral phase, where activity is first and foremost aimed at pleasing the experts of the discourse (e.g. the teacher), to an explorative phase where new mathematical narratives are produced by oneself for the sake of the activity itself. In contrast to explorative participation, the focus of ritual participation is activity itself, not the mathematical narrative produced by it. Ritual participation is often characterized by *syntactic mediation*, where instead of using mathematical signs as signifiers of objects, these signs are manipulated according to prescribed (often memorized) rules.

Ritual and explorative participation are governed by certain meta-rules or "patterns in the activity of the discursants" (Sfard, 2008, p. 201). These rules can be divided into mathematical meta-rules, which dictate how mathematical narratives are to be derived from each other (for example, by proof or by computation) and *subjectifying* meta-rules which govern the actions of people (e.g. asking questions, giving directions, talking with each other). To capture the fact that meta-rules have a certain structure and coherence, Heyd-Metzuyanim, Munter & Greeno (in review) suggested the term "frames", borrowed from socio-linguistic and socio-cognitive research to describe "a set of expectations an individual has about the situation in which she finds herself that affect what she notices and how she thinks to act" (Hammer, Elby, Scherr, & Redish, 2005, p. 9). Accordingly, they defined "frames" in mathematical classrooms as a set of meta-rules, both mathematical and social, which includes appropriate questions, answers, justifications and other discursive actions in a situation of solving a mathematical problem or performing a mathematical task. Whereas frames of explorations would be sets of meta-rules that cohere around the goal of producing mathematical narratives based on logical justifications, ritual frames would be more aligned with goals such as performing a procedure accurately according to a prescribed set of steps, and adhering to external authority.

Method

The case of Mr. M is taken from a larger study, where we followed 7 teachers and 5 teacher leaders throughout 8 months of professional development (PD) during 2014 – 2015. The PD was led by Margaret Smith and Victoria Bill from the University of Pittsburgh. It centered on Smith & Stein's (2011) "5 Practices for Orchestrating Productive Mathematics Discussions" and on Accountable Talk™ (see http://ifl.pitt.edu/index.php/educator_resources/accountable_talk). Teachers were supported via four full-day PD sessions and in-school individual coaching sessions.

Mr. M was chosen for closer inspection because he showed a steady growth in the "implementation" score of the Instructional Quality Assessment tool (Boston, 2012), used in the larger study (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016) to score lessons for cognitive demand. Mr. M's lesson 1 scored a 1, lesson 2: 2, lesson 3 & 4: 3 (where 4 is highest). Though his lessons never achieved the highest level of cognitive demand, in the last two lessons cognitive demand was partially preserved.

Data collection included four cycles (September, December, February and May), each containing: 1. Pre-lesson and post-lesson interviews with the teacher; 2. Lesson recording and students' worksheets. Frames of mathematical instruction were searched in four data categories:

(1) The potential of the task to engage students with different realizations of mathematical objects and with mathematical meta-rules of justification and generalizations. (2) Teacher's plans as collected from the pre-lesson interviews, as well as his post-lesson reflections. In these we looked for evidence of mathematical and subjectifying aspects of explorative frames. (3) Students' work: evidence of engagement with different realizations of objects and with connecting between them. Evidence for justifications and explanations based on mathematical logic. (4) Whole-classroom discussions: evidence for ritual vs. explorative meta-rules. Is there only talk about routines for manipulating symbols detached from their meaning, or are the symbols also related to mathematical objects?

Findings

During the first, introductory interview, Mr. M declared himself to be aligned with what may seem to be explorative instruction. Describing quality instruction, he said:

> … where groups get together in giving a situation in which they come up with strategies on their own, as opposed to a teacher standing up in front of the room and just saying "okay, this is how you do this, this is how you do this, now just take this problem and solve it". So … I keep trying to look for my own stuff that I'm doing, so I would look to see that in other peoples.

Notably, in remarking that such instruction was "my own stuff that I'm doing", Mr. M attested for practicing quality instruction. To this, he added:

> one of my philosophies (is that) there's never, ever, only one way to solve a problem, that there's many different strategies and many different pathways you can take to find different problems …

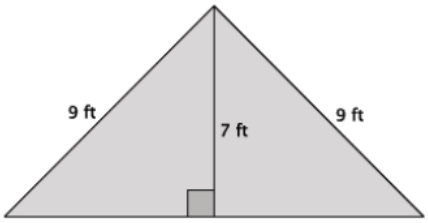
Mr. M also described his challenges in encouraging students to take on agency and not rely ritually on his guidance.

> I would say a majority of the students I've worked with over the last 7, 8 years pretty much when they were on a challenging problem, their first shot is to ask me to give them help and guidance along the way. I used to do that more often, …, so I tried to step back away from that for a moment, getting to know the kids a little bit better helps me understand those that are just being lazy and those that truly don't know it.

Thus, in his introductory interviews, Mr. M seemed to be well aligned with "reform" mathematical pedagogy, especially with regard to letting students struggle and encouraging them to look for "different strategies" for solving problems.

Lesson 1

Despite these declarations, the first lesson revealed Mr. M's actual practices were not well aligned with the explorative frame. For the main task of the lesson, Mr. M chose the task seen in Figure 1:



The triangle below is an isosceles triangle. Use what you know about the Pythagorean Theorem to determine the approximate length of the isosceles triangle (EngageNY, Module 8.7)

Figure 1 - Pythagorean Task given in lesson no. 1

The task was part of a module that introduced irrational numbers through the Pythagorean Theorem. The EngageNY[1] teacher guide encourages explicitly to present the task only after students had become familiar with this Theorem. The task was taken verbatim from the module, which assumed this previous experience, thus including the "use what you know" phrase. However, Mr. M. was well aware of the fact that his students were probably not familiar with the Pythagorean Theorem. In the pre-conference of the lesson he said:

[1] See https://www.engageny.org/resource/grade-8-mathematics-module-7

> The first challenge that I'm expecting is that we've not introduced Pythagorean Theorem at all last year. …. Um, I'd like to hear where their conversation goes and what they're focusing on –… when they're trying to find an opposite base, … given the $a^2+b^2=c^2$ idea, do they put the numbers in the right place – do they remember the square numbers, do they remember to take the square roots of numbers to get themselves back to what the – the single variable would be? … So I'm looking to see – what kind of challenges working with square roots and squares creates.

Though Mr. M was anticipating "challenges" in working with the problem (aligned with the PD encouraging letting students struggle), the way he talked about students coping with these challenges was through "remembering". Thus, meta-rules belonging to the explorative frame (student agency) were distorted through the ritual frame into meta-rules of fact-retrieval, and memorizing procedures.

During the lesson, the launch of the task proceeded as follows:

90. T Okay, so when we take a look at this (*pointing to a right triangle on the board, the sides of which are labeled 'a', 'b,' and 'c'*) I'm going to give you a little bit of a formula. And it may help you today in doing some of your work (*writes* $a^2+b^2=c^2$ *on the board*) ... Who can read this out loud for me, please?

91.-95. *Three students are prompted to read the formula "in different ways". One reads "a two plus b two equals c two". The second "a to the second power, b to the second power, and b to the second power", the third "a squared plus b squared equals c squared"*

96. T You see, all three of those ways that they said say the exact same thing. … So when we're looking at this, can you now start to see that knowing two of the three sides will allow me to figure out the third side? … I'm gonna have you work in groups today to see if you can't use …any knowledge that you have plus anything that you've seen today. Alright?

The subjectifying meta-rules enacted in this excerpt bore resemblance to explorative frames. Students were asked to provide "different ways" of saying the "exact same thing", and were directed to "figure out" and "work in groups" to "use any information" they have to solve the problem. Yet the mathematical meta-rules, that is, the ways to derive one mathematical narrative from another, were obscure at best. $a^2+b^2=c^2$ was written on the board, detached from the right triangle, with no indication of a^2, b^2, and c^2 signifying the geometrical squares adjacent to the triangles sides. By that, Mr. M was treating the $a^2+b^2=c^2$ Pythagorean Theorem syntactically, as a series of signifiers, detached from their meaning as signifying geometrical objects.

Not surprisingly, since students had no access to the meaning of the Pythagorean Theorem mediated by geometrical objects, they struggled over where to place the a, b, and c labels given by the formula, on the newly presented triangle. After some leading questions from Mr. M. students concluded they should label the base of the triangle as 'C' or 'C^2'. They subtracted $9^2-7^2=32$, but then were unsure as to what to do with that result. Some students thought it should simply be divided between the two halves of the base, labeling the whole base as '32' and halves of it with '16'. Others simply labeled the whole base as '16' or went another step and labeled the halves with '8'. Only 3 or 4 students, out of 24, figured out that the 32 should be square-rooted and multiplied by 2. However, even they did not label the triangle's base with the resulting number 11.4 (or estimated 12). The mathematical meta-rules governing the classroom activity were thus primarily ritual. They could be summarized as "apply a set of symbols, somehow related to a right-triangle, to a new triangle".

It was not that Mr. M was intending students to apply the 'a','b', and 'c' symbols randomly. He did have a certain type of reasoning he was looking for. After having asked a student to come to the board and present her solution, and while the student was labeling the triangle sides, he said:

289. T Yeah, please notice - … if we looked at that formula - A squared plus B squared equals C squared, and then I asked this question of some of you: Does it make a difference where C is? *…(Student answers C has to be bigger than A and B)*

291. T So if it's bigger than all the other ones, then the question becomes where does the C have to go - which side?

292. Student On the longest side.

293. T The longest side. Now the one misconception I saw that I didn't expect was that some of you thought that this (the base) was the longest side, the whole way across. Just be careful about that, you're focusing on just one of those two triangles, okay?

The meta-rules of the activity, as gleaned from Mr. M's words were thus "figure out where the C is according to it being bigger than a + b. Then figure out the longest side and label it as C". However, this was detached from the physical meaning of the Pythagorean Theorem. In relation to the confusion or "misconception" the students had, regarding where to place the C, Mr. M did not have much advice besides "be careful about that" [293]. There was no other indication why they should be focusing on one of the right triangles, and not the whole triangle.

In the post conference, Mr. M seemed reasonably content with the results of the activity. He explained that since students were unfamiliar with the Pythagorean Theorem, he expected them to struggle, but that that wasn't the focus of the lesson. Rather the focus was discussing imperfect squares, to which they got at the end of the discussion, when one student said 32 should be rooted and Mr. M led students to estimate the root between 5 and 6. Yet having detached the geometric, physical realization of the Pythagorean Theorem from its algebraic formula, the rationale behind the existence of imperfect squares had no way of being foregrounded. The fact that Mr. M was not concerned with students struggling considerably with something that was not "the focus of the lesson", indicates that through his frame, the whole process of working in groups and discussing solutions was not a central measure for achieving the mathematical goal of the lesson. Rather, it was a sort of a "side effect", performed for the sake of the lesson recording or for students to practice working in groups.

I now move to a similar examination of the last lesson. A description of the two middle lessons is out of the scope of this paper. However, evidence for movement from a ritual frame to a more explorative frame were starting to show during the 3rd lesson, where Mr M received, together with the rest of the teachers of the PD, a high-level task selected by the PD leaders and directed explicitly on the different solution paths that should be sought in the lesson. Lesson no. 4 seems to have reaped the benefits of this process, though as we shall see, the shift between frames was still very fragile.

Lesson 4

The first indicator that Mr. M's practice was starting to align with an explorative frame could be seen in his choice of a task. This task involved modeling of real-life processes with a quadratic equation. Following is the task as presented on the worksheet (taken from EngageNY, Algebra 1):

> The baseball team pitcher was asked to participate in a demonstration for his math class. He took a baseball to the edge of the roof of the school building and threw it up into the air at a slight angle

> so that the ball eventually fell all the way to the ground. The class determined that the motion of the ball from the time it was thrown could be modeled closely by the function: $h(t) = -16t^2 + 64t + 80$, where $h(t)$ represents the height of the ball in feet after t seconds.

Students were then asked to find the behavior of the function (maximum, minimum, vertex), to graph it and to indicate how many minutes passed until the ball fell to the ground and what was the meaning of h(0). The task thus had ample potential for using different realizations for a mathematical object – namely the quadratic formula.

The next indication for Mr. M's movement between frames was his talk about expectations for students' work during the pre-lesson interview. This is how he presented the choice for the task:

> I've been working on questions … like "how do you know that that's where your graph crosses the X axis?" So more than just– what is that value, how do you know what it is. How did you figure it out. Um, why does your graph continue past zero? In the context of the problem, what does that mean? … And try to gain some understanding of their understanding.

Instead of "remembering", Mr. M now talked about students' "understanding". He was also working hard on preparing questions that would assess students' understanding, a skill specifically taught in the PD sessions.

Examination of students' worksheets showed that most students performed the calculations involved in the problem correctly. 4 out of the 27 students wrote that h(0) indicated the height in which the ball was thrown was 80 feet, and 3 more related h(0) to the initial stage, before the ball was thrown in the air. Other students, however, either left this question blank, or stated that h(0) means "the ball is on the ground" indicating incongruence between the physical situation and the graphical realization. Thus, in contrast to the first lesson, there was evidence that a small part of the classroom was able to flexibly move between different realizations of the quadratic function.

During the whole classroom discussion, students' explanation on the board were mostly concentrated on the calculations involved in the problem. Mr. M encouraged this by calling on students to present "different ways to solve the problem" which referred to factoring by dividing the expression by 16 or by 8. He also referred to finding the value of h(0) by substituting t=0 or by just "looking" at the last term in the equation, as two different ways to find h(0). Yet these two "ways" were, in fact, both tending to the algebraic realization and could be considered as "different" only when looking at the function's expression syntactically. There was no mentioning in the whole classroom discussion of the graphical realization of the function or the physical "real life" story modeled by it. It was thus still mostly characterized by meta-rules of carrying out prescribed procedures using syntactic mediation, rather than meta-rules of exploring mathematical objects. However, some slight changes could be observed. One of the students who talked in the whole classroom discussion did, even if very briefly, mention the physical "real life" situation that the function was modelling. He talked about the ball ("the ball hasn't been thrown yet") and hinted at the height of the roof in "throw the ball at 80". Also, there was brief mentioning of the physical realization when Mr. M elicited from one student that the maximum was at "2 minutes".

In terms of social meta-rules, there were many more prompts for students to "restate" what other students have done. Mr. M made every effort to use this talk move, taught in the PD, albeit somewhat inflexibly, whenever a student made a mathematical statement he deemed as important.

Discussion

In this paper, I have used the concept of "framing" to examine subtle changes in one teacher's practice over a period of one school year, through which the teacher was receiving support both through PD sessions and through in-school coaching. These changes, including the different aspects of frames, are summarized in table 1.

Table 1 - Social and Mathematical aspects of frames in two lessons

	Social (subjectifying) meta-rules	Mathematical meta-rules
Lesson 1	**Beginnings of explorative social meta-rules** through teacher's declarations (students expected to persevere, work together). In practice, most of the talking is done by the teacher.	**Ritual:** Meta-rules only have to do with recalling facts and procedures from memory. Mathematical signifiers are detached from the object they are representing. Students' work indicates syntactic mediation with no connection to geometrical realizations.
Lesson 4	**Explorative social meta-rules are more dominant.** Students are asked to restate each other's ideas. Two students explain their thinking to the whole class. Evidence that some student listen to each other.	**Beginnings of explorative mathematical meta-rules:** Teacher seeks students "understanding", not just rule following. A small portion of students' work indicates connections between multiple realizations of the quadratic function. Still ritual meta-rules in teacher's discourse still dominate instruction.

As can be seen from Table 1, the shift from ritual to explorative frames in Mr. M's case was not uniform. The shift occurred first in the social meta-rules and only later in mathematical meta-rules. Also, both shifts seemed to occur first at the level of declarations and only later in practice. Thus, talk about social meta-rules aligned with explorative participation was evident already in the first interview, but enacted mostly in the last lesson. Talk about expectations for explorative mathematical meta-rules was evident in the last lesson pre-conference interviews, and only very slightly evident in the enactment of the lesson. This finding corroborates earlier findings showing that teachers are quicker to adopt declarations about explorative instruction than they are to enact it (Cohen, 2001) as well as our previous findings regarding social meta-rules of teachers trying to enact "dialogic" or "reform" instruction being more aligned with explorative instruction than mathematical meta-rules (Heyd-Metzuyanim et al., in review). Most importantly, the findings reveal that teachers' learning of new practices is not a matter of acquiring new practices on a "tabula rasa" of non-existing former practices. Rather, at first, new practices are seen through the old frame. As such, they gain unpredictable "twists", such as perceiving different syntactic procedures as "different solutions" sought after, or encouraging students to restate unimportant mathematical ideas.

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Situation-specific diagnostic competence of mathematics teachers – a qualitative supplementary study of the TEDS-follow-up project

Jessica Hoth[1],

[1]University of Vechta, Germany; jessica.hoth@uni-vechta.de

Abstract: One main facet of teachers' professional competences is diagnostic competence. While diagnostic competence of teachers becomes relevant in several situations within teaching and learning, this paper focuses on situation-specific diagnostic competence required by teachers within class. In a qualitative supplementary study of the TEDS-Follow-Up project, this situation-specific diagnostic competence is analysed using the video instrument of the TEDS-Follow-Up study. 131 primary level mathematics teachers participated in the primary study of this project and examined specific learning situations in the video instrument. These instances were analysed using qualitative text analysis (Mayring 2015). Results indicate that teachers notice very different aspects in those teaching situations. Two different diagnostic types can be differentiated: content-related and judging diagnostic type versus student-related and action-oriented diagnostic type.

Keywords: Professional competences of mathematics teachers, diagnostic competence, mathematics teachers' knowledge, situation-specific skills of mathematics teachers, video study.

Introduction

Teachers are faced with many challenges in the context of teaching and learning. In order to plan and to conduct teaching sequences that enable students to achieve their best possible learning results, teachers need several characteristics such as professional knowledge and situation-specific skills. According to Weinert et al. (1990, p.172), diagnostic knowledge is one of the "four areas of knowledge […] as constituting the cognitive components of teacher expertise" in addition to classroom management subject matter knowledge and instructional competence. With this perspective, this article focuses on the diagnostic competence of primary school mathematics teachers, more precisely on their diagnostic competence that becomes relevant during class – the so-called situation-specific diagnostic competence.

Theoretical background

Theoretical background of the study is the discussion on mathematics teachers' professional competences, which will be described specifically focusing on their diagnostic competence.

Professional competences of teachers

According to Weinert (2001), competence involves two main facets: a cognitive facet as well as an affect-motivational facet. The cognitive facet is often differentiated following Shulman (1986, 1987) into (Mathematics) Content Knowledge (MCK), (Mathematics) Pedagogical Content Knowledge (MPCK) and General Pedagogical Knowledge (GPK). Several empirical studies that deal with teachers' professional competences use this conceptualization of competence (see for example Mathematics Teaching in the 21st Century (MT21); Blömeke et al. 2008, Teacher Education and Development Study in Mathematics (TEDS-M); Blömeke et al. 2014 and Cognitive Activation in the classroom (COACTIV) Kunter et al. 2011). In addition to teachers' knowledge, these studies assess affective-motivational aspects such as epistemological beliefs, motivational aspects and those about the teaching profession (cf. Blömeke et al. 2008, Blömeke et al. 2014, Baumert and Kunter 2011). Thus, cognition and affect-motivation are hypothesized to build the basis of competent performance in classroom situations.

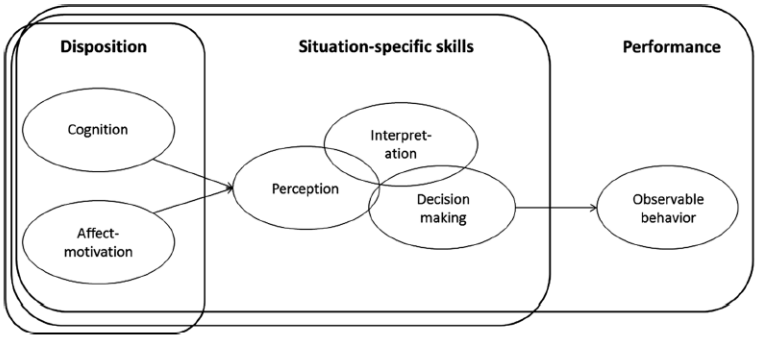

Figure 1: Modeling competence as a continuum (Blömeke et al. 2015, p. 7)

More situated approaches to assess teachers' professional competences are based on models of competence that also include more situated facets such as teachers' perception, interpretation and decision-making that become especially relevant during classroom interaction. According to Blömeke et al. (2015), competence is a continuum from personal traits such as cognition and affect-motivation that underlie and affect situation-specific skills which again determine the actual performance or behaviour in specific situations. Here, the perception, interpretation and decision-making in concrete situations "mediate between disposition and performance" (ibid., p. 7). A content-based specification of this broader concept of competence is the diagnostic competence (Abs 2007).

Diagnostic competence

Diagnosis and diagnostic tests are usually associated with medicine. Doctors need to diagnose illnesses on the basis of symptoms. However, teachers need diagnosis in their profession as well. They may use diagnostic (clinical or standardized) tests to detect learning disabilities (Ketterlin-Geller and Yovanoff 2009) but they also need to diagnose students' achievements and learning processes during class without using standardized educational or psychological tests. Both described instances require teachers' diagnostic competence. However, we describe the process of diagnosing in the course of teaching and learning as teachers' situation-specific diagnostic competence as opposed to the facet of diagnostic competence that becomes relevant in adequately choosing, using and evaluating diagnostic tests (see Hoth et al. 2016).

For this situation-specific diagnostic competence, teachers' situation-specific skills become relevant as proposed by Blömeke et al. (2015) in their model of competence (see figure 1). On the basis of this theoretical background, the study presented in this paper focuses on the following research questions:

1. How do the perception, interpretation and decision-making of primary mathematics teachers differ?
2. Can different diagnostic types be reconstructed?
3. How do these diagnostic types relate to their professional knowledge regarding the three knowledge facets mathematics content knowledge (MCK), mathematics pedagogical content knowledge (MPCK), general pedagogical content knowledge (GPK)?

To address those research questions, the methodological approach that was used to analyze the data, will be described.

Methodological approach

The study presented in this paper is a qualitative supplementary study of the TEDS-Follow-Up project (Follow-Up to the international Teacher Education and Development Study in Mathematics, TEDS-M). For the specific purpose of analysing teachers' situation-specific diagnostic competence, the focus of the supplementary study lay on the video instrument of the TEDS-FU study. TEDS-FU, its conceptualisation and design will be described in the following, prior to the outline of the methodological approach that was realised in this specific supplementary study.

The TEDS-FU study

The TEDS-FU study is the German Follow-Up-study of the international comparative study about mathematics teacher education TEDS-M. A subsample of primary and secondary school mathematics teachers who participated in TEDS-M was reassessed after four years of work experience. A total of 300 mathematics teachers participated in the primary school study, including 131 primary school teachers who are at the focus of this paper. Therefore, the TEDS-FU study analyses the teachers' development in their first years of work experience.

The study is based on an understanding of competence as a continuum (Blömeke et al. 2015, figure 1) and closely refers to research in the field of teachers' expertise (cf. Li & Kaiser 2011) and the concept of 'Teacher Noticing' (cf. Sherin, et al. 2011). In order to assess more situated facets of teachers' professional competences, three situated facets are distinguished in TEDS-FU in addition to knowledge-based facets of teachers' professional competencies (MCK, MPCK, GPK):

"(a) *Perceiving* particular events in an instructional setting, (b) *Interpreting* the perceived activities in the classroom (c) *Decision making*, either as anticipating a response to students' activities or as proposing alternative instructional strategies" (Kaiser et al. 2015, p. 373).

Therefore, different test instruments are used in the study. An online-survey assessed different contextual components such as beliefs, working conditions and school characteristics, a newly developed video analysis instrument assessed teachers' situation-specific skills and a shortened version of the TEDS-M proficiency test was used to assess teachers' MCK, MPCK and GPK. In addition, a time-limited test was included where teachers had to identify typical student errors (see Pankow et al. 2016).

With the aim to analyse teachers' situation-specific diagnostic competence, all tasks were focused from the TEDS-FU primary school study that required situation-based diagnostic competence. This was ensured by the video analysis test instrument as well as in some verbally described situations of the reduced proficiency test. The video test consisted of three short video clips of a primary school

mathematics classroom and corresponding questions concerning didactical and pedagogical aspects of the teaching sequence. A total of 19 questions were selected for the analyses, 14 questions from the video analysis test and five questions concerning verbally described situations in the MPCK proficiency test. Teachers' answers to the selected questions were analysed using qualitative text analysis (Mayring 2015; Kuckartz 2014). All answers of all teachers to all 19 selected questions were analysed using reducing and structural procedures (Mayring 2015).

To exemplify the coding process, one example item will be introduced as well as teachers' responses to the item and the result of their analysis. The example item refers to the video analysis instrument, more specific, it refers to the video 'real world problem' that shows a third grade mathematics classroom in Germany dealing with a real world mathematics problem that is shown in figure 2. In the video, three students' working groups are shown who discuss their working results. The students use very different approaches, one student produces a symbolic result while another student solves the task using a visual drawing of the situation. Referring to this scene, the teachers were asked to characterise the two solution approaches contrastingly. Teachers' responses to this task were coded using reducing processes of qualitative context analysis, capturing the content of the teachers' answers. In this regard, the following teacher's answer was categorised as 'contrasting the students' form of representation':

Figure 2: Real world mathematics problem that is discussed in the video vignette

Anna and Felix want to go to the movies. They take the bus together. Felix pays the bus tickets for both of them. One ticket costs 2€. Anna pays both cinema tickets. One ticket costs 7€. At the end of the day, the children want to share the costs equitably. How can they do that?

> **Teacher 1:** *"Lea does mental arithmetic and uses a symbolic approach. She does not have to use an iconic or enactive solution while Kim needs an iconic approach. She should try an enactive approach to see that Lea's approach is correct as well."*

This coding approach categorized each of the teachers' answers with regard to their content. In another coding process using structuring procedures, the teachers' answers were coded with regard to judgments. A teacher's response that contained a judgment of the two students' solutions is the following:

> **Teacher 2:** *"Lea's approach is more practical and more advanced than Kim's approach because she already subtracted the amount that the girl has to give to the boy.*

Finally, in another structuring coding procedure, all teachers' responses to the selected questions were coded with regard to proposed alternatives or continuations. A teacher's response that proposes a possible continuation of the presented situation is the following:

> **Teacher 3:** *"Lea does mental arithmetic and uses a symbolic approach. She does not have to use a visual or acting solution while Kim needs a visual approach. She should try an acting approach to see that Lea's approach is correct as well."*

These coding processes resulted in a category system that built the basis for type-building text analysis (Kuckartz 2014). Here, three dimensions emerged which further constructed a feature space to generate diagnostic types: the perspective that teachers' chose on the teaching sequences, their tendency to judge and their tendency to propose alternatives and continuations. In order to reconstruct ideal diagnostic types, connections between these three dimensions were analysed and idealised. Finally, these diagnostic types were interrelated using contingency analyses in a Mixed-Method-Design (Kelle & Buchholtz 2015) with the teachers' knowledge scores that resulted from the reduced proficiency test of the TEDS-FU study. The teachers' mathematics content knowledge, their mathematics pedagogical content knowledge and their general pedagogical knowledge is given in scale scores resulting from the TEDS-FU proficiency test. Contingency analyses between the teachers' knowledge facets and the dimensions presented above give insight into connections between teachers' situation-specific skills and their professional knowledge. Table 1 shows this contingency analysis between the perspective chosen and the professional knowledge for the example item. However, since the perspectives were coded in every selected question, these connections were also analyzed independent of specific teaching sequences.

	Didactical perspective	Didactical AND mathematical perspective	Mathematical perspective
Average MCK	519	567	597
Average MPCK	536	552	543
Average GPK	647	640	683

Table 1: Contingency analysis between teachers' professional knowledge and their perspective

Results

Resulting from the reducing and structuring procedures, teachers' responses differed with regard to several aspects. On the one hand, teachers chose different perspectives on the classroom incidents. While some teacher focused on the content, other teachers focused on the students, their understanding, motivation, behaviour etc. On the other hand, it became obvious that teachers had varying tendencies (a) to judge the classroom events which they observed and analyzed and (b) to anticipate teaching alternatives or continuations.

Relating the three dimensions that resulted from the coding processes–(1) the perspective chosen (2) the tendency to judge (3) the tendency to anticipate teaching alternatives or continuations and–showed that teachers who often chose a content-related mathematical perspective in classroom situations also often judged these incidents. In addition, the more often teachers chose a student-related perspective, the more teaching alternatives and continuations were anticipated. These connections between the three dimensions formed the basis for building diagnostic types. In this regard, the following two ideal diagnostic types could be identified:

> **"Content-related and judging:** This diagnostic type is characterized by a content-related perspective in the phases of perceiving and interpreting relevant incidents. These noticed criteria are subsequently used to judge the relevant incidents. The phase of decision-making is also characterized by a content-related focus. Here, the teaching continuation is conducted by the subject's (here mathematical) content.

> **Student-related and action-oriented:** This diagnostic type is characterized by a student-related perspective in the phases of perceiving and interpreting relevant incidents. This means that the students, their learning processes, understanding, motivation and behaviour are the noticing focus. If classroom situations show deficits with regard to students' understanding and learning, this phase is automatically followed by a phase of deciding on alternatives that improve the given situation or possibilities to optimally continue the situations. The phase of decision making is primarily characterized by considering teaching methods and the instructional organization." (Hoth et al. 2016, p.50)

Connecting to these results concerning different diagnostic types, a Mixed-Methods Design was realised that interrelated the different perspectives chosen with the teachers' knowledge that was assessed by the proficiency test in the TEDS-FU study. In this regard, connecting the perspectives chosen by the teachers to their professional knowledge showed the following results:

- Teachers who often choose a mathematical perspective on teaching situations have average or above average mathematics content knowledge while their mathematics pedagogical content knowledge is below average.
- The more often teachers choose a didactical perspective on teaching situations, the higher is their mathematics pedagogical content knowledge.
- The general pedagogical knowledge of teachers who often choose a pedagogical perspective exceeds their content specific knowledge.
- Teachers who often judge teaching instances have high mathematics content knowledge.
- Teachers who often propose teaching alternatives and continuations possess high mathematics and mathematics pedagogical content knowledge while teachers who seldom do this have below average general pedagogical knowledge.

With regard to specific teaching instances, the results indicate that teachers with comparatively high content-related knowledge (MCK and MPCK) plan their teaching with regard to the content while teachers with comparatively high general pedagogical knowledge focus to a greater extent on pedagogical facets while planning their teaching. Furthermore, other connections indicate that teachers with only little content knowledge more often miss aspects in teaching and learning that are relevant for the students' learning processes but focus on behavioural aspects if they are very striking. Teachers who focus on aspects of understanding and learning despite of the striking student behaviour have above-average mathematics content knowledge.

Summary and discussion

In this paper, mathematics teachers' situation-specific diagnostic competence is analysed. This is the diagnostic competence that teachers require during class. For this purpose, the answers of 131 mathematics teachers are analysed who took part in the TEDS-Follow-Up study. In this video-based study, the teachers are asked to answer questions referring to video scenes of mathematics classroom. Analyses showed that teachers focus on very different aspects in the same teaching scene and two diagnostic types were differentiated: the content-related and judging type on the one hand and the student-related and action-oriented type on the other hand. In addition, contingency analyses showed that there are connections between the teachers' professional knowledge and their focus on and analysis of teaching sequences.

The results enrich the already existing findings in the field of teacher noticing (cf. Sherin et al. 2011). The proposed connection between teachers' noticing and their knowledge is verified empirically. As a consequence, teachers' practice is essentially influenced by their professional knowledge which in turn emphasizes the importance of mathematics teacher education. However, following questions arise about further connections between teachers' identification, their beliefs and the perspectives that were distinguished in this paper. In addition, developing and implementing teacher education courses to foster teachers' situation-specific diagnostic competence may give further insight into the development of this specific facet of teachers' diagnostic competence.

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Pre-service mathematics teachers: How to make them ready to be ready

Veronika Hubeňáková[1], Ingrid Semanišinová[2] and Dušan Šveda[3]

[1,2,3]Pavol Jozef Safarik University, Faculty of Science, Košice, Slovakia;
[1]veronika.hubenakova@upjs.sk, [2] ingrid.semanisinova@upjs.sk, [3]dusan.sveda@upjs.sk

The changes concerning the final state examination, determined by the novelty of Slovak university law, stimulated us to make serious changes in the examination model of Didactics of Mathematics at our university. Moreover, we have observed significant gaps in pedagogical content knowledge (PCK) of our secondary pre-service teachers during the last years. These two stimuli led us to an improvement of the course of Didactics of Mathematics. In this paper, we present our new approach which was mainly focused on the assessment of the lesson plans presented by the pre-service teachers. The assessment was based on the rubrics which were developed through the course. After the application of the new approach, we have observed growth in PCK. We have partially confirmed reliability of the rubrics as well.

Keywords: Secondary pre-service teacher, pedagogical content knowledge, lesson plan, rubrics for PCK assessment.

Introduction

Before we introduce our new approach to secondary pre-service mathematics teacher PCK development, we bring a short description of the system of pre-service teachers' preparation at our university. First, all the secondary pre-service teachers (PSTs) study two disciplines in various combinations, including combinations of natural and humanistic disciplines. Second, pedagogical, psychological and didactics studies are reduced to the minimum for the bachelor level. Therefore, we speak about the joint degree study of the particular two subjects (e.g. mathematics and physics) at the bachelor level. Most of the pedagogy is concentrated at the master level of university studies, where we talk about the teacher preparation program. Third, complex state exam from both disciplines and their didactics, Pedagogy and Psychology and the thesis defense are required for successful finish of the study. The aim of our university was to make state exams more efficient and foster mutual relationships of all parts of state exams. Moreover, we had found out serious gaps in the PCK of our PSTs, so the bigger pressure for its growth was necessary. Implementation of the PCK assessment into the state examination seemed to be a way, because "what you test is what you get," is the broadly accepted quote (attributed to Lauren Resnick). As we realized, when the assessment is changed, the instructions should be changed as well. Therefore, we have prepared a new model of state exams and apply inevitable changes at the classes of Didactics of Mathematics. To be sure that the new approach has the positive impact on the PSTs' learning, we posed following research questions: **(1)** Are the rubrics reliable tool to assess PSTs? **(2)** Was the pre-service teachers' PCK developed effectively? Which of its components were developed?

The novelty of the approach can be understood geographically - there are no serious studies on PCK development in Slovakia or even in neighbor countries (Depaepe, 2013). Moreover, we have not found any study which would consider utilization of general – not thematically specific – rubrics for

development of PCK of secondary PSTs. The new approach is described and discussed below; however, some theoretical underpinnings are necessary to be stated first.

Theoretical background

Knowledge areas, in which should a good teacher of mathematics systematically grow and achieve a certain standard, are named for several decades. Shulman (1986) explained pedagogical content knowledge (PCK) which is beyond subject matter knowledge and

„includes an understanding of what makes the learning of specific topics easy or difficult. Teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank slates" (p. 9).

Furthermore, Shulman (1986) emphasized the importance of curricular knowledge, familiarity with the instructional materials and tools available for teaching distinct concepts at different levels. PCK in the domain of mathematics comprises mathematical knowledge a different kind to that used in everyday life and in other professions which need mathematics (Ball & Bass, 2000).

Hill et al. (2008) built on Shulman's (1986) definitions of different types of knowledge and proposed the model with three types of content knowledge and three types of PCK:

Subject matter (content) knowledge
(1) Common Content Knowledge (CCK) – knowledge which is used in the classroom in ways common with how it is used in other professions that also use mathematics. **(2) Specialized Content Knowledge (SCK)** – knowledge how to accurately represent mathematical ideas, provide mathematical explanations for common rules, procedures, understand unusual methods of solution. **(3) Knowledge at the mathematical horizon** – awareness of the large mathematical landscape in which the present experience and instruction is situated (Zazkis & Mamolo, 2011).

Pedagogical content knowledge
(1) Knowledge of Content and Students (KCS) – knowledge how students in general learn a concept, what mistakes and misconceptions are common, it involves understanding of students' thinking and what makes the learning of particular concepts easy or difficult. **(2) Knowledge of Content and Teaching (KCT)** – knowledge about how to develop students' thinking and how to deal with student errors effectively, ''knowledge of teaching moves'' (p. 378). **(3) Knowledge of Curriculum (KC)** – knowledge about the content of curriculum and knowledge how to utilize the content of curriculum to present.

In our approach we suggest an authentic assessment in order to make bigger pressure for the growth of each component of PCK and to move forward PST beliefs about mathematics. There are four features of authentic assessment (Darling-Hammond, Ancess & Falk, 1995): (1) "they are designed to be truly representative of performance in the field" (p. 11); (2) "the criteria used in the assessment seek to evaluate "essentials" of performance against well-articulated performance standards." (p.12); (3) "self-assessment plays an important role in authentic tasks" (p.12), (4) "the students are often expected to present their work publicly and orally." (p.12).

The course design

Ten PSTs attended mandatory class in Didactics of Mathematics led by the second author and assisted by the first author. Total time of every week meeting was two hours and 15 minutes, the course lasted for 10 weeks, 5 of them were designed as follows:

(1) All PSTs picked randomly one of the topics which were prescribed by the second author. They were asked to create a lesson plan due to a particular date. The lesson plan had to be focused on the mastering a new curriculum. The template they were asked to fill as the front page of the preparation contained following items: topic, grade, goals, necessary entry knowledge, didactics problem and misconceptions, tools, methods and forms. Only the front page was fixed, the rest of the preparation was in the hands of the PSTs. **(2)** The PST who was up to present his/her preparation the next week, sent his/her lesson plan in one week advance to the whole group. Everybody was expected to raise some questions. **(3)** The PST had 30 minutes for the presentation of the lesson plan. Next 20 minutes were reserved for discussion. The PST explained the main line of the new curriculum mastering, underlined the important connections between the information from the front page and the tasks and subsequently, in the discussion, clarified inconsistencies pointed out by the teachers or the PSTs. **(4)** Everybody who was actually present at the class was asked to assess the presentation and provide the feedback and score for the PST who presented the preparation. To minimize subjectivism of the assessment, following rubrics (Table 1) were created and used for the scoring. The PSTs were instructed how to use it, examples for the particular levels were supplied.

Rubrics for lesson plan assessment

The base for the initial rubric version was the observations which were conducted by the second author in the previous academic year. The PSTs were asked to fulfil the similar assignment. Nevertheless, the scoring was not conceptualized; everybody was expected to assess presenting PST an appropriate number of points. Such assessment did not produce pressure for specific part of PCK growth. That was the reason why we, for the next academic year, developed five rubrics that we believe to enhance PCK of PSTs during the preparation of the lesson.

(1) Rubric **Learning Objectives** is defined as the ability to formulate and clarify essential objectives of the unit and to link them with the rest of the preparation. Learning objectives result from curriculum. The rubric is connected with KC. We assess whether PST applies this curriculum content to appropriate learning activities for students. **(2)** Rubric **Motivation** shows PSTs' potential to present mathematical ideas in an attractive way, to provide reasons and mathematical explanations for the topic. It is linked with SCK. **(3)** Third rubric – **Correctness** - pertains to mathematical correctness of the lesson plan. It is associated with CCK and SCK. **(4)** Rubric **Didactic Means (Tools)** includes chosen didactic method, tasks and materials. Using this rubric we assess PST knowledge about how to build on students' thinking and how to address students' errors effectively. The aim is to identify the level of KCT. **(5)** Last Rubric we called **Didactic Problems** and it contains assessment of the level of PST knowledge of how students think about the topic, how students typically learn a concept from the topic, what mistakes and misconceptions are common. Rubric is designed to determine the level of KCS.

Level	Learning objectives	Motivation	Correctness	Didactic Means	Didactic problem and misconceptions
0	A PST cannot explain the objectives of the unit, or the explanation is only formal (there are only weak connection between the objectives and the preparation and/or the objectives are not appropriate for the students age group).	There is no explicit motivation within the lesson plan, or the stated motivation is very formal (neither students' activity nor questions cannot be expected).	The unit curriculum is introduced incorrectly (mathematics terms definitions, mathematics statements or tasks assignments are not formulated correctly or comprehensively or they are not appropriate for the students age group,) and/or the learning trajectory is not respected at all.	Chosen didactic means clearly support instructive approach to mathematics education, there is no prompt for students' activity and/or the tasks are chosen superficially.	A PST does not realize didactic problem and misconceptions connected to the unit topic or he/she realizes only marginal or general problems and misconceptions.
1	A PST explains the objectives of the unit partially, and/or he/she does not propose certain of the important goals, and/or certain objectives are not appropriate for the students' age group.	The motivation stated in the lesson plan probably would be interesting only for a few students, and/or there are no tight relations between the motivation and the objectives.	Language inaccuracies in oral or written communication are observable and/or the learning trajectory is respected only partially.	Chosen didactic means lead to rather instructive approach, there is small prompt for students' activity. The structure of the teaching unit is not well thought out and/or some important cognitive phase is missing and/or selection of the tasks is only partly thought out.	A PST can name the didactics problem and misconceptions connected to the unit topic and he/she resolve them within the preparation just partially.
2	A PST formulates and clarifies essential objectives of the unit, links them with the rest of the preparation and the objectives are appropriate for the students' age group.	The motivation stated in the preparation probably engages most of the students and it is linked with the objectives. If possible, the motivation suggests connection of mathematics and everyday life.	The unit curriculum is introduced correctly (mathematics terms definitions, mathematics statements and tasks assignments are formulated correctly and comprehensively and they are appropriate for the students age group,) and the learning trajectory is respected.	Chosen didactic means leads to a creative environment where activity of students dominates and the proposed teaching unit has a coherent structure, no important cognitive phase is missing. Task selection is thought out.	A PST can name the key didactics problem and misconceptions connected to the unit topic and he/she resolve them within the preparation.

Table 1: Rubrics for PCK assessment

In the rubric Correctness, the term learning trajectory is not conceptualized, it is used in simplified meaning concerning the important entry knowledge before the new one is going to be taught and learnt.

The presented lesson plans were not the first ones created by the PST. They had prepared and taught 18 lessons during their practice teaching before the course started. Additionally, they developed 15 lessons plans for the state examination, which was scheduled 5 months after the course finished. Some of PSTs worked on these lesson plans in groups. All these lessons plans were taken in account to track the PCK development.

Each of the rubrics also provides three general development levels: Beginning, Developing, and Advancing (see table 1). In order to explain more precisely authors' approach to assessment, we present two examples for the rubric Motivation from level 0 and level 1.

Level 0 - Example for "the stated motivation is very formal (neither students' activity nor questions cannot be expected)": Lesson plan for the Topic: How to multiply decimal by decimal numbers

PST: For the motivation, I chose the following task: A farmer stored fuel for the tractor in canisters of 0.5 hl. He has a) 10, b) 5, c) 2, d) 1, e) 0.5 canisters full of fuel. How much fuel does he have for the tractor (write result in hl)?

T (teacher – the author): Explain the reasons why you consider this task as motivating one.

PST: I think, in fact, this is not very motivating task. Maybe students could solve the last case of the task and found out how to multiply decimal by decimal number.

Level 1 - Example for "there are no tight relations between the motivation and the objectives": Lesson plan for the Topic: Definition of the concept Limit of a sequence

PST: Motivation task: In the hotel we have an infinite number of single rooms. Rooms are sequentially numbered by natural numbers. The hotel is fully booked. To the hotel, however, added three other tourists who would like to stay. Is it possible to accommodate them?

T: What is the connection between this task and the definition of the concept Limit of a sequence?

PST: The task is about infinity and when we count limit of a sequence we work with infinity.

Preliminary results and discussion

We try to answer the research questions mentioned in the Introduction.

(1) The concept of inter-rater reliability was used to find the answer to the first research question. The Table 2 depicts numbers of consistent and inconsistent decisions for each particular rubric.

Only the authors' assessment was taken in account because the PSTs' assessment was obviously loaded by the social norms and the relationships within the group. Some inconsistencies between the authors were caused by the usage of the rule, that for one deficiency only one point should be get off and the authors included the same mistake in the different rubrics. As we can see, the rubrics worked well and after precise preparation of their user, they can be considered as reliable tool. The certain vagueness of the developing levels formulations does not seem to be a problem when comes to its identification within the actual presentation.

The rubric	Match	One level differences	Two level differences
Learning objectives	9	1	0
Motivation	9	1	0
Correctness	9	1	0
Didactics Means	7	3	0
Didactics Problem	8	2	0

Table 2: Inter rater reliability

(2) At the beginning, PSTs filled the front page formally (see The course design). Most of the PSTs had no idea about how the front page tailors to remaining part of lesson plan. As the course continued, we could see how PSTs were moving forward in the development of their PCK. We explain it using examples from 3 rubrics (Learning objectives, Didactic Means, Didactics Problem).

(2a) Development of PCK within the group of PSTs - Learning objectives

At the beginning of the course, PSTs did not formulate any learning objectives or formally mention some objectives of prepared lesson but were not able to explain how to achieve it. Also, some of PSTs did not meet the learning objectives they have formulated.

Example from the lesson plan for the topic: The Binomial Theorem

PST: The student is able to formulate the binomial theorem and to write the binomial theorem by using the summation operator.

T: Formulate the binomial theorem and write it by using the summation operator.

PST: *(Started to write on the blackboard, but did not remember correct formulation, then started to look for the correct formulation in the hard copy of the lesson plan.)*

T: The student should be able to formulate the theorem and the teacher is not?

At the end of the course, PSTs started to formulate learning objectives in connection with chosen tasks and their solutions and they also explain more precisely what they expect.

Learning goals for the topic: Increasing and decreasing function

The student is able to identify whether the function is increasing or decreasing from the graph, from the table. The student is able to draw graph of increasing and decreasing function. The student is able to define increasing and decreasing function. The student is able to prove from the formula whether the function is increasing, decreasing or neither one nor the other.

Additionally, in the prepared lesson plan we can found tasks as means to meet the formulated objectives.

(2b) Development of PCK within the group of PSTs - Didactic Means

In the first stage, PSTs chose the tasks superficially, e.g. the PST prepared lesson plan for the Topic: Law of sines and he explain the task selection as follows:

PST: *(Presents some task where students have to calculate side or angle of triangle using law of sines.)*

T: Explain us, why did you choose exactly these tasks.

PST: I found them in the textbook.

T: Did you solve them?

PST: Only the last one because it looked hard.

T: Do you think these tasks will help your students to understand deeply methods of solution using Law of sines?

PST: Maybe some types of tasks are missing. (He started to draw on the blackboard.)

At the last stage the PST who prepared for the topic Definition of the concept Limit of a sequence explained precisely the reasons for each selected task.

T: Why did you solve the absolute-value inequality:$|x - 4| \leq 2$" at the beginning of the lesson?

PST: I chose this task because I wanted to recall the geometric properties of absolute-value which students meet in the definition of the limit.

T: You suggest dividing students into 6 groups and each group will work with different sequence. Explain your reason.

PST: Two groups will get increasing sequence, two groups decreasing and two groups oscillating sequence. I chose these sequences in order to prevent the following misconception: Only decreasing sequence has limit. Oscillating sequence cannot have a limit.

During the presentation of the lesson plans on the state exams the most of students showed that they are better able to intertwine all the items from the front page of the lesson plan with the tasks and their solutions, activities, mathematical explanations. Students were also able to explain better their reasons for selecting the particular tasks and activities for the lesson.

(2c) Development of PCK within the group of PSTs - Didactics Problem

Firstly, most of PSTs did not see any didactic problem and misconception with the most of the topics, some PSTs wrote the most common misconceptions, such as problem with negative sign during working with algebraic expression, or they formulated didactic problem very generally such as students have problems while working with fractions. Later, when PSTs were trying to identify didactics problems and misconceptions, they started to utilize experience from their life (this task/concept/method was a problem for my sibling, friend, me) and also from their teaching practice. The next example illustrates formulation of the misconception based on PST experience.

PST: Students think that the following scalar products $(1,1) \cdot (3,2)$ & $(2,2) \cdot (3,2)$ are equal. To prevent this misconception I formulated the following task: Find out if following scalar products have the same value:

A. $\vec{u} = (1,1); \vec{v} = (3,2);\ \varphi = 60°$ B. $\vec{u} = (2,2); \vec{v} = (3,2);\ \varphi = 60°$

Although the task is incorrect it shows PST's effort not only to formulate the misconception but also to look for the way how to prevent it.

Our experience indicates that the authentic assessment focused on PSTs' lesson plans and objectivized by five rubrics, tied with the content knowledge and three types of PCK, can help PSTs to develop their PCK. Presented examples demonstrate development of KCS (identification of didactic problems and misconceptions), KCT (precise selection of tasks) and KC (formulation of learning objectives).

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Measuring beliefs concerning the double discontinuity in secondary teacher education

Viktor Isaev[1] and Andreas Eichler[2]

[1]University of Kassel, Germany; isaev@mathematik.uni-kassel.de

[2]University of Kassel, Germany; eichler@mathematik.uni-kassel.de

In this paper, we discuss a research project to measure prospective mathematics teachers' beliefs towards their perceived disagreements between mathematics at university level and school mathematics that is known as double discontinuity. Firstly, we introduce the double discontinuity problem. Secondly, we refer to the construct of beliefs as the main part of our theoretical framework. Afterwards, we outline our method including a brief discussion on our approach of bridging the double discontinuity problem with so-called teacher-oriented tasks that are appropriate to illustrate connections between university mathematics and school mathematics. Furthermore, our results of pilot studies aiming to measure prospective teachers' beliefs are provided.

Keywords: Beliefs, double discontinuity, prospective teachers, teacher education.

Introduction

Over 100 years ago, Felix Klein coined in the preface of his textbook "Elementary Mathematics from a Higher Standpoint" the term of a "double discontinuity" (Klein, 1908, p. 1). This notion embodies the challenges of transitions in the mathematical socialization of mathematics teachers. The first discontinuity, i.e. the transition from secondary to tertiary education, became a main topic in research on university mathematics education during the last decades (e.g. Gueudet, 2008; Thomas et al., 2015). A challenge that accompanies the second discontinuity is the transformation of academic mathematics gained at university into educational forms of school mathematics (Prediger, 2013; Winsløw & Grønbæk, 2014). As a consequence of both discontinuities, teachers may lose sight of academic mathematics after university studies and, thus, teach on the basis of experiences from their own schooldays (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013).

Although mathematics instruction in schools as well as teacher education in universities changed considerably since Klein's claim, the phenomenon of a double discontinuity still seems to exist and prospective teachers nowadays frequently believe that the topics of university mathematics do not meet the demands of their later profession in school (cf. Ableitinger, Kramer, & Prediger, 2013; Hefendehl-Hebeker, 2013). The prospective teachers' perception of the mentioned relationships between school mathematics and university mathematics could be regarded as part of teachers' beliefs (Eichler & Isaev, 2016). Considering this background, we primarily rely on teachers' beliefs as the main construct of our theoretical framework in the next paragraph. Subsequently, we outline the method for our research project with a specific focus on pilot studies. Finally, we use the results of these pilot studies to explain the development of our instrument aiming to measure prospective teachers' beliefs concerning the double discontinuity.

Theoretical framework

Prospective teachers' beliefs

On the construct of beliefs we refer to the definition of Philipp (2007, p. 259), who defines beliefs as "psychologically held understandings, premises, or propositions" affecting an individual's view of the world or a special part of it, such as filters for receiving information. Beliefs can be considered as a component of teachers' mathematics related affect (Hannula, 2012), which itself can be defined as "a disposition or tendency or an emotion or feeling attached to an idea or object" (Philipp, 2007, p. 259). Thus, in our research we regard prospective teachers' beliefs as their propositions concerning the relationships between school mathematics and university mathematics. Following Calderhead (1996), teachers' beliefs are understood not only to impact on teachers' professional knowledge but also to potentially have an effect on teachers' classroom practices (cf. also Skott, 2009). Therefore, it seems essential to tackle the issue early and to investigate, how beliefs concerning the double discontinuity develop in secondary teacher education.

In related literature, beliefs are partially seen and sometimes even defined as relatively stable dispositions. In contrast, several studies provided a change of beliefs due to interventions, especially concerning prospective teachers (e. g. Decker, Kunter, & Voss, 2015). In this paper, we aim to provide an instrument in order to investigate, whether and how prospective mathematics teachers change their beliefs concerning the double discontinuity phenomenon based on our intervention project. From this background, we address beliefs being changeable and consider change as "a natural part of the development of beliefs and the reaction of beliefs in the face of experiences" (Liljedahl, Oesterle, & Bernèche, 2012, p. 35).

Related Research

Although numerous projects and institutions across the world aimed to overcome the perceived gap between school mathematics and university mathematics in recent years, not much research has been done on the perception of prospective teachers' concerning the double discontinuity problem. Winsløw and Grønbæk (2014) distinguished three dimensions of Klein's double discontinuity, which are not independent but important to separate: the institutional context (i.e. school vs. university), the difference in the subject's role within the institution (i.e. student at university or school vs. teacher of mathematics), and the difference in mathematical contents (i.e. elemantary vs. advanced). In our research, we primarily refer to the content aiming to figure out possibilities of "building bridges" (Winsløw & Grønbæk, 2014, p. 64) and to investigate the effect of bridging activities on the prospective teachers' beliefs referring to the double discontinuity.

Becher and Biehler (2016) used narratives in order to ask prospective secondary teachers in their third year of university studies about what benefits they see in learning university mathematics for their future career as a school teacher and which aspects are articulated by the prospective teachers in their evaluation of benefits of university mathematics. The results revealed a wide range of prospective teacher's beliefs on benefits of learning university mathematics with regard to school mathematics. Most of the statements can be matched with one of four levels of mathematical content knowledge based on Krauss et al. (2013), i.e. "A deep understanding of the content of the secondary school mathematics curriculum (e.g., 'elementary mathematics from a higher standpoint,' as taught at university)" (Krauss et al., 2013, p. 155). Taking these studies into account, we developed a

questionnaire for measuring prospective mathematics teachers' beliefs towards their perceived disagreements between mathematics at university level and school mathematics.

Method

The institutional frame

Referring to this theoretical framework, the main target of our project is to investigate changes in prospective teachers' perception of a double discontinuity that could be explained by our approach of building bridges. Prospective secondary mathematics teachers in Germany are usually enrolled in the same mathematics courses as mathematics majors (e.g. analysis), particularly in the first semesters. A big challenge for all students in the initial phase of the studies is the task to complete a range of exercises every week as homework (Ableitinger & Herrmann, 2013). These tasks can be solved on the basis of the plenary lectures (by usual four hours per week) and are reviewed in additional small courses (by two hours per week) which are organized by student assistants. Our aim is to develop and establish the desired bridges in these introductory mathematics courses for prospective secondary teachers. Our focus is here to enrich the set of tasks for homework with so-called "teacher-oriented tasks" that are appropriate to illustrate connections between university mathematics and school mathematics to prospective secondary teachers.

Teacher-oriented tasks

We conceptualize specific tasks which potentially demonstrate bridges between school mathematics and university mathematics to a model of domains of teacher knowledge according to Ball, Thames, and Phelps (2008). More precise, we differentiate our teacher-oriented tasks referring to the subdomains of specialized content knowledge (SCK), knowledge of content and students (KCS), knowledge of content and teaching (KCT) as well as curriculum knowledge. One example that represents specialized content knowledge (SCK) within the notion of mathematical tasks for teaching is provided below. In this exercise, "Evaluating the plausibility of students' claims" and "Giving or evaluating mathematical explanations" (Ball et al., 2008, p. 400) are requirements which can be used well to describe the setting.

In the subsequent task from a mathematics contest for students ("Känguru der Mathematik 2009") the participants were asked to solve which of the following figures is the greatest one.

(A) $\sqrt{2} - \sqrt{1}$ (B) $\sqrt{3} - \sqrt{2}$ (C) $\sqrt{4} - \sqrt{3}$ (D) $\sqrt{5} - \sqrt{4}$ (E) $\sqrt{6} - \sqrt{5}$

A student in grade 12 chose answer (E) and stated:

"$\sqrt{6} - \sqrt{5}$ is the greatest figure, because roots are monotone. So, the greater is x, the greater is f(x). Thus, their difference is the greatest, as well (by going more to the right)."

1. Analyze the student's answer. Where do you see problems with the argumentation?

2. Provide an own student-oriented answer to this topic.

3. Show in general: $\lim_{n\to\infty} \sqrt{n} - \sqrt{n-1} = 0$

Figure 1: Task "roots" for prospective secondary mathematics teachers

The design

In order to gain empirical evidence for the efficiency of our method, prospective teachers in the relevant mathematics courses are assigned at random to a treatment group and a control group. While the control group is taught traditionally, the treatment group gets an extra teacher-oriented task for homework every week that focuses on bridging mathematics at university level and school mathematics. Our main research question is whether and how prospective teachers change their beliefs about the double discontinuity phenomenon based on our intervention in the introductory mathematics courses. In this paper we only refer to pilot studies where prospective teachers got homework including an extra teacher-oriented task on a trial basis. The main aim of these pilot studies was to develop an instrument for measuring teachers' perception of a double discontinuity.

During the winter semester 2015/16, two basic mathematics courses at the University of Kassel were selected in which the mentioned teacher-oriented tasks bridging mathematics and mathematics education was administrated: "principles of mathematics" and "analysis". Prospective mathematics teachers attend these courses usually in the first or in the third semester of their university studies. Three prospective teachers from the third semester were interviewed at the end of the semester. The data was analyzed by qualitative content analysis. We also developed a questionnaire with 16 items for measuring prospective teachers' beliefs concerning their perceived disagreements between mathematics at university level and school mathematics. The questionnaire was piloted in a mathematics course for prospective secondary teachers (N = 60) and is outlined in the following paragraph.

Discussion of results

The analyses of the interviews revealed that all prospective teachers took the contents dimension into account when reflecting their university studies. For example, the first prospective teacher (PT_1), believes, that in university mathematics, there are too little relations to school with regard to the content.

PT_1: "There are not many connections - direct contents connections, so - as well as in the other lectures. And also in analysis I notice, that it - that the university stuff of the mathematics lecture - almost simply goes beyond school and it is more or less by chance, if there are contents, which fall together with school mathematics - I have the feeling."

Further, two of the prospective teachers also mentioned another aspect of the double discontinuity problem, i.e. their beliefs concerning the relevance of learning university mathematics with regard to school mathematics, such as can be found in the proposition of PT_2.

PT_2: „If I finished school, I would have the same status, which I want to teach the students. And if now deeper questions arise, I would not be able then to answer them, for instance, because I myself have never had this and then - such a teacher one also did not want formerly, who could only tell, what he has just done."

Indeed, Klein (1908) addresses both aspects in the double discontinuity phenomenon since on the one hand the problems at university may not suggest the things at school, and on the other hand, university studies may remain only a memory with no relevance upon teaching.

The collected data from the interviews also supported the development of our current questionnaire. Our first version (cf. Eichler & Isaev, 2016) contained 9 items using a 6-point Likert scale to assess students' beliefs about the double discontinuity problem. The questionnaire was piloted in a mathematics course for prospective secondary teachers (N = 60) and seemed to provide good internal consistency (Cronbach's alpha 0.782). Interestingly, a higher reliability (Cronbach's alpha 0.831) was achieved when regarding only the prospective teachers in the course (N = 35) and not all university students in the mathematics course. As a possible reason for this phenomenon, we identified two theoretical discernable domains in our questionnaire. On the one hand, we asked items which contained a personal statement including the words “I” or “me” like “I think that I require a deep understanding of mathematics in order to teach mathematics in school.” On the other hand, we provided a few statements such as “University mathematics has mostly little relation to school mathematics”, which were rather matter-of-fact. These items might have led to a different extent of identification within the different groups of university students. To provide useful information, we applied both dimensions in our current questionnaire containing 16 items.

Taking into account the four levels of mathematical content knowledge based on Krauss et al. (2013), we derived further items from the interviews such as “By the use of university mathematics, gaps are filled in the mathematical knowledge that is required in school”. We grouped all items into three subscales which we identified through our development process: “contents relationship”, “relevance for profession” and “higher standpoint”. In a further pilot study with prospective secondary teachers in higher semesters (N = 24), we approved these subscales to be internal consistent[1] with a total reliability value of Cronbach's alpha .911. Moreover, we used our qualitative approach in order to validate our survey afterwards.

Contents relationship – beliefs concerning the connections between university mathematics and school mathematics on the contents dimension (4 items; Cronbach's alpha .821):	
4.	University mathematics offers many parallels to school mathematics with regard to contents.
9.	School mathematics and university mathematics are applied to each other in contents.
11.*	In university mathematics, there are too little relations in contents to school.
12.*	School mathematics and university mathematics are two different worlds with regard to contents.
Relevance for profession – beliefs concerning the relevance of university mathematics for the later profession as a school teacher (6 items; Cronbach's alpha .814):	

[1] 7 Personal items with Cronbach's alpha .845 and 12 non- personal items with Cronbach's alpha .858.

1.	University mathematics is very useful for the teaching profession.
3.*	I will hardly ever need university mathematics after studying.
5.	By the use of university mathematics, I am well prepared to the job profile of a mathematics teacher.
10.	Without university mathematics, I could hardly teach mathematics in school.
15.*	Learning mathematics at university is not so important for the teaching profession.
19.	The relevance of university mathematics for the teaching profession is[2]
Higher standpoint – beliefs concerning the usefulness of university mathematics as a higher standpoint for elementary mathematics (6 items; Cronbach's alpha .818):	
2.	University mathematics helps me to get deeper into school mathematics.
6.	By the use of university mathematics, gaps are filled in the mathematical knowledge that is required in school.
7.	By the use of university mathematics, I gain a deeper understanding of concepts in school.
8.	By the use of university mathematics, I understand relationships within school mathematics much better.
14.	Learning mathematics at university promotes me to be in thinking "one step ahead" of the students.
16.	As a mathematics teacher, an in-depth mathematical content knowledge is required.

Table 1: questionnaire for measuring prospective teachers' beliefs concerning a double discontinuity

The possible range of scores for each component is between 1 and 6. Higher scores correspond to more positive beliefs (by reversing the responses to the negatively formulated items indicated with an asterisk*).

Concluding remarks

The main topic of this paper was to discuss our approach of measuring prospective mathematics teachers' beliefs towards their perceived disagreements between mathematics at university level and school mathematics that is known as double discontinuity. In order to be able to investigate changes in prospective teachers' beliefs referring to the double discontinuity problem, we chose a mixed methods design and developed a questionnaire including 16 items that actually seems to measure these beliefs. Grounded on our preliminary results, the following steps of our research will be a comparison between two groups of prospective teachers - one group in a traditional course and one group in a course using the mentioned teacher-oriented tasks to prove if the type of the course has an effect of the prospective teachers' beliefs. Since a variety of other factors may be related to our outcomes, we also collect among others data to study interest (Schiefele, Krapp, Wild, & Winteler,

[2] Whereas in the previous items the prospective teachers may choose an option in a scale from "strongly disagree" to "strongly agree", the last item refers to a scale from "very low" to "very high".

1993) and study satisfaction (Dargel, 2005) in a pretest and a posttest as well as additional items referring to relevant demographic and academic background information.

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Using a fraction learning trajectory as a tool to develop pre-service primary teachers' noticing of students' fractional reasoning

Pere Ivars, Ceneida Fernández and Salvador Llinares

Universidad de Alicante, Spain

pere.ivars@ua.es; ceneida.fernandez@ua.es; sllinares@ua.es

Since noticing has been identified as a critical skill that teachers must develop, research on how pre-service teachers develop this skill in teacher education programs has emerged. In this study, we focus on how pre-service teachers notice students' fractional reasoning through a task designed taking into account a students' Learning Trajectory of fractional reasoning. Our results show that pre-service teachers' learning of the Learning Trajectory helped them to notice students' fractional reasoning in a structured way: identifying important mathematical elements of the problems and, establishing relationships between the mathematical elements and students' fractional reasoning levels of the Learning Trajectory to help students progress in their fractional reasoning.

Keywords: Noticing, fractional reasoning, learning trajectories.

Noticing and learning trajectories

Noticing has been shown as an important skill for teachers. This skill has been conceptualised from different perspectives (Jacobs, Lamb, & Philipp, 2010; Mason, 2002, 2011; Sherin, Jacobs, & Philipp, 2011) but all of them emphasise the importance of identifying the relevant aspects in teaching and learning situations and interpreting them to make teaching decisions. Mason stated that "noticing is a movement or shift of attention" (Mason, 2011, p. 45) and identified different ways in which people can attend (p.47):

> *Holding wholes* is attending by gazing at something without particularly discerning details.
>
> *Discerning details* is picking out bits, discriminating this from that, decomposing or subdividing and so distinguish and, hence, creating things.
>
> *Recognizing relationships* is becoming aware of sameness and difference or other relationships among the discerned details in the situation.
>
> *Perceiving properties* is becoming aware of particular relationships as instances of properties that could hold in other situations.
>
> *Reasoning on the basis of agreed properties* is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms.

This perspective emphasises the importance of identifying the relevant aspects of the teaching-learning situations (discerning details) and interpreting them (recognising relationships) to support instructional decisions (perceiving properties).

On the other hand, research has shown that when pre-service teachers attend to students learning progressions in a particular mathematical domain, they are better able to make decisions about next

instructional steps (Wilson, Mojica, & Confrey, 2013). In this context, students' learning trajectories (Battista, 2012) can assist pre-service teachers in identifying learning goals for their students, in anticipating and interpreting students' mathematical reasoning and in responding with appropriate instruction (Sztajn, Confrey, Wilson, & Edgington, 2012). Our study is embedded in this line of research and analyse how pre-service teachers' learning of a fractional reasoning Learning Trajectory supports their development of noticing students' fractional reasoning. Our research question is: how do pre-service teachers interpret student' fractional reasoning and respond with instructional actions using a learning trajectory of fractional reasoning?

A learning trajectory of fractional reasoning

A Learning Trajectory consists of three components: a learning goal, learning activities, and a hypothetical learning process (Battista, 2011). A Learning Trajectory includes descriptions of learning activities that are designed to support students in the transition through intermediate stages to a more sophisticated level of reasoning.

The learning goal of the fractional reasoning Learning Trajectory used in this study is derived from the Spanish Primary Education's curriculum: the meaning of fraction and its different representations and, the meaning of fractions operations. This learning goal highlights two key aspects: a) the transition from an intuitive meaning of splitting into equal parts to the idea of fraction as part-whole taking into account different representations, and b) the construction of the meaning of operations with fractions.

The student's learning process takes into account how the student reasoning about fractions develops over time (Battista, 2012; Steffe, 2004; Steffe, & Olive, 2009). We have considered six different levels of students' fractional reasoning (learning trajectory proficiency levels): **at level 1**, students have difficulties in recognising that the parts of the whole must be congruent; **at level 2**, students recognise that the parts could be different in form but congruent in relation to the whole. This allows them to identify and represent fractions in a continuous context but they have difficulties with discrete contexts. They also begin to use unit fractions as an iterative unit (i) to represent proper fractions (although they have difficulties with improper fractions) and (ii) to solve some fraction addition problems with the same denominator; **at level 3**, students identify and represent fractions in discrete contexts recognising that the groups must be equal. They also recognise that a part could be divided into other parts. When comparing fractions, they recognise that the size of a part decreases when the number of parts increases. They can use a part (not necessarily the unit fraction) as an iterative unit to represent proper ($f<1$) and improper ($f>1$) fractions. They can also reconstruct the whole using any fraction as an iterative unit (continuous and discrete contexts). In addition, they use intuitive graphical representations to add/ subtract fractions with different denominators; **at level 4**, students can solve simple arithmetic problems with the help of a guide or support. They can do equivalent fractions so that operations can be graphically represented. When they add or subtract fractions with different denominators, they understand that the parts must be congruent to join/separate although they need a guide that allows them to choose the unit correctly. When they multiply, they understand the fraction as an operator "a/b of c/d" and when they divide, they develop two types of reasoning; (i) division as a measure and (ii) division as a partition; **at level 5**, students can operate and solve arithmetic problems symbolically, identifying patterns. They can graphically justify what they do but only in simple situations. At this level, they

are able to interpret the remainder of a division of fractions; **at level 6**, students can explain operations graphically. They do not need a guide to represent fraction operations.

Method

Participants and context

Participants were 31 pre-service primary school teachers (PT) enrolled in a degree to become primary school teachers. They were enrolled in a subject of 150 hours (60/90 attendance/nonattendance) related to teaching and learning of mathematics in primary school. In previous courses, these pre-service teachers had participated in a subject related to Numerical Sense and in a subject related to Geometrical Sense.

Instrument: The task

The task consists of three pairs of primary school students answers, with different learning trajectory proficiency levels of fractional reasoning, to a problem that implies the identification of a fraction (adapted from Battista, 2012) (Figure 1). These answers reflect characteristics of the first three levels of the Learning Trajectory. The answers of Xavi and Victor show characteristics of the level 1 since they are not able to identify that the parts of a whole must be congruent. The answers of Joan and Tere reflect characteristics of the second level since they are able to identify that the parts of a whole must be congruent in continuous contexts but they still do not recognise that a part can be divided into other parts. This last characteristic is evidenced when they say that Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded. Finally, the answers of Álvaro and Félix show that not only they are able to recognise that the whole must be divided into congruent parts but also they acknowledge that a part could be divided into other parts.

Pre-service teachers had to answer the next four questions. To answer them, we provided pre-service teachers with theoretical information about the mathematical elements of the fraction concept and about the Learning Trajectory of fractional reasoning used in this study.

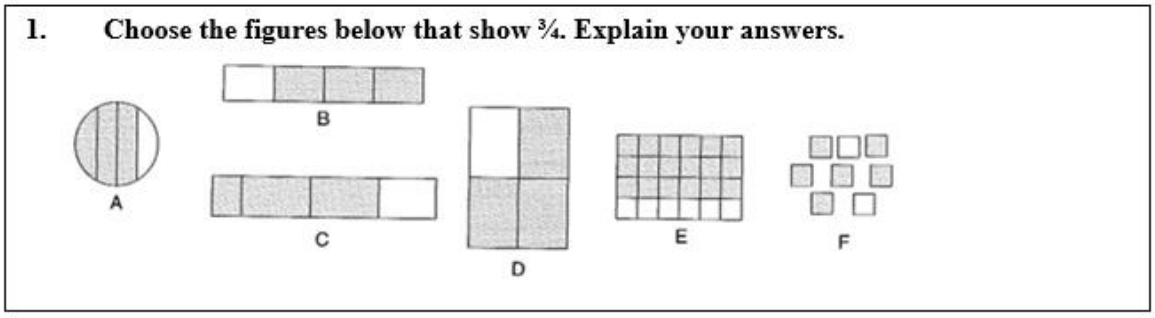
Q1- Describe the problem taking into account the learning objective: what are the mathematical elements that the student needs to know to solve it?

Q2- Describe how each pair of students has solved the problem identifying how they have used the mathematical elements involved and the difficulties they have had with them.

Q3- What are the characteristics of students' reasoning (Learning Trajectory) that can be inferred from their responses? Explain your answer.

Q4- How could you respond to these students? Propose a learning objective and a new activity to help students progress in their fractional reasoning.

These questions and the theoretical information given (Learning Trajectory of fractional reasoning) focus pre-service teachers' attention on relevant aspects of students' answers (discerning details) identifying the relevant mathematical elements; on interpreting these answers (recognising relationships between the mathematical elements and students' reasoning) and on supporting instructional decisions (attending students' mathematical reasoning).



Xavi and Victor's answers

Víctor: Mmmm, well we think Figures A, B, C and D represent three-quarters.

Teacher: Xavi, do you agree with Víctor?

Xavi: Yes, A, B, C and D are divided in 4 parts, and 3 are shaded.

Joan and Tere's answers

Tere: We believe that Figures B and D are three quarters because they are divided into four equal parts and three are shaded. Figures A and C have 3 parts of 4 shaded, but the parts are not equal...

Teacher: And Figure E? What do you think about Figure E?

Joan: Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded.

Tere: Sure, it is not three-quarters.

Teacher: And the F?

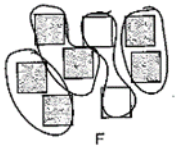
Both It is not a fraction. In figure F, there are only 6 shaded squares.

Felix and Alvaro's answers

Félix: Well ... yes. We agree with Joan and Tere answers related to figures A, B, C, and D but we think differently about figure E...

Teacher: What do you think? Could you explain your answer?

Álvaro: Well ... mmm sure. If you look each line of Figure E, each line has 6 squares, and as there are 3 lines shaded of the 4 total lines then it is three quarters. In addition, Figure F also represents three quarters because if you group the squares in groups of 2, you get 4 groups of 2, and there are three groups shaded.



Álvaro and Félix answer to Figure F

Figure 1: Task to support pre-service teachers' learning of a fractional reasoning Learning Trajectory to notice students' mathematical reasoning

Analysis

Taking into account Mason's work and the Learning Trajectory of fractional reasoning, we analysed pre-service teachers' answers according to if they had (i) identified relevant elements of fractional reasoning in the student's answers (discerning details); (ii) interpreted the student's reasoning considering the characteristics of students' fractional reasoning from the Learning Trajectory (recognising relationships between the elements identified and the different levels of students' learning progress of fractional reasoning); (iii) made instructional decisions (reasoning about next steps providing different activities that promote students' progression in the Learning Trajectory).

To carry out the analysis, initially a subset of pre-service teachers' answers was analysed by three researchers independently considering the points mentioned above. Then, we put together our respective analyses and compared and discussed our discrepancies until reaching an agreement. Afterwards, new data samples were added to review our allocation.

Results

From the analysis, we have identified three groups of pre-service primary school teachers according to the way that they used the Learning Trajectory to interpret students' fractional reasoning and make teaching decisions. These results show that 20 pre-service teachers were able to use the Learning Trajectory to interpret students' fractional reasoning, while the other pre-service teachers (group 1) had difficulties in using the Learning Trajectory to interpret students' answers. The characteristics of the different groups of pre-service teachers are:

- Group 1. Pre-service teachers who used some mathematical elements of the Learning Trajectory but in rhetoric way or without sense (11 PT).
- Group 2. Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, but they were not able to propose new activities considering the learning trajectory proficiency levels (11 PT)
- Group 3. Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, and proposed new activities to help students progress in their fractional reasoning taking into account the learning trajectory proficiency levels (9 PT)

Group 1: Pre-service teachers who used some mathematical elements of the Learning Trajectory but in rhetoric way or without sense

Pre-service teachers of this group used the mathematical elements implied in the problem (the parts of the whole must be congruent and a part can be divided in other parts) in a rhetoric way when they described students' answers but they did not recognise characteristics of the different Learning Trajectory proficiency levels in students' answers. For instance, the pre-service teacher E27 answered question 3 of the task, pointing out (emphasis has been added underlying the mathematical elements):

> Víctor and Xavi: They are at Level 1 of the Learning Trajectory because they do not know the concept of congruence and they do not know that a part could be divided in other parts
>
> Joan and Tere: They are at Level 1 because they have difficulties in recognising that the part must be congruent and they do not recognise that a part could be divided in other parts.
>
> Félix and Álvaro: They are at Level 1 because, related to congruence they know the same that Joan and Tere, although they recognise that a part could be divided in other parts in continuous and discrete contexts.

This pre-service teacher did not recognise differences between students' fractional reasoning saying that all pairs of students have difficulties with the mathematical element *the parts of the whole must be congruent* although he used the mathematical elements to describe students' answers.

Group 2: Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, but they were not able to propose new activities considering the Learning Trajectory proficiency levels

Pre-service teachers of this group used the mathematical elements of the Learning Trajectory that correspond with the problem (the parts of the whole must be congruent and a part can be divided in other parts) to recognise the different levels of students' fractional reasoning. However, these pre-service teachers did not justify a new activity taking into account the students' fractional reasoning. For instance, the pre-service teacher E09 answered to question 2 and 3 for each pair of students (emphasis has been added underlying the mathematical elements):

> Víctor and Xavi have difficulties in recognising that the parts must be congruent as they identify as a ¾ figures A and C whose parts are not equal. Another characteristic that we can identify is that they have difficulties in recognising that a part could be divided in other parts. They do not notice that figures E and F are divided in 4 parts, maybe they notice that E has 24 squares and F has 8 squares. Thus they do not realise that both are equivalents. So, these students are at Level 1.
>
> Joan and Tere are able to identify and represent fractions in a continuous context recognising that the parts must be congruent as they recognise that, although figures A and C are divided in 4 parts and 3 are shaded they do not represent ¾ because the parts are not congruent. They also identify that B and D are ¾. They are not able to recognise that a part could be divided in other parts/consider a group of parts as a part since they do not identify that even though E and F are divided in more parts, they represent ¾. So, these students are at Level 2.
>
> Félix and Álvaro agree with Joan and Tere about figures A, B, C, and D, thus they recognise that the parts must be congruent. Furthermore they recognise that a part could be divided in other parts and they identify fractions in discrete contexts since for figure E they say that, although it is divided in 24 squares, it represents ¾ because there are 4 lines with 6 squares each and 3 of those 4 are shaded (they recognise the equivalence 18/24=3/4). Besides of that, in figure F they group in pairs the eight squares of the whole to represent the ¾. So, these students are at Level 3.

Nevertheless, this pre-service teacher was not able to propose a specific activity considering the Learning Trajectory in order to help students progress in their conceptual reasoning. For instance, this pre-service teacher proposed for the first pair of students: *"With Víctor and Xavi we would work with the recognition that the parts must be congruent. To do that, we could propose the same task but with other figures and they (students) could represent 4/6".*

The answers of this group of pre-service teachers indicated the difficulty of making instructional decisions considering the Learning Trajectory proficiency levels.

Group 3: Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, and proposed new activities to help students progress in their fractional reasoning taking into account the learning trajectory proficiency levels

Pre-service teachers of this group, after using the mathematical elements (the parts of the whole must be congruent and a part can be divided in other parts) to recognise different levels of students' fractional reasoning, proposed new activities focused on helping students progress in their fractional

reasoning according to the learning trajectory proficiency levels. For example the pre-service teacher E25 proposed the next objective and activity to help Victor and Xavi progress in their fractional reasoning:

> Objective: In order to progress from Level 1 to Level 2, students have to recognise that the parts of a whole must be congruent (although they could be different in form).
>
> Activity: Represent in the following figure (square) 2/4 in three different ways

This group of pre-service teachers used their knowledge of the Learning Trajectory to interpret students' fractional reasoning, and proposed new activities to help students develop their fractional reasoning.

Discussion and conclusions

The aim of this research was to analyse how pre-service teachers' learning of a Learning Trajectory of fractional reasoning supports their development of noticing students' fractional reasoning. We focus on how pre-service teachers interpret student' fractional reasoning and respond with instructional actions using a learning trajectory of fractional reasoning.

Twenty out of thirty-one pre-service teachers who participated in the task were able to use the mathematical elements to interpret students' fractional reasoning considering the characteristics of the students learning progression of fractional reasoning and identifying different levels of students reasoning. This result indicates that the information about a Learning Trajectory of a particular mathematic topic can be used by pre-service teachers to begin to notice features of students' mathematical thinking in a particular domain and therefore, to develop the skill of noticing. The Learning Trajectory can be seen as a powerful tool that help pre-service teachers focus their attention on important mathematical aspects of the problem, on the students' mathematical reasoning and on making instructional decisions on the basis of students' mathematical reasoning. The other eleven pre-service teachers had difficulties in using the Learning Trajectory to interpret students' answers. This result is in line with other studies that have shown that interpreting students' mathematical reasoning is a challenging task for some pre-service teachers (Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015).

However, only nine out of these twenty pre-service teachers could use their interpretations of students' fractional reasoning to propose new activities according to the Learning Trajectory in order to help students progress in their fractional reasoning. Previous research has pointed out that the skill of making instructional decisions is the most difficult one to develop in teacher education programs (Callejo & Zapatera, 2016; Ivars & Fernández, 2016; Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015). Nevertheless, approximately one third of the participants, in our task, were able to design an activity to promote students' progressions of fractional reasoning according to the Learning Trajectory. Therefore, we think that the task of our study, designed according to a Learning Trajectory, seems to have a relevant paper in the development of the skill of providing activities that could help students progress in their learning. The Learning Trajectory could be seen as a referent or guide for pre-service teachers that could help them to link the mathematical domain (mathematical elements), the student's reasoning and the instruction that considers students' learning progressions.

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Collaborative research in in-service teacher professional development

Jónína Vala Kristinsdóttir

University of Iceland, School of Education, joninav@hi.is

Teachers in Iceland are faced with challenges to differentiate teaching as they implement a policy of inclusive education. This collaborative inquiry into teaching of mathematics aims at learning to understand how teachers develop their mathematics teaching through participating in a developmental research. Seven primary teachers worked at improving their mathematics teaching and researched their practice together with a teacher educator for three years. Narrative inquiry was used as an analytical tool to study the teachers' learning. In this paper the focus is on one of the teachers and her learning from participating in the project. The results indicate that she gained confidence in teaching mathematics in diverse classrooms while participating in workshops and that collaborative research can support teachers in developing their practice when meeting new challenges in their work.

Keywords: In-service teacher education, developmental research, collaborative inquiry.

Introduction

This paper reports on findings from a three-year qualitative collaborative inquiry into mathematics teaching and learning with the purpose of deepening our understanding of how teachers meet new challenges in their classrooms. The aim was to learn about the processes that emerge through collaborative inquiry between classroom teachers and a teacher educator. In this paper the focus is on one of the teachers', Pála, and her development in teaching mathematics while participating in the project. The research question that will be answered is:

> In what way did Pála affect the learning developed within the project and how is her participation reflected in her mathematics teaching?

The study built on earlier research on teacher development in mathematics teaching in Iceland that revealed that teachers take a passive role in their mathematics teaching and lack experience in creating meaningful learning environments for all children (Guðjónsdóttir, & Kristinsdóttir, 2011; Savola, 2010). They have particularly focused on instrumental understanding as opposed to relational understanding (Skemp, 1976) and emphasised that their pupils learn to carry out the steps of the 'traditional' algorithm (Fosnot & Dolk, 2005). My fellow teacher educators and I have found that if teachers are given opportunities to collaboratively investigate 'with' mathematics and solve mathematical problems, they discover how the different experiences they bring into the community can contribute to their own understanding of the mathematics involved, as well as how individuals learn mathematics (Guðjónsdóttir & Kristinsdóttir, 2011; Gunnarsdóttir, Kristinsdóttir, & Pálsdóttir, 2013). In our work with pre- and in-service teachers, we found that they must be offered opportunities to experience learning that enhances inclusive education. Our results correspond with those of Bredcamp (2004) and Moore (2005), who emphasised that if teachers' work is expected to be aimed at diversity and mutual understanding, they require the opportunity to develop and enhance their knowledge about teaching and learning in an environment that reflects the very same aspects that they are expected to foster in their own classrooms.

Teacher development in inclusive settings

Teaching children mathematics requires teachers to understand how their students learn mathematics and they need to be skilled both in mathematics and pedagogy as well as the knowledge that combines knowing about teaching and knowing about mathematics. In recent years the Nordic countries have emphasized mathematical competences of which eight specific mathematical competences were identified. These form two clusters; the ability to ask and answer questions in and with mathematics; and the ability to deal with mathematical language and tools (Niss & Højgård, 2011). Niss & Højgård also outlined a model for mathematics teacher competency where the ability to develop one's competency as a mathematics teacher as well as the competencies of working with students and others towards professional development were identified. It is important to note that development of teaching in classrooms is dependent both on the teachers' knowledge and their ability to learn together with others, both their students and colleagues.

Attention and awareness are important features of mathematics learning. Mason, (1998) holds that teaching is fundamentally about attention and teachers can enhance pupils' attention by attending to their own awareness. When someone else points something out to us our awareness changes slightly; we become more explicitly aware of some features, and less aware of others. Thus in collaborating with colleagues, teachers are afforded the ideal conditions in which work on their own awareness, which can provide conditions for their students to experience them too.

When gaining competence in teaching mathematics teachers build on their knowledge and experience and an essential factor in this process is the participation in learning communities. In order to be able to support learners in their classrooms in acquiring mathematics competence, teachers need to urge their pupils' to ask probing questions, take risks and learn from their mistakes.

In communities of learning the individual learner draws on knowledge in the community as well as on personal knowledge. Nevertheless the main emphasis has been on individualized learning in response to diversity in classrooms. Schools have thus adopted what Ainscow (1995) called integration by making only a limited number of arrangements for including all learners in classroom activities. Askew (2015) argued that learning communities are more inclusive than taking the individual as the starting point for planning learning experiences. In these communities teachers work with the collective construction of mathematical knowledge while still ultimately addressing the needs of the individuals within that community. This is the same position I took in working with teachers, attending to their diverse needs for improving their teaching and finding ways to work in inclusive ways with diverse groups of learners.

Through collaborative activity a community learns from the thinking, practices, and development of the individual. Important features of such communities are discussions about the mathematics attended to in the classroom. In the communities learners listen to each other's solutions and think about connections to their solutions while helping each other refine their methods and explanations. When learners participate in mathematical practices in whatever way they can diversity is no longer an obstacle to classroom talk. It is thus being enriched through the diversity of learners' contributions (Askew, 2015).

Methodology and methods

The study is a collaborative inquiry into mathematics teaching and learning (Goos, 2004), and the aim is to build a co-learning partnership between teachers and a researcher in order to support classroom inquiry (Jaworski, 2006). The methodology of developmental research (Gravemeijer, 1994) and the 'developmental research cycle' (Goodchild, 2008) guided the cyclic process of the research.

In an attempt to make explicit the 'practice' in which teachers and researchers participate when collaborating, Jaworski (2003) suggested shifting from the notion of community of practice (Wenger, 1998) to that of 'community of inquiry', where teaching is seen as learning-to-develop-learning. In such a community, teachers and researchers both learn about teaching through inquiring into it. In this project the vision was that all the participants would learn about teaching mathematics in diverse classrooms.

For three years I worked with seven teachers at 90-minute workshops on monthly basis. They taught 10 to 12 years old pupils in two neighbouring schools, four were homeroom teachers and three were support teachers that joined them in mathematics classes. The focus of the workshops was on reflection on mathematics, and on mathematics teaching and learning. To help the teachers develop their own understanding of mathematics, we worked with problems that had the potential to promote mathematical activity and thinking as well as to stimulate collaboration where discussions and sharing thinking were meaningful. We also discussed new research on mathematics education and stories from the teachers' classrooms, reflected on their pupils' mathematics learning and considered how their mathematical thinking developed. To learn about the teachers visions for the project and the cultures in their mathematics classrooms I interviewed them and observed their classrooms at the outset of the project, after the first year, and one year after the last workshop. Data was collected of videotapes from workshops, audiotapes from interviews and notes from classroom observations.

Narrative inquiry was used as an analytical tool to study the teachers' learning in participating in this project. It is a way of understanding and researching experience through collaboration between a researcher and participants and to research with practitioners their lived experience as a source of their knowledge and understanding (Clandinin, 2013). The stories the teachers told about their work, at the workshops and in interviews, are the basis of the narrative inquiry. The teachers read the drafts of their narratives and commented on them, and then on the final version.

Findings

Pála had been a general classroom teacher over 30 years in grades 5-7 when she participated in the collaborative project. In her teacher education her focus was on language skills and she had attended many in-service courses about language teaching but only a few about mathematics teaching. As a classroom teacher she taught mathematics to 10-12 year old children.

Emphasis on instrumental understanding

When I observed Pála's classroom at the outset of the study her emphasis on carefully describing the steps of algorithms was dominant. She started the lesson by reviewing homework and then

discussed the content of the lesson. She described carefully to her pupils how to work through the problems in their textbook that she wanted them to solve.

Pála worked closely together with her colleague Dóra, at teaching pupils in their 5th grade classrooms. At our first workshop Dóra wanted to discuss the teaching of 'traditional' algorithms (Fosnot & Dolk, 2005). She had discussed the algorithm of long division with Pála and questioned her belief that is necessary for their pupils to learn the steps of the algorithm. Pála added:

> What we have been reflecting on is, is it bad, does it spoil anything for them? Does it destroy their thinking process, does it stop anything?

Pála was eager to learn more about how to teach children to calculate. She had emphasised the memorising of facts and at her school children were regularly tested on multiplication facts. Dóra had also questioned this tradition and Pála was starting to review her beliefs about instrumental understanding (Skemp, 1976).

Reviewing her own way of calculating

Pála was eager from the beginning to improve her own way of solving mathematical tasks. When we at our forth workshop discussed how many cans there were needed to build a ten storey tower of cans she said:

> There would be 10 here [points to the bottom row of the 10 storey tower she drew]. Then I would count 9 and 1, 8 and 2, 7 and 3, 6 and 4. Then I have 10, 20, 30, 40, 50 and then add these 5 [points to her drawing for each step] and have got 55. I do this to be quick at counting.

Pála was reflecting on her own way of calculating when she said that she did this to be quick at calculating thus attending to her own awareness of learning (Mason, 1998).

As the project developed Pála brought in problems she had been solving with her pupils and wanted to discuss her understanding of the problems with us. At Workshop 15 she told us about her discussions with her pupils about how many handshakes there would be in their class if they all shook hands with each other. The children decided to try this and were quick to realise that they would only shake hands once with each person. They developed a rule that could be used to calculate the handshakes in their group of 15 pupils: 14+13+12+ … +2+1. They then split into smaller groups to test if their rule could be applied to a group of any size. Pála had not thought about the solution of this problem before it was discussed in her class and therefore took an active part in the solution process. By comparing the total handshakes for different number of pupils, they then had developed a formula together. Pála was keen to discuss with us whether the formula n(n-1)/2 could be applied to calculate the handshakes for a group of any size. Pála said:

> I do not understand why this equation works, why this connection. I know it works, we have tried it for many cases. Can you help me to understand why it works? I would like to proceed to work with the children in this way.

By asking us to discuss her experience with us Pála was adding to her competence of learning together with colleagues and in discussing with her pupils she was developing her competence in learning with them. She was also supporting her pupils in developing their the ability to ask and answer questions in and with mathematics (Niss & Højgård, 2011).

I reminded Pála on her earlier addition of consecutive numbers in relation to the tower of cans. Pála said that she remembered it but she still could not understand why the formula she had developed with her pupils worked. We then discussed their formula and why it could be used to calculate the handshakes and in doing so we were inquiring into our own mathematics learning (Goos, 2004) and cultivating our learning community (Jaworski, 2003). I pointed out that she took an active part in the learning process in the classroom. Not only did she learn about the children's thinking but also about her own thinking about the problem. She had given them a problem that neither she nor they knew beforehand how to approach. Then they all started to investigate and look for patterns and developed a rule together. Through these discussions our co-learning partnership was cultivated as we focused on classroom inquiry (Jaworski, 2006).

Learning together with her pupils

Pála was starting to learn together with her pupils by exploring with them in the classroom as opposed to the beginning of our collaboration when she had carefully explained to her pupils, how to solve problems. At our final workshop she shared with us her discussions with her pupils. They had worked with different kinds of word-problems in their textbook. They were required to write their solutions to the problems with algebraic expressions. She gave examples of the pupils' discussions about the problems and how they wrote the expressions. She had recorded these examples in her notebook and now wrote on the whiteboard to show us how the pupils calculated and how she interpreted their thinking about the problems.

We discussed two of the problems:

> Klara is 4 years younger than her brother Kári. Their total age is 18 years. How old is Kári?
>
> A large apple costs 11 ISK more than a small apple. The total price of a small apple and a large apple is 59 ISK. What is the price of a large apple?

Pála had solved the problems herself and her thinking was different from her pupils' but they all came to the same conclusions. She wanted to discuss this experience with us and hear my interpretation of the different ways they solved the problems. She was particularly keen to hear my opinion with regard to the way she had accepted her pupils' way of solving a problem instead of telling them to think about it in the same terms she did.

> Jónína: Pála, you said that the children wrote x+x+4=18 and you wrote x+x-4=18.
>
> Pála: Yes. And for the apples they wrote x+x+11=59 and I wrote x+x-11=59.

We discussed how the value of the unknown variable in Pála's equation was different from the value in the children's equation. Still in both cases they came to the same conclusion about the age of the siblings and the price of the apples. Pála said that all the children in her class were able to solve the word problems by first trying some numbers and then adjusting them until they found the right numbers. Many of them could write the equations and they then supported each other in doing so. Finally Pála concluded: "These were just my thoughts. I found it interesting to see how they understood and thought about this".

When Pála shared this story with us she was cultivating our learning community (Askew, 2015; Jaworski, 2006). But she had also attended to her pupils' way of learning and was now focusing on their way of expressing themselves instead of describing carefully to them the steps they needed to

take as she did to begin with thus making herself aware of her pupils diverse ways of learning (Mason, 1998).

Grouping pupils into ability groups

In Pála's school it had been the custom for many years to group children into groups in mathematics classes based on the outcomes of an end of term test. When the project began Pála and Dóra had divided the 43 children in fifth grade into three groups in mathematics classes. A special education teacher taught the pupils who got the lowest grades, Dóra taught those who got the highest grades and Pála taught the middle group. With this arrangement they were responding to diversity by making only a limited number of arrangements for including all learners in classroom activities (Ainscow, 1995). To begin with Pála was concerned that the pupils in her group were not capable of solving problems without her leading them step by step. Gradually as she became more confident with exploring with mathematical problems herself she started to listen to them and allow herself to join them in their explorations with problems as discussed above.

When Pála shared her experiences of working with her pupils with us we discussed how her approach supported the children's learning like the case with the handshake problem. She told us that some of her pupils understood why the formula could be applied to solve this problem and others did not. They though all understood that they could calculate the total number of handshakes by adding (n-1) + … + 1. We then related to our former discussions of tasks that can be solved at many levels and are therefore suitable to work with in diverse classrooms. Pála was satisfied with this experience and found that she was beginning to trust that all her pupils were capable of more in-depth learning than she had realized before thus acknowledging that diversity is no longer an obstacle (Ainscow, 1995; Askew, 2015).

The final year the project was running Pála and her close colleague, Dóra, had decided not to group their pupils into ability groups any more. They had become confident in investigating in mathematics with their pupils and found that all the children in their classes were capable of learning together and gained from sharing experiences with each other.

Professional development and influence on our project

Pála took an active part in using the tools for professional development that I offered the teachers in our learning community. She visited her colleague's classrooms and discussed with them what they learned from their visits and she recorded her lessons to learn from her communication with her pupils. She also shared her experience from her learning in the classroom with us and gradually started to lead what to focus on at our workshops. Not only did she share this experience with us she also brought in problems she had found elsewhere and asked us to solve them with her.

The project was only planned for one year to begin with. As we approached the end Pála expressed her wish to meet for a second year. She felt that she and the other teachers were just starting to develop their teaching and could not stop when they felt that they were gaining so much from our collaboration. The other teachers agreed with her and our project ran for three years as the teachers wished to extend it for the third year. With her willingness to share her thinking with us and take lead in what to focus on at the workshops Pála shaped the developmental process of the project and affected the 'developmental research cycle' (Goodchild, 2008).

Conclusions

Based on the narratives of Pála's participation in the collaborative project I have concluded that she gained confidence in teaching mathematics in diverse classrooms and that collaborative research can support teachers in developing their practice when meeting new challenges in their work. The sketches from our collaboration are representative for the learning that emerged during our collaboration. In the communities of inquiry we managed to build at the workshops we supported each other in learning-to-develop-learning (Jaworski, 2003) by reflecting collectively on the stories the teachers told of their classroom experiences. From the stories Pála told us we learned how her pupils' competences in dealing with mathematical language and tools were developing as well as their ability to learn about their own learning in working with their pupils (Niss & Højgård, 2011).

By offering the teachers opportunities to experience learning that enhances inclusive education Bredcamp, 2004; Moore, 2005), the teachers were empowered to develop their teaching as was reflected in Pála's learning.

During our three years of collaboration I, as a teacher educator and a researcher learned about teachers' capabilities to develop their own teaching if they are supported in reflecting on their learning of mathematics as well as their pupils' learning. In reflecting on their learning about mathematics teaching my understanding has deepened of the opportunities and challenges teachers meet when including all learners in meaningful mathematics learning.

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Integrating teachers institutional and informal mathematics education: The case of 'Project ArAl' group in Facebook

Nicolina A. Malara[1] and Giancarlo Navarra[2]

[1] Modena & Reggio Emilia University, FIM Department, Italy; malara@unimore.it

[2] Modena & Reggio Emilia University, FIM Department, Italy; giancarlonavarra@gmail.com

After a sketch of our ArAl project devoted to teaching/learning early algebra, we introduce our 'Progetto ArAl' group in Facebook, conceived not only to share and discuss among teachers didactical experiences, theoretical questions and materials but, more in general, to educate in informal way teachers in early algebra. For its features it can be said a non standard group (NSG) in Fb. The main question we put ourselves is: may a NSG become a community of practice*? To find an answer to this question we compared our group with a larger Italian group devoted to mathematics at primary school analyzing the interactions in the two groups launched by some common members. On the base of this comparison we delineate some hypotheses for the management of a NSG as a community of practice, where well known mentors and transparent theoretical guidelines allow the teachers consciously to approach the theory for the practice.*

Keywords: Early algebra, community of practice, informal on line education, teachers' professional development.

Introduction

The ArAl project belongs to the stream of studies devoted to the renewal of the teaching in the arithmetic-algebraic area in the perspective of early algebra. It is characterized by the intertwining among: a) the activation in the classes of innovative didactical paths on early algebra; b) educational processes of teachers based on the critical analysis of the mathematical discussions developed inside the didactical paths. It promotes a relational approach to arithmetic of linguistic and metacognitive type, to be realized through socio-constructive modalities. The classroom activities are based on the negotiation of a didactical contract for the solution of problems according to the principle: "first represent, then solve". For room questions we cannot discuss deeply our theoretical frame (we refer to Cusi et al. 2011 and related references), here we simply recall some key aspects of it: (i) the plurality of representations of a given quantity, beyond the canonical decimal representation[1]; (ii) the identification and making explicit algebraic relationships and structures underpinning concepts and representations in arithmetic[2]; (iii) the initiation to the essential algebraic cycle: *representing*,

[1] For instance, the number twelve, has a canonical representation in base 10, i.e. 12, but expressions such as 3×4, (2+2)×3, 36/3, 10+2, 3×2^2 are other ways to express the same quantity, we call them non-canonical representations of 12, each has its sense related to the process that characterizes it and offers pieces of information about the number. Being able to fluently shift among these forms allows pupils to easier recognize structural similarity among different numbers and to build the basis for understanding scriptures as a.b, -4p, x^2y, k/3).

[2] For instance to see the equal sign, in writings such as 3+4=7, not only in its *procedural* sense of connection between an operation and a result, but in its *relational* sense, as an indicator of equivalence between *two different representations of the same number*).

transforming, interpreting (Bell 1996) through *the devolution to the students* of*: a) the formalization of verbal relationships* individuated during explorative numerical activities (process named by us *algebraic babbling)*; (b) *the interpretation of simple algebraic sentences* both in themselves and with reference to a given context[3]; (v) the stress on natural language as didactical mediator in the slow construction of syntactic and semantic aspects of algebraic language[4].

Our work with and for teachers has always been realized in a community of practice, or better of inquiry in Jaworski sense[5], where the practice of the researchers and the one of the teachers meet, compare and develop in co-partnership and where, in addition to theory, methods and aims, values and expectations are shared (Cusi & Malara 2015). Because of the teaching in an early algebra perspective requires in the teachers, mainly the ones of primary school, a deep rebuilding of knowledge, beliefs, behaviours, and manners in the class, we have conceived specific modalities and apposite tools for teachers education. We simply recall here our *Multicommented Transcripts Methodology*, we have enacted to promote in teachers awareness of their own ways of being in the class and to guide them in managing mathematical discussions. Key tools of this methodology are the teacher's transcriptions of the classroom discussions enriched by written multiple comments (by tutors, maths educators and other teachers), *the MTs*. The joint reflections on each MT attain a shared development of the theoretical frame, of the classroom methodologies and of the teaching materials that shall create the basis for the teachers' professional evolution. The productions of MTs became in the time a distinctive character of the Project Aral membership.

The ArAl Project Group in Facebook

Along the years many times we have been asked to make available ArAl materials to a greater number of teachers; for this in 2014 we opened the 'ArAl Project' group in Facebook. Fb is mainly used as a way to share experiences, practices and materials among teachers and other professionals (see for

[3] For instance, to recognize that the sentence 85=4×21+1 represents 85 through the quotient and the remainder of its division by 4 (or by 21), and in the same time - looking for the letter which stays at the 85° place in a sequence generated by the ABCD module – to recognize that the same sentence allows to understand what the letter is (the pupils have to interpret the term 4×21 as the part of the sequence done repeating 21 times the module and the remainder as the number of place of the letter in the successive module).

[4] For instance, two pupils express in natural language, and then translate in mathematical language, their different ways to calculate the number of pearls in the necklace ○○●●●○○●●●○○●●●○○●●●○○●●●○○●●●: the first pupil says "I counted white and black pearls and I added them" and translates: 2×6+3×6; the second says: "I saw that there are 6 groups, each group has 2 white and 3 black pearls and I multiplied 2 plus 3 by 6 and wrote (2+3)×6. The comparison of the two sentences allow the pupils to gain experience about the distributive property.

[5] We recall that a community of practice (CP) is constituted by a group of people who share a craft and/or a profession. The group can evolve because of the members' common interest in a particular domain or area, or it can be created specifically with the goal of gaining knowledge related to their field (Lave and Wenger, 1991). Jaworski (2003), referring to the joint work developed between maths educators and in service teachers about classroom teaching-learning processes, introduces the construct of Inquiry Community (IC) and underline that what distinguish a CI from a CP is that all the participants engage with inquiry as a tool to develop meta-knowing, a form of critical awareness that manifests itself in inquiry as a way of being.

example Bodell & Hook 2011, Manca & Ranieri 2014), but in the last years it has also been used in educational activities for teachers (Staudt et al. 2013, Van Bommel & Liljekvist 2016). In our case, the initial idea was to spread themes and principles of early algebra among teachers and to motivate and help them in approaching it in the class but also to observe new, spontaneous didactical experiences, arising under the stimuli offered by the ArAl institutional courses. We believe that the Fb group can be a way to integrate institutional and informal education offering the teachers new occasions to promote their professional development. We started inviting expert teachers collaborating since long time in our project to become supporters of the group and to share their experiences with the teachers, recently involved in ArAl courses promoted by the schools, who have been invited to become followers of the Fb group. The fundamental methodological choices in managing the group are: our daily on line presence and prompt reactions to the teachers posts; the stimuli offered by the expert teachers posts through videos or pictures of classroom activities; our periodical posts about: mathematics questions and related theoretical references; examples of innovative activities, equipped by MTs, papers, powerpoint presentations for deepening the discussed questions and stimulating free experiments among the followers. The posts in Fb are classified in: 'like-agree' interventions; 'propositive-constructive interventions', doubtful-skeptical interventions; moreover meaningful sets of interventions related to interesting mathematical teaching questions are collected and commented in files put in our website. Periodical analysis of the data allowed us to highlight the interplay between our interventions and the teachers' ones, and to reflect on the teachers change. We discuss their evolution according to three temporal phases.

First phase (scholastic year 2014–15**).** In this first period, in front of a small group of teachers (in most part coordinators in the schools of the ArAl project activities), who were very active in posting documents related to their class activities as well as in commenting other posts, the other members were not so active, and their comments often were short and superficial. These teachers appeared *awed*: the most part of them had a feeble or null control over the early algebra topics and the strong difference among the competence of the expert teachers in the group and their knowledge in the arithmetic-algebra area did not encourage them to do more 'important' interventions. At the same time every day new members enrolled to the group. Some more expert teachers, members both in our group and in other groups for maths teaching, suggested us to visit them and in particular invited us to take part into the group 'Mathematics at primary school' (one of the most numerous and active Italian groups on maths teaching in the web, more than 5000 members), to offer our interventions whenever we seemed appropriate to do so. We call this last group a Standard Group (SG), in the sense that there are not pre-established leaders and that the exchange takes place freely through the sharing and negotiation of the individuals' knowledge. The comparison with the SG and other groups dealing with teaching issues brings in evidence that ArAl Project group is different from them, mainly for two reasons: (a) it deals with a well defined subject area, early algebra, it is structured according to a clear theoretical perspective for facing it, and it proposes methodologies, problematic situations, tools fitting with this framework; (b) it is daily supported by us and it is animated by experts teachers who may act as mediators among the members. Therefore we call it a non-Standard Group (NSG).

Second phase (scholastic year 2015-16): In this period we had continued to enter, as previously, examples of didactical activities, MTs, papers, powerpoint presentations but, at the same time - on our initiative or invited by the teachers - we had become more active in intervening on posts both in SG and in NSG. By way of example of this change in our strategy we focus on an episode: a post

inserted by a teacher which received great attention (154 likers and 75 comments), started in the SG and developed, through reciprocal sharings, also in the NSG. The initial post contained a link to a note inserted in the Unit 12 of ArAl project and presented in the form of FAQ in www.progettoaral.it site. In this post it is developed a critical analysis of a typical Italian school practice, supported also by many textbooks, for introducing in primary school the decimal system of representation of the natural numbers: the indication of the units with the letter 'u', the tens with 'da', the hundreds with 'h' and so on (the so-called 'marks'); thus there follow improper equalities such as 653=6h+5da+3u. Because of the impasse generated in the SG, a follower - a member of both groups - asked us for an intervention on this topic. In a comment of a theoretical and linguistic type Malara wrote:

"The symbols h, da, u represent words of Italian language. They are categorical terms that refer to orders of magnitude and they are used as 'indicators of quantity'. They are useful for bringing the pupils to shift from the experience with the abacus - where an assigned quantity is split into opportune multiple of powers of 10, operating for successive groupings of 10 – to the representation of the result of this operation through a string of symbols, each between 0 and 9 (extremes included), from which the name of the given quantity was born. This means that, for example the string 6h, 5da, 3u synthetizes the verbal sentence 'the quantity is constituted by six hundreds, 5 tens and 3 units' which generates the name of the number 653. The translation into the arithmetical language of this verbal sentence requires the conversion of the term 'hundreds' in the arithmetic operator '×100', the term 'tens' in '×10' and the 'units' in '×1' and the conversion of the connector 'and' in the operation of addition '+'. So, the total verbal sentence is translated into '3×100+5×10+3×1'. The sentence 653=6h+5da+3u is improper because it mixes the two languages, verbal and arithmetic, and confuses the metacognitive plan with the operational one".

While the debate on this issue was developing in the GS, many teachers did not understand why in the ArAl project sentences as 653=60+50+3, 653=6×100+5×10+3×1, $653=6\times10^2+5\times10^1+3\times10^0$ were proposed as correct and not the one they used, and opposed resistance to accept the explanation that 653=6h+5da+3u is to be discarded because it is not a correct representation in mathematical language. To facilitate this understanding, the improper mingling between verbal and arithmetic languages has been pointed also using examples of verbal sentences with words in two languages; the discussion then focused on the correct and incorrect representations of a natural number, the concept of 'equality' and on the meanings of the symbol '='.

Third phase (June 2016 to now) The analysis of the dynamics arisen and the kind of the comments posted in NSG and SG led us to the identification of some thematic questions who have given us valuable indications on a question that we did with increasing frequency: may a group with the characteristics of ArAl group become a *community of practice*? If the answer is yes, in which ways may this happen? How may a gradual constitution of a library of shared knowledge be put in place? This leads us to identify some answers to these questions concerning the prevailing attitudes of teachers who enroll in these groups. We discuss them articulating in the following points.

Features of a SG and of a NSG

Members of a SG feel all *equal*: they exchange information, questions, requests without demanding to receive in-depth and substantive answers; they hope to share with their peers working suggestions which are at the level of their knowledge and of their willingness to get involved. Individual growth stems from the strength of exchanges and the wealth of experiences put into circulation. Internal

leaders emerge, who often are recognizable more by the diligence than by the quality of interventions; they often are the most convincing not for their knowledge but because they expose themselves more than others, writing frequently comments. Members may find appealing ideas for new activities but their enthusiasm is not supported by an adequate knowledge; they express insecurity when discussing their colleagues' proposals of those embryos of new ideas. Everyone feels free to comment on impulse. On the contrary, a NSG as 'Progetto ArAl' gives the majority of subscribers some (cultural and psychological) constraints that limit them in exposing their contributions. The same dynamics occur in a working-group in which an expert is present. But then: if it is understandable, for the reasons explained, that a GS exceeds 5000 subscribers, how has to be interpreted the success of our NSG that in two years is approaching 1000 members? The answer could be given with a metaphor: the members *have the impression of living a moment of institutional training*. They know that in the NSG there are experts involved in the discussions, extemporary comments should be avoided and the participants are invited to put forth questions and to interact with others. At the same time they know that there are not 'free rounds' (as often happens in the SGs, where a rich variety of cues are offered but they often remain at a messy, unspecified, superficial level) and should deal with the theoretical aspects through an *individual study*. In fact, at the base of ArAl Project there is an organic vision that aims to propose a framework on early algebra, offering the participants opportunities to reflect on knowledge, beliefs, stereotypes. They accept a commitment which attracts them: to avoid free, trivial conversations or Pindaric flights.

How can personal experiences, beliefs, inclinations be influenced by interventions based on strong theoretical references?

The interventions on SG highlight different objectives between mathematics educators and teachers: basically, specialists focus their interest on the discipline, the teachers on their pupils. These different perspectives can create misunderstandings or misinterpretations. Then, in the NSG, mediations between them are necessary, that is: on one side the founding principles of mathematical knowledge – in our case of early algebra - have to be respected, but on the other side, at the same time, it has to be offered to the teachers a certain 'serenity' about the fact that deepenings and changes of perspectives in teaching do not affect learning, but on the contrary pave the way for subsequent extensions of mathematical concepts. There is a strongly felt concern that pupils do not understand or that a concept is too difficult or inappropriate (of course this concern is correct because teachers have the responsibility of the learning of their students, so they constantly consider the difficulty and feasibility of new proposals).

Limited capacity to distinguish between different types of knowledge

The posts and comments put in evidence that most teachers, along the years, reach their convictions grounding them more on the accumulation of heterogeneous strategies, methods, tools than on their consistency. One of the consequences of this behavior is that teachers confront themselves superficially with the theoretical references. For example:

> Elena: I think sometimes that famous 'didactic contract', of which we all partake the negative effects on pupils, has been moved up on teachers: "It is so, Tom said, Dick reiterated it"; someone makes it [i.e. the didactical contract] arguing and expressing his/her opinion (experience counts,

anyway!); someone else makes it ‘getting on the chair'. But: be they teachers or pupils or propagandists or colleagues, always 'didactic contract' is.

To what is Elena referring when she writes “the negative effects on pupils”? Her so peremptory statement was not reconsidered in the later comments: what does this mean? It could indicate that it has not been understood, or that it has been read superficially, or that it is not shared, or that it is an unfamiliar concept and no colleague wants to explore it. This short episode shows indeed that there are interactions between members, but in general they go on without reaching a real conclusion; at most, members achieve a superficial agreement, or a generic praise, or they remain on their positions. It would have been important to ask Elena what she means with this term (originally it is a theoretical construct by Guy Brousseau). Probably such statements would not have been made in our NSG. This might be a limit for the group because many convictions would not be expressed for a kind of compliance towards the coordinators experts. A low understanding of the key aspects of mathematics education (at the primary-secondary school level) favours the choice of cues - references, materials, paths, methods - that fit with the convictions and the personality of the teachers more than with the organicity of the knowledge taught. In this way, those facilitators that favour the perspective of *making* are privileged. The weak capacity to connect effectvely the suggestions of experts and mentors implies that one prefers a ‘do-it-by yourself’ shared with those who are felt as fellow-travellers: if an activity, a text, a method are exalted or defended by other members of the group, they may be adopted, or at least tested. Often, the length of an experiments is short because the activities are heterogeneous, have ‘little oxygen’ (the interest on them goes out early); almost immediately they are put aside without any reflection in general terms, mostly on the basis of local success achieved by pupils (or, more trivially, because they appear ‘nice’).

New characters emerging in NSG

The dense interactions developed in the NSG together with the offered theoretical and practical supports brought some new attitudes and awarenesses in the members of NSG. The members begin to understand that a new approach to the arithmetic and algebra teaching lies on a different role of the teacher. As to this a decisive importance assumes what J. Mason has called *the art of noticing* the classroom micro-situations for being ready to adopt the opportune micro-decisions (Mason 2002), intertwined with the attention to the *languages* and to the continuous recourse to the *argumentation*. Thanks to our frequent interventions where we underline that: a) a math teacher has to control a *plurality* of languages and that also a formal language must be monitored at two levels, the *semantic* one about the meanings and the *syntactic* one about the structure of the sentences into play; b) the weak control over grammatical/syntactical aspects of a sentence in mathematical language leads to temporary and unstable jargons in which the meanings assigned to the symbols are dictated by an apparent common sense that reduces the difficulties, promotes an immediate but feable understanding that leaves the problem unsolved; we observed in the activities posted by many members a bigger attentions forwards the translations questions between verbal and fromal languages and the increasing use of argumentations in their students. From a methodological point of view, thaks to our suggestions,when the teachers publish at the NSG the post of an activity, they begin to understand that it is not enough to insert some captivanting images, but that it is necessary to equip them with a presentation that synthetically shows the activated competencies and that includes the most meaningful protocols, the path in which the activity is inserted, how it develops in the next steps, the

theoretical references (ArAl Units, items on the website, Powerpoint presentations, papers). Our idea is to slowly bring them to approach the MTs methodology. An important contribution in this sense is offered by an increasing number of members the NSG, who are not involved in ArAl experimentations but following the project in a convincing way (teachers educators, mentors, collaborators of publishing houses, members of other research groups). Thanks to this people the posted comments begin to be richer and meaningful; the authors express their ideas also asking for experts' suggestions aimed at promoting new and more adequate behaviours for teaching arithmetic/algebra in a *relational* perspective. So, posts and comments begin to produce virtuous relations which gradually enhance the system: the posts induce comments of increasing quality, which generate important feedbacks in the organization of the succesive posts.

The recent mutations observed in the NSG members' posts delineates a new character of their participation which appears in tune with our aim to build a shared identity in the NSG and effective in offering contributions which can bring it to become *a community of di practice*. As to this, particularly meaningful appear the recent initiatives generated by the NSG discussions concerning the publication in the ArAl project website (http://www.progettoaral.it/) of two documents, respectively devoted to: (1) the most interesting classroom episodes presented by the NSG members, with the main related comments; (2); the early algebra papers written by members external to the project and inspired by our previous productions. Next to this we have to consider the request expressed on the web by several members of NSG to organize some ArAl meetings of one or few days to allow the participant know themselves *de visu* and to plan some common work. It seem us that these new tends in NSG may generate inside the group, mainly with the more sensitive and expert members, an embryo of a *community of inquiry*. In this frame institutional and informal ArAl educational initiatives are developing important merging points.

Final considerations

A NSG as 'Progetto ArAl' may initially disorient new participants, but its own structure can be considered its force because many of them declare that they appreciate the possibility to join to a group where experts favor an organization of knowledge according to transparent and shared principles. On the base of the observations made, we formulate some key points related to early algebra for the management of the NSG so that it can become a significant community of practice in this field: (a) to help teachers understand not only merits and limitations of instruments and didactical strategies that they implement along the years, but above all the importance of their coherence and adherence to a set of theoretical principles, such as: the importance of languages and, consequently, of the translation between them; (b) to bring teachers to consider the perspective of the generalization since the first years of primary school, highlighting the structural analogy between representations of the various occurrences of a phenomenon and guiding their modeling; (c) to propose any time, during the discussion on the issues raised by the members, gradual general frameworks, accompanying them with clarifications, insights, extensions which give answers for doubts, perplexities, conflicts emerging from the discussion. The basic idea is that the theory should be gained through a gradual process of refinement of knowledge in a continuous exchange among the members of the group, adapting explanations and deepenings to the difficulties or to the resistances and injecting now and then proposals of mini-workshops.

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Prospective teachers interpret student responses: Between assessment, educational design and research

Maria Mellone[1], Roberto Tortora[1], Arne Jakobsen[2] and Miguel Ribeiro[3]

[1]University of Naples Federico II, Italy; maria.mellone@unina.it, rtortora@unina.it

[2]University of Stavanger; Norway; arne.jakobsen@uis.no

[3]Research Centre for Spatial and Organizational Dynamics, University of Algarve, Portugal; UNICAMP; Brazil; cmribas78@gmail.com

We present a study about the development of interpretative skills in prospective teachers. In particular we discuss a kind of tasks designed by us for teacher education, containing the request of interpreting students' answers. The task in this study was built on an item concerning the sum of powers of 10 and was proposed to a group of prospective secondary teachers who were attending a Math Education course. The task was first faced individually by them and then discussed in group. We present the interpretations proposed by two prospective teachers before and after the collective discussion, in order to reflect on the differences in terms both of mathematical knowledge put in play and of attitudes exhibited.

Keywords: Teacher education, prospective secondary teachers, interpretation of students' answers, arithmetic and algebra.

Mistakes are the portals of discovery (James Joyce)

Introduction

In their daily practice, teachers are required to continuously interpret students' responses and productions. This not only assists them in evaluating their difficulties and achievements, but also allows to plan the next steps of the teaching activities. Therefore, this "interpretation activity" is one of the most crucial (and often most difficult) tasks teachers perform. Empirical evidence suggests that the true quality of a mathematics teacher stems largely from his/her ability to interpret students' productions, along with a flexible attitude to redesign the teaching approach based on them. However, the ability to make sound and accountable interpretations is rarely recognized as a crucial goal in teacher training. Moreover, in a previous research study (Ribeiro, Mellone, & Jakobsen, 2013), we observed that teachers do not naturally develop this ability as they gain work experience. So, presently, our research questions are the following. Is it possible for a teacher to acquire these interpretation skills or is it a matter of innate talent? In the first case, how can a prospective teacher develop this ability? Should we engage prospective teachers in mathematical discussions concerning the interpretation of students' reasoning?

Guided by our conviction that it is possible for a teacher to develop this ability, as a part of a joint research project, we explored a particular type of tasks we have conceptualized. In one part of these tasks, we asked the prospective teachers to interpret some students' responses to a problem and reflect on possible feedback they could provide to each student. This exercise had a twofold aim: to support prospective teachers in developing the skills required for interpreting and commenting on student work, and to investigate to what extent the particular mathematical knowledge and skills possessed by the (prospective) teachers support or hinder them in their interpretations and

"constructive" reactions. Our analysis shows that the abilities to interpret and to design an educational activity based on students' productions are inhibited in prospective teachers with a poor mathematical knowledge, due to their limited understanding of the subject and lack of appreciation of various ways that problems can be solved (Ribeiro, Mellone, & Jakobsen, 2013). In particular, this is true for prospective primary teachers, many of whom have poor mathematical knowledge. However, we have also observed that prospective secondary mathematics teachers—who have studied more advanced mathematics during the three years of their Bachelor in Mathematics degree (Jakobsen, Mellone, Ribeiro, & Tortora, 2016)—also struggle with this kind of work. This led us to posit that the ability and knowledge to interpret student work depends not only on (prospective) teachers' mathematical knowledge and its components, but also on their attitudes and beliefs toward mathematics and its teaching.

The tasks described herein are presently used in our Mathematics Education courses in three different modalities. First, we ask prospective teachers to individually solve the problem, before interpreting and reflecting on some selected students' productions, and finally engaging in group discussions on the mathematical aspects involved in these students' productions. Given that the prospective teachers (both primary and secondary) have difficulties in interpreting and in giving meaning to some students' answers, it was necessary to first assess their ability to solve problems that these students are given. This was informative, as some prospective teachers struggle with providing constructive feedback to the students even when they do not encounter any difficulties in solving the given problems for themselves. The findings yielded by this first phase of our research were utilized in the subsequent mathematical discussions of students' solutions and corresponding teachers' interpretations. These group discussions were helpful to most prospective teachers, as they were able to gain new perspectives on students' work and strategies that can be employed in teaching.

In this paper, we present the interpretations, given by a group of Italian prospective secondary teachers, of students' responses concerning a problem where sums of powers of 10 are involved. (see Jakobsen et al. (2016) for details). Here, we present analysis of the interpretations teachers gave before and after the mathematical discussions, in order to document their progress, as most demonstrated more sophisticated attitudes and greater mathematical knowledge following group discussions.

Theoretical framework

In order to characterize and study the features of teachers' interpretations of students' productions, in some of our previous work (see for example Ribeiro, Mellone, & Jakobsen, 2016), we have introduced the notion of *interpretative knowledge*, framed within the general Mathematical Knowledge for Teaching (MKT) framework (Ball, Thames, & Phelps, 2008). We define interpretative knowledge as the knowledge that allows teachers to give sense to pupils' answers, in particular to "non-standard" ones, i.e., adequate answers that differ from those teachers would give or expect, or answers that contain errors. We posit that interpretative knowledge is closely related to the ability of teachers to support the development of pupils' mathematical knowledge, starting from their own reasoning, even if students' ideas are incomplete or non-standard. Some similar ideas are implied in the notion of *discipline of noticing* (Mason, 2002). In particular, our construct encompasses the idea of teachers working "on becoming more sensitive to notice opportunities in

the moment, to be methodical without being mechanical" (Mason, 2002, p. 61). The development of pupils' mathematical knowledge starting from their own reasoning is, in our view, only possible if the teacher activates a real process of interpretation, shifting from a simple evaluative listening to a more careful hermeneutic listening (Davis, 1997).

In this sense, the notion of interpretative knowledge incorporates into the MKT framework the idea that errors and non-standard reasoning are considered as learning opportunities (Borasi, 1996). Moreover, the content of interpretative knowledge shapes teachers' ability to make informed choices in contingency moments (as defined by Rowland, Huckstep, & Thwaites, 2005), in order to respond to and deal with non-planned situations. In that sense, we felt the need to incorporate the role of beliefs and attitudes pertinent to the use of mathematical knowledge (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013).

With the goal of better understanding (prospective) mathematics teachers' act of interpretation, we characterized their interpretations of students' productions and attitudes, using the following three categories: (i) Evaluative interpretation: a process through which the teacher determines congruence between pupils' productions and the mathematical scheme of correct answers he/she has; (ii) Interpretation for the educational design: the manner in which the teacher designs educational steps based on the work produced by the students; (ii) Interpretation as research: teacher's willingness and ability to revise his/her mathematical formalization in order to ensure that it is coherent with students' productions (even when these seem in conflict with the traditional mathematics taught in school).

In Webster dictionary, "interpretation" is defined as "The act of interpreting, explanation of what is obscure"; however, it is also defined as "An artist's way of expressing his thought or embodying his conception of nature." This last definition stresses the potential creative nature of the act of interpreting that is in our context perceived as the potential new mathematical knowledge that can be developed owing to the process of analyzing students' productions.

Context and method

For several years, we have been studying the nature of (prospective) teachers' interpretative knowledge (e.g., Ribeiro et al., 2013; Jakobsen et al., 2016) by exploring the manner in which prospective teachers respond to specific interpretation tasks. In this design study the tasks are developed after the typical cycles of redesign of the design study method (Cobb et al., 2003). In their present form, essentially consist of three steps: (i) the teachers are initially required to solve a mathematical problem by themselves; (ii) they are given several students' productions in response to the same problem, some containing errors and some mathematically valid but following less standard procedures, which they are asked to interpret; and (iii) teachers are prompted to provide what they deem would be appropriate feedback to these students based on their solutions. The teacher trainees are asked to address these requests individually and in paper format (they are usually given 90 minutes to complete all three steps). In the next phase of the study, the educator engages all prospective teacher participants in a collective mathematical discussion (which again typically lasts about 90 minutes). The framework of the mathematical discussion is based on that proposed by Bussi (1996), as the aim is to allow the group of prospective teachers create a polyphony of articulated voices on the mathematical object starting from the interpretation of a

student's production. Upon completion of the group discussion, the prospective teachers are asked to provide in writing a new individual interpretation of the students' productions, allowing the researchers to determine if any progress has been made.

The task utilized in the present study is depicted in Figure 1, and was adopted from the annual Italian national assessment (2010-2011) for grade 10 released by INVALSI (Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione). A group of 34 fourth-year master students of mathematics enrolled in a Mathematics Education course took part in this investigation. Since most of these students are going to become secondary school teachers, we consider them prospective secondary teachers.

In our previous study, we focused on the interpretation these prospective secondary teachers gave to their students' productions (Jakobsen et al., 2016). Our analysis revealed that they experienced problems in mobilizing their mathematical knowledge for interpreting students' work. Indeed, while they were able to "see" some of the mathematical aspects involved in the solutions to the problems their students proposed, they seemed unaware of many important aspects relevant to mathematics teaching and problem solving.

The expression $10^{37} + 10^{38}$ is also equal to:

A. 20^{75}

B. 10^{75}

C. $11 \cdot 10^{37}$

D. $10^{37 \cdot 38}$

Figure 1: Item given to students to solve

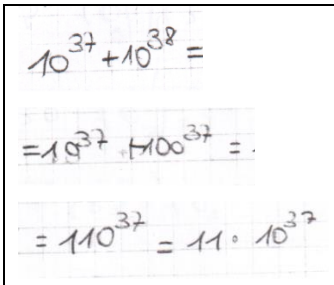
In the next section, we will present two out of seven students' productions that were included in the task given to the study participants (for their selection, see Mellone, Romano, and Tortora (2013)). This will be followed by the interpretations of these students' productions, provided by two prospective teachers—to whom we refer as Rossella and Gennaro (pseudonyms)—before (BF) and after (AF) the mathematical discussion.

Interpretation of two students' productions

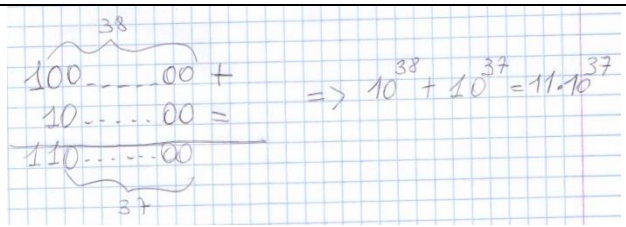
The following brief analysis of the students' productions included in the task aims to elucidate our reasoning behind the decision to deem these two students' productions effective for exploring prospective secondary teachers' ability to interpret the work of others.

Emanuela (Figure 2a) obtained the correct result, despite making three errors in her work: the first and the last can be described as lack of use of parenthesis and the second can be seen as a wrong application of linearity. Ciro (Figure 2b) arrived at the right answer using the arithmetical algorithm of the arrangement of the decimal representation of the numbers in column. In his responses, we can also recognize his perception of the algebraic structure connected with more general ideas implicit in calculus. Indeed, Ciro's use of the ellipses reveals the potentiality of generalization of his

response to other two consecutive powers of ten, and not only to the particular case given (as one of the mathematics students that took part in our study noted during the collective mathematical discussion).



(a) – From Emanuela's protocol



(b) – From Ciro's protocol

Figure 2: Two out of seven students' productions

Prospective teachers' comments on students' productions before and after mathematical discussion

Reflections on Emanuela's work

When individually interpreting Emanuela's response, prospective teachers seemed to experience difficulty in trying to understand the steps she used in arriving at the solution (see Jakobsen et al., 2016). Here, we focus on the interpretations given by a secondary prospective teacher —Rossella— before (BF) and after (AF) the discussion[1]:

The second prospective teacher, Rossella, shared the following:

Rossella (BF): There is no application of rules; it is pure invention. I don't know what I would say to the girl, but I would think that she had copied the solution and then tried to invent a justification.

Rossella (AF): Even if Emanuela's answer is correct, her arguments are far from being mathematically founded. Still, we can observe an interesting aspect in them, namely that if we repeat the steps with two powers having the base different from 10 and the exponents differing by one, we get the correct result. I followed this approach using different numbers, just to test this reasoning, which allowed me to assert that Emanuela's thought process appeared to work. More specifically, when you change the bases and use two consecutive numbers as exponents, or even not consecutive, her logic gives the correct result.

$$37^{25} + 37^{26} = 37^{25} + (1369)^{25} = 1406^{25} = (38 \cdot 37)^{25} = 38 \cdot 37^{25}$$

$$\left(\frac{1}{10}\right)^3+\left(\frac{1}{10}\right)^4=\left(\frac{1}{10}\right)^3+\left(\frac{1}{100}\right)^3=\left(\frac{9}{100}\right)^3=\left(\frac{9}{10}\right)^3\left(\frac{1}{10}\right)^3$$

$$2^{-3}+2^{-2} = 2^{-3}+4^{-3} = 6^{-3} = 3 \cdot 2^{-3}$$

$$x^n + x^{n+1} = x^n + (x^2)^n = (x+x^2)^n = ((x+1)x)^n = (x+1)x^n$$

[1] Of course, the students' words were translated from Italian into English. We tried our best to retain the exact expressions, including some errors, but some nuances are inevitably lost.

> In other words, the above three errors, made in sequence, yield a correct result that does not depend on the particular numbers used. This observation has prompted our reflection on how mathematics is usually managed. We are used to judge, without hesitation, arguments such as those mentioned above as wrong and completely invented. However, after our discussions, we were of view that errors should be seriously considered and, if possible, exploited as a stepping stone toward the construction of new knowledge. This is exactly what I tried to do in order to bring out something new from Emanuela's reasoning. Indeed I also tried to build a new system of rules for powers according with her reasoning. Several attempts have convinced me that this is not possible. Thus, my conclusion is that no new knowledge about the rules of powers can be derived from Emanuela's suggestions. However, her errors can be an invaluable tool to stimulate discussions and to highlight the need for a true comprehension of the rules of powers, which are often hard to grasp for students.

Reflecting on Rossella's BF and AF words, we can observe a change in the attitude when interpreting student's work. The first interpretation is an evaluative one—there is no effort to understand the rationality of Emanuela's steps. This results in Rossella's bias towards a solution like that Emanuela offered (referring to an her possible unfair behavior), which can be due to fear of moving toward an educational path she cannot (immediately) control.

The interpretation given by Rossella after the discussion is markedly different. She not only made an effort to derive a generalization from the errors in Emanuela's production, but further concluded that the three steps Emanuela used in solving the problem will give the right answer with other bases and exponents. Rossella thus went beyond the simple observation that Emanuela's steps are not mathematically sound, as she investigated the possibility to build "a new system of rules for powers according with her reasoning." For this reason, her second interpretation can be considered a form of interpretation as research.

Reflections about Ciro's work

> Gennaro (BF): Ciro reached the correct answer by a more practical method than those employed by his peers. In addition, the formalism seems original. He appears to have a strong expertise in the calculations with powers of 10, which highlights their significance and the importance of handling them correctly. Still, his method seems limited to powers of 10. It would be interesting to see how Ciro would proceed if presented with a different base. I think that Ciro's protocol could be used as an opportunity to explore differences between the properties of powers of 10 and those of other bases.

> Gennaro (AF): Ciro's argument is of an arithmetic character. Nonetheless, it allows us to appreciate some deep algebraic insights. Moreover, although it seems confined to powers of 10, it can actually be generalized to any base, if one represents the number in the base of the power. Hence, from Ciro's production going further, it would be possible to study the tables of operations in different bases, or even the divisibility rules in bases other than 10.

In his first interpretation of Ciro's protocol, Gennaro appreciates the originality of his method, while noting that it is limited to powers with base 10. Based on this observation, Gennaro proposed possible questions and issues that could be explored with Ciro and the rest of the students in the classroom, starting from his production. For this reason, Gennaro's first interpretation is aimed at educational design.

As with Rossella, whose interpretations we analyzed previously, the comments Gennaro gave on Ciro's protocol after the discussion shifted in focus. First, there is a subtle distinction between arithmetic and algebra that could be investigated and debated endlessly. Moreover, Gennaro's comments reveal his awareness that Ciro's method can be applied to other bases (indeed, 100…0 always represents the *n*-th power of the base). This fact was observed during the collective discussion by another prospective teacher, and for Gennaro, this discovery was so important that it became part of his new written interpretation. In other words, Gennaro's knowledge and interpretation benefitted from the mathematical discussion on Ciro's production. He reconceived the systems of representing numbers in different bases, which motivated him to explore the true meaning of digits, as well as of strings of digits. For this reason, Gennaro's second interpretation is perceived as interpretation as research.

Conclusive remarks

We started this paper by asking if mathematics teachers can develop the ability to interpret their students' productions in order to flexibly redraw the mathematical learning path, or if this should be considered as an innate talent. We are convinced that it is not only possible to develop this skill but is highly desirable. The observed difficulties these prospective teachers experienced when giving sense to student productions, along with the findings yielded by extant studies, indicate that the development of this ability requires a special attention in teacher education. These first results about our proposed method of working with prospective teachers appear to support its effectiveness. It stimulates prospective teachers' interpretive and critical skills and increases knowledge they must possess in order to teach effectively, taking into consideration the specificities of such knowledge. The value of our method stems from the nature of interpretive tasks involving student productions, as well as subsequent discussions among peers under the guidance of an expert on these interpretation tasks.

Our analysis of the interpretations given by two prospective teachers, before and after the collective discussion led by the educator, clearly demonstrates changes in terms of both their attitude and mathematical knowledge or awareness. We can hypothesize that the collective discussion mobilized mathematical knowledge that was previously present, but probably not put in play, and it also support the development of new mathematical knowledge, like for Gennaro. However, the improvements we witnessed were also due to the change in attitudes and beliefs supported by the discussion, and of course by the attitudes and beliefs incorporated in the educator's practice.

Still, our work leaves many questions to be answered in future research. It would be interesting to evaluate the sustainability of these changes, for example, by following the work of these prospective teachers in their future educational practices. Moreover, analysis of the mathematical discussion on interpretative task needs to be developed in order to clarify its features and dynamics.

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Investigating potential improvements of mathematics student teachers' instruction from Lesson Study

Reidar Mosvold[1], Raymond Bjuland[1], & Janne Fauskanger[1]

[1]University of Stavanger, Faculty of Arts and Education, Stavanger, Norway;
raymond.bjuland@uis.no

This paper reports from a project where Lesson Study (LS) was implemented in the field practice component in four subject areas of two teacher education programs at one Norwegian university. Previous analyses of data from interviews and mentoring sessions indicate that mathematics was a challenging case that makes it interesting to investigate further. In the present study, we analyze classroom observations with the Classroom Assessment Scoring System (CLASS) in order to investigate potential improvements from the intervention. The results indicate that there were no significant differences between the control group and the intervention group. Possible explanations for this are discussed and implications for future implementations of LS in field practice are suggested.

Keywords: Mathematics teacher education, field practice, Lesson Study.

Introduction

This paper has a focus on developing student teachers' ability to teach mathematics through LS. When Stigler and Hiebert (1999) published the results from their comparative study of mathematics teaching in Japan, Germany and the USA, they argued that there was a "teaching gap" among these countries. The teaching quality of the Japanese classrooms appeared significantly higher than in German and US classrooms, and Stigler and Hiebert suggested that a main explanation for this teaching gap could be found in the incremental developments of teaching through LS in Japan. In the aftermath of this study, LS continues to gain popularity as a practice-based approach to professional development outside of Japan. There is also growing interest among researchers to adapt and use LS in teacher education (e.g., Ricks, 2011).

In mathematics education research, numerous studies focus on how LS might increase mathematics student teachers' knowledge and understanding of the mathematical content. For instance, Cavey and Berenson (2005) argue that their adapted version of LS has a potential to increase student teachers' understanding of the mathematical content. Drawing upon the idea that knowledge for teaching must be learned in and from practice (Ball & Cohen, 1999), one might argue that LS has a potential to serve as a "professional development tool when faced with the challenge of providing high-quality learning experiences for student-teachers" (Murata & Pothen, 2011, p. 104).

From an ongoing review of literature on LS in mathematics teacher education, we notice that, while several studies attempt to measure the effects of LS on student teachers' knowledge and understanding, few studies report on effects of LS on the quality of instruction. Chassels and Melville (2009) suggest that LS "provides opportunities for teacher candidates to build professional learning communities, to deepen understanding of curriculum and pedagogy, and to develop habits of critical observation, analysis, and reflection" (p. 734). When investigating mathematics student teachers' development of lesson plans, Fernandez (2010) suggests that implementation of LS appears to influence their development of professional knowledge. Jansen and Spitzer (2009) focus on

mathematics student teachers' reflective thinking, and their study includes analyses of student teachers' own interpretations of their teaching. Although the issues raised in these studies are of importance, neither of them focus directly on effects of LS implementations on the quality of teaching. Leavy's (2010) study includes a focus on observing teaching, but the analyses emphasize student teachers' reflections and development of knowledge rather than their actual teaching. Ricks (2011) reports from an intervention study, but his focus is on mathematics student teachers' reflections rather than on their teaching. With this as a background, the aim of this paper is to investigate possible effects of LS implementation on the quality of student teachers' mathematics teaching. We consider the following research question: What potential influences can be observed from a LS intervention on the quality of classroom interactions in the field practice of mathematics student teachers? In order to approach this research question, we analyze classroom observations from a time-lagged design experiment where LS was implemented in the field practice of two Norwegian teacher education programs. Videos of classroom teaching are analyzed by using the Classroom Assessment Scoring System (CLASS).

The Classroom Assessment Scoring System

CLASS scores are related to students' academic performance (Teachstone, 2012), and research indicates that substantial gains in measured student achievement is mediated by teacher-student interaction qualities (Allen, Pianta, Gregory, Mikami, & Lun, 2011; Teachstone, 2012). An important mediator for academic outcome is the extent to which the students' interactions with their teachers motivate them (Pianta & Allen, 2008). Based on this, student-teacher interactions in the classroom are the focus of attention when observing classrooms using the CLASS instrument. This instrument is designed to assess the fit between teacher-student interactions and students' developmental, intellectual, and social needs, i.e. elements of high-quality teaching that have been identified as central to student achievement (Allen et al., 2011). The CLASS instrument consists of three major domains that provide behavioral anchors for describing and assessing critical aspects of classroom interactions (Teachstone, 2012): 1) Emotional Support, 2) Classroom Organization, and 3) Instructional Support. Student Engagement is also included, due to the importance of observing student behavior in addition to behavioral anchors on the classroom and teacher level.

The first domain, Emotional Support, relates to students' social and emotional functioning in the classroom and is highlighted in the CLASS instrument because "relational supports and connections, autonomy and competence, and relevance are critical to school success" (Teachstone, 2012, p. 2). Second, Classroom Organization is included in the instrument based on research, highlighting the relationship(s) between aspects of organization and students' opportunities to learn. The foundation for the third domain, Instructional Support, is constituted by the following teaching strategies that enhance learning: "consistent, process-oriented feedback, focus on higher-order thinking skills, and presentation of new content within a broader, meaningful context" (Teachstone, 2012, p. 4).

These three domains can be further divided into twelve dimensions or CLASS indicators that are defined in the CLASS manual (see Tables 1, 2 and 3). In addition to these observable indicators of effective interactions, the CLASS manual includes behavioral markers that provide clear examples of how teacher-student interactions in the classroom can be observed and assessed. These descriptions are specified and examples of justifications are provided on the basis of concrete classroom videos, coded by CLASS experts.

Method

This study is situated within the larger, cross-disciplinary project, Teachers as Students (TasS), which involved mathematics, science, physical education, English as a second language, as well as pedagogy. The TasS project (2012–2015), supported by the Norwegian Research Council (grant number 212276), investigated student teachers' learning during field practice, aiming at learning more about how student teachers develop the knowledge and skills required to promote student learning in schools. LS was used in a time-lagged design experiment (Hartas, 2010) in two Norwegian teacher education programs, both four-year integrated programs, one for grades 1–7 and one for grades 5–10. Subject matter and didactics (pedagogy) should thus be integrated in all subjects, and there should be a close relationship between what was taught on campus and in schools when student teachers had field practice (100 days within the four years). The mentor teachers have an important role and are considered teacher educators in field practice.

The student teachers were organized in groups of three or four during a three-week period of field practice both in the Business as Usual condition (BAU) and in the LS intervention (INT). The TasS study recruited student teachers during the spring term of their fourth semester (except the science groups in the BAU condition, who were in their sixth semester). The TasS project includes data with two groups of student teachers from the four subjects in both data collection periods (see Munthe, Bjuland & Helgevold, 2016 for an overview). In this paper, we mainly report from analyses of classroom recordings of lessons taught in mathematics from the BAU and INT condition, using the CLASS (Classroom Assessment Scoring System) observational instrument (Allen et al., 2011; Teachstone, 2012). We also draw upon findings from previous analyses in the discussion section based on conversations in mentoring sessions and pre- and post interviews (before and after the field practice in both conditions).

In the BAU condition, the mentor teachers were asked to conduct their mentoring sessions the way they normally did without any influence from the researchers in the project. In the LS condition (INT), it was crucial that mentor teachers were introduced to essential principles about LS since "they played the role as facilitator and knowledgeable other for the group of student-teachers who made up the Lesson Study group" (Munthe et al., 2016, p. 145). This required another approach. Three afternoon seminars were organized (from November 2012 to January 2013) for mentor teachers and the research group in order to discuss important characteristics within the LS cycle and to establish a shared understanding of implementing LS in student teachers' field practice. An important component of these afternoon sessions was to develop a "Handbook for Lesson Study", which included a text about important principles in LS and a list of questions which could support both the mentor teachers and the student teachers through pre- and post-lesson mentoring sessions.

Three researchers (the authors of this paper) took part in the coding of videos from the four lessons in both conditions (INT and BAU). We divided the videos into 19-minutes sections. This resulted in 22 sections (12 BAU and 10 INT). After watching a video section, we started the scoring for each dimension individually, using the 7-point range that is described in the CLASS manual, Low (1, 2), Mid (3, 4, 5) and High (6, 7). We made our judgements based on the general scoring guideline. Our scores were then discussed before we started to observe a new video section. The results that are presented in the three tables below illustrate the scores given by the three researchers (see Tables 1, 2 and 3). Where two scores are given, our individual coding differed.

Results

The CLASS domain of Student Engagement intended to capture "the degree to which all students in the class are focused and participating in the learning activity presented or facilitated by the teacher" (Teachstone, 2012, p. 109), was coded as Mid for both BAU and INT. This code means that either the students are listening to, or watching the student teacher, rather than actively engaging in classroom discussions and activities, that there is a mix of student engagement, or they are engaged part of the time and disengaged for the rest of the time. Across all the videos, there is a lack of off-task behavior and the students appear to be engaged.

Emotional support

The domain of Emotional Support is divided into three dimensions (Table 1)[1]. Across the first two dimensions, the code Mid was given by all three coders. A Mid score on the first dimension, Positive climate, indicates that the student teacher and students sometimes provide positive comments and appear quite supportive and interested in one another. A Mid score on the second dimension, Teacher sensitivity, indicates that the student teacher sometimes monitors students for cues and generally attempts to help students who need assistance, but these attempts are not always effective in addressing student concerns. This code also indicates that some of the students sometimes seek support, respond to questions and share their ideas.

CLASS dimension	BAU	INT
Positive climate	MID	MID
Teacher sensitivity	MID	MID
Regard for adolescent perspectives	LOW/MID	LOW/MID

Table 1. Results from CLASS analysis of Emotional Support.

The third dimension, Regard for adolescent perspectives, was coded as Low/Mid, illustrating that individual coding differed among the coders. Low to Mid on this dimension indicates that the teaching is the teacher's show. The students are rarely provided opportunities for autonomy and leadership.

As can be seen from Table 1, the LS intervention did not affect the coding for any of the dimensions included in this domain.

Classroom organization

The domain of Classroom Organization is composed by three dimensions (Table 2). Behavior management encompasses the student teacher's use of methods to maximize the learning time for the students. The code Mid indicates that there is some evidence that the student teachers encourage desirable behavior and prevent misbehavior. Productivity, the second dimension, does not relate to quality, but rather deals with the students' opportunity to get involved and the extent to which the teacher makes sure that everyone has something to do. A Mid score on Productivity indicates that most of the time there are tasks for the students and some routines are clearly in place. However, transitions could be more efficient and the student teachers could be better prepared.

[1] The CLASS domains are written with capital letters in both words (e.g., Emotional Support), whereas the CLASS dimensions are written with one capital letter (e.g., Positive climate).

CLASS dimension	BAU	INT
Behavior management	MID	MID
Productivity	MID	MID
Negative climate	LOW	LOW

Table 2. Results from CLASS analysis of Classroom Organization.

The dimension Negative climate is scored in reverse. A low score indicates that the overall level of negativity among student teachers and students is low or absent. This code counts for more than the others in this domain, indicating that the classroom processes related to the organization and management of time, student behavior and attention in the classroom provide the students with opportunities to learn. As can be seen from Table 2, the LS intervention did not affect the coding for any of the dimensions included in this domain.

Instructional support

The domain of Instructional Support is divided into five dimensions as shown in Table 3. The first dimension, Instructional learning formats, was coded Mid in both conditions, illustrating that learning objectives may be discussed, but they are not clearly communicated in order to support student attention which is an indicator for a high score.

It is only in the second dimension, Content understanding, that there are indications of a possible effect, from discrete pieces of depth of lesson content to sometimes finding meaningful discussions in order to help students comprehend the mathematical content. The three last dimensions, which emphasize higher-order thinking among the students with a purposeful use of a content-focused discussion in the classroom, are all coded Low or Low/Mid. The focus on mathematical content is not strong in the teacher-student classroom interactions. A Low score on Analysis and inquiry indicates teaching that does not let the students think, or that students are neither engaged in higher-order thinking, metacognition, nor have opportunities for novel application. The teaching is in a rote manner. A Low/Mid score on the dimension of Quality of feedback indicates that the feedback provided to the students neither expands or extends learning nor encourages student participation. A Low/Mid score on the dimension of Instructional dialogue indicates that the student teachers do not involve the students in content-based discussions in class.

CLASS dimension	BAU	INT
Instructional learning formats	MID	MID
Content understanding	LOW/MID	MID
Analysis and inquiry	LOW	LOW
Quality of feedback	LOW/MID	LOW/MID
Instructional dialogue	LOW/MID	LOW/MID

Table 3. Results from CLASS analysis of Instructional Support.

The results illustrate that our LS intervention had little effect on the quality of classroom interventions (Table 3). Across the videos, and in both BAU and INT, the classes are mostly dominated by student teachers' talk.

Concluding discussion

Many studies suggest that LS has potential to contribute to mathematics student teachers' development, but few studies analyze potential effects of LS interventions in mathematics teacher education on the quality of classroom interactions. CLASS analysis of classroom videos from a control group and an intervention group indicates that our LS intervention did not increase the quality of classroom instruction. One might argue that the challenging results that arise from this study are problematic since one would hardly expect the results to show any variance based on one LS cycle only. However tempting it is to bypass the reporting of such challenging results, we do the opposite. The majority of research reports in mathematics teacher education appear to be success stories, but we suggest that it is also important to discuss results that were not as positive as desired. In the following, we highlight three issues that might have influenced the results of this study: 1) experience and time, 2) lack of focus on critical aspects of LS, and 3) personal factors.

First, we discuss the issues of experience and time. Stigler and Hiebert (1999, p. 109) describe LS as a "system that leads to gradual, incremental improvements in teaching over time." Japanese improvements in teaching happen through systematic work over several decades, and it is unfair to expect significant improvements in teaching from groups of student teachers who have just been introduced to LS. Most implementations of LS in mathematics teacher education seem to involve participants with little or no previous experience with LS (e.g., Bjuland & Mosvold, 2015; Leavy, 2010). In addition, most of these studies are short-term studies that often report results from participants who have completed one LS cycle only (e.g., Bjuland & Mosvold, 2015; Chassels & Melville, 2009; Leavy, 2010). It is not realistic to expect significant long-term effects from studies like this.

Second, a possible explanation for the challenging results in this study might be that the participants failed to implement some important aspects of LS. Previous analyses of data from mentoring sessions and interviews support this. For instance, Bjuland and Mosvold (2015) identified four indicators of why the implementation was challenging in mathematics. First, the student teachers reported about a lack of emphasis on pedagogical content knowledge on campus before field practice, and they called for more focus on students' difficulties and teaching strategies. A second indicator was related to a lack of formulating a research question. In a LS cycle, the student teachers should collaboratively plan, conduct and evaluate a research lesson with a focus on students' learning, but they should also formulate a research question that focuses on their own learning. No signs of this were found in the mentor sessions in the mathematics groups (Bjuland & Mosvold, 2015). Third, there was little focus on student learning and structured observation – both of which are decisive in LS. The mentor teachers' questions did, however, focus more on planning, observation and student engagement in the LS intervention (Bjuland, Mosvold, & Fauskanger, 2015). A fourth indicator was that student teachers organized research lessons around individual work with textbook tasks – making observation of student learning difficult (Bjuland & Mosvold, 2015). These observations may explain why the LS intervention was not successful and why the quality of classroom interactions did not increase.

Third, other factors like the student teachers' background, motivation and support may have influenced the results of this study. From analyses of data from mentoring sessions as well as focus-group interviews, we learned that one group of student teachers in the intervention may have had a lack of motivation for participating in the study. In the other group of student teachers from the

intervention, the mentor teacher was absent for a period of time, and the resulting lack of support from the mentor teacher might have influenced the quality of classroom interactions. Similar factors were also observed in the BAU groups. For instance, one of those groups struggled to collaborate (Bjuland & Mosvold, 2014), and we cannot revoke the potential influence of such problems. Although many problems can be avoided or taken care of in a research project, there will always be a potential influence of human factors that cannot be controlled by the researcher.

Further long-term studies are called for in mathematics teacher education to investigate participants who have completed more than one LS cycle, emphasizing that teaching develops through incremental improvements over time (Stigler & Hiebert, 1999). We observe that many studies focus on potential effects of a LS intervention on student teachers' understanding of the mathematical content. More studies are called for to investigate effects of LS implementations on classroom instruction.

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Ritual towards explorative classroom participation of pre-service elementary school mathematics teachers

Talli Nachlieli and Yafim Katz

Levinsky College of Education, Israel; talli.nachlieli@gmail.com, katzyafim@walla.co.il

In this paper we follow preservice elementary school mathematics teachers learning processes in a course that was organized around problem solving and aimed at providing opportunities for students to participate exploratively. The goal of this study is to characterize pre-service teachers' participation on the "ritual-explorative" continuum, to understand better what opportunities for explorative participation are given and taken up by students. Findings show that the request to suggest various solution paths seems to help students focus on the explorative question of "where do I want to get at?" rather than at the ritual question of "how do I proceed?".

Keywords: Rituals, explorations, preservice mathematics teachers.

Introduction

This paper focuses on learning processes of pre-service elementary school mathematics teachers during a course whose goal was to promote students' mathematical thinking by engaging them in a discourse which is closer to that of mathematicians, and thus to provide our pre-service teachers opportunities to participate exploratively in doing mathematics. This resonates with Blanton (2002): "The development of a cadre of classroom mathematics teachers whose practices reflect current research on teaching mathematics rests in part on how pre-service teachers, as students, experience mathematics" (p. 117).

In a former study about opportunities for learning in a prospective mathematics teachers' classroom (Heyd-Metzuyanim, Tabach & Nachlieli, 2015), we found that despite what seemed to be an explorative environment, pre-service students still participated mostly ritually in a mathematics course. Since we believe that the opportunities to learn that preservice teachers provide their future students should be more explorative, it is obvious that they themselves should participate exploratively. We therefore designed a course that would provide opportunities for explorative participation. The course was taught in two separate groups by two instructors simultaneously. We are now starting to learn about what actually happened in those courses - could the participation of students be characterized as, at least sometimes, explorative? What characteristics of the course design and instruction seems to provide explorative learning opportunities? In the current study we focus only on one group of students, working on one type of tasks during two lessons, and follow learning processes in an attempt to characterize pre-service teachers' participation on the "ritual-explorative" continuum.

Theoretical framework

In this study we adopt Sfard's socio-cultural approach to conceptualize and study learning – the communicational framework (Sfard, 2008), and refer to the on-going development and refinement of the discourse on rituals and explorations (Sfard & Lavie, 2005; Nachlieli & Tabach, 2012; Heyd-

Metzuyanim, Tabach & Nachlieli, 2015; Heyd-Metzuyanim & Graven, 2015; Sfard & Lavi, in process).

One of the main characteristics of a discourse is the routines participants perform. Routines are repetitive patterns that are repeated in similar situations. That is, when one views a situation as similar to one he had participated in, and performs the same action. This could be social – e.g. when entering your home and placing the keys at a particular place. It could also be a cognitive action – e.g., in a mathematics classroom, when a student refers to a certain problem as similar to one performed earlier, and adopts the same procedure to solve the problem. The participant is not always aware of this repetitiveness. That is, a routine is such in the eyes of the researcher. Sfard & Lavie (in process) define the term *explorations* as "routines whose success is evaluated by answering the single question of whether a new endorsed narrative has been produced". That is, the task of an exploration is to produce new "historical facts" or a new "truth" about mathematical objects. Exploration is hence an act of production. Performers of explorations focus on the question: "what it is that I want to get?".

Rituals are "routines performed for the sake of social rewards or in an attempt to avoid a punishment." (Sfard & Lavie, in process). Ritual performance is usually initiated by, and addressed at, somebody else. Usually, the performance is an imitation of someone else's former performance. The procedure is rigid and the performer of the ritual never tries to make independent decisions. Performers of rituals ask themselves: "how do I proceed?".

It is important to stress that the same procedure performed (even simply multiplying 26 by 31) could be an exploration or a ritual, depending solely on whether the participant is engaged in trying to produce a new narrative, or, simply socially engaging in class, doing what she is expected to do using given procedures.

Former studies suggest that while explorative participation is desired, rituals are inevitable. Especially in the process of objectification, of developing new mathematical objects (such as the development of numerical discourse by children (Sfard & Lavie, 2005; Sfard & Lavie, in process) or the development of the discourse on function of 7th grade students, (Nachlieli & Tabach, 2012).

In mathematics classrooms, students' participation is usually neither purely ritual nor explorative. Those could be seen as ends of some continuum that differ in the performer's ability to separate the procedure and the task. As long as the performer does not just strive to arrive at a particular outcome but also feels compelled to do this by performing a specific procedure, the routine cannot count as a pure exploration.

The goal of this study is to characterize pre-service teachers' participation on the "ritual-explorative" continuum so that we could understand better what opportunities for explorative participation are given and taken up by students.

Method

Data collection

The data for this paper are taken from a course about "promoting mathematical thinking", for prospective elementary school mathematics teachers studying at a college of education in Israel. The course was a one-semester course which was taught in 2014. It included 13 lessons, each lasting an hour and a half. During the first lesson, the project was described to the students and consent forms

were collected and hence the lesson was not videotaped. The remaining 12 of the 13 lessons were videotaped and transcribed. Lesson plans and all of the students' written work (exams and the planned unit) were collected. The language of the data was Hebrew. This data was analyzed in its original language and parts were translated to English by the authors. To learn about students' ritual and explorative participation, we focus on their studying a specific type of problems - serial tasks (calculating sums of sequences). This topic was discussed in lessons 2 and 3. Assuming that students' participation may change when shifting to a new subject, we chose to focus on one specific topic in its entirety. The data analyzed include all whole class discussions that took place during each of the lessons.

Participants

The research participants include a group of 18 prospective elementary school mathematics teachers. The students are studying at their final academic year in a college of Education in Israel. The course instructor, the first author, has a B.Sc in mathematics and PhD in mathematics education. She had been teaching in this college for 15 years.

The course

Over the past two decades, accumulating evidence has shown that classroom environments that support "explorative" participation, that is, that encourage students' authority (Herbel-Eisenmann, Choppin, & Wagner, 2012); engage students in tasks that are cognitively demanding and are open to different solutions and procedures (Boston & Smith, 2009); and foster a community of learners that listen to each other and build on each other's ideas (Resnick, Michaels, & O'Connor, 2010) promote conceptual understanding. The aim of the designed course was for students to deepen their mathematical thinking by working on high cognitive-demand problems (Smith & Stein, 2011), solving problems in various ways and making connections between the different solutions as well as between the mathematical ideas related to the problems and the solutions. During the lessons, students worked in small groups to solve the problems and were encouraged to come up with as many solutions as possible. Whole-class discussions about the different solutions followed. The instructors chose to provide students with as much time as needed to work on certain problems alone or in groups, focusing entire lessons on discussing different solution paths suggested by the students. The students had to take three exams during the course and were required to plan a 3-lesson unit about any topic for elementary school students, which aim was to promote their students' mathematics thinking. Two groups (of around 20 students each) were taught simultaneously – by each of the two authors. This study refers to Talli's group only.

Data analysis

To identify whether students' participation is more ritual or more exploraitve, and to identify shifts in participation we followed all whole-class discussion around a specific type of tasks. The discussions were analyzed by addressing the questions in Table1.

		Ritual	Exploration
1.	What is the question the performer is trying to address?	How do I proceed?	What is it that I want to get?

2.	How does the performer evaluate its' success?	Performing a specific task-related procedure	A new narrative had been produced
3.	By whom is the routine initiated? To whom it is addressed?	Initiated by former performer of a similar task. Addressed at the teacher (or other superior).	oneself
4.	Flexibility	Applies a rigid routine. Seldom makes independent decisions	Could consider various routines. Makes independent decisions on the way.
5.	Separation between procedure and task	Not separated as the main task is to perform a (certain) procedure.	Separated
6.	Authority	The teacher	One's own

Table 1: Rituals and explorations

Findings and discussion

To understand what opportunities for explorative participation were given and taken up by students, we present our findings about students' participation while working on a specific type of tasks (calculating sums of sequences). This learning took place during lessons 2 and 3.

The first problem that the students solved was calculating the number of Hanukkah candles one lights during the 8 days of the Holiday (2 candles on the first evening, 3 on the second, … and 9 candles on the eighth night): 2+3+4+5+6+7+8+9 = (2+9) + (3+8)+ … = 4 · 11 = 44 (following Gauss's idea of pairing elements of the sequence: first element with the nth, second element with the one in the (n-1) place, and so on. The sums of each of the pairs are equal.). Then, after quickly calculating the sum of integers between 1 and 99, the students were asked to calculate the following sum: 1+3+5+7+….+997+999 = .

The following conversation took place:

1	Maya	I remember that there's something, a formula, I don't remember it now. To find the element in the middle.
2	Inst.	To find a formula for finding the middle element?
3	Maya	no, no, there is a formula
4	…..	[the class discusses a solution path suggested by a student]
42	Sonya	it could be factorial, right?
43	inst.	Factorial?
44	Sonya	yea, factorial, I remember something, I think this could be related.
45	inst.	[to the class] do you what factorial is? Remember what it is?

The first student's (Maya) saying refered to the existence of a formula that could, perhaps, be helpful in this situation. Shse also talked of remembering. That is, Maya's first reaction was to seek a formerly learned routine that could be helpful in this situation. Maya followed Gauss's idea to add pairs of

numbers. As the number of elements in this sequence is even (500 numbers), Maya's search for the "middle number" is surprising. It is possible that either Maya mistakenly thought that there actually is an odd number of elements in the sequence. A different interpretation is that as all the problems that the students have worked on in the course so far have been of an odd number of elements, Maya looked for the "middle element" as part of performing a given routine practiced earlier.

After Maya's remark, a whole-class discussion arose about the number of elements in the sequence, whether it was even or odd. After agreeing that there are 500 elements (half of the elements in the sequence 1, 2, 3, … ,1000), the students used Gauss' idea and calculated: (1+999)·250 = 250,000. The instructor asked for a different solution when Sonya replied: "it could be factorial, right?" she then added: "I remember something, I think this could be related". That is, similar to Maya, Sonya seeks a formerly learned and used routine to be applied here. She does not remember the routine or what the idea behind "factorial" is, but something about this problem reminds her of this formerly learned idea. Considering Maya and Sonya's communication about the given problem, it has strong ritual characteristics:

		Ritual	Exploration
1.	What is the question the performer is trying to address?	How do I proceed? *There is evidence of the students' attemps to seek a ready-made formula that could help them proceed. In Maya's case, the formula is not needed as there is an even number of elements. In Sonya's case – the idea she thiks of (factorial) is not relevant. Sonya is not sure what it actually is, but she does find some connection between the procedure and the problem at hand.*	What is it that I want to get?
2.	How does the performer evaluate its' success?	Performing a specific task-related procedure *Although not yet performing the procedure, Maya & Sonya seek a task-related procedure to follow. Both turn to memory (1, 3, 44).*	A new narrative had been produced
3.	By whom is the routine initiated? To whom it is addressed?	Initiated by former performer of a similar task. Addressed at the teacher (or other superior). *The routine was learned sometime in the past (not during this course). They do not try to develop a routine for the specific task by analyzing where it is they want to get at (1, 3, 44).*	oneself
4.	Flexibility	Applies a rigid routine. Seldom makes independent decisions *The students do not make any decisions at this moment. They turn to the instructor to remind them of the formula / concept that they thought was relevant (42).*	Could consider various routines. Makes independent decisions.
5.	Authority	The teacher *(42)*	One's own

Table 2: Analysis of Maya & Sonya's participation

During the rest of the lesson, the students worked in groups and came up with three ways to solve the problem. Later in the second lesson, the students suggested and discussed different solutions paths different series.

During the lesson the students learned: (1) that a problem could be solved by different solution paths; (2) that Gauss' idea could be helpful to solve sums of certain sequences; (3) to apply Gauss' idea, one needs to know the number of elements in the sequence. If the number is odd, then either the element in the middle should be identified or, the first or last element should be sided and later returned to the series. The students were encouraged to come up to the board to suggest solution paths, to make certain that they understand others' ways of solving the problem and ask questions when things were not understood.

At the beginning of the 3rd lesson students were asked to suggest ways to calculate:

$$1+2-3-4+5+6-7-8+...+2009+2010-2011-2012+2013$$

Solution paths suggested by two students (Nur and Sara):

1	Nur	so, I put the 1 aside, and saw that (in) each two pairs, the first gives me minus 1 and the second pair gives me plus 1. So I have 2012 numbers here and then, if I divide this to pairs, I have 1006 pairs. So half of them give me 1, 503 altogether, and the other half gives me minus, so it's (-503) and it cancels and so I have 1 left.
2	Inst.	Do you see what Nur did? Any questions?
3	Ziv	no. it's perfectly clear
4	Nur	now I found another one. After... I left the 1 aside here too, then I saw that if I take the 2 and the 2013, it gives me 2015. If I take then the minus 3 and the minus 2012 it will give me (-2015). So I have the sum of two negative numbers that give me (-2015) and the sum of two positive numbers give me (+2015). So it still cancels out and I have the 1 left.
5	inst.	What do you all say? yes?
6	Sara	I have another one. I saw here, my language is not that well, correct me, ok? like a sort of continuum, that every four, every three operations give (-4) each time, then its repeated.
7	Inst.	Every three operations, you mean, the sum of four numbers?
8	Sara	yes, the sum is (-4). So it's repeated till number 2012. So 2012 is divided by 4 and I get 503 times that it's repeated. … then I multiplied 503 by (-4) cause every such part is, mm…. (-4). And this is the result so far [-2012]. Then I have plus 2013 and it's 1.

The analysis of Nur and Sara's suggested solution-paths is in Table3.

		Nur (2 solution paths)	Sara
1.	What is the question the performer is trying to address?	*What is it that I want to get?* The student is trying to solve the given problem and create a new solution path.	
2.	How does the performer evaluate its' success?	By creating narratives of two types – (1) the sum of the series (equals 1), and (2) new solution paths to	By creating narratives of two types – (1) the solution (the sum is 1). However, this

		solve the problem. In both, the first number of the sequence is left aside. In the 1st solution, adjacent numbers are paired. The sum of each pair is 1 or (-1) alternately. Therefore the solution is 1. In the 2nd solution, Gauss' principle is applied to create pairs of numbers whose sum is 2015 or (-2015) alternately. Therefore, the total sum is 1.	narrative is already known from previous answers. (2) a new solution path to solve the problem: the sum of every 4 adjacent elements is (-4). The sum of all quadraplets is 503·(-4)= (-2012). The last element of the sequence (2012) is added. Therefore, the sum of the sequence is 1.
3.	By whom is the routine initiated? To whom it is addressed?	The routine used by the student to solve the problem (placing the first or last number aside and checking sums of sets of numbers), as well as Gauss's principle were used by the class in the previous lesson. However, this is not simple mimicking of previously performed routines by others, as those routines have not been used together for the same problem yet. Some adaptation needed to be done. Therefore, the routine is initiated by the student.	
4.	Flexibility	The student used previously perfomed procedures to create a new solution path to solve this problem. Considering the students' decision making – the student made all decisions for adapting previously used routines to this problem.	
5.	Separation between procedure and task	It is not clear whether for the student the procedure is a part-and-parcel of solving this problem or not.	
6.	Authority	One's own	

Table 3: Analysis of Nur and Sara's participation

It seems that Nur and Sara participated exploratively: they produced narratives that include new solution paths for the task and reaching a solution. They articulate their solution path in a way that makes clear that they have made independent decisions while adapting different formerly performed routines to solve the given problem.

Discussion

In this study we tried to characterize students' learning while solving a specific type of tasks on the ritual towards exploration continuum. In the discussed course, in which students were constantly asked to work on problems on their own or in small groups, and suggest various solution paths to each problem, there is evidence of students shifting from more ritual to more explorative participation. It may seem obvious - the course was designed in a way that would invite students to engage exploratively. Yet, studies show that even when teachers design lessons that aim at certain opportunities to learn, this does not always happen. We found that when faced with a task that is of a new type (to the learner), the learner's immediate response is of a ritual type - to seek related procedures that would assist her in solving the task. The request to suggest various solution paths seems to help students focus on the question of "where do I want to get at?" and not remain focused on the question of "how do I proceed?". Also, once a student chooses a certain routine as a solution-path, it is sometimes followed blindly, ritually, thus getting farther away from the task at hand.

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Co-disciplinary mathematics and physics Research and Study Courses (RSC) within two groups of pre-service teacher education

María Rita Otero[1], Viviana Carolina Llanos[1], Marcelo Arlego[1], María Paz Gazzola[1]

[1]Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN), Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina; rotero@exa.unicen.edu.ar; vcllanos@exa.unicen.edu.ar, marlego@exa.unicen.edu.ar, mpgazzola@exa.unicen.edu.ar

In this paper, we present results of an inquiry based teaching implementation carried out on a teacher training course in the University. The framework of the Anthropological Theory of Didactics (ATD) is adopted, and a co-disciplinary Research and Study Course (RSC) whose generative question requires studying physics and mathematics together is carried out by N=25 training teachers of Mathematics at University. Some conclusions concerning on the conditions, restrictions and relevance of introducing the RSC in teachers training courses at the university are performed.

Keywords: Inquiry based teaching, pre-service teachers training, modelization; research and study course.

Introduction

The Anthropological Theory of the Didactic (ATD) has proposed the Paradigm of Research and Questioning the World (Chevallard, 2012, 2013 a) advocating an epistemological and didactic revolution (Chevallard, 2012) of the teaching of mathematics and school disciplines, where knowledge should be taught by its usefulness or potential uses in life. The present work shows results obtained in two courses of pre-service mathematics teacher education (N=25) during a teaching inspired in the paradigm of questioning the world, by means of a Research and Study Course (RSC). To learn what an RSC is, and which kind of teaching is involved in, the trainee teachers (TT) must deeply experience a genuine RSC. Thus, a physics and mathematics co-disciplinary RSC is designed, implemented and analyzed with the students. Co-disciplinary means that in this case, physics does not only trigger the study of mathematics, but rather that both disciplines play a central role, being necessary to study both as well. The starting point of the RSC is the question Q_0: Why did the Movediza stone in Tandil fall? Which, to be answered – in a provisional and unfinished way- needs study Physics and Mathematics jointly. The rationale of the paper is to describe the trainee teachers' activities and their difficulties when they must experience a genuine RSC and to face a strong question. Some reflections on the ecology an economy of this kind of didactic devices for the pre-service teachers training are performed.

The research and study courses (RSC)

The ATD defines the RSC as devices that allow the study of mathematics by means of questions. The RSC establish that the starting points of mathematical knowledge are questions called generative questions, because its study should generate new questions called derivative. Teaching by means of RSC is complex and demands rootle changes in the roles of the teacher and students. The RSC are defined by the *developed Herbartian model* (Chevallard, 2013 b):

$$[S(X;Y;Q)\rightarrow\{R^{\lozenge}_1, R^{\lozenge}_2, R^{\lozenge}_3,\ldots, R^{\lozenge}_n, Q_{n+1},\ldots, Q_{m+1},\ldots,O_m, O_{m+1},\ldots,O_p\}]\rightarrow R^{\heartsuit}$$

Where Q is a certain generative question; S is a didactical system around of the study of Q. S is formed by a group of people trying to answer the question (X) and by people helping the study (Y). In classrooms of mathematics, X represent the students and Y represent the teacher and other instruments helping in the search of answers to Q. S has to build a didactic medium M to study Q, whereas M is composed by different knowledge, expressed by $R^{\Diamond}_i$, Q_j and O_k. The $R^{\Diamond}_i$ are any existing answer or "socially accepted answer", the Q_j are derivative questions of Q, and the O_k are any other knowledge that must be studied developing the answers. Finally, $R^{\heartsuit}$ is some possible and partial response to Q given by S. In the a priori analysis stage, the specific and didactic knowledge which could be involved within an RSC is set up and the Praxeological Reference Model (PRM) is elaborated. The researchers analyze the potential set of questions which the study and the research into Q might encompass together with the knowledge, mathematics and physics in this case, necessary to answer those questions (Chevallard, 2013). The PRM underlies the whole of the teacher, student and researcher´s activity, being always likely and desirable to identify and clarify it, emphasizing the dynamic nature of the PRM.

Methodology

This is a qualitative and exploratory research that aims to carry out inquiry based teaching as it is proposed by the ATD, in a mathematics teacher training course at the University. The RSC was implemented in a state university, in the city of Tandil, Argentina, in a discipline which is part of the didactic studies within the Mathematics Teaching Training Course, in which the researchers are also the teachers, where N=12 and N=13 students from the last year (4th), aged 21-33 took part in it. The students had studied the ATD in two Didactic courses; however, they had problems to understand what an RSC is, and how it works? To emphasize the inquiry dimension of the RSC, the lessons were carried out in the University Library, given the wide availability of books and internet-based searching, during 10 weeks (the half of the course extension), with a total of 7 weekly hours provided in two lessons. Six work groups were organized with approximately 4 members each. During the lessons students and teachers interacted permanently. In a RSC, the generative question Q_0 has to be pointed out by the teacher, and this was made in the first lesson. Then, the students started their research in the library, by selecting some texts, documents etc. as possible $R^{\Diamond}_i$. At the end of this class, each group presented and discussed with the teacher and the other groups their findings and possible ways to face Q_0. In the second class, many emergent questions Q_i were made explicit, and the teacher and students groups selected which questions Q_i and their related knowledge O_k were to be studied. This was the regular dynamic during the RSC. Recordings of each meeting were obtained and the students' productions were digitalized and returned in the subsequent meeting. The data analysis was performed by using the categories provided by the *developed Herbartian model* (Chevallard, 2013) summarized before, and all derivative questions Q_i, all "socially accepted answers" $R^{\Diamond}_i$ found by the students, together with the O_k studied were described and analyzed.

The Praxeological model of reference (PMR) and the RSC

The starting question Q_0 is: Why did the Movediza Stone in Tandil fall down? This enormous basalt stone has remained the city's landmark, providing it with a distinctive feature. Many local people

and national celebrities visited the place to watch closely the natural monument. It was a 248-ton rock, sitting on the top of a 300-meter-high hill (above sea level), which presented very small oscillations when disturbed in a non-arbitrary spot, as shown in Figure 1. Unexpectedly, on February 28, 1912, the stone fell down the cliff and fractured into three pieces, filling the town with dismay at the loss of their symbol. For over 100 years the event produced all kinds of conjecture, legends, and unlikely scientific explanations for the causes of the fall. Within the two groups where the RSC has been performed, there existed a certain curiosity and interest in finding a scientific answer to this question.

Once in contact with the available information, the question evolved into: What conjectures are about the causes the Movediza Stone fall, and which is the most likely from a scientific viewpoint? Considering that the fall can be explained by means of the Mechanical Resonance phenomenon, several questions Q_i arose which are linked to the physical and mathematical knowledge necessary to understand and answer Q_0.

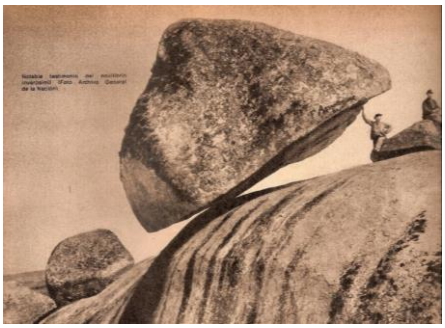

Figure 1: Photography of the Movediza Stone (Photo Archivo General de la Nación Argentina, available in: http://bibliocicop.blogspot.com.ar/2012/02/piedra-movediza-100-anos-de-su-caida.html)

If we consider that the real system is an oscillating system, the study can be carried out within the Mechanic Oscillations topic, starting from the spring or pendulum models which are ideal at the beginning. In this case, frictionless systems are used, in which the only force in action is the restoring force depending in a linear way on the deviation respect to the equilibrium position, and which produces oscillating systems known as harmonic, whose motion is described by a second-order linear differential equation, called by the same name. In the case of the pendulum, the restoring force can be considered depending on the oscillating angle (for small angles).

Progressively, the system becomes more complex. If friction-produced damping is considered, it provides a new term to the differential equation connected to the first derivative of the position (speed). Finally, it is possible to study systems that apart from being damped, they are under the action of an external force, and therefore called driven systems. Whenever this given force is periodic and its frequency coincides with the natural (free of external forces) frequency of the oscillating system, a maximum in the oscillation amplitude is produced, generating the phenomenon known as mechanical resonance. By increasing the complexity of the model, it is possible to consider a suspended rotating body, instead of a specific mass. This leads to the study of the torque and the moment of inertia of an oscillating body. Here again, the linear system is for small amplitude oscillations and the damped and driven cases can be also considered, corresponding to the same mathematical model, but in which the parameters have a different physical interpretation.

However, as it refers to a suspended oscillating body, this is not a suitable physical model for the Movediza stone system. Since that the base of the Stone was not flat, it is necessary to consider more precise models of the real situation. This leads to the mechanics of supported (and not hanging) oscillating rigid solids. In this case, we consider a rocker-like model in which the movediza stone base is curved and it lies on a flat surface, where the oscillation is related to a roto-translation motion (Otero, Llanos, Gazzola, Arlego, 2016 a, b). The application of Newton Laws to the rocker model of the movediza stone leads to a differential equation of the type harmonic oscillator, where the parameters are now specific of the movediza system: mass, geometry, inertia moment, friction in the base, external torque, etc. It leads to the following *effective* Harmonic oscillator mathematical model of the movediza physical system:

$$\ddot{\varphi} + \gamma\dot{\varphi} + w_0^2\varphi = (M_0 / I)\cos(wt) \qquad (1)$$

The solution to equation (1) is $\varphi(t) = \varphi_M \cos(wt - \psi)$, being the amplitude φ_M and the phase ψ

$$\varphi_M = \frac{M_0/I}{\sqrt{(w_0^2 - w^2)^2 + w^2\gamma^2}} \qquad \psi = tg^{-1}\left(\frac{\gamma w}{w_0^2 - w^2}\right)$$

The maximum of φ_M is for $w = \sqrt{w_0^2 - \gamma^2}$. The parameters: M_0 (external torque), I (inertia moment), w_0 (natural oscillation system frequency) and γ (friction coefficient), must be estimated. Detailed data about the shape, dimensions and center of mass position of the movediza stone are available (Peralta, et al 2008) after a replica construction and its relocation in 2007 on the original place (although fixed to the surface and without possibility to oscillate). These data bring us the possibility to fix some parameters in our model, as e.g. mass, inertia moment, and the distance of 7.1 m, from which the external torque could be exerted efficiently by up to five people (per historical chronicles) to start the small oscillation. By using these values, it is possible to study the behavior of the $\varphi_M(w)$ function for w_0 in a range of frequencies between 0,7 Hz and 1 Hz, historically recognized as the natural oscillation frequencies in the movediza stone system and calculate for each case the maximum amplitude $\varphi_M(w_m)$ that occurs for $w_m = \sqrt{w_0^2 - \gamma^2}$ .The Stone would fall if $\varphi_c \le \varphi_M(w_m)$, being $\varphi_M(w_m) = M_0 / w_0 I\gamma$ the maximum value of the amplitude function, that is to say $\varphi_c \le (M_0 / w_0 I\gamma)$. The value of φ_c can be determined by an elementary stability analysis, which according to the dimensions of the base of the stone and the center of mass position is estimated to be approximately of 6°. In the present model, we cannot estimate γ. If we adopt "ad doc" for this parameter, a magnitude order $\gamma \ge 10^{-2}$, we obtain in the various situations considered with different torques and the interval frequencies previously mentioned, that all the scenarios support the overcoming of the critical angle, i.e., predict the fall. Later, in search of a more appropriate approximation of the physics model for the damping that is clearly not due to air, we can consider the stone as a deformable solid, where the contact in the support is not a point, but a finite extension. Therefore, the normal force is distributed on such a surface, being larger in the motion direction and generating a rolling resistance, manifested through a torque contrary to the motion due to friction. The rolling resistance depends on the speed stone, giving a physical interpretation to the damping term. Therefore, the physics behind the damping is the same that

makes a tire wheel rolling horizontally on the road come to a stop, but in the case of the stone, the deformation is much smaller. Although the deformable rocker model has extra free parameters, tabulated values of rolling resistance coefficient for stone on stone, which are available in the specialized literature, allowed us to estimate and justify the damping values that we incorporate otherwise ad-hoc in the rigid rocket movediza model.

Some results obtained in the two implementations

During the implementations, the students aim at answering how and why the stone fell down the cliff. At the beginning, the TTs searched in the elementary physics textbooks for an "already-made" mathematical and physical model, which allowed them to solve a differential equation in a specific way. In both implementations, several physical and mathematical questions arose; the main preoccupation of the TTs was to study the oscillations subject, because it was a new knowledge to them. The implementation was carried out in parallel with a Differential Equations course, and none of the groups seemed to have difficulties with the underlying mathematics. In both implementations, the TTs tried to find a physical model suitable for the situation and they decided on the physical pendulum model initially, whose mathematical model might be adequate to the problem, although physically inadequate. However, the path performed in the first implementation was different from the second. In the first case, the TTs did not question the physical model, and spent most of the time to the study of inertia moment concept and their calculation for regular solids, which would result in an appropriate model for the irregular shape of the stone. Several interesting questions were therefore generated, for which the answers were provided by the teacher and the students together. After, the students calculated by themselves the characteristic frequency of the system, making use of the moment of inertia previously obtained. Thus, only a parameter resulted undetermined: the damping. But in the end, the physical pendulum model became an obstacle, because the stone was a supported body, and not hanged. On the other hand, the damping they considered was due to air-friction, whereas in the case of the movediza stone the main source of friction is the contact with the support surface.

At this point, the external torque (there were different trials to analyze and estimate it) and the solution of the equation remained unstudied. Until this moment, the solution for the differential equation did not seem to present any obstacles to the students, who considered they were facing an initial value problem. Once they had obtained the parameters, which they considered fixed, the solution seemed simple. However, they had problems to arrive at a final solution, even though this can be found in the physics textbooks (without its deduction). For this reason, it was discarded and they decided to do the calculation on their own. This event complicated the quantification they aimed to obtain, as well as the physical interpretation. Some groups in this cohort removed the term of damping, to reduce degrees of freedom; thus, the stone would have been in perpetual motion. This did not create any contradiction to them. Other ones adopted a damping value due to air-friction, which also led to wrong results. In summary, instead of adopting and adapting the solution that was presented in physics books, the TTs in this cohort dismissed it and did not interpret the answer in the texts concerning the Stone. The decision of the teachers to delay their intervention was with the purpose to make students live the dynamics of progress and drawbacks typical of the research and study courses. In addition the TTs had problems to understand the utility and necessity

of mathematical models, due to an epistemological conception close to pure or formal mathematics. The TTs did not understand how to use the mathematical models, neither the role that the parameters could play, which were considered as fixed and universal. In consequence, they failed to establish different sets of parameters and did not generate the feasible families of functions and values, whose compatibility with the physical situation could have been analyzed. These difficulties were considered for the second implementation of RSC. In the second cohort (TT2), the teachers had already perceived that the fundamental problem seemed to be in the models and in the modelization. For this reason, it was decided to devote 8 sessions to the development of two intra-mathematical RSCs (Chappaz & Michon, 2003; Ruiz, Bosch, Gascón, 2007), that the TTs could experience by themselves, therefore emphasizing the role of the modelization and the use of devices as spreadsheets and plotters. Besides, in this case, the teachers intervener as soon as the students proposed the physical pendulum and spring models. One group studied the AMS for the simple pendulum, the spring, and the physical pendulum, another group studied the spring model in all its possibilities and the third one did not develop further than the AMS in simple pendulum and spring. The synthesis stage corresponding to that class allowed the production of a complete answer for the three models and its possibilities, from which the TTs and the teacher arrived at the conclusion that the same mathematical model represented (9) nine different physical systems (Figure 2). A large amount of time was devoted to pondering on the differences and similarities between the mathematical and physical models and their connection with the real system we aimed at modeling. Then, the answers to the equations presented in the books were checked out.

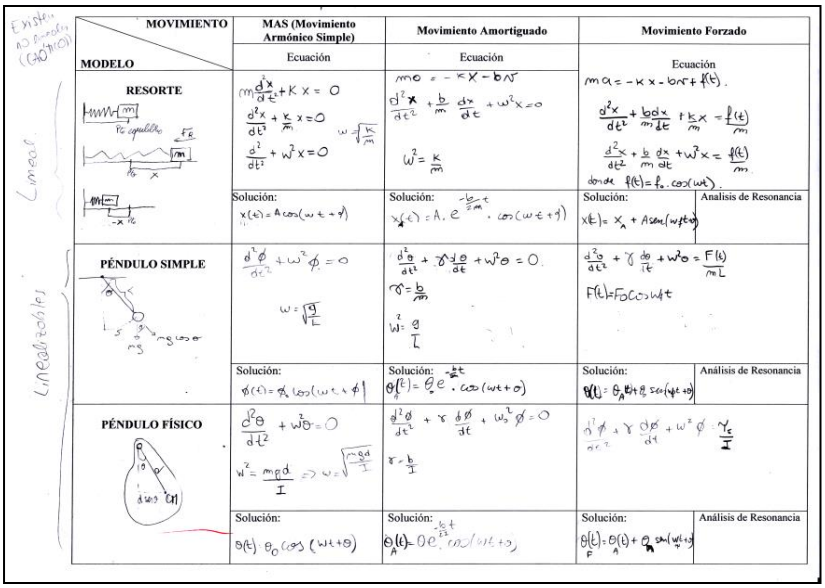

Figure 2: Protocol of the student E17. Implementation 2

In both cohorts, as a fixed route that is inevitably set by the books, the TTs came across the physical pendulum. However, in the second cohort some students presented strong objections to the possibility of using it in the case of the stone, not so much in relation to a body that is supported but as an "inverted" pendulum. This drew the discussion once more towards the real system and the standing point, so that the path went through the models which refer specifically to the system and that are not, usually in elementary books, like a rocker.

Firstly, the equilibrium was analyzed and the critical angle was calculated, and then, the model of the base of the movediza stone was sophisticated. For the study of the rigid solid physical model, the teacher proposed to the students a little text, as a new O_k that could be introduced into the didactic medium M. Finally, the students and the teacher calculated and estimated the parameters of the differential equation solution, and the classroom elaborated an answer that allows the explanation, by means of a model, of the plausibility of the fall.

Conclusions

Despite of the difficulties, the TTs experienced a genuine RSC within its means. There is a visible initial reluctant attitude on the part of the TTs: Why physics should be studied if we are teachers of mathematics? Later, it was gradually understood that the idea was to experience a genuinely co-disciplinary RSC, to analyze it and comprehend the teaching model supporting an RSC.

Even though the TTs had studied the ATD and other didactic theories, they did it in a traditional way comparable to the traditional training they get. This is reflected in the difficulties they have to understand and to use both physics and mathematical models. It was not expected that the TTs developed the models by themselves, but it was expected that they used the mathematical answers presented in the physics textbooks in a pertinent and exoteric manner. This fact did not occur in the first group and improved in the second one from the didactic decision to make a previous incursion into mono-disciplinary RSC particularly suitable for evidencing the role of the functional modelization. In addition, this allowed teachers to discuss the relationship between the mathematical model and the physical model and the meaning and role of the parameters.

The TT's behavior is interpreted from the fact that although they have experienced four years of "hard" university studies, the utility of the science they aim at teaching had never been visible. The epistemological conception about the mathematics produced by the traditional paradigm is so ingrained, that it is complex to reverse it. This would be, in our view, the most relevant drawback to permit the TTS at least understand what an RSC is and how the modelization activity works? However, it is important to notice that the sporadic incursions in the modelization activity do not seem enough to allow the TTs develop such school practices. Although the predominant teaching is mainly traditional, the TTs will face increasing demands for a change to a mathematic teaching based on the research, questioning and modeling. It is unlikely that a teacher whose training has been answers-based teaching can teach by means of questions. Then the training of teachers has to change profoundly.

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Writing fictional mathematical dialogues as a training and professional advancement tool for pre-service and in-service math teachers

Tikva Ovadiya

Oranim Academic College of Education, Israel; tikvao@actcom.net.il

Pre-Service and In-Service Math Teachers are often surprised to find themselves at a loss for words in the mathematics classroom. This feeling is not limited to the first day of class or to beginning teachers. Even experienced teachers describe unexpected classroom situations in which they cannot find the proper words to respond or to explain or mediate ideas. The teaching routine is fraught with on time decisions teachers must make. The objective of the current study was to promote the development of spoken and written mathematical discourse among pre-service and in-service math teachers in the context of classroom scenarios they considered unexpected and complex. The training was directed toward developing argumentative mathematical discourse skills through writing, with emphasis on writing fictional dialogues. The research focuses on the characteristics of fictional mathematical dialogues written by pre-service and in-service math teachers and seeks to show that these dialogues can used as a professional advancement tool.

Keywords: Writing fictional mathematical dialogues; professional advancement tool; pre-service and in-service math teachers.

Introduction

Teachers make decisions based upon knowledge, goals, beliefs and orientations. Accordingly, developing all of these factors can help promote decision making in the mathematics classroom.

For many years, I have been seeking creative ideas that will enable in-service and pre-service teachers to predict scenarios and unexpected situations in the mathematics classroom. Thus, they will be able to practice mathematical discourse before coming to class and to learn to provide argumentative responses that are quick, accessible and flexible. Teachers' responses in class and their responsibility in developing mathematical discussions and discourse have been the topics of much investigation (e.g., Schoenfeld, 2008-2011). The literature has placed less emphasis on examining training methods for developing discourse management for predicting unexpected classroom situations in advance, particularly training all that through writing.

Zazkis and Koichu (2015) describe a fictional dialogue on infinitude of primes between Euclid and Dirichlet and use this as a research method. The current study focuses on pre-service and in-service math teachers who write "fictional dialogues" as part of their training. The goal of this writing is to develop their ability to explain, respond and engage in argumentative mathematical discourse in a learning situation characterized by unexpected situations. The results of the current study indicate that the task of writing fictional dialogues has several advantages. One advantage relates to professional development and renewal. Veteran teachers tend to feel less challenged and less interested in preparing lessons in advance. Writing fictional dialogues challenges them to formulate unexpected mathematical situations for mathematical topics and ideas that for them are seemingly

simple and trivial. In writing fictional dialogues, they discovered both mathematical and didactic innovations. Another advantage applies to training. In writing the dialogues, beginning teachers learned to develop written mathematical discourse that explains the essence of mathematical terms. Further, they learned to use visual or other representations in context and practiced giving explanations to learners with a variety of learning styles.

Theoretical background

Unexpected situations in the mathematics classroom differ from teacher to teacher due to differences in the extent and depth of their mathematical knowledge, their ability to identify such situations and their ability to make decisions in real time about the didactic concepts appropriate for each situation. Hence, I examined the research literature on major topics related to the current research. These include training pre-service and in-service math teachers by means of writing, the role of the teacher in discourse development and management in the mathematics classroom, mathematical argumentation as a teaching tool and interaction in the mathematics classroom (Malaspina, Mallart, Font, , & Flores, 2016). The conclusions of these studies led me to formulate ideas for a unique intervention "training" program with the potential to promote mathematical discourse in the classroom in general and argumentative mathematical discourse in unexpected situations in particular. In the following sections, I review the relevant literature in these fields and explain how these studies relate to the current research.

Professional development and learning through writing

Teacher training usually incorporates writing through writing assignments about ideas learned in class or as reflection on learning (Korkko et al., 2016). Turning writing into a goal in and of itself is an innovation in the training of mathematics teachers. Therefore, in order to construct an intervention program that emphasizes writing, I surveyed and studied research that examines the advantages of writing in teaching math and of pedagogy based on writing in general.

In the study by Bostiga, Cantin, Fontana and Casa (2016), the students learned by writing diaries on mathematical argumentation. The research indicates that the process of writing develops students' in-depth thinking about mathematical concepts as well as underlining erroneous or other perceptions of concepts or phenomena. The writing process and the accompanying feedback prompted the students to write more precisely about mathematics, directed them to give arguments, explanations and reasoning in their writing and taught them to edit and rethink mathematical ideas. From this study among students, I decided to try to generalize the method for adults and to examine the results. Adults with a common professional interest often write together in a process that advances their shared understanding and learning in the field (Lowry et al., 2004).

Griffin and Beatty (2010) examined the attributes of shared writing among adults with a common professional interest. Their research pointed to several advantages, including professional and personal growth among the writers, a greater degree of creativity, the generation of new ideas and understandings, diversification in areas of specialization, increased documentation and output abilities, and shared knowledge. Shared writing generates a unified voice, increases feelings of satisfaction and pride in integrating the personal voice into the voice of the group and expresses respect for individual knowledge. Therefore, in this study the writing took place in pairs or in small groups as part of the process of developing skills in argumentative mathematical writing.

Read (2010) proposed the IMSCI model for supporting the writing process, with writing serving as a pedagogical tool for assimilating learning. In the IMSCI acronym, "I" stands for inquiry, "M" for modeling, "S" for shared writing, "C" for collaborative writing and "I" for independent writing. This scaffolding model was integrated into the intervention process in the current study.

Spoken or written mathematical discourse

According to Sfard (2008a; 2008b), discourse has four characteristics: vocabulary, visual mediators, unique routines and customary utterances. In the communicative approach, thinking constitutes an individual's discourse with the self. Such a discourse can yield ideas that express the thinking of those participating in the discourse. In contrast to those who talk, some people express themselves through writing and symbolic mathematical representations and have difficulty expressing their ideas verbally. Such individuals may eventually become teachers whose skills in developing and conducting mathematical discourse are not sufficiently developed. In most cases, this does not point to a lack of mathematical knowledge but rather to the difficulty teachers experience in translating this knowledge, which perhaps is represented in their minds through nonverbal symbols, into verbal tools. Mathematics teachers must generate significant discourse in their classrooms. Such discourse constitutes an organized and connected collection of all their students' and their own intellectual ideas. The job of the teacher is to conduct a discourse that reflects ideas and encourages participants to discuss these ideas, to endorse or refute them and to arrive at valid and agreed-upon mathematical rules that can be implemented in new situations that are similar or different. How can we promote and cultivate teachers who have the awareness and skills to cultivate this type of classroom reality?
Wagganer (2015) proposed five strategies for supporting meaningful math talk in class. First, teachers must talk with their students and arrive at common insights regarding the importance of math talk in the classroom. Second, teachers are responsible for teaching their students to listen and respond appropriately to one another. Third, teachers must teach their students to write sentence stems to emphasize their responses. Fourth, teachers must teach and demonstrate the difference between explaining and justifying what someone else says. Finally, teachers must provide examples of all these actions in class. The current study implemented all of Wagganer's ideas with pre-service and in-service teachers in the general context of group mathematical discourse and the particular context of written mathematical discourse in unexpected situations in the mathematics classroom.

Methodology

Participants

The research participants included undergraduate students taking a course that taught didactic and pedagogic skills for teaching math in elementary and junior high school and graduate students in mathematics education who teach math to all ages and at all levels. The two groups together totaled 35 students, as half of them were teachers were in fact teachers.

The research tool

Intervention design –The two courses comprised the same several stages. First, the students read the article by Zazkis and Koichu (2015) about fictional dialogues in order to understand and define fictional dialogues in the context of their unique methodological role in the original article. Next, we adopted the skill of writing fictional dialogues as a tool for developing spoken and written

mathematical discourse in lesson planning for unexpected situations in the math classroom. We embraced the following quote with the understanding that we as students also seek interesting learning methods. "People are eager for stories. Not dissertations. Not lectures. Not informative essays for stories" (Haven, 2007, p. 8).

Third, we defined and formulated conditions determining whether a potential fictional dialogue met the objective. In this stage, we read mathematical dialogues from various sources that resembled fictional mathematical dialogues and we reworked their mathematical discourse so it matched our definition of a fictional dialogue. Fourth, the students independently wrote fictional mathematical dialogues. In the fifth and final stage, the students showed their dialogues to their classmates. This generated an evaluative argumentative discussion and, if necessary, led to redesigning the dialogues. Throughout the course, we documented the sessions and their outcomes *focuses on fictional Mathematical dialogues.*

Definition of "fictional dialogue" in the current study

The definition of fictional dialogue emerged from agreement among all course participants and included the following characteristics: The dialogue must take place between two people with some sort of major gap between them. This gap may be rooted in culture, age, expertise, historical period (e.g., one speaker lives in contemporary times and the other lived 700 years ago), mathematical knowledge and more. One speaker is an expert in the field and should be able to bridge the gap through argumentative dialogue that leads the two speakers to understanding, definition and agreement on the mathematical topic they are discussing. The expert presents the mathematical explanation using formal intra-mathematical tools and extra-mathematical or other simple, practical and concrete examples and explanations. The non-expert participant's dialogue develops in unexpected directions, so that this participant can surprise the expert with questions or examples that seemingly contradict the mathematical concept under discussion or that present a challenge to the clear, simple and popular explanation. In the dialogue, the two participants express their perceptions of the mathematical topic being discussed, and each attempts to enrich the other's world through the mathematical knowledge at his or her disposal. Through the dialogue, the gap between the speakers becomes smaller in that all the relevant mathematical nuances in the field find expression in the dialogue.

Data analysis

The data analysis focused on the process of establishing the conditions for fictional dialogue.

Findings

In this paper, I describe one mathematical event representing two stages of the intervention period. Because the research focuses on the final product—"writing"—I give two examples of writing and discuss the processes involved in creating them. These two examples show that writing fictional mathematical dialogues can serve as a training and professional advancement tool for pre-service and in-service math teachers. The first finding refers to the third stage of the intervention period, in which we redesigned a dialogue and rewrote it as a group fictional dialogue. At this stage, each student individually redesigned the dialogue by writing a new dialogue based on the existing dialogue and thus creating a new personal product that conformed to the required conditions. In the next stage in the joint group work, the students showed their dialogues to their classmates for

evaluation, leading to writing an agreed-upon group product. The dialogue is the unified product after the group discussed their differences and went through the entire learning process.

Group design of a given dialogue and its transformation into a fictional dialogue

The given dialogue is from an Abbott and Costello movie titled *Buck Privates*:

Abbott: You're 40 years old, and you're in love with a little girl, say 10 years old. You're four times as old as that girl. You couldn't marry that girl, could you?

Costello: No.

Abbott: So you wait 5 years. Now the little girl is 15, and you're 45. You're only three times as old as that girl. So you wait 15 years more. Now the little girl is 30, and you're 60. You're only twice as old as that little girl.

Costello: She's catching up?

Abbott: Here's the question. How long do you have to wait before you and that little girl are the same age?

Costello: What kind of question is that? That's ridiculous. If I keep waiting for that girl, she'll pass me up. She'll wind up older than I am. Then she'll have to wait for me!

In order to determine whether this qualifies as a fictional dialogue, we mapped it to see whether it fulfills the conditions for fictional dialogues formulated in the second stage of the course. The mapping results indicate that the dialogue does not meet the conditions to qualify as a fictional dialogue. Hence, we redesigned the dialogue to fulfill the necessary conditions. Each course participant individually designed and wrote a fictional dialogue. In the next stage, the students as a group combined these individual dialogues into a fictional group dialogue. The group dialogue features an expert "player" called Achilles, provides intra- and extra-mathematical explanations, stresses the perceptions of each of the speakers so that it is clear who represents the erroneous perception and who represents the appropriate perception and stresses the unexpected situation. Using the ideas from the individual dialogues, the group wrote an argumentative fictional dialogue that gap the discrepancy between the speakers to the point of generating an unexpected situation in which the speakers "reverse" their roles, so that the rookie, Costello, triumphs over the expert, Achilles. The following lines from the dialogue demonstrate compliance to condition (4) as written by the group.

……..

Achilles: I, Achilles, run at a speed of 10 meters per second. My friend the turtle runs 1 meter per second. I decide to give the turtle a head start of 100 meters at the beginning of the race.

Costello: Wait a minute. This is a fable, right? So I want to convert it to apply to me. I gave the girl a forty-year head start. Wow, that's a lot. I am four times older than she is! And you run ten times faster than the turtle. Great, I get it.

……

Costello: So let's assume I'm 240 years old. How old will she be??? She will always be 30 years younger than me, so she'll be 210 years old. So her age will be seven-eighths of my age. It appears we are slowly advancing to the point where we're the same age.

Achilles: No. That's not right. Let's go back to my turtle.

Costello: I'm not going back to your turtle because I've discovered the problem and also the solution. The girl and I will never get married because there will always be a fixed difference of 30 years in our ages. But there is not a fixed difference of 100 meters between you and the turtle because around 12 seconds after the beginning of the race you will already catch up with the turtle. In ten seconds you run 100 meters and in another two seconds you run another ten meters, so the race is over because the turtle continues to trail behind you.

Achilles: Does that mean that the age difference problem is not representative of infinity.

Costello: Now we've switched roles. I'm the mathematician. What are you? The concept of infinity can be represented if the beautiful girl and I live forever and do not die. From a situation in which I'm four times her age and then three times her age, we get to a situation where the ratio is 7:8, and we can go on to 8:9 and even further. While the discrepancy in our ages is still thirty years, with time the relative difference in our ages gets smaller. In contrast, when you race against the turtle, a quick calculation tells me that you'll overtake your opponent after 12 seconds. The 100-meter difference between you is not fixed because you "grow at different rates" That is, you each run at a different speed and you are ten times faster than the turtle.

Dialogue analysis "Age difference problem"

In the above dialogue, the students completed all the conditions that were missing from the original given dialogue. They created two fictional characters and delineated a significant historical and mathematical gap between them. They defined an expert speaker who led the dialogue. They formulated intra-mathematical explanations (e.g., speed as the ratio between distance time) and extra-mathematical explanations (e.g., representing the concept of infinity by means of the girl and Costello, who grow forever and never die) for the age problem and for the paradox of Achilles and the tortoise. Furthermore, they created two unexpected situations in the dialogue. One was the comparison between the age problem and the Achilles paradox. The other was that Costello understood the difference between the problems and claimed that the turtle problem differs from the age problem ("Now we've switched roles. I'm the mathematician. What are you?"). They created a specific explanation for the problem and its concepts and accurately differentiated between the two problems. Using the dialogue, they understood that the age problem demonstrates Costello's misconception about the age gap, as he thought the gap would decrease over time.

In contrast, the turtle paradox shows that the gap between the turtle and Achilles is not fixed and that the distance decreases with time. Using numbers, the students demonstrated the two situations, showing that the gap in the age problem remains constant while the distance between the turtle and Achilles continues to diminish. At this stage, they reduced the gap between the speakers' dialogue.

During the group formulation, the students explored ideas and mathematical explanations. They designed and formulated the dialogue as a group exercise, so that in cases of disagreement they stopped and sought a consensus in the group.

Discussion and conclusions

The current study is a pioneer in this field. The research was inspired by studies that examined student writing in math classrooms (Bostiga et al., 2016) and writing-based pedagogies (Korkko et al., 2016; Zazkis, at el 2009). The study implemented Read's (2010) method using the IMSCI

model. Implementing this model one-step at a time was found to be effective and to validate the results of studies claiming that only theories that are practically applied in the training process can be properly implemented in the field (Anderson & Stillman, 2013; Bråten & Ferguson, 2015; Cheng, Tang, & Cheng, 2012; Gomez Zwiep,. & Benken, 2013). That is, it would have been more effective to teach the theory of fictional dialogue in the course and then to practice it step by step (IMSCI) through actual writing.

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Tensions in the role of mathematics coaches

Annette Rouleau

Simon Fraser University, Canada; annetterouleau@gmail.com

The use of mathematics coaches as a means of professional development for teachers is an increasing phenomenon in North American schools. The research presented here identifies tensions experienced by mathematics coaches and how they cope with those tensions. Utilizing a framework that characterizes tensions as dichotomous pairings, the results indicate that there are tensions that are unique to mathematics coaches. This adds to a growing body of research into the role of mathematics teachers.

Keywords: Tensions, mathematics coaches, professional development.

Introduction and background

In their search for school-wide models that support improvements in the teaching and learning of mathematics, districts are, "embracing coaching as a model of authentic professional development wherein teachers can learn in the context of their schools and their instructional practice" (Campbell & Malkus, 2014, p. 213). Underlying this development is the recognition that schools need to become places where both students and teachers can learn (Hawley & Valli, 1999). With this in mind, districts have begun placing mathematics specialists in their schools to work directly with practicing teachers (Anstey, 2010).

Bearing a variety of labels, such as mathematics specialist, numeracy specialist, lead teacher or learning coach, a mathematics coach is generally a highly knowledgeable teacher hired to support the improvement of mathematics teaching and learning within a district (Anstey, 2010). An effective mathematics coach would have a deep understanding of mathematical content combined with pedagogical expertise and strong interpersonal skills. Usually they are former classroom teachers, recognized for their abilities and promoted from within (Campbell & Malkus, 2011).

Mathematics coaches are responsible for providing ongoing professional development of the inservice teachers in their districts by "advocating for their change, nurturing their performance, advancing their thinking, increasing their mathematical understanding, and saluting their attempts" (Campbell & Malkus, 2011, p. 459). To reach this goal, mathematics coaches' work varies from modeling mathematics lessons in a teacher's classroom to observing and supporting a teacher as they teach (Campbell & Malkus, 2014). This is a varied, demanding role that Campbell and Malkus (2011) suggest "the profession does not understand and is only beginning to examine" (p. 449).

The tensions experienced by mathematics coaches is one such unexamined area. A review of the literature reveals little information about the dilemmas mathematics coaches face. Literature regarding generalist coaches is much richer and suggests several common tensions. For instance, Neufeld (2003) identified a series of tensions experienced by generalist coaches that begins with a lack of time. This is a frequently experienced tension, whether it is a lack of time to conference with teachers or lack of time to prepare for working with teachers. Leaving the classroom environment causes tensions for some generalist coaches, as does the switch to working with adults. Tensions also occur when teachers are slow to uptake change or are actively opposed to its implementation. Finally, Neufeld (2003) suggests tensions for generalist coaches arise from working with uncooperative

school cultures or administration and from a lack of opportunities for personal professional development. While there are perhaps some commonalities between these tensions experienced by learning coaches in general and those experienced by mathematics coaches in particular, it would be of benefit to identify whether there are any tensions specific to mathematics coaches. Jones (1995) suggests that "members of the mathematics education community, whether in schools, colleges, or universities, have a responsibility to help one another recognize and deal with tensions in a productive way" (p. 233). The intent of this paper then, is to identify some of the tensions experienced by mathematics coaches.

Theoretical background

Endemic to the teaching profession, tension encompasses the inner turmoil teachers experience when faced with contradictory alternatives for which there are no clear answers (Adler, 2001; Berry, 2007). Building on the work of Berlak and Berlak (1981) who identified sixteen dilemmas that illuminated the relationship between everyday school events and broader social, economic, and political issues, it was Lampert (1985) who first suggested the notion of teachers as dilemma managers who accept conflict as useful in shaping both identity and practice.

For the purposes of this study, I turn to the work of Berry (2007) whose self-study of tensions in her role as a teacher-educator resulted in a binary categorization of tensions. Seeking to depict the inner turmoil she experienced from the competing pedagogical demands in her practice, she proposed a framework for both identifying and understanding tensions. Isolating the following six pairs of interconnected tensions, Berry used these as a lens to examine her practice: (1) Telling and growth–between informing and creating opportunities to reflect and self-direct (2) Confidence and uncertainty–between exposing vulnerability as a teacher educator and maintaining prospective teachers' confidence in the teacher educator as a leader (3) Action and intent–between working towards a particular ideal and jeopardising that ideal by the approach chosen to attain it (4) Safety and challenge–between a constructive learning experience and an uncomfortable learning experience (5) Valuing and reconstructing experience–between helping students recognise the 'authority of their experience' and helping them to see that there is more to teaching than simply acquiring experience 6) Planning and being responsive–between planning for learning and responding to learning opportunities as they arise in practice (Berry, 2007, pp. 32–33).

Although initially used as a framework to isolate tensions in the work of teacher education of pre-service teachers, Berry's (2007) framework has been used in other contexts as well. As part of a larger, ongoing project, of which this paper is a part, Liljedahl, Andrà, Di Martino, and Rouleau (2015) applied Berry's tensions framework to a fictional composite of a mathematics teacher that comprised a collection of data sets. Their work expanded Berry's framework by identifying new tension pairs and they concluded that some tensions may be the driving force behind a teacher's pursuit of professional development by fueling a desire for change in practice.

While considered relative newcomers, mathematics coaches are part of a mathematics community that includes both pre-service and inservice teachers. Given that developing a shared understanding of the tensions teachers face gives them the power to shape the course and outcomes of their teaching practice (Adler, 2001); it is likely the same would be true for mathematics coaches. Bringing the

challenging aspects of their work to light would offer mathematics coaches the opportunity to recognise, talk about, and act on the tensions in their practice.

My goal then, in this article, is to isolate some of the tensions experienced by mathematics coaches. Specifically, using Berry's (2007) framework, I will identify and describe the tensions they face and how they cope with them. Thus, my research questions are as follows: (1) What are some of the tensions experienced by mathematics coaches? (2) How do mathematics coaches cope with those tensions?

Context and method

This study is part of an ongoing research project regarding tensions in teaching. In particular, it is the first look into the tensions experienced by mathematics coaches. This is a small scale, qualitative study that involves only three participants. As such, I am focusing on proving the existence of a phenomenon rather than its prevalence. It is important to note, however, that I chose to report only on those tensions that were experienced by more than one participant. While aware that a single instance of a tension can be as revealing as multiple instances, it is less likely to be seen as representative of a generalizable pattern. In keeping with that, the data corpus comprises interviews with three mathematics coaches working in three separate school districts.

Tara is employed by a small urban school district that employs 430 teachers in 17 schools. She has been in the role of K-12 District Numeracy Coordinator for 4 years. Prior to that, she worked as an elementary classroom teacher for 18 years. Having had negative experiences as a learner of mathematics, Tara's interest in math was only ignited 14 years ago after attending a mathematics professional development workshop. During the ensuing years, she attended every mathematics professional development opportunity offered and developed a passion for the teaching and learning of mathematics. She is about to begin a Master's degree program with a numeracy focus.

Pam works for a small rural school district with 23 schools and 320 teachers. Employed as a classroom teacher for over 30 years, she has taught all grades K-7. Pam completed a Master's degree with a numeracy focus in 2011 and left the classroom in 2012 to take on the role of Numeracy Helping Teacher. She has always enjoyed and had a passion for math.

Ray is employed by a large urban school district that employs over 1000 teachers in 49 schools. He worked for his district for 18 years as a secondary math teacher before taking on the role of Math and Science Program Consultant 4 years ago. During his time in the classroom, Ray completed a Master's in secondary mathematics education and was involved with his province's math teachers' association. Ray went on to serve a term as the association's president, while working as his district's math consultant. Like Pam, Ray has always enjoyed math and wants to recreate that experience for the teachers and students in the classrooms he supports.

Data was collected from the participants during semi-structured interviews that were transcribed in their entirety. The data corpus was then scrutinized using Berry's (2007) framework as an a priori frame for identifying and coding tensions. To begin this was done by searching the interview transcripts for evidence of tensions. In particular, I looked for evidence of utterances with negative emotional components such as "I think what's been difficult…" or utterances that conveyed doubt or uncertainty such as "I wasn't 100% sure, but…". The identified tensions were then grouped according

to the pairings described by Berry. Additionally, the framework was extended to encompass a tension that did not fit within her established framework.

Analysis

In the following analysis, Berry's (2007) framework will be used to identify and analyze the tensions experienced by the three participants in their roles as district mathematics coaches. During the analysis, the following four tensions pairs were evident.

Safety and challenge - Unwelcome in the classroom

All three of the participants mentioned the conflict they experienced between their desire to be working with teachers in their classrooms and not having that support seen as threatening. They describe the teachers as uncomfortable in having someone observe them and therefore are unable to utilize this valuable learning opportunity. A tension arises for the mathematics coaches who, like Berry (2007), want the teachers to feel safe, but who also recognize the value in challenging the teachers to open their doors. This is evident in the following excerpts:

> Ray: And I think that teachers are a little reluctant to have people in their classroom and do, sort of team teaching or have someone observe them… that hasn't happened as much as I kind of thought it would or as much as I'd sort of like.
>
> Pam: The kind of biggest piece, I think, for us, is how do you support those teachers that are too nervous or too anxious about having someone come in?
>
> Tara: If I get invited in, I'm in, I go. Absolutely. But unless I'm invited in, it doesn't, like, I don't just, well, I shouldn't say I don't just show up…. but to be actually modelling in a classroom and doing observations, that's all by invite.

As a rationale for the teachers' reluctance, both Pam and Ray offer related possibilities. Pam suggests that the teachers' reluctance stems from a fear of being evaluated even though she feels she makes it clear that her role is one of mentorship and has no evaluative elements stating, "They haven't shifted away from that fear yet, that I'm there to judge. I'm not, I'm there to support them." Ray suggests that the teachers are concerned with the overall quality of their lesson, which then becomes a barrier to observation, "When it comes down right to it, you know, we're all a little bit unhappy with every lesson we ever do so I don't really want you seeing me because, you know, it's got its warts and all that stuff. And so, a lot of good people, but not necessarily wanting people in their classrooms."

For all three, this appears to be an unresolved, ongoing tension in that none have successfully found ways to make classroom visits an accepted part of their roles. Pam, in particular, mentions that this tension leads her to consider ways of presenting this opportunity to learn as risk-free noting, "Well, I'd really like to be in more rooms and influencing more teachers. I'm trying to think of ways I can do that to support them."

Valuing and reconstructing experience - Resistance to change

Another of the tensions that was apparent for all three participants was similar to Berry's (2007) tension of valuing and reconstructing experience. The mathematics coaches experienced a dilemma between acknowledging the authority of the teachers' experience and helping them to see that there

is more to teaching than simply having acquired a requisite amount of experience. This is best exemplified by Ray in the following excerpt:

> Ray: I think the biggest barrier tends to be, as teachers, we've gone through a system a certain way that we can visualize how it looks in the classroom. We've taught that way and we see successes in that, in either ourselves or some students, and we hang onto those successes as sort of validation for doing what we do. And we tend to say, 'Well, those other kids just aren't being successful. They're not working hard enough. They're not trying hard enough. They need to do things differently. They need to change.' And I don't think a lot of teachers are as good at saying, 'Well, what do I need to do differently? What do I need to do to change?'

The mathematics coaches value the experience their teachers have, but know that experience can always be broadened and improved. None of data from the participants suggest they use a deficit model approach to coaching teachers, but rather they believe there is always room for growth. This belief perhaps stems from their own experience with life-long learning. They want the teachers they work with to consider which areas of their practice would benefit from further learning and support. As Ray suggests, "And, I think if teachers just come out a little bit more with the willingness that, you know, as good as I am, (laugh) I probably could be a lot better. That would be very helpful."

The data suggests that all three mathematics coaches see this tension as a resistance to change and this manifests in different ways. For Tara, who described her own career in terms of ongoing growth and change, the tension stems from the assumption that her colleagues would be open to similar experiences. She finds it difficult to accept that change is slow stating, "So I made the assumption that was once other teachers kind of have these a-ha moments [as she did], they would just fly and I've come to realize that's not the case."

Ray also experiences tension in slow change, but notes that, while "teachers can be very confident about some things and don't necessarily challenge themselves as much as they could", it is possible that "as much as we sometimes want to change, it's a lot of work to change and people only have so much time in the day so they sometimes just don't even get started." Ray's view suggests that outside influences play a role in teachers' readiness or willingness to change.

Pam views the resistance to change as more of a readiness factor. Her tension lies in the fact that the teachers she works with are not as ready to reconstruct their teaching experiences as she would like them to be. She recognizes that she "wants them to try more than they're capable of trying" and is aware that she's "not giving them time to slowly implement what they're comfortable with". She values the experience they bring, but struggles to encourage them the consider new practices.

This too appears as an unresolved tension that all three mathematics coaches deal with on an ongoing basis. Pam was the only participant to offer a partial solution, albeit unsatisfactory to her. She approaches this tension with perseverance tinged by frustration saying, "Well, I think, you kind of got to persist, but it can kind of get a little frustrating at times."

Confidence and uncertainty - Questioning role and ability

A tension that surfaced for both Tara and Pam correlates closely with Berry's (2007) tension pairing of confidence and uncertainty. Both coaches mention having had colleagues question their role and their qualifications. This created a tension between the necessity of exposing their vulnerability and

maintaining the confidence of the teachers they mentor. Tara mentions, “You get the naysayers in the room that might, you know, question you on things. The biggest thing I get is what are your qualifications to do this job. That's what I get all the time.” And Pam adds:

> Pam: I've had people that have said to me, I know enough that I don't really need you and I don't understand why the district is wasting money on your job. It's the senior math people, the 10, 11, and 12, that are the hardest to influence and they don't want to be influenced by me. I've been told many times by them that I have not the experience.

For both Pam and Tara, this appears to be a managed tension. Although the questions regarding their qualifications continue to be asked, neither seem to regard it as an ongoing source of tension. Both admit to limiting their role to elementary and junior high school and, for Pam, holding a Master’s in Numeracy was perhaps sufficient to manage any remaining tension. Tara chose two other methods, which appear to offer the credibility she needs to answer any questions—she outlines her credentials and acknowledges the research behind best practice in mathematics:

> Tara: So what I started to do more of after that was, whatever I was giving a recommendation for, I always had research to back up my recommendation. So I was always presenting what the research was saying. Always. […] I lay out what courses I've taken, the journey I talked to you about, and why it's a passion. They seem to be a little better once they hear that story.

Initiative and systemic barriers - Working with learning assistants

This is a tension pairing that extends Berry’s (2007) framework as it does not have a counterpart within her original set of tensions. It surfaced when the mathematics coaches were asked what they would like to implement in their role but have not been able to as of yet. Both Pam and Ray mentioned working with learning assistants. A strong desire to support every adult involved in the learning of students in their district drives them to want to work more closely with the learning assistants. Yet to do so would disrupt an existing functioning system. Ray expresses this clearly in the following excerpt:

> Ray: I’ve got a few things that are sort of happening, but not as deeply as I’d like. One of them is the learning support group in our district. They all work a little bit differently and it’s kind of hard to connect with them the way we’re set up in the system.

Their initiative meets with resistant in the form of systemic barriers. In Pam’s case, it is a result of an administration system that limits her contact with colleagues to only classroom teachers. For Ray, it is a result of different priorities. Like Pam, the learning assistants in his district provide both numeracy and literacy support — and that support tends more towards literacy. Ray notes, “They tend to be very heavily focused on reading recovery/writing kind of stuff over the years and they just haven’t had a lot of time to get together and talk about anything around math.”

Both Pam and Ray mention wanting to circumnavigate the systemic barriers and provide professional development to the assistants, who, in their respective districts, tend to work one-on-one in pullout environments with students. Ray wants the opportunity to offer more effective resources. Pam agrees, adding, “They are sending these support people out to work with students, but they’re working same old, same old. The child gets the same kind of repetitive practice over and over again and it never moves them forward.”

This is an unresolved tension that has both mathematics coaches searching out solutions. Ray offers the vague "hope" that he will be able to connect with the learning assistants this coming school year, but does not go deeper into his plan. Pam plans on speaking with her assistant superintendent to seek her assistance in convincing the learning assistant teachers that she is capable of providing them with support.

Discussion and conclusion

The first goal of this study was to identify tensions experienced by mathematics coaches. Three tensions emerged that closely aligned with the tension pairings in Berry's (2007) framework. The fourth was a tension the mathematics coaches experienced in their desire to support learning assistants. With no obvious parallel in Berry's work, likely due to her role as a pre-service teacher educator, the presence of this tension requires the framework to be extended when considering tensions experienced by mathematics coaches.

The findings also revealed tensions that could be considered unique to mathematics coaches, as there were two tensions they experienced that were not included in the list of tensions identified by generalist coaches. The first finding suggests that mathematics coaches may experience tension regarding their qualifications. This could be explained by the expectation that a specialist in one subject would be expected to have specific skills that a generalist, who works across all subjects, would not be expected to have. Additionally, given that many of the mathematics coaches are pulled from teaching positions within their districts (Campbell & Malkus, 2011), their former colleagues might question their abilities. Interestingly, Ray, who was a secondary mathematics teacher has never experienced this tension. Despite having no elementary experience, he stated that he has always been "well received" by the elementary staff. His status as a high school mathematics teacher appears to offer him credibility across all grade levels. The second tension experienced by mathematics coaches, but not generalists, was working with learning assistants. This might be the result of the relative newness of the role of mathematics coaches (Anstey, 2010). Districts are still in the process of determining the scope of the responsibility of the mathematics coaches in their employ. As Tara suggests, "As for the job itself, it was pretty much I just had to build the airplane as I was flying it."

The second goal of this study was to identify how the mathematics coaches coped with the tensions they experienced. The findings suggest that they appear to fit Lampert's (1985) image of dilemma managers who accept and cope with continuing tension. This means that the mathematics coaches initially manage the tensions that surface while never fully resolving their competing conflicts. What was interesting was the managed tension that Pam and Tara, both of whom are elementary trained teachers, experienced regarding questions about their role and qualifications. Their method for managing this tension was avoidance of interactions at the high school level. Similarly to the finding in Liljedahl et al. (2015), this suggests that this tension is managed on some levels but there is a possibility it could resurface at some point. While both are only required to work with teachers who volunteer and are willing, both of their roles encompass grades K to 12.

While the small number of participants in this study may limit its generalization, the findings do indicate the presence of tensions experienced by mathematics coaches. This language of tensions could be useful as a means for discussion and reflection on the practice of mathematics coaches. Whether managed or unresolved, identifying and describing these tensions will contribute to a small,

but growing body of research into mathematics coaching. If employing mathematics coaches in schools is to be a viable complement to professional development, more study will be necessary.

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How to deal with learning difficulties related to functions – assessing teachers' knowledge and introducing a coaching

Ute Sproesser[1], Markus Vogel[1], Tobias Dörfler[1] and Andreas Eichler[2]

[1]Heidelberg, University of Education, Germany; sproesser@ph-heidelberg.de; vogel@ph-heidelberg.de; doerfler@ph-heidelberg.de

[2]University of Kassel, Germany; eichler@mathematik.uni-kassel.de

Students often have difficulties with the content area of functions. If their teachers are not aware of these problems and lack of adequate teaching methods, they cannot counteract pointedly *in their classrooms. This paper presents a project developing and evaluating a coaching to foster teachers' pedagogical content knowledge about several learning difficulties with functions and about how to respond to them. As this work is still in progress, we here focus on the project description as well as on the development of the survey to measure teachers' corresponding knowledge.*

Keywords: Mathematics education, teachers' professional development, functions, learning difficulties, teacher survey.

Introduction

Being able to adequately reason with functions is considered to be a central goal of mathematics education (e.g. Eisenberg, 1992; KMK, 2003; NCTM, 2000). More precisely, reasoning with functions characterizes a specific way of thinking in interdependencies, relationships or changes (Vollrath, 1989) that is especially required when working on inner- and extra-mathematical problems (Hinrichs, 2008; NCTM, 2000).

However, several studies show that learners have particular difficulties in this domain (see for an overview Nitsch, 2015 or Vogel, 2006). For instance, they may experience problems with the meaning of the parameters (e.g. Schoenfeld et al., 1993), conceive graphs as pictures (e.g. Monk, 1992), confound the slope and the height of graphs (e.g. Hadjidemetriou & Williams, 2001) or have difficulties with word problems in the sense of the word-order-matching-process (e.g. Clement, 1982). Often, their teachers are not aware of these difficulties (Hadjidemetriou & Williams, 2002; Sproesser et al., in press) and therefore cannot counteract explicitly. Moreover, the study of Nitsch (ibid.) revealed systematic differences between school classes referring to learning difficulties with functions. She concludes from this finding that some teachers are more successful than others in responding to such difficulties.

Theoretical background

The findings mentioned above raise the assumption that teachers' professional development (TPD) focusing on such typical learning difficulties may enhance teachers' pedagogical content knowledge (PCK, see e.g. Shulman, 1987), their instruction and mediate also students' learning in this field. This is also in line with the general understanding that teachers need TPD in order to meet the challenges that they encounter in their professional lives as university studies cannot satisfy all of demands from practice (cf. Mayr & Neuweg, 2009). To our best knowledge, there is no empirical

evidence about the effects of a TPD related to learning difficulties with functions, especially taking into account the interplay between the teacher and student level, yet.

Particular TPD-characteristics that have already proven to be effective in general can be implemented in a TPD referring to dealing with learning difficulties in the domain of functions. In this context, Lipowsky (2013), for instance, found that TPD should be related to one specific domain instead of focusing on different domains. Furthermore, long-term TPD courses enable to integrate input, practice and reflection phases. The study of Lipowsky (ibid.) additionally confirmed that giving feedback (e.g. Shute, 2008) supports learning also in the context of TPD. Teacher coaching represents a specific form of TPD that can also implement the mentioned characteristics. In adaptive (teacher) coaching (Leutner, 2004), the coach refers back to the teachers' statements and activities. If teacher coaching focuses on a concrete classroom situation, it is, for example, possible to encourage teachers to reflect on this situation (West & Staub, 2003) or to train them in giving supportive feedback to students showing a particular learning difficulty.

In several studies, such a focus on responding to students' difficulties or errors (e.g. through giving feedback) has already shown to be useful in order to measure or promote teachers' PCK concerning different mathematical content areas: For instance Biza et al. (2007) propose to measure (pre-service) teachers' PCK by requesting them to analyze wrong student solutions and to formulate supportive feedback. The study of An & Wu (2012) revealed that teachers' PCK can be fostered through asking them to analyze students' errors and to develop approaches how to correct them.

Research goals

The teacher coaching developed in this project aims at building up teachers' pedagogical content knowledge related to learning difficulties with elementary functions and hence to support also their instruction and students' achievement in reasoning with functions. As a narrow content focus has proven to be a characteristic of effective TPD (Lipowsky, 2013), we here refer to specific PCK components as defined by Ball et al. (2008): In the case of 1) knowledge of content and students (KCS), we focus at fostering teachers' knowledge about typical learning difficulties and about students' thinking related to functions; in the case of 2) their knowledge of content and teaching (KCT), we train them in "adequately" responding to such specific learning difficulties. Within the content area of functions, we concentrate on linear functions and on the understanding of the concept of a bivariate functional relationship in order to assure a narrow content focus. The emphasis on this subdomain also takes into account that viable concepts about elementary functions appear to be crucial for understanding higher-order functional classes later on.

As the majority of existing TPD courses is not carried out in an experimental design, it cannot be clearly identified which of their characteristics would be responsible for a certain effect (e.g. Yoon et al., 2007). Therefore, this teacher coaching is brought out via two variations, namely with and without focus on feedback. This procedure takes into account findings from other studies showing positive effects of giving feedback (see above) but additionally evaluates the effectiveness of this characteristic (explicitly training to give feedback to students showing concrete learning difficulties). In this sense, the main goal of the project described in this paper is to prove what effective aspects of an adaptive teacher coaching are.

More precisely, we evaluate the following research questions:

- What do teachers know about typical learning difficulties in the domain of functions and what ideas do they have how to react to them (pretest)?
- To what extent can teachers' KCS and KCT related to functions be fostered through two variations of teacher coaching (pre- and posttest)?
- Which impact do the coaching treatments have on students' domain-specific competence?

Methods

Pilot study

The content of the coaching was identified via a pilot study in the academic year 2014/15 (see Figure 1 for an overview of the project's structure): Part I of the pilot study revealed that all of the learning difficulties derived from the literature (see Introduction) occurred among students within our learning settings (paper-and-pencil-tests in 4 classes of grade 7 and 8). Moreover, we found that their teachers only knew some of these learning difficulties and that their knowledge about them and about how to respond to them was very heterogeneous (interviews with 4 teachers). Therefore, TPD in this domain appears to be useful. A summary of these results can be found in Sproesser et al. (in press). As to our knowledge there is no consensus about how to "accurately" respond to such learning difficulties or how to largely prevent them, teacher trainers and university educators were interviewed about these issues within part II of the pilot study. Via these expert interviews, we collected and further developed teaching ideas, methods and material for the coaching.

Main study

Within the main study, the teacher coaching (3 modules) accompanies the instructional unit of linear functions in grade 7 or 8, respectively: Module 1 is held before, module 2 during and module 3 after this unit. This structure enables to implement the content of the coaching in the teachers' classroom as well as to reflect on the teachers' experiences within the TPD. About 60 teachers of grade 7 or 8 are assigned to one of two treatment groups or to a control group. Both treatments contain input, reflection and activity phases in order to foster teachers' KCS and KCT related to learning difficulties concerning elementary functions. Only in one of the two treatment groups, teachers are specifically trained in giving supportive feedback to students facing a particular learning difficulty.

In order to gain empirical evidence about effective characteristics of the coaching, the teachers' PCK as well as their students' knowledge related to elementary functions are assessed before and after the coaching / teaching unit. This data structure allows using analysis tools such as multilevel analyses and hence to evaluate the interplay between the two levels. The student survey (pre-, post- and follow-up-test) contains large parts of the test instruments developed by Nitsch (2015): Via a number of tasks referring to elementary functions, several learning difficulties (see above) can be identified. Moreover, covariates such as students' cognitive abilities (Heller & Perleth, 2000) or motivational variables (Pekrun et al., 2002) are gathered.

In order to measure KCT and KCS of the participating teachers, we developed a survey that particularly refers to several tasks of the student test. The development and the structure of the teacher survey will be presented in more detail in the next section.

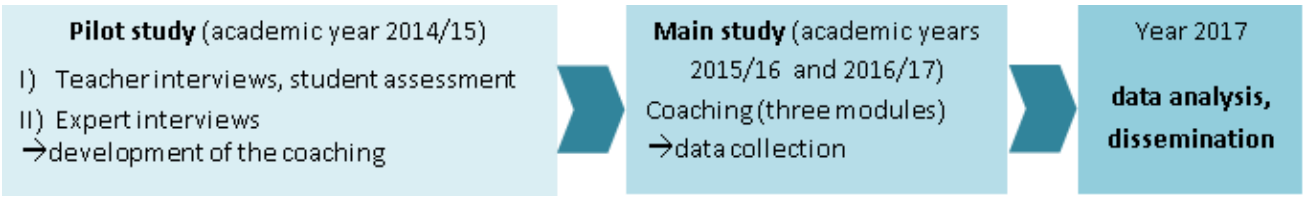


Figure 1: Outline and content of the project

Teacher survey

The participating teachers are requested to complete before and after the coaching a structurally identical paper-pencil-survey. This procedure allows directly investigating teachers' KCS and KCT developed in the course of the coaching. The PCK items of the teacher survey are all structured in the same way (see Figure 2 for a sample item): The teachers are shown a task of the student test and they are asked about typical mistakes or learning difficulties referring to this task (questions a) and b) in Figure 2) and how they would respond to them (question c) in Figure 2). Hence, according to the classification of Ball and colleagues (2008) the questions a) and b) are part of the knowledge component KCS as "Teachers must anticipate what students are likely to think and what they will find confusing" (ibid., p. 401). These authors propose to measure teachers' KCS for instance via questions about what students may find difficult or about interpreting students' thinking. Within our survey, teachers in question a) are asked which mistakes and learning difficulties they had already noticed concerning the given type of task; in question b), on the basis of a wrong student solution they have to put theirselves in a student's position in order to make transparent his or her thinking process when working on the task. Hence, these tasks require knowledge of typical student (mis)conceptions and errors as well as about students' thinking. The third PCK item (question c) in Figure 2) corresponds to the knowledge component KCT (Ball et al., ibid.): Teachers need to know about mathematics and about teaching in order to sequence their instruction and hence to promote students' understanding. For instance, they need to know different methods and procedures and choose appropriate ones for their instruction. This means that KCT is particularly relevant when teachers respond to students' mistakes and difficulties or when they aim at building up viable concepts through their instruction. Ball et al. (ibid.) propose to measure KCT e.g. by asking for examples for simplifying particular content or how learning of a specific content can be facilitated. As displayed in Figure 2, such KCT items are also included within our test instrument: In question c), teachers are asked to outline how they would react to a concrete student mistake.

Within the whole survey, the sequence of PCK items is always as displayed in Figure 2: The first question a) is open-ended in order to collect the teachers' ideas and experiences without being influenced by specifications of the survey. Afterwards (question b)), teachers are confronted with a concrete students' mistake referring to this task and they are requested which (mis)conception could cause the mistake. As in real classroom situations, responding to a student mistake (cf. KCT) happens after its noticing (cf. KCS), the question sequences are always ended up by the KCT item (question c) in Figure 2).

This sequence of questions (a) open-ended, b) referring to a concrete mistake) was chosen to gather data about teachers' knowledge and experience concerning several student problems in general and related to specific mistakes. Within the teacher interviews of the pilot study, this sequence was also

used and proved to provide essential findings. However, one particular limitation of this sequencing should not be disregarded: Teachers could add the mistakes and learning difficulties displayed in b) to the open-ended question in a) even if they had not thought of them without the indication of the survey. We decided to accept this possible drawback that may occur in field studies as the our rather than in laboratory studies because interviews instead of the paper-pencil-survey would be extremely time-consuming for the numerous participants of the main study and could irritate them; furthermore, a digital survey with time markers could hardly be implemented as the coaching is brought out in different schools where we cannot count on a safe internet-connection. In the teachers writings it can mostly be identified if they have come back to a previous item or not.

We consider the relevance and the validity of these PCK items as relatively high because of several issues: First, empirical studies show that the presented learning difficulties are common among students and hence they are relevant for teachers. In their research Ball et al. (2008) similarly have drawn typical student mistakes and learning difficulties from the literature. Furthermore and as pointed out above, the kind of questions that we use are also proposed by these authors. Hence, our approach is not arbitrary but systematical and can also be applied in other content areas.

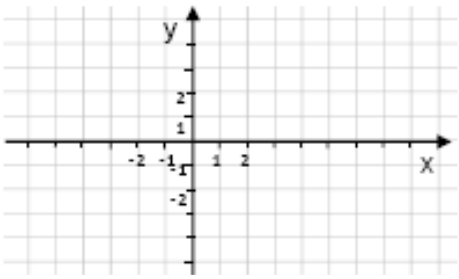
Student task

Draw the graph according to the functional equation

y = 5x – 2

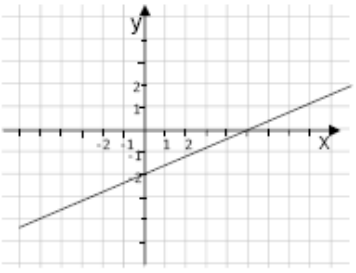
in the given coordinate system.

Explain briefly how you proceeded.



a) Which typical mistakes or learning difficulties would you expect from your experience in this student task?

b) A student solved the task as displayed on the right. Which concept could underlie this solution? Please justify your answer.



c) Imagine you would be confronted with this learning difficulty. How would you respond to it in your mathematics classroom?

Figure 2: Sample item of the teacher survey

In addition to the mentioned PCK items, the teacher survey contains covariates for instance about their professional background (e.g. university degree, teaching experience), beliefs related to mathematics education (e.g. their constructivist conviction (Stern & Staub, 2002), assumed determinants for mathematical ability (Stipek et al., 2001) or their experience with and motivation for TPD (see several scales in Jerusalem et al., 2007).

Current status and future steps of the project

As mentioned above, the pilot study has already been carried out in the academic year 2014/15 and its evaluation is almost concluded. Student assessment and teacher interviews revealed that a TPD referring to dealing with learning difficulties related to elementary functions would be useful for teachers within our learning settings. Moreover, the expert interviews were helpful to gather "best-practice-methods" and material for the coaching.

Concerning the content of the coaching, both treatments focus on the same learning difficulties (problems with the parameters, graphs-as-picture-mistake, slope-height-confusion, emphasis on the word-order in word problems). Teachers get information about their prevalence in empirical studies. Moreover, we illustrate the best-practice-methods and material how to prevent or overcome them that we have gathered through the pilot study. There are also active phases for the teachers: On the basis of the presented methods / material, they are asked to further develop tasks and material for their own classroom. Moreover, based on described classroom-situations showing concrete student mistakes, they are requested to think up a reaction to support the student to overcome his problem. In these tasks, the variation with / without focus on giving feedback comes into play: In the treatment with focus on feedback, teachers are asked to concretely formulate the feedback and explicitly explain the hints that they would use when being confronted with the corresponding student difficulty (e.g. "How would you respond to this learning difficulty in your mathematics classroom? Please be explicit: Verbalize your feedback and illustrate other ways to support the student."). In the treatment without focus on feedback, teachers are simply requested to mention adequate ways of responding to these learning difficulties in a more general way (e.g. "How would you respond to this learning difficulty in your mathematics classroom?"). Furthermore, in both treatments teachers' experiences in the course of the learning unit are discussed and reflected as well as their classes' results - if the teachers agree with students testing before and after the unit.

The coaching has already been carried out in the academic year 2015/16 and it is still offered in the year 2016/17. Hence, the main study is in progress at the moment and data will be gathered at the student and teacher level. Results are expected from the end of the academic year 2016/17 onwards.

Acknowledgment

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Math centers: A pedagogical tool for student engagement in intermediate math class

Amanjot Toor[1] and Joyce Mgombelo[2]

[1]Brock University, Faculty of Education, St. Catharines, Canada; aman.toor@peelsb.com

[2]Brock University, Faculty of Education, St. Catharines, Canada; jmgombelo@brocku.ca

This paper draws from a qualitative exploratory case study that aimed at exploring the learning experiences of teachers as they engage in professional learning project. The case study involved three elementary school teachers' professional learning experiences as they engaged in developing a practical, research-based approach to differentiated instruction using a flipped classroom and student-centered pedagogical approaches that would result in enabling students to be engaged with mathematics..

Keywords: Professional learning, flipped classroom, student-centered, engagement, intermediate mathematics.

Introduction

Research indicates that professional learning, which is job-embedded (Joyce & Showers, 2002), collaborative (Garmston & Wellman, 2003), occurs over time, and is driven by the needs of the teachers involved (Fullan, 1995; Lawler &King, 2000; Little, 2002), is effective. Furthermore, effective professional learning is focused on student outcomes, integrated into the teacher's day-to-day culture, and often tied to the school's improvement process (Way, 2001). The paper draws from a study that aimed at exploring the learning experiences of three teachers engaged in a professional development project in Ontario at an Intermediate level (grade 6, grade 7 and grade 8). The professional development project is part of an initiative of The Elementary Teachers Federation of Ontario (ETFO). ETFO invited and provided support for teams of teachers from the same school or in similar roles at different schools to come together and conduct professional learning projects relevant to their specific professional needs, circumstances and interests. In addition to the three teachers a university researcher was invited to collaborate with the group and conduct a case study of the professional development project. The three teachers in the professional development project collaborated in developing a practical, research-based approach to differentiated instruction *using a flipped classroom approach and student-centered pedagogical approach* that would result in enabling students to be engaged with mathematics. This flipped classroom approach and student-centered approach involved the use of grade-appropriate math centers where students would engage in a variety of math problems and/or topics; have opportunities to practice and consolidate basic facts and operational skills; use technology and manipulative as learning tools; become efficient communicators in math; and develop a sense of self-awareness toward their own math skills. Students would also grow in their ability to work independently and cooperatively as they work through various math centers, allowing the teacher(s) to conference with individuals and small groups of students. In this paper we describe some of the findings from the case study. The case study is guided by two questions: How did the professional development project facilitate teachers' understanding of the use of math centers in a flipped classroom and student-centered approach for

teaching and learning mathematics? How did the teachers negotiate constraints and possibilities as they engaged in their professional development project? .

Research has demonstrated that engaging students in the learning process increases their attention and focus (Jonathan & Aaron, 2012). Further it motivates students to practice in higher-level of critical thinking while promoting meaningful learning experiences. Chickering & Gamson (1987.), states that

> "learning is not a spectator sport. Students do not learn much just by sitting in classes listening to teachers, memorizing pre-packaged assignments, and spitting out answers. They must talk about what they are learning, write about it, related it to past experiences and apply it to their daily lives. They must make what they learn part of themselves".

In this sense, a flipped classroom and student-centered learning is essential. Educators who adopt to a flipped classroom and student-centered approach as a pedagogical method find that it increases student engagement, which allows for learners to successfully achieve the learning objectives (Jonathan & Aaron, 2012). For a flipped classroom and student-centered mathematics class to be effective, a shift in the role of the teacher and students in the classroom must be adapted. On the one hand, teacher's role is viewed as a guide for students' constructive processes towards mathematical meanings and mathematical ways of knowing. On the other hand, learning is viewed as an active, constructive activity in which students wrestle through problems that arise as they participate in the mathematical practices in the classrooms (Cobb, Yackel & Wood, 1992). Recently, there has been an upsurge of interest in instruction that focuses on flipped classroom approach for teaching mathematics.

Type of Math Centers

There were various types of math centers that each participating teacher used in their classroom. The choice of a math center was based on teachers' professional judgment of the students and students' needs.

Inquiry based Center: - A group of 4-5 students rotated from one station to another to learn about various topics. Examples of these topics included: explore and connect station, "what happens when…", various word problems, "Reflection on this", "Test your knowledge." Each station had the option of students either working by themselves and/or in their respective groups. Teachers used as one or more of inquiry based centers to develop curiosity of a given topic among the students.

Resource Center: - This center was available for students all the time. This station consisted of graph papers, blank papers, mathematics dictionaries, mathematics textbook (e.g. *Math on call*, *Math on hand*, and *Math makes sense 8*). This center also included measuring tools such as meter -sticks, rulers, weight measuring scale and measuring tape. This center gave student the opportunity to select an appropriate tool for themselves in order to learn a topic/concept at hand.

Online Research Center: - Students had the opportunity to use their own technology or/and the computer station located in class to deepen their understanding of topic/concept at hand. Students also had the option of exploring a given topic at home via online research.

Debriefing Center: - This is usually available at the end of a lesson, where students come back into their respective groups. Here students are given the opportunity to consolidate their learning as a group, clarify any misunderstandings and learn from one another.

An example of inquiry based math center

Students were given the opportunity to solve a real life problem, using a task called, "how big is a trillion?". In this problem, students were asked two open-ended questions: 1) Is it possible for trillion rice to fit into this room? Explain your solution 2) How much distance around the Earth can the rice cover if each rice is lined up in a line? Compare this with another non-metric unit. Students were asked to solve this problem in group of 4-5. Each group was given 6-7 classes (40 minutes each) to solve the problem in class. Students were allowed to do any background inquiry that they thought was necessary at home. The purpose of working on this problem in class was to have a common working place as a group, where they negotiated their learning and solution. At the end of the 7th class, groups were asked to submit their solution in form of poster which each group presented in the following class. Mathematically, this problem required students to learn about how to measure a unit in real life, length, metric versus non-metric units, volume, and capacity. Other than the mathematical knowledge, students had to discuss their ways of solving the problem, which means selecting appropriate tool to solve the problem in most efficient way, while self-regulating their learning and progress as a group. By the time this particular problem was given, students often became competitive. Teacher noticed that while students helped members of other groups with background mathematical knowledge, groups often tried to keep their solution a secret. This was because groups often wanted their solutions to be a unique solution.

Preparing the students for flipped classroom with math-center included teacher presenting and discussing math center code of conducts, where the purpose and the importance of self-regulation for one's own learning was discussed in depth as a class. In order to utilize various centers, students were divided in groups of 4-5 students, these groups were often changed and redesigned either by the teacher or the students. Students were informed that they may seek help support from their teacher at any point, however they are encouraged to first discuss it with the members of their respective groups.

Flipped classroom

Flipped classroom approach for teaching mathematics is considered as an effective way for engaging students in active learning as well as in meaningful peer-to-peer and peer-to-teacher interactions during the in-class learning process (Forsey, Low, & Glance, 2013; Pluta, Richards, & Mutnick, 2013). Moreover, Bergmann and Sams (2012) indicated that flipped classrooms enable teachers to take individual students' needs into account as well as to facilitate more interactions among peers and teachers in the classroom. The teaching and learning context of flipped classroom approach consists of two kinds of activities: in-class and out-class. In-class time is utilized for inquiry, application and assessment in order to better meet the needs of the individual learners. Technology-assisted out-of-class time involves personal instruction, where students acquire responsibility for their own learning, through studying course material on their own, using various sources (self-discovered and/or provided by teacher). The main goal in flipping a class is to cultivate deeper, richer, and active learning experiences for students where the instructor is present to coach and guide them. Further, emphasis is on higher-order thinking skills and application to complex problems, and which might include collaborative learning, case-based learning, peer instruction and problem set. In this sense the role of the teacher is to facilitate the learning process by helping students

individually and in groups. According to Bergmann & Sams (2012), there are many ways of implementing a flipped classroom approach. For this study, the participating teachers utilized various math-centers (discussed previously in the article) where students self-regulated their own learning in a math class.

Methodology

A qualitative research methodology was used to conduct this case study. According to Yin (2003) a case study design should be considered when: (a) the focus of the study is to answer "how" and "why" questions; (b) the behavior of those involved in the study cannot be manipulated; (c) you want to cover contextual conditions because you believe they are relevant to the phenomenon under study; or (d) the boundaries are not clear between the phenomenon and context. A case study was chosen because the study could not be considered without the context of flipped classroom approach, and more specifically the math centers classroom settings.

This case study involved three mathematics teachers at a Canadian middle school, who taught intermediate level (grade 6, 7 and8) students. Two of the teachers were intermediate level mathematics teachers with their own classroom. One teacher was the resource teacher assigned by the district school board. The role of the resource teacher was to support the two classroom teachers by finding necessary resources needed to run the project. The two classroom teachers applied the flipped classroom approach in their mathematics classes. In preparation for the project, all three teachers sought opportunities to enhance their professional knowledge of using technology in mathematics teaching and learning. Further, each teacher read a number of monographs provided by the Ontario Ministry of Education in order to develop efficient knowledge of the Ontario elementary school mathematics curriculum, in particular knowledge of the mathematical processes such as problem solving, communicating, reasoning and proving.

Data was collected from:

1) Participating teachers' observation of their teaching and learning experiences. Each teacher recorded field notes based on their own reflections as well as observation of their students in-class events related to math- centers (e.g., counting the number of students being engaged per center, how teachers guided the off track students to get back to work, etc.). In addition to these data was collected from teachers' notes of their conferencing with students as individuals and in groups.

2) Transcriptions and field notes of group meetings (selectively audio-recorded). Each teacher's observations were shared, discussed and reflected upon by teachers as a group for professional growth while focusing on their own professional growth and their students' engagement in respective mathematics class.

3) Teachers' artifacts such as lesson plans and assessment rubrics, and the teacher team's final project report.

Findings and discussion

All transcribed data, field notes and teachers' artifacts were analyzed by the university researcher and one teacher independently to identify major themes related to the guiding questions— How did the professional development project facilitate teachers' understanding of the use of math centers in

a flipped classroom and student-centered approach for teaching and learning mathematics? How did the teachers negotiate constraints and possibilities as they engaged in their professional development project? The findings will be discussed in the following themes which emerged from the analysis: time for teachers to meet; awareness of initial resistance from parents and students; enhancement of student learning; and challenges. Findings from the study suggest that in order to sustain a collaborative professional development project teachers need time and need to meet. Teachers in this study were able to plan collaboratively and develop a practical, differentiated math program based on flipped classroom and student-centered pedagogical approach using math centers. The flipped classroom approach using math centers allowed students to engage in purposeful practice while freeing up the teacher to meet with individual and/or small groups of students for teaching and/or learning. However the two participating teachers taught different grades in different buildings of the same school, which became somewhat challenging to coordinate schedules for sit-down meetings. Often, the conversation between the three teachers would occur either in between classes (as the teachers would pass by each other's classroom) or through email (keeping each other informed on their status with the project). Although the teachers planned collaboratively, they had fewer time to compare what they had initially planned. This however, had an unexpected positive outcome which ended up by providing teachers with two different ideas of math- centers and student engagement. Both classroom teachers did utilize flipped classroom, student-centered pedagogical approach, however during their final group meeting, teachers discovered that they had each taken a different approach to the math center idea. This allowed for each teacher to talk about their thought process behind their choice for developing the math centers the way they did. Further, this provided both teachers to learn from one another's professional learning experience with their respective math centers. As different as each participating teachers' math centers were, the participating teachers observed that there were common themed categories to the math centers (these categories were presented earlier in this paper).

Another theme that emerged from the case study is that the professional development project provided opportunity for participating teachers to be aware of and understand about the initial resistance from the students and their parents toward flipped classroom and student-center pedagogical approach. Teachers developed awareness of the fact that both the students and their parents perceived mathematics teaching and learning in a traditional manner. And that for both the students and their parents, mathematics was a subject where the teacher taught a lesson, the students completed assigned tasks like doing practice questions from a mathematics textbook related to the lesson, followed by an assessment in form of a test. The students in participating teachers' classes and their parents' perception of how mathematics should be taught presented with complex challenges. This resulted in the professional development project allowing teachers to learn about strategies for alleviating these challenges including having to do a lot of community building exercise in class, while also having conversation with parents through emails, phone calls and/or one-on-one meeting, about the importance of math centers for their child's learning. These conversations with parents often revolved around the topic of how math centers not only helped students to become more engaged with mathematics but also helped to develop importance skill set of becoming more self-regulated toward their own learning.

Another theme that emerged is how the professional development provided opportunity for teachers to enhance students' learning and development through using math centers. Teachers noticed that

after the initial resistance, the students began to be engaged with math centers and by the end of school year, they began to self-regulate their learning. One teacher noted about a grade 8 student who reflected on his journey with math centers and stated that it helped him to become more resilient to mathematics learning.

> First, I did not know why we were doing math centers. I felt that the teacher did not want to teach anything…. but now, when I go through different centers in class I know that I am able to do things on my own…I feel happy…. I have done these many [math centers] today…which means I can do math…I just have to take my time with each center and not worry about how much time my group members are taking with centers.

In relation to the same theme, another teacher expressed how her grade 8 student commented,

> I used to think that my teacher should know everything…you know, like all the answers…but now I know that I can find all the answers…and if I am stuck, I can take help from my friends…which is okay, because we are learning together

Both of the participating teachers experienced a sense of fulfillment in terms of their professional development experiences in relation to their impact on growth in their students in terms of both mathematical understanding and self-regulation toward their own learning. Teachers noted that many of their students grew stronger in their ability to self-regulate, as they had to make choices toward their learning in terms of what to work on, how long to work for and with whom. Teachers expressed satisfaction on how the one-on-one time with the teacher allowed the struggling learners to take risk and seek clarification without feeling restricted by the classroom environment.

Given the many positive outcomes of the professional development project that focused on flipped classroom and student-centered pedagogical approach through math centers, there were some challenges. These were mainly due to the fact that teachers became aware that some students needed more time with this approach, which was not possible given that there was a limited time that these students were with their mathematics teachers and that the teachers were expected to cover the curriculum expectations. Also teachers in the professional development project realized that for a small number of students, it was extremely difficult to adapt to this approach, even if they loved mathematics. This was because these students had only experienced learning only from a textbook teaching approach in mathematics, and flipped classroom and math centers approaches were a significant departure from their past mathematics learning experiences.

Implications

This project utilized a case study research design and was conducted at one Canadian middle school. Hence, the findings of the study should not be read in terms of generalizability, but of transferability to other cases (Creswell, 2008). Recently, mathematics educators have realized the potential for a flipped classroom and student centered pedagogical approach for enhancing student engagement and learning. However, very little is known in terms of the implementation of this approach in elementary schools. This study explored the mathematics professional learning experiences of elementary school teachers as they implement the flipped classroom and student centered pedagogical approach. The professional development project provided opportunity for teachers to enhance their understanding of flipped classroom and its impact on students' learning. Teachers noted that their students became engaged with mathematics and self-regulated toward their own

math learning. The findings suggest that given opportunity to learn in a professional development setting that ensures autonomy, teachers learn and are capable of teaching through flipped classrooms and student centered pedagogical approach. The study also suggests that professional development project provides opportunity for teachers to be aware of the need for communication and collaboration among teachers, parents and students regarding the benefits and implementation of flipped classroom. As a result, further research is needed on how professional development can facilitate teachers' learning about how to communicate and collaborate with parents and students in flipped classrooms.

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Noticing aspects of example use in the classroom: Analysis of a case

Pessia Tsamir[1], Dina Tirosh[2], Ruthi Barkai[3] and Esther Levenson[4]

[1]Tel Aviv University, Israel pessia@post.tau.ac.il

[2]Tel Aviv University, Israel dina@post.tau.ac.il

[3]Tel Aviv University, Israel ruthibar@post.tau.ac.il

[4]Tel Aviv University, Israel levenso@post.tau.ac.il

This paper investigates promoting knowledge of example use in mathematics education by way of analyzing a case using theoretical tools. Participants were both prospective and practicing teachers attending a university course. An event taken from a tenth grade geometry class was analyzed in terms of example use, and then discussed. Participants related to the type of example given, the timing of the example, agency, what the example was an example of, and the aim of giving the example.

Keywords: Examples, case analysis, teacher education, theoretical tools.

Introduction

Examples and non-examples are an integral part of learning and teaching mathematics. They are used in concept formation, when seeking relations between elements, abstraction, and generalization (e.g., Smith & Medin, 1981; Watson & Chick, 2011). Acknowledging the importance of example use in mathematics education, Shulman (1986) included knowledge of examples within the category of pedagogical content knowledge (PCK). This knowledge, he claimed, is essential for representing the subject, so that it will be comprehensible to others. Rowland, Huckstep, and Thwaites (2005) also included teachers' use of examples in their description of 'the knowledge quartet', a framework for thinking about the ways subject matter knowledge comes into play in the classroom. In their framework, the ways teachers use examples and the types of examples they use, are manifestations of teachers' own content knowledge, their meanings and descriptions, being transformed and presented in order for students to learn the mathematics. Ball, Thames, and Phelps (2008) also noted the importance of teachers' knowing how to sequence examples.

Although example use is a complex matter and promoting teachers' knowledge of example use could be quite challenging, few studies specifically investigated this aspect of teacher education. This study proposes fostering prospective and practicing teachers' knowledge and awareness of example use by applying research and theoretical tools to analyze cases, helping to bridge theory and practice for mathematics teachers. The material, which is the focus of this paper, is a case based on a classroom event, and used as an exemplar. The case material consisted of a classroom transcript, along with guiding questions to be answered by each participant. This was followed by a group discussion with the teacher educator. Specifically, we ask: What aspects of example use do individual participants notice when studying an authentic case? Can those aspects be traced back to theories learned during the course? What additional aspects arise during group discussion?

Example use in mathematics education

There are many aspects of example use which have been investigated. To begin with, there are different types of examples. Arcavi (2003) discussed visual representations, while Tabach et al.

(2010) focused on numerical examples. In addition to the form of the example, studies have categorized examples by how they are identified. For instance, there are intuitive and non-intuitive examples and non-examples (Tsamir, Tirosh, & Levenson, 2008). Intuitive examples are those examples which students immediately identify as such and are often derived from practical experience. Likewise, intuitive non-examples may encourage visual, rather than analytical thinking. On the other hand, it seems that non-intuitive examples and non-examples can encourage students to use reasoning based on critical attributes. Tsamir, Tirosh, and Levenson (2008) also discussed the sequencing of examples and non-examples and its effects on students' learning.

The difference between examples and non-examples is dependent on the mathematical lens one is looking through. When looking for polygons, a triangle is an example; when looking for quadrilaterals, a triangle is a non-example. In other words, another aspect of example use to consider is the way an example is being used. Zazkis and Chernoff (2008) introduced the idea of pivotal examples which can cause the learner to change his or her cognitive perception or way of approaching a problem. Rowland (2008) differentiated between two uses of examples in teaching. The first is when examples are used to motivate generality. In this case, the examples are examples *of* something where the aim is to teach a general procedure or to support abstraction and concept formation. The second type of example use is for students to practice what was taught. That is, students are given many examples to practice some procedure. This type of example use allows students to experience variation and can lead to additional awareness and understanding. Watson and Chick (2011) found that examples can be used as templates for dealing with other class members, indicating a relation between classes or to express equivalence.

If we take into account that the way an example (or non-example) is used by a learner is determined by the focus of that learner, then it becomes the teacher's task to choose examples and set them up in such a way that students will view those examples through the intended lens and focus on the intended pedagogical aim. Zodik and Zaslavsky (2008) identified six types of considerations employed by teachers when selecting or generating examples: starting with a simple or familiar case, attending to students' errors, drawing attention to relevant features, conveying generality, including uncommon cases, and keeping unnecessary work to a minimum. They found that the most frequent consideration was choosing to begin with the simple or familiar example. They also found that on the spot, teachers often choose an example that will attend to an error which arose in class, whereas pre-planned examples tend more to consider uncommon cases.

To notice something means to make a distinction, to stress some perceived feature and ignore others (Mason, 1991). As shown above, various aspects of example use can be considered when analyzing a classroom event. In this study we investigate which of these aspects, and perhaps other aspects of example use, practicing and prospective teachers notice when analyzing a case.

Methodology

The participants in this study were 13 practicing teachers (denoted as T) with between 1 and 11 years of experience (mean years of experience was four) and 10 prospective teachers (PTs) who had completed their first degree in mathematics or a mathematically rich field of study, such as engineering. The specific course which is the context of this study, aimed to promote participants' knowledge of explanations and examples, and the relationship between them in mathematics

education. The course consisted of a total of 13 lessons which took place once a week for a period of 90 minutes each. The first six lessons were devoted to explanations in mathematics education. Different types of explanations were reviewed, such as conceptual, procedural, mathematically-based and practically-based explanations (e.g., Levenson, Tsamir, & Tirosh, 2010), and theories concerning the roles of explanations in mathematics education were discussed (Levenson & Barkai, 2008). The last seven lessons were devoted to examples in mathematics educations. Theoretical perspectives of examples, like those described in the background of this paper, were read and discussed.

During the fourth course lesson on examples (lesson #10), the teacher educator handed out a transcript of a geometry lesson which took place in a tenth grade classroom. The geometry lesson had taken place within a few weeks of the course lesson and was observed by three of the prospective teachers attending the course, lending the case authenticity and relatedness. The overall aim of the geometry lesson was to introduce students to Thales' theorem and show its connection to similar triangles. The students had previously learned about similar triangles in ninth grade. The case transcript began with the classroom teacher stating Thales' theorem. This was accompanied by a drawing on the board, made by the teacher, of two similar triangles, under the headline: Geometry – Thales' theorem. Next to the drawing it says that if the given is BC ‖ DE then from Thales' theorem we conclude proportional line segments (see Figure 1).

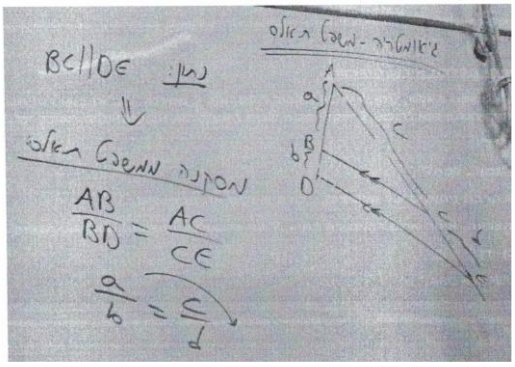

Figure 1: Presenting Thales' theorem on the board

The case transcript was handed out in the beginning of the course lesson with instructions to read it from beginning to end, without interruption, in order to understand the context and get a feel for the classroom. After reading through the transcript, participants were asked to reread the transcript and fill out a worksheet with the following questions: What did you learn about the use of examples during mathematics instruction from the examples given in the case presented? What would you do the same as the teacher did with regard to examples? What would you do differently from the teacher with regard to examples? After the participants wrote their answers and handed them in, a discussion followed. This discussion was audio-recorded and transcribed.

Data analysis

When analyzing participants' responses to the worksheet, our guiding question was, what aspects of using examples did the participants relate to when studying the case. We then used the literature background to help form a categorization scheme of those aspects. For example, we examined participant's responses for comments related to various types of examples that were discussed in class and were present in the case, such as intuitive examples, non-examples, familiar examples, and uncommon examples. Participants also commented on didactical aims of giving examples, such

as responding to a student's error. Because participants wrote freely, it happened that one sentence could encompass more than one aspect of example use. For instance, one teacher (T12) wrote "the teacher gave a numerical example, which in my opinion served the purpose of making it easier for the students to understand and generalize the idea." T12 refers to the type of example (numerical) as well as the purpose of the example (to help students generalize). Participants' comments that were thought not to be related to the giving and use of examples were categorized as 'unrelated'. Table 1 lists the categories along with examples from the data of each category.

Aspect of example use	Sample data
Type: What type of example is given?	The example uses simple numbers; it is a visual example of Thales' theorem.
Agency: Who is giving the example?	Only the teacher gives examples.
Aim: what seems to be the aim of giving the example?	The example is given to explain it again; the example shows the students how the theorem works.
Timing: When is the example given?	The example is good for the beginning of a lesson.
Example of: What is the example an example of?	The teacher gives examples of proportions.
Unrelated	The teacher presents a dry definition of Thales' theorem.

Table 1: Categorizing participants' comments

Findings

This section analyzes participants' comments from the worksheet on two segments of the case along with excerpts from the discussion which focused on those case segments. Thus, we review what participants noticed individually and what they discussed collectively. Analysis of participants' comments is carried out according to the aspects of examples listed in Table 1.

Segment one- Introducing Thales' theorem

The case transcript began a picture of the teacher at the board presenting to his students an example (see Figure 1). Accompanying the picture were the following lines from the case transcript:

1 T: In Thales' theorem, it is given that BC is parallel to DE. The conclusion is… wait, guess. If the length here (pointing to AB) is 6 and here (pointing to BD) is 2. And let's say that the length here (pointing to AC) is 12, what is the length of CE?

2 S: 4!

3 T: Right!

There are several ways to look at the examples in the above segment. One way is that there are two explicit examples. First, there is the drawing on the board (see Figure 1). Second, there is the numerical example given by the teacher in Line 1. However, the drawing on the board can be an example of Thales' theorem or an example of similar triangles, and indeed, that is what the teacher is trying to convey. In addition, depending on one's point of view, the example on the board and the

numerical example may be considered one complete example, with the example on the board written in a general matter, using parameters (a, b, c, and d), and the oral example, a specific example given with numbers.

The most frequent aspect of examples mentioned by participants was example *type*. Participants used such descriptions as: a simple example (PT2), an unfinished example (T8), a visual example (T12), a numerical example (T12, T13), and an intuitive example (PT 22). Although most participants wrote that the teacher gave the example, we do not take this as commenting on the *agency*, but rather as a description of what is going on. On the other hand, for a different segment, one participant wrote, "Only the teacher gives examples… but he should have requested examples from the students." This participant is not merely describing the situation, but commenting on who is and who should be giving the examples. The *aim* of giving the examples was mentioned by T8 who wrote, "In Line 1, the teacher gives an example that students have to finish. He is checking to see if the students are listening and if they understand." T13 wrote, "The teacher gave a numerical example so that the students could understand the example and draw on their previous knowledge of similarity and proportional triangle sides." *Timing* of the example, that the example was appropriate for the beginning of the lesson, was noted by PT2. Two participants commented on what the example was *an example of* – T12 wrote that it was an example of the theorem, meaning that the example showed how the theorem could be applied. PT22, referring to the numerical example in Line 1, wrote that it was an example of equivalent ratios.

Segment two – Proportional segments

The following case segment is a direct continuation of the first one:

4-6.		(The teacher and students review the concept of similar triangles.)
7.	T:	So, what is the ratio of their corresponding sides (referring to the example given in Line 1)?
8-10.	Students:	1 to 3. 3 to 1. It depends on how you look at it.
11.	T:	The ratio is … 3 to 4 because BD and CE are not sides of the triangle. So 6 is to 8 like 12 is to 16. Now, …, what came first in Euclidian geometry? First, there was Thales' theorem and only after that came the similar triangles theorems. So, let's say we are in ancient Greece and we don't know yet about similar triangles, but we do know Thales' theorem. With that theorem, we can prove proportional sides in similar triangles.

In lines 4-11, no new examples are given. Instead, the teacher and students still refer to the first examples given on the board. Like the comments on the first segment, here too, most comments related to the *types* of examples being given: simple numbers (PT7), a non-intuitive example (T11), a general example with parameters (T17), and a numerical example with familiar numbers (T19). None of the participants mentioned aspects of *agency*, *aim*, or the *timing* of examples. T11 noted that the example was an *example of* ratios. Two PTs wrote remarks connected with the story of ancient Greece. PT23 wrote, "There is an example from real life that I like – in ancient Greece." PT16 wrote, "In Line 11, the teacher made the material come alive when she told the story about Thales' theorem from ancient Greece ... This example raises the question of why the students first learn about similarity and then about Thales." T23 calls it an example from real life. Yet, it is not a

mathematical problem related to real life. The term ‘example’ when describing the ancient Greece story is not in line with the notion of examples discussed in the course.

Discussing the case

After the worksheets were handed in, the teacher educator (TE) opened up the discussion by asking who wished to comment on the case. After nine minutes of discussing general ideas, the discussion turned to the example given in Line 1 of the case transcript. Note that PT21 was present during the geometry lesson as part of their field work.

TE: Let’s look at Line 1.

PT21: But it’s not complete… He wanted to give an example of a ratio, but… Instead, he asked the students... He told the students to guess. And they did. He didn’t really give an example, in my opinion. But, in the next example...

TE: Where is the next example, in your opinion?

PT21: Line 11… In my opinion, it’s an explanation with a few numbers and that’s so you can see… I can’t decide. It’s, like, a numerical example. Here (in Line 11) is what was missing beforehand (in Line 1).

TE: Is it an explanation or an example? Can an example also be an explanation?

PT21: No. An explanation can be accompanied by an example. There can be an example and then the explanation generalizes it.

T8: I felt that way about Line 1. It feels like an explanation, and also like an example. On the one hand, there are numbers. On the other hand, it’s not complete.

The above excerpt gives us a glimpse into PT21’s and T8’s concept image of an example. Both infer that an example has numbers, but it must be complete, without any missing parts. There is also the question of the different roles an example may play in the classroom. Can an example be an explanation? Must all examples be specific and only the explanation can generalize it? These are some of the questions that the participants are grappling with. In the following excerpt, PT22, who was also present as an observer of the geometry lesson, tells what he observed.

PT22: He was trying to show how Thales’ theorem is really intuitive. That is, he gave an intuitive example.

TE: An intuitive example of what?

Many voices: Equivalent fractions.

PT22: And most answered correctly. The teacher was trying to show how easy it is.

TE: But what was it an intuitive example of?

T12: Of proportions. I know that it’s proportional because I know, I recognize it. It doesn’t really have anything to do with Thales’ theorem. Simply, 6 is to 2 as 12 is to 4. That’s it. That’s the example.

As stated in the background, what an example is an example of, depends on your focus and point of view. PT22 claims that the examples in Line 1 and Line 11 are examples of Thales’ theorem and

that the teacher used intuitive examples to simplify the concept for his students. However, other participants, those not necessarily present during the geometry lesson, see those examples as intuitive examples of equivalent fractions or of a proportion.

Discussion

During teacher preparation and professional development, participants are introduced to various theories. While field work is important, theories can help prospective and practicing teachers make the most of their field work by focusing their attention on different elements of practice. Findings showed that analyzing the case gave participants a chance to apply their knowledge of example theory when examining a classroom situation. Findings also showed that participants did not necessarily draw on the same theories when analyzing the same event. The example given in Line 1 of the case transcript was described alternatively as simple, numerical, visual, and intuitive. Each of these types can be traced back to different theories discussed in the course, but they focus on different issues. An intuitive example may also be numerical, but if the prospective teacher specifically comments on its intuitiveness, then that participant is remarking that a student will easily recognize it as an example (Tsamir, Tirosh, & Levenson, 2008). In other words, that participant is integrating knowledge of students (Ball, Thames, & Phelps, 2008) when analyzing example use and has appropriated a specific theory to accompany this integration. As teacher educators, we wish to encourage such integration of knowledge.

During the discussion, additional aspects of example use arose. Participants grappled with the nature of examples, and whether or not an example can be an explanation. This link between examples and explanations could have stemmed from the first half of the course which dealt with the topic of explanations in mathematics education. In any event, this question can help teachers and prospective teachers focus on the roles examples may play in the greater picture of teaching and learning mathematics. Finally, we also note how the integration of an authentic case, one that at least some participants actually observed, can help bridge theory and practice. In the discussion, participants held different views regarding what the example in Line 11 was an example of. Those who had actually observed the lesson had a chance to review the lesson again, focusing now on example use. They were also able to share with others some background of the lesson, perhaps adding to everyone's sense of 'being there'. This excerpt illustrates how fieldwork may be integrated into course work. It also reminds us, as teachers and as teacher educators, that it is not enough to offer examples. As Goldenberg and Mason (2008) said, exemplification is dependent on one's point of view. Analyzing a case using theories, and then discussing these analyses with participants, can raise awareness of how students may view examples and encourage planning example use in mathematics classrooms.

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Researching the sustainable impact of professional development programmes

Stefan Zehetmeier

University of Klagenfurt, Austria; stefan.zehetmeier@aau.at

Impact analyses and evaluations of professional development programmes are mainly scheduled during or at the end of a particular programme. They aim at and provide results regarding immediate and short-term effects. However, apart from and beyond that, an analysis of sustainable effects is crucial. To address this issue, this contribution deals with the central question: What is the sustainable impact of professional development programmes? Theoretical models and empirical findings are introduced. In particular, this contribution provides two case studies' results regarding Austrian professional development programmes' impact. Here, the factors which foster or hinder the sustainability of impact are in the focus. Finally, implications for professional development programmes' implementation and research are discussed.

Keywords: Impact, sustainability, professional development programme, case study.

Introduction

The question of how to promote mathematics teachers' professional development has been discussed in various papers (e.g. Krainer & Zehetmeier, 2013; Loucks-Horsley et al., 1996; Sowder, 2007; Zehetmeier, 2010, 2014a, b; Zehetmeier & Krainer, 2011). In this context, the question of impact is of particular relevance. Evaluations and impact analyses of professional development programmes are mostly conducted during or at the end of a project and exclusively provide results regarding short-term effects. These findings are highly relevant for critical reflection of the terminated project and necessary for the conception of similar projects in the future (Fullan, 2006). However, apart from and beyond that, an analysis of sustainable effects is crucial (Loucks-Horsley et al., 1996). Despite its central importance for both teachers and teacher educators, research on sustainable impact is generally lacking within teacher education disciplines (Datnow, 2006; Rogers, 2003). This kind of sustainability analysis is often missing because of a lack of material, financial and personal resources (McLaughlin & Mitra, 2001; Hargreaves, 2002).

Theoretical framework

The expected impacts of professional development programmes are not only focused on short-term effects that occur during or at the end of the project, but also on long-term effects that emerge (even after some years) after the project's termination. Effects that are both short-term and long-term can be considered to be sustainable. Sustainability may refer to both system and/or individual level. Sustainability can be defined as the lasting continuation of achieved benefits and effects of a project or initiative beyond its termination (DEZA, 2002).

Empirical evidence concerning the impact of professional development programmes points to the finding that "prior large-scale improvement efforts (…) have rarely produced lasting changes in either teachers' instructional practices or the organization of schools" (Cobb & Smith, 2008, p.

232). Thus, it seems reasonable to focus on factors which might foster the broad effects and scale-up of professional development programmes' innovations. Cobb and Smith (2008) highlight networks, shared vision and mutual accountability as key factors.

Teacher networks are described, for example, as groups of colleagues who provide social support in developing demanding instructional practices; this affords time built into the school schedule for collaboration among teachers and access to colleagues who have already developed relatively accomplished instructional practices.

Moreover, a shared vision of high quality instruction fosters the scale-up of impact: this includes a shared vision concerning the question of instructional goals (what pupils should know and be able to do) and the question of how pupils' development of these forms of knowing can be supported.

Another key factor which fosters the scale-up of innovations and impact in teacher education is mutual accountability. This means, for example, that if school leaders hold teachers accountable for developing high-quality instructional practices, then – in turn – school leaders are mutually accountable to teachers for supporting teachers' learning.

Examples of Austrian professional development programmes

This paper deals with the analysis of sustainable impact of professional development programmes. In particular, two Austrian professional development programmes are in the focus: the IMST project and the PFL courses:

IMST project

In Austria, a national initiative with the aim to foster mathematics and science education was launched in 1998: the IMST project. Since then, this initiative has undergone several adaptions and is still running.

IMST was implemented in three steps:

1. The task of the IMST research project (1998–1999) was to analyse the situation of upper secondary mathematics and science teaching in Austria and to work out suggestions for its further development. This research identified a complex picture of diverse problematic influences on the status and quality of mathematics and science teaching: For example, mathematics education and related research was seen as poorly anchored at Austrian teacher education institutions. Subject experts dominated university teacher education, while other teacher education institutions showed a lack of research in mathematics education. Also, the overall structure showed a fragmented educational system consisting of lone fighters with a high level of (individual) autonomy and action, but little evidence of reflection and networking (Krainer, 2003; see summarized in Pegg & Krainer, 2008).

2. The IMST² development project (2000–2004) focused on the upper secondary level in response to the problems and findings described. The two major tasks of IMST² were (a) the initiation, promotion, dissemination, networking and analysis of innovations in schools (and to some extent also in teacher education at university); and (b) recommendations for a support system for the quality development of mathematics, science and technology teaching. In order to take systemic steps to overcome the "fragmented educational system", a "learning system" (Krainer, 2005) approach was taken. It adopted enhanced reflection and networking as the basic intervention

strategy to initiate and promote innovations at schools. Besides stressing the dimensions of reflection and networking, “innovation” and “working with teams” were two additional features. Teachers and schools defined their own starting point for innovations and were individually supported by researchers and project facilitators.

3. The IMST3 support system started to continuously broadening the focus to all school levels and to the kindergarten, and also to the subject German language (due to the poor results in PISA). The overall goal of IMST is to establish a culture of innovation and thus to strengthen the teaching of mathematics, information technology, natural sciences, technology, and related subjects in Austrian schools (see e.g. Krainer et al., 2009). Here, culture of innovation means starting from teachers’ strengths, understanding teachers and schools as owners of their innovations, and regarding innovations as continuous processes that lead to a natural further development of practice, as opposed to singular events that replace an ineffective practice (for more details see e.g. Krainer, 2003).

For the future, the ministry expressed its intention to continue IMST. The overall goal is setting up and strengthening a culture of innovations in schools and classrooms, and anchoring this culture within the Austrian educational system.

PFL courses

In Austria, in-service professional development courses (PFL - Pedagogy and Subject-specific Methodology for Teachers) support teachers in developing their teaching skills and updating their knowledge of the subject they teach. The participants systematically reflect their professional work. PFL started in 1982, has undergone several adaptations, and is still running (for more detail, see Rauch et al., 2014). The programme is designed for teachers from all types of schools across the nation, including all age groups of pupils. The overall focus of PFL is on the professional development of teachers in the fields of content, didactics and pedagogy. School development plays a central role without losing sight of classroom instruction. The PFL concept is based on the implicit knowledge, which teachers possess concerning their work in class, their experience and their individual strengths. The course is intended to contribute to the further development of the teachers as professionals. Teachers are introduced to the methods of action research (Altrichter & Posch, 2009). They investigate different aspects of their teaching by defining research questions of relevance to their work, by collecting data, interpreting and drawing conclusions and writing down their findings in reflective papers.

The major goals of the teaching process should be primarily achieved through – and not detached from – the subject-related design of teaching and learning. PFL takes two years and focuses on the individual teachers’ own reflective practice using action research methods (Altrichter & Posch, 2009). By the end of the course, each participant is obliged to write a reflective paper using the data he/she has gathered throughout the process using qualitative and quantitative research methods. Participants are part of a community of practice (Wenger, 1998), since their work is embedded in a structure of mutual assistance and external support.

Case studies

Within both professional development programmes (IMST and PFL) several case studies were conducted, with the aim to research the sustainability of the programmes’ impact. The case studies

presented here were based on data from various sources and time periods to gain validity by "convergence of evidence" (Yin, 2003, p. 100): data collection contained documents (e.g. teachers' project reports, which were written during and at the end of teachers' participation in the project) and archival records (e.g. author's field-notes, which originate from author's activities as teacher educator in the project). Moreover, interviews were conducted from an ex-post perspective with former participating teachers, teachers' colleagues, principals, and project facilitators and teacher educators. Data analysis included both inductive and deductive elements (Altrichter & Posch, 2009) to analyse both the impact and the respective fostering (or hindering) factors. For example: document analysis aimed at gathering information concerning short-term impact which (a) occurred during and/or at the end of the teachers' participation and (b) might hold the possibility of sustainability and scale-up. Subsequently, this document analysis formed the basis for the interviews series. The interviews were semi-structured, since they were based on the analysis of existing data (document analysis), which identified various levels of short-term impact which occurred during and/or at the end of teachers' participation. The interviews were designed accordingly (a) to gather data concerning the sustainability and scale-up of impact and (b) to reveal other types of impact which were not already coded. Data was analysed by qualitative content analysis (Mayring, 2003) in order to identify common topics, elaborate emerging categories, and gain deeper insight into teachers' professional growth over time. The case studies' results were validated by means of member checking.

Exemplary case study from IMST

The case of Barbara, a former participant of the IMST² project, provides exemplary results concerning the issue of sustainable impact:

Barbara's beliefs regarding inquiry based learning (IBL) and open learning environments were changed during her participation in IMST²: Due to her participation in the project, she regularly used IBL settings (which she did not before her participation) and experienced positive effects on students' content knowledge, as well as on their self-confidence. In particular, she stated that there were positive changes regarding low-performing students' self-esteem, as well as the further development of high-performing students' competencies. This change was evidenced by data (both document analysis and interviews). In the interview, Barbara highlighted that this impact was sustained and enabled her to create and implement innovative teaching methods in a long-term way.

Besides this impact, she also developed (due to her participation in the IMST²) an inquiry stance towards the content and the method of her teaching. This inquiry stance was mirrored by her new belief about the value of feedback: due to the teacher education programme, topics such as classroom atmosphere and teaching quality were discussed with and evaluated by her students on a regular basis. This impact on her own IBL was sustained: Barbara stated in the interview, that she was convinced of the importance of critically evaluating her teaching. Even after the programme's termination, she continued to actively facilitate her students' communication and discussion about her teaching practices.

One of the central factors that fostered the sustainability of impacts was the engagement of the school's principal. The school had an efficiently organized management and school development structure, which represented another fostering factor. Additionally, Barbara experienced personal

benefit, which also helped the impact persist after the programme's termination. Both the teacher and principal highlighted (in the interviews) the role of the IMST² project facilitator as a fostering factor. Yet another fostering factor was represented by the IMST² workshops and seminars, where Barbara (according the interview data) got support and opportunities to share her experiences and to make her success and remaining challenges visible.

Exemplary case study from PFL

Eve participated in the PFL course and had the goal to promote open learning settings by implementing new teaching approaches in her mathematics classes. Document analysis showed that she aimed at enhancing pupils' inquiry-based learning opportunities. During her participation in PFL, Eve changed her teaching practices and implemented innovative teaching approaches to enhance her pupils' self-directed and independent learning. Interview data clearly shows that this impact was sustained: the changes in Eve's teaching practices stayed effective even after the termination of PFL. Core fostering factors were the school principal's support and a high level of mutual appreciation within the school staff, and pupils' benefit. In particular, Eve highlighted in the interview that the pupils' joy and success are core reasons for her to keep this impact sustained.

Document analysis further showed that Eve conducted various self-evaluations during her participation in PFL and gained new knowledge concerning action research methods. In the interview, she stated that she continued to reflect on her teaching practices, even after the end of her participation. This impact was sustainable, due to Eve's direct advantage (by getting information on her classroom performance) and the support of the school's principal (who was convinced that reflections and self-evaluations are important steps on the journey to school quality). This impact was also fostered by Eve's colleagues' joint reflection and communication. Interview data shows that teachers cooperated beyond school subjects and held similar values and standards concerning pedagogical or subject-related issues. The school's principal showed great interest in, and provided support for, the teachers' activities. He participated in the school's mathematics study group and shared his perspective with the teachers.

Discussion

The factors that fostered the sustainability of the case studies' impacts are mirrored by the theoretical framework (see above):

IMST and PFL enabled networking (Cobb & Smith, 2008) by community building, mutual appreciation and joint reflection. A particular factor was the principals' content knowledge (Cobb & Smith, 2008). Teachers' colleagues provided communication and social support in developing and reflecting instructional practices (Cobb & Smith, 2008). Moreover, a shared vision (Cobb & Smith, 2008) of values and standards regarding high quality mathematics instruction was established. In particular, the case studies' results highlight that the promotion of reflection and networking as key interventions (Krainer, 1998) turned out to be supportive for the sustainability of the professional development programmes' impacts.

NOTE

Parts of this paper are based on Krainer & Zehetmeier (2013), Zehetmeier (2015) and Zehetmeier, Erlacher, Andreitz, and Rauch (2015).

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The impact of professional development on teachers' autonomy-supportive teaching practices

Janette Bobis[1], Jenni Way[2] and Judy Anderson[3]

[1]*The University of Sydney, Australia; janette.bobis@sydney.edu.au*

[2]*The University of Sydney, Australia; jennifer.way@sydney.edu.au*

[3]*The University of Sydney, Australia; judy.anderson@sydney.edu.au*

Keywords: Professional development, autonomy-supportive practices, teacher beliefs.

Description of the research topic

The benefits to student academic and dispositional outcomes when exposed to autonomy supportive learning environments have been acknowledged for more than a decade (Assor, 2012; Reeve, 2009). Autonomy supportive teaching practices nurture students' internal motivations to learn, resulting in learning that is self-directed and both cognitively and emotionally engaging (Wolters & Taylor, 2012). While research affirms the benefits of instruction incorporating autonomy-supportive practices (e.g., Assor, 2012) it also shows that the mathematical beliefs of teachers can be an impediment to their commitment and enactment of such practices (Bobis, Way, Anderson, & Martin, 2016). With this in mind, an intervention study was conducted with mathematics teachers (grades 5-7) that aimed to enhance their use of engagement supportive teaching strategies in their mathematics classrooms. The intervention was a year-long professional development program that focused on shifting teachers' beliefs about student engagement and building knowledge of instructional strategies for promoting student autonomy in the mathematics classroom. The specific research question addressed was: What impact did the professional development program have on teachers' beliefs and practices that promote learner autonomy in mathematics?

Theoretical framework and methodology

Self-determination theorists (SDT, Deci & Ryan, 1985) advocate that autonomous motivation will improve students' academic and dispositional outcomes because activities undertaken for autonomous reasons are likely to increase students' willingness to apply effort when learning. According to SDT, students will be more intrinsically motivated to learn when teachers adopt autonomy-supportive pedagogy rather than controlling pedagogical approaches. SDT was used to guide our examination of self-reported data regarding mathematics teachers' instructional beliefs and practices as a result of their involvement in the professional development program.

Pre- and post-intervention data were collected from 32 grade 5 to 7 teachers of mathematics from four secondary and ten elementary schools located in Sydney, Australia. Participants included five male and 27 female teachers. Data were collected via focus groups and a 20 Likert-type item questionnaire that measured the extent to which teachers were committed to instructional beliefs and practices considered supportive of student engagement, including learner autonomy.

Findings and conclusion

Dependent T-tests were used to determine whether there were significant differences between teachers' pre- and post-intervention responses on each dimension of the questionnaire. Results for two dimensions pertaining to teachers' autonomy supportive beliefs and practices—discovery (the construction of ideas through student discovery) and teacher's role (co-learner and constructor of a learning community) are presented on the poster. During the pre-intervention focus groups, most teachers described their roles as a 'giver' of knowledge to students. However, in the post-intervention focus groups, teachers reported how they now tried to develop more autonomous learning strategies in their students and to encourage them to take greater responsibility for their own learning. The results indicate that teachers expressed beliefs and practices that were more supportive of student autonomy at the end of the intervention than prior to undertaking the professional development program.

Presentation of the poster

The poster is structured in four major sections: Section one provides a succinct introduction to the literature, providing a justification for the study and presents the research question. Section two presents a visual representation of the theoretical framework (SDT) underpinning the study and our analysis of results. Section three presents results from the questionnaire and focus groups. The final section presents implications of the findings and argues that such shifts in teachers' beliefs/practices can have practical consequences in terms of improving students' autonomy for learning mathematics.

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Supporting teachers' practices: A teacher educator-embedded professional development model

Scott A. Courtney

Kent State University, United States of America; scourtn5@kent.edu

The poster describes preliminary results from ongoing professional development with two U.S. mathematics teachers, one Algebra 1 and one eighth grade mathematics teacher, designed to increase and enhance teachers' content knowledge and transform their classroom instruction by embedding the author (i.e., researcher) in teachers' practices. The poster also articulates the embedded PD model. Preliminary results show participating teachers are engaging their students in more rigorous mathematics, teachers are demonstrating increased self-efficacy and are more frequently engaging students in mathematical sense making, reasoning, modeling, generalizing, and communicating.

Keywords: Mathematics instruction, mathematics teachers, theory practice relationship.

Description of research topic

The poster describes ongoing professional development (PD) in the United States with one eighth grade math teacher (students 13-14 years) and one ninth grade Algebra 1 teacher (students 14-15 years) designed to increase and enhance teachers' content knowledge and transform their classroom instruction by embedding the author (i.e., researcher) in teachers' practices. The poster focuses on the following research questions: How does embedding a mathematics teacher educator in a mathematics classroom (embedded PD) impact (1) participating teachers' content knowledge, (2) participating teachers' instruction, (3) students' self-efficacy, and (4) student achievement?

Theoretical framework

The embedded PD model is grounded in a constructivist approach to learning mathematics and aligns with Simon's (1995) Mathematics Teaching Cycle and Thompson's (2013) interpretive framework for the development of powerful Mathematical Meanings for Teaching (MMT).

Method

Throughout embedded PD, the author (i.e., researcher) and participating teachers relied on: participating teachers' prior assignments, assessments and notes; textbooks and district generated documents (e.g., curriculum maps); state-level standards and documents; and a variety of Internet resources (e.g., GeoGebra, NRICH Project). Throughout the planning of a lesson, which may take multiple in-person or online meetings (or both), the researcher attempts to motivate the teacher to make explicit (and objects thought, discussion and subsequent reflection) her (i.e., teacher's) understandings of: the mathematics inherent in the lesson, hypotheses of their students' knowledge, theories of mathematics learning and teaching, activities and assessments (Simon, 1995; Thompson, 2013). As such, notions of meanings, ways of thinking and the need for the teacher and her students to articulate their meanings, thinking, and reasoning are consistently addressed (Thompson, 2013).

Participants

Participating teachers reported on the poster involve Tami (eighth grade math) and Jeremy (Algebra 1). Tami's eighth grade class contained 15 students of both genders and multiple races. Five (of the

15) students were on individual education plans. Jeremy's Algebra 1 class contained 25 students of both genders and multiple races.

Data and analysis

Data consisted of: (1) video- and audio-recordings of and physical documents related to lesson co-planning sessions; (2) video- and audio-recordings of lesson implementations (i.e., co-teaching); and, (3) video- and audio-recordings of and physical documents related to teachers' reflection on student work and classroom instruction. Two embedded co-teaching descriptions will be described on the poster, one involving a co-planning session with Tami, the other involving co-teaching with Jeremy. Analysis will serve to characterize some of the differences exhibited in tasks, activities, and classroom interactions highlighted as a result of the embedded model.

Preliminary results

Teachers engaged in embedded PD have indicated their participation has provided them the support to do what they believe is best for their students and their practice while not feeling constrained by district and state demands. Rather than feeling the need to rush through content and focus on skills and procedures, embedded co-teaching has allowed participating teachers to focus on understanding, coherence, and discourse. Preliminary results show participating teachers are engaging their students in more rigorous mathematics and both students and teachers are demonstrating increased self-efficacy and are more frequently engaging in mathematical sense making, reasoning, modeling, generalizing, and communicating.

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Teachers learning through participatory action research – developing instructional tools in mathematics primary classrooms

Margareta Engvall

Linköping University, Department of Behavioural Sciences and Learning, Linköping, Sweden; margareta.engvall@liu.se

Keywords: Mathematics education, action research, teacher professional development, multilingual classrooms, instructional tools.

Introduction

In many classrooms, from first grade through the whole school system, there are many students whose mother tongue is not the same as the teachers' language used for instruction (Khisty, 2001). In Sweden, newly arrived immigrant children with limited Swedish language knowledge are learning Mathematics together with children who have spent their entire life in a Swedish context. Considering the large number of students with limited knowledge in the language of instruction one of the most important tasks for teachers in Swedish primary education is to create conditions to support the development of mathematical knowledge in these students.

Lately, the role of language in mathematics education has received a profound interest in educational research. Researchers have emphasized the importance of teachers using specific strategies to facilitate the classroom communication to and support students' mathematical thinking (O'Connor & Michaels, 1993).

Specifically, in the syllabuses (Skolverket, 2011) it is particularly prominent that mathematics is dominated by discourse-intensive approaches, and the use of instructional tools such as talk moves, give ample opportunities for student learning (Chapin & O'Connor, 2007). Similar strategies for supporting students' learning in mathematics have received attention among effective teachers of second language learners in mathematics (Khisty, 2001).

This study uses action research which is characterized by ongoing processes of self-reflection, which can be thought of as a spiral of self-reflective cycles on planning a change, followed by acting and observing the process and reflecting on the process and then re-planning and so forth (Kemmis & Wilkinsson, 1998). Using PAR gives an attempt "to help people investigate and change their social and educational realities by changing some of the practices which constitute their lived realities" (Kemmis & Wilkinsson, 1998, p.22).

Method

The poster gives a brief presentation of a one-year research project where four primary teachers at the same school (year 2, 4 and 5) have been working together with a researcher, using participatory action research (PAR) (Kemmis & Wilkinsson, 1998) to develop their instructional tools in order to support students' mathematical development in multilingual classrooms. Data collection has continued throughout the whole action research process during the academic year. The empirical data includes teachers' logs, teacher questionnaires with open answers, researcher's notes, audio-

recorded discussions from the meetings twice a month in the project group and 3-4 video-taped mathematics lessons in each classes, 14 lessons altogether

Results

Although the focus in this project has been on instructional tools for supporting students' talk in order to enhance their development in communicating and reasoning mathematically, it is noteworthy that the teachers express their development, not only in terms of (1) instructional tools but also regarding other aspects such as (2) classroom organization and (3) focus on mathematical content.

Methods structured in these three themes above constitute a teacher *tool kit* to support students' learning mathematics in multicultural classrooms.

Conclusions

By using PAR, the teachers had the opportunity to reflect critically, analyze and act as coparticipants in the challenge to change the practices in which they interact, which also challenged their approach to teaching.

When teachers act and reflect on their use of specific strategies of classroom talk they also start reflecting and acting on other aspects of teaching, such as classroom organization and how to keep attention to the taught content. Thereby, the change in practice became more than just temporary changes.

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TWG19: Mathematics teachers and classroom practices

Introduction to the papers of TWG19: Mathematics teachers and classroom practices

Charalampos Sakonidis[1], Ove Gunnar Drageset[2], Reidar Mosvold[3], Jeppe Skott[4] & Rukiye Didem Taylan[5]

[1]Democritus University of Thrace, Greece; xsakonid@eled.duth.gr

[2]University of Tromsø, Norway; ove.drageset@uit.no

[3]University of Stavanger, Norway; reidar.mosvold@uis.no

[4]Linnaeus University, Sweden; jeppe.skott@lnu.se

[5]MEF University, Turkey; tayland@mef.edu.tr

Keywords: Mathematics teachers, classroom practice(s), teaching.

Introduction

TWG19 is one of three TWGs at CERME10 dealing with issues related to teaching and teacher education (the others being TWG18 and TWG20). The group is particularly interested in studies that aim to understand the development of classroom practices and teachers' contributions to them. This includes the roles of other factors such as the available teaching-learning materials, modes of teacher collaboration at and beyond the school(s) in question, and school cultures as they relate to, for instance, teacher-student relationships, teachers' individual and communal responsibilities, and the role of curricular materials and testing. Finally, studies concerned with how micro-level interactions are informed by macro-level structures (society, culture and the political) are also of TWG19's interest.

A total of 27 contributions were initially submitted (26 papers and 1 poster), involving authors from at least 14 different countries, primarily in southern or northern Europe. The papers were grouped in five thematic areas, each assigned to one of the five team leaders for overseeing the reviewing process. Each proposal was reviewed by a team leader and two authors, and 24 contributions (22 papers and 2 posters) were accepted for presentation at the conference. Participants were expected to read the papers before the conference sessions. In the time slot allocated to each group of papers, the authors each gave a short presentation (5–7 minutes), sketching the key ideas of the work reported. This was followed by a reflection by the group leader on common themes and cross-cutting issues (15 minutes), which were subsequently discussed by the participants in small groups. Ultimately, 22 papers and 1 poster are included in the conference proceedings.

The five thematic areas according to which the TWG19 contributions were grouped are: (a) problem solving and general issues related to teaching practice, (b) lesson planning, lesson study and curriculum, (c) instructional quality and assessment, (d) in-the- moment teaching actions and decision making and reflection and (e) instructional practices. In the next two sections, we discuss the contributions first within and then across these thematic areas.

Substantive issues

In this section we present research considerations and concerns in the papers in each group, including the range of theoretical perspectives and methodologies employed.

(a) Problem solving and general issues related to teaching practice

Dominant perspectives on the teachers' knowledge have changed and now focus on content knowledge closely connected to the profession. This is fueled by current reform initiatives that emphasise processes of mathematics, for instance in terms of problem solving.

Three of the four papers in this section deal with problem solving. The paper by Kleve and Ånestad concerns a Norwegian teacher, who seems inspired by a process view of mathematics. The class works with subtraction and initially uses informal mental strategies. The authors argue that there is a need to link such methods to a standard algorithm, that the flexible use of both methods is needed, and that it should become socio-mathematical norm that both are acceptable. However, the teacher's mathematical knowledge for teaching seems too weak for her to support the students in the transition from informal strategies to standard procedures. This makes it impossible for her to support socio-mathematical norms in line with the reform.

Odindo's study from Kenya is based on the expectation that problem-solving approaches may alleviate secondary students' difficulties with their final exams. He uses Learning Study (LS) to support teachers in using problem solving when teaching algebra. The paper to some extent focuses on the students, but mainly asks what the learning opportunities are for the teachers. Odindo argues that the LS format allows teachers to consider general issues of, for instance, time management, but also to focus on patterns of task variation closely related to the contents. At least implicitly, then, the paper is concerned with how LS may support mathematical knowledge for teaching.

The study by Villalonga and Andrews is also on problem solving, but less on mathematical knowledge for teaching. In fact, the teacher is conspicuously absent, as the paper deals with how Catalan students may self-scaffold when engaged in problem solving. The students use a resource, an Orientation Base (OB), which is to help them monitor their problem solving. The teacher is almost obsolete, and the OB may be read as a response to the problem that many teachers find it difficult to support their students' problem solving. OB may be seen as an attempt to circumvent this difficulty and transfer responsibility to the students.

The last paper in this group, by Mosvold and Hoover, report on a literature review of studies on mathematical knowledge for teaching. The 12 studies in the review address questions of what, how and why such knowledge plays a part for the quality of instruction. The studies argue that mathematical knowledge for teaching is important for instance for teachers' selection and adaptation of tasks, for their planning of instruction, and for how they listen to students and pursue student thinking. Based on the review, however, Mosvold and Hoover argue that the results are mixed and that there is a need to shift the emphasis towards more dynamic understandings of the relationship between mathematical knowledge for teaching and teaching.

The discussions of these papers focus on what it may mean to adopt a dynamic perspective on the knowledge-teaching relationship. How, then, we may change the emphasis from teacher characteristics (e.g., their knowledge or beliefs) to the acts of teaching? The latter perspective

requires greater attention to issues of context, to what we mean by practice, and possibly to alternative frameworks that allow us to reconsider what we mean by knowledge.

(b) Lesson planning, development and curriculum

This group of papers focuses on planning for teaching, utilizing lessons learned in other contexts, and dealing with curricula and textbooks. Four papers approached these issues.

In her case study of a Swedish mathematics teacher, Grundén targets the practice of planning. Reflecting on her own planning, the teacher conceptualizes planning as making informed decisions regarding the teaching of the mathematical content in different contexts, and she relates planning to other practices of the teacher as well as practices of other teachers. Relating to the practice of other teachers is also an issue in the study of Runesson Kempe, Lövström and Hellqvist, who investigate how experiences from a Learning study can be shared and used by teachers in other contexts. Applying the results from a previous Learning study in new classrooms, the authors investigate some necessary conditions for learning about negative numbers and indicate possibilities for cultivating more effective professional practice in mathematics classrooms.

Although development and change in mathematics teaching might be teacher-driven, it is sometimes prescribed by curriculum reforms. Klothou and Sakonidis investigate the implementation of a new curriculum reform among primary mathematics teachers in Greece. They argue that contradictions in teachers' own discourses can be explained by recontextualization procedures that appear when teachers attempt to implement the reform, and inconsistencies may abide in the very discourses that teachers draw upon. Whereas some countries have official textbooks that everyone must use, French teachers are free to decide if they want to use a textbook and how – if they adhere to the national curriculum. In their study on how two experienced French teachers use and adapt the content of mathematics textbooks and teacher manuals, Priolet and Mounier analyze how the teachers use the same textbook when teaching the same mathematical content. None of them follow the recommendations from the textbook completely in their lesson, and the adjustments they make tend to reduce the difficulty of textbook tasks.

These four papers provide compelling glimpses into the complex work of teaching mathematics. They discuss how authorities, schools and other teachers may provide resources that are intended to support the work of teaching, and how adapting and using such resources introduces mathematical, didactical and social demands on teachers' work. Mathematical knowledge for teaching can be described as knowledge required to deal with such demands (Ball et al., 2008), and these four papers thus indirectly contribute to investigating more dynamic relationships between knowledge and teaching.

(c) Instructional quality and assessment

The five papers in this thematic group describe qualities of teaching. Three papers concern instrument development for assessing qualities. One such instrument is the Realization Tree Assessment tool (Weingarden, Heyd-Metzuyanim & Nachlieli). This tool is particularly interesting in the way it reduces the complexity of the lesson into a picture of the mathematical concepts discussed in order to describe qualities of the lessons. This picture organizes mathematical ideas related to what Sfard (2008) calls saming and objectification. A second instrument presented is an innovative tool to observe, describe and evaluate metacognitive practices in mathematics

(Nowinska & Praetorius). Six out of the seven dimensions developed had highly reliable ratings. A third instrument developed by Jentsch and Schlesinger starts from three established dimensions (classroom management, personal learning support, cognitive activation) and aims at adding a subject-specific dimension. This dimension includes nine characteristics – such as teachers' mathematical correctness, explanations and mathematical depth – and produces results with good interrater agreement and satisfying reliability measures.

In addition, two papers study qualities of teaching using observation and interviews. Tuset investigates pre-service teachers trying ambitious teaching, allowing students to exercise authority while staying accountable to the discipline. The study finds that the pre-service teachers are able to create opportunities to engage in explorative discourses, but that their talk moves seem to be ritualized and therefore constrain students' participation. Kaldrimidou, Sakonidis and Tzekaki attempt to identify crucial elements shaping classroom mathematical meaning construction. To achieve this, they study three highly motivated and professionally active teachers' instructional practices and reflections. Findings reveal that the teachers' choices restrict the mathematical meaning because they desire to provide an easy, safe and pleasant learning environment.

These five papers illustrate two main issues for further research. The first issue regards the challenges of low and high inference observations. Low inference observations, like talk moves, explain little in themselves. On the other hand, high inference observations require extensive rater training that might result in simplification and even ritualization of the rating. What could we lose then? The second issue is that these articles illustrate the need for an instrument review. Which instruments are available for assessing qualities of teaching, what do they intend to measure, what theories do they build on, how reliable are they, and how much data and extent of rater training is needed to make them reliable?

(d) In-the-moment teaching actions and decision making and reflection

The studies in this group address mathematics instruction in a variety of ways: as teachers' management of actions and moments determining students' learning (Ferreira & da Ponte); as an activity shaping and being shaped by teachers' professional enactment in intervention or reform settings (Stouraitis; Sterner); as a practice being intentionally problematized to support teachers to develop (Potari & Psycharis; NicMhuirí). A different approach is to distinguish between papers that look at mathematics instruction as a learning-to-teach site through scrutinizing yourself or others acting it (Ferreira & da Ponte; Potari & Psycharis), as a professional activity developed through collaborative action (Sterner; Stouraitis), or through individual reflection via literature (NicMhuirí).

A range of theoretical or conceptual frameworks – mostly of sociocultural origin – are at work in the studies reported, and reflection (on teaching practice) and collaboration are at heart of these frameworks. In particular, NicMhuirí employs a reflective practitioner's perspective in combination with a model allowing for levels of teachers' reflection to be identified. Reflection is also of concern in Potari and Psycharis' work operationalized through the construct of 'critical incidents' within a community of inquiry framework. The theoretical underpinnings of the community of practice approach are adopted by Steiner, with reflection being seen this time as a professional learning enterprise developing collaboratively. Drawing on a CHAT perspective, Stouraitis views reflection as an aspect of teachers' decision making which interacts with teaching activity. Finally,

the study by Ferreira and Ponte employs features related to tasks assigned to the students and the communication established in the classroom to evaluate teaching actions.

Most papers in the group report on small, qualitative studies. Empirical data include observations of teaching, meetings and/or interviews audio-taped and transcribed. These data are predominately analyzed based on categories indicated by the literature (content analysis) or by the data themselves (grounded theory like analysis). One study uses no data, but analyzes two empirically tested constructs to exemplify the tool indicated (NicMhuirí). The results of the studies highlight various levels of mathematics teaching interacting with teachers' professional activity.

Overall, the studies in the group seek to understand how mathematics classroom teaching feeds teachers' professional practice, focusing on teachers inquiring into specific aspects of it. The relevant discussions carried out during the conference sessions raised concerns about the clarity of the terms and constructs used, the appropriateness and the functionality of the theoretical frameworks employed, and the boundaries between teaching action/practice and teacher practice.

(e) Instructional practices

The four papers in this thematic group approach instructional practices from different sociocultural perspectives. Two papers investigate teacher change during professional development programs. In particular, Venkat and Askew employ variation theory and example spaces to understand how teachers mediate primary mathematics, mainly how they generate and validate solutions as well as build mathematical connections. Şeker and Ader, on the other hand, focus on maintaining the cognitive demand of mathematical tasks, teachers' attention to students' mathematical ideas and intellectual authority in the classroom. Using the aforementioned frameworks makes it possible to illuminate different aspects of teacher practices that seem to improve based on research recommendations. Future research concerning both papers may entail a close look into the nature of professional development that influences instructional practices.

The paper by Baldry focuses on the development and viability of an analytical framework aiming to understand a 'typical' mathematical classroom in the United Kingdom. The analytical framework Orchestration of Mathematics portrays the quality of mathematics orchestration combining several theoretical constructs including cognitive demand of mathematical tasks, sociomathematical norms, hypothetical learning trajectories and professional noticing. The framework thus seems closely related to the scheme of analysis adopted by Şeker and Ader. A common theme of the two studies is teachers' difficulty in noticing and building on student thinking as well as maintaining cognitive demand of the tasks.

The importance of discourse in understanding instructional practices is evident across all papers. Drageset and Allern use drama as an innovative tool, allowing students to gain mathematical authority and engage in productive discourse patterns in making sense of mathematics. Instructional practices shaped and were shaped by student participation and responses. Future consideration for this work is to explore how teachers might implement such drama-based interaction patterns in their classrooms with the help of teacher educators.

A general trend identified in this group is using interviews together with observational data to understand teacher instructional practices, including teacher decisions and professional noticing abilities, closely connected with student participation and sociocultural norms. It would be

productive to define boundaries of instructional practices and how to incorporate pre- and post-observation interviews with teachers in analysing relevant data.

Trends and developments

The studies in TWG19 address a wide range of features and factors that regulate the quality and development of classroom mathematics teaching as well as its relation to teachers' professional growth. A systematic attempt to understand, assess and trace contributions to teacher and classroom practices can be identified across the papers. Issues pursued along each of these three directions are discussed below. The section concludes with some critical considerations related to the studies hosted by TWG19.

In trying to understand teachers and classroom practices, some of the studies look at mathematics teaching in challenging circumstances (e.g., Kaldrimidou et al.; Kleve & Ånestad; Priolet & Mounier). Their results suggest that adapting teaching to effectively respond to such occasions is a difficult endeavor often leading to poor and even contradictory teaching practices. Teaching is also seen in relation to teachers' professional knowledge and practices/perspectives, with the relevant studies indicating a complex but dynamic and fertile relationship (e.g., Mosvold & Hoover; Grundén; Stouraitis). Finally, some studies consider the influence of contextual factors upon teaching practices (e.g., Baldry; Venkat & Askew). The findings reveal teachers' difficulties in administering the mediational role of these factors in order to develop effective teaching practices.

The qualities of teachers and classroom practices are assessed by focusing quantitatively or qualitatively on subject-specific rather than generic features. Studies adopting quantitative instruments highlight the value of such approaches when able to assess high inference valuations with the necessary inter-rater agreement (e.g., Jentsch & Schlesinger; Nowinska & Praetorius). The qualitative instruments, like interviews and observations (e.g., Weingarden, Heyd-Metzuyanim & Nachlieli; Tuset), found to face the same challenges of assessing high inference valuations. To do this with trustworthiness, the qualitative research typically focuses on depth of scrutiny rather than inter-rater agreement.

Certain ways of contributing to teachers and classroom practices are identified in the papers, mainly in some form of collaboration, reflection or intervention. In particular, opportunities to collaborate with other teachers to explore various aspects of teaching mathematics seem to offer possibilities for teachers to develop their professional practice (Sterner; Runesson Kempe et al.; Odindo). Teachers' reflection on classroom practice is a central constituent of this collaboration facilitated by inquiry tools. When exercised on varied levels and at specific aspects of classroom practice, affordances and constraints of this practice become evident (NicMhuirí; Psycharis & Potari). The latter appears to be also the outcome of intervention studies supporting teachers to transfer more learning responsibilities to students (Drageset & Allern; Şeker & Ader; Villalonga & Andrews).

The research activity on teachers and classroom practices included in TWG19 reflects some interesting steps forward, but it also reveals at least five sets of issues in need of critical consideration. Firstly, issues related to the methods adopted, for instance, the issue of generalizing across contexts, the role of using multiple methods, and the (dis-)advantages of different teacher-researcher relationships. Secondly, concerns about the theoretical frameworks employed, for example, selection criteria, levels of generality targeted, issues of compatibility, questions (not)

addressed, ‘own’ frameworks. Thirdly, there is a need to carefully consider concepts and terms used. For instance, terms like ‘practice’, ‘context’ and even ‘teaching’ need clarification, whereas constructs like ‘stability of knowledge’ require further consideration. Fourthly, it is important to adopt clearly defined criteria for assessing the quality of teaching with reference to the learning of mathematics achieved as well as the wider educational goals targeted. Finally, it is necessary to consider quality criteria for research adopted, contribution to theory or practice, coherence, and sufficiency of evidence to warrant an empirical claim.

Concluding remarks

The work reported in TWG19 is part of the research on mathematics teachers and classroom practices developed in recent years employing a predominately sociocultural perspective. Within this perspective, teaching is seen as a social practice in which teachers are practitioners (Jaworski, 2006). Classroom practices are viewed as regular activities and norms continually developed by teachers’ involvement in multiple simultaneous activities, “taking into consideration working contexts, meanings and intentions (…) the social structure of the context and its many layers – classroom, school, community, professional structure and educational and social system” (Ponte & Chapman, 2006, p. 483). These activities mutually structure and frame each other to constitute the practice of the classroom (Skott, 2013).

There is a range of issues addressed by the studies reported that concern teachers’ contribution to classroom practice in various contexts, mostly related to critical aspects of instruction-in-action and teachers either inquiring into their own teaching or working towards developing it. Collectively these contributions appear to suggest that it is valuable to shift the emphasis in this line of research from teachers to teaching. Several theoretical and analytical frameworks are used often in combination (rather than in coordination) providing multiple lenses through which certain constructs (rather than structures) are examined within particular contexts. Along the same line, different methodological approaches are pursued, mainly qualitative, seeking to capture the complexity and richness of the practices unfolding within mathematics classroom life shaping students’ learning of mathematics and teachers’ learning to teach mathematics alike. The findings of the studies offer some notable insights into this shaping, highlighting the importance of focusing on the micro-level of classroom practice, on the resources teachers draw on as they engage in it, and their (intentional or unintentional) professional activity.

The plurality of theoretical perspectives, constructs and analytical tools employed in the studies of teacher and classroom practice reported in TWG19 underline the dynamics of the research activity aiming at ‘unpacking’ teaching practice. It might be the time for the research community working in this area to consider what is already known, what is to be further examined and to develop on the basis of this evaluation a research agenda to fill the gaps. How different tools may be used considering different theoretical perspectives, decisions of whether to use an existing tool or to develop a new one and how to report the added value of using different tools require special attention in moving forward. To this end, the emphasis should be on teaching rather than on teachers, as suggested by the work presented and discussed in the context of the conference sessions.

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A teacher's orchestration of mathematics in a 'typical' classroom

Fay Baldry

University of Leicester, UK; fb128@le.ac.uk

This paper explores the complexity of interpreting teachers' practice in relation to mathematical meaning making in 'typical' classrooms in England. An observation framework has been developed to interpret classroom activities that draws on a range of theoretical perspectives, including variation theory (Marton & Pang, 2006) and notions of classroom norms (Cobb, Gresalfi, & Hodge, 2009). This paper focusses on the analysis of two lessons in order to discuss the potential of this framework to foreground the mathematics made available to students and the pedagogical moves made by the teacher to bring this about. In England, class composition for secondary mathematics is usually decided by measures of prior attainment, with students of similar 'ability' grouped together. A wider study is exploring shifts in teachers' pedagogical approaches when they teach classes with different attainment profiles. Consideration here is given to whether observation of a few lessons using this framework can identify stable mathematical characteristics, so that in future the framework would allow comparisons between classes to be made.

Keywords: Observation, teaching methods, sociomathematical norms, variation theory.

Introduction

In England, ongoing attention is paid by politicians and educators to the mathematical attainment of students, and this includes analysing the progress of different groups (Easby, 2014). At secondary level, class composition in mathematics is often determined by measures of prior attainment, with students grouped with others of similar 'ability' and is referred to as *setting* (Ireson, Hallam, & Hurley, 2005). Whilst findings from research into the impact of setting do vary, the predominant view is that setting does not improve overall attainment and may indeed act as a suppressant (e.g. Hattie, 2002). Moreover, setting does increase the spread of attainment, resulting in students placed in lower attaining sets being disproportionally affected. Issues of equity are raised further as students from lower socioeconomic groups are over-represented in low sets, even when prior attainment is taken into account (Muijs & Dunne, 2010). In spite of these concerns, setting in secondary mathematics classrooms appears to be firmly entrenched, with many teachers perceiving setting as the only practical way to teach a range of 'abilities' (Wiliam & Bartholomew, 2004).

There is a tendency for the teaching of sets with different attainment profiles to have distinctive pedagogical characteristics; for example, contextualised language is more common in low attaining sets, shifting to more formalized mathematical language in higher attaining sets (Dunne et al., 2011). The aim of a wider ongoing study is to explore how individual teachers shift their pedagogy when they teach classes with different attainment profiles. It is anticipated that this will offer insights into how they tailor practice for different sets and the impact this has on the mathematics made available to students. In that larger study, a conceptual framework has been developed to interpret teachers' actions in relation to their orchestration of mathematics (figure 1); in this study the framework is employed to capture practice that the teacher considers typical for them with a particular class (set).

Theoretical framework

Classrooms are widely acknowledged as dynamic environments, where the complexities cannot be captured in a simple model (e.g. Potari & Jaworski, 2002). Goodchild and Sriraman (2012) argue that the didactic triangle, where vertices represent teachers, students and mathematical content, "serves as a starting point to theorise the dynamics of teaching–learning" (p. 581). For researchers, this raises a question as to how that practice could be understood without taking into account the actions of individual students. This study utilises the notion of classroom norms as a mechanism for taking into account student activity whilst maintaining a focus on the teacher.

Underpinning much of the recent research relating to teachers' practice appears to be the notion that good practice is related to an inquiry orientation (Boesen et al., 2014; Schoenfeld, 2013). Termed the 'reform agenda' in the US, an inquiry orientation is associated with the development of conceptual understanding through the use of rich mathematical tasks, discussions and problem solving approaches (Stein, Engle, Smith, & Hughes, 2008). This is often contrasted with a 'traditional' approach, characterised as students working on individual tasks that focus on the efficient application of algorithms, and delivered through transmission style teaching. Many of the existing analytical frameworks are linked to inquiry-oriented goals for the professional development of teachers (e.g. Boesen et al., 2014; Schoenfeld, 2013). Others focus on particular aspects of classroom practice that are considered important, such as the design of rich tasks and the management of discussion (M. Simon et al., 2010; Stein et al., 2008). However, with evidence that traditional approaches to teaching are still common in England (S. Watson & Evans, 2012), the applicability of affordances of inquiry-oriented frameworks to the analysis of 'typical' lessons has to be questioned. For example, inquiry-orientated contexts place more value on discussion, explanation and justification, with the potential to make students' meaning making more visible, than do more traditional approaches.

The Orchestration of Mathematics Framework (OMF) was developed as a tool to build a picture of teachers' classroom practice (figure 1). Whilst there is insufficient space here for a detailed discussion, an iterative process of development was undertaken, where concepts with traction in interpreting classrooms were considered from the teachers' perspective, and their relationship to each other. For example, Stein, Grover, and Henningsen (1996) tracked the cognitive demand of tasks as lessons unfold; a notion that has subsequently been drawn on by many researchers (e.g. Schoenfeld, 2013). Here, the focus was on linking the teachers' activities with other theoretical perspectives, such as relating problems with multiple solution strategies with variation theory (Marton & Pang, 2006) or the press for explanations with patterns of discourse (Imm & Stylianou, 2012). In addition to demonstrating that a wide range of classroom features can have a critical effect on the learning of mathematics, research has indicated that it is not the presence or absence or particular features *per* se that influences the mathematics experienced by students, but rather the nuances of implementation and interdependency (Hiebert et al., 2003). The OMF has been designed to offer a range of lenses that can be brought into play as classroom activity unfolds. As part of this development process, the OMF was used to analyse three publically available video lessons from the TIMSS studies. Whilst not reported on here, the OMF orientated the data analysis and distinct lesson profiles were identified.

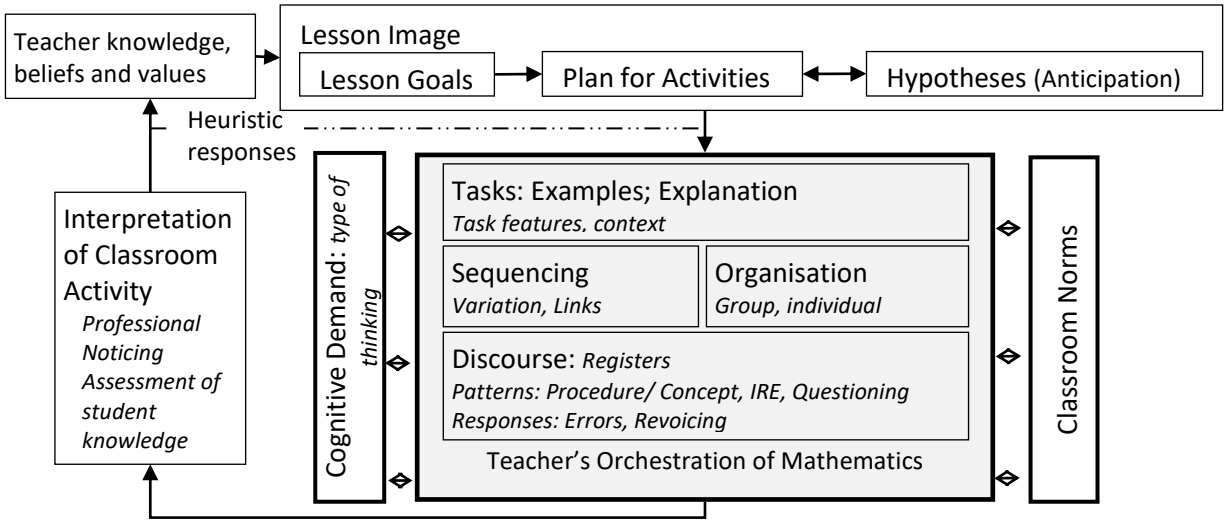

Figure 1: Orchestration of Mathematics Framework

The central dimensions relate to in-class activity, and in particular the teacher's orchestration of mathematics: that is, the mathematics made available in the shared space of the classroom and the actions taken by the teacher to bring this about. Two significant elements of teachers' practice are the selection of tasks and the management of classroom discourse (e.g. Ainley, Pratt, & Hansen, 2006; Stein et al., 2008). Within each dimension, there is a range of significant elements, such as the role of multiple representations and management of student responses. Variation theory draws on the idea that learning requires variation set against a backdrop of invariance (Marton & Pang, 2006); A. Watson and Mason (2006), amongst others, have drawn on this theory to explicate how the sequencing of questions or activities can make visible critical features of a concept and hence support generalisation. Moreover, variation theory also offers a mechanism as to how other previously identified beneficial features could support learning. For example, multiple representations and multiple solution strategies could be seen as holding the concept constant whilst varying representations and processes.

The notion of cognitive demand offers a way to categorize the "the level and type of thinking that a task has the potential to elicit" (Boston & Smith, 2009, p. 122). Stein et al. (1996) introduced a rubric where 'memorization' and 'procedures without connections to concepts' are classified as low demand, whereas 'procedures with connections to concepts' and 'doing mathematics' are classified as high demand. As such, this can be viewed as potential of the teacher's orchestration of mathematics to influence student learning. Cobb, Stephan, McClain, and Gravemeijer (2001) offer an interpretative framework that coordinates social and psychological perspectives, where the social aspect is framed in terms of norms. All lessons and all interactions are unique, but norms offer a way to interpret interactions as typical or atypical, which offers the possibility of generalising beyond particular incidents. In particular, their notions of sociomathematical norms and mathematical practices can identify what is considered legitimised mathematical activity in that particular context.

The lesson image, activity and interpretation cycle draws on Simon's (1995) work on hypothetical learning trajectories, but extends the focus to include performance and engagement goals, as evidence indicates that not all teachers focus on learning (Amador & Lamberg, 2013). This captures

the iterative planning, activity and interpretation that starts before the lesson and continues as the lesson unfolds. Teachers' interpretations of mathematical activity are predicated on what they notice; the conceptualisation of professional noticing by Jacobs, Lamb, and Philipp (2010) is one construct drawn on here. Taken together, the dimensions offer a way to build a picture of the classroom based on the features as they occur and interact in the course of a teacher's normal practice.

Methodology

This paper reports on a section of a pilot study, undertaken is part of a larger ongoing qualitative case study, following the interpretative tradition. The research question explored in this paper is how viable the OMF is as an analytical tool for charting a teacher's pedagogical approaches and the mathematics made available to students.

For a pilot study, two secondary mathematics teachers known to the author have been recruited. Decisions regarding selection of classes and the timing of observations resides with the teachers, in order to minimize the imposition on them and their schools. So far, data have been gathered from one teacher and two lessons with a year 8 class (12 to 13 years old), taught ten days apart. The author acted as a non-participant observer, with the OMF used as an observation pro forma. The lessons were recorded by two static video cameras and lesson artifacts, such as students' work and teaching resources, were collected. Pre- and post- lesson semi-structured interviews were conducted with the teacher, focusing on how and why the lesson was planned in the manner chosen, and on moments the teacher thought important for learning, including whether these were as anticipated or unexpected.

The audio data from lessons and interviews was transcribed. Lesson activities were coded as being mathematically relevant, organisational or not mathematically relevant. The mathematically relevant sections of the videos were then reviewed; interactions were mapped to the OMF and cross-referenced with the observation notes. The interviews were analysed for evidence of the teacher's interpretation of classroom activities and their lesson image. The analyses of the two lessons were compared; consistencies and contradictions in the dimensions of the OMF were sought, to see whether there was evidence that observation of a few lessons could build a sufficiently representative picture of a class to allow comparison with others (Staub, 2007). All the analysis in this pilot study has been conducted by the author; it is anticipated that another researcher will review the dimensional analysis before the next stage of the wider study.

Findings and discussion

In this section, two linked extracts from the first lesson will be used to illustrate how the dimensions of the OMF were populated (figure 3). Then the comparison of the two lessons will be discussed.

Lesson 1

The students individually attempted to calculate the areas of six different shapes. The numerical answers were shown, in a mixed order, alongside the questions at the start. After nine minutes solutions were discussed as a whole class.

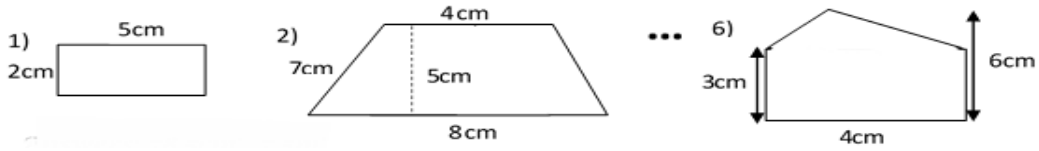

Figure 2: Questions projected onto classroom whiteboard

78 Teacher: OK number one, what did you do and what answer did you get?

79 Azariah: I got five times two.. ten meters.. centimeters meters squared

80 Teacher: Ten centimeters squared, perfect, units are really important ok. Finley second one then.

81 Finley: I did four add eight, twelve divided by two, six.. times five makes thirty centimeters squared.

⋮ [the last question]

113 Teacher: Sixteen centimeters squared, what did you do to get it?

114 Sam: It was just the last…

115 Teacher: Just the last answer, did anyone manage to do it with the maths? Raj

116 Raj: Three times.. three times four so it's the bottom rectangle

117 Teacher: Yep

118 Raj: and that's 12 meters squared, but then you triangle the top bit, which is three times four meters and divide by two

119 Teacher: So what Raj did, and I'm guessing what most people did who managed to get that was to draw a little line there to do our 3 times 4 which is 12 meters squared and work out the triangle on top which was 4 meters squared ok good.

Analysis of Lesson 1

Line 78 is an example of the teacher indicating that a procedural explanation was expected as part of a response. When answering subsequent questions, most students provided a procedural explanation without prompting but other types of explanations were not offered, indicating an established sociomathematical norm. Line 78 was also the start of an IRE sequence (teacher initiation, student response, teacher evaluation), which was the predominant form of teacher-student interaction in the whole class context. In lines 80 and 113, the teacher repeated correct answers, and in line 119 reworded a more complex student explanation, claiming understanding of student reasoning. In terms of social norms, this contributed to accountability residing with the teacher.

<table>
<tr><td rowspan="2">Interpretation of activity: errors, explanations not explored</td><td>Tasks: Examples; Explanations: Multiple solution strategies possible but no acknowledgement. All questions standard format.</td><td rowspan="3">Classroom Norms: Social Norms: Agency and accountability: resides largely with the teacher. SM norms: procedural explanation counts as explanation. Mathematical practices: area equates to multiplication; units vital</td></tr>
<tr><td>Sequencing: Unsystematic variation and links were not explored.</td></tr>
<tr><td>Organisation: Individual working</td></tr>
<tr><td>Cognitive Demand: Potential- high; As enacted- low</td><td>Discourse: Teacher led; Teacher requested (procedural) explanation (line 78), followed up when not provided (line 113) IRE with teacher evaluation (lines 78-80, 80-81, 113-119) Teacher re-voiced contributions; repeating correct answer (lines 80, 113), rewording more complex explanations (119)</td></tr>
</table>

Figure 3: Extract from lesson 1 OMF summary

Integrated OMF for Lesson 1 and 2

When the two lessons were compared the overall profiles were very similar, and the differences in use of context and mathematical practices could be explained by the different lesson topics. More important is the fact that contradictions between the two lesson analyses were not apparent. From these profiles a summary OMF was formed (figure 4) specific to this class. The central core, consisting of tasks, sequencing, organisation and discourse, located specific instances of the teacher's activities, with their impact interpreted in the wider framework. For example, questions were posed that could have been solved in multiple ways or with the integration of multiple representations. However, the IRE pattern of interaction, focusing on a single procedural calculation, was mirrored by the students in their work and led to the categorization of low cognitive demand.

Links between dimensions did emerge. For example, there were two occasions where pseudo-contexts were used: that is real-world objects such as cars or apples were introduced, but in contrived and unrealistic ways. Students made errors in the whole class discussion that were not explored; the teacher focused on explaining the abstracted mathematical procedure whilst the students involved focused on interpreting the context. Follow-up questions were asked by the teacher, but when students' responses did not conform to the abstracted mathematical solution the teacher moved on by offering a direct demonstration of the 'correct' procedural answer. In the post lesson interview, the teacher expressed surprise that errors were made on those questions and was unclear as to why this had occurred. The activities related to the use of pseudo-context can be traced through the dimensions, contributing to the conclusion that the teacher did not explore student thinking and that mathematical competence equates to efficient production of standard solutions.

<table>
<tr><td>OMF</td><td colspan="3">Lesson Image</td></tr>
<tr><td rowspan="3">Interpretation of classroom activity:
Professional noticing - no evidence of exploration of student thinking when it was not directly relatable to a standard solution.</td><td>Goal: Performance</td><td>Plan: Exam style questions</td><td>Hypotheses: Familiarisation</td></tr>
<tr><td colspan="2">Tasks: Multiple solution strategies were possible, but rarely explored. (4 occasions students offered alternative calculation- not evaluated or compared) Lesson 1: No context; Lesson 2: Pseudo-context 2/6</td><td rowspan="4">Classroom Norms:
Social Norms: Agency and accountability: resides predominantly with the teacher.
Sociomathematical norms: procedural explanation counts as explanation. Mathematical competence equates to obtaining correct answers efficiently (errors to be avoided).
Mathematical practices: Lesson 1: Area equates to multiplication. Lesson 2: Proportional reasoning equates to multiplication</td></tr>
<tr><td colspan="2">Sequencing: Questions sets unsystematic variation. All questions dealt with in isolation; links between questions were not explored [Dimensions of variation and range of permissible change not made explicit]</td></tr>
<tr><td rowspan="2">Cognitive demand:
Potential - high
As enacted - low</td><td colspan="2">Organisation: Individual working- tables in groups of four; peer to peer discussions were had.</td></tr>
<tr><td colspan="2">Discourse: Registers: teacher used colloquial language with no evidence of inducting students into a 'vertical discourse'. Patterns: IRE dominant form of interaction. Correct answers acknowledged, often repeated or extended. Errors often ignored; when acknowledged focused on moving to standard solution, reverting to direct explanation if initial follow-up failed. Extended student explanations taken over by the teacher.</td></tr>
</table>

Figure 4: Summary OMF for lesson 1 and 2

Conclusion

When the dimensions of the OMF are considered, distinctive patterns of discourse consistent across both lessons were identified, and the classroom norms indicated that these were regular patterns of interaction. Moreover, the same restricted range of task features were utilised throughout both lessons and the sequencing of questions was classified as unsystematic variation. This provides some evidence that the framework was effective in characterising a teacher's pedagogical approaches. However, whilst the teacher indicated that these lessons were 'typical', the timing of the data collection meant that they were focused on reviewing previously met material. As such, there may be features that are part of the teacher's usual repertoire but are not captured here. For the larger study, when comparisons between classes is sought, collecting data at the same time of year and when the classes are being taught similar material could ameliorate some of these issues.

All elements of the lesson that were classified as being mathematically relevant were mapped to the OMF. It may be tempting, therefore, to say that the model is sufficient to capture mathematically significant classroom events. However, the dimensions of the framework have been populated by features identified as significant in the literature and these have orientated the data collection and analysis, therefore a complete mapping could be anticipated. Instead, the question is whether the orientation this framework offers provides insights into the characteristics of teachers' pedagogical moves more powerful than a list of features and with sufficient validity to allow comparison. The evidence presented here indicates that this may indeed be the case, but further research is needed.

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Using drama to change classroom discourse

Ove Gunnar Drageset[1] and Tor Helge Allern[2]

[1]UiT – The Arctic University of Norway, Norway; ove.drageset@uit.no

[2]Bergen University College, Norway; tor.helge.allern@hib.no

This article provides insights about a project studying how process drama can be used to change classroom discourse in mathematics. The idea is to use process drama to help students practice with roles relevant for mathematics discourse, and then use these roles to help students become active participants of learning during regular mathematics lessons. This article reports from the first lesson conducted after the process drama, and finds a lesson where the students dominate talk, especially asking for and explaining methods and reasons, and where the teacher acts as a moderator.

Keywords: Classroom discourse, process drama, communication.

Introduction

Based on our own studies (Drageset, 2014; 2015) we have observed that mathematics teaching in our area is typically teacher dominated with students rarely contributing to the discourse beyond answering questions. We wanted to find ways to activate students as participants of the mathematical work of the classroom. In drama methods we found tools to develop roles and then rehearse them during a process drama. To develop and rehearse, we designed a process drama called Out of Syria with the students grouped as families. The play was set in Damascus and the families had to escape the war. Although process drama may be used as a subject of its own, emphasizing the artistic process more than the learning outcome, change of attitudes and knowledge will always be important. Thus, applied drama often deals with topics in need of change, and the purpose of drama is to empower the participants (Bolton, 1998; Landy & Montgommery, 2012; O'Neill 1995).

During the travel the families had to solve both practical and mathematical tasks. Simple scenery, sound effects, music, costumes and the physical actions of the participants were used to create fiction. Different rooms were used for home, bus, apartment in Egypt, border crossing and boat travel. Dramatic effects were used by lightening contrasts, sound effects and by the researchers playing different roles such as police, smugglers and coastal guards. An important goal for the process drama was to create a reference among the pupils to role aspects such as the authority, the skeptical, the curious, and the mediator. Each member of the families was assigned one of these roles. We wanted to examine if and how the reference to these roles would contribute to a change in classroom discourse, i.e. to create discussions, argumentation and reasoning. The elder should always ask all members of their meaning, and try to enlighten their reasoning for this, before deciding. The curious should ask why as often as possible, while the skeptic should try to oppose and suggest alternative solutions or decisions. Using the situation of refugees from Syria as our topic might seem insensitive. It was, however, conducted with care and related to learning about other areas than mathematics.

A few weeks after the process drama, the researchers returned to the classrooms to use the roles in ordinary teaching. An important point is also to change the teacher role, avoiding to play the elder (authority) and thus open a space for the students to use. The aim with this article is to do an analysis of the first lesson after the process drama in order to characterize the lesson as a whole and describe qualities of the discourse.

Theoretical framework

Studies related to classroom discourse often describe different ways teachers dominate the discourse. Normally these fit into a pattern described as IRE (initiation-response-evaluation) where the teacher initiates the questions, the students respond to them, and the teacher evaluates the response (Cazden, 1988; Mehan, 1979). IRE is often related to procedure-bound discourse, with little emphasis on 'students explaining their thinking, working publicly through an incorrect idea, making a conjecture, or coming to consensus about a mathematical idea' (Franke, Kazemi, & Battey, 2007, p. 231). Other teacher-dominated patterns are described, such as uni-directional communication (Brendefur & Frykholm, 2000) and conventional text-book classroom culture (Wood, Williams, & McNeal, 2006). However, the teacher domination appears in different ways, which might be illustrated by the four types of communicative approaches suggested by Mortimer and Scott (2003). Three of these approaches can be seen as illustrations of different types of teacher dominance. In B – the non-interactive / dialogic approach – several points of view are paid attention to but without allowing others to participate. This could occur when a teacher presents several points of view and discusses these without allowing students to participate actively. In C – the interactive/authoritative approach - the participants are allowed to participate but only one point of view is paid attention to by the teacher. In D – the non-interactive / authoritative approach - only one point of view is attended to and the teacher does not allow others to participate.

The alternative to teacher domination is found in A – the interactive / dialogic approach - where several points of view are paid attention to and people are allowed to participate actively. Others have also described communicative approaches where students are actively engaged. One such type of approach focus on sharing ideas, such as strategy-reporting classroom culture (Wood et al., 2006, and reflective communication (Brendefur & Frykholm, 2000). Another type of classroom communication reported is where the students work alongside the teachers solving problems, such as inquiry/argument classroom culture (Wood et al., 2006) and instructive communication (Brendefur & Frykholm, 2000).

While the above are examples of characterizing entire practices, other scholars have studied single utterances in more detail. In one such approach, Alrø and Skovsmose (2002) identified eight communicative features: getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging and evaluating. While all these eight communicative features were present in both student-student and teacher-student interactions, others describe different types of student participation, such as Drageset (2015) describing five types of students interactions; initiatives, explanations, partial answers, teacher-led responses, and unexplained answers. Students explanations might be particularly interesting, consisting of explaining reason (why), explaining concept and explaining method (how and what).

There also exists frameworks and concepts describing how teachers orchestrate discourses in the classroom. Ponte and Quaresma (2016) suggest a framework to analyze discussions that distinguish between management actions and actions related to mathematics. The four teacher actions related to mathematical aspects are inviting, informing/suggesting, guiding, and challenging. Fraivillig, Murphy and Fuson (1999) also focus on similar teacher actions by describing how a teacher use students' ideas to lead them towards more powerful, efficient and accurate mathematical discussion by eliciting, supporting, and extending children's mathematical thinking. While Fraivillig et al.

(1999) developed the concepts from a study of a particularly skillful teacher, Drageset (2014) developed a framework based on a study of five rather ordinary teachers. The result was that the redirecting, progressing and focusing framework (Drageset, 2014) that describes actions where the teacher actively involves the students (such as enlighten details and justify), and also describes actions where the teacher is more authoritative and less interactive (such as simplifying, closed progress details and demonstrating) than described by Fraivillig et al. (1999). Teacher dominance is also described by others, such as through funneling (Wood, 1998), the Topaze effect (Brousseau & Balacheff, 1997) and guided algorithmic reasoning (Lithner, 2008).

Method

This study is part of a larger study on how the use of drama and roles can change the classroom discourse. The study includes classes from primary, secondary and pre-service teachers. The data in this article comes from a primary classroom.

The long term goal of this project is to develop knowledge that can be used to educate teachers in changing classroom discourse. With the lesson reported in this article, we wanted to try out a lesson with the students to understand what is possible using roles exemplary. To achieve this, the lesson was led by one of the researchers. This means that we are exploring the potential to change students' participation using roles, analyzing qualities of students' participation. We will also look at the teacher's (researcher's) actions in order to understand the reason for any changes in student participation. To see how students normally participated, we filmed a few lessons before the process drama.

The selected classroom was chosen of convenience and consisted of 17 students from a typical upper primary school in Norway aged 11-12. None had been refugees or had any first-hand knowledge of Syria or refugees. Their teacher has a typical Norwegian four-year teacher education with some specialization in mathematics.

The lesson was filmed and transcribed. The data was then analyzed using conversation analysis (Linell, 1998), describing qualities of single utterances from student and teachers, grouping them together to establish categories.

Findings

The lessons before the process drama showed a traditional IRE-classroom. The students answered the questions they were asked, often answering tasks and sometimes explaining concepts and methods. The teacher dominated the talk.

The lesson in focus here is the first lesson after the process drama. The teacher of this lesson, which was one of the researchers, started by discussing the roles learned during the process drama. The students suggested important features from each role, especially the elder, the skeptic and the curious. They then agreed to use these roles actively during the coming lesson. Sometimes during the lesson, the teacher reminded the students of their roles, or challenged some of them to use one specific. The following excerpts are carefully chosen to illustrate the typical discourse of the lesson.

The students were given two tasks about decimals and told that they should use two different methods to solve them. Prior to this, the students had worked with informal methods to operate on decimals. The tasks used names from the process drama, Omar and Samira, and told them that Omar had 3,2

liters of water while Samira had only 2,6 liters during a travel in the desert, and asked how much Omar had to give to Samira to get level. The other task was similar, but with Omar and Samira having 0,7 and 5,3 liters, respectively. Early in the lesson, one student volunteered to tell how he solved the first task of sharing water (figure 1):

Student: Here I only thought, first I made a number line, and then I thought… what was it? Two comma … six? (draws a number line with 2,6 to the left and 3,2 to the right)… And then I took as many markers as it was… between the two (points out 2,6 and 3,2, and draws a marker for every tenth, see figure 1)… three comma zero, three comma one, and then three comma two (points)… and then… and then I just jumps like this, this, this, there (draws three jumps from each side towards the middle, marks the middle with a vertical line). And then I see how long it was from the middle to there (points out 3,2), it was three… or it was three to the middle. And then I thought that it just were zero comma three.

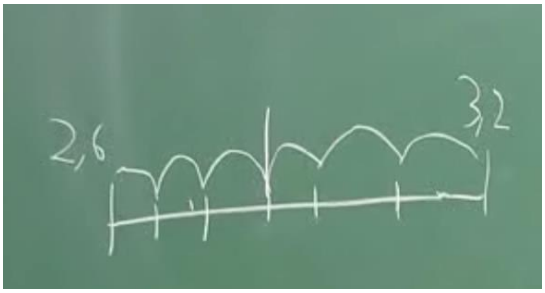

Figure 1: Solving the first task – first strategy

When explaining an alternative method for solving the same task as above, a student said this:

Student: 2,6 and 3,2 (writes them on the blackboard, look at figure 2) then I just jumped… then it became 3,0 there (writes it below 3,0) and 2,8 there (writes it below 2,6, and draws and arrow from 3,2 to 2,6 with 0,2 above the arrow). And after that I just jumped one there (draws a line between 2,8 and 3,0). And then it became 2,9 and 2,9 (writes 2,9 below both 2,8 and 3,0, and writes 0,1 above the line between 2,8 and 3,0). And then I just added these two (0,2 and 0,1) and then I got 0,3 (writes =0,3 to the right).

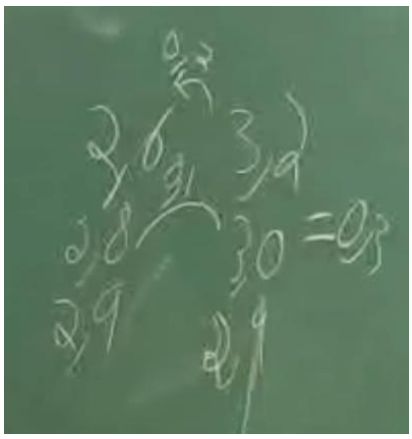

Figure 2: Solving the first task – second strategy

These two explanations are similar as they seek to explain every step from task to answer, telling others how the answer was found. Such explanations naturally occur following a request to tell what had been done or how a solution were found, and were the most common type of explanations during the lesson.

However, this was not the only type of explanation observed during the lesson. An interesting exchange of meaning came when a student disagreed to the first solution (figure 1):

Student A: Yes, but I found another answer

Student B: What did you get?

Student A: 0,6

Student C: Me too, but then I found out that I had got wrong

Student D: 0,3

Student B: Why did you get 0,6?

Student A: Because I counted on both sides

Student E: Me too

Student B: Yes, but in the middle. Thus when you jump with both of them then you should not count it and, because, the question was thus, the question was not how much was between them, it asked how much it was, how many liters Omar had to give Samira then. You should not count how far, it was only like, when they met. Where they met you should stop counting, you should not continue counting… and down. That was only therefore.

There are several things worth noticing in this discourse. First of all, a student challenges another. When the two opposing answers are presented, student A is asked why she got 0,6. The answer is short, but explains the reason for the answer clearly as the student points at the drawing on the blackboard. We can also see two instances of support for the answer 0,6, even though one had changed meaning. In the end student B explains the reason why he means 0,3 is correct and 0,6 is incorrect. It is a long explanation, and the student struggles to find the words, but ends up convincing the other students that 0,3 is correct. And during all this, the teacher is not participating.

Explaining reason is different from explaining method (Figure 1 and 2) as explaining reason seeks to argue for and justify the solution, explaining why instead of how or what. Explaining reason occurred frequently during the lesson. Sometimes the reason was explained quite clearly, other times more struggling (like the long explanation of 0,3 being the correct answer), and some times the explanation was insufficient. Explaining reasons naturally came as a result of someone asking why a solution is correct, or as above as an argument when challenged with an alternative solution.

The examples above illustrate the discourse of the lesson, where the students frequently ask questions to each other (why, how, what) and frequently explains method and reason. Also, student challenges and clarifications are observed.

The teacher did participate during the lesson in different ways, mainly by asking questions such as these five examples from different parts of the lesson:

1) On the first task, is there anyone that wants to tell the method used on that one?

2) Can you show us on the blackboard?

3) Were there anyone else that got three comma zero?

4) Okay. Are there anyone that has done it in another way?

5) Yes. And what was the answer then?

The first two is about enlightening details, either by asking a student to tell the method or by using the blackboard so that it is easier to follow the line of thought. The third and fourth are examples of how the teacher asked for alternative strategies, and the fifth is an example of how the teacher sometimes requested clarifications. Evaluations or support from the teacher were observed, but rarely. Funneling and guided algorithmic reasoning were simply not observed at all.

Conclusion

This first lesson after the process drama contained considerably more student than teacher talk, as exemplified by the excerpts. The most frequent types of student interactions were questions (why, what, how) and explanations of method and reason. The questions can be related to the roles of skeptic and curious, and the result of these were explanations and arguments. These questions and explanations played out as a discourse where the teacher acted as a moderator. The teacher did not take the typical role of the elder (authority) but instead allowed the students to take this role. This type of discourse has similarities with both interactive/dialogic communication (Mortimer & Scott, 2003), strategy-reporting classroom culture (Wood et al., 2006) and reflective communication (Brendefur & Frykholm, 2000) as the focus is on sharing different strategies. But the lesson goes beyond these where the students' request explanations and the discourse goes on with explanations, questions and clarifications without teacher participation. In these cases, the lesson might be defined as instructive communication (Brendefur & Frykholm, 2000), where the teacher requests details, asks for alternative strategies, and requests clarifications. However, it is not an inquiry/argument classroom (Wood et al., 2006) as the teacher is not really working together with the students to solve problems.

Reduction of complexity, such as funneling, guided algorithmic reasoning and the Topaze effect, which are the most frequently used teacher actions in other classrooms at upper primary level in the same area (Drageset, 2015) were not observed at all. It is also evident that the lesson does not follow an IRE-pattern, as students both initiate (ask questions) and evaluate (agrees, requests explanations, and challenges).

A majority of the student interactions, especially their ability to ask, explain and challenge, is similar to the roles practiced upon during the process drama, especially the curios through the students' use of questions (why, how, what), but also the elder through the willingness to listen to alternative strategies and assess which is the best, and occasionally the skeptic through questions and challenging.

The role of the teacher was withdrawn and might seem of little importance. But this change does not happen by itself, the teacher is the key to the change. First by leaving the typical 'elder' role and inviting students to fill this role by asking them to decide right and wrong and assess each others'

suggestions. Secondly, by acting as a moderator to encourage the use of other roles such as being curios and skeptic.

Further study of the process drama itself, and the lessons filmed before the process drama, is needed to understand how much the students' involvement changed and if it is possible to explain any changes by our use of process drama. In general, there is a need for research related to the use of roles to establish different types of discourse, and also how teachers can learn, rehearse and use roles to develop their practice.

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Actions of a prospective teacher in different moments of practice: The case of Berta

Nadia Ferreira[1] and João Pedro da Ponte[2]

[1]Instituto de Educação, Universidade de Lisboa, Portugal, nadiadferreira@gmail.com;

[2]Instituto de Educação, Universidade de Lisboa, Portugal, jpponte@ie.ulisboa.pt

In different moments of teaching, such as in launching, exploring and discussing a task, teachers carry out different actions, which have a critical influence on the classroom dynamics. Our aim is to identify and understand the actions that Berta, a prospective teacher, undertakes in instructional practice to promote students' conceptual learning of rational numbers. For this paper, data were collected and analyzed from classes observed and videotaped. To promote and organize an exploratory environment, Berta launched the task, strived to promote interactions, organized students' solutions, and attempted to promote a discussion environment with inviting, supporting and challenging actions in order to lead students to learn rational number concept.

Keywords: Teachers actions, moments of practice, prospective teachers, rational numbers.

Introduction

Rational numbers are a topic that raises many difficulties for students. Teaching rational numbers leading to conceptual learning is a challenging task for teachers. This topic, very important in the elementary mathematics curricula, requires the exploration of different representations to support students' learning. In fact, the complexity of these numbers is related to the different meanings (such as part-whole, quotient, measure, and operator) and representations (such as decimals, fractions, percent, active and pictorial representation) that they may assume (NCTM, 2007). To foster students' understanding of mathematics concepts teachers are challenged to promote an exploratory learning environment (Ponte, 2005). Research has given attention to prospective teachers' knowledge about how and why to teach rational numbers in different ways. For example, a study by Isiksal and Cakiroglu (2011) indicates that prospective teachers have different perceptions of children's mistakes and different suggestions of strategies that may be followed including using multiple representations, using problem solving strategies, making clear explanations of questions, and focusing on the meaning of concepts. However, it is important to understand the practice that prospective teachers, with these teaching views, accomplish and what kind of action takes place in the classroom. There are few studies focusing on the teaching practice of prospective teachers, paying attention to their actions in different moments when they explore different tasks with their students and to the communication that entails. So, in this paper we aim to identify the actions that a prospective elementary school teacher (Berta) accomplishes in different moments of instructional practice, as she strives to promote students' conceptual learning of rational numbers.

Prospective teachers instructional practice

Teachers' practice could be analyzed with different approaches (cognitive or sociocultural). Ponte, Quaresma and Branco (2012) reconciling the two approaches attend to curricular and social context, teacher knowledge, actions and teachers reflection. In this regard they propose to analyze teachers' practice with reference to two main aspects: the tasks proposed to students and the communication

established in the classroom. In respect to tasks, teachers may choose to just offer simple exercises or also propose challenging exploratory tasks, problems and investigations in which the students need to design and carry out solving strategies based on their previous knowledge (Ponte, 2005). Communication may be oral or written, and it includes both linguistic and mathematical representations. One important aspect of communication is questioning, involving confirmation, focus, and inquiry questions (Ponte et al., 2012). Communication also includes those representations that are used to aid in solving a task, such as building or illustrating objects, concepts, and mathematical situations (NCTM, 2007). Another important aspect of communication is explanations. Instructional explanations may have different purposes and characteristics and may be carried out at different times during a lesson (Charalambous, Hill & Ball, 2011).

Shaped by communication we have the teachers' actions that influence the classroom dynamics. In different moments of work the actions address the development of mathematical concepts and processes (Ponte, Mata-Pereira & Quaresma, 2013) and management of the learning environment (NCTM, 2007). In an exploratory environment the work with a task may develop in three fundamental moments: Launching, exploration and discussion (Stein, Engle, Smith & Hughes, 2008). In launching the teacher organizes the students and the materials (NCTM, 2007) and proposes the *task informing, inviting and guiding* (eliciting) the students to solve it (Lobato, Clarke, & Ellis, 2005). The students have to understand the task, identify conditions and data. To promote the quality of the discussion of the task and knowledge that may be built, the teacher must discuss key aspects of context and verify if the students recognized them. Further, the teacher must discuss and relate mathematical ideas with key contextual characteristics and build a common language clarifying the unknown or confusing vocabulary for students. Note that teachers should take care of maintaining the challenge of the task (Jackson, Garrisson, Wilson, Gibbons & Shahan, 2013). When students begin their work, in pairs or groups, it is important that the teacher *guides and challenges* them to build together their solutions, promoting productive interactions, and register solutions in an organized way (NCTM, 2007). During this time the teacher monitors students' work, questions them and clarifies doubts related with content or the task context *informing*. Again, the teacher supports students, questioning or even explaining, but must not decrease the level of demand of the task and give the opportunity to different solutions to emerge by challenging students. During this moment the teacher selects solutions and structures the sequence of the discussion with a focus on the purposes of the task (Stein et al., 2008). The third moment is the discussion of the task where the teacher orchestrates students' ideas, organizes and supports oral and written communication, promotes interaction between students, connects ideas and guides students towards a powerful mathematical solution (Stein et al., 2008). Addressing the development of mathematical concepts and processes Ponte, Mata-Pereira and Quaresma (2013) consider four main types of actions: *inviting*, to begin the discussion; *supporting/guiding*, leading the students through different kinds of questions; *informing/suggesting*, giving information or validating students' ideas; and *challenging*, encouraging students in interpreting situations, finding new representations, making generalizations and justifications, making connections, and evaluating their work. At the end of the discussion, the teacher must promote reflection on the work accomplished, the new concepts and procedures that emerged, and institutionalize the expected learning. The teacher may also review other concepts and connect with other situations reinforcing the main ideas (Stein et al., 2008).

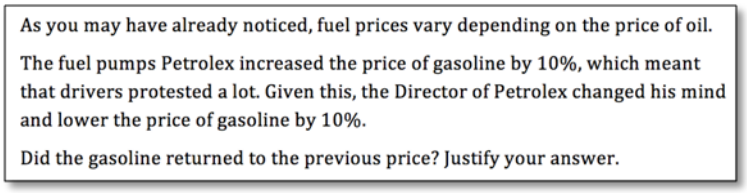
Research methodology

This paper emerged from a larger study of an exploratory nature and takes a qualitative and interpretative approach, following a case study design (Stake). We analyze the supervised practice of three different prospective elementary school teachers to characterize it and to understand the challenges and options that they face in different moments of instructional practice, as they strive to promote students' conceptual learning of rational numbers. Data were collected from semi-structured interviews and video-stimulated recall interviews after class. Four lessons were observed, videotaped and fully transcribed. We also collected documents produced (lesson plans and written reflections). For this paper we analyze one prospective teachers and her instructional practice in one of her lessons. Data were analyzed based on categories that emerged from the above framework, namely teachers action in launching, exploring the task and in discussion of the task combining the actions *invite, inform*, *guide and challenge*. Bertas' case, a prospective elementary school teacher, could represent the teachers that suggest an exploratory approach to develop conceptual knowledge of their students. So, we aim to identify the actions that she accomplishes in different moments of instructional practice, as she strives to promote students' conceptual learning. In the Teacher College last semester, Berta considered important that students understand the concepts and develop mathematical communication through an exploratory approach, valuing contextualized situations and open tasks. In the mathematics education course she analyzed the video of a class and its teacher's plans and reflections. When she was about to teach the notion of percentage selected the task of the video to explore with her students. We should note that her practicum takes place in a 6th grade class with 20 students who don't have experience in discussing open tasks. Comparing the knowledge and skills of her students with the students of the video, she defined her teaching purpose and decided to promote her students' conceptual learning of percentage, the development of mathematical communication, and problem solving skills.

Actions of a prospective teacher in different moments of teaching practice

Launching the task

Berta introduced the task "Petrolex" distributing the worksheet to each student and asking one student to read it (figure 1). In order to promote "a short discussion with students about the thematic of fuels price" (WR), she highlighted key aspects of the context of the task, ensures that the students recognize them, and reinforces the main question:


As you may have already noticed, fuel prices vary depending on the price of oil.

The fuel pumps Petrolex increased the price of gasoline by 10%, which meant that drivers protested a lot. Given this, the Director of Petrolex changed his mind and lower the price of gasoline by 10%.

Did the gasoline returned to the previous price? Justify your answer.

Figure 1: Berta Percentage Task

Let's see. The gasoline has a price… Next you will discuss that but… The gasoline has a price, right? You will increase the price in 10%. But the drivers don't like the new price and protest… Then the director decreased again the price in 10%. What does this mean… Diana [as others] is telling us that gasoline returned to the same price … And they say no… Who thinks that the price returned to the same value?

Berta invites, informs and guides the students in understanding the task and feel challenged. In response to some students said that the price went back to the original price and others disagreed. The discussion began and Berta provided information about the organization of the work and stressed the importance of making written records of the solving processes to support the discussion. The prospective teacher invited the students to solve the task, organized students' time and reinforced the importance of recording their different ideas about the challenge proposed. Berta ended the initial discussion and emphasized that it has to be made in pairs:

> So what will you do? We will not say more and join your partner and try to understand what would be the price of gasoline be… Will it return or not to the initial price? We have five minutes and I will circulate between the tables to try to understand… Attention, what you have written. Say yes, say no, do not erase anything! Leave everything as it is! Then we will see what you thought, what do you see and what is the final decision.

Exploring the task

The students began to work in the task and Berta circulated around the pairs, observing solutions and supporting struggling students. One student requested her help and she highlighted the main question posing different questions:

Student 1:	This is a tricky question!
Berta:	Why? I do not say anything... I'm only asking if it is the same value of the beginning? There is no trick!
Student 2:	It returns to the initial price!
Berta:	So we can see it... Can you try a way to view it?
Student 2:	Calculating!
Berta:	You can do it!
Student 2:	Can I make up a price?
Berta:	You can make up a price! But look carefully what the price is...
Student 1:	60 euros.
Berta:	You did not have to say an actual price but do you think that this price is so...?

In this dialogue Berta posed an inquiry question reinforcing the main question. The students' idea was not right and the prospective teacher guided students' work by suggest/informing them to try different values and verify the conjecture. The students suggested a price, she confirmed the idea and reinforced and focused students on the nature of the price. Before Berta letting the students work on their own, she tried to promote students' interactions and warned the class about the importance of the written record of the solution:

> Okay... So you are working in pairs and therefore you can talk about that and make up a price . . . [To the class] Attention of what you write down... Do not erase so we can see how you thought!

Berta wanted the students to discuss their options and insisted that they had to work in pairs, promoting interactions. Anticipating and preparing the moment of whole class discussion, she

emphasized the importance of recording all the ideas. As she asked not to erase any solutions we can assume that she wanted the wrong solutions so that the errors and different ideas could be discussed. Berta observed the students' work and supported and guided, in a provoking way, other students. However the class ended and the discussion happened in the next day.

Discussing the task

Berta began the class by organizing the students in the same pairs and distributing back the solutions that she had already analyzed at home. One pair had a good solution but it was not well organized. The prospective teacher numbered the different steps and checked with the students her proposed organization. She informed the pair of the importance of organizing written communication to clarify their ideas to others and skipped the opportunity to discuss it with all students. After this, she began the discussion of the task.

To promote a discussion environment and to invite students to the work, Berta recalled the first idea of many students. She asked a student to present his first idea and then the second solution he built when he realized the error. We notice the sequence of solutions presented:

> Most of you [initially] considered that the price of gasoline would return to the initial value. After the increase and after the decrease! So your colleague will start by explaining the first part. Why did you think that the price returns to the same? . . . Do not copy anything, first let's see...

After the students registered their solution in the white board, one student explained their idea (figure 2):

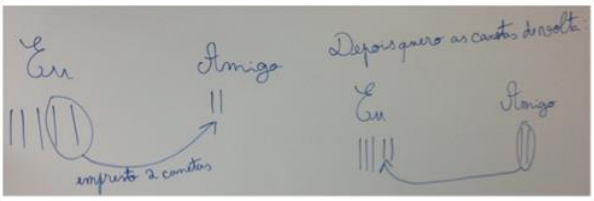


Figure 2: Tomas first solution

> I did... a friend and I... I lent pens [to a friend] and I went back and ask him for the pens. [It is the same situation] and so the price is back to the same!!

Berta guided the presentation and asked the students if they tested their idea with a value. She challenged the students by asking "Have you verified if it gives you the same value!?" establishing connection with the second solution of the same pair. The student recorded the second solution with 10 euros as a starting value and explained his ideas with the support of Berta:

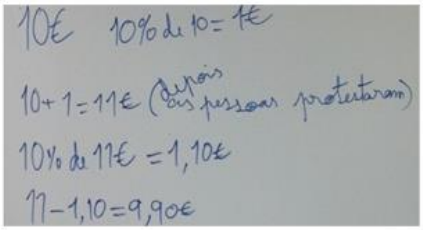


Figure 3: Tomás' second solution

Tomás: The gasoline costs 10 euros and 10% of 10 euros is 1€.

Berta: . . . Why did you do 10% of 10?

Tomás: To know how much is the value we have to add up to 10 €.

Berta: Exactly! And then?

Tomás: Then I added 1 euro to 10 and gave 11 euros. And then people protested...

Berta: . . . How much was the increased value of fuel? . . .

Tomás: It increased 1 euro.

After Berta made a supporting question the student replied:

Tomás. Then it decreased by 10% and 10% of 11 is 1.10 euros.

Berta: And here Tomás did very well... Because Tomás calculated 10% of 11 euros and did not calculate (as some students did) 10% of 10 €? . . . Because we had to see 10% of what we had again! Right? So the amount of the discount is 1.10 €. OK? So the fuel is 9.90 €. If you were a driver, would you prefer this to happen or not?

The students presented their work and to support their presentation Berta posed an inquiry question, challenging Tomás to explain why he and his partner did multiply 10% of 11 euros. After that she focused the students to the specific procedure. At the end she recalled Tomás' explanation and reinforced the students' focus on the main question. Others pairs of students presented their work with a more realistic value. To help students' presentations, Berta organized their computations and clarified issues related to mathematical language. At the end of the presentation, she recalled the task and the two solutions and focused students' attention on the reference unit. For that, she built a representation in hard paper and related it to the students' solutions. However she struggled with the students' difficulties in "seeing" the difference between before and after the rise of gasoline. Berta first explained the conceptual idea of the difference between 10% of the first unit (brown) before ~~of~~ the increased price and the 10% of the second unit (pink) after the decrease of the price:

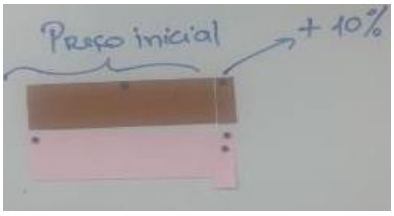


Figure 4: Final synthesis scheme

The 10% will now relate to another unit, not the same as before? Now we have a different unit . . . And this 10% is bigger than that one . . . We have a new price with plus 10%. You are thinking well! I stay with this unit but now I have to withdraw the 10%. But the 10% are not the same! . . . Because the value of the unit is different.

Given the difficulty of some students in understanding the explanation, Berta assumed that maybe other percentages would allow the students "to see better" and presented a new representation with 20% making more evident the difference between the increase and the decrease in the final price. In the synthesis she said:

Berta: If you noticed the starting price was up here (bigger heavy paper). But the final price is already here (less heavy paper). What does it mean? The final price will always be lower than the initial price, right? . . . We had seen it here for € 1.5 that gave us € 1.48, right?

Student1: And this part here, brown, is the 0,15€.

Berta: Yes! This part here, brown, is the 0,15€…

Student1: And this one is 0,02€…

Berta: And this piece is the 0,02€! Very well!

For the synthesis, to guide students on building the percentage concept, Berta prepared a hard paper in order to represent the problem situation. She struggled with the representation prepared and students had difficulties in realizing the difference between the rectangles. Some students understood the explanation but others did not. The prospective teacher was referring to the problem posed and connected her representation with the procedural solution of the students. We notice that she was returning to essential aspects like reasonability of the price value and the influence of the unit of reference on percentage.

Conclusion

Bertas' lesson was based on launching, exploring and discussing one task. She organized students and materials, encouraged and valued students' ideas and discussion among pairs. She combined invited, informed and guided actions reinforcing the main question and focusing students; attention on the fuel value and the impact of the increase and decrease of fuel by 10%. During the exploration, she suggests students to try real values of the fuel, supported, guided and challenged students' ideas and finally promoted solution recording. Berta also highlighted the importance of the discussion among students but, in this lesson, we couldn't see students posing questions to each other or actions consistent with the idea of developing mathematical argumentation. So, during students' autonomous work, Berta mainly challenged students. Berta organized and selected several solutions sequencing them for the discussion. We cannot say that Berta accomplished a dialogic discourse and students really discussed the task but as it is her first teaching practice, her effort can be valued with respect to sequencing students' presentations to enable connections among solutions (incorrect, correct and mathematically powerful). During the task exploration and discussion she posed put different questions with different focuses: from more open and challenging to more supportive and focused. In the final synthesis, although the main ideas were not recorded on the board, she focused the students' attention on the solutions of the task, establishing connections and explaining the main idea but not discussing the percentage concept. So, at different moments of the instructional practice, Berta carried out different management of the learning actions and of the learning environment actions. Berta's actions in launching and exploration moments created opportunities to learn when she supported students in solving the task and didn't decrease the task demand. As a result different solutions emerged to be presented and analyzed in whole-class discussions (Jackson et al., 2013). In different moments of practice we can identify different actions combined. Bertas' aim was to challenge students to build their knowledge together. However maintaining the mathematical demand during questioning and promoting the task discussion could be a complex practice for prospective teachers as we can see with Berta's case.

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Practice of planning in mathematics teaching – meaning and relations

Helena Grundén

Linnaeus University, Växjö, Sweden; annahelena.grunden@lnu.se

Understanding the complexity of teaching also means understanding issues outside classrooms, including planning in mathematics. Although planning is part of a mathematics teacher's everyday life, there is no shared understanding of it, and little is known about how teachers' planning is related to other practices. In response, to explore what planning means to mathematics teachers and planning's relations to other practices, interviews were conducted with teachers and their contents analyzed in several steps to generate a story of each teacher's experiences with planning. For one teacher, Fia, planning meant decisions and considerations about mathematical content and teaching situations, as well as navigating the decisions and opinions of other actors. Fia's planning is related to practices of management, mathematics teaching, and mathematics teachers, all of which influenced her planning and how her students encountered mathematics in the classroom.

Keywords: Planning, teaching, meaning, practice, interview.

Introduction

Teaching in mathematics is complex and cannot be isolated from students' learning or context. To acknowledge that complexity, "deeper explanations of teacher's decisions and actions" (Potari, Figueiras, Mosvold, Sakonidis, & Skott, 2015, p. 2972) are necessary. The decisions that mathematics teachers make and the considerations they take into account before entering teaching situations influence what happens in the classroom and thereby students' opportunities for learning. To explore those considerations and decisions, the planning in mathematics teaching is vital in order to understand the complexity of teaching and to improve students' possibilities to learn. However, research has demonstrated a lack of consensus about the concept of planning in mathematics. Former studies focus mainly on the planning itself and not how planning in mathematics is a part of a complex whole and related to other practices. The lack of shared understanding, as well as of studies addressing the complexity of teaching and planning, are thus inevitable stumbling blocks when studying planning in mathematics. Still, knowledge about what teachers actually do when they plan and their constructed meaning about planning might be important information that contributes to an understanding of teachers' work. However, examining only individual aspects of teachers' work is not enough to overcome gaps in research. Teaching is complex not only because it consists of many parts, but also because those parts are framed by "contextual, epistemological, and social issues" (Potari et al., 2015, p. 2972). To elucidate those issues, it is helpful to conceive planning in mathematics as a practice in which individual teachers act in a specific time and place, and in which their habitual ways of acting are related to other practices within "a network of practices" (Chouliaraki & Fairclough, 1999, p. 23). By acknowledging planning in mathematics as a practice, the aim of this study was to explore that practice as a part of a network of practices. To that end, it was necessary to know how teachers construct meaning while planning and reflecting upon their planning in mathematics, to what practices they relate planning and reflection, and how they express them in stories about their planning. An interview study with six teachers was designed to answer those questions. This paper recounts the story of one teacher, Fia, that includes several of the aspects and relations described by the other teachers interviewed.

Background

Planning in mathematics teaching

Although planning done by mathematics teachers bears consequences for students' learning, studies in the topic remain scarce. Research about planning in mathematics that does exist address three chief aspects: design research including learning and lesson studies, teachers' mathematical knowledge in relation to their planning, and models and templates for planning. Few studies, none of them conducted in Sweden, have focused on what teachers do in their everyday lives as teachers. National documents based on the Swedish curriculum advocate that planning should be done in a systematic way (Skolverket, 2011) similar to that in models emphasized in research (e.g. Gomez, 2002; Superfine, 2008). However, conversations with Swedish teachers and student teachers have indicated that planning means deciding what to do with a focus on activities, not goals or mathematical content. Such a focus on activities also appears in international research on the topic (Akyuz, Dixon, & Stephan, 2013; National Research Council, 2001). In sum, when people talk about planning in mathematics teaching, they demonstrate no shared understanding of what it means.

Practice

One way of describing the complexity of teaching is to use the term *practice*, which captures both individual people's actions and more habitual, common ways of acting within a practice (Chouliaraki et al., 1999). Accordingly, conceiving planning in mathematics as a practice can afford a way to conceive teachers' actions both as individual actions and actions shared by other mathematics teachers, as well as a way to conceive the relationship between those actions and abstract structures, that is how social structures govern people's possibilities to act (Lund & Sundberg, 2004). Since each practice is determined from others within a network of practices and since power relations always are present (Chouliaraki et al., 1999), knowing more about teachers' planning practice becomes a way of knowing more about how the process of planning is related to other practices and how power relations are working within the network of practices. In that sense, using practice as a concept to explore planning in mathematics is a way of considering "contextual, epistemological, and social issues that frame mathematics teaching" (Potari et al., 2015, p. 2972).

Meaning

In this study, meaning referred to "a (collectivity of) subjects' way of relating to – making sense of, interpreting, valuing, thinking, and feeling about – a specific issue" (Alvesson & Karreman, 2000, p. 1147). How teachers relate to planning, makes sense of planning, interprets planning, values planning, and thinks and feels about planning were thus of interest in interviews and their analysis. The meaning that teachers expressed was both transient – that is, constructed and emergent in interactions in interview situations – and durable – that is, connected to cultural and individual ideas. By conceiving meaning as partly durable, it was possible to explain how previous experiences and more habitual ways of acting formed part of the meaning that teachers expressed in interviews.

The study

An important starting point when designing the study was an interest in considerations that teachers' have and the decisions that they make that precede and influence what happens in the mathematics classroom, here called *planning in mathematics teaching*. With the notion that planning is both a

focused, time-bound activity and what emerges from reflections and thoughts that can occur at any time, as well as given the aim to explore a concept about which there is no shared understanding, it was necessary to approach the phenomenon as unprejudiced as possible. Since all teachers have heard about and applied the concept of planning it was important to listen to the voices of teachers, hence the decision to use interviews as a method. In the larger study, from which Fia's story was taken, teachers' reflections on planning were explored. Each of six participants was asked to keep a notebook for a period of two weeks before the interview in which they were asked to record actions, reflections, and thoughts that for them were related to planning in mathematics. In the interviews, teachers referred to their notebooks and chose topics to talk about. The interviewer's role was to listen affirmatively by uttering encouragement and nodding, asking for clarification when something was unclear, and asking follow-up questions. By not using predetermined questions, teachers were afforded freedom in the discussion about planning, which made it possible to see beyond pre-understandings and the normative speech of planning that dominates mathematics education research. The use of notebooks provided a possibility for each teacher to return to the notebook on several occasions, and the interview situation where the notebook was used as stimuli, was a way to experience meaning as durable. In the interview situation, transient meaning was constructed and emerged in interactions both with the interviewer and with the notebook.

Analysis

Conducting interviews with notebooks as stimuli was a way of foregrounding teachers' experiences and meaning. To continue in that spirit, analysis needed to be based on the material, not predetermined categories. Along with reviewing the stories of each teacher separately, aspects hidden in stories as a whole were also sought. Those somewhat contradictory motivations required staying close to the material and keeping a distance from it. Consequently, analysis proceeded in several steps, the first of which involved reading each utterance per se, and noting what discursive action was performed by making the utterance. During that initial coding memos were written to record spontaneous reflections and ideas, as inspired by Charmaz (2014). In another version of the transcripts, meaning units (that is, units considered relevant to considerations and decisions that preceded and influenced what happens in the mathematics classroom) were marked. Each unit was paired with the activity belonging to the unit in the first transcript, and by interpreting the meaning unit and the activity together an aspect of planning, considering, or decision-making emerged (Table 1).

Meaning unit (from step 2)	Activity (from step 1)	Aspect
We have a template that we should stick to.	Expresses requirements from school administration	Formal requirements
…that talented students and parents would say that it [special educational approach] was wrong… or the parents of those students	Expresses unspoken expectations from parents and students	Discourse of mathematics education
Why have I not talked about… talked with colleagues about this movie before?	Reflects on telling each other	Colleagues

Table 1: Examples from analysis step 1

To see aspects hidden in the stories as a group, distance from the material was necessary. Inspired by Szklarski (2015), meaning units were therefore transformed from the first- to the third-person perspective (Table 2).

Meaning unit	Transformed meaning unit
When will I be able to plan with my colleagues? The work turns into working alone although I don't want it to. It is a lot… We make these big, long-term plans, but we never have time to see each other once we've started [implementing the plans].	When will she be able to plan with her colleagues? The work turns into working alone although she doesn´t wants it to. It is a lot… They make these big, long-term plans, but they never have time to see each other once they've started.
Do I dare consider it [special education approach] from the beginning? Do I have the energy? Do I have the time?	Does she dare consider it from the beginning? Does she have the energy? Does she have the time?

Table 2: Examples from analysis step 3

The transformed meaning units were organized so that units dealing with the same aspect were grouped and read as a whole. As a result, meaning was identified and could be expressed as a product of synthesis (Table 3).

Sorted and transformed meaning units (per aspect)	Synthesis of transformed meaning units
Content	In her planning Fia sees that there are several parts to decide upon: what mathematical area the planning should cover, how to connect that to the everyday lives of students, and how to work with and make assessment in relation to students. Good activities can be reused with different foci. Fia thinks about how she will be able to apply an overall special education approach to her long-term plans.
Area [mathematical]	
Planning in detail and thinking… how	
Relation to everyday life: How to get it [mathematics] related to the students.	
At examination, it [thoughts about students] comes	
How does Fia apply a special educational approach in her long-term plans?	
They watched a movie again. Fia has done that several times… The story of 1[name of the movie], but that one focuses on different things. Finding… find activities that you can say, that you can get the most out of… and maybe dare to weed out those that really do not give anything.	

Table 3: Examples from analysis step 4

The synthesis of the transformed meaning units were organized and assembled into stories, one of each teacher. This paper presents the story of Fia, since it emphasizes several aspects visible in the other stories.

Fia's story

Fia is a mathematics-, science-, and technology teacher in compulsory school grades 7–9. She used the notebook for reflections on her planning, teaching, and decisions. In the interview, she referred to her notebook and chose topics to discuss. At several times, she also reflected upon her reflections and reached new insights – for example, when she referred to "pedagogical plans" in her notebook and discussed how constrained she feels when she has to do her planning with a template.

> Fia: I have had exactly that [referring to the template] content before, but it has not been so formal… That formality… everything has to look the same. It makes me constrained, or I don't feel free to think, or… 1,2,3,4: that must come first, then that, then that… But actually… It is also up to me! I can start to think about paragraph 4 if it's about how we should work.

In her story Fia referred to the template several times. She discussed how her school management has decided that a specific template has to be used when planning, largely to be able to collect the plans and thereby "see what is happening". Time is another constraining factor for Fia, regarding both

individual and collaborative planning. Colleagues are resources in Fia's planning, and she would like more cooperation with them, also in her short-term planning. She also highly values spontaneous exchanges of ideas and experiences.

Fia's work with planning varies throughout the year. At the beginning of the school year she generally has more energy, but in the final weeks, particularly for the for the ninth graders, Fia tends to perform what she calls "spontaneous planning" – that is, decides immediately before lessons what she will do. Fia argues that "spontaneous planning" can be good; the creativity that she sacrifices with templates can bloom in "spontaneous planning", and this also affords greater opportunities for student participation.

Having two groups of the same grade level at different times of the day has made Fia aware of how much the schedule influence her planning. She has also experienced how other activities planned for the students (including field days, theater visits, project periods) steal time from her mathematics teaching and thereby affect her planning. If Fia were allowed to decide upon the schedule, then she would plan for more teaching situations with one or two students. One year she had opportunities for such occasions in her schedule and felt that they helped the students very much.

The availability of materials is another factor that influences her planning. Fia discussed an occasion when she was sitting with a small group of students in a room beside the main classroom while the rest of the class was supposed to work in pairs with problem solving and show their solutions on small white boards. However, since there were enough white boards for all students, each student took one and worked individually instead. For Fia, that occasion exemplified how a teaching situation is a meeting between planning and reality and how the outcome can differ from what was intended.

When planning, Fia makes several decisions, including what mathematical content to cover, how to relate that content to students' everyday lives, and how to work with the content. Referring to a film that she has shown several times, she expressed how good activities can be used several times with different focuses. Besides decisions directly related to the concrete teaching situation, Fia has considered how to adopt an overall special education perspective in her long-term planning, which she thinks can benefit all students' learning. Planning can help to "play it cool" and break norms about mathematics teaching. Fia gave examples when she has planned for working with a couple of students at a time although she had a lesson for the whole class, or when she has used the same film on several occasions for the same students. For Fia, planning is always about prioritizing and is related to feelings. She expressed how care for the students and their learning is critical when making decisions and how her own fears and energy level influence how the planning is done. Fia is constantly reflecting on previous experiences and how those experiences can be applied in future plans. She believes that even more reflection for example, after spontaneously planned lessons can be a good way to take advantage of good experiences instead of letting them go to waste.

Fia describes a practice in which the way of talking about planning in mathematics is frustrating. Some ways of planning are more valuable than others, and Fia almost excuses herself for sometimes doing what she calls spontaneous planning. Knowing that other actors might have comments influence her way of thinking, and, according to Fia, changes in teaching can lead to questioning from for example parents and students. In her notebook, she had written about how she wanted to include a special education approach in her long-term planning. She had also written: "Do I dare? Do I have

the energy?" Fia said that those considerations represented fears that not all students will be challenged and that she will have to argue for her choices. Fia thinks that she has the authority to make decisions about the teaching, but that exercising that authority takes energy.

Analysis and discussion

Reading Fia's story in the light of *meaning* defined as "a (collectivity of) subjects' way of relating to – making sense of, interpreting, valuing, thinking, and feeling about – a specific issue" (Alvesson et al., 2000, p. 1147) makes it possible to conceive how she constructs meaning and makes sense of planning in mathematics by choosing topics to discuss, by interpreting the practice of planning, and describing what she is doing and what is important for her. She discussed how choosing mathematical content and ways to present and work with that content always are part of her planning, as well as how good activities can be re-used with a different focus. Also related to the *practice of planning* is how she uses, and wants to use, reflections to benefit from past experiences, and how planning can help her break norms about mathematics teaching.

Besides describing the practice of planning itself, Fia constructed meaning by, for example, discussing how decisions by school management regarding schedule, availability of materials, and templates influence her planning. From her story it is clear that she perceives models often emphasized as a support for planning (Goméz, 2002; Superfine, 2008) as constraints. This is possible to interpret as she is referring to a *practice of management*.

Visible in Fia's story are also norms about how mathematics teaching "should be done." Those norms influence her considerations and decisions and emerged in her story as an invisible idea of what counts as teaching in mathematics, but also as concrete examples related to opinions of students and parents. Interestingly, Fia has ideas that she thinks would benefit students' learning, but she contemplates to abandon them because she worries about parents' and students' reactions. When referring to thoughts of mathematics teaching, Fia relates the practice of planning to a *practice of mathematics teaching*.

Closely connected to practice of mathematics teaching are the colleagues that Fia referred to several times. She talked about a desire to make more collaborative planning, and how colleagues can be resources also in spontaneous exchange of ideas and experiences. Those parts of the story can be interpreted as her referring to a *practice of mathematics teachers*, in which she sees herself as a part. In reality, the practice of management and the decisions made there to some extent determines how she can participate in the practice of mathematics teachers. The degree to which co-planning is valued in the practice of management clearly affects Fia's schedule and how much time she has with her colleagues.

Fia's story makes it clear that her actions and reflections within the practice of planning relate to other practices in different ways. Some of those relations are constraining and hinders Fia from planning the way that she wants, whereas others contribute positively to her planning in mathematics. Since those other practices influence her planning they also implicitly influence her students' possibilities to learn. Although the teacher is ultimately responsible for the teaching, results show that there are other aspects that influence the planning and, in turn, what happens in the classroom. That dynamic needs to be taken into account when discussing mathematics teaching and forming development initiatives. Viewing teachers and situations in the classroom as isolated entities poses the undesired

consequence that other important aspects that also influence students' possibilities to learn mathematics are neglected.

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Measuring instructional quality in mathematics education

Armin Jentsch[1] and Lena Schlesinger[1]

[1]University of Hamburg, Germany; armin.jentsch@uni-hamburg.de

Instructional research has recently become more important within the framework of teaching effectiveness research. Various instruments have been developed within this research discipline in order to gain a better insight of what is really happening in the classroom. Most of these instruments mainly focus on generic aspects of instructional quality. In this paper we first describe a subject-specific dimension of instructional quality. Second, we show how these subject-specific aspects could be measured empirically with a standardized observational instrument. The results point out both good interrater agreement and satisfying reliability measures. The presented observational instrument has been developed within the study TEDS-instruct in which relations to teachers' competencies and students' achievement are analyzed.

Keywords: Mathematics instruction, instructional quality, classroom observation techniques, instructional improvement.

Introduction

In German instructional research, subject-specific aspects of instructional quality have been investigated rarely until now. Three generic dimensions have been introduced which are classroom management, personal learning support and cognitive activation (e.g. Lipowsky et al, 2009). Although these basic dimensions were developed for mathematics instruction in the first place, they are now deemed relevant for every subject at school (Baumert et al., 2010; Helmke, 2012). Other important aspects of mathematics instruction were apparently disregarded then (e.g. representations, examples, modelling, proof). Blum and others (2006) therefore ask for a high *mathematical* quality of the lesson beyond the three basic dimensions.

At the same time, several instruments for measuring subject-specific aspects of instructional quality have been developed within the American debate (e.g. Learning Mathematics for Teaching Project, 2011). However, some of these instruments do not contain generic aspects even though the prognostic validity of the three basic dimensions of instructional quality has been shown empirically more than once (see Baumert et al., 2010; Kunter et al., 2013; Lipowsky et al., 2009). In conclusion: to our knowledge there is no standardized instrument in existence that is based on a sound theoretical framework (e.g. the three basic dimensions) covering also subject-specific aspects regardless of the mathematical content discussed in class. Starting from these three basic dimensions, we would like to point out subject-specific characteristics of instructional quality that are deemed relevant in the literature and show how these characteristics can be measured with a standardized instrument. The purpose of this paper then is to describe the development of an observational protocol that is used to assess instructional quality in secondary mathematics classes and to present first empirical results.

Theoretical framework

In recent years educational research has shown a great scientific interest in teacher knowledge and instruction (see Hattie, 2012). At the same time, the relation of teacher competencies and students'

achievement has been analyzed (Baumert et al., 2010; Hill, Rowan & Ball, 2005; Scheerens, 2004). The so-called process-mediation-product-paradigm is regarded as a theoretical framework in which these research questions are grounded. This framework describes a relation between students' learning and instruction which has to be *offered* by teachers and *used* by students (Brophy, 2000; Brophy, 2006; Helmke, 2012; Oser, Dick & Patry, 1992).

Three basic dimensions of instructional quality

As mentioned before, three generic dimensions of teaching quality have been developed in recent instructional research which are classroom management, personal learning support and cognitive activation. These dimensions were shown to have a positive impact on both students' learning and their motivation in class (Baumert et al., 2010; Lipowsky et al., 2009).

Classroom management focuses on quality-oriented learning time provided for students and on how effectively the teacher deals with disciplinary conflicts (Brophy, 2000). Effective classroom management is also characterized by a lesson that is organized well and has clear routines (Lipowsky et al., 2009). The second dimension, *personal learning support*, includes the individual support provided by the teacher, the relationship between students and teacher as well as constructive feedback (e.g. Rakoczy, 2008). Finally, *cognitive activation* refers to how problem-solving tasks are used to activate learning processes (Baumert et al., 2010; Brophy, 2000). This dimension includes the activation of previous knowledge and whether challenging tasks and questions are presented that foster students in high-level thinking activities (Lipowsky et al., 2009; Praetorius et al., 2014).

Although the three basic dimensions focus on generic aspects of instructional quality, the question remains whether they could be operationalized in subject-specific way. This holds specifically for cognitive activation (Drollinger-Vetter, 2011; Schlesinger & Jentsch, 2016). Moreover, it is not clear which subject-specific aspects are missing in this framework and how its dimensionality changes when generic and subject-specific aspects are measured simultaneously (Drollinger-Vetter, 2011). Due to these concerns, most instruments that have been developed for the analysis of instruction are only suitable for a very small number of situations (e.g. regarding the mathematical content, see Steinweg, 2011; Schoenfeld, 2013).

Measuring instructional quality

We will now focus on the question how instructional quality can be measured both reliably and validly. Praetorius and colleagues (2012) see classroom observations as a straight-forward way to do so, especially compared to the analyses of material or minutes conducted during the lesson. Helmke (2012) even claims that one can only speak of instructional research in a narrower sense when classroom observations are performed. The reliability of classroom observations is always an issue since observer ratings are often heavily biased and the stability of the measurement is sometimes questioned (for an overview see Praetorius, 2014). Most authors suggest analyzing variance components beyond measures of interrater agreement to understand better which sources of error have added to the variance of the observed score (e.g. Praetorius et al., 2012; Praetorius et al., 2014).

Observational instruments may contain both low and high inference items (Praetorius, 2014). Codings with low inference are operationalized in a way that is strictly observable. High inference

items, in comparison, need the observer to interpret what he or she sees which makes the observation much more complex (Hugener, 2006). However, at the same time one gains a higher validity because instructional research has shown that low inference items explain only little variance when students' achievements are measured as outcome variables (e.g. Baumert et al., 2010). This is because the surface structure of instruction (e.g. which method is used by the teacher) and its quality may sometimes vary independently from each other (Kunter & Voss, 2013).

Developing an instrument for measuring instructional quality

The observational instrument that is presented in this paper was developed within the study of TEDS-Instruct which is a Follow-Up Study of TEDS-M (Teacher Education and Development Study in Mathematics). The main goal is to empirically investigate teachers' competencies in mathematics education at the secondary level and their influence on students achievement mediated by instructional quality. As a matter of fact, students' achievements will be collected to describe the prognostic validity of both teachers' competency tests and the observational protocol that is presented here.

Subject-specific aspects of instructional quality

For developing a subject-specific dimension of instructional quality we first analyzed which subject-specific aspects of instruction are assumed to have an impact on students' learning which has to be examined empirically. The main goal for developing a fourth dimension with subject-specific aspects was hence to extend the existing generic theoretical framework of instructional quality. Such an extension with subject-specific aspects is not established until now (e.g. Steinweg, 2011). During the development of the fourth dimension it became apparent that it is necessary to discuss the subject-specifity of some aspects that are included within the former three-dimensional generic framework. This discussion leads to the assumption that the three basic dimensions of instructional quality are not completely generic. However, in the fourth dimension there were included only such subject-specific aspects that were not already used to operationalize the other three dimensions.

For conceptualizing this dimension, a systematic literature survey within the databases of Web of Science, ERA and ERIC was conducted (see Schlesinger & Jentsch, 2016 for more detail). At the same time, the national debate on mathematics education and the German common core standards was reviewed. Based on the described approach, the following aspects were operationalized for the observational instrument:

- the teacher's mathematical correctness
- the use of representations
- mathematical competencies (modelling, problem-solving, the use of mathematical language, argumentation and proof, training mathematical tools and operations)
- a constructive approach to students' mathematical errors
- the quality of exercises and tasks
- sense-making
- teachers‘ mathematical explanations
- appropriate examples
- mathematical depth (e.g. generalizations)

Method

Based on the three basic dimensions of instructional quality and the subject-specific aspects that were condensed into a fourth dimension, we developed an observational instrument that can be used for in vivo ratings without needing videos of the lesson and that can be utilised independently from both the specific mathematical content and the academic year. Instructional quality was rated by assessing the items on a four-point scale (*1="not at all true"; 4="completely true"*). The instrument consists of four dimensions which are classroom management (five items), personal learning support (seven items), cognitive activation (five items) and mathematics educational quality of instruction (nine items). The data for TEDS-Instruct was collected in Hamburg from a sample of 38 teachers at the secondary level. The teachers participated on a voluntary basis. Therefore it can be assumed that they were greatly motivated to have their lessons observed. Each teacher was observed for two lessons (90 min each). Within one lesson, the instructional quality was assessed four times (every 22.5 min).

Example items	Indicators
Mathematical depth	• The teacher provides generalizations • The teacher provides mathematical connections • The teacher deepens and structures mathematical knowledge
Representations	• The teacher provides various representations for mathematical objects • The teacher illustrates the linking between different representations

Table 1: Two example items for the subject-specific dimension

Altogether, there were six observers involved in the classroom study, all of which held at least a university degree from a mathematics teacher program. The observers were trained for the classroom observations in advance which took around 20 hours. The training had three main goals: 1) a joint understanding of the theoretical underpinnings of each rating dimension, 2) familiarizing with the observational protocol, 3) ensuring a satisfactory amount of interrater reliability. By doing so, all items and indicators were discussed thoroughly with the help of a rating manual. The goal of a joint theoretical understanding also involved the *object of measurement*, i.e. instructional quality. Based on the process-mediation-product-paradigm, instruction is regarded as a learning opportunity that is individually adapted to students' skills and dispositions. Even though the focus of the observations lies mainly on the teachers' behaviour, the latter is dependent of the students' behaviour and student-teacher-interactions. Students' reactions to the learning opportunities are crucial to understand and assess the *quality* of instruction and are hence part of the observation, too. Nonetheless, due to pragmatic reasons no student self-reports or cognitive tests have been collected.

Before stepping into real classrooms, the observers trained their skills on videotaped lessons until they reached a certain amount of interrater agreement. Finally, a pilot study was conducted with 13 teachers in three German federal states. After each observation, the ratings were discussed intensively between the two raters. For the data collection the lessons were observed directly without using videotapes, i.e. the raters assessed all items within the lesson in vivo. Two raters were chosen randomly and rated the lesson independently from each other. For this reason it was possible to avoid systematical agreement between certain raters. In addition to these ratings, there were also produced minutes for every lesson. These minutes included teaching methods, teacher-student-

interactions, students' behaviour and reactions, the mathematical content and provided materials and tasks from the lessons for a detailed description of the learning opportunities.

Results

As a first step we calculated descriptive statistics for the data that was collected by external observers. The following table contains the results of all items from the three basic dimensions and the newly developed subject-specific dimension. For ensuring interrater reliability *Spearmans* ρ was calculated. This is a common measure since in educational research one is more often interested in relative than in absolute decisions (Praetorius et al., 2012; see also Shavelson & Webb, 1991). In the present study we reached satisfying results of $.75 \leq \rho_{inter} < .97$ which can be interpreted as high or very high correlations between both observer ratings. In order to calculate the descriptive statistics the data was aggregated to a single datum per person ($N = 38$). By doing so, we first took the average rating of both observers and then calculated the mean of all eight measurement points per teacher.

Items	*M*	*SD*	r_{it}
Classroom management (α = .83)			
Effective use of lesson time	3.58	.33	.59
Clear rules and routines	2.97	.19	.66
Preventing disruptions	3.39	.45	.83
Advance organization	2.89	.49	.55
Working atmosphere	3.23	.51	.77
Personal learning support (α = .714)			
Students' individual support	2.05	.45	.37
Approach to heterogeneity/differentiation	1.26	.38	.64
Self-regulated learning	1.48	.35	.58
Teacher's feedback	3.07	.37	.49
Teacher approval	3.10	.38	.49
Students' feedback	1.05	.11	.22
Fostering cooperative learning	1.75	.46	.27
Cognitive activation (α = .821)			
Challenging tasks and questions	2.54	.47	.79
Supporting metacognition	1.25	.29	.42
Activating prior knowledge and co-construction	2.66	.37	.76
Quality of teaching methods	2.81	.41	.66
Securing knowledge	2.43	.48	.50
Mathematics educational characteristics (α = .820)			
Constructive approach to students' errors	2.79	.56	.69
Teacher's mathematical correctness	3.64	.37	.54
Representations	2.29	.65	.39
Exercises and tasks	2.37	.52	.63
Examples	2.99	.42	.54

Mathematical competencies	1.62	.15	.44
Sense-making	2.09	.49	.32
Teacher's explanations	2.93	.54	.62
Mathematical depth	2.34	.40	.69

Table 2: Descriptive statistics for all items

When looking at the measures in table 2, we see that correctness has reached the highest values in the subject-specific dimension. Even though this could be seen as a ceiling effect, the statistical discrimination is quite high. The same holds for the items in the first dimension. Nonetheless, these ceiling effects are not surprising as the sample consisted of professional teachers only (Baumert et al., 2010; Blömeke et al., 2010). On the other hand, the average individual support that was observed in the lessons is quite low which is also supported by the low measures of the items "Self-regulated learning" and "Differentiation". Finally, the standard deviation of most items is high enough to conclude that a decent amount of variance was measured.

Altogether we can conclude that an acceptable internal consistency could be reached for all four dimensions. When r_{it} = .25 is regarded as a threshold for acceptable measures of statistical discrimination, the item "Student's feedback" did not reach acceptable values and was thus excluded from further analyses which is also due to a floor effect. All other items show at least mediocre correlations to the corresponding scale which supports the claim of three generic dimensions. This is, however, supposed to be confirmed by factor analyses. Recent both exploratory and also confirmatory approaches once again support the hypothesis of three generic dimensions but suggest dividing the subject-specific dimension into two sub-dimensions which will be discussed in more detail in the presentation (Blömeke et al., submitted).

To sum up, this present study has mainly an explorative character concerning the mentioned subject-specific aspects. However, from a more content-related standpoint one can conclude that fostering specific mathematical competencies like modelling or proof has often been disregarded during lessons. Precise analyses of the used material might then be fruitful to understand better what has happened in the classroom.

Discussion and outlook

The presented instrument for measuring instructional quality shall finally be discussed concerning advantages and disadvantages compared to other instruments in the field. Since this instrument has been developed in order to be used in classrooms without analyzing video there is a chance that it could possibly be used in a broader way than instruments from video studies. Second, measuring instructional quality more than once in a given lesson may describe the learning process in more detail and can lead to more reliable data because certain aspects of instructional quality may change a lot during the lesson. The ratings then tend to be biased heavily since the observer has to give a single rating for the whole lesson (Praetorius et al., 2012). Third, the instrument is suitable for most mathematics classes, academic years and mathematical contents. Finally, in this instrument generic and subject-specific aspects are combined which, in addition, can then be analyzed on their relation.

The question remains whether the present instruments' prognostic validity can be shown by analyzing the relation of instructional quality and students' achievements. It should be tested whether instructional quality can be seen as a mediator variable between teachers' competencies and

students' learning, too. This might especially be interesting for mathematics educational scholars since the impact of generic aspects of instructional quality has already been shown in some studies (Baumert et al., 2010; Helmke, 2012; Lipowsky et al., 2009). The important mathematical or mathematics educational aspects of instructional quality and their impact on both learning and other outcome variables as motivation or metacognition have still to be found. Here, our study could help to gain a little more insight.

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Teachers' voices related to the mathematical meaning constructed in the classroom

Kaldrimidou, Maria[1], Sakonidis, Charalampos[2] and Tzekaki, Marianna[3]

[1]University of Ioannina; mkaldrim@uoi.gr

[2]Democritus University of Thrace, Greece; xsakonidis@duth.gr

[3]Aristotle University of Thessaloniki, Greece; tzekaki@auth.gr

Research on classroom teaching practices is mainly focused on teachers' knowledge, beliefs and practices paying limited attention to a crucial aspect of the instructional activity, that is, on the mathematical meaning constructed in the classroom. The present study examines three highly motivated and professionally active primary teachers' instructional practices and their reflections on them in an attempt to identify critical elements shaping classroom mathematical meaning construction. The results show that all three teachers, intentionally or not, make instructional choices, which tend to restrict the mathematical meaning under negotiation. These choices could be attributed to their desire to provide children with an 'easy', 'safe' and 'pleasant' learning environment.

Keywords: Teacher practices, teachers' reflections, epistemological features

Introduction

Significant research has been carried out on classroom teaching practices. An important part of this research examines teachers' mathematical knowledge, beliefs and practices employing different theoretical as well as empirical lenses and aiming to understand its impact upon students' learning. Although the importance of the mathematical meaning constructed in the classroom is widely recognized and implicitly implicated in all the different approaches, relatively little attention has been paid to a detailed analysis of this construction.

In previous studies, we attempted to systematically examine this important aspect of mathematics teaching and learning, analyzing teaching episodes of different teachers, mathematical topics or age students. Lately teachers' reflections on their teaching decisions and practices drew our attention, because of the interest in understanding the value teachers attribute to the nature of the mathematics knowledge shaped in the classroom (Linares and Krainer, 2006). Thus, in this study, after analyzing three primary teachers' classroom practices with respect to the mathematical meaning constructed, we focus on their reflections on this construction. Our aim is to examine each teacher's decisions and practices related to the mathematical meaning construction process.

Classroom mathematical meaning construction

We first attempt to define "*classroom mathematical meaning construction*" and then present framework related to the epistemological features of this meaning used to analyze teaching practices.

Mathematics is a very special discipline and the nature of the mathematical knowledge and the way it operates are among the main objectives of the subject matter curriculum worldwide. The mathematical way of developing (ideal, theoretical) objects and processes reveals ideas' attributes and relations as well as definitions and theorems to identify objects, to produce new, to relate them and to justify properties and relations. One of the most significant aspects of mathematics education is the understanding of this mathematical way of operating to which students gradually become acquainted by the way teachers manage mathematics within the classroom. Thus, the *mathematical meaning construction* concerns the significance that teachers' classroom management attaches to different mathematical contents and procedures. *How does this construction appear to be omitted from different studies?*

The considerable body of research focusing on the so called 'mathematical quality' of teaching employs terms such as '*connection to worthwhile mathematical ideas'*, '*richness of the mathematics*', '*accuracy*' and so on (Ball, et al., 2008) to describe various dimensions of the mathematical meaning shaped in the classroom. Kilday and Kinzie (2009) report on different dimensions related to the 'quality of mathematics instruction' for classroom observation verifying the absence of clear designation for these terms. They report on various tools used to examine this quality summarized along the following dimensions: *teaching aspects* (roles, strategies, classroom setups, tasks, time, etc.), not necessary related to mathematical content; *teachers' knowledge* (e.g., mathematical content); *teachers' and students' functioning* (for example, interactions, behavior, engagement, expectations, etc.) and *learning aspects* (such as cognitive demands). Only one rather unclear term can be identified among the above referring to the mathematical meaning constructed in the classroom that could be seen as relevant, '*clarity and correctness'*, without though any specific relation to the mathematical content.

Within our perspective (Kaldrimidou et al, 2008; 2013), the term '*mathematical meaning construction'* is oriented to the epistemological features of mathematics. These features concern 'definitions' to identify and differentiate the theoretical mathematical objects, 'attributes and relations' to study them and special 'processes' for the management of these objects and relationships. Students are expected to gradually approach these ideas, objects and properties through the meaning assigned to them by the teacher's classroom management. This aspect of the classroom '*mathematical meaning construction'* is only partly approached by the research, which examines the mathematical quality of teaching.

Theoretical Perspectives

Most of the studies examining the mathematical knowledge at play in the classroom focus indirectly on this knowledge, often adopting a teachers' knowledge perspective. This section discusses some of these studies with respect to the way these examine the shaped mathematical meaning.

Ball et al (2008) studied teachers' mathematical knowledge for teaching (MKT) arguing that this special knowledge, in addition to other, like the knowledge of content and students, content and teaching, content and curriculum and so on determine the quality of teaching and thus the learning outcome. They propose a framework for examining this quality including features like *richness* and *rigor* of the lesson, presence of *mathematical explanation* and *justification, mathematical representation*, etc. Their approach allows the study of certain features of the mathematical

knowledge present in teaching practices but not the exact mathematical meaning shaped in the classroom.

Turner & Rowland (2010) focus on teaching practices and examine teachers' mathematics knowledge based on their instruction. Four categories of situations revealing teachers' mathematics knowledge are identified: *'foundation'* referring to the knowledge, beliefs and understanding acquired during teachers' education, '*transformation*' and '*connection*' explaining the ways teachers present the relevant mathematical topic and '*contingency*' related to the ways teachers react to 'unanticipated events'. Their framework named Knowledge Quartet (KQ) mostly relates the mathematical knowledge in teaching to teachers' mathematical expertise or to principles of classroom management in different types of situations. Thus, it is less connected to the mathematical meaning shaped in the classroom as a result of teachers' management of this meaning.

Some studies attempted to examine teachers' knowledge based on teaching practices, errors and teachers' reflection on these, the use of representations and examples (Lin & Rowland, 2016) without deepening, however, into the impact of this knowledge in the classroom management of the mathematical meaning. Also others examined teachers' knowledge from a cognitive point of view, without concentrating on specific contexts and the nature of mathematics (Davis & Simmt, 2006).

From a decision making perspective, Schoenfled (2013) proposed a framework for classroom observations related to effective instruction analyzed along three basic dimensions*: access, accountability and productive dispositions*. Here the focus is on mathematics and opportunities for their learning, thus on mathematical meaning construction: "Students are given a chance to learn mathematics… This requires making mathematics learning practices explicit and accessible … Mathematical exploration and discussion should be accurate. Reasoning and justification should be tied to mathematics" (p. 611). Terms like 'mathematical reasoning', 'mathematical accuracy', 'richness and integrity' are used to describe the mathematical character of the knowledge built. However, although the framework is thoroughly and accurately presented, it leaves unclear the meaning of each term and its connection to the epistemology of mathematics.

Mathematical meaning and the understanding of the nature of the mathematics constructed in the classroom have been also seen as an important aspect of classroom management encountered as a complex multi- dimensional phenomenon and studied in varied ways. Relevant research indicates that teachers make decisions based on multiple perspectives, often less mathematical and mostly pedagogical or didactical (Bednarz & Proulx, 2009).

For a number of years, our studies have been looking at teachers' classroom management, teaching practices in various mathematical contexts and students' learning in relation to the epistemological status of the knowledge under construction in the classroom. The results indicate that "in most cases the activity developed in the classroom had none of the epistemological features characterizing mathematics, thus affecting students' mathematical understanding" (Kaldrimidou, Sakonidis & Tzekaki, 2013, p. 306). Below, an episode from our data is analyzed within different perspectives to exemplify the aspects of the mathematical meaning under consideration. In this episode, a primary teacher offers an introduction to fractions and deals with definitions:

T(eacher). … Tell me, what is the difference between fractions and natural numbers? … How do they differ? … Are they the same numbers?

S(tudent). The fractional numbers ... can b… That is, we have a cake and we cut it in six pieces and take one. This is 1/6. The natural numbers are 1, 2, 3, … up to infinity!

T. Good! …

The student presents fractions using a specific example making reference to descriptive characteristics and then simply names natural numbers; the teacher accepts his answer (although a description rather than a definition is provided) and even praises him. What is the meaning of definitions constructed by the students? Is at all connected to the mathematical meaning of definition? The teacher's urge to offer a familiar context to the students destroys the accuracy of the definition, and, thus students' understanding of it.

What could we detect examining this episode through the *mathematical quality* lenses? Error, richness, rigor, or presence of mathematical explanation (Ball, et al., 2008)? There is no error, while the student's explanation (accepted) has no rigor or any other mathematical quality. Similarly, using Schoenfeld's framework we could identify less *opportunities for mathematics learning*. However, both analyses cannot explain the meaning constructed by the students. In an analogous manner, the KQ lenses would examine the connection between the initial question, the specific response, the descriptive explanation and the teacher's decision to accept it, but wouldn't explain the meaning constructed by the students. Examining the episode from the teacher's knowledge perspective (Davis & Simmt, 2006), her management provides no hints about this knowledge, because her decisions are consciously aiming to create a familiar environment for the students.

The above suggest that studies examining teachers' classroom management of the epistemological features of mathematics as well as the ways in which they understand and interpret this management play a central role in the improvement of mathematics teaching and learning. In the present study we look at teachers' reflections, interpretations and justifications related to the teaching decisions shaping the mathematical meaning constructed in the classroom.

Methodology

The data presented here come from a large project following the development of a new mathematics curriculum promoting mathematical literacy through active learning in social contexts. Here the focus is on three primary teachers, members of a small group chosen on the basis of their substantial teaching experience and promising professional development record, implementing units of the new syllabus over a school year. There were all females with more that fifteen years of experience each, teaching in an experimental primary school in the northern part of the country. Over the year, they discussed, designed, implemented and evaluated a series of lessons in collaboration and under the supervision and support of an advisor/consultant. The lessons, the meetings as well a number of interviews were taped and transcribed providing the data for the study. For the purposes of this report, three transcribed lessons and a follow up semi-structured interview on certain aspects of the teaching session for each teacher are considered.

The research problem pursued was to explore different meanings constructed in these teachers' classrooms related to mathematical objects, their definitions, attributes and relations to other mathematical objects based on their teaching actions/ management as well as reflections on them.

A combination of content analysis and grounded theory techniques were used to analyze the transcribed lessons and the discourse developed in the interview. In particular, we first identified critical episodes in the teachers' practices related to the above mentioned features and then analyzed their reflections on these, seeking to identify teachers' acknowledgement of the mathematical nature of this knowledge.

Results - Analysis

In this section, the results for each teacher participant according to the aforementioned scheme of analysis are presented. Due to the limited space, for each teacher, a critical episode is first provided and commented and then her reflection on it is discussed.

(1) Teacher A (5th graders): The episode below concerns the notion of percentage. Classroom activity concentrates on the completion of a 2x2 table, its rows representing games and its columns the number of students out of 100 voting favorably for each game, in three forms (fraction, decimal or percentage), partly completed. Teaching management focuses on the calculation procedure needed to move from one number representation to the other, especially on division, paying no attention to the equivalence of these representations.

T(eacher):	Because, 100 divided by 4 makes 25!! Hence, we have 25 out of 100! … Do you agree?... She had the fraction ¼ and wanted to turn it to decimal… Because here we have 100 students.
T:	Danae said before that the decimal fraction is what?
Danae:	A fraction with denominator 10, 100, 1000 …!
T:	And since I want 100 as denominator, what am I going to do?
Thanassis:	I will multiply it by …
S(tudent):	By hundred!!
T (to Thanassis): By what? (She writes on the board simultaneously)	
Thanassis: …	(noise increasing in the room) … By 25! I will do the same with the numerator
T:	(She writes on the board) That is, 25/100! ...
T:	How did you come up with this 0,25? Thano?
Thanos:	We got 0,25 from the fraction 25/100
T:	It was very easy for you to do the decimal number from the decimal fraction …
Adriana:	We will perform the division of 1 by 4 and we will find 0,25!
Teacher:	Why shall I divide 1 by 4?
S:	Because, if I divide the numerator by the denominator, I make decimal!
T:	Because it is very easy to make decimal fractions, but it suits me to get numerator with denominator, to divide them, because I am very good at division! … The percentage! ... Have we met the percentage only in graphs so far, eh? We haven't really worked with percentages … What does 25% mean? This is the new idea that came up there!

Teacher A claims that the mathematical focus of her session is on % and then on pupils becoming able to see the three different number representations (fraction, decimal, percentage). However, the way in which she manages group work and outcomes (dominance of question - answer practice and vague transitions between representations) destroys the mathematical equivalence between representations envisaged by the task. Nevertheless, in her reflection on this she appears unaware that this equivalence should be at the heart of her teaching. At the end of the interview, explaining why

children tended to ‘calculate the decimal to be able to deal with the number’s, she even argues that this might be her fault as she also does this in everyday life.

(2) Teacher B (2nd graders): The episode comes from an introductory session on fractions. The task here focuses on fair/even sharing of certain objects and materials depicted on paper, including a loaf of bread, a lolly-pop, some candies and a pizza. Children adopted strategies of folding and measuring, with the latter being mostly praised by the teacher.

Fotini:	I took the ruler and I measured it! I found its half!
T:	That is, how much is the biscuit … Take first the biscuit … Let your co-students see… The way the biscuit is, what did you measure? (She shows) … Aaaa! You measured this side from above! And how much did you find that the biscuit is?
Fotini:	Twelve centimeters!
T:	Oh, you found that the biscuit is 12 centimeters!
Fotini:	I cut it into six! …
T:	Ah! Go and bring your notebook and show us how you shared your loaf! … Because I haven’t seen many to share this way! … Look, please how did George share his loaf!! Do you agree?
S:	Yes!
T:	Did he share into two equal pieces?
Students:	Yes!
Joanna:	Yes! And then I cut it in the middle!
T:	How did you cut it in the middle? What did you think? …
Joanna: …	I cut it! …

Teacher B explained in her interview that she wanted pupils to ‘explore and discover’ for themselves how this even sharing is carried out, almost forgetting that this was all about fractions. Although we tried to draw her attention to the interesting strategies pupils came up with while trying to share, she insisted in the interview on the importance of children getting familiar with the ‘sharing procedure’ which they had recently discussed in the class. She was stuck with material and kinesthetic activities promoting no connections and generalizations related to the idea of fractions, because “they were familiar, manageable by the children” and therefore appropriate. This interpretation is apparently context-specific, that is, concerns this particular occasion of the teacher’s management. Nonetheless, it is difficult to deny that this occasion can frame students’ conceptions specifically and possibly inappropriately with respect to the mathematical meaning under construction. Even if this is seen as an 'effective' introduction to the concept of fractions, it is possible to keep both the teacher and the students stuck to this action driven approach in the future, which allows for a partial conceptualization of the concept at hand.

(3) Teacher C (5th graders): The episode selected comes from a teaching session on comparing fractions. The teacher is closing the lesson by attempting to help students generalize and draw a conclusion. However, both she and the students remain faithful to referring to pizzas and to the quantity “we eat”.

T:	But there must be something in order to be able to compare! What have you noticed? How did I place the fractions in order to be able to compare? What is common in each case?
S:	Either the denominators or the numerators are the same!

T.	When the numerators are the same, which fraction is larger?
Spyros:	When the numerators are the same, you eat more when there are fewer pieces!
T:	Listen … When the denominators are the same, when do you eat more?
S:	When the denominator is smaller!
T:	Smaller!! Whereas, when the denominators are the same, when do you eat more?
S:	You look whether the numerator is bigger!

The episode above is typical of what we would call a classroom 'destruction of the mathematical meaning' case. While working on ordering fractions, teacher C (a Mathematics degree holder, actively involved with research) keeps holding on to pieces and pizzas. Reflecting on this in her interview, she appears aware of the epistemological issues related to the knowledge managed in the classroom, but she is prepared to "sacrifice them", to deal loosely with these, because of her priority to motivate students, to allow them accessing the mathematical idea to "any cost really" even though distorted (Kaldrimidou, Sakonidis & Tzekaki, 2013, p. 309).

Discussion and concluding remarks

We presented the cases of three professionally active and highly motivated teachers with different, however, mathematics education background and varied awareness related to the nature of the mathematical meaning emerging in the classroom during instruction. The first of these teachers seems to be unaware of this aspect, while the second attempts to allow for mathematical elements to emerge, but through teaching practices of practical and partial character. Teacher C knows the importance of the mathematical content but prioritizes accessibility and manageability. In other words, all three teachers, intentionally or not, make choices concerning tasks, elements to highlight and approaches to manage which tend to reduce the mathematical meaning under teaching negotiation. These choices could be attributed to their desire to provide children with an 'easy', 'safe' and 'pleasant' learning environment. Their reflections on their teaching practices indicate that these decisions are strongly influenced by their own experience, regardless of their training and involvement with the pilot project and in accordance with Ponte & Champan's (2006) position that "teachers eventually develop their own PCK … shaped by their own experiences" (p.469).

In concluding, we would highlight two issues. First, that an analysis revealing classroom construction of the mathematical meaning requires, in addition to the study of teachers' knowledge, of the mathematical content elaborated and of the tasks employed (Ball, et al., 2008) as well as of the management of students' actions and thinking (Turner & Rowland, 2010; Davis & Smitt, 2003), a detailed analysis of the epistemological features of the content under construction (Kaldrimidou Sakonidis & Tzekaki, 2013). Teachers need to be aware of the importance of such an analysis of the mathematical meaning construction because they tend to either ignore or underestimate it. It is imperative to become aware that their management of the mathematical objects within the classroom connects or dissociates students from what should be at the heart of their instruction, that is, learning epistemologically legitimate mathematics.

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The interplay between sociomathematical norms and students' use of informal mental strategies or standard algorithms

Bodil Kleve and Gerd Ånestad

Oslo and Akershus University College of Applied Sciences, Norway; bodil.kleve@hioa.no

In a year 4 classroom, we studied students' presentations of their solutions of problem-solving tasks. Sitting in pairs (learning-partners), they solved the tasks before presenting their solutions orally in class. Based on transcripts from video recordings of the lesson, students' written notes and post interview with the teacher, the role of sociomathematical norms related to students' use of informal mental strategies and standard algorithm for subtraction is discussed. For students flexibly to carry out arithmetic operations, we suggest to develop switching between informal mental strategies and standard algorithm as a sociomathematical norm. In that respect, attention is put on mathematical knowledge for teaching (MKfT) and emphasis on place value system is suggested as amalgam between different strategies.

Keywords: Subtraction, place value system, learning-partners, mathematical knowledge for teaching,

> «Ah, mental (informal) algorithms are all very well, but they must learn the standard methods sooner or later» **Or must they?** Plunkett (1979, p 4).

Background and introduction

This paper is based on a study which purpose was to identify situations in a classroom where development of existing sociomathematical norms or establishing new norms may create potential for students' learning. A video research study was carried out in a year 4 classroom (9-10 years). An earlier publication reported situations in the classroom related to argumentations, and development of existing sociomathematical norms and establishing new norms were suggested in order to increase the potential for students' learning. (Kleve & Ånestad, 2016).

Based on students' (learning partners') written and oral presentations of a problem-solving task, where a three digits subtraction had to be carried out, sociomathematical norms (Yackel & Cobb, 1996) are identified and we discuss development of sociomathematical norms in light of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Rowland, Huckstep, & Thwaites, 2005). Our research question is: What role does mathematical knowledge for teaching play in order to develop sociomathematical norms, which can bridge the gap between informal mental strategies and standard algorithm for subtraction so the children can flexibly switch between different strategies?

Raveh, Koichu, Peled and Zaslavsky (2016) presented an integrative framework of knowledge for teaching the standard algorithm of four arithmetic operations. Studying connections between the four basic algorithms for arithmetic operations, they encouraged teaching the standard algorithm with emphasis on conceptual understanding, putting weight on connections between the four algorithms. In our study, the focus is on the relation between informal mental strategies and the standard algorithm for subtraction.

Torbyns and Verschaffel (2016) analyzed students' use of mental strategies and standard algorithm on subtraction. They found that when having been introduced for the standard algorithm:

> Children presumably became gradually more efficient in this algorithm, while their mastery of mental computation in general, and compensation in particular, may have stagnated or even *declined* (p. 112, italics in original).

Even when numbers involved were suitable for and "strongly invited to mental computation strategies", (ibid.) they found that students, when first having been introduced to the standard algorithm for subtraction, used this instead of mental strategies.

Torbyns and Verschaffel (2016) suggested that when the standard algorithm was introduced, the students would think that this newly introduced method was "the superior way to subtract larger number" (p.112). Referring to Yackel and Cobb (1996) they linked this to socio-cultural classroom norms in the classroom. Furthermore they referred to international efforts to reform elementary mathematics education which emphasizes children's abilities to flexibly apply informal mental strategies before they are introduced to the standard algorithm, and the claim that children then will continue to efficiently apply informal strategies. Torbyns and Verschaffel therefore encouraged research in more reform-oriented classrooms, comparing children from these with children taught in traditional oriented classrooms.

The classroom, in which our study took place, was reform oriented. There was a strong focus on children's use of own informal strategies and an extensive use of learning partners. The focus in this paper is on sociomathematical norms related to their use of informal mental strategies and/or use of standard algorithm for subtraction. Related to sociomathematical norms we also discuss what role mathematical knowledge for teaching play in the development of children's flexibility in using different mental strategies and the standard algorithm.

Theoretical perspectives

"Sociomathematical norms are normative aspects of mathematical discussions that are specific to students' mathematical activity" (Yackel & Cobb, 1996, p. 458). Yackel and Cobb focused on sociomathematical norms when studying "how students develop specific mathematical beliefs and values and consequently become intellectually autonomous in mathematics [ ] how they come to develop mathematical dispositions" (p. 458). They distinguished sociomathematical norms from general classroom social norms in that they are specific to the mathematical activity carried out in a classroom. In our study, we focus on an episode where use of informal mental strategies and/or standard algorithm for subtraction as mathematical activity is discussed.

Sociomathematical norms can be what counts as an efficient mathematical solution, different mathematical solution and a sophisticated mathematical solution (Cobb, Stephan, McClain, & Gravemeijer, 2001). Cobb et al. emphasized that both social norms and sociomathematical norms are dealing with what is «Taken as shared» in the classroom.

Classroom norms are developed in collaboration between the teacher and students or between students. However, the teacher is the key person when norms are changing or new norms are established (McClain & Cobb, 2001). In our study, we consider that the teacher has great influence on sociomathematical norms, whether the norms are already established, in the process to be

weakened or under development in the classroom. We therefore want to link development of sociomathematical norms to mathematical knowledge for teaching.

Ball, Thames and Phelps (2008) developed a framework for mathematical knowledge for teaching, MKfT. They distinguished between *Common content knowledge*, which is mathematical knowledge possessed not necessarily for teaching, and *Specialized content knowledge for teaching*, which is about the teacher's way of 'unpacking' mathematics, neither necessary nor desirable for others to do. They also included a third category, *Horizon content knowledge*, which is about making connections between different areas and topics. "Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum [ ]. It also includes the vision useful in seeing connections to much later mathematics". (Ball et al. 2008, p. 403).

In order to investigate how aspects of the teacher's mathematical knowledge surfaced in the lesson observed, the Knowledge Quartet (KQ) developed by Rowland, Huckstep and Thwaites (2005) has been a valuable tool. The KQ has four dimensions: Foundation, Transformation, Connection, and Contingency. Foundation is informing the other three dimensions, and Connection is the dimension, which we see as linked to "Horizon content knowledge".

Plunkett (1979) discussed pros and cons with regard to use of informal mental strategies and standard algorithm in school, and questioned whether standard algorithms necessarily have to be taught and learned. He claimed that unlike standard algorithms, which only deal with separated digits, informal mental strategies are holistic in that they work with complete numbers and thus requires understanding. Liping Ma (2010) emphasized regrouping rather than technical use of standard algorithm. When regrouping, the subtraction algorithm will be understood on a holistic number level, rather than as separate digits. False mathematical statements as "we can't subtract a bigger number from a smaller" will be avoided. Such false statements are related to teachers' horizon content knowledge (Ball et al., 2008).

Anghileri, J., Beishuizen, M., & Putten, K. v. (2002) compared the Dutch approach to written division calculations in school, which extensively built on children's own informal strategies, to the English approach which was schematic and focused on separate digits. Based on the results from their study Anghileri et al. (2002) warned against replacing informal strategies with standard algorithms. Rather one should give support to structuring informal approaches in a written record.

Referring to Plunkett (1979) and that calculations are carried out in real life, Anghileri (2006) emphasized children's mental strategies as a starting point for developing more formal methods.

Based on the above, one can suggest that when introducing a standard algorithm, teachers should focus on regrouping numbers and take children's informal mental strategies as a starting point. This puts demands on the teacher's mathematical knowledge for teaching, which again will constrain the sociomathematical norms in the classroom.

Referring to among others Ball et al. (2008), Raveh et al. (2016) proposed a framework consisting of four components: Procedural Knowledge (PK), Knowledge of Underlying Concepts (KC), Knowledge of Similarity between the algorithms (KS) and Knowledge of Representations (KR). In our analysis, we will use components from this framework, mainly PK and KC, and some of KR. PK is about carrying out the steps correctly in the (subtraction) algorithm, while KC refers to

knowledge about mathematical concepts underlying different algorithms as place-value and number regrouping. Rather than analyzing different representations of the subtraction algorithm as KR refers to in Raveh et al's framework, we emphasize the relationship between informal mental strategies, and the standard algorithm for subtraction.

In our study, we will not argue for not introducing the standard algorithm. However, we consider bridging between informal mental strategies and standard algorithm as valuable features of mathematics, which may influence students' mathematical beliefs. Development of flexibility and children's ability to switch between informal strategies and standard algorithm are linked to sociomathematical norms and to the aspects pf teachers' mathematical knowledge for teaching.

Methods

We observed two mathematics lessons in a 4th grade (mixed ability, 9-10 years old) classroom. Prior to the classroom observations, we had come to know the teacher. Her educational background was a pre-school teacher. She described her teaching as being reform oriented and that her students performed well on "transition tests". She had established an extensive use of learning partners in her classroom. She put her students together in pairs at random, same partner in all subjects, and changing partners every week. According to the teacher, the students never complained or protested against whom they received as their partners. This was established as a social norm in the class. With regard to Eli's view on mathematics teaching and learning, she emphasized the process rather than the product, saying: "For me it is not so important if the answer is correct. I am more interested in the strategy they use, that they have understood the principle behind solving such tasks". She also told us that she encouraged students to develop their own strategies in solving arithmetic problems and to discuss their strategies with their learning partner. Against this background, we wanted to study sociomathematical norms in Eli's class.

During our first visit in the class, we observed and wrote field notes. The second time we video recorded a 90 mins mathematics lesson. We used two cameras. The students were sitting in pairs and should solve different tasks, which were written on the board. After having solved the tasks, and written down their solutions, they presented their solutions orally. Towards the end of the lesson, they had some ('warm ups') whole class activities where they "worked with concepts" (teacher's expression). One of the activities was linked to the place value system.

Our analysis here is based on transcriptions of video recordings of their oral presentations and on their written work, which we collected. We also studied the video recordings together with the teacher several weeks afterwards. We interviewed the teacher and asked her to comment the different pairs' written sketches and oral presentations.

The task, on which we base our analysis, was written on the board:

> Tobias has two 200 NOK notes and six 10 NOK coins. He spends half on a gift, 142 NOK on a book and then he buys "pig ears" to Doffen. One ear costs 11 NOK. How many ears can he buy?

Analysis

The subtraction 230-142 had to be carried out to solve the task. Eleven out of twelve pairs of students had used the standard algorithm, however with different degree of detail in their oral presentations. The standard algorithm had become the "taken as shared" method (Cobb et al.,

2001). First, we discuss four pairs' written calculations (sketches) together with their oral presentations, illuminated with the teacher' comments to the presentations, then we go further into the teacher's comments from the interview. The teacher did not comment on any of the presentations in the lesson.

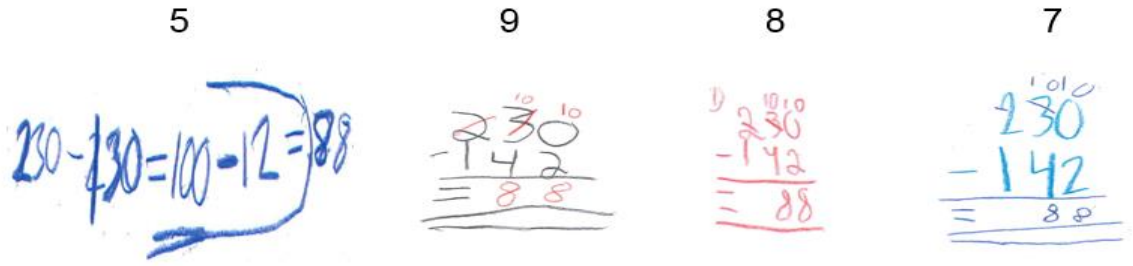

Figure 1: Four pairs' written calculations

The pairs' oral presentations:

Pair 5: 230, it was easier to do 230 minus 130 equals 100 and then we did minus 12 because 130 plus 12 is 142, and that made 88

Pair 9: 230 – 142 is zero minus two, doesn't work. We have to exchange from the three, and ten minus two is eight and then we have two left there. So then we take, however four minus two doesn't work either, so then we will have to exchange from the two. Ten minus four is six plus two is eight and then we have only one left, which makes one minus one is zero. So then it is 88".

Pair 8: We did 230 minus 142 using the standard algorithm. We got 88

Pair 7: 230 minus 142 is 88

Studying Pair 5's calculation, both their written work and oral presentation, we see that they did not use a standard algorithm for subtraction. The students regrouped the subtrahend 142 into 130 and 12. This way of calculating is flexible and requires understanding and a holistic way of thinking. They treated the complete numbers rather than separated digits (Plunkett, 1979).

The interview with the teacher Eli, with regard to Pair 5, revealed that she did not see the way of solving and presenting this as a potential for further development. The teacher expressed her acknowledgement of different ways of doing subtractions, but that this was a cumbersome and much lengthier way. She considered one of the students in the pair as a "funny one", and that "you need to be much sharpened to follow his thoughts. However, I keep telling him that he ought to start using another strategy in order to make things go faster. So after a while you' will have to do that". This is in line with Plunkett's (1979) characterization of informal mental strategies: "often difficult to catch hold of "(p. 3). This comment revealed that the teacher did not see the potential in her students' mental calculations for further development. She now looked upon the standard algorithm as the most efficient and sophisticated way of carrying out subtraction, and using the standard algorithm was in the process of being established as a new sociomathematical norm.

As we can see from figure 1, the three other pairs used the standard algorithm for subtraction. All correctly written out, displaying decomposition (exchange or borrowing). This can also be interpreted as regrouping of the minuend based on the place value system. However, the students did not express any regrouping. Studying their oral presentations reveals that the students were on different levels in using the algorithm. "Standard algorithms are *not easily internalized*. They do not correspond to ways in which people tend to think about numbers" (Plunkett, 1979, p. 3, italics in original). The pairs only referred to digits between 0 and 10. We suggest that this is why Pair 9 presented a detailed explanation of the algorithm as such. Their claims: "zero minus two, doesn't work" and "four minus two doesn't work reveal either", reveal a misconception or a "false mathematical statement" (Ma, 2010, p. 3). These students have not been presented for negative numbers yet, and false mathematical statements like these may lead to later misconceptions about negative numbers. The students in this class (except Pair 5) used the term "exchange" when regrouping the minuend 230 into 220 + 10, and when they later regrouped 220 into 120+100.

A question here is whether the students know what they are doing. According to Plunkett, use of standard algorithms encourage suspended understanding. Pair 9's explanation reflects procedural knowledge in carrying out an algorithm rather than conceptual understanding. Ma (2010) encourages regrouping rather than exchanging or borrowing (decomposing) when introducing the standard algorithm. Because then they will be working on a holistic number level rather than with separate digits.

With regard to Pair 8, they only said they had used the standard algorithm, whereas Pair 7 only presented the answer. They can be considered as having internalized the algorithm, and as Plunkett (1979) puts it: "While the calculation is being carried out, one does not think much about why one does it in that way" (p. 3).

In the interview with the teacher, Eli said that when starting a new arithmetic operation, she encouraged everyone to do it his or her own way, and that she used to present all students' different informal strategies on the board. She expressed a great concern about these differences when a new arithmetic operation was being introduced. Thus, we consider that use of mental strategies for subtraction was earlier established as a sociomathematical norm. However, after having introduced the standard algorithm, this sociomathematical norm was in the process of disappearing, or at least fading, and a new sociomathematical norm was about to be developed. About the introduction of the standard algorithm, Eli said, "We practiced memory numbers and exchange in detail". Consequently, we consider such detailed explanation as a new sociomathematical norm. This sociomathematical norm is also in the process of disappearing. Everybody was now expected to use the algorithm without further explanations or comments. As we see from our data, some students used the standard algorithm naturally without further explanation, while others still explained the procedure in detail. Only one pair (5) explained the subtraction as use of mental strategies without mentioning the standard algorithm. Hence, eleven out of twelve pairs looked upon the newly learnt algorithm "as a superior way" (Torbeyns & Verschaffel, 2016, p. 112).

Discussion

Our findings suggest that although Pair 5's way of doing subtraction was not acknowledged ("he ought to start using another strategy in order to make things go faster"), the students displayed both

number sense and a well-developed subtraction concept. Of those who had used the standard algorithm, some explained the procedure in detail, Pair 9, while others just referred to the algorithm. Although they might have had conceptual understanding, they did not display it. Their focus was on the skill carrying out the subtraction *procedure*. According to Eli, the students who used the standard algorithm had developed a more mature number sense than those still using mental strategies. Although being influenced by reform-oriented working methods, Eli expressed the necessity of learning the standard algorithm, both as a tool, an assurance to always have a method to use, and as an efficient way of doing subtraction. She looked upon standard algorithms as a supplement to informal mental strategies. However, she was not aware what research has shown; when first have been introduced to the standard algorithm for subtraction and exposed to instruction emphasizing mastery of the standard algorithm, children will gradually become more efficient in using the standard algorithm, while their use of informal and mental strategies will fade (Torbeyns & Verschaffel, 2016). The challenge is to bridge or close this gap. A goal must be to develop sociomathematical norms where students are able to switch between different strategies dependent on the nature of the numbers involved. This puts demands on the teacher's MKfT, and especially the Horizon knowledge.

There was no indication in what the teacher said in the interview that the informal mental strategies the children earlier had used had been taken as a starting point when introducing the standard algorithm. The teacher's mathematical knowledge for teaching seemed too fragile to give her courage to rely on students' mental strategies, and thus to bridge the gap. The mathematics presented for her students seemed to be fragmented. During the place value activity towards the end of the lesson students should answer questions as "what value does 1 have in 5129?" If this had been linked to the standard algorithm for subtraction, regrouping, based on place value system, could serve as an amalgam between informal strategies and the formal algorithm. This refers to Raveh et al.'s (2016) KR, which we see as knowledge about connections between informal strategies and mental strategies. Attention could here be brought to the foundation and connection aspects of the teacher's knowledge (Rowland et al., 2005). Knowledge of the mathematical concepts underlying the algorithm, KC (Raveh et al., 2016) did not surface in what she said. However, she demonstrated procedural knowledge (PK) related to correct computations and the steps in the algorithm.

Our findings suggest that the teacher did not see the potential in taking earlier established sociomathematical norms about students' use of mental strategies as a starting point. We claim that linking informal mental strategies to the place value system in introducing the standard algorithm for subtraction would enhance students' ability to switch between informal strategies and the standard algorithm, dependent on the numbers involved. Regrouping based on the place value system could then serve as an amalgam between informal mental strategies and the standard algorithm.

Based on our data, we cannot say anything about the sociomathematical norms related to other areas of the mathematics in this classroom. However, the students had not yet been introduced to the standard algorithm for division. We saw that in carrying out the necessary division operation to solve the task (how many ears can he buy?) 88:11, the students used either repeated subtraction or repeated addition as (informal) mental strategies.

Encouraging teachers to rely on and see the potential in earlier established sociomathematical norms where students use informal mental strategies, are important issues for further research. In that respect, attention must be directed towards to teachers' mathematical knowledge for teaching, with special focus on Horizon knowledge and the connection dimension of the teacher's mathematical knowledge (Ball et al., 2008; Ma, 2010; Raveh et al., 2016; Rowland et al., 2005).

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‘Routes’ of recontextualizing mathematical processes through a curriculum reform

Anna Klothou[1] and Charalampos Sakonidis[2]

[1]Democritus University of Thrace, Department of Primary Education, Greece; aklothou@eled.duth.gr

[2]Democritus University of Thrace, Department of Primary Education, Greece; xsakonid@eled.duth.gr

The recontextualizationstaking place during the implementation of a new curriculum are identified through different discourses teachers draw on in order to attribute meaning to the ideas promoted by the new curriculum. The paper studies the ways in which thirteen Greek primary school teachers who participated in aone-year pilot implementation of a new mathematics curriculum recontextualized its innovations through the discourses they drew on. The analysis of the data revealed contradictions within the teachers’ discourse that can be attributed to the recontextualization procedures activated during the implementation of the new curriculum and indicate inconsistencies within or/and between the various discourses available to them.

Keywords: Curriculum, mathematical processes, reform, recontextualization, teachers.

Introduction

A number of studies examining mathematics curriculum reforms reveal the complex nature of the change in theteaching practices and the agentsoperating at the institutional level (State) as well as at the individual level (teachers), constituting obstacles to the implementation of the innovations promoted. The results of these studiesindicate distortion of the innovations promoted, mainly because of the dominance of traditional teaching approaches and only partial adoption of innovative features, usually of those that can be painlessly integrated into existing teaching practices. While teachers’ beliefs can be consideredcritical for the implementation of a mathematics curriculum, there are difficulties in using beliefs to interpret effectively the outcomes of the implementation. According to Lerman (2002), research should keep distances from such “personalized” views on the relations between beliefs and practices, so that the “conflicting” ways in which promoted reforms are reflected in the practices of teachers can be understood and interpreted as social phenomena (e.g. Morgan, Tsatsaroni&Lerman, 2002). Research to this direction, althoughstill very limited, is of particular importance for making sense of the problems arising while implementing a mathematics curriculum that goes beyond the concept of belief. We argue in this paper that Bernstein’s concepts of recontextualization and recontextualization fields have much to offer in understanding how intervention programs employing socio-cultural approaches succeed or fail because of teachers’ ways of interacting dynamically with curriculum reforms (e.g. Jaworski, 2007).

Implementing a new mathematics curriculum: recontextualization in practice

Each new curriculum is unavoidably being ‘altered’ during its implementation by the action of institutional factors as well as of the teachers themselves. This situation does not differ from that of the distortion of academic mathematics during their transformation to school mathematics and of the academic knowledge related to mathematics education when ‘translating’ it the knowledge needed to

support mathematics teaching in practice (Morgan, 2010). More specifically, the actions of those involved with school mathematics aim to develop students' relevant knowledge rather thantoextend mathematical knowledge itself. Moving from academic mathematics (production field) to school mathematics (reproduction field) presupposes a reframing process of the discourse established both in the first and in other areas (e.g., in relation to learning theories) [the term 'discourse' refers to a social construction that establishes a person's social hierarchy (Koulaidis&Tsatsaroni, 2010)]. This is similar to the transition from mathematics education research and theory production field (academic) to the reproduction field (school). Teachers are generally not engaged in producing new knowledge about teaching and learning, but are expected to acquire knowledge and skills that enable them to teach effectively. The development of curricula, the production of teaching materials that support a curriculum and the professional development activities play an important role to this. The field of reproduction here is embedded in many fields of teachers' professional life, creating a more complex environment for developing discourses related to their practices.

The adoption of a sociological agenda for examining the process of implementing a new curriculum, which prioritizes the notion of 'recontextualizing knowledge', could provide an operational framework for understanding and interpreting problems arising because of the complexity of the task. Such a framework is being proposed by Bernstein (2000), who focuses on a pedagogical mechanism that structures and organizes the educational contents and their distribution based on the dominant process of recontextualizing. That is, the transfer of knowledge through selection procedures from the fields where it is produced (e.g., universities) to areas of formal or informal training (e.g., classroom). Bernstein (2000) distinguished the Official Recontextualizing Field (ORF), established and dominated by the State for the construction and supervision of the State pedagogical discourse, and the (official) Pedagogic Recontextualizing Field (PRF), in the formulation and management of which agents such as teachers' trainers, more or less independent of the State, are involved. The relationship between the two fields and the curriculum reform processes depends on the degree of the PRF autonomy, the extent to which the discourses produced by these fields differ and the source of reproduction of the revised curriculum (Morgan, 2010). Teachers act as agents in the PRF, reproducing the official pedagogical discourse established in the ORF. However, their practices cannot be completely regulated externally. What is being reproduced at school and in classroom depends on the principles of recontextualization that arise from "the specific context of a particular school and the effectiveness of the external control in the reproduction of the official pedagogical discourse" (Bernstein, 2000). That is, from the operation of the complementary resources that are being produced locally (Local PRF, in distinction to Official PRF). This suggests that, despite the independence between them, the recontextualization fields influence each other, with the agents playing an important role in more than one fields (e.g., researchers who teach in universities, in professional development programs, in the formation of a new curriculum etc). This complex relationship creates differences between pedagogical discourses established in different fields of recontextualization and, consequently, between the practices adopted on the basis of these discourses. The interaction between the fields of recontextualization on the one hand and the interpretations of the discourse developed in these fields on the other generates the resources used by teachers to legalize their practices in classroom.

In the Greek educational system the State has almost absolute control over the curriculum, its implementation at school level, the production of educational materials and teachers' professional

development. There isnot, therefore, a truly independent PRF. However, there are differences between individual fields of recontextualization and the discourses these produce, providing diverse resources of curriculum interpretations for teachers in the field of reproduction. Teachers draw on discourses arising from teaching and learning processes experienced during their own mathematics education and within the context of their professional activity. The absence of a discernible pedagogical recontextualization field, independent of the Greek State's control, influencing mathematics classroom teaching practices and the meaning attached to them by teachers as well as the occasion of a new mathematics curriculum being implemented in the country, offered a situation seen as worth studying.

A new mathematics curriculum in Greek schools

Studying the curriculum reform of Mathematics undertaken in Greece, various levels of the development and the structure of the administrative control exercised by the State can be identified. It appears that different agents occupy different positions within this structure and are involved in recontextualizing the curriculum, having different interests and different relationships with schools and teachers. The Greek recontextualization field can be considered to consist of three official sub-fields: (a) the Official Recontextualization Field (ORF) - the curriculum itself and its authorities are set up at a national level by the Ministry of Education, (b) the Official Pedagogical Recontextualization Field (OPRF), where the production of all educational materials like textbooks is directly controlled by the Ministry of Education, (c) the Local Pedagogical Recontextualization (LPRF) field related to the implementation at the school level, where schools and teachers interpret the new curriculum through training and additional resources produced locally. Although both the OPRF and the LPRF have a degree of autonomy, this is quite limited. The discourses produced by the agents in these two fields (e.g., in the form of mathematics textbooks, public examinations or teachers' training programs) should be approved by the Ministry of Education. Unlike the educational systems in most Western countries, there is no Recontextualization Pedagogical Field independent from the Greek State to influence the adoption of instructional practices. However, even when the State does not encourage the development of independent discourses relating to the curriculum, the different individual fields that make up the field of recontextualization and the generated discourses act as sources on which teachers can draw to 'interpret' the curriculum in the field of reproduction. In addition, teachers can draw on previous discourses related to mathematics, for instance produced locally, at school and in the wider community.

As it has been already mentioned, the Greek educational system is highly complex, its administration is being exercised centrally and the hierarchical structure of its organization and management are characterized by high concentration of powers at the top. Every educational activity, be it the identification of educational objectives, the implementation of the curriculum, the educational materials used to support teaching and learning or the evaluation of teachers' work, is centrally controlled. The heavily complex and centralized nature of the Greek educational system creates differences between the pedagogical discourses produced at different levels of the recontextualization fields. In turn, differences between the pedagogical texts, between the practices adopted by these different pedagogical discourses and between the ways in which teachers might legitimize their practices are evident. Morgan (2011) states that the interaction between recontextualization fields and the interpretations resulting from the production of different discourses create space for teachers to

place themselves in the position of a ‘good teacher’, legitimizing a number of different classroom practices.

The above situation is quite different from that met in well knowneducational systems like the British, where the curriculum is strongly controlled by the State and is being regulated by tests and inspections. There is an organized pedagogical recontextualization field that holds a high degree of autonomy and is being developed mainly at universities and educational communities. The Pedagogical Recontextualization Field produces alternative discourses related to the curriculum by setting up different sets of recontextualization authorities and plays an important role in reframing the discourse of the curriculum produced in the Official RecontextualizationField (Morgan & Hu, 2011). Despite the fact that the two fields are independent in structure, they affect one another (for example, mathematics education academics participate in working groups for the design of the mathematics curriculum or in national exams committees using State-defined standards). Given the tight control exercised by the Greek State on the whole educational process, no independent Pedagogical Field can be ever expected.

The study

The context of this study is provided by a recent reform effort in the Greek compulsory education system concerned with the development of new curricula, completed in 2011, piloted for two years and presently utilized primarily in experimental schools. A central innovative element with respect to mathematics education has to do with students’ active involvement with activities that enhance processes such as a) mathematical reasoning and argumentation, b) creating bonds between concepts, c) communicating through the use of different tools and d) promoting metacognitive awareness (Institute of Educational Policy, 2014). Utilizing Bernstein’s theoretical framework, the study focuses on the impact of this reform on teachers’recontextualization practices. In specific, the aim of the study is to examine the ways in which primary teachers participating in a one-year pilot implementation of the new curriculum recontextualized its innovations, especially the mathematical processes promoted, based on their pedagogical discourse.

The sample consisted of 13 primary teachers who worked in three schools in the north-western part of the country. Most of the teachers had considerable teaching experience (10–25 years) and were involved in professional development activities, such as participating in in-service training programs and research projects. During the study, five teachers were teaching to upper, four to middle and four to lower primary school classes. The teachers were all involved in the implementation process of the new mathematics curriculum because their schools were selected by the State as pilot units. The extent of their involvement was determined by their sense of ‘duty’ tomeet the requirements set by the new curriculum but also by their interest to participate in activities promoting their students’ learning and their professional development.

The study included two phases. In the first, a non-participant observation of teachers’ reformed mathematics teaching took place. The teachers were observed for two teaching sessions and were then interviewed mainly in relation to the ways they exploited the mathematical processes promoted by the new curriculum in designing and implementing their instruction. The second phase included a semi-structured interview with each of the teachers aiming to study the pedagogical discourse developed and through that the recontextualized process that possibly took place in relation to the

mathematical processes promoted by the new curriculum. Each teacher was interviewed for four hours in four meetings.

To describe the procedures of recontextualization of the mathematical processes taking place, techniques of Grounded Theory and Content Analysis were utilized. In particular, articulations associated with each of the recontextualization fields were identified, coded, grouped and merged, providing meaning to its content and structure. Specifically, the analysis of the data from the semi-structured interviews was carried out at two levels. First the researchers held multiple and careful readings of the interview transcripts, identifying phrases for each mathematical process within each recontextualization field. The recorded phrases in each recontextualization field were then coded and those indicating a 'special' aspect of the corresponding field were identified and marked with a new code. When a phrase that reflected a field reappeared, it was also noted. So, gradually, the phrases that were part of each first level domain were organized into second level categories, based on the 'special' theme. In the following, some results from the second phase of the study are presented and discussed.

Results and discussion

In the following, the participant teachers' conceptualizations of each of the four mathematical processes promoted by the new curriculum as emerging through their discourse are briefly presented and discussed.

(a) Mathematical reasoning and argumentation: Teachers interpreted mathematical reasoning and argumentation in a way that is consistent with the official discourse, as aiming to promote classroom interaction. This interpretation is related to the professional official discourse and is supported by tasks demanding cooperation and communication included in the mathematics textbooks produced and distributed by the State. Consistency in the recontextualization between the Official Pedagogical Recontextualization Field and the Local Recontextualization Field with respect to this process is identified (e.g., difficult to apply in practice).

> But there is a problem ... it is the textbooks we use, they do not help us at all. We must follow the material and we do not anticipate. Of course, I give more exercises; beyond the textbook ... the textbook does not cover us. And moreover, you know something? We have the pressure from parents who want more exercises. They are pushing us and they do not understand what we do (teacher with 11 years teaching experience, moderate professional activity, teaching to a lower primary class).
>
> Time often forces me not to be able to give the time needed so that the student loses the opportunity to communicate, but communication is essential (teacher with 19 years of teaching experience, moderate professional development activity, teaching to a middle primary class).

The 'objections' raised drew mainly on a local discourse concerning classes, especially size of classes, the time pressure and parents' demands. Teachers also drew on elements from the conventional discourse facing mathematics as a 'discipline' which is not always available for discussion because of its 'absolute nature'.

(b) Creating bonds between concepts: Teachers appeared to be aligned with the official discourse with respect to this process, giving however little value to it. They only deviated from it when referring to opportunities of exploiting this process in classroom. To this direction they made reference to a pedagogical recontextualization often governed by informal, local factors (e.g., the importance of knowing 'good' mathematics, of repetition, of continuous emphasis on conceptual understanding) and not by official "routes" (e.g., attending courses on issues of teaching mathematics).

> I'm not sure about what you mean ... relate to other mathematical concepts, that is what? Now we do fractions, what should I do, then? It sounds important …. Solving exercises, repetition, this must be done, otherwise mathematics would be forgotten (teacher of 21 years of teaching experience, moderate professional development activity, teaching to a lower primary class)

(c) Communication through the use of different tools: Teachers' discourse concerning mathematics classroom communication utilized different tools mainly related to the importance of tasks requiring the use of manipulatives, through which students are able to use their hands to 'do things' (for example, to cut a piece of paper, to take cubes from a bag etc).

> We are working with manipulatives in the class every day. But if you do maths in an experimental way, when will you understand that maths is not a game? We need to be accurate (teacher of 11 years of teaching experience, moderate professional development activity, teaching to a lower primary class)

The same idea seems to prevail when reference is made to the use of technology and digital materials. No distinction was made between the presence and functionality of colors, virtual objects, sounds and other digital 'goodies' that do not have themselves any mathematical meaning but exist to invigorate the general 'elegance' of software (Institute of Educational Policy, 2014). As argued in the "Teacher's Guide" (2014), in cases where digital technology 'provides' a patchwork of mathematical and non-mathematical representations, the importance of mathematical representation deteriorates.

> I personally deal successfully with technology because I like it. It facilitates our work when we are going to teach difficult mathematical concepts. It is impressive with all these shapes, colors, the environment, but also it is functional, a tool to work (teacher of 21 years of teaching experience, moderate professional development activity, teaching to an upper primary class)

Teachers' discourse drew also on their own mathematical education and the experience gained from the previous curriculum.

> We are the generation of the book, but we can do things with the computers. It would be nice for us to be familiar with it, but so far it was not provided, nor from the previous curriculum (teacher of 17 years of teaching experience, moderate professional development activity, teaching to an upper primary class)

This mix of 'conformity' to the official discourse with opposition to it was expressed implicitly by teachers who were resisting to the official discourses (but one teacher), although they seemed to be in favor of it.

(d) Metacognitive awareness: Teachers argued for this process and 'wished' they could foster more of its components into their teaching. On the whole, they aligned with the relevant official discourse.

However, they deviated from it when referring to the conditions set for the use of this process in class, moving to a pedagogical recontextualization determined predominately by informal, local factors (e.g., the effectiveness of mathematics learning and the coverage of the appropriate content) and less by official agents.

> For me it is important for students to really have the desire to explore and then teachers will guide them effectively…However, for students who are less 'comfortable' with this, it is a great challenge, but I'm disappointed...If these conditions are met, then it is a good idea, but I can say that especially for my class, to be honest, it is difficult (teacher with 16 years of teaching experience, university degree, moderate professional development activity, teaching to a lower primary class).

Concluding remarks

Teachers drew mainly on the resources provided by the official discourses when arguing about the four mathematical processes promoted by the new curriculum and tended to align to these discourses. However, they deviated from them when interpreting these processes in relation to their own classes. The official discourses shape the range of the new curriculum interpretations in relation to teaching and learning, while the conventional and local discourses fuel teachers' pedagogical discourse with further resources for understanding the new mathematics curriculum having an impact on the options available to teachers and adopted by them in the classroom practice. This finding raises the question why some discourses are more powerful than others regarding the influence they have on teachers. The strength of the local school discourse of other teachers, parents and pupils can also be related to the regulatory effect exerted by the school administration, which requires certain standards in student performance. As the official discourses, such as the discourse promoted by a curriculum, do not 'recognize' the difficulties in implementing key curriculum principles or the recontextualizations activated during classroom implementation in different students or groups of students, teachers need a 'way of facing' the difficulties experienced in their classroom, seemingly provided by the resources of the local discourse.

In the Greek educational reality is evident the parallel operation of recontextualization fields, often mutually incompatible, which influence contradictory ways in which teachers select and transform the official discourse. Teachers utilize the discourses produced in these fields and place themselves in relation to them, being affected by their official or non-official nature, thus creating the potential for resistance and for the use of alternative discourses. The study of the implementation of a new mathematics curriculum will benefit from the analysis of the structures that rule the recontextualized fields and their discourses as this will help us understand the ways in which the choices and the transformations taking place within these discourses shape teachers' professional practices in educational contexts similar to but also beyond the ones studied here.

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Mathematical knowledge for teaching and the teaching of mathematics

Reidar Mosvold[1] and Mark Hoover[2]

[1]University of Stavanger, Faculty of Arts and Education, Stavanger, Norway; reidar.mosvold@uis.no

[2]University of Michigan, School of Education, Ann Arbor, MI, USA; mhoover@umich.edu

This paper reports on a review of 12 empirical studies framed to address the problem of whether and in what ways mathematical knowledge for teaching influences teaching practice. From a larger review of literature on mathematical knowledge for teaching, this qualitative synthesis examines the theoretical foundations, methods applied and claims made. Most of the studies reviewed are small-scale qualitative studies. There is variability in the language to describe teaching and in how focused the studies are on teaching. We suggest that the tendency has been to focus on the question of the knowledge that teachers need, but that it would be more useful to focus on the mathematical entailments of doing *teaching, which will require more detailed and shared conceptualizations of the mathematical work of teaching.*

Keywords: Mathematical knowledge for teaching, teaching practice, literature review.

Introduction

In a review of literature on mathematical knowledge for teaching from 2006 to 2013, with colleagues we identify a number of studies that argue that mathematical knowledge for teaching influences teaching practice (Hoover, Mosvold, Ball, & Lai, 2016). The connection between mathematical knowledge for teaching and the quality of instruction is, however, complex. Hill, Umland, Litke, and Kapitula (2012) report evidence that weak mathematical knowledge for teaching predicts low instructional quality, and strong mathematical knowledge for teaching predicts high instructional quality, yet they also report that there is much more variation in teaching quality as well as in student achievement with teachers who perform in the midrange on measures of mathematical knowledge for teaching. Likewise, Hill et al. (2008) suggest that professional development, supplemental curriculum materials and teacher beliefs are all factors of potential influence, but these factors may cut both ways depending on the teachers' mathematical knowledge for teaching. In addition, efforts to clarify the conceptualization of mathematical knowledge for teaching continue to be concerned with the dynamic nature of mathematical knowledge for teaching, the usefulness of knowledge, and whether, when, and how it plays in teaching (Ball, 2016; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012). From these different lines of work, it seems clear that decisions about teacher education or policy cannot be made simply by establishing *that* mathematical knowledge for teaching correlates with teaching practice. Although these correlational studies are arguably important, in this review we draw attention to the need to refine our understanding of *how* mathematical knowledge for teaching influences teaching practice.

The literature review

The present paper draws on results from a larger literature review of empirical research on mathematical knowledge that is specific to teaching (Hoover, Mosvold, Ball, & Lai, 2016). That

review included a total of 190 articles that were coded for the following categories: 1) genre of study, 2) research problem used to motivate the study, 3) variables used, 4) whether or not and how causality was addressed, 5) findings. A research problem is an issue, topic, or question that motivates a study, indicating why the results would be of interest and how an investigation is linked to the literature. In most instances, the research problem was the same as what was specifically investigated, but at times there were tensions between the research problem and the research questions or the specific focus of the analysis. Distinguishing between research problems and the genre of the study helped us understand what a paper argues and how. In coding the research problem, we paid specific attention to the introduction and conclusion as opposed to the statement of the research questions or the specifics of the research design. In general, differences between the research problem and the research genre or design reflected inevitable tensions in the interrelated components of empirical research publications and provided useful insight into the approaches used in the study and the arguments made in the article. When considering the research problems that motivated the studies in our larger review, 12 studies focused on the ways in which teachers' knowledge contributes to practice. In this paper, we analyze these 12 studies with a particular focus on the theoretical frameworks of the studies, the methods applied and the claims made.

Studies of the influence of mathematical knowledge for teaching on teaching

The 12 studies framed to address the problem of whether and in what ways mathematical knowledge for teaching influences teaching practice have different characteristics. Seven studies investigate effects of mathematical knowledge for teaching on teaching practice (absent a specific intervention), one is an intervention study and four studies investigate mathematical knowledge for teaching as a construct in relation to teaching. Only one of the studies is quantitative, whereas most studies are small-scale qualitative studies. The participant teachers in the studies teach mathematics in primary, middle, and secondary schools, as well as at the university level — most of them are practicing or experienced teachers (see table 1).

Study	Sample size	Type	Causal design	Experience and level of teachers	Region	Teaching studied
Bansilal (2012)	Small (n=1)	Effect	Qualitative	Practicing secondary	Africa	Identifying key ideas, organizing explanations, listening to students
Cengiz et al. (2011)	Small (n=6)	Effect	Qualitative	Experienced primary	North America	Extending student thinking
Charalambous (2010)	Small (n=2)	Effect	Qualitative	Practicing primary	North America	Using representations, giving explanations, Interpreting and responding to student thinking
Choppin (2011)	Small (n=1)	Nature	None	Experienced middle school	North America	Engaging students with challenging tasks
Izsák et al. (2008)	Small (n=1)	Effect	Qualitative	Practicing middle school	North America	Using number lines for fraction addition
Johnson & Larsen (2012)	Small (n=1)	Effect	Qualitative	Practicing tertiary	North America	Listening to student thinking

Nardi et al. (2012)	Medium (n=11)	Nature	None	Practicing secondary	Europe	Identifying task objectives, interpreting and responding to student thinking
Rowland (2008)	Medium (n=24)	Nature	Qualitative	Future primary	Europe	Selecting and using examples
Speer & Wagner (2009)	Small (n=1)	Nature	Qualitative	Practicing tertiary	North America	Social and analytic scaffolding
Steele & Rogers (2012)	Small (n=2)	Effect	Qualitative	Practicing secondary	North America	Integrating different ideas of proof and positioning students as observers, creators, and explainers
Sullivan et al. (2009)	Large (n=97)	Intervention	Statistical	Practicing all levels	Oceania	Converting tasks to lessons
Tanase (2011)	Small (n=4)	Effect	Qualitative	Practicing primary	Europe	Connecting place value to other mathematical concepts, setting objectives, challenging students

Table 1: Studies investigating influences of mathematical knowledge for teaching on teaching

Next, we describe these studies with a focus on what they investigate, their methods, how teaching is conceptualized, and what we can learn from them. Cengiz, Kline and Grant (2011) focus on how teachers' MKT supports their teaching. They develop an extending-student-thinking framework based on analysis of instructional actions within episodes. In their investigation of six experienced elementary teachers, they draw upon Ball et al.'s (2008) conceptualization of mathematical knowledge for teaching. It is assumed that the participating teachers, due to their experience, have well-developed MKT. From analysis of video-recorded classroom observations and teacher interviews, these researchers provide detailed accounts of teaching and "demonstrate that MKT matters in the way teachers pursue student thinking" (Cengiz et al., 2011, p. 372). Their analysis of data from one of the participating teachers "provide evidence that a lack of certain aspects of knowledge can negatively impact a teacher's pursuit of student thinking" (p. 372). Similarly, Izsák, Tillema and Tunç-Pekkan (2008) provide fine-grained details in their analysis of the cognitive structures used by a teacher and her students when using number lines as a representation for fraction addition. Audio- and video-recorded interactions of a practicing middle-school teacher and her students formed a starting point for interviews with three pairs of students. Excerpts from lesson and student interviews were then used in a video-elicited interview with the teacher. They argue that subtle variations in the teacher's approach to partitioning unit intervals matter for the students' opportunities to learn.

Several studies are situated in the teaching of particular mathematical content. Steele and Rogers (2012) investigate the relationship between mathematical knowledge for teaching proof and teaching practice by combining clinical assessments with classroom observations of two secondary teachers — a novice and an expert teacher. Data collection included lesson observations, pre- and post-lesson interviews, written assessments and semi-structured interviews after the observation. The authors argue that the more experienced and MKT-knowledgeable teacher not only enacts a stronger and more nuanced lesson on mathematical proof, but her students end up having more mathematical authority. They argue that their use of MKT as a frame for examining practice provides an innovative

method for investigating both MKT and features of instruction, such as student positioning as a key, mediating factor between MKT and opportunities to learn.

A study by Tanase (2011) investigates the connection between four Romanian first grade teachers' mathematical knowledge for teaching place value and their classroom practice. The participants are selected from a well performing and an average performing school in Romania. One experienced and one less experienced teacher from each school is selected for participation, and data collection includes teacher interviews, classroom observations and student assessments. Although all four teachers display good understanding of place value, Tanase suggests that teachers' knowledge goes beyond their own mathematical understanding. Differences are observed in teachers' ability to make connections between place value concepts and other mathematical concepts, how they set different objectives for students as well as the extent to which they challenge students in their mathematical work. Tanase also observes that although teachers have strong mathematical knowledge for teaching, and this knowledge impacts the quality of their instruction, their students may still not perform well. She suggests that student achievement is also influenced by external factors inside and outside of school.

Among these initial investigations of the specific influence of mathematical knowledge for teaching on teaching, most studies focus on teachers in primary, middle or secondary school. Two studies focus on mathematics teaching at university level. Speer and Wagner (2009) examine one undergraduate instructor's use of constructs of social and analytic scaffolding as a frame, the authors argue that aspects of pedagogical content knowledge are important for helping students find productive ways of solving particular problems and for understanding which student contributions, whether correct or incorrect, are important to emphasize in a discussion. They trace ways in which teachers' particular knowledge of students' understanding aids them in assuring that the lesson reaches intended mathematical goals and in understanding the role of particular mathematical ideas in students' development.

Another example is Johnson and Larsen's (2012) study of how a university teacher's mathematical knowledge influences her ability to listen when teaching abstract algebra. Their investigation focuses on how this particular aspect of mathematical knowledge for teaching supports mathematics teachers' listening when implementing a reform curriculum. Their theoretical framework distinguishes among three types of teacher listening: hermeneutic, interpretative and evaluative. Drawing on Speer and Wagner's (2009) argument that teacher listening requires particular types of mathematical knowledge for teaching, Johnson and Larsen examine the role of knowledge of content and of students in hearing tertiary students as they engage in reinventing the group concept in abstract algebra. Based on analyses of three teachers' classroom interactions when implementing a particular reform curriculum, Johnson and Larsen report on a teacher whose classroom interactions contained several episodes where the students were confused and the teacher was unable to make sense of their struggles. They observe that this teacher's ability to listen to her students draws on her knowledge of content and students. Johnson and Larsen posit that teachers need not only knowledge of students' misconceptions, but also knowledge of when and why students are likely to be confused and display misconceptions and of the consequences of such misconceptions when students engage in new activities.

The focus on teacher listening is also prevalent in Bansilal's (2012) investigation of how a South-African mathematics teacher's poor mathematical knowledge influences her classroom interactions. In this case study, the focus is on the process-object understanding of ratio. Based on narrative analysis of field notes and transcripts from five lesson observations with interviews, Bansilal organizes her claims around three emerging themes. First, she argues that the teacher displays limited understanding of ratio in her teaching. Second, she argues that the teacher fails to identify key ideas and organize her explanations in a way that enables the students to notice the big ideas involved in the mathematical task. Third, Bansilal points to the stressful environment that the teacher experiences in this classroom and suggests that this environment is caused by her lack of knowledge of the students as well as her preference for evaluative rather than interpretative listening.

In his study of mathematics teacher knowledge and its impact on how teachers engage students with challenging tasks, Choppin (2011) explores pedagogical content knowledge as situated in an instructional sequence. From his study, he aims at exploring teachers' "local theory of instruction". Choppin investigates an experienced middle-school mathematics teacher while she is teaching a particular curriculum unit over two years. In order to investigate the teacher's knowledge, interview data are analyzed with a focus on her articulation of "(1) how student thinking develops over time, (2) the process by which that thinking develops, and (3) the resources that facilitate the development of student thinking" (p. 12). Based on his analysis of data, Choppin claims that the teacher develops her local theory of instruction from teaching. The teacher's knowledge appears to influence her teaching in several ways, for instance in her adaptation of tasks.

Engaging students with challenging tasks is an important component of the work of teaching mathematics, and so is the selection and use of appropriate examples. Rowland (2008) focuses on mathematics teachers' purposes for using examples in elementary mathematics teaching. Video recordings from 24 lessons taught by 12 pre-service elementary teachers are analyzed from a grounded approach, and codes are developed that focus on aspects of their teaching practice. The resulting 18 codes — one of the most common codes is "choice of examples" — are then placed in four overarching categories that constitute Rowland's conceptualization of mathematical knowledge in teaching, commonly referred to as the knowledge quartet.

Although eight of the studies reviewed investigate effects of mathematical knowledge for teaching on mathematics teachers' classroom practice, only one applies standardized measures of mathematics teacher knowledge. In his exploratory study, Charalambous (2010) investigates the connection between two primary teachers' mathematical knowledge for teaching and their use of mathematical tasks. The two primary mathematics teachers had different levels of mathematical knowledge for teaching — as measured by MKT measures — and notable differences were found in how they planned, presented and implemented mathematical tasks. Charalambous applies Stein and colleagues' mathematical tasks framework to examine the cognitive level of enacted tasks, and he formulates three tentative hypotheses about mechanisms of how mathematical knowledge for teaching impacts teachers' selection and use of mathematical tasks. First, he hypothesizes that strong mathematical knowledge for teaching may contribute to a use of representations that supports students in solving problems, whereas weaker mathematical knowledge for teaching may limit instruction to memorizing rules. Second, he proposes that mathematical knowledge for teaching appears to support teachers' ability to provide explanations that give meaning to mathematical procedures. Third, he proposes that

teachers' mathematical knowledge for teaching may be related to their ability to follow students' thinking and responsively support development of understanding.

The study of Nardi, Biza and Zachariades (2012) differs from many of the other studies on how teachers' knowledge influences their teaching practice in that they do not study observed teaching. Instead, these researchers analyze teachers' argumentation about hypothetical classroom scenarios in task-based interviews. From their analysis of eleven teachers, they suggest that the teachers' warrants for the claims made about these classroom scenarios are not always mathematical. Their argument, which has potentially interesting methodological implications, is that analysis of the argumentation provided by teachers in such task-based interviews may provide insight into how the teachers' knowledge and beliefs influence their classroom interactions.

Sullivan, Clarke and Clarke (2009) also investigate the influence of teacher knowledge on the planning phase of teaching. In particular, they investigate the assumption that teachers are able to convert tasks to lessons easily. From their analysis of 107 primary and secondary teachers' responses to questionnaire items — and interpreting the responses by using the subcategories of MKT — they observe that many teachers find it difficult to translate tasks to lessons. For instance, many teachers find it difficult to convert the task of determining which of $\frac{2}{3}$ and $\frac{201}{301}$ is larger into a worthwhile learning experience for students.

Discussion

With regard to research design and choice of methods, we observe that most of the studies are small-scale qualitative studies that explore the connections between mathematics teacher knowledge and teaching practice in different ways. Although many studies draw on a similar conceptualization of mathematical knowledge for teaching, only one study applies one of the existing standardized measures of such knowledge (Charalambous, 2010). Several studies present innovative methods to investigate contributions of teachers' mathematical knowledge to teaching practice, such as video-elicited interviews and hypothetical classroom scenarios in interview prompts. As we have argued elsewhere (Hoover et al., 2016), given that research is this arena is in early development and to date we lack clear, replicable methods, scholars' efforts to innovate seem well placed. Ideas proposed in these dozen papers contribute to that development.

Each of the 12 studies reported is concerned with uncovering what, how, and why mathematical knowledge for teaching matters for teaching, yet the overall picture is unclear. One issue may be that an effort to show *that* mathematical knowledge for teaching matters (a focus on impact) may lead to holding knowledge and teaching at arms length in ways that obscure the dynamic nature of the role of that knowledge in teaching. For instance, several papers argued that teachers' lack of knowledge constrained what they were able to see, hear, and do, without taking the additional step of elaborating what knowledge arises in the work, when, where, and how. We suggest that the field would profit from studies that examine the interplay between knowledge and teaching practice and that impact studies are better conducted at a larger scale once clear conceptual and measurement tools are in place. Another issue may be that the conceptualization of and focus on *teaching* in these studies is underdeveloped. Some of the studies examine what might be better described as features of instruction than as teaching practice. For instance, Steele and Rogers (2012) examine the degree to which different ideas of proof are integrated into instruction and how students are positioned in

relation to mathematical explanation. We agree that these are important, but would like to understand more fully what it is that teachers need to do to integrate ideas and position students and what the mathematical entailments are for doing so. Some of the studies address constrained, specific tasks of teaching (cf. Hoover, Mosvold, & Fauskanger, 2014), such as selecting and using examples, while others are broad and general, such as engaging students with challenging tasks. What is meant by "teaching" and its role in these studies vary.

Progress on the problem of whether and in what ways mathematical knowledge for teaching influences teaching practice will require building more shared language for talking about teaching, starting with more explicit attention to how it is conceptualized and continuing through the development of more widely shared conceptualizations of the work of teaching. It will require more focused examination of what it takes to do teaching, conceptualized as meaningful work, supportive of learning and doing the work in professional community. As we have argued elsewhere (Hoover et al., 2016), this may need to go hand in hand with developing the theoretical foundations of research on teaching. Teaching is a professional practice engaged in human improvement work. While there are other important aims of education, teaching is centrally about supporting the learning of subject matter. Understanding the theoretical implications of these observations and acting on them may strengthen research and practice.

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Considering research frameworks as a tool for reflection on practices: Grain-size and levels of change

Siún Nic Mhuirí

Dublin City University, Ireland; siun.nicmhuiri@dcu.ie

Reflection is widely recognized as essential to teachers' learning but questions remain about what exactly teachers should reflect on and how effective reflection might be facilitated. This paper considers how research frameworks might be used as a tool to facilitate reflection on mathematics classroom practices. It will be argued that frameworks which facilitate planning and analysis of classroom practices at different levels of specificity may also target reflection at different levels. This will be illustrated with reference to frameworks which relate to talk in the mathematics classroom. It will also be argued that for research frameworks to be effective in facilitating reflection on mathematics classroom practices, they must target different levels of reflection.

Keywords: Reflection, research framework, classroom practice(s), grain size.

Introduction

A growing body of literature recognizes the importance and complexity of the practices involved in effective mathematics teaching (e.g., Ball et al., 2009; Potari et al., 2015). Classroom practices have been described as "the repeated actions in which students and teachers engage as they learn" (Boaler, 2002, p. 114). Dooley, Dunphy and Shiel (2014) identify five overarching practices as essential in promoting mathematical thinking and understanding. These meta-practices are the development of a productive disposition; emphasis on mathematical modelling; the use of cognitively challenging tasks, formative assessment, and the promotion of mathematical talk. In each of these areas, research frameworks exist which may hold potential for teacher development. By 'research framework', I mean either a theoretical framework or methodological tool which 'frames' or structures a coherent set of understandings about a particular theme. For example, consider how the Math Talk Learning Community (MTLC) framework (Hufferd-Ackles, Fuson & Sherin, 2004) might offer a coherent set of understandings about the nature of talk in classrooms or how the Mathematical Tasks Framework (Stein, Grover, & Henningsen, 1996) offers a similar set of understandings for the use of cognitively demanding tasks. Such frameworks illustrate teacher actions that contribute to desirable meta-practices for mathematics teaching and learning.

The growing emphasis on mathematics classroom practices is occurring at a time when reflection has been widely established as a guiding principle within teacher education (Hatton & Smith, 1995; Zeichner, 2008). Mathematics education research frameworks have been used to facilitate teachers' reflection (e.g., Stein & Smith, 1998). However, there is little literature to offer guidance on how frameworks might be used as a tool for reflection on mathematics classroom practices, or to guide researchers investigating this area. This paper offers a consideration of this theme. First, I will outline some key issues from the literature on reflection and will discuss the theoretical perspective from which teachers' use of research frameworks is understood. Then I will outline Korthagen's (2004) model of levels of reflection and change. Finally, I will argue that grain-size, or level of specificity, is of key importance when research frameworks are used to facilitate teachers' reflection on practices. I will illustrate this with reference to two frameworks relevant to the practices involved the promotion

of mathematical talk. I will argue that for frameworks to be effective in facilitating reflection on classroom practices, they must target different levels of reflection.

Reflection and research frameworks

It is generally accepted that experience alone may not result in learning and reflection is considered to be integral to learning from practice (Loughran, 2002; Schön, 1983; Zeichner, 2008). The widespread adoption of reflection in teacher education programs has resulted in conflicting conceptualizations and many efforts to develop reflective practitioners are often underpinned by different ideological stances (Hatton & Smith, 1995; Zeichner, 2008). Despite the multiplicity of interpretations, there is some agreement that reflection should involve finding solutions to real problems of practice (Hatton & Smith, 1995) with some authors suggesting that the initial stimulus should be a problem arising from practice (e.g., Jaworski, 1998). Loughran (2002) maintains that learning arises from framing and reframing the initial problem. Research frameworks may be helpful in the reframing process and towards the goal of problematizing teaching, a key component of reflective practice (Jaworski, 1998). This problematizing of teaching may arise from considering a situation from another point of view (Loughran, 2002). Research frameworks can facilitate consideration of alternate or multiple viewpoints. For example, the MTLC framework (Hufferd-Ackles et al., 2004) addresses both the student and teacher experience.

The reflective practice literature includes much attention to the process of reflection. Eraut (1995) critiques and extends Schön's (1983, 1987) stages of reflection-in-action and reflection-on-action to also include reflection-for-action. The extent to which research frameworks might inform teachers' reflections-in-action is limited due to the many pressures of real-time teaching but frameworks can be used to support reflection at both the planning and post-teaching phase either as a tool for formal analysis, or as a means of troubling accepted understandings of practice. The literature also highlights links between reflective practice and action research. Jaworski (1998) describes a mathematics teacher research project where the theory of the teachers' research activity aligned with a view of action research being connected to critical reflective practice. Critical reflective practice has also been conceptualized in different ways (Hatton & Smith, 1995). Larrivee (2000) maintains that critical reflection is an essential part of becoming a reflective practitioner and defines it as encompassing critical inquiry and self-reflection. Critical inquiry is described as "the conscious consideration of the ethical implications and consequences of teaching practice" (p. 293).

Theoretical perspective

Zeichner (2008) suggests that often the obligatory content of reflection is around how well practice conforms to what is expected. This issue might arise when using research frameworks for the purposes of teacher reflection. If a research framework is considered to be an example of what has been sanctioned as 'acceptable,' then teachers' interactions with frameworks might actually serve to undermine their agency. This is not the stance that I adopt. Instead, I understand reflective practice as "the making explicit of teaching approaches and processes so that they can become the objects of critical scrutiny" (Jaworski, 1998, p. 7). I view the teacher as knower and agent for educational and social change (Cochran-Smith & Lytle, 2009). The research framework is understood as a tool for teacher inquiry and reflection rather than a prescription for action. Inquiry is taken to be "a critical habit of mind that informs professional work in all its aspects" where the data arising from practice

is continually interrogated (Cochran-Smith, Lytle 2009, p.121). It is from this perspective that a teacher's use of research frameworks is understood.

Levels of reflection

Models of reflection often emphasize process or chronological phases, e.g., Gibbs (1988). In contrast, Korthagen's (2004) model (Figure 1) emphasizes the teacher as person. Korthagen describes the levels as "different perspectives from which we can look at how teachers function" (p. 80). This model should be of interest to researchers seeking to investigate the nature of teacher reflection and teacher-educators seeking to support the same process. For teacher-researchers, it may offer an overview of the landscape of reflection and serve to frame and contextualize the inquiry process. The umbrella-nature of the model is powerful because the individual levels commonly exist as distinct research domains within mathematics education. This scope is challenging to address and only key issues and contradictions are highlighted here. Korthagen explicates his model largely with reference to the literature of psychology. Some concepts may not align with sociocultural approaches present in much current mathematics education literature.

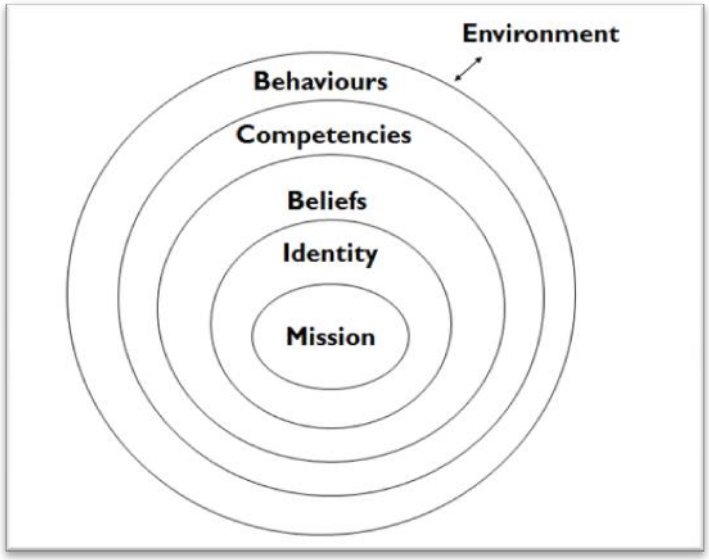

Figure 1: Korthagen's (2004) model of levels of change

Korthagen maintains that the inner levels influence the outer levels just as the outer levels can influence the inner levels, e.g., *behavior* can be influenced by external environmental factors as well as personal *competencies, beliefs, identity* and *mission*. A teacher's *competencies* will determine the *behavior* he/she is able to show but Korthagen suggests that *competencies* contain also the potential for behavior though this may not be enacted. He maintains that competencies are determined by *beliefs*. The mathematics education literature has also explored links and discontinuities between teacher beliefs, competencies and classroom practices (e.g., Stipek et al., 2001). Korthagen suggests that compete alignment between the levels may take a lifetime to achieve and is unlikely to occur without careful reflection on practice and self. Recently, the mathematics education literature has questioned the value of research on beliefs without due attention to teachers' participation in social practices (Skott, 2013). Though Korthagen's model does not address this directly, he does situate teachers' beliefs within a complex framework, with the levels of *environment* and *mission* in particular premised on the individual's engagement in a social world.

Recent research emphasizes the importance of understanding mathematics teacher competencies as personally, situationally and socially determined (Blömeke, 2016). Where 'situational' might be connected with the *environment*, 'personal' and 'social' suggest links with the *identity* and *mission* levels respectively. The identity level is concerned with the personal singularity of the individual, and the mission level (or spirituality level in earlier versions of the model) is intended to acknowledge the individual's participation in both local and global communities. Korthagen (2004) describes *mission* as being "about becoming aware of the meaning of one's own existence within a larger whole, and the role we see for ourselves in relation to our fellow man" (p. 85). Reflection at this level necessarily encompasses consideration of the short and long term influence of teaching on students and the larger goals of mathematics education or education more generally. This level must be considered to be important in relation to critical reflection (Larrivee, 2001). Some commonalities also exist with critical mathematics education or values education (e.g., Bishop, 2008).

While noting the importance of *identity*, Korthagen admits to a certain 'vagueness' around the definition of professional identity as the concept has been informed by many different research traditions. The concept has also been understood in different ways in the mathematics education literature though Boaler (2002) and others contend that the identities students develop are strongly related to the classroom practices they have opportunities to participate in. In relation to teacher identity, it is likely that a more complete understanding of teachers' engagement with research frameworks might arise from a perspective which foregrounds the social and situated nature of identity, and acknowledges that teachers engaged in such work may be working at, or across, the boundaries of various communities of practice (Lave & Wenger, 1991).

Korthagen positions his model as particularly useful to the teacher-educator. He suggests that a teacher's behavior may imply reflection is needed on a particular level and the teacher-educator can orchestrate his/her interactions with the teacher accordingly. It is likely that relevant research frameworks could be introduced to focus a teacher's reflection on a particular level. It has long been accepted that research frameworks may provide an impetus to question taken-for-granted assumptions and a language to describe, analyze, and interpret practice (e.g., Erikson, 1986). The novel element here is that the grain size, or level of specificity of a research framework, becomes important as this may determine which of Korthagen's levels the framework will relate to. Existing mathematics education research frameworks range from broad general theories about (mathematics) learning to very finely grained, highly structured frameworks concerned with the teaching and learning of particular mathematical content. Korthagen's model provides a structure for considering how such differing frameworks might be used to facilitate effective reflection.

The model is particularly useful in the specific case where research frameworks are being used to facilitate reflection on and development of mathematics classroom practices. Despite classroom practices being enacted at the outer level of *behavior,* they are connected to both inner and outer levels and arise from a complicated interaction between *mission, identity, beliefs, competencies* and *environment*. Larrivee contends that a "deliberative code of conduct" (2000, p. 293), or conscious adoption of particular practices, results from the infusion of personal beliefs and values into a professional identity. If research frameworks, or teachers' interactions with research frameworks, confine reflection to the outer levels of Korthagen's model, there is a danger that rationalization of practice (Loughran, 2002) may occur rather than inquiry and development. While an appropriate

research framework may provoke reflection across and between levels, frameworks which focus attention only on the outer levels are likely to have limited effectiveness. Teachers should have opportunities to question the relationship between inner and outer levels and reflect on how their practices align (or not) with inner levels such as *beliefs, identity* and *mission*.

A researcher interested in investigating teacher reflection who adopts the perspective outlined by Korthagen is likely to be interested in the relationships between the different levels of reflection and change. For example, if reflection appears to be occurring as a result of a perplexing situation arising from practice, then the researcher may be interested in attempting to track this to a particular level or a possible conflict between levels (e.g., a teacher is perplexed because she is struggling to implement in practice (*behavior*) what she believes (*beliefs*) to be true about 'good' mathematics teaching or in line with her *mission*. Similarly, a researcher investigating the extent to which research frameworks facilitate reflection may be interested in considering the extent to which a teacher's interaction with the framework(s) aligns with the various levels of Korthagen's model.

An illustration

Two research frameworks that might be used to develop the practices involved in the promotion of math talk are discussed below. This topic has been chosen as classroom interaction emerged as a strong theme in TWG19 at CERME9 (Potari et al., 2015). The MTLC framework arises from research which tracked a classroom community transitioning from a traditional model to one in which students helped each other learn by engaging in meaningful talk about mathematics (Hufferd-Ackles et al., 2004). The framework describes four levels of mathematical talk and the overall community trajectory is described as growing "to support students acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking" (p. 88). Associated with the levels are developmental trajectories for teacher and student actions across the areas of questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. Table 1 shows teacher actions at the highest level of the framework.

The MTLC framework may facilitate reflection at all levels but it raises particular challenges at the levels of *beliefs, identity* and *mission* because it presents an alternative to traditional teaching. It may challenge the idea of teacher as sole-mathematical authority and it disrupts traditional conceptualizations of teacher and student roles by emphasizing student agency. In doing so, it may provoke reflection on personal *beliefs* about mathematics and the teaching of mathematics. It may also foreground issues of teacher *identity* and *mission*. I used this framework in previous research and while I never formally stated my 'mission*',* it did help me clarify what I wished to achieve as a teacher of mathematics: students who could think mathematically, and who valued their own thinking and their responsibilities within the classroom community. The framework details teacher actions and gives some direction as to how this mission might be achieved. However, used on its own, it is unlikely to provide sufficient support for developing the complex network of teacher *competencies* and *behavior* necessary for the promotion of productive math talk.

Description of Teacher Actions
Questioning: Teacher expects students to ask one another questions about their work. The teacher's questions may still guide the discourse.
Explaining Mathematical Thinking: Teacher follows along closely to student descriptions of their thinking, encouraging students to make their answers more complete. Teacher stimulates students to think more clearly about strategies.
Source of Mathematical Ideas: Teacher lets students explain and "own" new strategies. (Teacher is still engaged and deciding what is important to continue exploring.) Teacher uses student ideas as the basis for lessons or mini-extensions
Responsibility for Learning: Teacher expects students to be responsible for co-evaluation of everyone's thinking. She supports students as they help one another sort out misconceptions. She helps and/or follows up when needed

Table 1: Teacher Actions at level 3 of the MTLC framework (Hufferd-Ackles et al., 2004, p. 88 -90)

Teacher Question Categories		
1. Gathering information, leading students through a method	4. Probing, getting students to explain their thinking	7. Extending thinking:
2. Inserting terminology	5. Generating discussion	8. Orientating and focusing:
3. Exploring mathematical meanings and/or relationships	6. Linking and applying	9. Establishing context:

Table 2: Boaler and Brodie's (2004) Teacher Question Categories

The second framework I will discuss is Boaler and Brodie's (2004) teacher question categories (Table 2). This framework arises from analyses of practice and was developed to allow researchers investigate multiple lessons at a relatively fine grain-size. The framework directs attention to the level of *behavior*, specifically the questioning practices of the teacher, with scope for consideration of the learning opportunities that arise for the student. It may allow for consideration of the emphasis on relational (type 3 questions) and instrumental understanding (type 1 questions) or the opportunities created for student discussion (type 5 questions). This framework might be used as a tool for reflection, either to support planning of effective questions or to support analysis or reflection on practice. However, it is unlikely that this framework alone will hold meaning for the teacher unless its use is mediated by consideration of the 'bigger picture'. Teacher questions can only be considered effective or ineffective with reference to the overall goal for learning. This refers both to the specific mathematical goals for a lesson or unit of work, as well as goals that might be considered part of *mission*, such as the development of student agency and mathematical authority.

The frameworks described above are of different levels of specificity. The MTLC framework attempts to describe teacher and student interactions at a broad level while Boaler and Brodie's framework facilitates fine-grained analysis of teacher questions. For this reason, they offer different affordances and constraints when considered in relation to Korthagen's levels of reflection. It is argued, that for the purposes of facilitating teacher reflection, the use of either of the frameworks alone has limitations. Potential for reflection is maximized when such frameworks are combined. It is in the interplay between reflection at the inner levels and reflection at the outer levels that more profound development may occur.

Conclusion

Many national curricula have begun to emphasize the practices, as well as the content, of mathematics education (e.g., Dooley et al., 2014). In this context, it is worth considering to what extent research frameworks might be used to support and develop teachers' reflection on practices. I have argued that any such work must strive to take into account the level(s) of reflection that a research framework might target. I also contend that the perspective outlined is of relevance to both teacher-educators and researchers. Further consideration of the methodological implications of a researcher adopting such a stance on teacher reflection is necessary. Theoretical and empirical work is also needed on teachers' use of broad and finely grained research frameworks. Such work should seek to identify the characteristics of research frameworks, and teachers' interactions with frameworks, that enable their use as tools for effective reflection.

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Evaluation of a rating system for the assessment of metacognitive-discursive instructional quality

Edyta Nowińska[1] and Anna-Katharina Praetorius[2]

[1]Osnabrück University, Osnabrück, Germany; enowinska@uni-osnabrueck.de

[2] German Institute for International Educational Research, Germany; praetorius@dipf.de

Metacognition plays an essential role in learning mathematics. However, due to the lack of observational systems for evaluation of metacognition in mathematics instruction, rarely anything is known about how metacognition is practised and fostered when teaching and learning mathematics in class. This paper presents an observational system (a rating system) developed to reliably assess metacognitive activities in mathematics instruction. It also explains the methodology used to evaluate the reliability of ratings achieved with this tool and to investigate the stability of metacognitive-discursive practices between lessons of an individual teacher/class. Despite the high inference of conclusions needed to assess metacognitive-discursive instructional quality in seven dimensions, highly reliable ratings have been achieved for six dimensions. The paper discusses reasons for and consequences of the high reliability.

Keywords: Metacognition, discourse, rating system, generalizability study, decision study.

The role of metacognition in teaching and learning mathematics

Metacognition has been ascribed an essential role in regulating students' cognitive processes in problem solving as well as in learning mathematics in general, in particular when constructing, organising, systematising, and connecting (pieces of) knowledge (cf. Schraw & Moshman, 1995; Wilson & Clark, 2004). However, hardly anything is known about how metacognition is practised and fostered in mathematics instruction. Assuming that enhancing learners' metacognition is essential for promoting learning, research on metacognition in this area definitely merits future research (cf. Mevarech & Kramarski, 2014; Depaepe et al., 2010). For this kind of research a tool is needed that allows to *reliably* assess metacognitive practices when teaching and learning mathematics in a class. This paper reports on a research project that aimed at developing such a tool, named the *r*ating *s*ystem for analysing and assessing the *m*etacognitive-*d*iscursive instructional *q*uality (Nowińska, 2016). This tool can be used to first *describe* metacognition during class discussion, and to second *evaluate* how metacognitive activities are used to foster understanding in mathematics, in particular by elaborating students' ways of thinking and reasoning, and by discussing them in a coherent and comprehensible way. One important research question underlying our work on developing and evaluating this tool was how stable metacognitive-discursive instructional quality is across lessons of an individual teacher in one class. In addition to advancing our knowledge about the occurrence of metacognitive-discursive instructional quality, investigating its stability allows identifying the number of lessons per teacher/class which would be needed to reliably measure metacognitive-discursive instructional quality.

Metacognition is often understood as *knowledge* about cognition and *regulation* of cognition (Flavell, 1976; Schraw & Moshmann, 1998). The groundwork for investigating metacognition in the domain of teaching and learning mathematics in a class has been done by Cohors-Fresenborg and

Kaune (2007) as they developed a *c*ategory system for an interpretative, transcript-based analysis of *m*etacognitive and *d*iscursive *a*ctivities (CMDA)[1]. This category system decomposes metacognition in *planning*, *monitoring* and *reflection*. Examples are *planning* the structure of a proof or definition; *monitoring* the correctness of an argumentation; and *reflecting* on misconceptions or on difficulties experienced in interpreting a definition or in solving an equation. According to studies suggesting that the effects of metacognition on students' understanding when learning in class seem to depend on the quality of the class discussion (cf. Mevarech & Kramarski, 2014; Depaepe et al., 2010), it was necessary to combine the analysis of metacognition with a deep analysis of precision, coherence and accuracy of teacher's and students' contributions. For this purpose, CMDA also includes the categories *discursivity* and *negative discursivity*. *Discursivity* means activities enhancing precision, accuracy and coherence in a class discussion, e.g. by making connections between different answers, or between external concept representations and students' conceptions. *Negative discursivity* means activities with a negative influence on precision, accuracy and coherence. The results of many years research conducted on metacognitive and discursive activities led the authors of CMDA to the conviction that discursive ways of practicing metacognition are crucial for supporting students' understanding when learning mathematics in class. The term "discursive" does not simply mean "in a discourse" but is meant as a characteristic of discussions elaborating, explaining and linking students' ways of thinking in a coherent and comprehensible way.

The category system CMDA allows a detailed *interpretation* and categorization of local, single metacognitive and discursive activities, but it does not provide any additional tool for the *global, comprehensive assessment* of their instruction-related quality, thus of the extent to which they facilitate understanding of the mathematical subject discussed in class. The new rating system discussed in this paper is a result of extending[2] the category system CMDA to a video-based observational system aimed to analyse and measure 'metacognitive-discursive instructional quality' in a comprehensive way. For this aim, several dimensions of the metacognitive-discursive instructional quality as well as evaluation criteria to rate them have been developed (for details see Nowińska, 2016). To allow its application, the rating system needed to be valid and reliable despite the complexity and high inferences required for rating metacognitive-discursive instructional quality.

In the following, we first explain the design of the rating system. Second, we describe the methodology used to evaluate its reliability (G study), present the achieved results, and discuss their consequences for generalizable evaluation of metacognitive-discursive instructional quality. Finally, we discuss consequences of our study for further research aimed to deepen our understanding of metacognitive-discursive instructional quality and to improve teaching and learning practices in class.

[1] The complete German version of CMDA is presented in Cohors-Fresenborg, Kaune, & Zülsdorf-Kersting (2014).

[2] www.mathematik.uni-osnabrueck.de/fileadmin/didaktik/Projekte_KM/Kategoriensystem_EN.pdf

The design of the rating system

Our conceptualization of metacognitive-discursive instructional quality and its decomposition in seven dimensions is based on research literature concerning relations between metacognition and learning gains (e.g. Mevarech & Kramarski, 2014; Depaepe et al., 2010), and on the preliminary research work related to the category system CMDA (e.g., Cohors-Fresenborg et al., 2010; Gretzmann, 2011). Furthermore, we analysed more than 20 videotaped lessons to deepen our understanding of these dimensions. Each dimension is described by means of a guiding question (GQ) focusing raters' attention on aspects to be analysed and evaluated, as well as of several answering categories. For each GQ, the answering categories describe particular aspects of classroom discussions that differ in quality. The categories are ordered so that they reflect increasing quality of the classroom discussion with regard to the relevant aspects, and constitute a rating scale. In the following, the seven guiding questions are described briefly (for the detailed version see Nowińska, 2016).

GQ 1 puts the focus on using metacognitive activities for an elaborate discussion of mathematical content and on supporting learners' autonomy in practicing such activities. To answer this GQ, the rater has to distinguish, among others, between metacognitive activities limited to monitoring results of calculations, on the one hand, and extended to reflection on mathematical ways of reasoning, methods, definitions, and conceptions related to them, on the other hand. Due to the essential role of argumentation in learning and understanding mathematics, GQ 2 focuses on justifications combined with metacognitive activities, and on supporting learners' autonomy in providing and analysing justifications. To answer this GQ, the rater has to distinguish between fragmentary justifications, on the one hand, and efforts made in class to orchestrate single justifications in order to produce precise comprehensive argumentations, on the other hand. GQ 3 aims at assessing to which extent the interplay of metacognitive and discursive activities foster students' understanding of subject-specific issues discussed in the particular lesson. The answering categories for this GQ distinguish among others between situations without any productive use of metacognitive and discursive activities, and situations in which (at least in the case of one single learner) metacognitive and discursive activities foster and express learners' understanding of the subject-specific issues discussed in the class. GQ 4 analyses the use of discursive activities in producing precise and coherent discussion. Such discussion is an essential precondition for an effective use of metacognition in class in order to foster learners' understanding. GQ 5, on the contrary, evaluates to what extent negative discursivity (e.g., not taking notice of inadequate mathematical vocabulary or of fragmentary answers as well as of answers not related to the discussed question) leads to ignoring students' cognitive and metacognitive processes, and hinders the reciprocal understanding in class as well as the understanding of subject-specific issues. GQ 6 evaluates to what extent metacognitive and discursive activities are used to build coherent and stringently guided discourse units (i.e., debates). The answering categories for this GQ distinguish between classroom situations without any debates, and situations with at least one remarkable debate led by the teacher or by students. GQ 7 aims at assessing to which extent metacognitive and discursive activities are related to challenging and complex subject-specific issues (e.g., related to meta-mathematics), used to elaborate such issues, and to foster learners' understanding of them.

To ensure reliable ratings despite the high level of inference needed to answer the guiding questions, the rating procedure was designed as a two-step procedure. In the first step of the rating process, the rater *categorises* each of the teacher and student contributions. Hereby, the rater uses the category system adopted from Cohors-Fresenborg and Kaune, and works with special software which at the end of the categorisation generates a graphic representation (i.e. category line; for more details see Nowińska, 2016). The category line includes all codes for metacognitive and (negative) discursive activities set by the rater, and distinguishes between codes for teacher and student activities. It serves as a basis for interpreting relations between teacher's and students' metacognitive and discursive behaviour, and for assessing students' autonomy in practicing these activities. Thus, the purpose of the first step is to get insight into the kind and quality of each single metacognitive and (negative) discursive activity, and to prevent the rater from rushed and inadequate ratings. In the second step, the rater uses the category line and the video transcript to evaluate the lessons by means of the seven rating scales elaborated on above.

In order to be able to carry out these tasks, three raters (students at the end of their master study course in mathematics education) participated in an intensive rater training (6 months, 180 h in sum). They were qualified to understand the purpose of the rating system, the foci of the seven rating scales, and the use the rating system. During the rater training (and also after it) the raters were obligated to justify their decisions regarding their interpretation of each single metacognitive and (negative) discursive activity as well as their final evaluation of the instructional quality. This allowed the trainer to discuss the answers given by the raters in detail, to discuss reasons for differences between the raters, and to provide each rater with detailed feedback. The videos and transcripts used during the training were separate from the ones used in the current study.

Methodology

In the current study, sequences from 24 videotaped mathematics lessons (6 teachers/classes with 4 lessons per teacher/class) were analysed. For each teacher, four lessons were videotaped within two weeks, and should represent "normal" lessons in these classes. From each lesson, a 10-minute video sequence showing a discussion in the class was chosen. This was done by two independent experts who first analysed each lesson, and suggested the sequence in which the main topic of the lesson was discussed, and in which the students actively participated in the discussion. Finally, the experts agreed on one sequence. In many cases, however, only one 10-minute discussion could be indicated, whereas in the remaining time the students worked individually or in pairs. Each video sequence was evaluated by three independent raters, who had taken part in the rater training.

Generalizability theory was used (Shavelson & Webb, 1991; for an application to the instructional context, see Praetorius et al., 2012) for assessing the generalizability (which can be interpreted similarly to reliability in classical test theory) of the rating instrument. The reported relative G coefficient can be interpreted analogously to a reliability coefficient in the classical test theory. Thus, a coefficient ≥ 0.7 is needed for a satisfactory reliability. In addition to providing these G coefficients, generalizability studies (G studies) allow decomposing the variance in rating scores into different components (e.g., teachers, lessons, and raters), their interactions, and measurement error. Therefore, G study results provide more detailed and precise information regarding reliability than reliability coefficients used in classical test theory. Furthermore, decision studies (D studies)

can be conducted to estimate the reliability under multiple hypothetical measurement conditions, thus also allowing to analyse numbers of lessons per teacher/class higher than the number actually evaluated by the raters in our study. In the present study, it was investigated how many lessons per teacher/class would be necessary for a reliable assessment of the aspects of the metacognitive-discursive quality determined be the seven dimensions.

Results

The results of the G studies indicated satisfactory reliability of ratings concerning six out of the seven dimensions of the metacognitive-discursive instructional quality (guiding questions 1 to 6), for which the relative G coefficient varied between 0,78 and 0,98 (see Table 1). The ratings concerning dimension 7 were not reliable, with a relative G coefficient of 0,38.

		GQ 1	GQ 2	GQ 3	GQ 4	GQ 5	GQ 6	GQ 7
Lesson-unspecific (stable) component	t	89	45	60	52	71	50	6
Lesson-specific component	l:t	0	46	21	39	22	30	14
Rater bias components	r	0	1	0	0	0	0	5
	r*t	3	0	0	2	1	3	51
Residual	r*(l:t); e	8	9	19	8	7	16	25
Relative G-coefficient		0,98	0,78	0,90	0,83	0,92	0,83	0,38

Table 1: Relative G-coefficients and variance decomposition (in %) for the seven dimensions

Based on the rating data, the variance in the ratings was decomposed in variance components attributable to the teacher/class (t), lessons nested in teachers (l:t), raters (r), the interaction between teachers and raters (r×t), and the unexplained variance, i.e. residual (r×(l:t),e). Table 1 shows the percentage of variance explained by the different variance components.

For dimensions 1 to 6, the amount of variance attributable to rater bias was very small (between 1% and 3% of the entire variance); this indicates that the raters do rarely differ in their ratings. However, further rater training would be needed to eliminate the very high amount of the variance (55%) attributable to rater bias for dimension 7 in order to get reliable ratings

The ratio of t to l:t, which describes the stability of the given dimension across lessons of an individual teacher/class, indicates partly very high stability (see e.g., GQ 1) and partly very low stability (see e.g., GQ 7).

To determine how many lessons per teacher/class are necessary to measure metacognitive-discursive instructional quality in a stable and reliable way, D analyses were conducted with the hypothetical number of lessons per teacher/class varying between 1 and 10. The number of raters was fixed to the actual number in the study (i.e., three). Figure 1 illustrates the results of the D study for each of the seven dimensions.

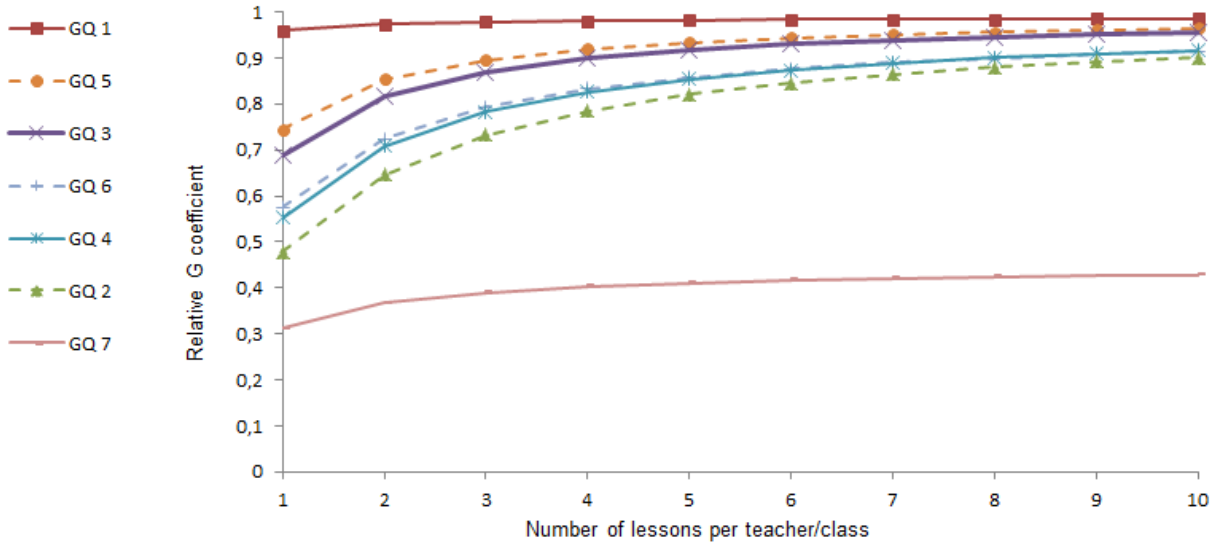

Figure 1: Relative G coefficients for D studies with 1-10 lessons per teacher/class for GQ 1–7

To obtain relative G coefficients greater than 0.7, one lesson is needed for the dimensions related to GQ 1 as well as GQ 5. Two lessons are needed for the dimensions related to GQ 3, GQ 4, and GQ 6, and three lessons for the dimension related to GQ 2. Thus, three lessons are sufficient to achieve a G coefficient of 0,7 for GQ 1 to GQ 6.), whereas 5 lessons per teacher/class would be needed for the reliability greater than 0.8. Due to the high amount of the variance attributable to rater bias for GQ 7, no satisfactory reliability concerning this dimension could be reached, even with 10 lessons per teacher/class without further rater training.

Discussion

The aim of our research project was to develop a reliable rating system for assessing *m*etacognitive-*d*iscursive instructional *q*uality. For this purpose, seven dimensions of metacognitive-discursive quality had been developed. Despite the high amount of inferences needed to rate these dimensions, highly reliable ratings were achieved for six of them. This rather unusual result (for an overview on the amount of rater effects found in prior studies, see Praetorius et al., 2012) can likely be explained, among others, with the intensive rater training, and with the two-steps procedure of the rating process. Both aspects prevented the raters from a superficial analysis, and instead forced well-reasoned scoring.

No satisfactory reliability could be achieved for the seventh dimension. The reliability analyses showed that this is due to high rater bias. Obviously, the meaning of “complex subject-specific” issues, which is at the core of the seventh dimension, has not been interpreted in the same way by all raters. Thus, additional rater training or other raters with a more substantial background in mathematics education would be needed to get reliable ratings for this dimension. Such efforts seem highly desirable as the seventh dimension plays an important role in a *long-term* evaluation of the metacognitive-discursive quality in a class. In general, complex subject-specific issues are discussed in mathematical instruction rarely. Discussing such issues indicates teacher’s efforts to deepen and systematise students’ cognition related to (meta-)mathematical questions, methods or ways of reasoning, and therefore it is a significant characteristic of instructional quality.

The seven dimensions of the metacognitive-discursive instructional quality vary in their stability between lessons in a particular class. The lowest variability could be determined for the dimension concerning the extent to which metacognition is practised in a class, in interactions between the teacher and the students (GQ 1), and the highest for this concerning metacognitive activities combined with justifications (GQ 2). The quite stable first dimension is based on GQ 1 which also takes some observable aspects of patterns in interactions between metacognitive and discursive teacher and students activities, whereas GQ 2 focuses more on metacognitive activities in relation to the content discussed in class and to students' reasoning concerning this content. The results indicate that the quite stable observational patterns in practicing metacognition do not necessarily imply the stability of metacognitive efforts to elaborate the mathematical issues discussed in class and to foster students' understanding. A deep analysis of videos is needed to explain this observation. Our preliminary analysis shows that in some classes, providing justification seems to be well established as a social norm. This means that the learners and the teacher are used to justify their answers, i.e. to practice monitoring or reflection. However, by doing so, not always the necessary attempts are made in the class to reflect on these justifications, to control and correct them, and to orchestrate *single* and fragmentary justifications in order to produce a coherent *global* explanation related to mathematical issues discussed in the class. This can be observed in particular when new concepts, definitions or strategies are introduced. Despite the high number of single justifications combined with metacognition, the lack of a well-orchestrated mathematical justification related to the new issues may hinder understanding. Consequently, this leads to a low score for the second dimension of the metacognitive-discursive instructional quality (GQ 2). The score can be higher when the tasks discussed in the class do not require a global well-orchestrated mathematical justification, and the lack of it cannot be evaluated negatively, with very low scoring. Thus, the variability of the second dimension seems to be related to the complexity of the mathematical content. This observation seems plausible but it must be investigated more deeply. Considering only the number of justifications combined with metacognitive activities would most likely enhance the reliability of the ratings in the second dimension but it would distort the validity of the instructional quality.

Our D studies are of crucial importance for further research on metacognition in mathematics instruction. Due to the variability of metacognitive-discursive practices between lessons in a particular class, at least three lessons per teacher/class and three qualified raters would be needed for reliable (generalizable) evaluation of the metacognitive-discursive quality with regard to six dimensions of this construct. Thus, given these relative small numbers, the rating system can be considered as a practicable research tool although intensive rater training is needed.

Given this result, a pivotal next step for research on metacognitive-discursive instructional quality is to investigate the effects of each of the six dimensions on students' mathematics achievement. In doing so, the empirical relevance of metacognitive-discursive instructional quality can be investigated, and implications for supporting metacognition to foster mathematical understanding can be suggested. Thus, continuing this research is highly desirable. It would shift the focus from measurement and evaluation to development and improvement. This would require the work *with* teachers, and not only research *on* teachers' instructional practices. Thereby, the rating system presented in this paper can be used as an analytical tool in teacher trainings for guiding teachers' reflection on their own practices and on learners' metacognitive and discursive behaviour.

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Teacher learning in teaching the topic of quadratic expressions and equations in Kenyan high schools: Learning Study design

Fredrick Odindo

University of East Anglia, School of Education and Lifelong Learning, Norwich, UK
F.Odindo@uea.ac.uk

Performance in mathematics national examinations in Kenya has been weak and raises questions about the pedagogical approaches adopted by teachers. The purpose of this paper is to report on teachers' experiences of the application of a problem-solving teaching strategy in a Learning Study design to teach factorization of quadratic expressions. The study follows a qualitative and interpretive research approach with data collected through questionnaires, classroom observations, video replays and reflection sessions after lessons. The findings reveal teachers' appreciation of the application of problem-solving teaching in a Learning Study design, saying it helped them to observe the difficulties students experience in learning algebra and the shortcomings in their lesson preparation. In addition, they noted the importance of reflection in that it helped them improve the second lesson.

Keywords: Learning Study, variation theory, problem-solving, teamwork.

Introduction

Teaching is an activity that involves engagement of the teacher, the student and the content (Loef Franke, et al., 2007; Stigler, & Hiebert, 1999). In mathematics teaching, this engagement needs to be effective to create a productive learning environment for both teachers and students (Loef Franke, et al., 2007). However, Stigler and Hiebert (1999) note that levels of engagement among the three is not uniform due to different classroom cultures and teaching strategies in different countries. They observed, for example, that in some US classrooms, teaching strategies are so procedural and teacher-directed that students are passive recipients of knowledge, and that there is little interaction between student and content.

Stigler and Hiebert's (1999) finding concerning US mathematics classrooms seems to be relevant in other classroom cultures in countries such as Kenya (CEMASTEA, 2010). Based on past performances, the Government of Kenya initiated an in-service education programme for high school teachers of mathematics and science in 1998. Among the strategies employed was Lesson Study approach, which implicitly applies constructivist theory (Elliot, 2014). However, a survey conducted to check the extent of the implementation of the programme, reported that about 65% of the teachers were not implementing the programme and were applying the teacher-directed method of teaching (CEMASTEA, 2010). Teachers cited lack of clear guidelines and workload among reasons. Students continued to post weak results in mathematics in 2012 and 2013 from the Kenya national examinations (KNEC, 2014). This study adopts a Learning Study (LS) design which explicitly applies Variation Theory to support classroom learning. Little is known about LS in Africa, and the outcome of this present study may support the further development of LS in Kenya in particular.

The purpose of this study is to solicit teachers' views and experiences of the application of a problem-solving teaching strategy in a LS design to teach factorization and solution of quadratic

equations, one of the topics poorly performed by students (KNEC, 2014). My intention is to eventually extend the approach to other mathematics topics. The research question of the present study is "What are the teachers able to learn about the students learning of factorization of quadratic expressions in a Learning Study design?"

Variation theory

Variation Theory is a theory of learning which asserts that to learn something entails experiencing it in a variety of ways. The theory proposes that learning is always directed towards an object, which could be a skill or a concept referred to as *the object of learning* in the '*what*' and '*how*' aspects of learning (Marton & Booth, 1997). The 'what' aspect is the content to be learnt while the 'how' aspect is concerned with the process of learning that enhances a student's ability to apply the learned concept in a new environment (Elliot, 2014; Lo, 2012). The theory postulates that learning takes place when students focus on a *critical feature* of *the object of learning*. For example, suppose the solution of simultaneous linear equations by elimination were the object of learning: a critical feature to be discerned can be the collating of the equations so that one variable has the same numerical coefficient in both equations. To achieve this, Lo (2012) and Marton (2015) propose that teachers need to create learning opportunities by explicitly or implicitly offering patterns of variation in which some parts remain invariant as others vary. Four patterns of variation are identified, namely: contrast, separation, generalization and fusion. For example, to discern the concept of quadrilateral, a type of quadrilateral would be kept invariant and contrasted with other polygons such as triangles and pentagons. To discern a particular type of quadrilateral, for instance a kite, it is kept invariant as other types such as rectangle, trapezium are varied. The kite is separated from the whole and focused on. To generalize that the total sum of interior angles of a quadrilateral is 360°, each type of a quadrilateral is varied with its sum of interior angles calculated. To compare properties of different types of quadrilaterals, such as a kite and a rhombus, the two and their properties are brought into focus simultaneously, the fusion pattern of variation.

Learning Study

Learning Study (LS) provides a framework for supporting learning in the classroom by applying aspects of Variation Theory, in which all the three categories of persons participating in the lesson (the teachers, the students and the researchers) could learn in the process (Marton, 2015; Pang, 2008). LS adopts a Lesson Study organizational structure in which a group of teachers prepare a lesson together, then one teaches the lesson while others observe as they collect research data and thereafter converge for a reflection session (Pang, 2008). LS focuses on *the object of learning* which points to the beginning of the learning process with learning 'what' and learning 'how' aspects (Lo, 2012). In this study, the topic of quadratic expressions and equations is the 'what' aspect while the 'how' aspect is addressed by small group discussion approach to learning. For ease of monitoring the learning process, the *object of learning* is categorized into: (a) *lived object of learning 1*, (b) *lived object of learning 2*, (c) *intended object of learning* and (d) *enacted object of learning*. Part (a) is concerned with prior experiences and awareness that students have about the concept and is monitored through a student's diagnostic pre-test or interview whose outcome is considered in lesson preparation. Part (b) is the acquired experiences after the teaching of the lesson and is monitored through a student's post-test or interview. Part (c) is the planned lesson and part

(d) is the taught lesson (Pang, 2008). Parts (c) and (d) are monitored through a post-test or interview and reflection session.

Methodology

The study on which this paper is drawn follows a qualitative and interpretive research approach in a LS design conducted in two classes in Kenya. Three teachers planned the lesson together as explained earlier. Prior to the first lesson, a diagnostic pre-test was given to the students in both classes; a post-test identical to the pre-test was given to each class at the end of each lesson. The teachers will be addressed by pseudonyms as Dominic – head of mathematics, Peter – teacher of the first lesson and John – teacher of the second lesson. The two classes were for third year high school students (age 16-18 years) comprising 68 students altogether.

Data were collected through classroom lesson observation, pre-post questionnaires, video recordings of the lessons and transcribed reflection sessions. Teaching was approached through a small group discussion by the students. The *object of learning* was factorization of a quadratic expression with a unit coefficient of x^2 (i.e. $x^2 + bx + c$). The *critical feature* was the identification of factors of the constant term of a quadratic expression that sum to the coefficient of x, often expressed in textbooks as "sum and product".

The items in the questionnaire, whose outcomes were considered in the preparation of the lesson, included: (1) Why is $x^2 + 5x + 6$ called a quadratic expression? (2) What do we consider in attempting to factorize a quadratic expression such as the one given above? (3) How many factors do we expect from a factorized quadratic expression? Frequent student responses included: (1) the given expression is called a quadratic expression because it has unknowns, (2) we consider like terms, (3) two factors (considered the correct answer). The questionnaire responses were scored 1 for a correct answer and 0 otherwise.

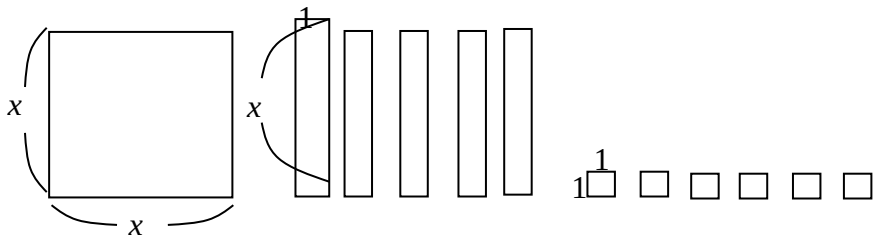

Figure 1: Paper cuttings for a hands-on activity aiming at the factorization of $x^2 + 5x + 6$

Based on the students' responses, the teachers prepared for *the intended object of learning* in a 40-minutes lesson, incorporating a hands-on activity intended to raise students' curiosity, and to motivate them to discuss in small groups as the approach was new to them. The first task on the activity was to form a rectangle using the pieces of paper shown in Figure 1, and to find the product of the sides of the rectangle formed. This was intended to lead to the factorization of $x^2 + 5x + 6$. The second task was to explain the relationship between the numerical terms from their expression of the area, $(x + 2)(x + 3)$; and the coefficient of x and the constant term in the quadratic expression.

First lesson

The first lesson was taught by Peter in his class of 40 students. Prior to this lesson, students had been taught the expansion of quadratic factors of the form $(p + q)(p + r)$. Peter introduced the lesson by asking the students to identify the coefficients of x^2 and x in the expression $x^2 + 4x + 3$. Whereas all the students could correctly identify the coefficient of x as 4, most of them were unable to correctly identify the coefficient of x^2. Students gave responses that included: $x \times x$, x and 2 - presumably from the exponent 2. The teacher asked them to discuss in pairs and seek a correct solution. After a while, a student correctly identified the coefficient as 1, but could not explain her answer. The teacher explained why it is 1.

Peter then asked the students to factorize the expression $x^2 + 5x + 6$. After about five minutes, he asked the students to form eight groups of five members each and he distributed the pieces of paper in Figure 1 to each group. He explained the first task of the activity and allowed 15 minutes for the task. After 15 minutes, only two groups had formed the rectangle. Peter allowed a further 10 minutes for discussion before calling upon groups to present their work. The teacher grouped the students' work in categories as shown in Table 1.

Category	No. of Group(s)	Rectangle	Area
One	4	Correct rectangle	Correct working
Two	1	Correct rectangle	Wrong working
Three	2	Correct rectangle	No area worked out
Four	1	No rectangle formed	No area worked out

Table 1: The categories of groups and how they carried out the task

One group from category one and the category two group were asked to present their work shown in Figure 2 and Figure 3 respectively.

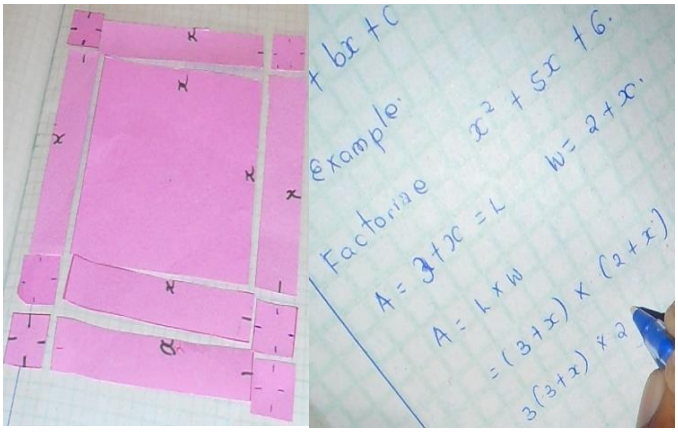

Figure 2: Correct working

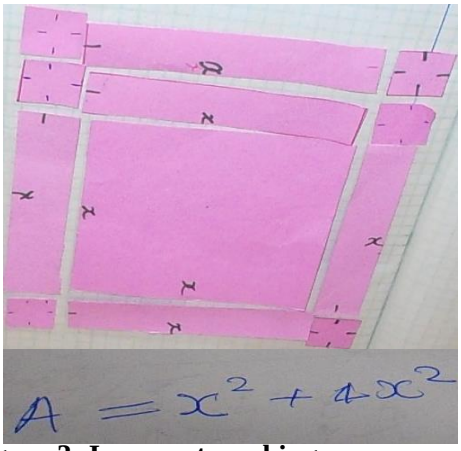

Figure 3: Incorrect working

The Figure 2 representative explained thus,

Student 1: Length, L = 1+1+1+x = (3+x) and width, W = 1+1+x = (2+x). Area, A = L×W = (3+x) (2+x)

The Figure 3 representative explained,

Student 2: The width has two pieces of x each giving an area of $x \times x = x^2$. The length has a piece of x at the bottom and the upper part has two pieces of x plus the big piece whose length is x giving a total of $4x^2$. Total area, $A = x^2 + 4x^2$.

Due to pressure of time, Peter summarized the lesson by explaining task two procedurally, thus, to factorize a quadratic expression such as $x^2 + 5x + 6$, identify the factors of the constant term, 6, that sum to 5, the coefficient of x. Thereafter he told the students to factorize $x^2 + 3x + 2$ as homework, before administering the post-test. The results of the pre-test and the post-test are shown in Table 2.

Items	**Percentages of correct responses**	
	Diagnostic pre-test	Post-test
Question 1	3	30
Question 2	3	10
Question 3	70	43

Table 2: The percentage correct responses of the pre-test and post-test from the first lesson

First lesson's reflection session

Peter felt that his introduction took more time than he had expected: as he remarked,

Peter: I would have taught how to get the coefficient first before I look at this lesson.

Other highlights during the reflection include:

Dominic: The students had some fear I do not know what they were fearing [...]

John: I think we should have done some peer teaching. We forgot how the students would present their work and this became unexpected challenge.

Researcher: Peter gave students a long time for discussion because he wanted everybody to obtain the correct answer not knowing that the strength of learning at times is in the few mistakes made by students.

Based on the reflections, the post-test results and the fact that the lesson was not implemented as planned, the teachers modified the lesson, which was retaught in John's class (second lesson).

Second lesson

John introduced his lesson by asking students to expand the expression, $(x + 2)(x + 1)$, which they did to obtain $x^2 + 3x + 2$. He then asked them to identify the coefficients of x^2 and x from $x^2 + 3x + 2$. The majority answered the question correctly. John then asked the students to factorize the expression $x^2 + 5x + 6$. After about five minutes, he asked the students to form seven groups of four members each. The class had 28 students present. He distributed the pieces of paper shown in Figure 1 to each group and stated the tasks for the activity as: (1) Form a rectangle with all the pieces of paper given and work out the area of the rectangle formed. (2) Find the relationships between the constant terms of the factors of your worked-out area and i) the coefficient of x in the expression $x^2 + 5x + 6$, ii) the constant term in the same expression $x^2 + 5x + 6$. This was part of modification made on the lesson. After 15 minutes, he stopped the group work and asked some groups to discuss their work with the whole class. Table 3 shows how groups carried out the tasks, with same categories as in Table 1.

Category	No. of Group(s)	Rectangle	Area
One	3	Correct rectangle	Correct working
Two	1	Correct rectangle	Wrong working
Three	3	Correct rectangle	No area worked out

Table 3: The categories of groups from the second lesson and how they carried out the task

One group from category one and the category two group were asked to present their work. The representative of the category one group explained the working thus,

Student 1: Width = $1 + 1 + 1 + x = 3 + x$, Length $=1 + 1 + x = 2 + x$

Area, A = L × W = $(2 + x)(3 + x) = x^2 + 5x + 6$

The representative of category two explained her work referring to a figure similar to Figure 3.

Student 2: The two strips above are multiplied to obtain x^2 and the four pieces on one side (pointing at the width with 1 unit by x units strip and the three 1 unit by 1 unit) are counted and multiplied by x to obtain $4x$. Area, A = $x^2 + 4x$.

After the presentation, John discussed task two with the whole class. With the help of the students and referring to student 1's expression, John simultaneously presented the factors of 6, {(1 × 6), (2 × 3), (-1 × -6), (-2 × -3)} and the addends of 5, {(0 + 5), (1 + 4), (2 + 3) and so on}. He introduced the second activity that asked the students to factorize, $x^2 + 3x + 2$. The students correctly factorized the expression. The teacher summarized the lesson and administered post-test questionnaire. Both pre-post results are shown in Table 4.

Items	Percentages of correct responses	
	Diagnostic pre-test	Post-test
Question 1	26	86
Question 2	0	71
Question 3	48	86

Table 4: The percentage correct responses of the pre-test and post-test from the second lesson

The teachers prepared the lesson to apply the generalization of patterns of variation and invariance. This was realized fully in the second lesson.

Varied	Invariant	Discernment
$x^2 + 5x + 6$ $x^2 + 3x + 2$	Working out the area of rectangles formed.	Factorization of a quadratic expression with a unit coefficient of x^2 depends on the factors of the constant term that sum to the coefficient of x

Table 5: Generalization pattern of variation and invariance applied in the enacted object of learning

The two different expressions were varied to help the students to generalize the process of factorizing a quadratic expression with a unit coefficient of x^2 such as $x^2 + bx + c$. The students applied the cuttings to form the rectangles whose areas represented the factorizations of the given quadratic expressions, that is, $x^2 + 5x + 6$ and $x^2 + 3x + 2$.

Discussion and conclusion

The activities proved challenging as students took time to discuss and explore ways of factorizing the expression $x^2 + 5x + 6$ (Lester et al., 1994). At the end of the discussion, eight groups out of 15 did not factorize the expression, and one group even failed to form the rectangle, as shown in Table 1 and Table 3. Peter even allowed more discussion time but still some groups could not form the rectangles. Peter therefore, did not apply fully the patterns of variation and invariance. Marton's (2015) cautioned teachers to take control of students' own work during learning to implement the planned patterns of variation. The effect was reflected in post-test results of Table 2. A comparison of pre-post results (*lived objects of learning 1 and 2*) of the two lessons, Tables 2 and 4, show that at the pre-test, Peter's class had higher scores in questions 2 and 3 but at the post-test, John's class, where the pattern of variation was fully applied, had a notable improvement in all the questions than Peter's class. Question 2 that addressed the *critical feature* had a slight improvement of 7% in Peter's class compared with 71% in John's class.

The content of the lessons addressed the "what" aspect of *the object of learning*. Eight groups had difficulty expressing the sides of their rectangles in algebraic form, thus failing to factorize the expression $x^2 + 5x + 6$ as shown in Tables 1 and 3. The expected algebraic expressions for the sides of the rectangle fall within the topic of formation of algebraic expressions taught in the first year of high school according to the Kenyan mathematics syllabus for high schools. Also, in the introduction of the first lesson, the students could not identify the invisible 1 as the coefficient of x^2 from the expression $x^2 + 4x + 3$, which suggested that students either did not understand multiplicative identity property of real numbers or they did not know the term coefficient. These cases show that students could not recall what they had been taught earlier. These were learning moments for teachers to realize that the problems that students experience in quadratic expressions and equations originate from the introductory contents of the algebra.

The planning gaps identified during the lessons contributed to the time management issues especially, the first lesson where students did not work on the second activity as was planned; and they also had a short time to discuss the second task of the first activity. The effect was reflected in the post-test outcome from the first lesson's class as shown in Table 2. From these observations teachers learned that LS design is helpful in focusing the teachers in every aspect of the lesson that improves students learning. They also learned that a good implementation of a lesson by applying patterns of variation can improve students' learning as suggested in the second lesson's post-test result Table 4. The teachers appreciated the implementation of the second lesson where students were able to generalize the conditions for factorization of a quadratic expression with a unit coefficient of x^2.

Peter: Yeah the lesson was good. I am sure now they are aware that they can use that formula without the cuttings and factorize any quadratic expression.

Dominic: The fact that the two activities were discussed helped the students to see the relationship and I am sure they can now factorize the quadratic expression without any problem.

The teachers also learned the need for explicit preparation of all activities and for good time management, as they stated during the reflection session after the first lesson and confirmed the

same after the second lesson through the post-test result. This supports Hiebert, Morris and Glass's (2003) suggestion that lessons should be treated as experiments with explicit preparation of all activities. The teachers learned that the LS design through its post-test aspect helps monitor students' learning progress, which enables a teacher to address students' errors/misconceptions in the immediate subsequent lessons.

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Mathematics textbooks and teaching activity

Maryvonne Priolet[1] and Eric Mounier[2]

[1]University of Reims Champagne Ardenne, France, Maryvonne.priolet@univ-reims.fr

[2]University of Paris-Est Créteil, eric.mounier@u-pec.fr

This article focuses on the relations between the activity of the teachers and the contents of textbooks and the teacher's manuals. Through the observation of lessons, we analyse and discuss how the teachers follow the recommendations written by the authors of teacher's manuals. We describe the adjustments made by these teachers, comparing them to the recommendations written by the authors of teacher's manuals. The observations lead us to point out some didactic obstacles and to mention the major role of an epistemological and didactic teacher training.

Keywords: Mathematics textbooks, primary school, didactic, resources.

Introduction

The present study constitutes a part of a larger research project investigating the place of mathematical textbooks in the French publishing market and the role teachers assign them in the daily practice (Mounier & Priolet, 2015). In France, teachers may decide for themselves and in each of their classes whether they want to resort to textbooks or not, and which resources they wish to use, on the condition of respecting the national curricula. The resources and their uses have been already the object of plentiful scientific literature (Pepin, Gueudet, & Trouche, 2013; Fan, Zhu, & Miao, 2013; Matić & Gracin, 2015; Lenoir et al., 2001). This paper examines how two teachers interact with mathematics textbooks in teaching the same topic; it focuses on the use of number lines. The choice of this theme seems to us relevant, with regard to the works of Hamdan and Gunderson (2017, p. 587) that show how "the number line plays a causal role in children's fraction magnitude understanding, and is more beneficial than the widely used area model".

Theoretical framework and research question

The teaching activity

The teacher has "to prepare the course", "to handle the class" and "to teach the class" (Amigues, 2003, p. 11). His activity results from a compromise between his objectives, his own purposes, his constraints, and the resources of his work environment (Goigoux, 2007, p. 47). So, it exceeds the context of the classroom. Then, the teaching activity has to be considered outside and inside the classroom, but, in this article, we mainly reserve the expression "teaching activity" for the activity of teaching in the presence of pupils.

Resources, textbook and teacher's manual

Within the framework of his teaching activity, the teacher interacts with a set of resources. To define the concept of resource, Adler (2010, p. 25) refers to two meanings of a word: a "reserve" from which the teacher can draw, and the action "to be nourishing again". We define the resource as "product of the human activity, developed to join a finalized activity" (Rabardel, 1995, quoted in Gueudet & Trouche, 2010, p. 58). As produced by an author, the textbook and the teacher's manual are resources for the teacher. The resource "Textbook" is intended for the pupils in the class and it is in connection with the curricula. The "teacher's manual" is the documentation annexed to the textbook, intended

for the teacher and who allows "to understand better the transactions of the teachers with the curricular resources in mathematics" (Remillard, 2010, p. 201).

Teacher-resources relation

The teachers are differently positioned with regard to the use of the resources according to "modes of commitment" (Remillard, 2010, p. 214) being able to be shaped by particular expectations, convictions, habits or past experiences. This positioning may have an important effect when the use of the resource leads to the adoption of a didactic device structuring the session. The didactic devices in a primary class come down to three devices (Rey, 2001, pp. 31–35). In the first one named "explanation-application", some part of knowledge, for example the definition of a mathematical object, is presented to the pupil. Practical exercises follow the presentation of this knowledge. In the second named "observation-explanation-application", in the first instance the pupil is asked to observe an object, for example a geometrical figure then to generalise from this observation. Practical exercises are then proposed. In the third named "problem-explanation-application", in the first instance the pupil starts with the active manipulation of material or conceptual resources is brought in to the apply to problem-solving. This phase is followed by the shaping of the knowledge and then by the series of exercises. Some teachers sometimes make some adjustments. If the teacher's degree of expertise and the level of training are not sufficient, these adjustments can lead to "problems of coherence between objects of teachings, processes and activities" (Arditi, 2011, p. 361). Besides, from a generative document, expert teachers can proceed to relevant adjustments whereas, for lack of self-important training, the novice teachers are sometimes going to bring modifications going against the intentions of the authors of textbooks, the specialists of didactics (Margolinas & Wozniak, 2009; Priolet, 2014).

Research questions

Considering the above research, we question the relations between the activity of the teachers and the contents of mathematics textbooks and of teacher's manual. Do teacher-users of a medium operate different types of adjustments during the activity of teaching in the presence of their pupils, and do they follow the model led by the teacher's manual?

Methodology

In order to answer our research questions, we provide a qualitative approach based on observation of practices used by teachers or semi-structured interviews with them.

Participants

This case-study involves two female teachers, Teacher B and Teacher A. Both of them teach at the 4th level of elementary school (9–10 year-old pupils), in two schools located in two small towns in the centre of France. Teacher B has been teaching for 15 years and Teacher A for 10 years. None of them has studied higher education in mathematics. They both teach all school subjects.

Both teachers belong to a sample of 10 teachers of the 4th level of elementary school who declared using mathematics textbooks and being volunteers to participate in our research. We had chosen this level regarding the introduction of fractions and decimal numbers. The ten teachers had agreed to be observed, by one of the two researchers, in their class, during a lesson of mathematics concerning the numbers, then to be interviewed during a semi-directive interview. For this case-study, we chose

Teacher B and Teacher A among these 10 teachers for two reasons. Firstly, Teacher B and Teacher A have the same textbook[1] in their class. Secondly, when we observed them teaching in their classroom, both presented a lesson on the theme of fractions and decimal numbers.

Method

In the classroom of Teacher B, the observation lasted 57 minutes and the interview 36 minutes. In the classroom of Teacher A, the observation lasted 33 minutes and the interview 44 minutes. We did not film the learning sessions, but some photographs have been taken, related to the use of the textbook or other artefacts. An observation table has been assigned in two parts :

- The observation of the classroom with identification of the different moments of the learning session (total duration, duration of each phase), duration of the phases of use of the texbook by the pupils, identify the moments while the teacher uses her teacher's textbook.
- The database about pupil and teacher documentation.

Following this observation of sessions, a second data collection was made through semi-structured interviews. An interview guide was set up on these subjects: preparation of the observed session, place taken by the manual during the session, manual's choice, general use of the teaching and pupil's guide and during the session, and finally, teacher training. The interviews often relied on the photographs that we had taken during the session concerning the use of the manual by the teacher or by the pupils. They can be linked in a methodological way with the self-confrontation method (Theureau, 2010).

For the "fractions and decimal numbers" topic, the classroom manual has eight sessions numbered from five to twelve in its summary. The selected lessons for the analysis are lessons 8 (Teacher B) and 9 (Teacher A), because both of these lessons refer to the use of number line.

Both interviews were transcribed. All the data collected through observation and interviews have been analysed (Bardin, 2007) in order to extract those concerning the presence and frequency of use of the textbook and the resources used by the teacher for the conception of his teaching. The times of effective use of the textbook by the pupil have been converted in percentages of the total duration of the lesson.

Results

For each teaching activity, we present below a lesson in which we can spot the relation that each teacher has with the textbook and with the teacher's manual in her teaching activity.

Teacher B

Teacher B herself chose the textbook given to each of the pupils of her classroom. She reports using it frequently in class, mainly for the exercises. The Teacher's manual is present in the classroom. Teacher B reports that she doesn't use it because she has been disappointed by its general contents. She organizes the distribution of the lessons of the year herself.

[1] Whereas in our study (Mounier & Priolet, 2015), there are at least 23 different textbooks in France for this level of teaching.

Today, she proposes the following situation: she shows on the board a big number line she has prepared herself (Figure 1).

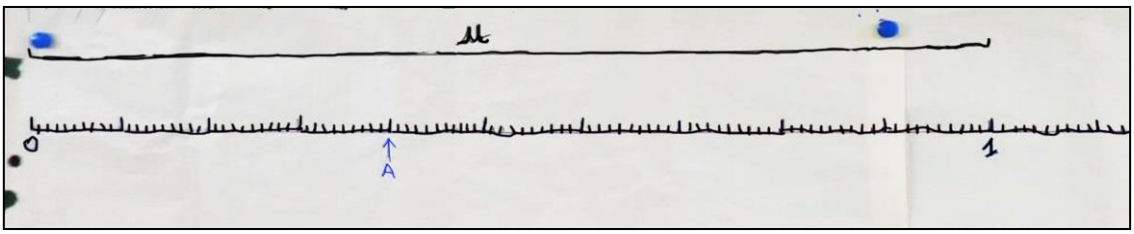

Figure 1: number line showed on the board (Teacher B)

In the first part of the lesson, she explains to the pupils how she made this number line: "the unit is here (u), so here between 0 and 1 there are 10 parts". She tells them that point A is equivalent to four-tenths of one. Then she asks pupils to write on their board the fractional numbers to which the points placed on the number line are associated. In the second part of the lesson, pupils open their textbook to do the 4th exercise (Figure 2).

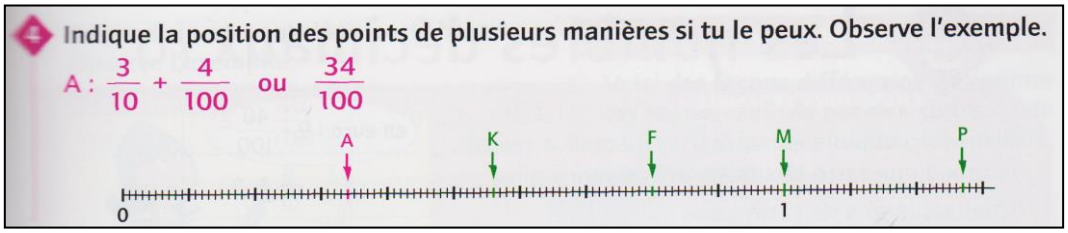

Figure 2: Exercise number 4, page 43 of the pupil's textbook

For this lesson related to session 8 of the manual (*À portée de maths CM1*) page 42–43 and named "decimal fractions", the teacher's manual first planned a research path with an individual preparatory report (Figure 3) to the activity "Let's look together" in the pupil's manual (Figure 3).

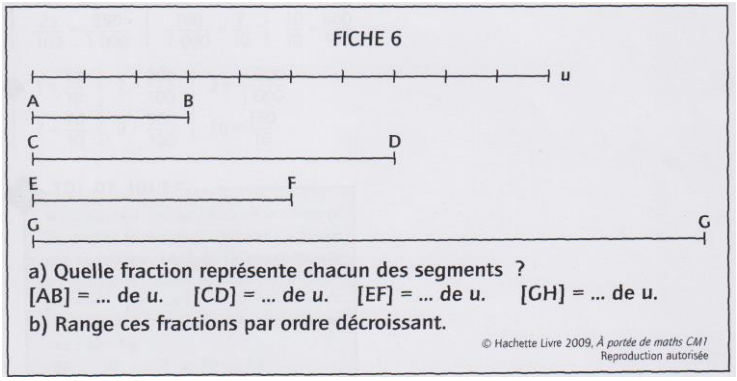

Figure 3: teacher's manual page 37

Teacher A

In Teacher A's classroom, each pupil has got a textbook. This textbook has been chosen by one of the colleagues predecessors of Teacher A in that school. She declares to have adopted this textbook which was already present in the class before she came. Teacher's manual is present in the classroom. Teacher A reports: "Mathematics is absolutely not my field. I refer a lot to the teacher's manual but

after this I try to appropriate it". She says that she follows the annual distribution of the lessons in the manual. She also uses the manual's exercises.

For this lesson related to session 9 of the manual (*À portée de maths CM1*) page 44–45 named "Decimal fractions", the teacher's manual first planned a research path with the number line drawn at the board (Figure 4) to prepare the activity "Let's look together" in the pupil's manual (Figure 4).

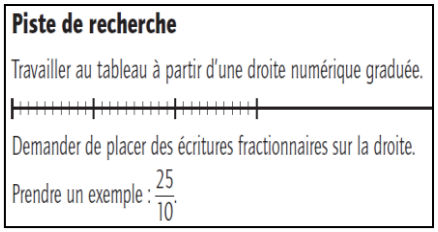

Figure 4: Teacher's manual p. 39 (Beginning of the research path)

Teacher A reports referring to the teacher's manual to build the "Let's look together". While the teacher's manual suggests as support for each exercise a number line increased without digital marks, this teacher writes a number line increased in tenth marks-units from 0 to 3, on the blackboard.

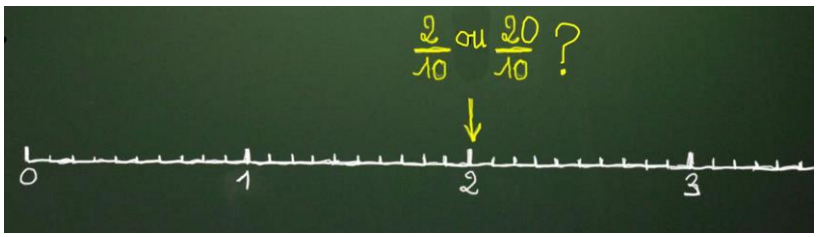

Figure 5: Photography of the board during the lesson (Teacher A)

Then she asks the pupils to indicate which fraction corresponds to such a graduation (yellow arrow). On the board, she writes two answers and $\frac{20}{10}$ (Figure 5) proposed by two pupils. She asks them to explain their process. Then Teacher A asks all the pupils to open their textbook to individual work on exercise number 2. In this second part of the lesson, and especially with this exercise number 2, the pupils have to use a number line (Figure 6).

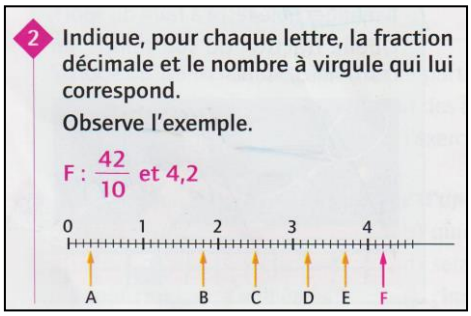

Figure 6: Exercise n°2 page 44 of the pupil's textbook

During the interviews, Teacher A and Teacher B report that they want to do the best to help their pupils to understand the fractions and the decimal numbers. So, Teacher A decided to write a number line increased in tenth marks-units on the blackboard instead of the number line increased without

digital marks which was suggested by the teacher's manual. Teacher B decided to explain to her children what each graduation means on the number line.

Analysis and discussion

We use Rey's model (2001) to analyse the didactic set up plan in each of these two classes. We compare it with the model underpinned by the instructions provided by the teacher's manual's research path.

Although the authors declare in the preface (page 3, pupil's textbook) that the "teacher is a professional that chooses and assumes his pedagogy" and in the preamble (page 3, teacher's guide) that "the guide is conceived in order to give the teacher the freedom of his own ways", the instructions which are supplied in the scenario of the teacher's manual about the research path of both consider lessons, seem to lead an approach of the "observation, explanation, application" type.

Our observations show that Teacher B operates the didactic device "explanation-application", whereas Teacher A tends to use the "observation-explanation-application" device. Teacher A modifies the starting situation support by converting the teacher's manual (number line increased without digital marks on a number line increased in tenth marks-units from 0 to 3).

This modification of the support does not favour the devolution (Brousseau, 1998) of this problem to the pupils. It has transformed, by reducing it, the difficulty of the task planned by the authors' textbook: to question on the density of decimal numbers which constitutes an epistemological obstacle to the pupils' understanding.

Although the authors of the textbook declared that teachers keep their pedagogic freedom, in both of the observed situations, both teachers do not commit the pupils in an approach of type "problem-explanation-application".

Conclusion

In order to analyse the relations between the teaching activity and the contents of mathematics textbooks and of teacher's manual, we have referred to the didactic model of Rey (2001). Our purpose was to detect the adjustments operated by two 4th level of elementary school teachers who use the same mathematics textbook. We observe that both do not follow all the recommendations of the authors of the teacher's manual. For example, the teachers redefine the task planned by the authors of the textbook, then changing the planned didactic device, from a model of "observation-explanation-application" led by the teacher's manual, into a model of "explanation-application" (Rey, 2001). This change may reduce, in a way, the pupils of the understanding of the density of the order of decimal numbers. Thus, our analysis reveals a problem of coherence, already pointed by Arditi (2011) between the adjustments operated by the teacher-user and the authors-designers of the textbook. This echoes the question of the validation of the collected knowledge (Bruillard, 2010), in particular in the context of the development of the recourse to the digital resources.

In conclusion, we notice that the logic of the teacher and the logic of the textbook cannot be the same. We observe that this gap can interfere with the aimed knowledge, from which we conclude in the necessity for the teacher to exercise an epistemological and didactic vigilance on pupils' understanding. It seems to us essential, following the example of Charles-Pézard (2010), to include this issue in the training of the teachers.

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Critical incidents as a structure promoting prospective secondary mathematics teachers' noticing

Giorgos Psycharis and Despina Potari

National & Kapodistrian University of Athens, Greece; gpsych@math.uoa.gr, dpotari@math.uoa.gr

This study builds on the idea of using "critical incidents" as a tool for inquiry and reflection in the context of mathematics teacher education. The analysis was based on 22 prospective teachers' portfolios reporting and interpreting selected critical incidents on the basis of their observations of mathematics teaching conducted by other teachers and by themselves in the context of their field experiences. The critical incidents addressed a multiplicity of issues related to mathematics teaching and learning. Prospective teachers' noticing developed in terms of what and how they notice indicating a more relational way of conceptualizing mathematics teaching and learning.

Keywords: Noticing, critical incidents, prospective mathematics teachers, teacher education.

Introduction

In this paper, we study prospective teachers' (PTs') noticing of mathematics teaching in their initial field experiences through their engagement in identifying and interpreting critical incidents taken from everyday classroom situations in the context of a teacher education course. Critical incidents are everyday classroom events which have significance for the teachers, make them question their practice and seem to provide an entry for their better understanding of teaching-learning situations (Hole & McEntee, 1999). To observe and question mathematics teaching is a rather demanding task for both practicing and prospective teachers. A number of research studies have indicated that PTs face difficulties in identifying salient aspects of classroom instruction. For instance, they tend to describe the lesson as a chronological order of disconnected events (Sherin & van Es, 2005), they keep their attention primarily on the teachers rather than on the learning students (e.g., Van Es & Sherin, 2002) and they have difficulties in developing interpretative analysis of classroom instruction (Jacobs, Lamb, & Philipp, 2010).

Research suggests the need for the development of structures that foster teachers' systematic reflection on teaching practice and help to make the act of noticing critical aspects of classroom interactions more concrete (Mason, 2002). Examples of such supportive structures are: the use of theoretical tools to code teaching (Mitchell & Marin, 2015); the decomposition of video lessons in small parts (McDuffie et al., 2014); the identification of critical incidents from classroom teaching (Goodell, 2006). These structures have been exploited in situations where prospective mathematics teachers analyze teaching of others mainly through video noticing (e.g., Sherin & van Es, 2005) while few studies refer to PTs' reflection on their own teaching (e.g., Goodell, 2006). However, there is an open discussion on if and how reflecting on other teachers' practice transfers for reflecting on PTs' own practice (Stockero, 2008). Many research studies prioritize helping PTs to focus on students' mathematical thinking (e.g., Jacobs et al., 2010) while few studies aim to facilitate PTs' attention to other important features of mathematics teaching and their interrelation to students' learning (McDuffie et al., 2014). Linking students' learning opportunities to teacher's

discourse moves is a rather demanding task and it poses a research challenge in the area of prospective mathematics teachers' noticing.

In our study, we attempt to explore how critical incidents can be used as a structure to support PTs in reflecting on mathematics teaching recognizing interrelationships between teaching and learning. Our research questions are: (a) What is the nature of critical incidents that PTs identify while reflecting on mathematics teaching conducted by other teachers and by themselves? (b) How does PTs' noticing develop when identifying and interpreting critical incidents related to students' mathematical activity?

Theoretical framework

Under a community of inquiry perspective, Jaworski (2006) introduced the concept of critical alignment, in which participants align with the practice of mathematics teaching while critically questioning aspects of it. Critical alignment is promoted through the tool of inquiry. Inquiry is a process of encouraging critical reflection and promoting critical alignment (Jaworski, 2006). In this perspective, reflection is considered as a tool that allows participants to be engaged in a continual reconstitution of the practice of teaching. The reflective process involves "firstly, a recognition of questions to address, identifying some perplexity, making some aspects of teaching problematic; and, secondly, through some processes of enquiry, to seek solutions, or resolutions to, or new ways of understanding, the problems identified" (Jaworski, 1998, p. 7). This perspective is close to our view of a critical incident as a continuum involving identification, interpretation and potential action where critical questioning is a constituent element of it.

Researchers have been concerned about the introduction of sufficient structures for making the act of inquiry into teaching practice more concrete. An example of a structured framework for reflection on classroom episodes, are critical incidents. In mathematics education, the idea of critical events/moments in mathematics teaching has been used as an analytical tool in studying mathematics teaching and learning. Skott (2001) used the term "critical incidents of practice" to describe moments of a teacher's decision-making in which multiple and possibly conflicting motives of his activity evolved that challenged the teacher's own school mathematics images and provided learning opportunities for students. As a developmental tool, critical incidents have been used by Goodell (2006) in pre-service mathematics teacher education. She analyzed PTs' reports of critical incidents and she found that the issues raised concerned: teaching and classroom management; student factors; issues concerning relationships with colleagues, parents and students; and school organizational issues. She also identified that PTs fruitfully addressed important aspects of teaching for understanding such as the necessary conditions, factors facilitating teaching for understanding and barriers to teaching for understanding.

Noticing has been introduced to mathematics teacher education to study shifts in the structure of teachers' attention and, through this, to address different levels of awareness both in mathematics and in mathematics teaching (Mason, 2002). According to van Es and Sherin (2002), noticing is a more complicated action than just observing teaching. Rather, it requires teachers to notice what is significant in a classroom interaction, to interpret this noteworthy incident on the basis of their knowledge and experiences, and to link these with broader principles of teaching and learning. Van es (2011) proposed a framework for learning to notice students' thinking constituted of four levels

of noticing according to "what teachers notice" and "how teachers notice." As regards to what the teachers attend to, the four levels include: making general observations about the whole class environment (Level 1 – Baseline Noticing); focusing on teacher pedagogy and begin to attend to students' thinking (Level 2 – Mixed Noticing); attending to particular students' mathematical thinking (Level 3 – Focused Noticing); and interrelating particular students' mathematical thinking and teachers' teaching strategies (Level 4 – Extended Noticing). When it comes to how the teachers notice and provide interpretations, the four levels include: providing general impressions and descriptive comments (Level 1 – Baseline Noticing); providing primarily evaluative with some interpretative comments and beginning to refer to specific events and interactions as evidence (Level 2– Mixed Noticing); providing interpretative comments, referring to specific events and interactions as evidence and elaborating on events and interactions (Level 3 – Focused Noticing); and making connections between events and principles of teaching and learning and suggesting alternative pedagogical actions (Level 4 – Extended Noticing). This framework provides a base for teacher reflection as well as a tool to describe the development of teachers' noticing. The above studies indicate that noticing critical aspects of mathematics teaching of others and prospective mathematics teachers' own teaching seems to constitute a basis for professional learning.

Methodology

The research took place in the context of a 14-week mathematics education undergraduate course (taught in one semester by the second author) included in a university program of a mathematics department leading to a first degree in mathematics. Enrolling in the course in which the study took place, PTs had already successfully passed at least four courses on pedagogy and mathematics education. The aim of the course was to engage PTs in critical consideration of aspects of mathematics teaching as they emerge from the complexity of teaching practice in schools. Every second week for the entire semester PTs were asked to participate in a number of field activities (over six field activities-weeks) while each week following the activities in schools included a three-hour meeting at the university. PTs' field activities consisted of observing other teachers' mathematics teaching for six hours in total (first three field activities-weeks), designing and teaching a lesson in one group of students outside the classroom for one teaching hour (fourth field activities-week), and designing and teaching lessons in the whole classroom for two teaching hours (fifth and sixth field activities-weeks). The 22 PTs (9 males, 13 females), who served as participants in this study, were divided into pairs and carried out collaboratively the field activities under the supervision of eight postgraduate students of mathematics education who acted as mentors.

Inquiry into mathematics teaching was a rather new practice for PTs and was supported through the discussions in the university meetings and the field activities. Critical alignment with the practice of the mathematics teaching in which they were engaged through observing and teaching, was expected to be developed through the process of inquiry and questioning aspects of practice. Critical incidents were expected to facilitate this process. PTs' field activities were based on the cycle observing-reflecting-designing-implementing-reflecting. For instance, PTs were asked to: identify the specific content of a lesson in the curriculum and to trace it throughout the different grades; look for possible research evidence related to potential students' difficulties; keep systematic notes from and/or record the lessons; reflecting on their classroom experiences; and analyzing lessons. In this context, PTs were asked to select critical incidents and provide a reflective account on the basis of

justifying their selection, interpreting them and proposing potential teaching actions. Instructional practice in the university sessions aimed to support PTs' reflection on their recent field experiences and to link emergent issues with existing mathematics education research in order to develop deeper levels of awareness. PTs were introduced to the idea of critical incidents through (a) a brief presentation of Goodell's (2006) study (including the meaning of critical incidents, the classification of them and examples from PTs' written reports), and (b) analysis of transcripts of lessons to identify critical incidents and discuss/justify in the class their criticality. The teacher educator facilitated the discussion, but also challenged the PTs to justify their selection of the critical events, to provide evidence of their claims, to make interpretations, and describe their potential teaching decisions. The PTs themselves presented the analysis of the critical incidents and their reflections in the university meetings. Overall, PTs' field activities and the discussions in the university meetings revolved around the idea of critical incidents and thus they were compatible with our research focus.

The data for this study consisted of: (a) PTs' personal portfolios including their written accounts of critical incidents, and material related to the design, implementation, and presentation of the field activities in the classroom (e.g., worksheets, lesson plans, presentation files); (b) video recordings of all meetings at the university (8 in total) and (c) researchers' field notes. In this paper we analyse the data from the PTs' portfolios. The analysis was carried out in three levels. In the first level, we adopted a grounded theory perspective (Charmaz, 2006) and indentified thematic areas indicating what the PTs noticed (first research question). In the second level, we analysed the critical incidents, their interpretation and the potential actions that PTs reported in their portfolios for each week's assignment in terms of the levels of van Es' (2011) framework. Finally, we traced PTs noticing over time looking for shifts in what they noticed in students' activity and how they interpreted it.

Results

The nature of critical incidents from PTs' portfolios

In Table 1, we present a categorization of the critical incidents that the PTs identified in their reports in two cases; one is while reflecting on the observations of other teachers' teaching and the second while reflecting on their own teaching. The total number of critical incidents in the first case was 72, while in the second 54. In both cases, the incidents reported most often were related to students' activity (35 out of 72 - 49% in the first case, and 21 out of 54 - 39% in the second) and in particular, to their conceptual difficulties. Another category of incidents focused on teaching - especially on the interaction between teacher and students (e.g., how the teacher responded to students' questions and answers). Thirty-three out of seventy-two (46%) incidents in the first case fell in this category and eighteen out of 54 (33%) in the second case. A third category appeared mainly when PTs reflected on their own teaching, concerned students' learning in relation to teaching (5% in the observations and 22% in the personal teaching). A fourth category that emerged only in the second case included three incidents focusing on epistemological issues.

Below, we present some illustrative examples of the above categories and we elaborate on the issues emerging from the analysis of the critical incidents in relation to our research goals. Focusing on students' activity, the PTs recognized misconceptions and difficulties in using mathematical language, performing procedures, connecting representations, and developing problem solving strategies. For example, the confusion between perimeter and area was noticed by one prospective

teacher, Marina, while observing a lesson in an eighth grade class: "The teacher asked the students to draw a triangle and then to name the sum of the sides. One student answered 'area' and another one 'perimeter.' The first one seemed to confuse area and perimeter". As regards to the unexpected students' responses, one prospective teacher, Leonidas, reported students' innovative approaches in finding triangular numbers in the Pascal triangle: "One student discovered a personal algorithm to calculate triangular numbers only by observing the arrangement of numbers in the Pascal triangle".

Incidents from classroom observation (72)		**Incidents from personal teaching (54)**	
Students' activity	**35 (49%)**	**Students' activity**	**21 (39%)**
Difficulties	29	Difficulties	18
Unexpected responses	5	Unexpected responses	3
Motivation	1	Motivation	0
Lesson planning and teaching	**33 (46%)**	**Lesson planning and teaching**	**18 (33%)**
Teacher-students interaction	19	Teacher-students interaction	10
Classroom norms	5	Classroom norms	3
Quality of tasks and mathematical content	8	Quality of tasks and mathematical content	2
Teaching versus planning	0	Teaching versus planning	2
Dynamic character of teaching	1	Dynamic character of teaching	1
Linking teaching and students' learning	**4 (5%)**	**Linking teaching and students' learning**	**12 (22%)**
Relating interaction and learning	0	Relating interaction and learning	8
Relating task and learning	3	Relating task and learning	3
Relating norms and learning	1	Relating norms and learning	1
Epistemological issues	**0 (0%)**	**Epistemological issues**	**3 (6%)**

Table 1: Categorization of the PTs' critical incidents

Concerning teaching and in particular teacher-student interaction, the PTs commented on positive and negative ways that the teacher or PT reacted to students' contributions. A positive example was when Vassilis noticed that the classroom teacher acknowledged different solution strategies and discussed those in the classroom. Stella referred to a negative example from her own teaching: "One student proposed to find the requested area through transformations, which is a good approach. However, I directed her to follow the approach described in the textbook". Stella also noticed the classroom norms and their effect on the mathematical communication: "Although the students provided repeatedly wrong answers, the teacher did not evaluate them and encourage further discussion". The quality of the tasks in relation to the mathematical content was related to the teacher's choices of the content, its integration in the designed tasks, and its transformation in the classroom teaching. Anthi reported: "I was impressed by the way that the teacher introduced students to the idea of limit in the context of geometry. … This experience can help students to get an intuitive sense of the idea of limit". By being involved in designing and teaching, PTs started to consider the complexity of teaching. In particular, they started to recognize the gap between planning and teaching and the dynamic character of teaching as it is indicated in the following example from Sofia's reflection: "Although I had designed a realistic problem with the aim of engaging students in making sense by themselves of the notion of circle, during the implementation, I ignored the design. Actually, I took a directive stance to secure that the task would lead the students to the expected conclusions".

Moreover, the PTs started to relate different aspects of teaching such as classroom norms, classroom interaction, and nature of tasks to students' learning. For example, Alexandros, recognized the mediation of digital tools in supporting students' understanding while reporting on his classroom observations: "I noticed a student who had difficulty realizing that the ratios in Thales' theorem remain constant independently of the position of the non parallel lines. She understood this property through dragging these lines in Sketchpad". Another example is about the relation between the presentation of a task and students' engagement. In an application of the Thales theorem, Leonidas noticed that the complexity of a geometrical figure in the task he designed posed barriers to students' participation: "Students' participation dropped vertically when they were asked to discern ratios of segments in the shape. So, the weak students could not consider at all even simple questions such as 'Show me a line that intersects the parallels'".

Finally, in the category "epistemological issues" we include critical incidents that refer to the nature of mathematical content from an epistemological point of view. For example, Anna noticed in her teaching that some students did not verify the validity of their findings, a process that she considers important in mathematics: "I chose to discuss this incident because verification constitutes an important process in mathematics. However, students often are not engaged in this".

The growth of prospective teachers' noticing

Here, we use the van Es' (2011) framework to trace PTs' development of what and how they notice when observing teaching and reflecting on their own teaching. The analysis of the portfolios indicated that most PTs progressed to higher levels of the van Es' developmental trajectory where relations between teaching and learning were noticed and connections between events and principles of teaching and learning were made. Below, we illustrate this shift through a representative case of a PT (Katia).

Katia provided a written account of the critical incidents she selected as part of the course assignments involving observations and designing and teaching. During the observations, Katia offered general descriptions of the whole class environment and incidents related to students' difficulties. She shifted from a baseline noticing in her first two observations (level 1) to mixed noticing (level 2) in the third one both in what and how she notices. For example, in her written account based on the second observation she gave as a critical incident the students' lack of motivation to participate in the lesson due to the fact that some of them would not have been examined in mathematics in the university entry examinations. As regards how she notices the above critical incident, she provided descriptive and evaluative comments considering teaching independent of students' behavior. In reflecting on her potential teaching actions, she mentioned that she would insist on inviting students to pay attention. In her account based on the third observation, Katia focused on students' difficulty to transform the formula of the area of a trapezium $E = (B+b)\times h/2$ to an equivalent expression in terms of another variable (e.g., the height h). This time she provided evidence of this difficulty by specifying students' errors in algebraic manipulations. She also noticed that the teacher used numerical examples with the same structure to address these difficulties. Commenting on this critical incident, she wrote: "Although students do well with numbers and equations with one variable, they get confused when more variables are involved and they panic". It appears that Katia begins to notice students' thinking and refer to teacher-students interactions in the teacher's attempt to address students' difficulty. While she was

challenged by the teacher educator to look for further evidence to support and interpret her observation (by discussing with the classroom teacher and one student who demonstrated this difficulty after the lesson, and by reading a relevant research paper), she still confirms students' difficulty without offering an explanation.

Katia's noticing was further developed while reflecting on incidents selected from her own teaching. She started to attend to subtle aspects of tasks and the way they influence students' activity, to develop interpretations based on her classroom experiences and research readings and to deviate from her planning at contingency moments. Our analysis provides evidence that while reflecting on her own teaching she was able to consider teaching and learning in a relational way and to provide justified arguments and alternative pedagogical solutions reaching focused noticing (level 3) and extended noticing (level 4). The following example illustrates this finding. Katia designed a lesson for the teaching of area measurement in grade 7 by taking into account research findings on students' strategies on area measurement. Her main goal was to engage students in calculating the area of irregular figures by developing as a main strategy the dissection of the shape in other shapes whose area could be calculated by the known formulas. The students were really engaged in the process and developed different strategies. Katia reported as a critical event the fact the use of the word "irregular" in the given worksheet raised a lot of questions in the classroom: "I did not expect that the word "irregular" would create questions and negotiations. However, I exploited to see how students think about these figures". In her analysis of the phenomenon, Katia refers to specific student's ideas about the meaning of the word "irregular" and how this influenced students' work.

Discussion

The critical incidents that PTs identified in their portfolios addressed a multiplicity of issues related to mathematics teaching and learning focusing mainly on students' activity and on student-teacher interaction. A similar picture was also formed in the study of Goodell (2006) where students' conceptual understanding and classroom interaction were the most dominant categories of the selected critical incidents. As regards the context in which the selected incidents emerged, there were not distinct differences in the nature of critical incidents that the PTs selected through their observations of other teachers' teaching and of their own. At the level of classroom management, the PTs found it more difficult to focus on the teacher-student interaction in their own teaching than in other teachers' teaching. Nevertheless, when PTs reflected on their own teaching, they started to see more clearly the impact of teaching on students' learning. One possible explanation could be that PTs' engagement in analyzing other teachers' teaching provided them a reflective stance towards their own teaching. A similar finding has been reported by Stockero (2008) who identified that PTs' experiences in analyzing video lessons of other teachers can enhance deeper levels of reflection on their own teaching. Tracing PTs' critical incidents, their interpretations and suggested teaching actions indicated shifts in their ways of noticing. Most PTs reached levels 3 and/or 4 of the Van Es' framework (2011) in terms of what and how they notice realizing interrelationships between teaching and learning. This finding adds to existing research on developing structures in teacher education facilitating PTs' noticing and enriches discussions that have taken place in previous CERME conferences (e.g., Potari et al., 2011). Integrating selection and reflection on critical incidents in teacher education provides a structured way that helps PTs to become aware of significant classroom interactions and to develop a critical way of addressing them.

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Exploring how 'instructional products' from a theory-informed Lesson Study can be shared and enhance student learning

Ulla Runesson Kempe, Anna Lövström and Björn Hellqvist

School of Education and Communication, Jönköping University, Sweden; ulla.runesson@ju.se.

In this paper, we present how experiences gained from a theory informed Lesson study in regard to a specific learning goal can be shared and used by other teachers in new contexts. A group of teachers worked together in a cyclic, iterative process of planning, evaluating and revising teaching. The aim was to provide possibilities for grade 2 and 3 students to become familiar with negative numbers. The teacher group draw the conclusion that the pupils needed to be able to differentiate some aspects of negative numbers. The conjecture was put to the test in a follow-up study with five new teachers and eight classes. One lesson was taught based on the empirical finding in the Lesson study. When learning gains from pre- to post-test in these classes were compared to those in the Lesson study, similarities were found.

Keywords: Lesson study, sharing instructional products, negative numbers, variation theory.

Introduction

Morris and Hiebert (2011) have presented Lesson study as a model for transforming teachers' craft knowledge into professional knowledge i.e. making it public, sharable, storable and verified as well as improved and should be organized around public changeable knowledge products. Their arguments are based on the necessity to reduce differences in classroom instruction (Morris & Hiebert, 2011, p. 5). The aim of this paper is to illustrate and discuss what an instructional product, generated on the basis of a pedagogical theory and empirically grounded, could look like and whether making use of such a product can be productive to enhance student learning. We report on how a theory-framed version of lesson study – Learning study – can produce 'instructional products' useful outside the specific context. Insights gained from one Learning study (LrS) about how to enhance the learning of negative numbers were communicated and used by new teachers in new contexts.

Lesson Study not just professional development

Stigler and Hiebert (1999) pointed out the effectiveness of Japanese Lesson study (LS) model for improving teaching and learning of mathematics. There are extensive reports on the effectiveness of Lesson study for teachers' improvement of teaching skills, how they learn to reflect, on changes in motivation and capacity to improve instruction and the development of content and pedagogical content knowledge (e.g. Lewis, Perry & Hurd, 2009). Furthermore, it is often pointed out that Lesson Study can promote the establishment of learning communities and teacher collaboration, a culture of mutual accountability, shared goals for instruction and a common language for analyzing instruction (e.g. Chichibu & Kihara 2013; Hunter & Back, 2011; Toshiya & Toshiyuki, 2013). To us, with these purposes, Lesson study will be restricted to a model for professional development only, not as a system that can generate new and relevant knowledge recognized as a legitimate knowledge source for professionals.

Hiebert and Morris (2011) take Lesson study further when they promote it as a system for "the creation of shared instructional products that guide classroom teaching" (p. 5). 'Instructional

products' should be designed with a specific learning goal in focus and detailed enough to guide classroom instruction. An instructional product is the current answer to common and shared problems on teaching and learning. It is tentative, changeable and thereby open to improvement. Therefore, such 'local theories' embedded in the instructional product must be communicated, shared and improved by other teachers in other contexts. In this way, they could be tested and verified under new and local conditions.

Learning Study

Learning study is a theory-informed version of Lesson study (Marton & Pang, 2003). It shares features with Lesson study, such as the collaboration among teachers and the iterative design of planning, implementing, observing and revising of the lesson, but it is framed by a theory of learning —variation theory (Marton, 2015). Just as with Lesson study, there are reports on the positive effects of Learning study on teachers' professional development (e.g. Lo, Chik & Pang, 2006). Learning study is also a model for constructing knowledge concerning the objects of learning as well as the teaching-learning relationship. It takes the professional task as the point of departure and generates public and sharable knowledge for the improvement of teaching and learning, exactly in line with what Hiebert and Morris (2011) advocate.

The knowledge produced in Learning study is an instructional product, not in terms of a lesson plan or specific teaching methods, but in terms of what is found to be necessary to learn in order to develop a specific understanding, skill or attitude. It is not about learning in a general sense, but in relation to specific learning goals (cf. Hiebert & Morris, 2011). In Learning study, variation theory serves as a tool for teachers to identify the necessary conditions of learning the object of learning. Learning, from this theoretical perspective, is seen as a change in one's way of experiencing something. How we experience something has to do with what aspects we notice and become aware of. For every object of learning there are certain critical aspects necessary to discern.

'Critical aspects' are dimensions of variation in the object of learning that the learner has not yet learned to discern and attend to. It has been suggested, however, that the critical aspects must be identified for every group of learners (Pang & Ki, 2016). Variation theory takes a relational perspective on learning, meaning that the critical aspects are not merely a feature of the content (a concept for instance), but a feature of the experienced object of learning. They cannot be derived at "from disciplinary knowledge alone or as taken-for-granted truths" (p. 333) Learners bring various experiences to the classroom and experience phenomena in different ways. Therefore, to identify the critical aspects, learners' ways of experiencing must be taken into account. In Learning study, this is done by carefully diagnosing—via interviews and/or written tests—before and after the lesson. So, 'critical aspects' should be defined in relation to the phenomenon in question as experienced by learners rather than in relation to what is deemed critical in the curriculum or subject discipline (Pang & Ki, 2016, p. 328)

Marton (2015) asserts that one cannot become aware of new concepts or aspects without becoming aware of differences (i.e. variation). Variation theory is used when the teachers explore students' prior understanding and to what extent the object of learning has been achieved by the learners after instruction. The exploration of teaching and learning in the Learning study entails identifying what aspects of the object of learning that are critical for learning and how to make it possible for the

learners to experience them. When planning the lesson, variation is used for creating problems, example spaces and choosing representations for example.

Teaching and learning negative numbers—some recommendations

Gaining understanding of the nature of negative numbers has been problematic for early mathematicians to comprehend (Bishop et al., 2014), as well as for teachers to teach and learners to learn (e.g. Ball, 1993). The difficulties have to do with the meaning of the numerical system and the magnitude and direction of the number, the meaning of arithmetic operations, and the meaning of the minus sign (Altiparmak, & Özdoğan, 2010). For Swedish students the meaning of the minus sign is probably particularly difficult, since in Swedish a number like –2 (in English: negative two) is pronounced as 'minus två' (minus two) and written –2. Thus, there is no linguistic and symbolic difference between the minus as a sign for the operation and as a sign for the number.

It has been recommended that teaching of negative numbers should take the point of departure in real life problems or situations known from the children's experience and transformed into mathematical models. For instance, using 'a house' with floors above and below the ground floor, or a bird flying/diving above/below sea level, has been suggested (Ball, 1993). Usually in the Swedish mathematics curriculum negative numbers are contextualized within discussions about temperature below and above zero and with the help of the thermometer. However, there might be a risk with this. The number system and the ordering of integers might not be visible when negative numbers are talked about as 'minus-degrees' (in Swedish: 'minus-grader'). Every child probably knows that it is colder when the temperature is –10 degrees C compared to a temperature of 3 degrees C. This may be confusing when they have to learn that –10 is a smaller number than 3. This was also found initially in the Learning study reported here. So, the teachers decided to use the number line only and talk about the numbers within a mathematical context instead of referring to temperature or depths.

A Learning Study on expanding students' number range from N->Z

In the Learning study (LrS) one of the authors of this paper worked in collaboration with two primary school teachers and 64 students in four different classes in grade 2 and 3 (8–9 years old) in Sweden. The teachers wanted to extend the students' experience of numbers to include the negative numbers also. In doing so, they explored what the students must learn—thus finding the critical aspects—in order to be familiar with integers and how to teach this in a way that would enhance the students' learning.

The LrS encompassed four cycles, that is, four lessons were taught with four different classes. A diagnostic pre- and post-test was given to the students. Results from this, together with a close analysis of the recordings of the lesson, gave insights into what is critical for learning and how the content must be handled to promote learning. Thus, when the learners failed to learn that which was targeted, they had to go deeply into the lesson and inquire how the content was handled and whether it was made possible to learn that which was intended. This analysis became the basis for the planning of the following lesson in the cycle, which was taught by a new teacher, and to *new students*, and again the recorded lesson and the diagnostic post-test are analyzed. The iteration proceeded until all classes were taught. Hypotheses about the critical aspects were tested in class. So, the critical aspects emerged as a result of trying them out in class and carefully analyzing students' learning outcomes and what was made possible to learn in the lesson. When it was found that the learning outcomes

were not as expected, the teachers had to consider the possibilities for learning during the lesson and, by being guided by variation theory, discuss learning in terms of discernment. As the process continued the critical aspects became more specified; from something to be discerned, to something that should be differentiated, namely:

- To differentiate the value of two negative numbers
- To differentiate the function of the minuend versus the function of the subtrahend in a subtraction
- To differentiate the minus sign for negative numbers versus the minus sign for subtraction

To get the students to discern the critical aspects, carefully constructed examples, based on the idea of variation/in-variance were used. So, for instance, the examples $3-2=$ and $2-3=$ ('3' varies; minuend/subtrahend) were contrasted as operations on the number line. The choice and character of the example space (Watson & Mason, 2006) was changed and developed during the process. It was not until lesson 4 that examples like $2-4=$ and $-2-4=$ were implemented in the lesson, for example. Since the results on the post-test after lesson/class 4 were significantly better compared to the previous lessons, it was concluded that the examples chosen and how they were sequenced seemed to be important for the possibility to discern the critical aspects.

Putting the conjecture to the test: The follow up study

Lövström (2015) concludes that when the critical aspect was phrased in terms of differentiation, that is what things could be compared, it indicated not just what dimension that must be opened up, but also what values in that dimension that were critical and needed to be contrasted (two or more negative numbers). Thus, critical aspects in terms of differentiation highlight a specific subject matter and students' experience of the content, and furthermore, provide directions for handling the content.

To put the conjectures of the critical aspects identified in the LrS to the test, a follow-up study (FS) with eight classes of new learners (N=116) and five (partly) new teachers were conducted. All the teachers had more than 15 years of teaching experience. All but one were primary school teachers that were not specialized in mathematics. Three of the teachers taught two classes each. One of them had participated in the LrS and is one of the authors of this paper. All except one were, to a varying extent, familiar with variation theory. The teachers were selected on the basis of previous interest in Learning study and variation theory and asked to teach one lesson (three of the teachers in two different classes) about negative numbers. One of the classes was grade 7, a group of learners with difficulties in mathematics; all the other were grade 2 and 3. Swedish was the first language for the majority of the students, but several other languages were represented in all classes. The guardians had given their written consents to student participation. The students were given a test (with a few exceptions identical to the test in LrS) before and after the lesson.

The FS was planned in a 3-hour meeting with the teachers and two of the authors of this paper. Results from the pre-tests in the eight classes were presented and discussed and it was found that the 'new' group of students had similar problems to the students in LrS. So, the critical aspects identified in LrS were assumed to be valid for the new group of students also. The results from LrS were presented to the teachers and the identified critical aspects were described and discussed. A video-recording of lesson 4 was observed by the group. Some sections were repeatedly paid attention to. It was specially observed and discussed in detail how the number line was used in the lesson. The aim was to conduct

the eight lessons as similarly as possible in terms of how the critical aspects were handled. Similar, it was important that all the examples presented and discussed in lesson 4, were present in all the 'following up-lessons' just as the usage of the number line. Except for these requirements, the teachers were free to arrange the lesson in their own fashion—to choose group- or individual work, for example.

The FS did not have the same iterative design as the LrS. It was conducted in parallel during the same week. The lesson was conducted mainly in whole class, intersected with individual and/or group-work. The interaction was more of a discussion between the teachers and the students with probing questions around the examples presented on the board. The examples used opened up dimensions of variation and were designed to make the critical aspects possible to discern. The teacher drew the learners' attention to differences in the midst of similarities and the students were required to justify their answers, sometimes after discussion in peers/groups. The lessons lasted about 60 minutes. In our experience, in Swedish schools it is uncommon that such a long period of time is allocated to whole class teaching of mathematics among younger students. Still, the students seemed to remain concentrated and focused.

The data consists of video-recordings of eight lessons, and results on four tasks (1, 3, 4 and 9) in the pre- and post-tests. Here, only results from pre- and post-test are drawn on. In task 1 (a–e), students should identify the biggest of five numbers. In a) all were positive numbers, b)-c) negative and positive numbers and zero, and e) negative numbers only. The object of learning was not preliminary to operate with negative numbers, but in order to test if the students were able to experience that there are numbers , operations with negative difference were chosen. So, task 3 (8 items) involved subtractions with positive or negative difference. The subtrahend and the minuend were positive numbers except in g) where the subtrahend was negative. Similarly, task 4 was a subtraction with negative difference. Here the students should also give a justification of their answer. Task 9 was about the difference of the meaning of the minus sign. The test comprised another four tasks not accounted for here.

Some preliminary results based on a measurement of correct answers on the pre- and post-test are presented here. Results on the pre-test were compared to the post-test on each task and on a group level. A comparison between the LrS-group and the FS-group was also made. Preliminary results are presented in this paper.

Results

Preliminary results from the analysis of two tasks for all the groups are presented in Table 1 and 2.

Item	a	b	c	d	e
Pre-test	113(97)	84(72)	104(90)	65(56)	34(29)
Post-test	112(96)	102(87)	112(97)	97(84)	85(73)

N=116

Table 1: Numbers (percentage) of students who answered correctly on task 1, ordering of numbers

In task 1 there were learning gains in terms of numbers and percentage of students who displayed the targeted experience of integers on all except one item. As can be seen from Table 1, the frequency of

correct answers was higher on all items on the post-test, expect for a) (which had a high rate from the beginning). The highest increase is on d): from 56% to 84% and e): from 29% to 73% who answered correctly. Item d (negative numbers and zero) and particularly e) (negative numbers only) were initially more difficult than the others (lower scores on the pre-test compared to the others). Although significant progress was made, item d) and e) have lower scores on post-test compared to a-c. There were still students (15–26 %) who did not manage to find the biggest number among negative numbers or negative numbers and zero after the lesson.

The analysis of task 3 and 4 (Subtraction pos./neg. difference) suggests improvement on all items except a) c) e) and f). These subtractions (positive difference) are well known to the students, but their encounter with subtractions with negative numbers might have confused some students.

Item	a	b*	c	d*	e	f	g*	h*	4
Pre-test	113(97)	15(13)	105(90)	19(16)	115(99)	113(97)	11(9)	17(15)	17(15)
Post-test	96(83)	57(49)	90(78)	62(53)	112(96)	110(95)	51(44)	52(45)	61(52)

N=116 *negative difference

Table 2: Numbers (percentage) of students who answered correctly in task 3 and 4

The frequency of correct answers on the subtractions with negative difference is particularly interesting. Item b), d), g) and h) show a similar result. About half of the students could solve these correctly after being taught just one lesson. Before they were taught, the average frequency of correct answers on these items on the pre-test was slightly more than 10 % (13.6). So, there was a significant improvement on the post-test.

Item g) $-2-2=?$ is perhaps the most interesting. This item had the lowest frequency of correct answers before the lesson. Only 9% managed this on the pre-test. 44% answered correctly on the post-test. To be able to solve $-2-2=?$, one must differentiate the minus sign as an operation sign and the sign for negative numbers. Task 9 was designed to test whether the learners could understand the two meanings of the minus sign. On the pre-test, 8 students (7%) could tell the difference between the minus sign in the operations $4-7=\mathbf{-3}$ and $4-3=\mathbf{1}$ respectively. After the lesson, 42% answered correctly.

Conclusions and discussion

What has been reported here is not a description of the 'best' lesson design or an answer about to how to teach negative numbers. It is a theoretically and empirically grounded description of some necessary conditions for learning about negative numbers among young students, generated by a group of professionals. Our interpretation of the analysis so far is that, *the simultaneous differentiation of the value of two negative numbers, the differentiation of the function of the minuend and subtrahend, together with differentiating the meaning of the minus sign*, seem to be necessary conditions for learning about the nature of negative numbers. Although learning was improved, still there was a fairly large group of students who seemed not to have learned that which was targeted. So, the 'instructional product' is open for development and improvement.

As was described above, finding the critical aspects is a transactional process comprising the learners, their learning (what they learn), what is targeted, or, using Dewey and Bentley's (1949) description: a transaction of the known, the knowing and the known. This was demonstrated in the reported LrS; what was found to be critical emerged as the teachers got deeper understanding of how the learners responded to instruction, what was made possible (and not possible) to learn in the lesson in relation to the targeted 'known'. The object of learning, in terms of what is critical for learning, is constituted in a transactional and continuing process in LrS. The instructional products produced in LrS are hypotheses of what is needed to learn, that can and must be tested and developed to deepen the understanding of teaching and learning. The object of learning and its critical aspects are dynamic and emergent, and this study supports this proposal. In this study, there are most likely things that have been taken for granted or even neglected that might be critical.

Hiebert and Morris (2011) call for a need to accumulate evidence about what works and what does not across different classroom settings (p. 5). Our analysis suggests that the results for the FS-group on the post-tests reflect the results for the LrS-group after the lesson. Our study supports finding from Kullberg's study (2012); when critical aspects generated in LrS become visible as dimensions of variation in new settings, similar learning outcomes are gained. This further suggests and points to possibilities that the development of effective ways of teaching could be shared among professionals.

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A case study of a first-grade teacher's quality of implementation of mathematical tasks

Vuslat Şeker[1] and Engin Ader[2]

[1]Middle East Technical University, Faculty of Education, Ankara, Turkey; vuslatseker@gmail.com

[2]Boğaziçi University, Faculty of Education, İstanbul, Turkey; ader@boun.edu.tr

The aim of this study is to investigate changes regarding a first-grade teacher's quality of implementation of mathematical tasks within the scope of a professional development (PD) program. The quality of implementation of mathematical tasks was analyzed through the instructional quality framework developed by Stein and Kaufman (2010) including maintaining cognitive demand, attending to students' thinking and the intellectual authority in mathematical reasoning. This is a qualitative case study focusing on the teaching of a first-grade teacher with 40-year experience. 13 lessons were video-recorded and weekly meetings were audio-recorded. All data were transcribed for content analysis. The results indicated that there was a slightly positive change in quality of implementation of tasks throughout the PD program.

Keywords: Mathematical tasks, cognitive demand (CD) of a task, intellectual authority, student thinking, quality of implementation.

Introduction

Mathematical tasks are the units of instruction that create an environment for students to work mathematically. To explore mathematics instruction and potential changes in it, an investigation of teachers' selection and implementation of mathematical tasks is necessary (Hsu, 2013). Stein et. al. (2000) characterized use of a mathematical task through four successive phases in Mathematical Task Framework (MTF); (1) task as appears in curricular materials, (2) task as set up in the classroom, (3) task as enacted in the classroom and (4) student learning. Teachers select or develop, present and implement mathematical tasks in their instructional processes.

Stein and Kaufman (2010) defined the high-quality mathematics lessons through the maintenance of high level cognitive demands (CDs), attending to student thinking and vesting intellectual authority in mathematical reasoning. The first is to maintain a high level CD through the steps of MTF. Doyle (1988) described cognitive demand of a task as the cognitive processes necessary for successful completion of the task. If mathematical tasks require high CD level, these tasks need to foster students' high level cognitive processes such as reasoning about the mathematics concepts involved, problem solving, making justifications, making sense of representations etc. Maintaining high CD of a task would require sustaining a focus on such processes through the steps of MTF.

Based on Doyle's (1988) work, Stein et al. (2000) divided mathematical tasks into two categories as high level CD and low level CD. Each category consists of two subcategories. Memorization tasks and procedures without connection to mathematical concepts tasks (P without C) belong to low level CD. The procedures with connection to mathematical concepts tasks (P with C) and doing mathematics tasks are high level CD (see Stein et. al., 2000 for a detailed explanation of these categories in Task Analysis Guide). Literature indicates that students show their best performance on reasoning and problem solving when tasks are maintained in high level CD (Stein et. al., 2000).

However, in most practices high level CD cannot be maintained and there is a decline in CD during set up and implementation of tasks (Tekkumru-Kisa & Stein, 2015).

Attending to students' thinking is about paying attention to what students tell regarding mathematics (Stein & Kaufman, 2010). The goal is to gather students' strategies, representations, understandings or thinking, and to use them in classroom discussion in order to support conceptual understanding. Vesting intellectual authority in mathematical reasoning emphasizes that the authority in the classroom needs to be in mathematics itself, especially in mathematical reasoning. Therefore, teachers are not acting as the judge to tell whether student responses are right or wrong (Stein & Kaufman, 2010).

Approaches to mathematics teaching have been going through changes in many countries in the last two decades. The emphases on problem-solving and reasoning for conceptual understanding, i.e. reflections of high levels of CD, are prominent in many countries' recently revised mathematics curricula and curricular materials. At times of change, teachers are the agents who enact curricular and pedagogical changes by arranging the learning environments according to the needs of students. However, teachers have difficulties while adapting new approaches in implementing new curriculum (Davis, 2003). Even though teachers are expected to implement tasks focusing on conceptual understanding, studies demonstrate that they continue to teach mathematics in a traditional way; focusing on practicing procedural skills (Hsu, 2013; Stein et. al.,2000). In Turkey, where there are similar changes in the curriculum, mathematical tasks in textbooks are aiming to reflect this focus (Ubuz et. al., 2010). Thus, teachers' implementation of mathematical tasks requires a close attention in this context.

In various international studies, the whole process in MTF has been investigated considering the change in curricular approach (e.g. Charamlambous, 2010; Stein & Kaufman, 2010). The studies mostly focus on how CD of a task is maintained through the phases of MTF (e.g. Charalambous, 2010). In the context of Turkey, studies related to investigating teachers' implementation of mathematical tasks are scarce (e.g. Ubuz & Sarpkaya, 2014). More specifically, according to our knowledge there has not been any study conducted considering all three components of instructional quality.

Due to demands of a new curriculum from teachers, we conducted a professional development (PD) program in a private primary school in Turkey. We aimed to investigate how the quality of teachers' implementation of mathematical tasks progressed through the PD program. The current paper reports a part of this project. The aim of this article is to examine the case of a first-grade classroom teacher's quality of implementation of mathematical tasks throughout a year. The research question is:

> How does a first-grade teacher's quality of implementation of mathematical tasks change during a professional development program?

Method

A qualitative case study approach was adopted, aiming for in-depth analysis of a particular complex situation in a realistic context. Neşe (pseudonym) was a first-grade teacher at a small private primary school in İstanbul and was one of the participants who joined the PD program. We chose the case of Neşe to further analyze in this study because of her experience in teaching and her initial resistance to changes regarding the quality of implementation of mathematical tasks.

The participant

Neşe had approximately 40 years of experience in teaching as a classroom teacher and had deeply rooted classroom practices. In her classroom, there were 16 students whose SES backgrounds varied from middle to high. At the beginning of the study, Neşe reported that she already taught in a way to foster high level cognitive processing. When she acknowledged that she could not follow the kind of teaching suggested by the quality of implementation of tasks framework, she highlighted that her first graders required detailed teacher directions of what needed to be done, tasks focusing on singular skills or knowledge at a time. She wanted to spoon-feed them since she believed the students needed experiencing success in tasks. She explicitly referred to her experience as a reason for her resistance to making changes in her practice. Neşe was a typical case of experienced teachers with deep-rooted beliefs and skeptical to new approaches (Ghaith & Yaghi, 1997).

Data collection

In the PD program, we adopted Borko's (2004) phase 1 teacher professional development research approach through the collaboration of teacher and researchers in one school. We, as two mathematics education researchers, aimed to create a community of learners where researchers and teachers discussed their ideas together. Classroom observations were done approximately twice in every month. While observing the classrooms, we were nonparticipant observers who took field-notes and video-recorded the lesson. The video recordings were used for two purposes; (1) collecting research data and (2) supporting teacher reflection. We had weekly meetings with teachers to discuss their implementation of mathematical tasks based on the videos. The teachers watched their videos before the meetings and reflected on them. There were 23 meetings lasted for 40 minutes and were audio-recorded. Furthermore, mathematics lesson plans for the coming week were also discussed. We gave suggestions for lesson planning and teaching but made sure teachers made the final decisions. Besides, in order to explore MTF's first step, before observation, we wanted teachers to send us their plans for the lessons to be observed. 13 lesson observations were conducted in Neşe's classroom. 44 mathematical tasks were implemented in these lessons. However, not all tasks were present in the lesson plans Neşe provided.

Data analysis

For the quality of implementation of mathematical tasks, all videos were transcribed and coded using the Classroom Observation Coding Instrument (Stein & Kaufman, 2010). Based on the instrument, we coded intellectual authority in mathematical reasoning ranging from 0 to 2 and attending to students thinking ranging from 0 to 3. While coding, the enactment episodes of instructional tasks were used as the unit of analysis. For the maintenance of CD, we used the paths provided by Charalambous (2010) (see Figure 1). For interrater reliability, two mathematics education researchers coded 4 of the 13 lessons including 11 of the 44 tasks independently. Cohen's kappa was calculated to check agreement between raters for coding CD of tasks, CD of task set-up, CD of task enactment, student thinking and intellectual authority. Cohen's kappa values were κ =.784, κ=.694, κ =.792, κ =1.00, and κ =. 792 respectively, which shows a high level of agreement between raters. Beyond the provision of descriptive statistics, we will present key episodes from her teaching practice and comments in the meetings to document teacher resistance to change and teacher change in terms of her quality of implementation of tasks.

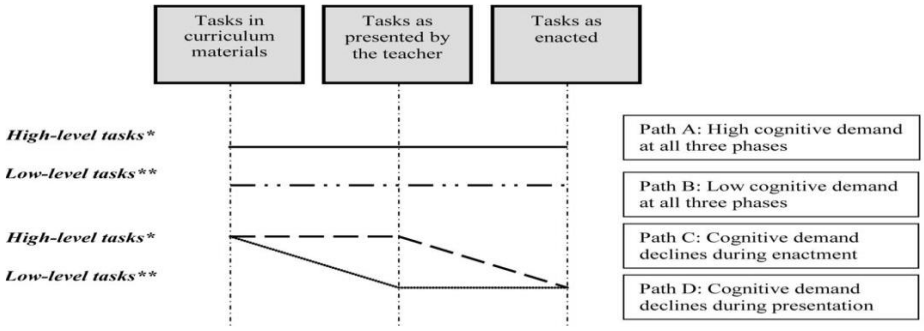

Figure 1: Paths for the maintenance of CD (Charalambous, 2010, p.258)

Results

Neşe used 44 tasks in total throughout 13 lessons. Sixteen of 44 tasks were not laid out explicitly in the lesson plans Neşe provided. Table 1 summarizes CDs of the tasks with respect to semesters. The table shows that about 61% of the tasks in the second semester were set up as cognitively demanding compared with only about 43% of the tasks in the first semester. In the first semester, about 57% of the tasks Neşe presented required low cognitive processes (i.e. recalling information, applying algorithms). Table 1 also indicates similar trends in enactment phase; Neşe implemented about 61% of the tasks at a high level in the semester 2 while she enacted about 33% of the tasks at this level in the semester 1. When the maintenance of CD was analyzed, it was observed that Neşe mainly maintained CD at its intended level for all phases for both semesters. While Neşe maintained about 59% of first semester's tasks at a low-level (Path B), she maintained 73% of second semester's tasks at a high level (Path A). For only two tasks throughout the whole year, she did not maintain cognitively challenging tasks; the decline occurred during enactment phase (Path C). Analyses showed that Neşe's choice of tasks determined the level of CD to be maintained.

	Semester 1						Semester 2					
CD levels of tasks	Planning		Set-up		Enactment		Planning		Set-up		Enactment	
	f	%	f	%	f	%	f	%	f	%	f	%
Memorization	3	14.3	3	14.3	3	14.3	0	0	0	0	0	0
P without C	7	33.3	9	42.9	11	52.4	3	13	9	39.1	9	39.1
P with C	6	28.6	7	33.3	7	33.3	6	26.1	11	47.8	11	47.8
Doing math	1	4.8	2	9.5	0	0	2	8.7	3	13	3	13
Not present	4	19	0	0	0	0	12	52.2	0	0	0	0

Table 1: Descriptive analysis of CD levels of tasks

Table 2 shows the limited work Neşe did to uncover student thinking in the first semester (i.e. she mostly asked for short or one-word answers). She did not connect students' responses in the discussion. However, there were slight differences in her use of tasks where she used students' answers to direct and connect the discussion on 8.7% of the tasks in semester 2. She demanded explanations from the students, called on certain students for directing the discussion to specific outcomes and connected the discussion for a fruitful experience for students as a classroom community.

	Semester 1		Semester 2	
Categories for attending to student thinking	f	%	f	%
(0) no attention to student thinking	6	28.6	6	26.1
(1) limited attention - some student explanation	11	52.4	10	43.5
(2) purposeful selection of responses, attention, but no connected discussion	4	19.0	5	21.7
(3) purposeful selection, attention and connected discussion	0	0	2	8.7

Table 2: Descriptive analysis of attending to student thinking

Table 3 shows that the nature of Neşe's practices on judging the correctness of students' work was slightly different in the second semester. She was in charge of deciding what was correct or not for most of the tasks in semester 1. Although she wanted students to prove or check the correctness via mathematical tools, she was the one confirming students' answers at the end. In semester 2, she continued with similar teaching practices; but she also experienced teaching episodes where mathematics was the tool students used to decide on the correctness.

	Semester 1		Semester 2	
Categories for intellectual authority	f	%	f	%
(0) judgments of correctness derived from teacher or text	11	52.4	7	30.4
(1) judgments of correctness sometimes derived from teacher or text, but also some appeals to mathematical reasoning	10	47.6	14	60.9
(2) judgments of correctness derived from mathematical reasoning	0	0	2	8.7

Table 3: Descriptive analysis of intellectual authority in mathematical reasoning

Illustrative episodes from Neşe's lessons

In this part, we will present two episodes from Neşe's quality of implementation of tasks. The first episode illustrates Neşe's common use of non-challenging, low level CD tasks throughout all phases of MTF. There were elements representing her resistance to changing her practice. The second episode presents how Neşe maintained a cognitively demanding task through letting her students struggle and encouraging their ideas to come out, practice emerging more prominently in the second semester. Neşe's comments from meetings about the tasks in the episodes gave insights about the nature of change in her practice.

Making 10. This episode is from a lesson on pairs of numbers that make 10, covered at the beginning of the PD program. In the plan of the lesson, Neşe stated the lesson goal as discovering the pairs of numbers that make 10. The task in the episode included nine possible pairs that would make 10. The students needed to find one of the pairs to make 10. The task and the coding decisions for this episode are presented in Table 4. While implementing the tasks, Neşe directed students to count the flowers as seen in the sample item. In doing so, she made available the unknown to the students. Neşe eliminated opportunities for students to explore the pairs that make 10 by focusing on the counting procedure and finding the answer. This led students not to use high cognitive processes, or think about the operation they were engaged in; counting the flowers was enough for the completion of the task. Neşe had similar trends in her implementations of the first semester's tasks.

The Task	Coding Decisions
How many more needed to make 10? Sample item: 2 + ☐ = 10  Sample episode: Neşe: Elif, there are 2 flowers. Which number should I add to make 10? I can count these flowers to find out. Let's count. Student:One, two, three, four, five, six, seven, eight. Eight. Neşe: Well done, this is it!	The task was coded as procedures without connections for task selection, set-up, and enactment. Neşe expected students to count the flowers to decide what would be the unknown of the pair that makes 10. Since the focus was on counting the flowers, students enacted the task by not relating with pairs of 10, but counting the flowers and writing the unknown number. During instruction, she did not attend student thinking; she asked for completion of the task. The judgments regarding correctness were derived from the teacher; she checked students' work constantly.

Table 4: Making 10 task and coding decisions of the making 10 task

In the follow-up meeting of the lesson, Neşe did not prefer to comment on the episode before we made any comments. We pointed out the discrepancies between the CD of Neşe's expectations from the students as reflected in her activity and the goal she noted down for this task. We discussed the importance of giving opportunities for higher-level thinking and mechanisms for shifting teaching towards this aim. After such comments from the researchers, Neşe wanted us to lower our expectations from her and emphasized her teaching habits by saying:

> Neşe: If I bring open-ended tasks to the classroom, the students could not complete the task. I need to use such repetitive activities for students to learn. I have been using teacher-centered approach for years. Do not expect me to improve my teaching. At most, two years later I will be retired from teaching. Do not try hard for me. Contribute to younger teachers (personal communication, November 26, 2014)

Yet we emphasized that we believed that there would be changes if she wished to work together. This extract shows that Neşe held on to her experiences in teaching and her expectations about student learning. Neşe was reluctant to change because she wanted to retire from teaching in two years. This reaction is an evidence of her resistance to change her implementation of tasks.

Subtraction Problem. This episode is from a lesson on problem solving using subtraction, covered towards the end of the PD program. The task and coding decisions are presented in Table 5. During the episode, Neşe expected students to analyze the problem and to explain their thinking by modeling the problem and writing mathematical sentences. Therefore, she maintained the complexity of the task by pressing for meaningful explanations so that the students realized the unnecessary information by asking, "Why do you think it is unnecessary information?", "Can you explain in more detail?", "Why don't I use money?" "Why is the result 11, not 14 or not 27?" In the post-lesson meeting, Neşe examined her lesson in detail before the researchers made any comments. She referred to the maintenance of CD in her comments. She had certain concerns about the set-up phase of the task after watching her practice.

The Task	Coding Decisions
Examine the following problem situation by modeling it. Write mathematical	Contextualized task was coded as procedures with connections for all phases of MTF. Applying general subtraction procedures were necessary with the need of

sentences and solve it. Explain your reasoning. **There are 6 roses in our vase. Dad brought 8 tulips too. Mom told that dad spent 10 ₺ for the tulips. Next day, mom realized 3 flowers faded. How many flowers are left in the vase?	conceptual connections to complete the task. Neşe expected students to connect multiple representations by asking for a model and mathematical sentences. She attended to student thinking by demanding justifications from students. She mostly directed students to the models for checking the correctness of their answers.

Table 5: Subtraction problem task and its coding decisions

In the meeting following classroom observation, Neşe stated that the task was too abstract and hard for the students. Then she pointed to getting students to struggle in order to construct meaning through abstraction as a necessary practice for learning:

> Neşe: It would not be easy for students, but it is good to present the task abstractly. It is challenging for them and for me too. I think I maintained the high level CD. I did not just expect them to apply the subtraction procedure. I wanted them to question the problem situation, and the unnecessary information within the problem. I wanted them to explain their thinking by using manipulatives. It took a long time, but it was necessary for students to experience high level cognitive processes. (personal communication, May 6, 2015).

This episode and Neşe's interpretations showed the importance she gave to supporting students' reasoning in her teaching practice. The focus on explanations and justifications helped the teacher implement the task with high quality. The post-lesson interview provided evidence for teacher's change in her emerging practice and comments during the PD program.

Discussion and conclusion

This study aimed to investigate the changes in a first-grade teacher's quality of implementation of mathematical tasks during a PD program. Neşe, our case, was one of the experienced teachers having difficulties with adapting educational innovations into their practices as illustrated in the literature (Ghaith & Yaghi, 1997). The PD program aimed to meet the needs of new approaches to mathematics education. The results indicated a slight positive difference in Neşe's practices between first semester and second semester based on maintenance of high levels of CD, attending to student thinking and intellectual authority. The teaching episodes showed that the PD program contributed to the teacher's approach towards implementing high quality of tasks that focus on problem-solving, reasoning and conceptual understanding (Stein & Kaufman, 2010). Neşe was resistant to change at the beginning of the PD program; her selection and implementation of tasks were low level CD in general (e.g. the making 10 task). However, high expectations from the researchers and the post-lesson interviews persistently focusing on the quality of implementation of tasks contributed to emerging changes in Neşe's teaching practice as well as the nature of her comments (e.g. the subtraction problem task). This study contributes to the existing body of literature on change of teachers' practices about the implementation of mathematical tasks within the context of a PD program. This study might inform future studies to explore facilitators' actions that lead to change in teacher practice. Further research might explore how change occurs in an experienced teacher practice to work with other experienced teachers. Especially in Turkey, there are a limited number of studies related to the classroom practices

of primary school teachers in a climate of change in curricular approaches (e.g. Ubuz & Sarpkaya, 2014). Results of this study provide information about mechanisms of change in context and contribute to the development of larger scale PD programs.

Additional information

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Long-term learning in mathematics teaching and problematizing daily practice

Helén Sterner

Linnaeus University, Växjö and Dalarna University, Falun, Sweden

hse@du.se

This paper stems from research on mathematics teachers' participation in a particular collaborative learning process that addresses the issue of mathematical communication and mathematical reasoning in relation to the teaching of algebra. Although results from the developmental research revealed changes in the working group's meaning making about mathematical communication and reasoning, whether these changes are long-term and influence the teachers' mathematics teaching over time remains unclear. The aim of this paper is to discuss possible theoretical frameworks and ways of understanding mathematics teachers' long-term learning about mathematical communication and reasoning by describing what they can learn in an organized community of practice (Wenger 1998) when working with key mathematical issues. I will use the data and results from the developmental research to design another study on long-term learning.

Keywords: Collaborative learning, long-term learning, mathematical communication, mathematical reasoning, mathematics teaching.

Introduction

Changing mathematics teaching is a complex process that requires the improved alignment of theory and practice (Sowder 2007). To that end research has failed to focus on answering questions about how mathematics teaching can change as a result of collaborative teacher learning projects (Sowder 2007). In this paper, I present an earlier study (Sterner 2015) as a background for a discussion of potential ways of conducting further research on understanding what a developmental research project can achieve in three years after its completion. The previous study addressed a school developmental project in mathematics in a middle sized community in Sweden. Figure 1 illustrates the background of the study and a possible direction for further research.

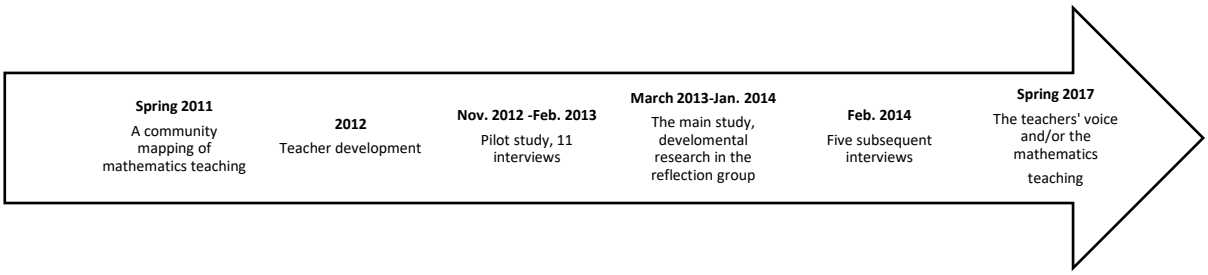

Figure 1: The study's chronological development.

The first section of this paper focusses on the learning process and results of a working group (i.e. the reflection group) that formed part of the developmental research study (Fig. 1). The second section comprises questions about possible ways to conduct further research three years after the study's completion. The bulk of research in the field of teacher learning and development has indicated the

failure of teachers to learn how to promote and support teaching and student learning (e.g. Borko 2004; Opfer & Pedder 2011; Sowder 2007). New issues emerged during and after the previous study, including ones addressing what happened in mathematics teaching after the completion of collaboration in the reflection group and how research on mathematics teaching can integrate the significance of context. Among other questions, whether we can listen to teachers' voices (Potari, Figeiras, Mosvold, Sakonidis & Skott 2015, p. 2972 - 2973), and what comes to mind when teachers listen to their own narratives three years after the completion of work in a reflection group are of particular importance (Fig. 1).

Background

As Figure 1 illustrates, results from a pilot study (November 2012 – February 2013) revealed teachers' difficulties with describing the concepts of mathematical communication and reasoning, as well as with applying those concepts in their teaching. Based on the results, the main study (Sterner 2015) was designed as a collaborative development initiative in a working group called *reflection group*. The author and five mathematics teachers in grades 1-6 collaborated on the key issue of mathematical communication and reasoning in relation to their teaching of algebra. Since the reflection group met monthly for a year the study can be characterised as *developmental research*, which Jaworski and Goodchild (2006) have defined as:

> Research which both studies the developmental process and, simultaneously, promotes development through engagement and questioning (p. 353).

The developmental work in this study addressed change achieved in an ongoing investigative process which occurred in parallel with the active creation of the participants' meaning making related to the key issue. However, as Goodchild (2008) pointed out, transformations through such dialectic cyclical processes of research and development are complex. In a literature review, Sowder (2007, p. 158) outlined 10 important issues facing mathematics teachers' development. Three of those issues were of specific importance to the study at hand and constituted the underlying questions addressed in the developmental research study:

1. How do teachers learn from their professional communities about teaching mathematics?
2. What can teachers learn from investigating their own teaching of mathematics?
3. What can be learned from research on teacher change?

In this paper, I discuss theoretical frameworks for understanding mathematics teachers' sustainable and long-term learning by describing what teachers can learn in an organised community of practice (Wenger 1998) addressing a key mathematical issue. The teachers in the reflection group wanted to develop their understanding of communication in mathematics teaching in order to stimulate mathematical reasoning in their own teaching of algebra.

Methodology

The study derived from a developmental project that adopted the perspective of collaborative learning among mathematics teachers and a researcher when designing tasks and environments to investigate

students' learning of mathematical reasoning related to algebra. A socio-cultural approach was adopted and the focus of the study was the learning process of the reflection group.

The theoretical perspective employed was that of communities of practice (Wenger 1998), in which learning is an aspect of participation in a social practice, whose participants engage in the negotiation of meaning (Wenger 1998). The theory of communities of practice (Wenger 1998) focusses on meaning making, participants' learning, and their reification of the key issue in a social context. Negotiating meaning is a central and dynamic process when teachers participate and reify. In that sense the reflection group's joint enterprise (Wenger 1998) was its members to understand more about communication and reasoning in their own mathematics teaching.

The process in the reflection group concentrated on two interacting parts: participation and reification (Wenger 1998). Framing the case as a community of practice shed light on the teachers' negotiation of meaning. At the same time the negotiation of their experience with teaching and learning related the key issue of mathematical communication and reasoning to their teaching of algebra.

The developmental research cycle method (Goodchild 2008) and the theory of communities of practice (Wenger 1998) together shaped the methodology of the study (Fig. 2). Developmental research and choice of methodology were intended to provide duality between a developmental and a research process over time and to enable a participant's perspective. Figure 2 illustrates my interpretation of Goodchild's (2008, p. 8) schematic figure of the developmental research cycle.

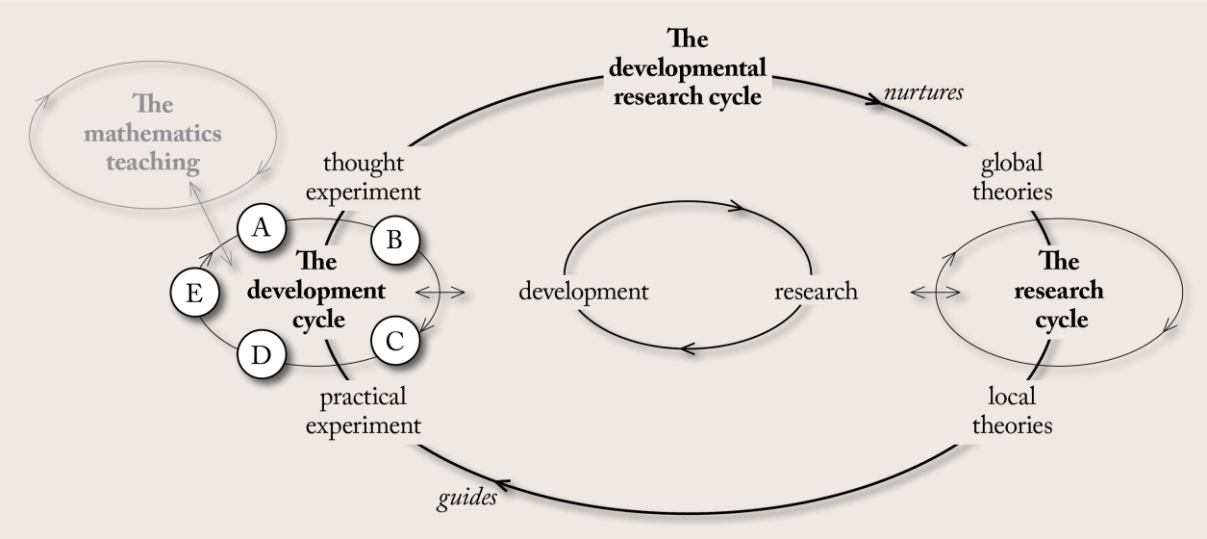

Figure 2: Interpretation of the developmental research cycle (Goodchild 2008, p. 208)

This study draws upon the idea that mathematics teachers' professional development should be based on their own classroom practice and students' learning (e.g., Broodie 2014; Goodchild 2014; Goodchild, Fuglestad & Jaworski 2013; Kazemi & Franke 2004; Matos, Powell & Sztajn 2009). The reflection group constituted a learning community that reflected on their teaching practices as well as on their students' mathematical communication and reasoning related to algebra. Working collaboratively, the mathematics teachers developed a shared repertoire (Wenger 1998) of the key issue, mathematical communication and reasoning in relation to their teaching of algebra. Developmental research represents a methodology based on interacting cycles of research and development (Goodchild 2008). As illustrated in Figure 2, the developmental research cycle constitutes the largest ellipse that spans the entire study since a cyclical process clearly exists between development and research. The ellipse on the left, representing the developmental cycle (A-E),

illustrates the work performed in and organisation of the reflection group (Sterner 2015). The developmental process appears as a cycle between a practical experiment and a thought experiment (Fig. 2). Every meeting of the reflection group were started at phase C (i.e. common reflections, challenges and questions and activities completed by the students when working on mathematical reasoning in algebra). This meetings were recorded. The ellipse on the furthest left, representing mathematics teaching (Fig. 2), illustrates the teachers' own practice in which they attempt to align and adjust common mathematical tasks and make individual reflections. Figure 3 illustrates systematic reflections in the process and the three levels of reflection in the reflection group.

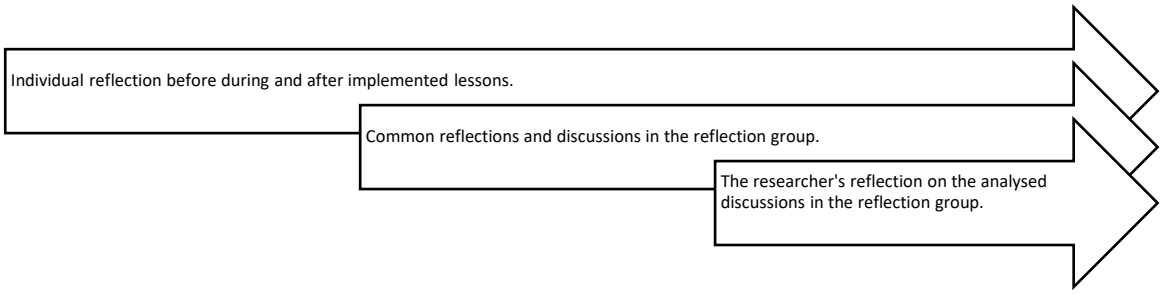

Figure 3: Systematic reflections among participants in the reflection group

The discussions in the reflection group provided empirical data that nurtured the research cycle. In Figure 2 the research process appears as a cyclical process between global and local theories. My interpretation of global theories (Goodchild 2008) is comparable to a theory-guided design research approach (Gravemeijer 1994; Gravemeijer & Cobb 2013) that in turn produces new theories (Gravemeijer 1994; Goodchild 2008). The research process guides the developmental cycle by means of local theories, which nurture the research cycle in the form of thought experiments and new questions. Reflecting together in the reflection group (phase C, Fig. 2) and the challenges of group members' own teaching resulted in problematizing questions.

Analysis and results

The analysis of the reflection group's discussions involved three steps. The first two continued throughout the developmental research process and constituted tools used for reflection in the reflection group (Fig. 3). The third step of analysis occurred following the completion of work in the reflection group. All three analyses were based on Wenger's (1998) concepts of participants' meaning making, reification and shared repertoire related to the key issue. The first two analyses and the preliminary results motivated the reflection group to negotiate their meanings of the key issue and wielded questions about what the group needs to discuss in terms of mathematical communication and reasoning. The reflection group returned to the preliminary results of analyses in order to identify further opportunities for development (Goodchild et al. 2013). As a participant researcher, I provided reflection to the group members with "findings of the research" and problematizing questions based on their own thoughts and questions.

During the ongoing data analysis from the reflection group discussions, new questions emerged when participants problematized daily mathematics teaching practices and became aware of new questions and challenges in their practice. The key principle in that process was reflection on three levels, as illustrated in Figure 3, since an essential component of developmental research is participants' interpretation (Kvale & Brinkman 2009). Wenger's (1998) modes of belonging (i.e engagement,

alignment and imagination) served as the participants' means for aligning and changing the discussions and activities of the reflection group. Those alignments and changes derived from participants' negotiation of meaning and reification of the key issue. The following dialogue from the initial analysis reveals that participants' shared repertoire concerns their frustration with failing to understand the meaning of reasoning in mathematics teaching.

> *Majken:* There is, generally speaking, no resistance among the students to conducting mathematical reasoning, but when we tell them to do so, they have no idea what it means.
>
> *Irma:* We need to provide them with tools that enable them to practise mathematical reasoning.
>
> *Majken:* But how can we do it, when we don't know the meaning of mathematical reasoning ourselves?

The dialogue led to consider textual content of mathematics as a science and in teaching from Lampert' (1990; 2001) and the National Council of Teachers of Mathematics (2008). Lampert (1990; 2001) described the science of mathematics as the formulation of assumptions followed by investigations to verify or refute them. When it comes to learning from a participant perspective, Lampert (2001) has outlined how she stimulated students' mathematical reasoning by encouraging them to make a mathematical assumption (conjecture) about, for example, a strategy or a solution. She also stressed the importance of advancing a plausible mathematical justification for the assumption that can be explored and verified. This dynamics exemplifies how reflection group members returned to the analysis and its results.

The textual content of strategies for mathematical reasoning in teaching suggested by from Lampert (1990; 2001) and the National Council of Teachers of Mathematics (2008) can be global theories transformed into local ones (Fig. 2). During discussions and activities participants aligned and developed local theories into a practical experiment (Fig. 2), which they sought to align for implementation in their own mathematics teaching. The teachers attempted to support their students in using the strategies for mathematical reasoning and to conceptualise mathematical reasoning as a cyclic process of exploration, conjecture and justification. The quote bellow is from the reflection group. The students have worked with equations and to concretize that x can have different values, the students used boxes with different amounts of beans.

> *Irma:* […] the strategies for "the reasoning cycle" (conjecture, justification and exploration) helped both me (in grade 4) to understand the students' mathematical thinking. The students worked with the equation $3x + 3 = 2x + 5$ and the students had to determine the value of x. I saw differences between students who made a wild guess and students who argued for their assumptions e.g. […] if we imagine that x represent the boxes with beans. In each box there is same number of beans, we don´t know the amount yet. We need to balance the left and right side… if we reduce the same amount of boxes ($2x$) from the both side of the equal sign, what will happen then?]

Irma gives a student example of an initial mathematical reasoning. Later on the reflection group discussed situations from their own mathematical teaching in terms of how and when mathematical reasoning occurred and interpreted why. In the reflection group the negotiation of meaning centered on teachers' awareness of stimulating students to "become involved in the reasoning cycle of"

exploration, conjecture and justification (Lampert 2001; the National Council of Teachers of Mathematics 2008). The teachers reflected on and interpreted their own teaching and used a thought experiment as a form of individual experience and reflection (Fig. 3). Ongoing analysis revealed how participants' discussions and shared repertoire about the key issue changed over time. As a participant researcher, my strategy was to focus on questions that arose in the reflection group and search for mathematical education theories that problematized the teachers' challenges and questions (Goodchild 2008) in thought experiments (Fig. 2).

Results and conclusions

I investigated how the reflection group developed their meaning making and shared repertoire related to mathematical communication and reasoning, which promoted a change in the members' ways of communicating about mathematics teaching in relation to students' mathematical communication and mathematical reasoning. Four relevant changes in the mathematics teaching were identified in the reflection group's discussions and learning. The changes ranged from understanding communication and reasoning to identifying, interpreting, applying and practising that reasoning. Teachers in the reflection group also changed their approach to discussion. In the initial stage, they achieved consensus, but gradually adopted a positive yet critical approach in which they problematized the process of learning in and from daily practice (Sterner 2015). The three levels of reflection (individual and shared reflections and the researcher's reflection on the preliminary outcome in the reflection group) resulted in discussions that promoted new and meaningful ways to communicate mathematically and stimulate mathematical reasoning in algebra. This methodology could be a way of linking the activities of students and teachers.

Ultimately, in response to Potari et al.'s (2015, p. 2,972) 'How can we link students' activity to teachers' activity', the present study demonstrates the importance of linking research and development in order to enable teachers to learn about their own mathematics teaching and students' learning. Moreover it provides a response to Sowder's (2007, p.158) questions; 'How do teachers learn from their professional communities about teaching mathematics' and 'What can teachers learn from investigating their own teaching of mathematics' by indicating the combined method of the developmental research cycle (Goodchild 2008) and the theory of communities of practice (Wenger 1998), along with reflection on three levels (Fig. 3) allowed using the results and questions that emerged in the reflection group.

Implications and further research

The main study, between March 2013 and January 2014 (Fig. 1) focused on a group's learning process, the group's meaning making of the key issue. The third question from Sowder (2007) 'What can be learned from research on teacher change' found a partial answer. Results of the study demonstrate what can happen in the change process when a reflection group begins to work actively. On that note, other questions are whether mathematics teachers' activities and shifts in collaborative learning change their mathematics teaching and whether teachers' meaning making and their shared repertoire about communication and reasoning in mathematics teaching influence their teaching and persisted three years later. Since I am curious about teachers' learning from a long-term, sustainable

perspective, one question I will continue to carry with me comes from the last meeting in the reflection group, when one of the teachers, Clara said:

> *Clara:* I'm worried about myself. It's very easy to sit back and fall into old habits when we no longer meet for reflection. What will my teaching be like now?

As Clara suggests, a question not answered in this study is whether teachers' activities and the shift in their approach in the discussion can change their mathematical teaching shortly after and also three years later.

Possible new routes and issues three years after the completion of the reflection group

The research in the reflection group involved the group's process of learning about the key issue of communication and reasoning. The teachers' meaning making and shared repertoire (Wenger 1998) about that issue shifted from understanding to identifying, interpreting, applying and practising mathematical reasoning. The present study does not provide answers about what happened in the light of the grey ellipse representing mathematics teaching in Figure 2 or what happened to the teachers' thoughts and their mathematical teaching three years later. What questions will arise when the five teachers listen to the interviews they gave in 2014, after the completion of work in the reflection group and what thoughts will they have on hearing their own narratives? Will it be possible to use the same theory of communities of practice (Wenger 1998) to analyse the teachers' individual reflections when they listen to their own voices from those interviews? Further research is necessary to understand sustainable, long-term learning in this case whether the mathematics teachers' activities and shifts in their collaborative learning actually changed their mathematics teaching over time. What roles, if any, do teachers' discussions changed meaning making and changed shared repertoire about mathematical communication and reasoning play in their teaching in a long-term sustainable perspective?

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Teachers' decisions and the transformation of teaching activity

Konstantinos Stouraitis

National and Kapodistrian University of Athens, Greece, kstouraitis@math.uoa.gr

In this paper I study teachers' decisions as a response to emerged contradictions in the context of enacting a new set of curriculum materials. The way these decisions are framed and the potentials they have to transform the teaching activity are analysed. Our data come from discussions in teachers' group meetings through one year. I use activity theory to capture social and systemic aspects of decision making and to interpret teachers' decisions. Future-oriented envisioning of the deliberate outcomes of teaching and action-based decisions about the actions to be undertaken are two different aspects of decision making. Both are traced in the data of this study as two necessary aspects for decisions that create possibilities to broaden the horizon of teaching activity.

Keywords: Activity theory, contradiction, decision making, teachers.

Last decades, in the context of curriculum reform efforts, teachers are seen as active agents and designers, whose instructional actions are influenced by curricular materials, but also shape the enacted curriculum alongside their students (Remillard, 2005). Considering teacher at the centre of the curriculum enactment, highlights the importance of teacher's decision making. Thus, a number of studies focus more or less explicitly on teachers' decisions. For example, Lloyd (2008) follows a teacher for two years and finds that his perception of students' expectations and his own discomfort associated with using the new curriculum were key factors in his decisions. A large number of studies are focused on in-the-moment decisions made by teachers. ZDM special issue 48(1–2) on teachers' perception, interpretation, and decision making, is indicative. Schoenfeld (2011) uses the notions of resources, goals and orientations to "offer a theoretical account of the decisions that teachers make amid the extraordinary complexity of classroom interactions" (p. 3).

The studies examining in-the-moment teacher decisions, focus on the classroom context and emphasize the individual dimension of deciding. Nevertheless, the broader social, temporal and cultural dimensions of decisions a teacher makes in his planning or in the classroom are not addressed. In this study I seek a better understanding of how decision making process develops and how is shaping the teaching activity, drawing on cultural historical activity theory. The study is conducted in two secondary schools in Greece at the time of the introduction of a newly prescribed mathematics curriculum. In Stouraitis, Potari, & Skott (2015) and Stouraitis (2016), we have analysed the contradictions emerged in this context and how teachers' decision-making is framed and develops considering social and systemic dimensions. In this paper I study how and why teachers' decisions, may or may not have a transforming effect on teaching activity. The decisions in focus are discussed in group meetings and refer to planed actions undertaken in the classroom. I examine three teachers' decisions, the different ways these decisions shape the teaching activity and I interpret these differences. Although sociocultural perspectives have been used in research about teachers' decisions in mathematics education (see for example, Skott, 2013), activity theory has not been used so far. Thus, although empirically based, the paper is methodologically oriented, giving an account about the affordances of using activity theory in studying teachers' decisions.

Theoretical considerations

Cultural historical activity theory (AT) offers a lens that tries to capture the complexity of teaching, by integrating dialectically the individual and the social/collective. The activity is driven by a motive and directed towards an object (Leont'ev, 1978). In our case, teachers (the subject) are involved in teaching activity with the motives of students' learning of mathematics and the fulfilment of other professional obligations. The unit of analysis is the activity system (Engeström, 2001a) incorporating social factors (rules, communities, division of labour) that frame the relations between the subject and the object with the mediation of tools (Figure 1). In our case, a tool with considerable influence in the teaching activity is the new curriculum.

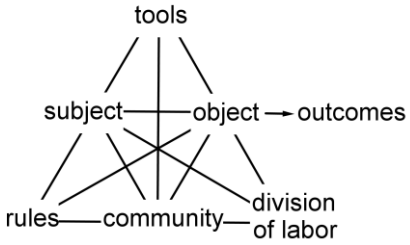

Figure 1: The activity system (adapted from Engeström, 2001a)

Activity is carried out through actions which are "relatively discrete segments of behaviour oriented toward a goal" (Engeström, 2001b). I conceptualise teaching action as discrete instructional acts or clusters of acts that constitute the teaching activity, e.g. the selection or creation of a task, the enacting of a lesson plan, etc.

Every activity system is characterised by contradictions which are the driving forces for the development of every dynamic system (Ilyenkov, 2009). They may create learning opportunities for the subject and may broaden the activity, for example leading to reconsideration of the actions and goals (Engeström, 2001a; Potari, 2013). In our study, the introduction and enactment of the new curriculum produced or revealed contradictions in teaching that emerged in group discussions (Stouraitis, Potari, & Skott, 2015).

Dealing with contradictions involves decisions about the goals and the actions to be undertaken. Of particular importance are decisions related to the "discrete individual violations and innovations" (Cole & Engeström, 1993), that is the search for novel solutions as the first, individual response to the emerged contradictions. Thus, although teachers' decisions are part of the teaching activity, they may have a transformational effect on this activity. Engeström (2001b) identifies four dimensions of decision making: social, temporal, moral and systemic. The systemic dimension is particularly concerned with the way “this [decision] shape the future of our activity?” (Engeström, 2001b, p. 281). This dimension is connected to expansive learning, the activity theoretical notion in which the learners are creating new ways to carry out the activity, reconceptualising its object.

Engeström, Engeström, & Kerosuo (2003) make a distinction between action-based decisions about the actions to be undertaken, and future-oriented envisioning, which is the imagination of the deliberate situation of the object as outcome of the activity. Drawing from their interventional study in health sector, they argue that intertwining these two aspects is necessary in any attempt to transform the activity stating that “history is made in future oriented situated actions” (p. 287).

Methodology

A new set of reform-oriented curricular materials was introduced and piloted in a small number of schools in Greece in 2011-12 and 2012-13. The new materials emphasize students' mathematical reasoning and argumentation, connections within and outside mathematics, communication through

the use of tools, and students' metacognitive awareness. It also attributes a central role to the teacher in designing instruction. In 2012-13 I collaborated with teachers in three of the lower secondary schools that piloted the new materials. The collaboration took place in group meetings at the respective schools, where the teachers discussed about their lesson planning and reflected on their experiences from teaching some modules of the designed curriculum. I participated in these meetings, providing explanations about the rationale of the materials, as I was a member of the team that developed the curriculum. I was also discussing their reflections provoking their explanations about the rationale of their choices. In this paper, I refer to two reflection groups, one of five teachers working in school A and one of two teachers working in school B.

School A is an experimental school with an innovative spirit. Our focus here is on the teaching decisions of two teachers, Marina and Linda. They both have more than 25 years of teaching experience and additional qualifications beyond their teacher certification, as Marina has a masters' degree in mathematics and Linda has one in mathematics education. They both have experiences with innovative teaching approaches and both have strong views about their instructional choices and a critical stance on teaching innovations and materials introduced by various agents.

School B is a normal school with a culture open to innovations. Peter, the teacher in focus, is teaching in public schools about 15 years. Before this, he was teaching in private education preparing students for examinations. Peter is assistant principal of the school, he attends master studies in education and he is educator preparing teachers to use digital tools in teaching mathematics. He is open to the new curriculum, but he bases his teaching on the old textbooks. In the year of the study Peter is questioning the teaching practices he was involved for many years.

The data material consists of transcriptions of audiotaped group discussions and interviews conducted with each teacher in the beginning of the study and six months after the end of the group meetings. The transcriptions were analysed with methods inspired by grounded theory (Charmaz, 2006). The initial open coding resulted in the identification of the discussion themes for each meeting and the corresponding contradictions that emerged in the context of enacting the curriculum. Seeking an understanding of these emerging contradictions I used AT which provided me a lens to study them and a language to discuss about their dialectical nature, integrating social, cultural and historical aspects. Analysing the ways teachers decide to deal with contradictions, I traced shifts in teachers' discourse across different meetings and interviews and I used AT and the relevant literature to interpret these decisions and the factors influencing them. In this paper I focus on the part of my analysis concerning the relations between the action based decision making and the future oriented envisioning, and the potential of transforming the teaching activity.

Results

Below I present two examples selected as illustrative cases for the relations of action based decisions and future oriented envisioning. In the first one, Marina and Linda make contrasting decisions, both addressing their perspectives about their students' learning. In the second one, Peter makes action based decisions without a clear articulation of his envisioning about his students' learning.

First example: teaching congruence involving geometrical transformations

Geometrical transformations are introduced as a distinct topic in the new curriculum with the rationale of supporting students' development of spatial sense and of using transformations when tackling issues of congruence and similarity. The use of transformations as a proving tool is an alternative to the Euclidean perspective in school geometry: the intuitive use of the moving figure is seen as incompatible with the rigorous deductive rationale of Euclidean geometry. This contradiction between the two proving tools is a manifestation of the dialectical opposition between intuition and logic. In Stouraitis (2016) I discuss in details the two contrasting ways Marina and Linda deal with this contradiction in the discussions in school A. Below, I briefly describe their decisions to highlight the different future-oriented envisioning they hold for the object of activity.

In the fourth meeting (A4), Marina discusses her thoughts to use geometrical transformations in teaching triangle congruence in grade 9. She considers using tasks with geometrical transformations in parallel to or in combination with criteria of triangle congruence. She describes her goal saying "I want them [the students] to understand that when we compare angles or segments, we have two tools. One is transformations and the other is the criteria of triangle congruence". On the other hand, although Linda appreciates Marina's approach as a "nice idea", she prefers not to intertwine the two topics. She refers to "the purpose [students] to learn how to write [a justification], to observe the shape, to distinguish the given data from the required claims, to make conclusions, and to prove", implying that these goals can be achieved through teaching congruence with a Euclidean perspective, without involving transformations. Although Marina's response is that the same goals are relevant in every geometrical topic, Linda states that in teaching congruence she wants to focus on Euclidean geometry and not transformations.

In next meetings (A5, A6) Marina describes how her students work with both geometrical transformations and congruence of triangles, discussing also emerging epistemological issues. She explains her decision as creating an "opportunity to change the framework [of proving] in grade 9" and to "get away from Euclidean geometry". Linda contributes to the discussion with her opinion and ideas, but she does not change her decisions. In other meetings, Marina mentions a seminar on transformations she attended three years ago in the university and her experimental teaching of transformations in a school she was previously working.

Analyzing Marina's and Linda's decisions across Engeström's (2001b) four dimensions, I conclude (Stouraitis, 2016) that, although Linda and Marina share similar experiences and perspectives and participate in the same school community and in the same reflection group, there are significant differences between the goals they set, the decisions they make and, consequently, the actions they undertake. Marina appears more fluent with the mathematics of geometrical transformations to use them as a proving tool alternative to Euclidean geometry, and this may possibly and in part be explained by her involvement in past activities like the seminar on transformations and her experimental teachings. Linda has not such experiences. Moreover, her goals are based on the affordances of the Euclidean perspective.

Focusing on the possibilities their decisions have to shape the future of teaching activity, I look at the way the teachers envision the future of their students learning. All the aforementioned extracts of Marina's discourse reveal a future-oriented envisioning of the object of the activity she is

engaged in: she imagines her students working fluently with both approaches and consciously about the differences between them and she notes their development in this direction. In the interview conducted in the next year, Marina says that she uses the same approach, with more elaborated tasks for her students. Like Marina's envisioning, Linda is showing her motive in the relevant extract: she imagines her students in the future to work having developed understandings and proving abilities based on Euclidean perspective in school geometry.

Second example: the use of modelling in teaching algebra

The new curriculum materials recommend mathematical modelling as an important aspect in students meaning making in algebra. Generating algebraic expressions and equations to represent realistic situations and problems is introduced in grades 7th, 8th and 9th. In group discussions about teaching polynomials in grade 9, a common contradiction was about introducing polynomials and operations in a formal, abstract way or involving realistic situations and modelling procedures. This contradiction is a manifestation of the dialectical opposition between the abstract and the concrete. Below I describe Peter's dealing with this contradiction as appeared in group discussions of school B with Manolis (Peter's colleague) and the researcher.

In the 3rd meeting (B3) Peter describes his introductory lesson of monomials using only definitions, examples and counterexamples. He says "we begin with the algebraic expression, they [the students] read the definition, and I give them examples to discuss … then to monomials [with the same way]". After researcher's and Manolis' questioning about the "why" of teaching polynomials, Peter refers to a similar student's question. He is reflecting that "he begins with the definitions", but "we must pay more attention … to the practical use of monomials". Again in the discussion with Manolis and the researcher about modelling, Peter starts thinking the potentials of it. After some turns, he says that he likes the word "modelling" because "it shows exactly what we are doing: we transform real situations to mathematics, verbal expressions to mathematical ones". With modelling "you give [the students] a motive, a goal. Ok, you must first pose the problem to create questions"

Although Peter finds modelling a useful idea, he is involved in a discourse emphasizing the role of mathematics and his own teaching but not the deliberate students' development. For example, he describes what "he did" and what he "usually does", and that modelling is what "we do in mathematics". In this discourse, no explicit or implicit longitudinal objective appears related to the way his students should deal with modelling. This can be interpreted as absence of any clear articulation of his future-oriented envisioning that could lead his decisions.

In another meeting (B7), Peter refers to classroom discussions about functions where students and teacher modelled realistic situations and phenomena (mostly from physics) leading to linear and quadratic functions. He says that his goal is "[the students] to understand that a function shows a relation between two interdependent things. And that everything is a potential function". These formulations reveal Peter's future-oriented envisioning about students understanding of functions and connecting them to realistic situations and also physics. But again, there is not any similar envisioning about students' work on modelling per se. The modelling processes Peter involved in classroom discussions were limited at the level of actions subordinated to his teaching of functions.

In the interview conducted in the next year, the researcher asked Peter if he uses modelling in teaching polynomials this year. Peter responded that although he thinks it is useful and keeps it in

mind, he “hasn’t the time to do all this”. This response shows that there is not any movement in the way Peter carries out the teaching activity about modelling.

Peter's adoption of the idea of modeling in teaching polynomials and functions can be interpreted as adoption of elements introduced by the new curriculum, based on Peter's reflection about teaching and on the group discussions with Manolis and the researcher. But this adoption did not gave rise to actions involving students in modeling procedures, especially in teaching polynomials. Peter's previous involvement in practices, like preparing students for examinations in the private education, seem to have strong influence on his decisions. Moreover, his decision about modelling had not any systemic influence on the teaching activity, since it was not connected with future oriented actions.

Discussion and conclusion

The introduction of the new curriculum created or revealed contradictions that provide opportunities for teachers to engage differently in mathematics teaching and learning. The analysis exemplifies these opportunities and the teachers' decisions to make or not shifts in their teaching.

In both provided examples, all three teachers seem to be aware of a contradiction of the introduction of the new curriculum. Marina and Linda appear to be more consciously aware of its epistemological and dialectical nature. Peter also shows an understanding of some aspects of the relevant contradiction. Teachers’ awareness of the contradiction is the necessary but insufficient driving force for the development of the teaching activity. From this point, teachers’ decisions can lead to one or the other direction.

On the contrary of "traditional views [that] locate decision making in the heads of individuals at a given point of time in a particular place" (Engeström, 2001b, p. 282), searching, under an activity theoretical view, what makes teachers to set goals and what creates the horizon for possible actions, contributes to our understanding of teachers' decisions. Although activity is collective and the object is socially formulated, different teachers can have "different positions and histories and thus different angles or perspectives on their shared general object" (Engeström, 2001b, p. 286). In the first example provided in this paper, Marina and Linda make different decisions about the same contradiction. The difference may in part be explained by their different histories, including Marina's attending of the seminar and her experimental teachings. In the second example, Peter’s decision seems to be influenced by his previous activities in the private education sector.

In the two provided examples three possibilities appear for teachers’ decisions and the way these decisions may or may not influence the future of the activity. Marina’s decision to combine geometrical transformations with Euclidean geometry is an attempt to overcome the contradiction synthesizing dialectically the opposing poles. On the other hand, Linda decides to keep the two opposing poles separated, pursuing the affordances of Euclidean geometry. Somehow in the middle, Peter decides to deal with the contradictions using aspects of modelling in teaching functions, but not to use modelling as meaning-making introductory activity in polynomials.

Marina’s decision has the potential to transform the teaching activity, broadening the horizon of the possible modes this activity is carried out. The dialectical overcoming of the contradiction is a discrete individual innovation, although its evolvement is not already known. Linda’s approach does not transform the activity, but clarifies and strengthens some objectives of teaching Euclidean

geometry. Linda's decision reinforces aspects of the existing way activity is carried out, showing that every learning is not necessarily expansive (Engeström, Engeström, & Kerosuo, 2003). Peter's decisions have not any shifting effect to the way the activity is carried out, neither reinforce any existing practice. Somehow this decision seems to have not the power to affect the activity.

What is the difference between Marina's and Linda's decisions on the one hand, and Peter's decision on the other, that provide the different power on them? The difference could not be the attempt to overcome or not the contradiction, since Marina's and Linda's decisions differ at this point although both are strong enough to have an effect on the teaching activity. The difference is grounded on the connections made between action-based decisions and future-oriented envisioning of the object. Marina and Linda underpin their decisions about the actions they undertake with a strong future-oriented projection of their students' understanding. This adds fluency in deciding among the possible actions realizing the relevant goals. At the same time it generates decisions with the potential to be stabilized, even if initially the stabilization refers only to individual modes of carrying out the activity. On the other hand, Peter's decisions seem to be restricted to action level, without a grounding on future envisioning of the object, namely the deliberate modelling processes his students should be able to involve as outcome of the sequential actions undertaken. The absence of future-orientation restricts the horizon of possible actions and reduces the potentiality of stabilizing them. Our conclusions appear in line with Engeström, Engeström, & Kerosuo (2003) who, researching developmental work in the health sector, write that "professionals make history in future-oriented discursive actions" (p. 286) and "to overcome the gap between action and imagination in history-making, it may be necessary to bring them closer to one another" (p. 305).

Summing up, one can argue that for decisions to affect the activity the following elements seem to be necessary: the emergence of a contradiction and some degree of awareness about it, a willingness to deal with it and a future-oriented envisioning about the outcomes of the activity. If there is to have a transformation of the activity, the decision must aim to a dialectical overcoming of the contradiction by searching new solutions. Although schematic and perhaps simplistic, this sequence may represent some crucial aspects of decision making, especially the relations between action-based decisions and the future of the activity.

Our developmental intervention was not designed on an AT basis. However, based on AT, our analysis traces aspects of the path leading from the contradiction to the transformation of the teaching activity. In this analysis, AT seems to offer two particularly important aspects. Firstly, the four dimensions capture social and historical aspects of teachers' decisions, which is critical in our interpretations. Secondly, the distinction between action-based decisions and future oriented envisioning, provides a lens to interpret the possible power of teachers' decisions. The not-predetermined nature of the intervention might be seen to provide the analysis with a potential to interpret more naturally some snapshots of the trajectory of transforming the teaching activity. More research could be useful for a more holistic, but also detailed view of this trajectory.

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A commognitive lens to study pre-service teachers' teaching in the context of achieving a goal of ambitious mathematics teaching

Gry Anette Tuset

Western Norway University of Applied Sciences, Norway; gry.tuset@hvl.no

This paper investigates how Sfard's (2008) commognitive framework might inform the investigation of pre-service teachers' (PSTs') teaching in the context of achieving a goal of ambitious mathematics (AM) teaching. In particular, I will show how the commognitive framework can be used to foreground the mathematical talk when analyzing qualities in PSTs' attempts in forms of opportunities and needs created and exploited. I use the commognitive framework and especially the distinction between rituals and explorations to conceptualize qualities of PSTs' attempts at AM teaching on two levels. The first level points to PSTs' use of teaching tools provided in a teacher education program in Norway to support AM teaching. The second level points to characteristics of the mathematical discourses manifested in the attempts of PSTs to utilize these tools. On both levels I investigate students' opportunities for explorations.

Keywords: Teacher education, mathematics teaching, commognitive framework, geometry.

Preparing for ambitious mathematics teaching

Over 25 years ago, Lampert (1990) described a possible approach to bring the practice of knowing mathematics in school classes closer to what it means to know mathematics within the discipline. Her vision is in contrast to what is viewed in most classrooms, where doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher (Lampert, 1990, p. 32). Sfard (2017, p. 125) calls this memorized-symbolic-manipulation type of activity *ritualized* participation in a mathematical discourse. Such participation is connected to ritualized instruction and it is characterized as performance for the sake of connecting with or pleasing others (Heyd-Metzuyanim & Graven, 2016).

The term *ambitious* is used by Lampert, Beasley, Ghousseini, Kazemi, and Franke (2010, p. 129) to distinguish this kind of classroom discourse from a classroom discourse that is more aligned with Lampert's vison. They define the work teachers need to do to manage the complicated interactions between the teacher, the students, and the mathematics in classroom environments as *ambitious mathematics teaching* (AM teaching). The most important and challenging work of AM teaching is allowing students to exercise authority for mathematical ideas while staying accountable to the discipline (Lampert et al., 2010). Students are expected to participate in a discourse in which they strive to learn more about the mathematics involved. Such activities are called by Sfard (2008) *explorative* participation in mathematical discourse, in contrast to ritual participation. Explorative participation is connected to explorative instruction and is characterized as performance for its own sake (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016).

In a practice-based teacher education program in Norway, pre-service teachers (PSTs) were provided among others the following teaching tools to support their attempts in AM teaching: A set

of talk moves to increase the mathematical talk in the classroom (Chapin, O'Conner, & Anderson, 2009), and some screen manipulable shape-making quadrilaterals constructed in GeoGebra with the potential to develop more sophisticated geometrical discourses (e.g., Sinclair & Moss, 2012).The figures (e.g., squares, rectangles, rhombi, and parallelograms) are constructed to hold certain properties of the diagonals (see Figure 1).

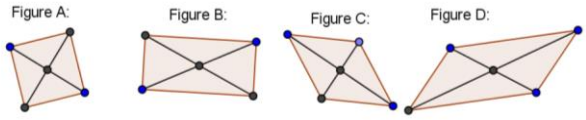

Figure 1: Four of the seven shape-making quadrilaterals

It is within the context of this teacher education program that I study the mathematical talk, the extent to which it becomes explicit for the students through the PST's use of the provided teaching tools, and how this affects student's opportunities to participate in the discussion. I explore Sfard's commognitive framework regarding these relations because of the comprehensive set of conceptual tools it provides for capturing both mathematical discourses and social participation patterns (e.g., Heyd-Metzuyanim & Graven, 2016; Sfard, 2017). This paper is a contribution to the general need to develop analytical approaches for conceptualizing and analyzing interactional and content-related aspects of classroom discourse (Sfard, 2008).

In the following, I will first briefly outline key concepts from the commognitive framework and report how they are used to conceptualize qualities of PST's attempts at AM teaching in discursive terms. Finally, I will show how the analytical framework is used to foreground the mathematical talk when analyzing qualities in one PST's attempt in the form of opportunities and needs created and exploited. The PST is named Andy and he is in his second semester studying mathematics in a Norwegian teacher education program for lower secondary school. He leads a classroom discussion about quadrilaterals in ninth grade (14–15 years). This is his first attempt to lead a classroom discussion in the context of achieving a goal of AM teaching.

Analytical framework

Sfard's (2008) *commognitive* framework is rooted in the claim that *thinking* is a form of communication and that mathematics is defined as a *discourse* where *mathematical objects* are abstract discursive objects. *Discourse* is a special type of communication, set apart by its objects (*word use*), all kinds of mediators created and acted on for the sake of communication (*visual mediators*), a set of meta-level rules followed by the participants (*routines*), and the outcomes of their processes (*endorsed narratives*) produced within the *community of the discourse* (Sfard, 2008, p. 93). Sfard does not explicitly define teaching. However, she talks about ways teachers can provide opportunities for different kind of mathematics learning and participation through their communicational activities (Sfard, 2017, p.44). In AM teaching the communicational activities aim to bring students' mathematical discourses closer to disciplinary mathematical discourses in such a way that they are able to participate in an explorative way.

Sfard (2017, p. 45) describes distinctions between ritual and explorative mathematical discourses when she talks about the historically established mathematical discourses we aim at in schools. I use the work of Wang (2016) and Sinclair & Moss (2012) in redefining and refining van Hiele's levels

of geometric thinking into discursive terms to make distinctions between ritualized and explorative geometrical discourses. In ritualized discourses (levels 1 and 2) a "square" is often used as a proper name on a concrete thing or a family name of disjoint discursive objects. Thus, in ritualized discourses it is difficult to call a shape by a different name, because names represent different families. In contrast, mathematical definitions guide the use of the name "square" in object-driven explorative discourses (levels 3 and 4). A shape can be called by several names if it has the necessary and sufficient properties described in several definitions. A characteristic in explorative discourses is therefore the possibility of hierarchic classification by definition. The identification routines are also different in the two discourses. In explorative discourses, one strives to use definitions as criteria to identify geometric figures instead of direct visual recognition or just checking some partial properties. In ritualized discourses, there is often no need for substantiation routines, because claims seem to be self-evident. If they are not, one tends to use the concreteness of the figures to endorse the narrative, for example, by using measuring and dragging routines to check and verify the sides and angles in a figure constructed in GeoGebra. In contrast, explorative discourses emphasize deductive reasoning to substantiate the endorsement of a narrative by using previously endorsed narratives. The more ritualized checking routines are still used, but mainly to modify the justifications. Engagement in explorative geometrical discourses can therefore be related to qualities in AM teaching.

Supporting transitions from ritualized to explorative geometrical discourses is central in AM teaching. It involves what Sfard (2008) calls *meta-level development*, so called because the meta-level rules change. The inevitable point of departure for meta-level development is imitation of the moves of an expert discursant (Sfard, 2017). If everything goes well, the participation will gradually become explorative. However, this transition is demanding for the students and for the teacher. According to Sfard (2017, p. 44), there are two ways in which the teacher can support such transitions. The first way is to take leadership in the new discourse in appropriate learning-teaching situations and model how words are used and what routines count as acceptable within the new discourse. For example, the teacher may demonstrate the use of definitions as a way to enhance direct visual recognition in appropriate situations (Sfard, 2008, p. 254). In order to succeed, the students need to show confidence in the expert, be willing to take the role of the learner, and make changes that bring them closer to explorative geometrical discourses. The second way is to explicitly encourage the desired discourse by using appropriate teaching moves. In order to succeed, the teacher needs to elicit contributions from the students to identify and analyze their geometrical discourses up against ritualized and explorative discourses. Then, the teacher must respond in such a way that students become aware of possible differences in the use of words and routines. In addition, the teacher may expose them to situations in which their discourses would prove insufficient and support them in the process of understanding the advantages of the new way of doing or saying things instead of the method with which they have been so familiar. For example, students may drag the shape-making figures, creating opportunities to widen the range of shapes students are ready to call the *same* (Sinclair & Moss, 2012).

The provided teaching tools have the potential to take into account the complexity of the goals in AM teaching and support such transitions (Lampert et al., 2010; Sinclair & Moss, 2012). However, the choice of appropriate teaching tools involves more than just the use of a tool. It involves making judgments about when and where it is appropriate to use the tool. The PST's use of the provided

teaching tools could therefore be characterized as ritualized or explorative (Heyd-Metzuyanim et al., 2016). A ritual performer would be concerned about how to proceed when a specific tool has been chosen (Nachlieli & Katz, submitted). The use is often rigidly defined and dependent on others decisions in order to achieve social goals. For example, talk moves may be used to reward ritual participation as appropriate behavior, or they may be used to follow a prescribed list of possible properties of diagonals regardless of students' responses. In contrast, an explorative performer is concerned about choosing the appropriate tool in order to achieve her intended goal (Nachlieli & Katz, submitted). For example, she may use the shape-making figures to create situations for potential explorations or use talk moves to explicitly support transitions from ritualized to explorative geometrical discourse. PST's explorative use of these teaching tools can therefore be related to qualities regarding AM teaching.

It is important to stress that the distinction between ritual and exploration is not a categorization of students' participation or PST's use of teaching tools as such. It is meant to serve as a way to better understand qualities in PSTs' attempts at AM teaching in learning-teaching situations. In this paper, I explore how the use of this distinction points to the different qualities of PST's attempts at AM teaching; the characteristics of ritualized and explorative geometrical discourses; students' opportunities for ritualized and explorative participation in these geometrical discourses; and PST's ritualized and explorative use of the provided teaching tools to create and exploit these opportunities.

An investigation of Andy's attempt

In this paper, I use the transcription of a video recording of Andy leading a ninth-grade classroom discussion about quadrilaterals. I use it to show how the analytical framework informs the investigation of qualities in PSTs' attempts at AM teaching. I also present findings from a three-tiered analysis design.

Tier 1: The transcription was first organized into mathematical episodes in which Andy and students discuss an endorsable mathematical narrative as a claim about one of the shape-making figures or a relation between them. Each episode encompasses the whole discussion around one claim, including the routines of construction and substantiation. I chose Andy's attempt out of four PSTs' discussions because of the nature of the claims in the ten identified mathematical episodes. In episodes 1, 2, 3, 7, and 9, the talk is mainly about identifying and describing potential properties of the shape-making figures. However, in episodes 4 and 8, Andy challenges students to explain how necessary conditions are linked to the naming process (e.g., "Is this (perpendicular diagonals) one thing that needs to hold for it to be a square?" [48]). In episode 5 a student identifies figure B in Figure 1 as a rectangle and Andy prompts the student for further explanations ("Why is it a rectangle?" [75]). Andy also challenges students in episodes 6 and 10 to extend their thinking about the possibility of a figure having several names (e.g., "But, here (figure A in Figure 1) are two and two sides equal as well (5s). Is it a rectangle too?" [85]).

These examples show that Andy's use of the provided talk moves (see Table 1) managed to create several opportunities for explorative participation in a geometrical discourse. The examples also revealed teaching-learning situations where Andy was given opportunities to support transitions from ritualized to explorative geometrical discourses. Understanding whether these opportunities

were exploited requires further investigation. The mathematical episodes were therefore examined qualitatively using the analytical framework on two levels described in tiers 2 and 3 and inspired by the work of Heyd-Metzuyanim et al. (2016).

Tier 2: The first level points to Andy's use of teaching tools and the opportunities created and exploited for students' participation. I separated Andy's and students' talk and coded their talk moves on a turn-by-turn basis. I used a modified coding scheme based on the set of teacher's talk moves provided by Chapin et al. (2009) and created codes for students' talk (see Table 1).

Andy's talk moves	Say more	Revoice	Repeat	Press for reasoning	Challenge	Agree/ Disagree	Add more
	14	2	7	1	6	1	6
Students' talk	Narrative	Justify	Explain	Judge	Repeat	Clarify	Question
	6	6	4	18	5	3	3

Table 1: The amount of talk moves used by Andy and students' talk

Table 1 presents an overview of the amount of talk moves used by Andy and students' contributions. The findings show that Andy makes use of the recommended talk moves to promote student engagement in the leading geometrical discourse and in relation to each other's contributions. The table also shows that students contributed to the geometrical discourse by constructing geometrical endorsable narratives (e.g., "It is a rectangle" [74]). They tried to verify narratives (e.g., "Because two of the sides are longer than the other two" [76]) and they contributed further explanations (e.g., "Yes, but all sides are equal there…In a rectangle there are only two and two equal sides" [88]). It is difficult to determine whether students' intentions were to produce new mathematical narratives for their own sake or to please Andy. However, they show examples of explorative use of talk moves. However, the amount of talk moves did not provide answers concerning whether and how Andy's use of talk moves supported students in the meta-level development towards explorative geometrical discourses.

Tier 3: The second level of analysis therefore points to opportunities and needs created and exploited for students to engage in explorative geometrical discourses. I screened the episodes for signs of ritualized and explorative geometrical discourses in the forms of word use and routines. The aim was to identify appropriate learning-teaching situations for meta-level development. I then investigated Andy's use of teaching tools in these situations in order to study how their use supported students in their transitions.

Due to limited space, I present an analysis of one episode regarding a more detailed investigation of the geometrical discourse and PST's use of talk-moves. In episode 1, Andy starts to drag figure A in Figure 1 and asks if anyone has something to say about the shape-making figure on the smartboard.

1	Andy:	We start with figure A (2s). Does anyone want to say something about figure A? (4s) Yes? (*Andy points at a student who has her hand up.)*
2	Student:	The sides are always of equal length
3	Andy:	The sides are always longer (.) of equal length?
4	Student:	Mmm.

5	Andy:	Yes (2s), can you explain why they are of equal length?
6	Student:	Because it is a square.
7	Andy:	Yes?
8	Student:	And if you change the size, the sides will change. They are still of equal length.
9	Andy:	Yes (2s), it is said it is a square. All sides are of equal length; therefore, it is a square. All sides are equal because (*Andy is waving his arm at the figure on the smartboard*) (4s) Does anybody disagree? Does anybody have anything else? (4s) Let's look at the diagonals (*Andy points at figure 1 on the smartboard*), the ones which intersect. We have four criteria here (*he removes the figures and puts a scheme up on the smartboard*). Now you have the figures on your computers, OK? What do you think? (5s) Can anyone repeat what has been said? (4s) (*Andy points at some students in the back.*) You in the back.
10	Student:	Oh OK, yes, it is a square=
11	Andy:	=Square, yes=
12	Student:	=and all sides are equal
13	Andy:	Yes? What do you want to say? (*Andy points at a student)*

Andy invites the students to participate in the discussion with an open question [1]. One student contributes with an identifying narrative about the length of the sides [2]. The student has therefore created an opportunity to discuss variant and invariant properties in figure A. Andy exploits the opportunity and prompts for an endorsement by asking why ("say more") [5]. Instead of relying on immediate visual recognition, the student provides the justification "because it is a square" [6]. The use of "it" refers to the concreteness of the figure on the smartboard. The identifying narrative "it is a square" [6] has not been previously endorsed by the use of definitions. It is established by the student as something that is already known. The student provides a correct deductive inference: If figure A is a square, then the sides are of equal length. Andy signals that he wants her to "say more" by asking "yes?" [7]. The student supports her justification by explaining a more ritualized checking routine [8]. The student draws on ritualized routines but her contributions also show explorative characteristics in the ritualized discourse.

This is a teaching-learning situation in which Andy has the opportunity to either demonstrate how definitions are used in explorative discourses or choose appropriate teaching tools to create a situation in which the suggested routines prove to be insufficient. Instead, Andy confirms the narrative "it is a square" [9]. He then restates the construction of the narrative suggested by the student but changes the premise and conclusion [9]. His deductive inference is: If all sides in figure A are of equal length, then figure A is a square. This endorsement would not have been accepted in explorative discourses because equal sides are not sufficient properties to define a square. Andy starts to explain why the lengths of the sides are equal, but he stops talking and waves his arms without touching the smartboard [9]. In this situation Andy shifts from being an explorative user of talk moves to a ritual user. Instead of trying to solve the problem or cope with the difficulties within the geometrical discourse, he chooses talk moves that help him to redirect the discussion in order to go on. In this situation, Andy uses "disagree" ("Does anybody disagree?") and "add more" ("Does anybody have anything else?"). The students do not respond. He then redirects the talk towards the

properties of the diagonals. He then uses "repeat" ("Can anyone repeat what has been said?") to activate the students. The student repeats what is said without adding anything new [10], [12]. Andy invites another student to participate ("add on") and the talk shifts to the properties of the diagonals. Thus, the opportunities that were given for students shifted from explorative to ritual participation in the geometrical discourse.

This analysis shows how Andy struggles to respond appropriately when he has created opportunities for students to participate exploratively in the geometrical discourse. The analysis of the other episodes revealed similar patterns. Andy manages to make use of the shape-making figures and talk moves to create opportunities and needs for explorative participation, which are important qualities in AM teaching. However, he struggles to stay accountable to the discipline and take leadership in the explorative geometrical discourse when needed. Instead of exploiting the teaching-learning situations to engage students in explorative geometrical discourses, he tends to use the provided talk moves such as "repeat," "agree/disagree," and "add more" as shortcuts to overcome the difficulties and keep the discussion going. In these situations, Andy's use of talk moves shifts from explorative to ritualized, which affects the students' opportunities to participate in the geometrical discourse. Even more importantly, students were not offered the necessary opportunities to engage ritually or in an explorative way in the explorative geometrical discourse in order to modify their thinking and bring them closer to accepted ideas of the discipline. Without knowing it, they were stuck in a ritualized geometrical discourse (Sfard, 2017).

Some concluding remarks

AM teaching is characterized by its deliberately responsive and disciplinary-connected instruction which complicates interactional and content-related aspects of classroom discourse (Lampert et al. 2010). The purpose of this paper was to show how the commognitive framework and particularly the distinction between rituals and explorations foreground the mathematical talk when investigating qualities of PST's attempts at AM teaching. Analyzing the mathematical episodes and the amount of talk moves helped to uncover qualities regarding students' opportunities for explorative participation in the PSTs' teaching attempts. Students' talk provided some evidence of students' uptake of these opportunities. However, it was the more detailed analysis of the geometrical discourses in the episodes that revealed if and how the PST's use of talk moves provided opportunities for participation in explorative discourses.

The analysis of Andy's attempt in AM teaching uncovers just some of the complexity that a PST must attend to in the challenging moment-by-moment interactions with students and mathematical discourses. It also shows the critical need for teacher education programs to provide PSTs with opportunities to learn and reflect upon their use of potentially powerful teaching tools in such learning-teaching situations. Sfard (2008, p. 223) argues that if one only focuses on how a teaching tool should be performed and neglects the question of *when* and *where* this performance is appropriate, it is most likely to result in ritualized rather than explorative participation.

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Focusing on the 'middle ground' of example spaces in primary mathematics teaching development in South Africa

Hamsa Venkat[1] and Mike Askew[2]

[1]Wits School of Education, University of the Witwatersrand, Johannesburg, South Africa & School of Education and Communication, Jönköping University, Jönköping, Sweden; hamsa.venkatakrishnan@wits.ac.za

[2] Wits School of Education, University of the Witwatersrand, Johannesburg, South Africa & Faculty of Education, Monash University, Melbourne, Australia;

michael.askew@wits.ac.za

Our focus in this paper is on locating ways of working with teachers' use of conventional example spaces to include a focus on structure, abstraction and generality in primary mathematics teaching in South Africa. We share the ways in which we have worked with trajectories of working with example spaces in two strands within our framework for analyzing teaching in terms of teachers: 'mediating primary mathematics'. We discuss the ways in which this work with example spaces differs from previous writing where emphasis is placed on moving beyond conventional example spaces towards the realms of boundary examples. The ways in which attention to structure and generality might be retained in the context of conventional example spaces are discussed, with focus too on why this might be useful to do in a developing country context marked by very low mathematical performance.

Keywords: Example spaces, primary mathematics teaching, South Africa, mathematical discourse in instruction, generalization.

Introduction

The background for this paper rests upon two bodies of writing that are interpreted as pushing in somewhat opposite directions in relation to the kinds of examples that are in focus. The first evidence base is located in mathematics education writing on 'example spaces' (Watson & Mason, 2005). These authors, over nearly two decades now, have produced an extended evidence base arguing for learning to be seen in terms of extensions in the example space that students are able to work with and construct. On the pedagogic side, they have focused on both teacher-set example spaces (Watson & Mason, 2006a) and student-constructed example spaces based on prompting for 'another example' of something or 'a different kind of example' of something (Watson & Mason, 2005). In both teacher-set and student-constructed cases, the focus on example spaces is driven towards attention to the abstractions and generalizations inherent in the idea of 'examplehood' – the attributes that make an example an 'example of' some class or category. Goldenberg & Mason (2008) describe such attributes as critical 'dimensions of possible variation', and the extent to which variation in the dimensions of these attributes can allow students to see the boundaries or extents of application of properties or structure defines the 'range of permissible variation' in their terms. As Watson and Mason (2005) put it:

> '[Example] spaces can be explored by finding out what can vary and how far it can vary, identifying new variables, working from first principles, building objects from definitions, and using alternative modes of representation to see what is possible in one and relating it to another and in other ways.' (p.52)

Within their work, critiques of pedagogy are directed towards what they see as the overemphasis on 'conventional' or 'central' examples that commonly contribute to unhelpful abstractions on the part of students – for example, that triangles are three-sided shapes with acute angles that have the base edge parallel to the bottom of the page, or multiplication only with positive integers. Instead, they call for a pedagogy in which there is encouragement for the construction of examples that extend students' existing 'boundaries' through attending to dimensions of possible variation and range of permissible change, and focusing discussion on the 'tipping points' at which 'examplehood' appears to end. Illustrating this approach are tasks relating to questions such as 'Can we multiply by a number and end up with an answer smaller than the one we started with?' And 'What kinds of numbers produce this result?' Pedagogy in mathematics, in this formulation, is consistently directed towards tasks functioning in ways that expand the boundaries of the example spaces associated with particular ideas:

> 'one of the important roles for tasks inviting learners to construct examples is to broaden their range of permissible change in the images they associate with concepts' (Watson & Mason, 2005, p.56)

In this pedagogy, the emphasis is on the boundaries of example spaces related to the concept in focus – on what Askew & Wiliam (1995) describe as 'only-just examples' on the example side and 'very nearly examples' on the non-example side.

Sitting with somewhat different emphases to this attention on example spaces is a second body of work, located in developing country contexts, but with parallels in contexts of disadvantage in developed country contexts as well, that is focused on learner performance. In this narrative, there is extensive evidence of a lack even of limited forays towards success in the restricted purview of the conventional or central example spaces associated with key concepts. Pritchett & Beatty (2012), working in the policy terrain and overviewing mathematics (and reading) performance data across India, Pakistan, Uganda and Kenya, identify what they describe as almost 'flat learning profiles' when looking at mathematical performance in items across grades. For example, in a study carried out in one state in India, Pritchett & Beatty note that while just over half of Grade 2 children were able to correctly answer 697+505 presented as a vertical addition, less than 10% of Grade 5 children were able to correctly fill in the missing number in this horizontal equivalence sentence: 200 + 85 + 400 = 600 + __. Further, it was noted that performance on 'mechanical' procedures was generally higher than performance on even-low level conceptually oriented items, with one of the overviewed studies showing that less than 30% of a Grade 4 learner sample were able correctly answer the question: 3 x __ = 3+ 3+ 3+ 3.

What interests us here is that the examples identified in this latter body of work fall well within the realms of 'conventional' example spaces, with the 'multiplication as repeated addition' example above also possible to interpret as a 'reference' example (examples that 'somehow contain information about a whole class of objects', Watson & Mason, 2005, p.84).

While Watson & Mason's pedagogic approach emphasizes the need to need to focus on expanding students' example spaces, which are viewed as personal, and locally situated, these authors' exemplifications of ways of working with sets of examples point to a relatively fast skip beyond the conventional and central example spaces towards the boundary examples.

But why is this difference of emphasis of interest to us? We answer this question in the next section.

Background

Our work over the last five years in South Africa has focused on primary mathematics knowledge and teaching development, set in the context of the Wits Maths Connect-Primary (WMC-P) research and development project, located in Johannesburg, and working in partnership with ten government primary schools. The South African primary teaching context is marked by an emphasis on oral, chorused responses to closed questions, weaknesses and gaps in the teacher knowledge base, and concerns related to coverage, connections & coherence, and pacing within primary mathematics teaching (Hoadley, 2012). The national context is one that is marked by low performance in mathematics at all phases, reflecting many of the concerns about limited progress and significant learning deficits raised by Pritchett & Beatty (2012). A key problem that has been widely written about in primary mathematics is the ongoing use of highly inefficient counting based methods for solving number problems well into the middle years (Fleisch, 2009).

In this context, we have worked on a combination of interventions aimed at supporting development in terms of both 'primary mathematics knowledge for teaching' and of primary mathematics teaching itself. Quasi-longitudinal data on learner performance in the early primary 'Foundation' years (Grades 1 -3) has pointed towards some improvements in early number learning, with broad evidence of moves from highly rudimentary 'count all' strategies used for early additive relations problems to the more efficient 'count on'/'count down' strategies underpinned by some initial reifications of number (Sfard, 2008). With interventions in place focused on working with teachers to develop number sense, our attention has started to turn towards working to understand whether, and, if so, how, changes in learning might be linked to changes at the level of teaching. We have found the work on example spaces, and seeing expansions in teachers' ways of working with example spaces useful for thinking about the changes in teaching that we have observed. But these explorations have focused firmly on the middle ground of conventional example spaces, rather than on the boundaries. In this sense, they are more aligned with supporting mathematical learning in the middle ground rather than at the boundaries of particular topic spaces, with the examples selected grounded in expanding the boundaries of personal and situated example spaces, rather than mathematical boundaries of concepts in any more disembodied, objective sense. Our considerations of changes in teaching have led to the development of a framework for exploring differences, focused on primary teachers 'mediating primary mathematics[1]' (MPM). Empirically, our attention within this framework centres on teachers' mediation of mathematics as enacted in the context of their selected example spaces across episodes in lessons through their use of artifacts, their inscriptions and their talk, with focus on the following strands:

1 Mediating with artifacts

2 Mediating with inscriptions

3 Mediating with talk & gesture in a) methods for generating/validating solutions; b) building mathematical connections; and c) building learning connections: explanations and evaluations of errors/ for efficiency/ with rationales for choices

The concepts and theories underlying the aspects in the MPM framework have been detailed in other writing (Venkat & Askew, under review). Expansions in teachers' personal, situated example spaces, as seen in observed lessons, feature within two key strands of our overall framework – 3a and 3b – and our attention in this paper is on detailing the ways in which expansions in example space are considered in relation to the two bodies of literature discussed at the start of this paper, with the illustration and discussion in this paper focused on our ways of thinking about expansions within these strands. We remain interested in the notions of abstraction and generalization that Mason (1989) has written about over an extended period of time, and have looked at ways of retaining a focus on these elements drawing from a base in highly conventional example spaces. Our illustrations of levels of mediation for structure and generality in these two strands draw from excerpts of teaching and teacher explanations seen across our work supporting teaching development in Foundation Phase classrooms, and also, in this paper (and for the purposes of exemplification of the levels), from teacher responses to tasks in our primary mathematics for teaching courses.

Analytical discussion

Strand 3a: Method for generating/ validating solution

We have formulated expansions in teaching related to this strand as follows:

	0	1	2	3
Method for generating/ validating solutions	No method or problematic generation/ validation (Mixing of knowns and unknowns)	Singular method/validation (provides a method that generates the immediate answer and can produce answers in the immediate example space)	Localized method/validation (provides a method that can generate answers beyond the particular example space)	Generalized method/validation (provides a strategy/method that can be generalized to both other example spaces AND without restriction to a particular artefact/inscription)

Our earlier work pointed to problems with the coherence of teachers' ways of generating solutions to the problems that they set in the context of their teaching or that were set to them in our mathematics knowledge for teaching courses. At the lower extreme, lack of coherence manifested itself in a range of ways. We saw the mixing of knowns and unknowns in teachers' solution processes, with unknowns sometimes drawn into the problem-solving process prior to their actual generation, and questioning sometimes treating 'knowns' as if they were unknowns. For example, we saw episodes where, in adding 10 to a number on a 100 chart, the teacher counted on from, e.g. 17, saying 18, 19, 20, etc, and stopping at 27, without any overt demonstration of keeping track of how many she had counted on. Knowledge of the answer – the unknown value – was assumed in this kind of working. There was also evidence of teacher talk that connected poorly with the artifacts and inscriptions in use during

the focal episode, disrupting possibilities for generating or validating a solution to the tasks in the example space being worked with.

In the middle ground of this strand, we focused on the extent of applicability of the methods that teachers communicated for solving problems in, and related to, the example space being worked with. An excerpt of teachers' responses to a task we used in our teacher development work helps us to illustrate some of the range in this strand. The question in focus was:

Is $-8 < -5$? What diagrams and explanations might you use to help to explain your answer?'

Almost all teachers from a group of 50 we were working with responded to this by drawing a number line and marking –8 and –5 in the correct positions on their number lines. The explanations linked to this inscription varied between two key justifications. Some stated that –8 was less than –5 because it was further away from 0, with many teachers marking 0 and annotating the two respective distances for the immediate example on their number lines. In some instances, this 'particular' statement was accompanied by more general statements like: 'Numbers further from 0 are smaller'.

Other teachers had annotated their number lines with the words: 'Further to the left, numbers are smaller. Further to the right, numbers are bigger.' The first formulation remains applicable if both (or all) of the numbers being compared are of the same sign, i.e. both negative or both positive, but is not necessarily true if the numbers under consideration have different signs. The latter statement, in contrast, is more generally true across the real number example space and the number line convention. We code the latter formulation as a Level 3 offering, but we still have questions about whether the first formulation should be coded as a Level 1 or Level 2 offering, and whether this should be interpreted in relation to the example space that is being worked with in the empirical terrain. If the example space only contains pairs of negative numbers, we could argue that while the method offered is 'localized' (in the sense that the method can also be applied to pairs of positive numbers as well), there are limitations in the example space that can be critiqued. The broader point remains valuable though: that it appears useful to think about teachers' offers of methods for solving or validating problems in relation to their extent of applicability to example spaces related to the concept in focus. Further, this appears important in a context of broad evidence of a lack of move beyond unit counting strategies – which we have usually coded at Level 1 due to their inefficiency for use as the number range being worked with increases.

In a sense, what we are focusing on in this strand relates to the range of permissible variation in the example space that can be managed within the constraints of the solution procedure that has been taught. This range is considered pragmatically rather than strictly mathematically: for example, at one level, even large additive relationship calculations can be carried out using the unit counting strategies that we have noted as prevalent. In practice though, such methods are both inefficient and error-prone, and therefore represent solution choices with pragmatic limitations as the number range increases. In larger number ranges, 'structured' representations of number – i.e. representations of number underpinned by properties and relations, offer much greater purchase for both range of applicability, and for flexible efficiency. While not the focus of this paper, we code 'structured' representations (representations underpinned by structural properties of number) more highly in the artifact and inscription strands than 'unstructured' representations of number. In the 3a strand, a focus on solution

procedures employing numerical structure provides an important feature within the possibility for attention to procedures with more extensive scope of applicability.

Strand 3b: Building mathematical connections

Expansions in relation to building mathematical connections have been conceptualized as follows:

	0	1	2	3
Building mathematical connections	Disconnected and/or incoherent treatment of examples OR Oral recitation with no additional teacher talk	Every example treated from scratch	Connects between examples or artefacts/ inscriptions or episodes	Makes vertical and horizontal (or multiple) connections between examples/ artefacts/inscriptions / episodes

This strand was built into the framework on the basis of common evidence of disconnections. These disconnections sometimes occurred, as noted in the previous strand, within teachers' handling of examples, leading to incoherent explanations. Further, in our observations, and noted as prevalent in the broader South African landscape, is evidence of extensive chorused oral recitation, for example of skip counting in multiple sequences, with no input from the teacher. Both of these phenomena were coded at the lowest level of the building mathematical connections strand.

Example spaces figure within this strand in terms of the nature and extent of the connections between the examples seen. Watson & Mason (2006b) have described these kinds of connections in terms of vertical connections between elements of examples – which they describe as 'going with the grain' and horizontal connections within examples (e.g. equivalence structures within relationships) – described as 'across the grain' connections. Linked focus on several examples allows for attention to invariances amongst the variation between examples, with these invariances forming the grounds for abstraction and generalization. Keeping vertical and horizontal, and more generally, multiple connections between examples in focus, allowed us similarly to pay attention to shifts in the possibilities for building awareness of generality in pedagogic mediation.

Our way of attending to example spaces in this strand contrasts with our focus in the previous strand where the emphasis was on the extent of 'reach' into other examples and example spaces of the methods for solving problems that are communicated within the focal example space. At Level 1 of this scale, we placed episodes where examples within the focal example space were dealt with as individual and separate instances. Venkat & Naidoo (2012) describe an episode involving finding pairs of numbers adding to 16, where each offered pair is verified as correct by concrete unit counting, with very limited reference to any of the partitions that were established previously in the episode as correct. In this 'extreme localization' there is no opening for a focus on 'examplehood' as thinking about instances as 'examples of' some property necessarily requires some invoking of either other examples from which abstractions can occur, or juxtaposing the instance with the property or generalization or definition of which it is an instance. In our observations and analyses, we noted that

this invoking could occur through teachers making connections between examples in the example space as noted here, but could also occur more multi-directionally through connections made between artifacts, inscriptions and episodes as well. Importantly for us, this way of considering the possibilities for generality can, once again, be worked with in the context of conventional example spaces. Askew (2015) has analysed one teacher's ways of working with multi-directional connections in lesson episodes focused on early place value, noting her working with the example space as a connected set of instances with fluid vertical and horizontal links, as well as links to place value artifacts involving ten strips and unit squares, and the inclusion of symbolic inscriptions. While the teacher's working with the examples was entirely coherent, the conventional nature of the example space contrasts, for example, with Watson & Mason's (2006a) 'stretching' of what looks, initially, like a conventional example spaces (related to co-ordinate plotting to fractional values) in the context of a single exercise, as a key aspect of hypothesizing a generality related to a given constraint.

Discussion

Working in this way with the concept of example spaces has allowed us to develop attention to structure and generality in somewhat different ways to those presented in Watson & Mason's writing. Specifically, the base for generalizations is located in conventional example spaces, with limited – if any – move towards boundary examples. We would acknowledge that concepts are less fully rounded out in this way of working. But we would also argue that these more mundane expansions are important to thinking about developing primary mathematics teaching in contexts of the flat learning profiles described in Pritchett & Beatty's overview. At the lower extremes of both of the strands we have delineated in this paper, there is attention to students being able to reproduce coherent procedures that have been presented or offered and accepted in class. This kind of move already represents some potential for forward moves in relation to students' existing problem-solving repertoires. Further moves upward in the strands discussed start bringing other example spaces into the realms of possibility for learners. Our early analyses show moves towards coherence and connection in teachers' work with example spaces, coupled with greater inclusion of structured artifacts and inscriptions. The concurrence of these fledgling moves towards coherence and generality in pedagogy with improvements in students' performance on conventional and central example spaces suggests that our ways of working with example spaces may be useful to carry further into our research and development activity.

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[1] In previous writing developing this work, we have used the term 'Mathematical Discourse in Instruction – Primary', or MDI-P. The history of this work is a co-development of MDI frameworks between Hamsa Venkat and Jill Adler, that shared roots in socio-cultural theory but differed in specific formulations across work in secondary and primary mathematics. In order to avoid confusions between the secondary and primary level models, we have changed our titling of the framework to MPM. Writing with Adler and her team is underway, detailing the histories and trajectories of development of both MPM and MDI.

Self-scaffolding students' problem solving: Testing an orientation basis

Joana Villalonga Pons[1] and Paul Andrews[2]

[1]Universitat Autònoma de Barcelona, Catalunya, Spain; juanamaria.villalonga@e-campus.uab.cat

[2]Stockholm University, Stockholm, Sweden; paul.andrews@mnd.su.se

It is known not only that young students have difficulty solving mathematical problems but also that appropriate scaffolding can support them in the process. In this paper we describe the development and pilot implementation of a device for self-scaffolding grade 6 Catalan students' mathematical problem solving. Called an orientation basis (OB), the OB, which addresses cognitive, metacognitive and affective aspects of problem solving, drew on research describing the actions of an expert problem solver. The evidence indicated that the OB, while in need of refinement, had a positive impact on the problem solving behaviours of many participant students.

Keywords: Mathematical problem solving, scaffolding, orientation basis, Catalonia.

Introduction

Mathematical problem solving is difficult, both for students (Mason, Stacey & Burton, 1982; Pólya, 1945) and teachers trying to create appropriately conducive classroom environments (De Corte & Verschaffel, 2004; Schoenfeld, 2013). When children solve problems without being conscious of the relationship between their actions and their solutions their ability to transfer their solution processes to new situations will be limited (Coltman, Petyaeva & Anghileri, 2002). However, appropriate adult intervention can help children become aware of these processes (Coltman et al., 2002). Such interventions, known as scaffolding, build on what learners already know in order to close the gap between current learner competence and task objective (Bruner, 1985; Greenfield, 1984; Wood, Bruner & Ross, 1976). Moreover, given time, scaffolding can be provided by peers and, ultimately, students themselves (Holton & Clarke, 2006). In this paper we describe the development and use of a device, which we have called an orientation base (OB), for use in Catalan sixth grade classrooms. The OB's role is to support the transition from where teachers scaffold learner's problem solving to where students scaffold their own.

Problem solving

As a human activity, problem solving can be understood as an example of goal-directed behavior (Schoenfeld, 2007). It is a dynamic, but not necessarily linear, activity requiring the organization and activation of multiple skills and strategies (Mason et al., 1982; Pólya, 1945). At the heart of problem solving lies an appropriate mathematical knowledge, an awareness and experience of solution strategies, self-regulatory or metacognitive competence and a belief not only that the problem is worth solving but also that the solver can solve the problem (De Corte, Verschaffel & Op't Eynde, 2000; Schoenfeld, 2007, 2013).

A key aspect of the above, at least from the perspective of providing scaffolding for the learner, is the encouragement of students' metacognitive competence. For example, expert solvers spend more time understanding and analyzing the problem and solution process than calculating, and they continuously reflect on the state of the problem solving process (De Corte et al, 2000), behaviours that are typically absent with weak problem solvers (De Corte et al., 2004). Such students need

scaffolded support with respect to interpreting a task, identifying its sub-objectives and planning a strategy (De Corte et al., 2000; Mason et al., 1982). They need to learn how to reflect on their existing knowledge and thought processes; that is they need to learn how to evaluate and regulate their own thinking (Sanmartí, 2007). This regulative competence is not acquired automatically but emerges over time (De Corte et al., 2004). Thus, with support in understanding how things work, students can become more efficient and self-regulated problem solvers (Schoenfeld, 2013).

Scaffolding

Scaffolding in an educational context

Drawing on Bruner's (1975) initial observations with respect to the ways that parents scaffold their infants' learning, Wood, Bruner and Ross (1976) argued that knowledgeable adults can scaffold students' problem solving activity. Here, the adult seeks to reconcile implicit theories of the task components, the necessary steps to solution, and the child's capabilities (Stone, 1998). In this way, acknowledging a socially imitative process, six ways of assistance were differentiated; recruiting the child's interest, reducing the degrees of freedom, maintaining goal direction, highlighting critical task features, controlling frustration and modelling preferred solutions paths (Wood et al., 1976). Recent work has continued this theme, examining how teachers can best provide (temporary) support that enables learners to complete tasks they would otherwise not have been able to complete independently (Smit, van Eerde & Bakker, 2013; van de Pol, Volman & Beishuizen, 2010). In this process, whereby the learner becomes incrementally independently functional (Smit et al., 2013), both teacher and learner actively share and build common understanding (Stone, 1998; van de Pol et al., 2010).

Scaffolding strategies

Scaffolding is not a 'technique' that can be applied in every situation in the same way (van de Pol et al., 2010). As in the construction industry, where each scaffold is unique to a specific building, learning scaffolding can be provided at different ages and in a variety of ways, addressing learners' knowledge gaps as part of an ongoing progress (Wood et al., 1976). Significantly, effective scaffolding is thought to comprise three components, involving the six processes of feeding back, giving hints, instructing, explaining, modelling and questioning (van de Pol et al., 2010), which are

- Contingency: Support should be adapted to the student's current level of performance.
- Fading: Support is gradually withdrawn over time.
- Transfer of responsibility: Task completion is gradually transferred to the learner.

Moreover, effective scaffolding not only promotes learners' cognitive and metacognitive activities but also positive affect. Finally, acknowledging different agents in the process, whether they are informed adults, a group of learners or the individual student, scaffolding is progressively relocated to the learner, whereby the external dialogue of scaffolding becomes the inner dialogue of metacognition (Holton & Clarke, 2006).

Orientation basis for problem solving

One means of encouraging self-scaffolding of students' problem solving-related self-monitoring skills is to use an orientation basis (OB) (Sanmartí, 2007). Here we understand a problem solving-

related OB to be a necessary sequence of actions based on the problem solving behaviour of experts. An orientation basis leads the learner to a solution in ways that structure an emergent independence and problem solving autonomy. An OB is not a 'one size fits all' tool but tailored according to learners' requirements and achievements. At every age and according to the learner's needs, an OB can be presented through different statements.

Dimensions	Actions	Track
I understand the problem	A1. I have read the question twice, at least.	
	A2. I understand what the question wants.	
	A3. I have identified and understood the data.	
I devise a plan	A4. I have played with the data from the question.	
	A5. I have prepared a strategy.	
	A6. I have checked that my strategy fits the data.	
I apply my plan	A7. I have implemented my strategy.	
	A8. I have recorded all my actions in ways that I understand.	
	A9. I have recorded all my actions in ways others can understand.	
I review my task	A10. When I get stuck I go back to the beginning.	
	A11. When I have finished I have checked my answer(s).	
	A12. I have checked for other answers or better solutions.	

Table 2: The orientation basis (OB)

In this paper we discuss the development and implementation of an OB for grade 6 Catalan pupils. At this age, pupils are typically expected to have acquired a minimum background in problem solving. However, experience has shown that they lack regulative and problem solving competence, especially in understanding and analyzing the problem, and planning and implementing a solution process. Therefore, drawing on Pólya's (1945) problem solving principles, the OB depicted in Table 2 was developed. Each of Pólya's four dimensions comprised three particular actions, which can be tracked in the right hand column. The inclusion of each action was a consequence of earlier observations of the problem solving behaviours of grade 6 Catalan pupils and the problem solving strategies found in the literature (e.g. De Corte et al., 2004; Mason et al., 1982). The OB shown in Table 2, translated from the original Catalan, was designed to be a contingent, hint-giving, feedback tool focused on facilitating both fading and transfer of responsibility (van de Pol et al., 2010). As indicated above, the aim of this paper is to present an initial evaluation of the efficacy of the OB shown in Table 2 for scaffolding grade 6 students' mathematical problem solving.

The study

The participants were students in a 6th-grade class of a Barcelona primary school. Their teacher was an experienced generalist primary school teacher. Such teachers, who receive relatively little subject knowledge instruction during their pre-service education, typically acquire their mathematical knowledge in practice, a situation much criticised (Egido, 2011; MECD, 2012). Data were collected during a regular, 50 minute, lesson at the end of the second quarter of the academic year 2015-2016. They derived from 22 students' initial use of the OB as they tried to solve the mathematical problem posed in Figure 1, which was originally posed in Catalan.

Problem: My grandmother's blanket

My grandmother is a crochet expert. She loves doing crochet blankets. I haven't seen yet the last one she made, but she told me how she did it: "I joined squared pieces. Their styles are different but all of them have the same size."

She explained me that the boundary of the blanket is made with a dozen of green squares with a white rhombus, and with a dozen of white pieces with green dots. She set them up alternately: one green piece with rhombus, one white piece with green dots,... and she managed so that in the four corners there is always a green dotted blank piece.

She also told me that the interior part of the blanket is really different. It is made with different coloured pieces, but all of them have the same size of the external ones.

So,

1. How could my grandma distribute the pieces to make the boundary of the blanket? Explain how you know it.
2. I wonder how many pieces she used to end the interior part of the blanket. How can I know it?

Figure 1: The Problem translated from Catalan

Before solving the problem, the teacher explained the purpose of the OB carefully and together with the class discussed and clarified the meaning and purpose of each element. This ensured, as far as is practicable, that students understood its vocabulary and overall purpose. Students were each given a copy of the OB's rubric, which included a grid in which they recorded their engagement with the OB as well as a paper copy of the problem on which their solution was to be written. Students were instructed to solve the problem, using the OB to guide their activity, and then record the OB actions they addressed. They were also told that their teacher would not intervene in the problem solving process but check, as they worked, that they completed their OB tracking.

Results

Table 3 shows the data from all 22 students' use of the OB as they worked on the problem. It can be seen that only one student, Student 21, failed to engage with the OB, while all others used it in varying degrees. Nine students obtained correct solutions for both parts of the problem, a further five managed just one part and eight failed to complete either, including the one who failed to complete any OB actions. Four students indicated some difficulty with respect to understanding some OB actions. In this respect, all four found A3, 'I have identified and understood the data', difficult to understand. The only other action that caused uncertainty was A6, 'I have checked that my strategy fits the data'. Thus, in the light of an OB being necessarily adaptive (Sanmarti, 2007), these issues would be addressed in the next iteration of its development. Importantly, even when faced with uncertainty, each of these students was able to continue the problem solving process to at least the next step. Student 12, the only student who found two statements difficult, completed 11 of the OB's stages but failed to complete either part of the problem. Importantly, from the perspective of the OB's development, Student 9 completed all the OB's actions but failed to solve either part of the problem, pointing, perhaps, to the need of cognitive interviews to determine in depth the nature of the difficulties encountered in completing the task.

	Student																					
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
A1	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X
A2	X	X	X	X	X	X	X	X	X	X	X	X		X	X	X	X	X		X		X
A3	X	X	X	*	X	X	X	*	X	X	X	*		X	*	X	X	X		X		X
A4	X		X	X	X	X	X	X	X	X*	X	X		X	X	X	X			X		X
A5	X	X	X	X	X	X	X		X	X	X	X	X	X		X	X			X		X
A6		X	X	X	X	X			X	X	X	*		X		X	X			X		X
A7		X	X	X	X	X			X		X	X		X		X	X			X		X
A8		X	X	X	X	X			X	X	X	X		X		X	X			X		X
A9			X			X			X	X	X			X		X	X			X		X
A10			X	X	X	X			X		X	X		X		X	X			X		X
A11			X	X		X			X	X	X	X		X		X	X			X		X
A12			X	X		X			X	X	X	X	X	X		X	X			X		X
Total	5	7	12	11	9	12	5	4	12	10	12	11	3	12	4	12	12	3	1	12	0	12
Correct	0	2	2	1	1	2	0	1	0	0	2	0	0	2	1	2	2	1	0	2	0	2

Table 3: OB-related data for each student. An asterisk shows a difficult but completed OB action

Was the OB effective and did students take it seriously?

As is typical of classroom interventions, eliciting evidence of their efficacy and their being taken seriously is not straightforward. With respect to its efficacy, it is interesting to compare the number of completed OB actions with the number of completed problems, as shown in Table 4. A Fisher exact probability, $p = 0.008$, indicated not only that the figures of Table 4 were unlikely to have been due to chance but, importantly, that students who failed to complete the OB tended not to complete the tasks. Indeed, Table 4 shows that a necessary but not sufficient condition for the completion of both tasks was that students completed seven or more OB actions.

		Number of correct solutions		
		0	1	2
Completed OB activities	0-6	5	3	0
	7-12	3	2	9

Table 4: Number of completed OB actions and number of successfully completed problems

When viewed as four dimensions rather than as individual actions the data offer further interesting insights. For example, while probably not surprising, the data of Table 3 show that as they move down the OB, the number of students completing each dimension gets smaller. With respect to the first dimension, 'I understand the problem', 21 students began its three actions, of which 19 (90.5%) completed all three. With respect to the second dimension, 17 began with the first action, of whom 12 (70.6%) completed all three. The third dimension, 'I apply my action plan', was begun by 13 students (one more than completed the second dimension), of whom 9 (69%) completed the dimension. Finally, of the 12 students who began the final dimension, 'I review my task', 11 completed (91.7%) it. These figures tell two stories. The first is that students who begin working on a dimension can typically be expected to complete it. The second is that once they reach the final dimension students seem almost guaranteed to complete it. In other words, an in-depth examination of the four dimensions can also inform future developments of the OB; the first and last dimensions seem less problematic with high completion rates in comparison with the middle two.

Looking at the data qualitatively, it can be seen that students' solution attempts tended to show that they took the OB seriously. Students were able to connect OB actions to their own activity, and did not confirm those actions until after they had been completed.

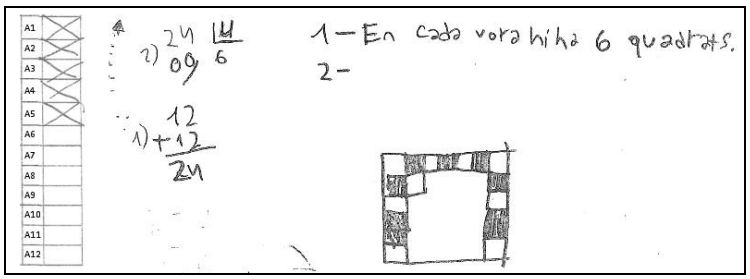

Figure 2: A solution of the problem and related OB tracking

For example, Figure 2 shows a student solution and his OB tracking. The picture confirms that he had read the problem and understood what the first question required. For example, his arithmetical operations and note, 'on each side there are 6 squares', indicate not only that he had identified and understood the required data but that he had also played with the data which let him to prepare a strategy for the first part of the problem. In short, the solution the student presented corresponded with the OB actions he claimed to have completed.

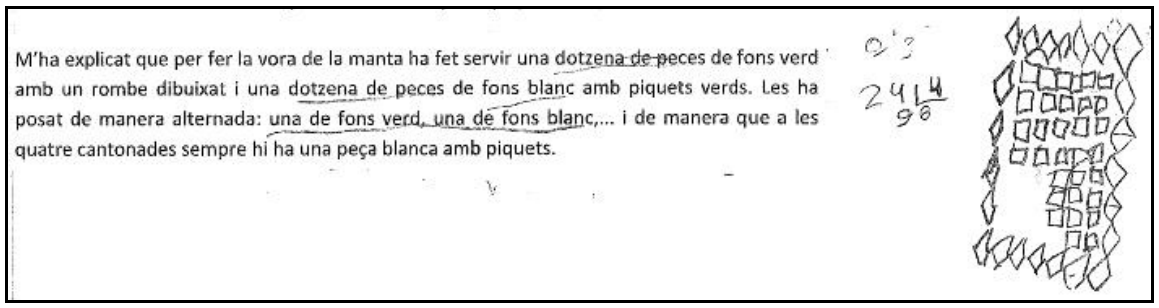

Figure 3: Solution of the pupil who failed to complete explicitly any OB action

Even when students failed to complete any OB action, there was evidence of its having influenced their solution attempts. For example, Figure 3 shows how the single pupil who failed to complete any OB action attempted to address the OB's first action.

Student response to the OB

Several students attempted to communicate with the OB, particularly when uncertain as to its intentions. Figure 4 shows, in the underlining of the word *quefer*, uncertainty as to its meaning and, essentially, an invitation for someone to explain. In similar vein, students annotated their OB in ways indicative of doubt or just a desire to comment on their response, both cognitively and affectively. Figure 5 shows comments inserted alongside the ticks indicating the student's completion of the various actions. The top two comments are the same and translate as, 'yes, but it takes me a great effort'. The lower comment, while similar in its intention, translates as, 'regular, because it takes me a great effort'.

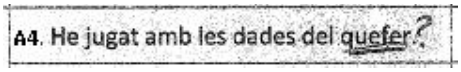

Figure 4: A student's doubts with respect to the meaning of the fourth action

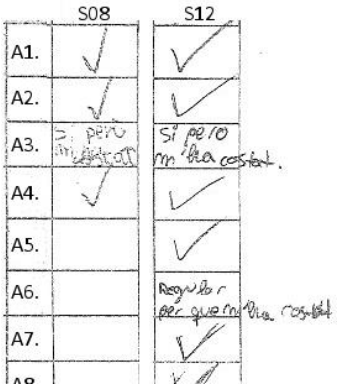

Figure 5: Two annotated record sheets

Discussion

In this paper we have outlined the development and trial of an orientation basis, designed to support 6th grade-students' problem solving-related self-scaffolding. Derived from the literature the four dimensions, and their respective actions, provided evidence suggesting that the OB has a role to play. The four dimensions and the means of their operationalisation make real for students the actions that guide problem solving (Holton & Clarke, 2006). The evidence supports earlier findings that appropriate scaffolding may have a beneficial impact on cognition, metacognition and affect (van de Pol et al., 2010). However, with respect to the extent to which the OB for problem solving is contingent, exploits fading and encourages transfer of responsibility (van de Pol et al., 2010) is variable. With respect to contingency, our view is that students were able to connect OB actions to their own activity and those who were affected by typically persisted until at least the next step. Also, students took the OB seriously, indicating initial support for both fading and transfer of responsibility, although a longitudinal study would allow these to be better examined. The dimensional structure and the ways in which students use the actions embedded within it point towards a productive cycle of refinement. Despite its linearity, based on the behaviours of an expert problem solver, students' engagement with the final dimension confirmed not only the cyclic nature of problem solving but also the role of the OB in supporting students' awareness of it. Finally, the OB comprised short statements written in the first person. Our view is that it helps learners' not only understand what problem solving expects of them but also anticipate possible actions.

Finally, this paper has reported on the first iteration of an emergent study. Since the completion of this first task students have solved two further problems using the OB. Their teacher has commented, anecdotally, that students are becoming more familiar with and confident in their use of the OB. Therefore, a longitudinal analysis of students' OB-related problem solving would seem an appropriate next step. As found with previous studies, the impact of scaffolding is difficult to evaluate (van de Pol et al., 2010) and this will remain a key objective of future work.

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The Realization Tree Assessment tool: Assessing the exposure to mathematical objects during a lesson

Merav Weingarden[1], Einat Heyd-Metzuyanim[2] and Talli Nachlieli[3]

[1]Technion, Faculty of Education in Science and Technology, Israel; merav.weingarden@gmail.com

[2]Technion, Faculty of Education in Science and Technology, Israel; einat.metz@gmail.com

[3]Levinsky College of Education, Israel; talli.nachlieli@gmail.com

We present a new tool – Realization Tree Assessment (RTA) for assessing the mathematical quality of lessons and the ways in which the whole classroom discussion expose students to mathematical concepts. The tool, built upon the commognitive framework, depicts the different realizations of a mathematical object treated in a lesson, and then uses different shades to signify who articulated the realization – the teacher or the students. We exemplify the tool on two lessons implementing an identical Hexagon pattern generalization task. The RTA visualizes the manner in which one lesson gave students sufficient opportunities to "same" different algebraic expressions, while the other lesson did not. We show how this visual presentation of the mathematical ideas complements existing assessment tools, particularly, the Instructional Quality Assessment and Accountable Talk. We conclude by discussing the potential of the tool as an aid for lesson planning.

Keywords: Teaching practices, Realization tree, commognition, cognitive demand, lesson assessment tools.

Introduction

Recent years have seen increasing efforts to train teachers to teach exploratively – provide students with opportunities to engage with cognitively demanding tasks, problem solve, and participate in rich mathematical discussions (Schoenfeld, 2014). Within such efforts an important role lies in the tools that are used to examine lessons enacted by the trained teachers (Boston & Smith, 2009; Schoenfeld, 2015). Scoring and evaluation tools (such as the Instructional Quality Assessment tool or TRU math) can be used both for evaluating lessons and thereby examining the effectiveness of the training program, as well as tools for teachers' professional development. A common difficulty with these tools, however, lies in operationalizing their criteria for evaluating the quality of mathematical ideas dealt with in the lesson. In this paper, we propose an analytical tool – the Realization Tree Assessment tool which is based on the "commognitive" framework (Nachlieli & Tabach, 2012; Sfard, 2008). This tool enables drawing a succinct yet sufficiently meaningful picture of the mathematical concepts surfaced in a specific lesson in such a way that lessons can be both compared with each other as well as planned ahead more accurately.

Theoretical background

Tools for the examination and evaluation of classroom instruction can be categorized into three types: scoring tools such as Instructional Quality Assessment, or IQA (Boston, 2012b) and the Teaching for Robust Understanding of Mathematics summary, or TRU math (Schoenfeld, 2014); "coding and counting" tools, such as Accountable Talk (O'Connor, Michaels, & Chapin, 2015); and qualitative analytical tools (e.g. commognition, Sfard (2008)). Scoring and coding measures have the benefit that they are quantifiable. They thus enable both the comparison of teachers with each

other, as well as comparison of within-teacher change from lesson to lesson, for example, as a result of professional development. Scoring tools, however, have a drawback. They are heavily based on extensive training of scorers for the development of inter-rater reliability. This, because of their high-inference nature. Coding and counting tools, which are based on coding talk moves, necessitate lower inferences and are therefore easier for achieving reliability. However, these tools are mostly good for capturing non-mathematical aspects of the discourse.

The difficulty in assessing the surfacing of "important mathematical concepts" (Boston, 2012a) or "important content and practices" (Schoenfeld, 2015) in a lesson is not surprising, given that the definition of "mathematical concepts" has been under much dispute for decades (Sfard, 2008). To our aid, we draw on commognition (ibid), which we have extensively used in the past as a qualitative tool for describing learning-teaching processes. In the present work, we simplify this tool, to attune it with the demands of coding and scoring schemes that seek to evaluate lessons in a relatively short period of time, for the goal of comparing large sets of lessons.

Realization trees

Mathematical learning, says Sfard (2008), is a process whereby students gradually become able to communicate about *mathematical objects*. These objects are produced by discourse (or communication), and are made up of different "realizations" (ibid, p. 165). The term realization is used by Sfard instead of the more common term "representation", to emphasize the fact that nothing is, in fact, "there" to be represented. All mathematical objects are products of human discourse and come to life by being different realizations being "samed" and alienated from human agency so that they are talked about as existing of themselves. For example, the signifier ½, the process of dividing a pizza into two pieces, and the process of shading 3 circles out of 6, are all samed into the object "one half". Children often learn each of these realizations separately and only later come to relate to them all to one object. This is the heart of a process Sfard calls "objectification". Objectification, or talking about mathematical signifiers as "standing for" mathematical objects that "exist" in the world, is a major and necessary accomplishment for advancing in the mathematical discourse. Sfard used the term "realization tree" to illustrate the fact that realizations are usually hierarchical. A half is made of different realizations (1/2, 0.5, 50%, 3/6 etc.) but the whole numbers making up these realizations also have endless realizations (3 apples, 3 fingers, etc.). Nachlieli & Tabach (2012) used realization trees to visually explain the complexity of the object *function* and to relate to the historical development of this object, as well as to make explicit students' development of the discourse of function. Before moving to explain our use of realization trees as tools for assessing the conceptual quality of a lesson, let us briefly describe the two other tools that have been serving us for quantifying and comparing mathematics lessons.

IQA

The IQA (Instructional Quality Assessment tool) has been designed by Boston and Smith (2009; Boston, 2012a) to evaluate the *cognitive demand* of mathematical lessons. This, based on the "task framework" put forward by Stein and her colleagues (1996), which differentiates between the cognitive demand of a task, the way it is presented to the classroom, and the way students eventually engage in it. Every rubric in the IQA is scored on a scale from 0 to 4. For reasons of space, we will concentrate here only on two rubrics: AR-2 (implementation) and AR-X

(mathematical residue). Regarding the implementation rubric, 1 means students engage only in rote memorization and producing facts, 2 means they engage in the application of procedures explicitly taught, 3 means cognitive demand is not lowered but mathematical reasoning is not sufficiently explicated, and 4 means full engagement in a cognitively demanding mathematical task. 'Mathematical concepts' or 'ideas' are mentioned almost in every rubric in the IQA. For example, in the rubric that refers to the mathematical residue, the highest score should be given when: "The discussion following students' work on the task surfaces the important mathematical ideas, concepts, or connections embedded in the task" (Boston, 2012b, p. 20). However, IQA does not provide any clear guidance on this matter, besides giving a few examples of high and low level lessons.

Accountable Talk

Accountable Talk coding (Resnick, Michales, & O'connor, 2010) is a tool originating in socio-linguistic analysis of classroom talk (O'Connor & Michaels, 1993). It provides teachers with a set of specific *talk moves* they can make during whole classroom discussions, to hold students accountable to the community, to knowledge and to reasoning. Our version of Accountable Talk coding (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016) includes eight codes for teacher moves (e.g. press for reasoning, revoice, restate, agree/disagree) and four codes for students' moves (e.g. student-agree, student-justification). These moves track the amount in which teachers attempt to make students' thinking public, help students to reason mathematically, and hold them responsible for attending to the reasoning of others. Though the manual does contain examples of mathematical statements, Accountable Talk's basic framework does not deal specifically with content. It has no clear indicator of what consists as more important or "conceptual" reasoning, and what does not.

The study

In what follows, we first describe the setting of the study on which we developed the Realization Tree Assessment tool (RTA). We then describe the results of analysis using the IQA and AT, showing what could be achieved by them and what was missing or difficult to agree upon. We follow this by describing the RTA results for the data, showing where they agree, complement and elaborate on the findings obtained by the IQA and AT.

Setting

The study reported here was performed in the context of a project for training Israeli teachers to implement explorative instructional practices in middle school mathematics classrooms, using methods inspired by Smith & Stein's (2011) "Five Practices for Orchestrating Productive Mathematics Discussions". In this report, we focus on two teachers: Dani and Sivan. Dani was teaching a 7th grade classroom in a school serving a community of middle-high socio-economic background. Sivan was teaching an 8th grade classroom in a school serving a community mostly from a low-middle socio-economic background. Both teachers participated in training sessions where the instructor planned together with each of them separately a lesson according to the "5 Practices". In both cases, the lesson centered around an identical task: the Hexagon Task. The main session in the task was to write description that could be used to compute the perimeter of any train in the pattern of hexagons (See Figure 1):

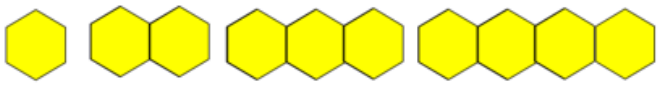

Figure 1: The Hexagons Pattern

The reason this task was used, was that it has proved in a previous study (Heyd-Metzuyanim et al., 2016) to be very productive for teachers who are beginning to implement the "5 Practices". We observed, video recorded, and transcribed both lessons. In addition, Dani and Sivan were both interviewed before and after the lessons, and their lesson planning sessions were recorded. In what follows, we present the IQA and AT measures of the two lessons, as well as what was still missing from them for a full understanding of the task implementation.

Findings

Accountable Talk in the two lessons. Both Dani and Sivan's lessons were conducted over a double period (90 Minutes) and both included work in groups (or pairs) where the teacher was walking between the groups, followed by a whole classroom discussion. The two whole classroom discussions took similar time (in Dani's classroom 28 minutes and in Sivan's lesson 26 minutes).

Overall, there were many more AT moves in Dani's lesson (98) than Sivan's (46). In particular, Dani's lesson had much more student talk moves coded as AT moves, either as student agree/disagree (N_Dani=22, N_Sivan=0), or as student justifications (N_Dani=20, N_Sivan=11). Dani was also higher than Sivan in pressing for students' reasoning (N_Dani=23, N_Sivan = 14). The overall picture drawn from the AT measure is, thus, that Dani's lesson had more accountability to reasoning and to the community than Sivan's lesson. Using the AT measure alone, however, does not enable learning about what mathematical concepts were dealt with, and which mathematical ideas surfaced through the discussion.

IQA scoring of two lessons. According to the IQA, Dani's lesson got higher scores then Sivan's lesson on all the rubrics, except the potential of the task, which was given in both cases by the teachers' trainer. In the Implementation rubric, we scored Dani's lesson as a 4, since multiple solutions were found and presented by the students; the teacher did not lead the students towards any particular solution; solutions were linked to each other both by the teacher and by the students; and there was no proceduralization of the task. In contrast, Sivan's implementation scored a 2. Though students generalized the Hexagon pattern into a expression, this was not done through the visual Hexagon's representation, only through the table; connections were not made with other algebraic expressions; in particular, students seemed to be well rehearsed in producing a table, algebraic expression from it and a graph of that expression, thus the task was proceduralized.

In the mathematical residue rubric, the results of the scoring were similar. Dani's lesson received a 4 since: the mathematical idea of equivalence of algebraic expressions was driven through the different algebraic solutions student presented. Evidence for students' understanding could be seen in one of the girls' exclamation "so they're all the same!" In contrast, Sivan's lesson scored a "2" on the mathematical residue rubric, since although the discussion dealt with some mathematical ideas, it did not touch upon the main idea behind the Hexagon task. The teacher did not focus on the different algebraic expressions but rather on the different representations of a linear function (graph,

table and algebraic expression). However, as will be shown later, even this idea was not treated fully and appropriately.

Of all the Academic Rigor rubrics, we found the "Mathematical Residue[1]" most difficult to operationalize. It appeared Dani and Sivan had different ideas regarding the mathematical goals of their lessons and this had consequence for the way they led the lesson. While Dani seemed to be well aligned with the goal of showing the equivalence of algebraic expressions, Sivan seemed as though she was mostly aiming at ideas related to linear functions (which are, indeed, part of the 8th grade curriculum). We therefore searched for a tool that would aid in explicating the mathematical ideas explored in the two lessons. For this end, we developed the RTA.

Realization Tree Assessment tool

The first step in RTA is examining the task and explicating the *mathematical object(s)* that can be surfaced through engagement with the task. This includes the different realizations that are reasonable to expect from students at a certain grade level. In our case, we built our realization tree based on a lesson plan provided by the Institute for Learning (http://ifl.pitt.edu/index.php/educator_resources), where the different solutions, expected from middle schoolers for this task were drawn out. This produced a "blank" tree, with nodes as seen in Figures 2 and 3. We then proceeded to shade the tree nodes with four different colors, as follows: Shade no 4: the student's explanation was complete and accurate; Shade no. 3: the student's explanation was not complete and accurate but the teacher helped explicating the idea; Shade no. 2: the student did not articulate the realization, but the teacher did; Shade no. 1: The realization was partially mentioned, but neither the student nor the teacher explained it fully.

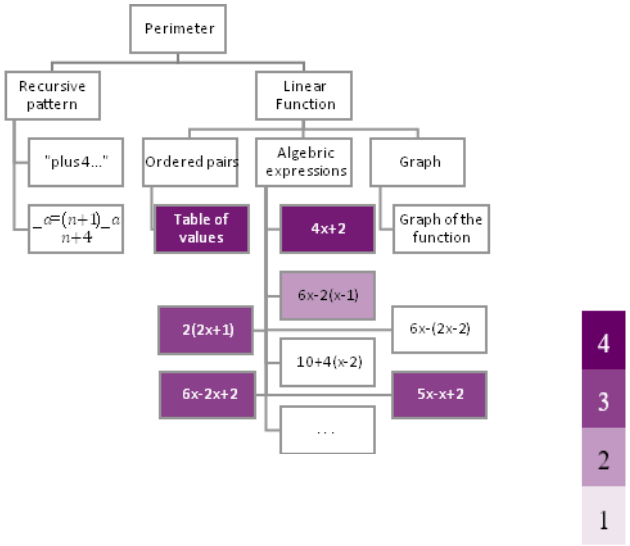

Figure 2: The RTA of Dani's lesson

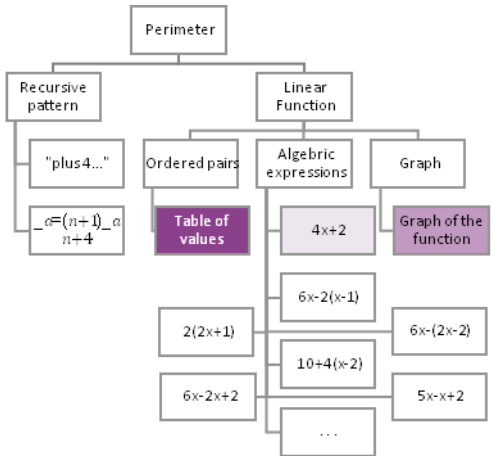

Figure 3: The RTA of Sivan's lesson

[1] The Mathematical Residue rubric appears in our manual as "under development".

Finally, if the realization was not mentioned at all, but was hypothesized to be relevant to the lesson and the grade level according to the lesson plan, it was shaded white (no. 0).

As can be seen in Figures 2 and 3, the main branch of our realization tree ("algebraic expression"), branches out on the multiple realizations of the algebraic expression . This, in accordance with the potential of the task to explore the different ways in which the visual representation of the hexagon sides can be generalized into a pattern and expressed algebraically.

Figure 2 describes the RTA for Dani's lesson. It shows that three realizations were explained fully and completely by students, three were explained by students, but the teacher filled in some gaps in these explanations, and one realization was explained only by the teacher. This full treatment of the "algebraic expressions" branch led students to endorsement of the narrative that "they all (all the algebraic expressions describing the pattern) equal to ", thus to the saming of different realizations, which was the goal of the lesson, as expressed both by Dani and by the teacher trainer.

In contrast, the RTA for Sivan's lesson (see Figure 3) is much lighter and sparser. It shows that only three realizations were treated in the lesson, and none of them was fully explained by the students. Moreover, the main branch of the tree – the "algebraic expressions" branch, is particularly empty. Only the realization was treated, and even that one was not explained accurately by the teacher or the students. The relative "emptiness" of Sivan's RTA corresponds well with the relatively low IQA and AT scores her lesson received. Still, it puzzled us, since Sivan was prepared in the PD very specifically for a lesson that was envisioned as similar to that of Dani. "What went wrong?", we asked ourselves. In order to answer that, we went back to the planning session, as well as to the post-lesson interview with Sivan, conducted right after the lesson. We found that, despite the PD instructor's conviction that she and Sivan were "on the same page", Sivan, in fact, had different goals for the lesson. She was focused on connecting the lesson to the previously learned unit on linear functions, where she had taught students to connect the concept of "slope" with the term "m " in $y = mx + n$, as well as connect it with the visual slope of a linear graph:

> "I wanted the students to see that every time it rises by four so that they will connect it with the slope that we have done with functions… I deliberately divided the board into three sections, to show the different stages in reaching the function itself - the graph that combines all the various representations of the function". (Sivan, Post-lesson interview)

It appears, then, that Sivan had a *different* mathematical object in mind (though probably only tacitly) when she planned the lesson – the "linear function" object[2]. Within the linear function object, the "slope" attribute of that object was her focus of attention. This could have been an appropriate goal for the lesson, had it been explicated and thought through. In particular, the following realization tree (see Figure 4) could have been appropriate for discussing slope and linear functions.

[2] Though she named it inaccurately simply a "function", we understood from the context and from the curriculum she was referring to linear functions.

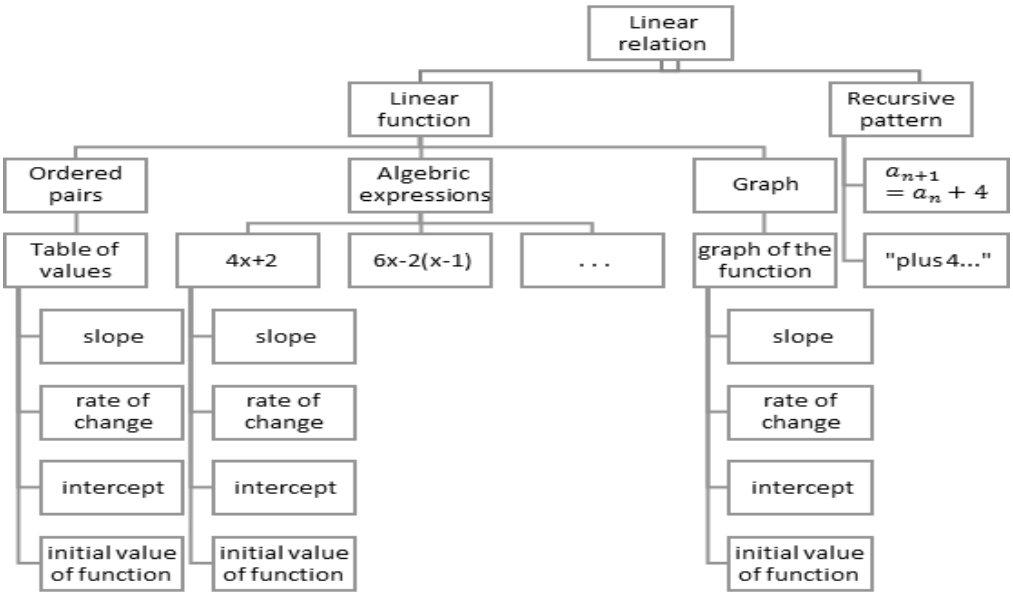

Figure 4: Alternative Realization Tree for discussing slopes and rate of change

However, the Hexagon task, especially as written for this lesson, was probably not the optimal task for talking about "slope". This, since it depicts a situation where the function is discreet and cannot be described using a linear line. In practice, Sivan neglected very early the connection to the Hexagons drawing. Thus even the "rate of change" (which could have been visualized as the addition of four sides with the addition of each hexagon) was not connected to the "slope" on the graph.

Discussion

Our goal in the present report was to present a new analytical tool for the evaluation of mathematical lessons – the RTA. Though this tool does not give a numerical value such as scoring and "coding and counting" tools do, it still enables relatively easy qualitative comparisons between lessons. We have used this tool to enable comparison between two more lessons that were performed on the Hexagon task, and the results give a quick overview of the mathematical opportunities to learn in each lesson. The RTA can also serve as an aid for determining the quality of mathematical content (or "mathematical residue") that is sought after in coarser grained assessment tools such as the IQA. In addition, the RTA can give us information about the potential of the task to engage students in explorative mathematical learning and about the relation between this potential and the actual implementation of the task in the classroom.

In the two cases reported here, the application of the RTA was done post-hoc, after the lessons were planned, implemented and recorded. However, we believe there is much potential for using this tool as an aid for planning lessons and training teachers for explorative mathematics instruction. Such a tool is particularly needed in light of previous findings which point to the difficulty of teachers to explicate to themselves the *mathematical goals* of the lesson (Heyd-Metzuyanim, Smith, Bill, & Resnick, submitted). We also believe that drawing realization trees with teachers will help them plan tasks and whole classroom discussions that provide sufficient opportunities for explorative participation. Often, when teachers talk about explorative instruction, their focus lies on the social or socio-mathematical norms of the classroom, such as students talking and listening to each other

(Heyd-Metzuyanim, Munter, & Greeno, submitted). We believe no less emphasis should be put on the nature of mathematical objects that students get exposed to, and on the paths for objectification that are opened through sufficiently rich mathematical discussions.

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A study of readiness for the integration of 21st century practices in mathematics classrooms

Aibhín Bray, Elizabeth Oldham and Katriona O'Sullivan

Trinity College Dublin, the University of Dublin, Ireland; braya@tcd.ie, eoldham@tcd.ie and osullik14@tcd.ie

This poster explores the complex process of integration of 21st Century (21C) teaching and learning practices into mathematics classrooms, reporting on mathematics teachers' perceptions of the predictors for, and barriers to, their integration in European post-primary schools. Data are drawn from teachers' responses to a questionnaire for an Erasmus+ project that addresses readiness for the integration of 21C practices. Responses from 52 Irish, Swedish, Estonian and German practitioners listing mathematics as one of their teaching subjects are considered. For quantitative data, descriptive and inferential statistics were used; a directed content analysis approach was taken for qualitative data. Findings indicate that system restrictions and resources are major barriers, and that classroom management and teacher beliefs impact on confidence with and frequency of use of 21C practices. We propose that this work form the basis of a broader study.

Keywords: Teaching practices, 21st century learning, mathematics education.

The perceived importance of a '21st Century' (21C) approach to teaching and learning is well documented (Dede, 2010; Voogt & Roblin, 2012). In terms of mathematics pedagogy however, while there is considerable research into the use of technologies for teaching and learning (e.g., TWG15 and TWG16), the broader field of 21C practices in the classroom is less considered. This research explores responses to a survey instrument developed for an Erasmus+ project, Teaching for Tomorrow (TfT). TfT is a partnership between institutions in four countries (Ireland, Sweden, Estonia and Germany) that are working to develop a model of 21C teaching and learning across subject areas. The poster reports on the responses of 52 teachers who name mathematics as one of their teaching subjects. The aim is to identify what they see as the predictors for and barriers to usage of 21C practices in the classroom, with a view to larger-scale research.

The theoretical framework underpinning the model for 21C practices being developed by TfT draws on the work of Ravitz, Hixson, English, and Mergendoller (2012), which emphasises a project-based, collaborative, and student-led pedagogic approach. "Readiness for integration" is taken as involving *confidence* in using and encouraging, and *frequency* of using, the 21C practices of: Critical thinking, Collaboration, Communication, Creativity & Innovation, Use of Technology, Self-direction, Global and Local Connections.

The questionnaire used to gather data was developed by the Irish partners, with items drawn from the validated instruments of Euler and Maaß (2011), Ravitz et al. (2012), and the OECD (2010). It involved 4 main sections: (1) *Background information*; (2) *Teachers' beliefs about the nature of teaching and learning (direct transmission versus constructivist)*; (3) *Orientation towards, usage of, and barriers to 21C teaching and learning*; and (4) *Confidence with and frequency of integration of 21st skills in practice*. Apart from section 1 and an open-ended item in the Barriers section, all items used 5-point Likert-type scoring system.

Multiple regressions were performed to identify whether the categories of *beliefs*, *opinions and usage*, and *barriers* had a significant bearing on teachers' *confidence* with, and *frequency* of, integration of 21CL practices in the mathematics classroom. Also, *t*-tests and one-way ANOVAs were used to compare the mean ratings across the four participating countries. Directed content analysis was undertaken for the qualitative data.

Results indicate that teachers' mean *orientation* towards 21C practices is quite high, with respondents tending to agree that 21C teaching and learning has a positive impact on student motivation. However mean levels of *confidence* are less positive, and mean *frequency of usage* is rather low, pointing to a lack of readiness for integration. Respondents' mean scores for self-reported direct transmission beliefs are lower than those for constructivist beliefs, the latter being predictors of confidence in 21C practices.

In the qualitative analysis, students' and teachers' direct transmission beliefs are reported by respondents as barriers to the integration of 21C practices, with "*teacher inertia and general reluctance to move from traditional methods*" emerging as a common issue. Barriers at the system level, particularly those associated with time, and curriculum and assessment, also appear important. In addition, both quantitative and qualitative analysis reflects that classroom management issues act as barriers to teachers' implementation of 21C practices: "*Students are not used to 21CL, because most of the time they do not have to do it, so at first it takes a lot of time.*"

In order to encourage teachers to integrate 21C practices in the mathematics classroom, it is essential to address some of the barriers identified. The features of the TfT model, outlined above, are intended to provide guidance for teachers and students, a structured approach to the development of 21C activities, and relevant assessment practices.

It should be noted that although the samples from each country are small and not representative, and that there were variations in the criteria for participant selection, the results across counties show surprising commonality. Thus, we propose to conduct a larger study, involving representative groups, uniformly selected in each country, to see if such trends arise outside the confines of TfT.

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TWG20: Mathematics teacher knowledge, beliefs and identity

Introduction to the papers of TWG20: Mathematics teacher knowledge, beliefs, and identity

Miguel Ribeiro[1], Fatma Aslan-Tutak[2], Sebastian Kuntze[3], Francesca Martignone[4], Kirsti Rø[5] and Amanjot Toor[6]

[1]State University of Campinas, Brazil and Portugal

[2] Bogazici University, Istanbul, Turkey, Turkey

[3]Ludwigsburg University of Education, Germany

[4]Università del Piemonte Orientale, Italy

[5]Norwegian University of Science and Technology (NTNU), Norway

[6]Brock University, Canada

Rationale

The variety of papers presented in TWG 20 at CERME 2017 connected to the growing field of teacher education. There were total of 27 papers and 4 posters that had been presented. The number and heterogeneity of the research foci, contexts, methodological and theoretical approaches, provided opportunity for in-depth discussions and reflections around the presented papers. Although the core topic proposals were expected to embrace three intertwined domains, the focus of the presented papers at TWG20 was mainly on teachers' knowledge, while the topic of teachers' beliefs and teachers' identity would appear implicitly or in the background of some of the research.

Main topics

The priority given to teachers' knowledge can be seen throughout the unbalanced number of proposals which included the 27 papers presented (30 were submitted). For the development of the work three thematic strands have been considered: (i) knowledge in mathematics education (3 papers); (ii) lesson study context and beliefs (5 papers) and (iii) teachers' knowledge" (19 papers). It is interesting to note that the differences between (i) and (iii) concerns the aims of the research (even if not perceived explicitly), and not necessarily the theoretical perspectives considered.

(i) Knowledge in mathematics education. This thematic strand included three papers, focusing on aspects different from the specificities of teachers' knowledge, even when the context was teachers' education. Although the theoretical dimensions of the presented researches addressed teachers' knowledge conceptualizations the research focused on the knowledge of prospective teachers' that is revealed at the very beginning of teachers' training.

(ii) Lesson study. The second thematic strand included five papers and one poster, dealing with mathematics teachers' professional development (PD), in the context of the development of Lesson Studies (LS), and (prospective) teachers' learning with regards their own teaching practice and students learning, as well as on how teachers perceive themselves as participants in such LS context. Although there is evidence that PD contexts can lead to improvements in teaching practices and students' learning, less is known about what and how teachers learn from PD and about its further impact on students' learning outcomes (Borko, 2004), and also some other intertwined variables.

Common to the presented research were the perspective of considering teachers as learners, where the research is looking at the relationships between these two elements. In this thematic strand, two particular issues arise; how can theories/approaches/perspectives on teachers' knowledge be used to analyse the impact of teachers' participation and involvement in LS for teaching practices and professional development; concerning questions around the incorporation of theory in a methodological approach for LS in order to analyse the different phases and cycles.

Further, the research developed in the context of the implementation of a LS (or a research on the LS process), needs to take into account the particularities and specificities of the cultural contexts in which it is implemented – in order to acknowledge, the differences of those cultural contexts and the one where it is originated. In, and for doing so, one needs to take into account different aspects, such as the type, nature and impact of the affordances and constraints that takes into consideration; the influence of the researchers' background in the implementation, development and design of the research in the LS context; to what extent, in the context of a mathematics education research, the *mathematics* features of teaching is effectively the focus of attention.

(ii) **Teachers' knowledge.** Similar to previous CERME's, the explicit focus on teachers' knowledge has been given a major importance in the context of most of the presented proposals. The papers were grounded on teachers' knowledge conceptualizations that have already been discussed extensively in previous CERME conferences (Mathematical Knowledge for Teaching (Ball, Thames & Phelps, 2008); Mathematics Teachers' Specialized Knowledge (Carrillo, Climent, Contreras & Munoz-Catalan, 2013); Knowledge Quartet (Rowland, Huckstep & Thwaites, 2005); Ontosemiotic Approach (Godino, Batanero & Font, 2007)). We observe that, within a period of four years (from CERME 8 to CERME 10) a certain shift related to the focus of attention occurred, namely a shift from discussing the need for different conceptualizations towards an effort to deepening on the nature of teachers' knowledge when assuming a certain conceptualization.

In CERME9, Ribeiro, Aslan-Tutak, Charalambous and Meinke (2015) suggested that the use and development of diverse conceptualizations could be perceived as both a richness of the research field and as a constraint. The richness concerns possibilities for gaining a better insight into factors that influence the development of teacher knowledge. However, there are challenges of finding a common ground for discussing the core aspects of the research field. At the current conference, the issue of a diverse conceptualization was addressed in discussions on how to investigate mathematics teachers' knowledge when assuming it to be in interplay with students' learning. In other words, the need to pay "close" attention to how we, as researchers, take into consideration the aspects of mathematics teaching and learning, being connected to teachers' intertwined knowledge as well as to the role and impact of teachers' knowledge in practice concerning the use of resources (where the focus was the teachers' knowledge involved in/for preparing and using such resources – in a broader sense – in practice and not the resources itself).

The research, and the associated discussions and reflections in the group, also bought forward one of the recurrent items in the group discussions: the need for the research on teacher education to move from a prevalent focus on what teachers do not have (the deficit perspective – in term of knowledge) to a focus on what teachers actually know, how they know it, and possible different hows that can contribute to the development of teacher's knowledge, specifically related with teacher's work of teaching. Along the discussions, and aligned with some of the presented papers, a

possible direction for future research emerged on area of teachers' knowledge sustaining teachers' noticing and "earing" ability. For example, future research focusing on how mathematic teachers' pay attention to and make sense of what happens in the complexity of instructional situations (see e.g. Sherin, Jacobs & Philipp, 2011). Also on what aspects of one's knowledge do teachers' ground their decision making – in order to develop mathematically demanding practices, aiming at developing critical mathematical thinking for deeper mathematical understanding. One other possible focus concerns on how and why (the impact) the teachers' and researchers' knowledge influence their foci of attention and awareness.

Emerged themes and future perspectives

We have considered three thematic strands for an operational reason, but one need to have in mind the intertwined nature of such strands. Thus, in our case, the connecting element was teachers' knowledge. Some of the discussions were grounded in ideas already discussed in previous CERME's, aiming at deepening the understanding on those aspects/dimensions while other discussions seek for an alternative and complementary path for getting such a broader understanding. A particular sensitive aspect was the need for a deeper understanding on the relationships between teachers' knowledge and practice, and for gaining such a deeper understanding some new approaches to research on teachers' knowledge were discussed, in particular studies that investigate how teachers use their knowledge to give meaning to others' solutions or to anticipate students' answers. Moreover, the role of task design in and for assessing, accessing and developing teachers' knowledge and improving practices was emphasized in the discussions. We have grouped the main research trends emerged in three groups:

- Deepening research into teachers' knowledge, beliefs, identity, and noticing

- Taking into account that in some contexts mathematics teachers knowledge specificities are perceived mainly in the domain of PCK, how is the "weight" of PCK perceived in the field of research in mathematics education and how it intertwines with the specificities of the teachers' content knowledge, beliefs and identity?
- How to take into account teachers' noticing?

- Research on interactions with fields of practice

- How can the focus of research be intertwined with practice and education in a more explicit manner, perceiving practice in a broader sense?
- How can we investigate whether and how teachers' knowledge affects students' learning and transitions throughout student's education? How to design and develop research aimed at approaching "simultaneously" teachers' knowledge and students learning?
- How to move from frameworks for analyzing, describing, understanding and/or evaluating teachers' knowledge, to the use of frameworks by teachers (for analyzing teaching practice)?
- How the teachers' knowledge conceptualizations take into account the notion of assessment, and how does knowledge on assessment contribute to students' learning in mathematics?
- What are the roles and knowledge (e.g., features, nature, content) of mathematic educators in teacher education (e.g. facilitator in LS; teacher trainer)?

- Research on methodological (and theoretical) challenges:

- How can we deal with (and what are the implications for) similarities and differences of aims and challenges in mathematics teacher education in different cultural contexts?
- How to clarify the findings we have, when using a particular theoretical lens for analyzing teachers' answers, comments and/or practices?
- How to develop research that emphasize teachers' potentials instead of teachers' deficiencies, and how to design approaches for grounding teachers' knowledge development in such potential?
- As many of the researchers developing research on teachers' knowledge – in multiple contexts, including lesson study – are educators, how do we deal with such fact (a recurrent issue); what significant does researcher's role as an educator play on the research itself and what is the actual impact of research in improving teachers' education (in what terms is the research one does implying on the ways teachers' education occur)?

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The reasons underlying mathematics teachers' decisions about the teaching objectives of a mathematical task

Miriam Amit[1] and Hodaya (Liora) Hoch[2]

[1]*Ben-Gurion University, Israel*; amit@bgu.ac.il

[2] *Orot Israel College, Ben-Gurion University, Israel*; liora_h@macam.ac.il

Responsibility for evaluation is assigned to teachers so that they can critique their own work and follow each of their students' progress. To succeed, teachers must build an appropriate means of assessment by themselves. This study examines the ability of pre-service and novice mathematics teachers in secondary schools to do so, using a questionnaire that requires them to make a connection between teaching objectives and mathematical tasks. The results show difficulties in matching teaching objectives to a given task and vice versa, with no significant differences between the pre-service and novice mathematics teachers. Interviews show how their beliefs, and their lack of mathematical and pedagogical knowledge, influence the teaching objective and assessment task they choose.

Keywords: Pedagogical mathematical knowledge, assessment, mathematical knowledge, pre-service mathematics teachers, novice mathematics teachers.

Introduction

The aim of evaluation is to provide teachers with reliable information so they can make decisions regarding their teaching methods as well as track their students' learning (NCTM, 2000). To achieve that aim, teachers must design and construct assessment tools on a routine basis (Magnusson, Krajcik, & Borko, 2002). This means that teachers should have the capacity to find or develop assessment tasks that align with their teaching objective, and that have the potential to accurately reflect their students' knowledge and understanding (Siegel & Wissehr, 2011). They should also have the reverse capacity: they must know how to find the teaching objective behind a given mathematical task. The two capacities integrate pedagogical knowledge with mathematical (content) knowledge, which has been defined as a part of pedagogical mathematical knowledge in assessment which teachers should have (Tamir, 1988). This study attempts to examine the pedagogical mathematical knowledge in assessment of pre-service and novice mathematics teachers by checking their ability to draw a connection between teaching objectives and mathematical tasks.

The theoretical framework

Teaching as a profession is characterized by the existence of a unique knowledge base for those involved in it. Shulman (1986) defined categories of knowledge that teachers need in order to be professionals. Among these, he listed content knowledge (in this case, of mathematics), pedagogical knowledge, and pedagogical-content knowledge. Toward the end of the last century, the assessment of students' achievement was placed in the teachers' hands, with the hope they would have the knowledge to develop a reliable and valid assessment methods aligned with the teaching objectives (NCTM, 2000). Thus, the topic of assessment became part of the knowledge base teachers have to have. Tamir (1988), followed by Magnusson, Krajcik, & Borko (2002), expanded the definitions of

pedagogical knowledge and pedagogical-content knowledge for the topic of assessment. **Pedagogical knowledge in assessment** means knowledge of general rules related to assessment that can be applied to every subject-matter, such as familiarity with the various means of assessment. **Pedagogical-mathematical knowledge in assessment** refers to the knowledge teachers need in order to implement the general assessment rules in mathematics, and also to the ability to build a means of assessment that aligns with their teaching objectives.

Connecting teaching objectives and mathematical tasks is the foundation upon which pedagogical-mathematical knowledge in assessment is based, and particularly all means of assessment, formal as well as informal. This capacity should be acquired by every teacher from the time he/she starts to work. To the best of our knowledge, research that examines teachers' ability to connect between tasks and objectives has yet to be done in any field. This study is the first attempt to explore this ability in general, and particularly in mathematics, among novice and pre-service mathematics teachers. This paper is aimed to check this ability in an innovative way.

Methodology

The data analyzed below derives from a research program about assessment that examines the pedagogical knowledge and pedagogical mathematical knowledge in assessment of pre-service and novice mathematics teachers for primary and secondary schools (Hoch & Amit, 2013; Hoch & Amit, 2011).

The research questions

The aim of this paper is to examine the ability of pre-service and novice mathematics teachers to find the most suitable teaching objectives to associate with a given mathematical task. The research questions are:

a. Given a mathematical task, to what extent do the participants know how to match it to a suitable teaching objective which can be checked by that mathematical task?

b. What are the sources of mismatching in selecting the appropriate teaching objective?

The population

The study focused on sixty-six participants: thirty-two pre-service teachers, who were taught in five teacher training colleges, and thirty-four novice teachers, who had also been trained in one of those colleges.

The pre-service teachers were near the conclusion of their studies, and about to become secondary school mathematics teachers (grades 7-10). The novice teachers were already teaching mathematics in secondary schools (grades 7-10), and had up to three years of teaching experience (Vonk, 1993).

All the participants had taken a one-semester course in student achievement assessment during their studies at the teacher training colleges. None of the novice teachers took any additional course besides the one in which they had trained.

The research approach

This study used the Mixed Method approach. First, quantitative data was gathered; then, on the basis of the gathered data, qualitative data was collected (see below).

The instruments

The research was conducted by means of a questionnaire, followed by interviews.

The questionnaire:

This study is innovative in that it examines assessment pedagogical-content knowledge in mathematics by checking teachers' ability to draw connections between teaching objectives and mathematical tasks. Specific questionnaires were created for this purpose. In this paper we shall focus on one of these, which checks the participants' ability to match a suitable teaching objective to a given mathematical task. Every question starts with a mathematical task followed by three teaching objectives that can assessed by that task. The participants were asked to rate those teaching objectives according to their suitability to the given mathematical task on the scale of 3 (= the most suitable) to 1(= the least suitable) (see Figure 1 below).

The questions focus on three central subjects in the curriculum of grades 7-10: algebraic expressions, equations and functions. All the mathematical tasks appear in the most widely used textbooks and in national tests. The teaching objectives were taken from the mathematics curriculum and from these textbooks, in order to simulate live classroom situations.

The questionnaire was given to a panel of judges which included seven experts in mathematics assessment. All have a PhD. in mathematics education, except for one who has an M.A. in mathematics education. All are familiar with the topic of assessment, and some of them even teach this topic in teacher training colleges. The panel of judges was asked to rate the teaching objectives in each question according to their suitability to the given mathematical task (just as the respondents would be, on a scale of 1-3). To ensure reliability (see Burstein et al., 1995/1996) each one of the judges was given the questionnaire discretely and did the assignment independently, without knowing who the other judges were.

After all the judges had completed the questionnaire, their ratings for each question were checked. Only questions that were given the same rating by at least five judges were left in the final version of the questionnaire. For the remaining questions, all the judges agreed which was the most appropriate teaching objective, but did not agree about the order of the other tasks.

The interviews:

Interviews were conducted with eight participants (four pre-service teachers and four novice teachers) a maximum of two days after they completed the questionnaire. The interviews were done in order to gain a deeper understanding of the participants' way of thinking and to find explanations to and elaborations for the results obtained from the questionnaire (see Burstein et al., 1995/1996; Ercikan & Roth, 2006; Luft & Roehrig, 2007). The interviewees were therefore chosen according to the degree of incompatibility between their own rating and the experts' rating, and subject to their willingness to be interviewed. The interviews were semi-structured so that for each item in the questionnaire the interviewees were asked: "Can you explain your considerations for the rating you gave?" Each interview was recorded rather than hand written so as to avoid interruptions and delays during the interviewing process. Later the interview was transcribed.

Data analysis

Quantitative analysis: For each question, each participant's rating was compared to that of the experts and received a mark indicating the degree to which it matched the experts' rating. The mark was on a scale from 4 to 0 following the rules: 4 points - the participant's rating is equal to the experts' rating; 3 points - the participant found the most suitable teaching objective but confused the order of the two other objectives; 2 points - the participant marked the most suitable teaching objective as the second and the second suitable teaching objective as the most suitable one; 1 point - the participant marked the most suitable teaching objective as the second and confused the order of the other two; 0 points – any other option. As a result of this process each question got a score, thus enabling the use of common statistic tests.

Interview analysis: The material obtained from the interviews was coded and analyzed using content analysis (Bauer & Gaskell, 2010), with the explanation for each question used as an analysis unit. At the initial stage of the analysis, the participant's rating was compared to that of the experts for each question and the explanation for the rating was checked. In cases where the rating did not match that of the experts, or the interviewee's explanation for the rating (even when it was in alignment with the experts' rating) did not correspond to the experts' reasons, the causes that led to those mistakes were identified. Finally, all the causes which were found to have a common source were grouped together, thus creating several categories. Since similar research has not previously been undertaken, there were not yet any known categories.

Findings and interpretations

General results for the questionnaire

On average, the participants obtained 2.5 points (out of 4) in this questionnaire. This means that they managed to answer correctly an average of 6 questions out of 9. No significant differences were found between pre-service and novice teachers.

"Substitute Set" Question (see Figure 1 below).

In this question the participants obtained 2.51 points (out of 4). No significant differences were found between the two groups. Less than 50% of the participants correctly chose the most appropriate teaching objective.

For this question the causes that led to a mistaken rating were divided into four categories.

Difficulties in content knowledge

Interviewer: Which algebraic expressions are suitable?

Sara: b, c, d

Interviewer: b, c, d, all right

Sara: And e

Interviewer: And e

Sara: Even though in e you can substitute (-5)

Interviewer: So what is the right answer?

Sara: I really do not know …..

The right answer is b and d, but many students of all ages have problems with the algebraic expression given in c. They need more than a few minutes to understand that c is a wrong answer. The one which is written in e caused problems for some interviewees, and one interviewee gave it as a correct answer without realizing that there was a problem. The fact that (-5) cannot be substituted, while 5 can, caused problems. This algebraic expression shows that the ability to find the substitute set is not sufficient for providing the right solution. Those interviewees who had difficulty with mathematical knowledge chose the second objective as the most suitable one or rated the teaching objectives in the same order as the experts.

In a test that was given to 7th grade students the following question was asked:

Find the algebraic expressions where the set $\{x| x \neq \pm 5\}$ is the substitute set.

(There is more than one correct option.)

a. (x-5)(x+5) b. $\frac{4}{(x-5)(x+5)}$ c. $\frac{x-5}{x^2+25}$ d. $\frac{x}{x^2-25}$ e. $\frac{7}{x-5}$

The teaching objective: To check if the student	Degree to which the objective fits the task
Knows how to substitute numbers in an algebraic expression	1
Knows how to find a substitute set to an algebraic expression	2

Figure 1: "Substitute Set" Question (with the experts' rating)

Difficulties in pedagogical knowledge

The results showed that the participants did not clearly and unequivocally understand how a teaching objective should be determined. As a result, they made their choices based on several different criteria. Some of the interviewees chose the suitable teaching objective according to the order of the syllabus. In this case, for instance, finding a substitute set to an algebraic expression topic is learnt after the students know how to substitute numbers and practice doing so. According to Orna, "Here the student does not try to substitute numbers because this stage is already known, it is something obvious". Thus the teaching objective that would be suitable in this case is the second one, with the additional reasoning "that is the way I would do it" (Orna). Osama's rationale, on the other hand, led her to choose the first teaching objective, saying "Here the student first has to substitute in order to know if the denominator is equal to zero… [Therefore] I think this task checks very well if the student knows how to substitute". Others, like Yelena, based their determination of the most suitable teaching objective on a strict adherence to the exact phrasing of the assessment task. She declared that "the student does not have to know the idea behind the term 'a substitute set' because no one asks for this".

She thinks that a teaching objective should be determined according to what is specifically asked for in the written question.

Difficulties in pedagogical content knowledge

Some of the interviewees did not take into consideration the difficulties students have when the substitute set of the algebraic expression is not the same as the given one. "Even those who do not understand the learnt material can substitute numbers and succeed in solving the question and the teacher will think he/she understands" (Osama). "In 'e' the student finds that the substitute set is all numbers except 5, so he will not mark this ['e'] because [it is given that] the substitute set should be different from 5 and also from (-5)" (Sharon). Knowing how to find the substitute set or knowing how to substitute numbers does not ensure that the student will come to the right conclusion because an additional step is requires, namely comparing the result obtained from the finding of the substitute set with the given substitute set and coming to a conclusion.

Beliefs/conceptions

The interviewees raised concerns about whether a task like this could in fact accurately reflect their students' understanding at all, or whether it just reflected their ability to follow instructions. Their responses addressed questions like: What is the nature of the rules that are provided by teachers or books? Do these rules replace the need for understanding the subject matter? Can students follow the rules and succeed in solving tasks without understanding? Anat, for example, pointed out that "If the teacher wrote down the rules on how to find a substitute set…all kinds of rules, the students may not understand what a substitute set means, but they will be able to solve exercises without such understanding." Anat expresses the belief that knowing the rules and using them correctly can cover up a lack of understanding, and that therefore this task cannot check the understanding of a substitute set.

Discussion and conclusions

This study attempts to give a preliminary idea of the mathematics teachers' pedagogical-content knowledge of assessment. It examined two groups: pre-service teachers in secondary schools just prior to their entering the educational system, and novice teachers in secondary schools. The example presented here shows the participants' difficulties finding the teaching objective behind the given mathematical task, dividing them into four categories: a. difficulties in mathematical knowledge, b. difficulties in pedagogical knowledge, c. difficulties in pedagogical mathematics knowledge and d. beliefs/conceptions teachers hold. The first three categories have already been discussed extensively by many researchers as an acknowledged part of the knowledge base all teachers must have (e.g. Shulman, 1986; Turner-Bisset, 1999). As this example shows, an incomplete knowledge base also influences the quality of the assessment teachers carry out. The application of beliefs in the teaching process has been well documented in the literature (Eren, 2010; James & Pedder, 2006; Turner-Bisset, 1999) and arose in the context of assessment as well. In this case, it was the teachers' belief that providing students with rules could cover their lack of understanding. This issue has been dealt with by many researchers for many years (e.g. Skemp, 1976). The belief that teaching students to follow a set of rules can be a substitute for teaching them the underlying ideas for those rules may be the result of what has, until recently, been the prevalent Israeli method of learning mathematics – a

method that demanded low levels of thinking from the students. Various researchers (e.g. James & Pedder, 2006) have recommended encouraging the pre-service teachers to express their beliefs by opening them up for discussion them. Hearing and addressing these beliefs that can influence future behavior, can lead the holder to undergo a process of professional development.

Finally, this study showed that the abilities of the novice teachers are no better than those of pre-service teachers. Both groups demonstrated the same problems, suggesting that experience in practice does not rectify problems that originate in training. This forces the assessment course's lecturers to focus on how to connect between teaching objectives and mathematical tasks. Moreover, every teacher educator should encourage preservice teachers to express their presumptions or beliefs, since discussing them can eliminate wrong beliefs and lead to a professional development (James & Pedder, 2006).

The study's limitations

Although the number of participating in this research is small, and despite the fact that this research was restricted only to novice teachers, it is nevertheless important, since there is no study to date that examines the ability of teachers to identify a suitable teaching objective for a given mathematical task. Understanding the sources of the mismatching in selecting the appropriate teaching objective can help teacher educators focus their efforts on these problems, thus improving the training program's changes of preventing them.

The study looked at the participants' ability to connect between teaching objectives and mathematical tasks in three themes, limiting each theme to very few questions. These alone could not encompass all the aspects to be addressed in each theme. Further studies of this ability (and its opposite) should be done, preferably with every study conducting an in-depth examination (including interviews) of only one teaching theme.

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Differences in the forms of content teachers are seen to offer over time: Identifying opportunities for teacher growth

Nick Andrews

University of Oxford, Department of Education, UK; nicholas.andrews@education.ox.ac.uk

In this paper, I report on a study into differences in how teachers make content available to learners over a series of lessons on a topic of their choice. The study highlights significant differences in the forms of content made available between lesson series, with the taught topic being a critical factor. It also highlights significant differences between the forms of content made available during classwork and seatwork, but these differences are independent of both topic and teacher. The implications of these findings are discussed, including their potential along with the analytic research tool for prompting teacher knowledge growth.

Keywords: Teacher knowledge, instructional design, teacher education.

Introduction

Findings of research into classroom practice have clear impact when they draw attention to features of a classroom situation that teachers might not otherwise attend. This impact can be particularly powerful if it affords deep psychological investment: if findings are presented in such a way as to promote an awareness of choice, if a reason to choose to act differently can be harnessed and if alternative actions can be imagined (Mason, 2002). In this paper I focus on an exploratory study of four cases of classroom practice over a series of lessons, and in particular differences in the manifestations of mathematical concepts offered by the teacher over time (a phrase I will define in due course) revealed through quantitative content analysis of teacher talk and classroom tasks. I draw on and adapt ideas from international comparative studies in order to inform the research design of this local study. I express the findings of my analysis as rhetorical questions about teacher knowledge that shapes decision making around the sequencing of content over time (a contributory code of the connection element of the Knowledge Quartet (Rowland, Huckstep & Thwaites, 2005)).

Background to the study

Differences in how content is made available to learners

The notion of differences in teaching methods is well established through comparative research (e.g. Stigler & Hiebert, 1999). Such studies have tended to use the lesson as the unit of analysis, yet Clarke, Emanuelsson, Jablonka & Mok (2006) propose a series of lessons on a topic as more appropriate for the purposes of comparison. The nature of teaching can change over the course of a lesson series, with the decisions that teachers make in planning and teaching individual lessons shaping the way they act over time but also being shaped by the classroom situation of which the teacher is part. Furthermore, teachers make decisions at the lesson series level when preparing to teach a topic and so analysis at this grain size affords focusing attention on such decisions as well as alternative choices that might be available. What is not known is whether particular forms of content are emphasised, and whether this emphasis might differ between series of lessons taught in different contexts.

Differences between lesson time given over to the teacher speaking with the whole class and to students engaged in seatwork have been researched (e.g. Stigler & Hiebert, 1999). Serrano (2012) underlines the emphasis placed on application/practice during seatwork in classrooms in USA and German, contrasting this with greater opportunities for exploration offered in Japanese classrooms. Hino's (2006) within-country comparisons offer an account for this, highlighting how seatwork activity frequently precedes explicit presentation of the main content in lessons in Japan rather than following it. From these studies, some variation in the forms of mathematics being offered during seatwork in different contexts might be anticipated, but are these differences as significant as those between the forms of content emphasised when the teacher speaks with the whole class and those emphasised during seatwork?

Case study design

I purposefully designed the current study so that it involved four cases of teachers teaching a series of lessons on a topic of their choice, which allowed me to explore the role of the teacher and the topic. In Clarke et al.'s (2006) study the length of a lesson series was set by the researchers as ten lessons (or equivalent), but in the current study the length was determined by how long the teacher intended to work on the topic and was between three and five one-hour lessons. This ensured that the stretch of lessons included a start, middle and end, and I report elsewhere on the notable lesson-to-lesson differences that were revealed in each case (Andrews, 2016). Figure 1 shows the convenient selection of three experienced mathematics teachers (Ashley, Bernie and Courtney) from two local secondary schools (King's Meadow School and Bishop Langton School) that generated the four case studies for my investigation.

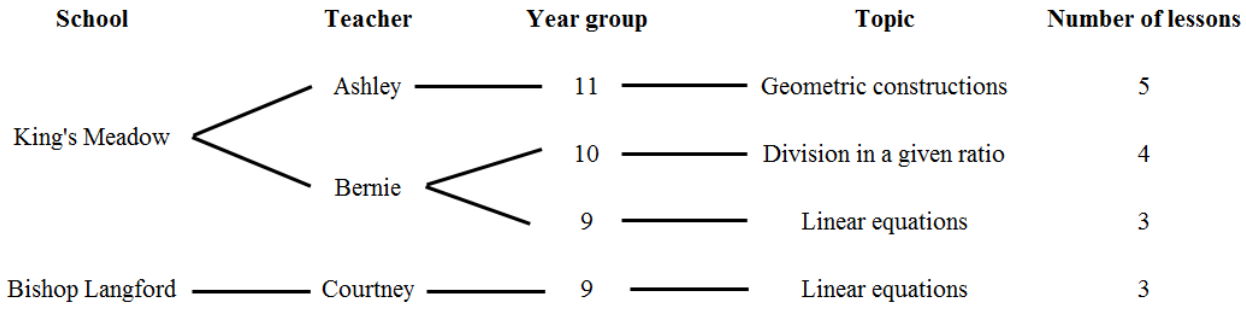

Figure 1: Tree diagram representation of the multiple case study design

Figure 1 also shows that classes were from school years 9 to 11 in England (learners aged between 13 and 16). Classes were set by prior attainment in both schools, with the average teacher-predicted attainment of learners in each class by the age of 16 being in line with the national average for mathematics. The broad topics that were the focus of each lesson series are also given in Figure 1. Thus cases involving Ashley and Bernie had the school context in common, the two cases involving Bernie had the teacher in common and the two cases involving linear equations had the age and predicted attainment of students and the topic in common. For convenience, I generally refer to cases through teacher and topic, but it is important to have in mind the significance of all four dimensions (teacher, class, school and topic) and not let this notation obscure the contribution of school and class to the classroom environment.

Conceptualising forms of content offered by teachers

In this section I clarify what I mean above by 'forms of content offered'. Firstly, I view the teacher's role within the classroom situation as making content available to learners, where by content I mean the material dealt with in teaching rather than its form; it is the content that brings teachers and learners together. In the context of mathematics teaching, content is made up of the mathematical phenomena that constitute a topic. But it is precisely the 'form' of content that I observe the teacher offering learners – rather than the content itself – that is of interest to me. The emphasis on "I observe" here is deliberate, since this is a researcher's stance. In the current study I am not seeking what the teacher perceives as the form of content they are offering, for which I might use the term *instantiation* of a mathematical concept. Rather it is my own perception of what is being offered from the position of observer, for which I use the term *manifestation* of a mathematical concept. This also explains why more familiar terms used to describe learner activity such as 'prepare, 'apply' and 'explore' (Serrano, 2012) and Clarke et al.'s (2006) codes are inappropriate from a methodological perspective, since attending to what the teacher says and makes available as tasks allows for neither assertions regarding teacher intentions nor learner activity.

The notion of manifestation requires further exemplification. When observing a teacher working with a class on linear equations, I might see the teacher offering diagrams of bars as in Figure 2, mathematical symbols such as "$5x + 3 = 24 - 2x$", or word problems such as "My father is currently three times my age. In five years' time the sum of our ages will be 50. How old am I now?"

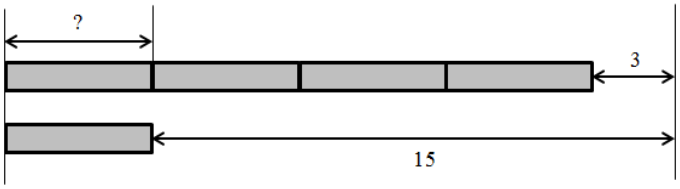

Figure 2: A problem featuring bars

I see each as a manifestation of a linear equation, but each is a different manifestation. I classify the manifestation in Figure 2 as *visual*, the mathematical symbols as *technical* and the worded problem as *functional* (in the sense that a solution is likely to be arrived at through modelling the problem mathematically). There are parallels here with the three strands of *Structure of a Topic*: awareness, techniques and emotion (Mason & Johnston-Wilder, 2004). These three components of manifestation may be combined, for example if the teacher was observed offering the worded problem above and heard saying "Solve this by forming and solving an equation," this would be both a functional and a technical manifestation of a linear equation. In other cases, the spoken instruction may not be so explicit but nevertheless clearly implied in the context of the lesson. Either way, I would classify what is offered as *functional-technical*. Similarly manifestations might be classified as *visual-functional* or *visual-technical*, or indeed *visual-functional-technical*.

Although this classification of manifestations has been established here through focusing on linear equations, it is transferable to the other topics under consideration in this study. For example, when teaching geometric constructions, the teacher may seek to evoke an image of a perpendicular bisector of two fixed points as the locus of points equidistant from the fixed points (visual) or ask learners to construct a perpendicular bisector using a straight edge and compasses (technical). Or when teaching

division in a given ratio, the teacher may offer a pile of coins for learners to experience sharing in a given ratio visually (visual) or pose a worded problem (functional) such as: "On starting up a company, Sasha invests £25000 and Tina £40000. At the end of the first year they make a profit of £19500 that they agree to share based on their original investment. How much does Sasha receive?"

To add texture to these descriptions and to consider implications for learners, I cautiously associate visual, technical and functional manifestations with opportunities to focus learners' attention on image having (Pirie & Kieren, 1989), procedural fluency (Kilpatrick, Swafford & Findell, 2001) and confident manipulation (Mason, 1980) respectively. I associate visual-functional, visual-technical and functional-technical manifestations with opportunities to shift the focus of learners' attention; these are likely to be formative, and afford learners new ways in which to encounter a mathematical phenomenon working from what is already familiar.

Focus of the investigation

The multiple case studies provide broad instantiations of teachers teaching over time, so afford localised responses to my research questions:

> When comparing series of lessons, what differences are discernible in the forms of content that the teacher is observed offering (manifestations) over time?
>
> Within and across series of lessons, what differences are discernible in the forms of content that the teacher is observed offering (manifestations) when speaking with the whole class (classwork) and through tasks (seatwork)?

In posing these questions, my intention is to explore features of classroom situations that may go beyond those which teachers routinely attend. But if they are features to which teachers *could* attend, then this opens up opportunities to act differently.

Method of analysis

The transcript of the teacher's voice, field notes, screen-shots of information displayed on boards, hand-outs, worksheets and text books were all used in order to infer how the teacher made content available to learners. In the process of coding, I considered the mathematics-related utterances made by the teacher when speaking with the whole class during classwork or with individuals during seatwork, which together I refer to as teacher-talk. I also considered whether learners were working on a particular task, which might have been made available for example through a worksheet or spoken instructions from the teacher. I refer to such tasks as given-tasks.

Each lesson was parsed into half-minute intervals, which were then coded for manifestation taking account of the teacher-talk and given-task present during the interval. Selecting half-minute intervals as the smallest unit of analysis afforded reliable and manageable coding while still allowing sufficient sensitivity to discern small differences in how content was made available. Each interval was coded with a Barycentric co-ordinate (x, y, z), where x, y and z represent the relative emphasis between visual, functional and technical manifestations respectively observed during the thirty seconds with $x + y + z = 1$ (c.f. Swan, 2006). For example, an interval featuring only technical manifestation was coded (0, 0, 1) and an interval featuring visual-functional manifestation was coded (½, ½, 0). Coding was based on presence of a manifestation in the half-minute interval rather than the proportion of time, and given-task and teacher-talk were weighted equally. Further, if in a particular half-minute

interval of seatwork the given-task was classified as visual-functional but the teacher made an brief articulation that was classified as technical, the overall interval would be coded (¼, ¼, ½), being the mean point of (½, ½, 0) and (0, 0, 1). More details of the coding rules and approach to analysis, including the strengths and limitations of this approach, are provided in Andrews (2016).

The *series centre* for manifestation for each case was calculated through finding the mean point over the whole lesson series. Further analysis allowed for the calculation of series centres for each case for classwork and for seatwork. Differences between lessons were explored by treating the *x*, *y* and *z* values as separate variables. Each variable is ordinal-valued, and as such only non-parametric approaches to statistical analysis are appropriate. The Wilcoxon rank-sum test (Wilcoxon, 1945) was selected for this purpose. It was found that this test was sufficiently sensitive in order to detect even small difference between cases, and so I use the term *materially different* to describe the situation where the effect size (*r*, calculated from the Wilcoxon test statistic) of case comparison on one of the variables satisfies $r > 0.3$.

The use of Barycentric co-ordinates as a method of coding half-minute intervals afforded representing series centres as points within an equilateral triangle. This may evoke a sense of the series centre representing a point of 'electromagnetic attraction' to three differently charged 'poles' – visual, functional, technical – positioned at the vertices of the triangle. With this imagery in mind, I refer to the triangle as the *tri-polar space for manifestation*.

Results

Comparing series centres for manifestation

The series centres for manifestation highlighted differences in how content was manifested across each of the four lesson series. Figure 4 presents the four series centres in the tri-polar space for manifestation, indicating clear differences between some of the cases.

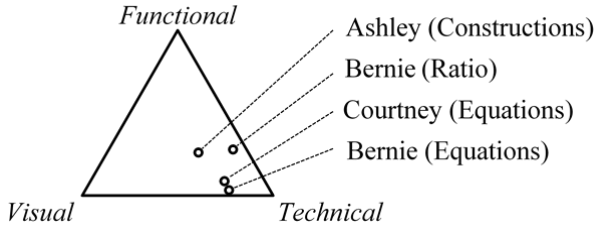

Figure 4: The four series centres plotted in the tri-polar space for manifestation

The three components of manifestation were most evenly stressed in Ashley's series on geometric construction. Functional and particularly technical components were stressed in Bernie's series on sharing in a given ratio, while visual and particularly technical components were stressed in the two series on solving linear equations (Bernie and Courtney). Table 1 quantifies the between-case differences seen in Figure 4 by presenting the effect sizes on the three components of manifestation when comparing lesson series.

Case comparison (Topic; half-minute intervals)	**Effect size on component of manifestation**		
	Visual	*Functional*	*Technical*
Ashley (Constructions; *n* = 488) to Bernie (Ratio; *n* = 383)	**-0.45**	0.03	**0.38**
Ashley (Constructions; *n* = 488) to Bernie (Equations; *n* = 289)	-0.10	**-0.57**	**0.57**
Ashley (Constructions; *n* = 488) to Courtney (Equations; *n* = 416)	-0.11	**-0.46**	**0.50**
Bernie (Ratio; *n* = 383) to Bernie (Equations; *n* = 289)	**0.42**	**-0.65**	0.30
Bernie (Ratio; *n* = 383) to Courtney (Equations; *n* = 416)	**0.41**	**-0.51**	0.17
Bernie (Equations; *n* = 289) to Courtney (Equations; *n* = 416)	-0.01	0.20	-0.14

N.B. Material differences are highlighted in bold.

Table 1: Quantification of between-case differences in the components of manifestation

The series of lessons taught by Ashley on constructions and Bernie on ratio were marked out as materially different to the two series on equations, and different to each other. However, the analysis did not highlight significant differences between the two series on equations. This led me to infer that within this study the topic had an impact on the forms of content the teacher offered.

Comparing classwork and seatwork

Combining the four cases, there were differences in the forms of content the teachers were observed offering during classwork and seatwork (see Figure 5). On average, the visual component was stressed more during classwork whereas the functional component was stressed more during seatwork, which aligns with Serrano's (2012) findings about American and German classrooms.

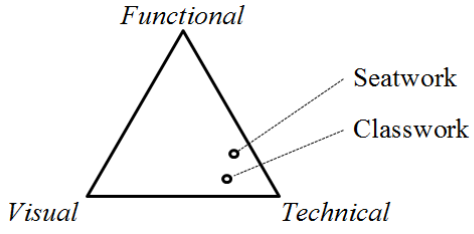

Figure 5: The across-case centres for classwork and seatwork plotted in the tri-polar space

Focusing now on each case, the difference in emphasis of the visual component between classwork and seatwork was statistically significant, although the difference was only material in Courtney's lesson series on equations (see Table 2). Greater emphasis on the visual component during classwork was observed regardless of the particular teacher, topic or class.

Case (Classwork intervals, seatwork intervals)	**Effect size on component of manifestation**		
	Visual	*Functional*	*Technical*
Ashley (Constructions; c = 77, s = 411)	-0.21	0.26	-0.02
Bernie (Ratio; c = 82, s = 301)	-0.17	**0.38**	-0.24
Bernie (Equations; c = 81, s = 208)	-0.18	-0.06	0.19
Courtney (Equations; c = 146, s = 270)	**-0.34**	0.21	0.17

N.B. Material differences are highlighted in bold.

Table 2: Comparison of classwork to seatwork across the four cases based on the relative stressing of the three components of the manifestation

Only in Bernie's series on equations was there no statistically significant difference in the functional component between classwork and seatwork. Yet the only difference in the functional component between classwork and seatwork to be material was in Bernie's series on ratio (see Table 2), suggesting that the teacher was not a critical factor here but that the class may be.

Discussion

This investigation has highlighted differences in the teaching of topics over a series of lessons. Material differences were discernible in the forms of content that the teachers were observed offering over time. In the current study, the topic was a critical factor and this contrasts with Clarke et al.'s (2006) study since although their analysis was of a series of lessons on a topic, the topic itself was not foregrounded in their findings. My findings raise the question of why the forms of content Bernie offered in a lesson series were so different when teaching ratio compared to equations. In particular, the findings lead me to ask whether opportunities associated with functional manifestations of equations were considered or not and, if they were, why these manifestation choices were not enacted.

Differences in the forms of mathematical content offered during classwork and seatwork gets to the heart of the purpose of these two types of engagement from a didactic perspective. From other perspectives, classwork may have a specific purpose such as offering learners extrinsic motivation to engage in mathematical activity. Yet this study prompts the question: what forms of content is it necessary for the teacher to make available through classwork rather than seatwork, if any? The small sample in this study raised the conjecture that making available visual manifestation might be better served by classwork rather than seatwork, and that the class – rather than the topic – being taught would shape the extent of this.

In the introduction, I spoke of how the impact of findings "might be particularly powerful if there is an awareness of choice, if a reason to choose to act differently can be harnessed and if alternative actions can be imagined." The above prompts along with the tri-polar space for manifestation offer opportunity for such activity, for the interior triangular space suggests the possibility of choice and invites actions associated with particular positions within it to be imagined. This activity in turn invites new research opportunities to explore teachers' knowledge, attitudes and beliefs.

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How do prospective teachers imagine mathematical discussions on fraction comparison?

Kristin Krogh Arnesen, Ole Enge, Yvonne Grimeland and Torkel Haugan Hansen
Norwegian University of Science and Technology, Department of Teacher Education, Norway;
yvonne.grimeland@ntnu.no

Prospective teachers plan a short mathematical discussion on comparing fractions by writing lesson plays. We analyse how their mathematical knowledge for teaching surfaces in their written scripts, using three dimensions of the Knowledge Quartet: foundation, transformation and connection. Our findings give insight into the prospective teachers' knowledge of fractions and comparison strategies, their perspectives on mathematics and mathematics teaching, and insight into how they transform their knowledge to make it accessible to middle school students.

Keywords: Mathematics knowledge for teaching, representations of teaching, classroom discussion, lesson planning, rational number sense.

Introduction

Several recent studies call for a practice-based approach to research on teacher education (e.g. Ball & Cohen, 1999; Grossman & McDonald, 2008). An integral part of teachers' work is planning for teaching. Recently, several researchers have advocated writing lesson plays as a means of learning how to plan for instruction (see Zazkis, Liljedahl, & Sinclair, 2009). Lesson plays are imagined mathematical discussions written verbatim. Zazkis et al. (2009) argue that lesson plays can give a "window" for researchers to investigate mathematical knowledge for teaching. We have collected and analysed prospective teachers' (PTs') planning documents for a practice assignment in their school placement. The assignment was to write a lesson play on fractions. Several studies show that many PTs struggle to understand fractions (Newton, 2008; Ma, 1999). Siegler et al. (2010) recommend, "[p]rofessional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them". With this in mind, our research focuses on the following question: *How does PTs' mathematical knowledge for teaching surface in their lesson plays on fraction comparison?*

Lesson plays

This study reports our efforts to develop practice assignments in our teacher education courses with respect to the notion of high-leverage practices. Ball, Sleep, Boerst, and Bass (2009) define "high-leverage practices" to be those that, when done well, are likely to lead to improved student learning. High-leverage practices are practices which novice teachers need to learn to do, and from which they will learn more about teaching (Lampert, 2009). Further, a high-leverage practice is such that novices can begin to master it (Grossman, Hammerness, & McDonald, 2009). Lampert, Beasley, Ghousseini, Kazemi, and Franke (2010) give several instructional activities that could be part of high-leverage practices in mathematics, which can be realised in a relatively short time, typically 10-15 minutes, through classroom discussions. The PTs used coursework literature to analyse and discuss a series of videos and transcripts of classroom discussions concerning computational strategies. We asked the PTs to plan a similar short mathematical discussion with a group of middle

school students, age 9 to 13. This was part of a practice assignment, intended to be implemented during their school placement towards the end of the term. The PTs were asked to plan for the mathematical discussion by writing a *lesson play*.

Lesson plays, introduced by Zazkis et al. (2009), are proposed as a way to plan teaching by writing a script for (part of) a lesson. An envisaged interaction between a teacher and a group of students is given verbatim, as an alternative for the traditional lesson plan. Zazkis et al. (2009) argue that lesson plays can give an opportunity for in-depth discussions of crucial aspects of mathematics teaching *before* the lesson, while such discussions can only take place after the lesson if the lesson is planned using a traditional lesson plan. As such, lesson plays are not affected by John's (2006) claim that traditional lesson plans do not give insight into "the substance of the particular activity" (p. 487).

Zazkis et al. (2009) typically give their PTs a prompt representing a mathematical error or misinterpretation, and ask the PTs to write the script of a discussion, which resolves the prompt. We asked our PTs to plan all aspects of the discussion, including formulating a mathematical aim for the discussion and a task, or a sequence of tasks, to achieve this aim. Our requirements for the discussion were that the mathematical topic was fractions, and that the aim should be to discuss some calculation or reasoning strategy on fractions. Furthermore, the script should include argumentation and some type of generalisation of concepts and/or strategies. Generalisation, argumentation and reasoning was a major focus in the coursework. The duration of the discussion should be 10-15 minutes.

Mathematical knowledge for teaching

We are interested in how PTs' mathematical knowledge for teaching surfaces in their lesson plays, and we use the Knowledge Quartet (KQ) (Rowland, Huckstep, & Thwaites, 2005) as a framework for our analysis. The KQ consists of four dimensions, three of them resting on the first, named *Foundation*. Foundation concerns the teacher's or PT's knowledge of mathematics and mathematics teaching as acquired in their education. It underpins a teacher's ability to make rational, reasoned choices and decisions about instruction based on knowledge of mathematics and mathematics pedagogy. The second dimension of the KQ, *Transformation*, is about how the teacher transforms her own subject matter knowledge of mathematics into forms which enable others to learn it. Such transformation is informed by the teacher's choice of examples and representations and how these support learning of the intended mathematical topic. *Connection* is about the choices a teacher makes in order to ensure the consistency of planning and teaching a topic or concept through a lesson or lessons. As such it concerns anticipation of what students will find problematic, and decisions about sequencing. Crucial is the teacher's understanding of connections between mathematical concepts and between concepts and procedures as well as anticipation of complexity when planning and teaching a topic. *Contingency* concerns a teacher's responses to events that were not anticipated or planned for. Since we are considering planning for teaching this dimension will not be relevant to our analysis.

As the lesson plays analysed concern comparing fractions, it is useful to clarify the implications this has for the foundation dimension of the KQ. When describing number sense, researchers state that it manifests in flexible mental computation, understanding number magnitude, making judgements about calculations, using benchmarks, and having an inclination to use and develop understanding

of numbers and operations (McIntosh, Reys, & Reys, 1992; Sowder, 1992). Researchers have identified several strategies for comparing fractions based on number sense (see Yang, 2007). The *parts* strategy can be used when comparing fractions with the same numerator or denominator. The *benchmark* strategy refers to comparing two fractions to some well-known third fraction, typically $\frac{1}{2}$ or $\frac{3}{4}$. When using *residual thinking* one builds up the fractions to 1. There are also "standard" ways of comparing fractions, which do not overtly depend on number sense, such as finding a common denominator or converting to decimals. Since the task for the PTs was to write a lesson play about reasoning with fractions, we expected the planned discussions to contain more than performing an algorithm. Using visual models of fractions can be a legitimate strategy, but researchers also warn about the limitations of relying on a visual strategy alone (Petit, Laird, Mardsen, & Ebby, 2015; Lamon, 2012). Thus, we expected strategies beyond visual strategies in the planned discussions.

Method

The participants in the study were PTs following a 4-year teacher education programme, for age 6-13, at a university in Norway. The data were collected from their responses to a coursework assignment given during their first mathematics education module, in their first year of study. Of the 178 PTs in this cohort, 32 had chosen tasks on comparing fractions for their written classroom discussion. These 32 scripts are the data analysed in this paper, in particular they were chosen for analysis because fraction comparison has good potential for reasoning based on number sense (Yang, 2007). The excerpts presented in this paper are chosen to exemplify general trends identified in the 32 lesson plays.

All four authors conducted the analysis. Together, we first analysed in detail two lesson plays, using the descriptions of the dimensions from the KQ in our analysis of the two scripts. We then individually analysed the rest of the lesson plays looking for occurrences of similar and contrasting forms of mathematical teacher knowledge related to the KQ. After this independent analysis, we compared similarities and differences in our analyses, and agreed on an interpretation of different aspects of the lesson plays, using notions included in the KQ framework.

Analysis

Of the 32 scripts, 6 used no strategies based on number sense, instead relying on visual "parts of shapes" strategies, or on algorithms such as finding a common denominator or converting to decimal numbers. In the remaining 26 scripts, the PTs used a number sense-based strategy at least once. 9 PTs used *benchmarking*, 13 used *parts* and 17 used *residual thinking*. In the following, we analyse some examples from the scripts in light of the mathematical content.

Anne was one of the PTs who based most of the imagined discussion on number sense-based strategies. Her stated goal of the discussion is to build understanding of the strategies of benchmarking and residual thinking. She gives two tasks designed to encourage the students to

utilise these strategies, and we quote here[1] two excerpts from the imagined discussion between Anne's teacher and her students while discussing the sorting of $\frac{7}{8}, \frac{3}{4}, \frac{2}{10}$ and $\frac{3}{6}$ by magnitude:

Ola: Yes, at least you see that $\frac{3}{6}$ is the same as the half of something, that was where I started.

Teacher: OK, so you believe that it is one half. But how can that help us?

Ola: Well, since it is one half, we also see that $\frac{2}{10}$ is less than one half.

Teacher: Per, can you try to elaborate Ola's thinking?

Per: $\frac{2}{10}$ is sort of lacking 3 parts to become one half. Because 5 parts is half of 10 parts. So then $\frac{2}{10}$ is less than $\frac{3}{6}$

Teacher: Right, do the rest of you agree? Yes. OK, what do we do next?

We see that Anne, in Ola's words, uses one half as a benchmark when comparing $\frac{2}{10}$ and $\frac{3}{6}$. We note that Ola's explanation is incomplete; it does not state why $\frac{2}{10}$ is less than one half. The PT seems to be aware of this, as she asks Per to *elaborate* Ola's thinking, from whom she receives the completed reasoning. In the next sample from Anne's script, a residual argument is pursued:

Teacher: All right, so now we know that $\frac{2}{10}$ is the smallest, and then comes $\frac{3}{6}$, but which is the biggest of $\frac{7}{8}$ and $\frac{3}{4}$?

Per: Since $\frac{1}{8}$ is smaller than $\frac{1}{4}$ then $\frac{7}{8}$ is the most

Mia: But why is that when $\frac{1}{4}$ is a lot bigger than $\frac{1}{8}$?

Teacher: Good question Mia, does anyone want to explain?

Ola: When we looked at $\frac{1}{8}$ and $\frac{1}{4}$ we were looking at how much was missing to fill one whole. The one that miss the biggest part is then missing the most, and therefore that fraction is the smallest. Because $\frac{7}{8}$ is only missing a small $\frac{1}{8}$ to become one whole.

Mia: Oh yes, now I understand, because we are looking at what is missing.

Anne does not acknowledge any students' claim without a justification. Through the whole of Anne's script, the teacher is encouraging the students to utilise their number sense when reasoning about the tasks. Similar approaches are also apparent in the rest of the 26 scripts, but not always as comprehensive as in Anne's case. One typical feature is that even though a mathematically correct conclusion is reached, there is no valid argument given by the students, and the PTs tend to accept this without comment. This is evident in Alice's script, when the students are comparing $\frac{4}{8}$ and $\frac{3}{4}$.

[1] The original scripts were written in Norwegian, with translation to English by the authors of this paper. In the translation we have retained linguistic inaccuracies and imprecise use of terms, as in the original Norwegian.

Fredrik: Yes, me and my group decided that $\frac{3}{4}$ is the biggest. I think that 3 is closer to 4, than 4 is to 8.

Teacher: That was good thinking. [Proceeds with a different task]

Our analysis shows a general tendency in the scripts that strategies based on number sense have some kind of justification, while strategies based on algorithms and rules are more likely accepted without justification. The above excerpt from Alice's script is one of few exceptions, where she gives a correct conclusion with an attempted justification that is not valid as an argument. Judging by the teacher's response, it seems that Alice regards Fredrik's argument as valid.

The following excerpt from Christine's script shows another problem.

Sindre: $\frac{3}{6}$ must be the biggest, because that fraction is only missing 3 parts to become one whole, while $\frac{3}{8}$ is missing 5 parts to become one whole.

Teacher: Yes, that's right Sindre. Did the rest of you understand what Sindre was thinking? Nina, can you explain what Sindre meant?

Nina: Yes, you can also say that $\frac{3}{6}$ misses one half to be whole, while $\frac{3}{8}$ lacks more than half to be whole, since it is lacking 5, and half of $\frac{8}{8}$ is $\frac{4}{8}$.

Teacher: That was a good explanation, Nina. Did the rest of you also understand what Nina meant? (The class agrees.)

In this excerpt, we notice that Sindre's argument is wrong even though the conclusion is correct. Christine (in the role of the teacher in the discussion) does not comment upon this, instead simply accepting Sindre's argument. Interestingly, Nina subsequently gives a valid residual argument, but Christine does not draw attention to the difference in the two arguments in her script. Our analysis shows that similar arguments that "work" on the fractions in question, but where a counterexample would prove the argument not generally valid, are typical for many of the lesson plays.

Another aspect of our findings was the PTs' choice of tasks used in the discussions. Returning to Anne's script, she uses only two tasks. They both underpin the strategy in focus, and have a natural progression in complexity: The fractions involved seem to be carefully chosen to make her target strategy suitable, and residual thinking is further highlighted by Anne asking the question "Which of the fractions are missing the most to become one whole?" at the start of the discussion. In contrast to this, Molly's choice of tasks and the sequencing chosen, seems less appropriate: Compare $\frac{3}{5}$ and $\frac{3}{7}$; $\frac{7}{7}$ and $\frac{7}{9}$; $\frac{2}{4}$ and $\frac{3}{4}$; $\frac{2}{4}$ and $\frac{4}{8}$; and $\frac{1}{2}$ and $\frac{3}{6}$. Molly does not state explicitly a mathematical goal for the planned discussion, and her imagined discussion covers several ideas in a brief way. Moreover, the progression of difficulty in the sequence of tasks does not seem to be well thought through: in the lesson script on the first tasks, Molly's fictive students use benchmarking with one half, indicating that one half is a well-known concept for them. To then proceed with the final three tasks focusing on equivalent fractions to $\frac{1}{2}$, seems exaggerated.

Discussion

We now relate the findings presented in the analysis to the dimensions of the Knowledge Quartet.

Foundation

For the foundation dimension, the most visible aspects are the PTs' mathematical knowledge of fractions and comparison strategies, as well as their beliefs about mathematics itself, and about mathematics teaching. In general, the PTs try to use strategies relying on number sense to compare fractions. This could indicate that the PTs value developing understanding rather than focusing on an algorithmic approach. The strategies attempted in the scripts are not always followed through in a mathematically valid argument, and some of the PTs fail to recognise the difference between valid and invalid arguments. Christine's script is an example of this.

Another finding is that very few scripts contain any attempt at discussing the *generality* of the strategies used. This indicates that the PTs' beliefs about mathematics might not include this as an important aspect of doing mathematics. Instead, the PTs seem to be satisfied as soon as the problem at hand is solved, as in Christine's script when neither Sindre's or Nina's arguments are investigated further from a general point of view. Recall that the task given to the PTs particularly required them to emphasise the development of their students' understanding and reasoning.

Transformation

The PTs' scripts afford good insight into their choice of examples to elicit an idea. With very few exceptions, the tasks chosen by the PTs are suitable comparison tasks where it is clear that there is at least one number sense-based strategy that could be applied.

We proceed to consider the PTs' use of questions. In the context of PTs writing an imagined discussion, we regard this as a form of teacher demonstration, and thus consider it a part of transformation. We find in most scripts a use of certain techniques and types of questions known from their coursework literature on orchestrating mathematical discussions. For example, in Anne's script, the teacher's questions structure what her students have discovered and then seek to develop their ideas further. When Anne's teacher asks Per to elaborate Ola's thinking, she succeeds in bringing to light an argument. In other scripts, the PTs seem to emphasise the use of discussion techniques in itself to such an extent that it suppresses the attention on connecting the mathematical ideas. This can be seen e.g. in the excerpt from Christine's script above, where the teacher asks a student to repeat another student's reasoning without connecting the different explanations. Sometimes the PTs fail to notice when a clarifying question is needed. This can be seen in Alice's script above, where Fredrik's attempted justification is an invalid argument in general, and yet the teacher accepts it and proceeds without further enquiry.

Connection

The *sequencing of tasks*, how one task should connect to the previous task, and the anticipation of what students will find problematic, is part of the connection dimension. We find that in most scripts, the sequencing of tasks is appropriate. However, we find examples of situations where the PTs do not seem to anticipate the complexity of the sequence of tasks. An example is Molly's script as discussed above. Other scripts seem to have too many tasks, given the time allotted. In these scripts the discussion moves forward smoothly with students giving the desired response quickly and effortlessly. This may indicate that these PTs do not anticipate complexity in the discussion and that the conceptual challenge for the students is underestimated. Thus, these discussions take more the form of numerous repetitions of the same procedure, which relates to the foundation dimension

of the KQ and perspectives on how mathematics is learned: These PTs seem to emphasize procedural repetition as an important aspect of learning mathematics, perhaps on behalf of unpacking the mathematics of the procedures. However, some scripts include deliberate mistakes and misconceptions made by the students, which are then discussed. We see this as an anticipation of complexity.

Conclusions

Following the discussion above, we claim that lesson plays encourage the PTs to use and develop several aspects of their mathematical knowledge for teaching. For the foundation dimension, we claim that the insight we get from the scripts, is more than what we would get from simply assigning the PTs fraction comparison problems for them to solve. We note that several PTs write discussions including both valid and invalid arguments and both are accepted without further probing. For instance, Christine knows what a valid argument for comparing fractions looks like, but at the same time she accepts an invalid one. Such inconsistency in the PTs' thinking might become more visible when they plan teaching by imagining a detailed mathematical discussion.

We also claim that our findings show the importance of emphasis in mathematics teacher education on generalisation and argumentation, and how classroom discussions concerning generalisations could play out. Our PTs were asked to have those aspects in mind when writing their lesson plays, and yet it is rarely found in the scripts. How to develop the PTs' ability to emphasise this aspect more needs to be studied further. Managing classroom discussions is a complicated task for novice teachers. However, due to its high leverage on students' development of mathematical understanding it is a critical factor in mathematics teaching, and thus in teacher education.

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A novice high school teacher's perception on the effect of content course taken as preservice

Fatma Aslan-Tutak[1] and Tugay Durak[2]

[1]Bogazici University, Istanbul, Turkey; fatma.tutak@boun.edu.tr

[2]Bogazici University, Istanbul, Turkey; tugaydurak1@gmail.com

The purpose of the current study is to provide findings of a follow-up study on a content course developed to improve specialized content knowledge of high school teachers. Elif was chosen to be interviewed among the participants of the content course according to certain criteria of sampling. Elif graduated within a year of taking the course and have been teaching in high school since. The individual semi-structured interview on her perceptions of how the content course influenced her teaching was conducted. The interview was transcribed and coded. Two themes emerged about the influence of the content course; using same examples or proofs, readiness to implement tasks for teaching. The findings show the participant's self-report of how she transformed her SCK experiences from university to her instruction. Furthermore, her answers revealed the relationship between SCK and PCK, how this unique knowledge of mathematics (SCK) forming basis for PCK.

Keywords: Specialized content knowledge, mathematical knowledge for teaching, novice teachers, content course, secondary mathematics teaching.

Introduction

Teacher knowledge has been in central position for teacher quality. Not only Shulman's (1986) work and emphasis on pedagogical content knowledge (PCK) but also further studies (e.g. Ball and her colleagues) on understanding nature of knowledge needed for effective mathematics teaching have been shedding light on role of teacher knowledge. In addition to studies to develop different models on teacher knowledge, there are many researches on understanding or improving teacher knowledge. However, most of those studies investigate either preservice teacher education (content or pedagogy courses) or in-service teachers (professional development). In this study the authors will share findings from a case study of a novice secondary school mathematics teacher's perception on influence of taking a content course focusing on Specialized Content Knowledge (Aslan-Tutak and Ertas, 2013) on her teaching. Authors will shortly review literature on mathematics teacher knowledge that guides this research.

Mathematical Knowledge for Teaching

The theoretical framework of this study is mathematical knowledge for teaching (MKT) model developed by Ball et al. (2008). According to MKT model, there are six sub-domains of teachers' knowledge for mathematics teaching. Three sub-domains of subject matter knowledge are Common Content Knowledge, Specialized Content Knowledge, Horizon Content Knowledge, and sub-domains of pedagogical content knowledge are Knowledge of Content and Students, Knowledge of Content and Teaching Curriculum Knowledge. Since this model is very well known, in literature review part of this paper, authors will focus on only Specialized Content Knowledge (SCK). Further information about these domains can be read from Ball et al. (2008).

SCK is a type of subject matter knowledge; it is defined as mathematics knowledge which is unique to teaching. Indeed, it is about knowledge about mathematics, not teaching, but for teaching. Ball and her colleagues stated the role of SCK as follows:

> What caught us by surprise, however, was how much special mathematical knowledge was required, even in many everyday tasks of teaching— assigning student work, listening to student talk, grading or commenting on student work. Despite the fact that these tasks are done with and for students, close analysis revealed how intensively mathematical the tasks were. (Ball et al., 2008, p.398)

Due to nature of SCK it may be omitted in university mathematics courses and also in pedagogy courses (Ball, et al., 2008). Especially in secondary school teacher education, mathematics courses are advanced and require teachers to study mathematics from a different perspective than teaching. Since MKT model was developed based on a study of elementary school teachers, it was necessary to revise definition of SCK for high school preservice teachers. In an attempt to improve high school preservice mathematics teachers the first author developed a mathematics content course. The course content was developed by addressing not only MKT but also Advanced Mathematical Knowledge model of Zaskis and Leikin (2010), Advanced Mathematics for Secondary School Mathematics Teachers by Usiskin and his colleagues (2003), and SCK task development perspective of Suzuka and her colleagues (2009). Usiskin and his colleagues emphasized three types of mathematics experiences for secondary school mathematics teachers; concept analysis, problem analysis and mathematical connections. Lastly, the content course practices were developed to allow preservice teachers to unpack their already existing knowledge and to develop a flexible understanding of concepts (Suzuka et al., 2009). How these models and perspectives were merged together to form a content course can be read in detail in author's previous paper for CERME 8 (Aslan-Tutak & Ertas, 2013).

Since our research is a follow up study, it is needed to mention about aforesaid content course's ingredients. The mathematical tasks which were used during the course can be categorized under four approaches; unpacking concept definitions, applications and modeling, procedures and generalizations, and historical perspective of concepts (Aslan-Tutak & Ertas, 2013).

Unpacking concept definitions practice was identified by participants as eye-opening experience. One example is the activity focusing on geometric definition of imaginary number "*i"*. Almost all of the students stated that they had seen geometric representation of "*i*" for the first time and they were really surprised. The instruction which focused on geometric representation of "*i*" was discussed in Aslan-Tutak and Ertas (2013) as;

> In order to explore number "*i"*, instruction started with discussion on the roots of an equation as its constant terms changes (Usiskin, et al., 2003). Then graphs of the equations were plotted on Cartesian plane (geometric representation of the equations) and their roots showed on a number line (geometric representation of real numbers). The number line was sufficient for the roots of first two equations but participants realized that they could not show the roots of third equation on number line. Then, the definition of the imaginary number *i* was discussed by rotation of 90° perspective (Lakoff & Núñez, 2000; Trudgian, 2009; and Usiskin et al., 2003) which provides a geometrical understanding of the imaginary number.

Second categorization of the course was applications and modeling. In the content course, there were some modeling problems for the topic of functions and also alternative definition of function. An example of modeling activity was about oil spill and its cleaning procedures. At the beginning of the problem there were some discussion about oil spill incidents in Gulf of Mexico and France and its effect on environment. After discussion part, the students were taken the problem as: *If there was 8000 gallon oil due to a spill, and the crew can clean only 80% of oil for a week. So how much oil would remain after a week? And how long will it take to clean until there is 10 gallon of oil is left?* Erbas et al. (2014) discussed two approaches in use of modeling: as a means of teaching mathematics and modeling as an aim of teaching mathematics. In this current study, the authors used modeling in content course as a means of teaching mathematics. This practice was aimed to introduce preservice teachers to use mathematical modeling in instruction while examining definition of functions. In addition to modeling, history of mathematics was used for examining functions. For example, history of logarithm and Napier's problem were studied. The historical approach was not limited to functions but almost all the topics, such as history of irrational numbers.

Participants of the content course stated that, the proof of equality 0,9999…=1 was also one of the surprising tasks of the course. Seven different approaches to show it were posed to preservice teachers and then the relationship between the aforesaid proof and converting infinitely repeating decimals to fraction without any formula were established. This mathematical task was used with preservice teachers in different settings. Conner (2013) used this task to promote argumentation with preservice teachers. She stated that "Engaging prospective secondary teachers in mathematical argumentation is important so that they can learn to engage their own students in creating and critiquing arguments." (p. 172).

The second author of this current paper was a student at the time of the study. He graduated from the program and has been teaching for three years. The purpose of the content course was to improve participants' SCK which can be used during teaching. Therefore, the authors examined perceptions of participants, actively teaching in high schools, about how their learning from the undergraduate course transferred into their teaching.

Methods

The two-year long study, SCK development of high school teachers, was conducted at a mid-size, western public university in Turkey. The content course was offered in 2011 fall semester and 2012 fall semester. Based on students scores on national central university entrance exam, this high school mathematics teaching program had the highest rank among all of the mathematics education programs in Turkey. Students of this program had to take advanced level mathematics courses such as calculus, linear algebra, and complex analysis. Seventy percent of the courses are mathematics courses. Pre-service teachers also took general pedagogy courses, mathematics teaching courses and school practicum courses. At the time of this SCK project, there was no content course in mathematics teacher education program. The content course that developed based on this project was therefore elective for students. There were 31 students who enrolled to this course in 2012 fall semester.

The sample of the current study was determined based on three criteria; graduation from the program and starting teaching in spring 2013, teaching in high school, and living in Istanbul. Some of the students did not pursue career in teaching but other professions such as economics. Also, not all

students graduated in 2013. This condition was necessary in order to allow minimum time between taking the course and starting teaching. The second condition of teaching in high school was used because the topics of the content course were high school topics. The topics of this course are not covered in middle school. In order to investigate how novice teachers' transfer their SCK learning into their practices, it was important to choose participants according to grade levels they teach. The last condition of living in Istanbul was for convenience, in order to be able to conduct interview easily. There are some novice teachers who fulfill the first two conditions but they were living in other cities. Considering all of these conditions, there are only two novice teachers. Since one of them is the second author of this paper, the only other novice teacher (Elif, pseudonym name) was the sample for this study.

Indeed, all these conditions are also helping researchers to establish trustworthiness of the study. The conditions allow clear examination of transferability of the study. The findings of the study can be transferred into settings where preservice teachers study mathematics content for SCK in context of teaching and start teaching afterwards. Furthermore, in order to improve trustworthiness of the study researchers did member-checking with the participant. In addition to asking Elif to confirm interviewer's interpretations during the interview, later researchers also contacted her to discuss about findings of this case study. The member-checking also aided for the conformability of the study. For conformability, after the completion of analysis done by the first author, the second author went through the analysis. After this two-stage of analysis, Elif's member-checking was helpful to finalize findings. Elif approved most of the findings and she improved what we called "completeness". This phenomenon of "completeness" is discussed in findings section.

Yin (2013) explained the rationale of using case study research methodology as explaining "how" or "why" questions in-dept. The purpose of this study is to investigate how a novice high school mathematics teacher uses her SCK experiences in the content course. How does this course influence her teaching practices? Since teaching is a complex practice, it was necessary to examine this research question through case study methodology. The authors conducted a semi-structured interview with the participant, Elif. The interview questions were in two parts (i) background questions (e.g. years of teaching), (ii) her experiences in the course as a student and how these experiences influence her teaching.

Elif's interview was first transcribed and the open-coded. After open-coding, Elif's answers on how the SCK tasks in the course influenced her teaching practice were emerged into two themes; (i) using same example, proof in the classroom, (ii) readiness for implementing tasks of teaching (understanding student misconceptions, answering student questions, material development).

Findings

The first part of the interview was to understand Elif's teaching experiences and the environment that she has been working in. The school environment and possible mentor-mentee relationship is important in novice teachers development of practice. Elif took the content course in fall 2012 and graduated from university on June 2013. Right after her graduation, she started to work at a private high school, Baris Schools. Baris Schools is a school of a foundation which has more than 100 years background in education. Baris Schools have three K-12 campuses in Istanbul. In other words, Elif started to work at a well established school, a good opportunity for a novice teacher. After her first

year, she transferred to another campus and has been teaching for two years at that campus. So, she has been teaching high school (9-12 grades) for three years. In the interview, she stated that her school is a good place to learn about teaching. Considering how she transferred what she learned in content course in her teaching, she stated that mathematical exploration in the course provided her "completeness" for the topics/concepts that were covered in the course.

Elif: You can guide, direct the students to understand concept because you know that concept.

Interviewer: How do you direct them?

Elif: You construct the concept together, like *i* square to be -1, by asking questions, when you are complete for content.

Especially as a novice teacher, she valued this feeling of completeness both when preparing instructional material and during the instruction.

Elif: When teaching, for the concept, you can be prepared. Your explanations are complete, select materials and put them in order.

According to Elif, completeness is a broad term which includes readiness to unexpected questions, constructing relationships between mathematical concepts and having deep understanding of the concepts. Her answers on how she transferred what she learned in the course to her teaching practice will be discussed in detail according to two themes that were emerged from data.

Using same examples or proofs

She stated that she kept all of the course materials, book, notebook, activities and presentations. She said, for some topics, she specifically used same examples. Her answers on what she kept same for her teaching reveal that she used alternative definitions (number *i*, mathematical functions), proofs (irrationality of $\sqrt{2}$, quadratic formula), and examples given for concepts (examples for exponential functions). For example, she had been teaching complex numbers for two years, she stated that she always introduced number *i* by the same geometric definition from the content course. But she made a differentiation about using materials from the course. Elif stated that while she was teaching geometric definition of number *i* for whole class instruction, she used many proofs only when certain group of students asked for further mathematical knowledge. She said that she decided to use a definition/proof/example based on students' mathematics level. In the following quote from the interview, she described how she used "deriving the quadratic formula" that she learned in the content course.

Elif: When I explain it (quadratic formula) to students, even the order of doing it was stuck in my mind because we teach it this way, we give formula at the end. Before that to make square formula…It make sense, you get quadratic formula from making square formula. Discussing $b^2 - 4ac$ from the beginning, why there are two roots, or one root…Yes, it's a terrible formula but we derive it together, of course with the ones who are good at math

Readiness to implement tasks of teaching

Content course, in other words SCK tasks that were done in the content course, aided Elif's teaching during the instruction when answering student answers. In the individual interview, Elif discussed about using her mathematical experiences from the content course. Many of Elif's answers to interview questions were emerged to form the theme, readiness for teaching. However, her answers were similar to mathematical tasks for teaching addressed in MKT model (Ball et al., 2008). So, in order to put emphasize on Elif's differentiation of demands of teaching, we merged her answers together and named them *readiness for implementing tasks for teaching*. As it was stated before in her feeling of completeness, she emphasized *knowing what to prioritize in teaching a concept*. She explained knowing and realizing the big idea in concepts several times. For example, while she was getting ready for teaching a concept, she knew what to emphasize in instruction even before thinking about how to teach it.

Elif: In complex numbers, maybe students didn't get why square of *i* is -1, but I checked my notes (from content course) in order to get 180 degree rotation thing in my mind so I can teach it. This was for me actually.

The second code is *building awareness*, and directing her to find other resources. Elif didn't have to use same examples from course in her instruction. Mathematical tasks provided some awareness to her and she did further research about certain concepts. She stated that based on mathematical tasks in the course, she searched and found other tasks/examples from Khan Academy or YouTube. The third code is *directing students to understand concepts*. Elif emphasized the role of mathematics that she learned in course to help her ask mathematical questions to guide student thinking. In a sense, this is related to first code, identifying and knowing big idea of a concept. Further to that identifying big ideas, in this code, she is using that knowledge for directing student thinking.

Elif: In the class, maybe I didn't use exactly same example but when there is a student question, if it is related to mathematical concept, I can just direct student to reach the concept. Because I learned that content.

The fourth code under readiness to implement tasks of teaching is *answering student questions and addressing their misconceptions*. This code is related to previous one but Elif made explicit differentiation between directing students and answering students' questions.

Elif: When someone asks, or what does quadratic formula mean, why do we need it. I know these. When there is a question like these I can answer them easily.

Here, Elif focuses on her state of being confident in her mathematics knowledge when answering student questions. The last code, fifth one is related to her *implementation of curriculum*. As a novice teacher Elif stated using textbooks and course materials from other experienced teachers. However, she realized differences between her and experienced teachers in terms of implementing curriculum. In her first year, the national curriculum was revised extensively. One of the changes was how functions were introduced in 9th grade. In previous curriculum, first mathematical relations were introduced and then set theory approach of functions was given. In revised curriculum, functions should be taught through mathematical modeling without an introduction with mathematical relations. The covariation definition gains importance in this new curriculum. Elif realized that knowing alternative definitions of function (including covariation) helped her to easily adapt her

instruction to new curriculum. The SCK experiences of the content course provided basis for her use of the new curriculum. Furthermore, she was able to implement certain features of the curriculum while experienced teachers omitted.

Elif: Maybe, they don't think it (proof of $\sqrt{2}$ irrationality) is important. Because other teachers also analyzed the new book, but I said we need to do this. They didn't want to spent time on it. This actually give information about what do we need to give importance conceptually.

Discussion

The purpose of this study was to investigate perception of novice teachers on how they transfer what they had learned in the content course into their practices. Elif, a former student of this course and with 3 years of teaching experince, was volunteer to participate this follow up study. She stated that the course and mathematical explorations (SCK tasks) clearly helped her in teaching. The most expected influence would be using mathematics examples, tasks from the course. Usiskin and his colleagues (2003) discussed that "Often the more mathematics courses a teacher takes, the wider the gap between the mathematics the teacher studies and the mathematics the teacher teaches" (p. 86). So, based on findings from Elif's case study, it should be explored further if a content course with SCK tasks provides mathematical explorations that preservice teachers will be teaching in their profession. Furthermore, these SCK tasks also allowed Elif to unpack her knowledge of mathematics (Suzuka et al., 2009) so she could identify mathematical big ideas and prioritize important concepts.

It is important to note Elif's feeling of readiness in her first three years of teaching. There are various demanding tasks of mathematics teaching such as selecting appropriate mathematical task, using proper representations, answering student questions, and leading student discussions. Elif mentioned five different tasks (themes emerged from data analysis) of teaching that influenced by her experiences in the content course. Her discussion of these tasks may be used to depict how SCK is taking role in classroom instruction. It can be used to discuss the link between SCK and PCK. For example, when Elif discussed role of knowing mathematics in directing students to understand concepts, she was actually talking about PCK. She explained and clearly discusses the role of specialized knowledge of a teacher. Without this type of mathematics knowledge, she will be lacking directing students. Similarly, in using curriculum materials, she uses her SCK knowledge.

There are some limitations of this study such as sampling only novice teachers in Istanbul, and also relying on participant's self report on how she transferred what she learned in the content course. Even though, interview questions were specifically asking about the content course, there is still a limitation of influence of all other teacher education courses/practices on Elif's transfer from university to her teaching. Furthermore, Elif also has been teaching at a prestigious high school which provides various resources for a novice teacher to improve herself. In order to investigate further how preservice mathematics teachers transfer what they have learned into their teaching, researchers are planning to extend the study to include other participants who possess different characteristics.

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Study of primary school teacher's practices for a lesson after a Lesson Study process in mathematics

Valérie Batteau

HEP Lausanne-University of Geneva, Switzerland; valerie.batteau@hepl.ch

This study presents the analysis of primary school teacher's practices in mathematics for a lesson that has taken place after a professional development training called lesson study (LS) in Lausanne, Switzerland. Practices are analysed in a double didactical and ergonomical approach. The methodology used is a case study of the particular teacher's practices. Results about the teacher's practices after the LS process are discussed.

Keywords: Teacher's practices, problem solving, lesson study.

LS is a field of research and professional development developed principally in Asia, in US and in Northern Europe (Lewis & Hurd, 2011; Yoshida & Jackson, 2011). LS is a collective and reflexive process that involves a group of teachers and facilitators meeting to improve instruction.

This study concerns practices of a teacher who participated to a LS process and this study falls within the "double approach" (Robert & Rogalski, 2002, 2005).

Theoretical framework: The "double approach"

Teachers' practices are analysed using the following theoretical framework: the "double approach" based on a French didactical approach (Robert & Rogalski, 2002, 2005) and an ergonomical approach based on activity theory (Leontiev, 1975; Leplat, 1997).

This "double approach" distinguishes the task, the "goal to be attained under certain circumstances" and the activity, what the teacher engages in during the completion of the task (Rogalski, 2013, p. 4). The prescribed work fits the *prescribed* task (in our context: what the teacher must do according to Teacher's Handbook and the official program) and the real work fits the *conducted* task (in our context: what the teacher does in reality during the lesson). To appropriate the *prescribed* task, the teacher should modify it. Thus, a gap exists between the *prescribed* task and the *conducted* task: the reasons can be a lack of the necessary competencies or an inappropriate representation of the task for example (Ibid.). Leplat (1997) adds two levels of tasks: the *represented* task (in our context: how the teacher represents the *prescribed* task and what he thinks we attend of him) and the *redefined* task (in our context: the teacher redefines his task according to the *prescribed* task and his own professional goals). These levels of tasks are neither hierarchical nor time: the teacher can represent the *prescribed* task and can redefine a new task before and during the lesson in taking into account different sources (students' activity, his own activity, institutional constraints).

The teacher combines professional acts and knowledge (mathematical, didactical, pedagogical) in his representation of the *prescribed* task and in his redefinition of the *represented* task. This study focused on these professional acts and knowledge at stake in these representation and redefinition. Thus, the teacher's activity is analysed as a process of modifications between the *prescribed* and *conducted* tasks (Leplat, 1997; Mangiante, 2007).

Research question

This case study aims to provide elements to respond to the following question: what are the sources of the process of modifications between the *prescribed* and *conducted* tasks?

Methodology and data

This qualitative study used a case study to analyse teacher's practices. A LS process (see Figure 1) can be decomposed four steps (Lewis & Hurd, 2011, p. 2): the group studies a mathematical subject, standards and sets instructional goals (step 1), the group prepares a research lesson based on their study of the topic and standards (step 2), the group selects one teacher to conduct the research lesson while others observe and collet student data (step 3), and finally, the group analyses and reflects on the research lesson (step 4), with the option of teaching it again (Batteau, 2016).

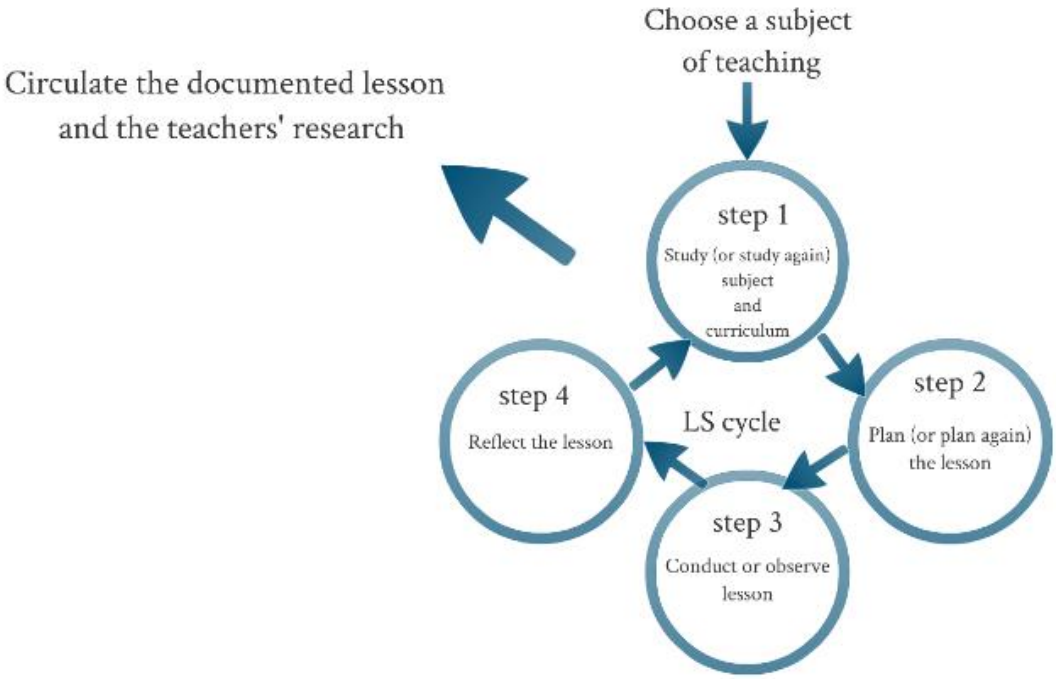

Figure 1: A LS cycle (Lewis & Hurd, 2011)

In the Swiss context, some researchers chose to implement this form of LS process without modify the structure in four steps because this model fits a French didactical point of view (Clivaz, 2015): the Theory of Didactical Situations (Brousseau, 1997; Warfield, 2014). In the TDS, the methodological research tool consists of an *a priori* analysis of the possible teaching of a mathematical subject: the steps 1 and 2 fit a deepen *a priori* analysis with a study of the mathematical subject, the didactical variables, the student's strategies, and difficulties. A second methodological research tool (in the TDS) consists of an *a posteriori* analysis, who takes place during the step 4, which compares what would be anticipated and what is happened. Thus, a LS process in mathematics began in Lausanne in September 2013 and occurred over two years with two

collective sessions occurring per month (Clivaz, 2016). The group consisted of eight primary school teachers ranging in experience, voluntary, and generalist[1] teachers, and two facilitators[2].

This study focused on Océanes' practices for one lesson observed after the end of this LS process. For this teacher, data were collective sessions during the LS process (cycles *c* and *d* about problem solving), one lesson after the LS process (about problem solving), informal meetings after this lesson, all written documents produced during the lesson, and student work. Video data (lesson and collective sessions) were transcribed. This teacher has seventeen years of experience and students eight to ten years old.

To operationalise the theoretical tools presented for this study, the *prescribed* task fit the aim of the problem chosen by Océane, the Teacher's handbook, and the planning material for this problem. The *prescribed* task is analysed *a priori*, which means the mathematical knowledge at stake in the problem, the possible resolutions, and the didactical variables were analysed. We analysed the modifications between the *prescribed* and *conducted* tasks. To explain these modifications, we analyse the representation of the *prescribed* task and the redefinition of the *represented* task with using the informal meetings and collective sessions.

This paper presents first the mathematical subject worked during the LS cycles *c* and *d* before this lesson.

Analysis

Cycles *c* and *d* of the LS process

During the cycles *c* and *d*, the group worked on problem solving and how to help students represent a problem. The group relied on an article (Julo, 2002)[3] in which the main idea was explained during a collective session.

> Facilitator: (quoting Julo) "this help doesn't give clues about the answer, doesn't guide to a strategy and doesn't suggest a modelling". But it's difficult to achieve, it's written just after that. It is an ideal […] but if we don't follow this ideal, it means that we do precisely a part of what students have difficulty to do.

The research lesson of the cycle *d* was based on this problem: examine the matchstick pattern represented below. How many matchsticks are needed to align 99 squares?

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[1] In the French speaking part of Switzerland, primary school teachers teach several school disciplines (mathematics, French, sciences...).

[2] In this particular LS process, the two facilitators were researchers in Mathematics Education and in teaching and learning (Clerc-Georgy & Clivaz, 2016). They had the role of trainers and « knowledgeable others » (Lewis & Hurd, 2011, p. 30;33).

[3] During the first collective session of LS process, teachers said to facilitators which subject they wanted to work according to their teaching difficulties and/or students' difficulties. The subjects were numeration (cycle *a*), isometries (cycle *b*), and problem solving (especially how to help students represent and model a problem, cycles *c* and *d*). Then, the facilitators proposed reading this paper to teachers, in order to find elements of answer to this issue.



Figure 3: Problem "99 squares" (Teacher's handbook of 6H, Danalet, Dumas, Studer, & Villars-Kneubühler, 1999, p. 187)

The mathematical function at stake was u(n)=3n+1, where n is a whole number. The group worked on this problem focussing on how to help students represent and model this problem.

Context of the lesson observed after the LS process

For this lesson, Océane chose to manage problem solving and she explained it during a collective session at the end of the LS process.

Océane: There is a lot of problem which I think oh I don't dare to try [...]

Océane: I think, this year with my students, I take the textbook and I do a lot of things I never did before.

Anaïs: Oh, you dared.

Océane: Yeah, I did.

The *prescribed* task: some elements of analysis

For this lesson after LS, Océane chose the problem "Fold": Fold a strip of paper in half, here are two parts. Fold a strip of paper in half, then a second time, here are four parts and so on. How many parts are there with a folded strip of paper ten times?

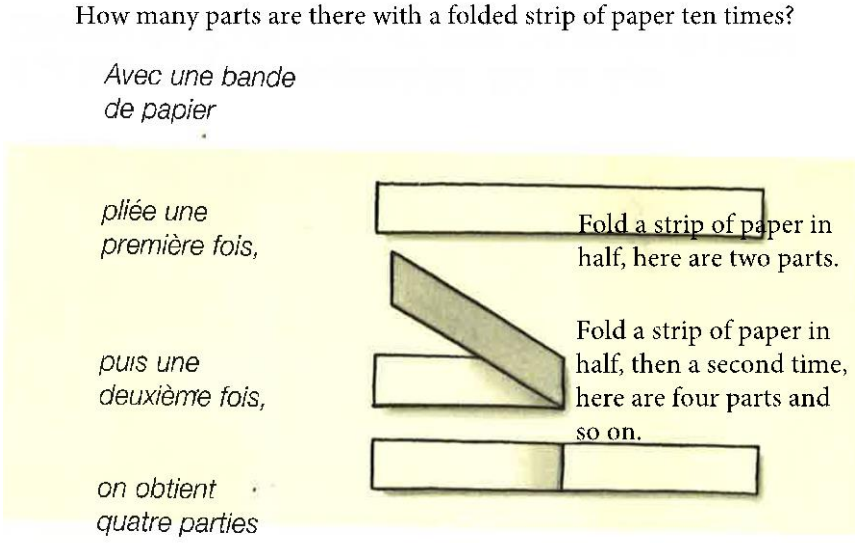

Figure 4: Problem "Fold" from (teacher's handbook, Danalet et al., 1998, p. 96)

The aim of this problem is to develop reasoning capacities and research strategies (Ibid.). In this problem, students should go from handling to representation in order to predict the result of acts (Ibid.). To determine the number of parts when the strip of paper is folded 10 times, we should

calculate 2x...x2 with ten factors 2 (or 2 power of ten). Thus, to find the number of parts with a strip of paper folded *n* times, it's not necessary to know the answer when the strip of paper is folded (*n-1*) times.

The problem solving "Fold" is similar to "99 squares" in that sense these two problems rely on functions (power function or affine function). The kind of functions and the context of these problems are different but the idea of function is the same and the idea that it's possible to determine the number of matchsticks whatever the number of squares or the number of parts whatever the number of times we fold the strip of paper.

Modifications between the *prescribed* and *conducted* tasks

Some significant elements of modifications between the *prescribed* and *conducted* tasks are summarized about the mathematics at stake in the problem. Océane took over the modelling of the problem: she realized a two-column table, then students had to complete it by calculating doubles. Thus, she modified the aim of the problem. During informal meetings, she said that she chose this problem to introduce the multiplication. The issue of the problem is not the same for her (involving the multiplication) and for the designer of the problem (modelling a problem). During the lesson, she took over the modelling of this problem instead of students. Furthermore, she reduced the problem to calculations of doubles of numbers as in this characteristic extract of the lesson.

> Teacher: doubles. Here, we double. We double every time. The double of two, four. The double of four, eight. The double of eight, sixteen. The double of sixteen, thirty-two. The double of thirty-two, sixty-four. The double of sixty-four? All right? So Nadège, the double of sixty-four is? It folds in seven times. […] It's as if we calculate sixty-four more sixty-four. Is it? (*Nadège looks all the folds in her strip of paper*).
>
> Nadège: one hundred twenty-six. One hundred twenty-eight.
>
> Teacher: great. […] Next, Luc?
>
> Luc: two hundred fifty-six.
>
> Teacher: very well. Yes? If we fold it nine times, it should be?
>
> Romuald: five hundred six.

When Océane prepared her lesson, she did not identify the mathematical knowledge at stake in the problem (power of two). She validated students' strategies only with calculations (see extract), and she did not link strategies together. In this extract, she said "the double of sixty-four is? It folds in seven times". However, she did not explain why it's necessary to multiply by two when the strip of paper is folded half. Her strategy of doubling could not allow to respond directly to the problem. With her strategy of doubling, in order to find the number of parts with a strip of paper folded ten times, it's necessary to know the answer when the strip of paper is folded nine times and eight times, …, until two times (see Figure 5). With the "expert" strategy, to find the number of parts when the strip of paper is folded n times, the students should calculate the product 2x2x….x2 with n factors 2.

Using a similar problem solving activity than for the research lesson of the cycle *d*, Océane could not identify the mathematical function at stake.

Another modification of the *prescribed* task was to propose to students to calculate the number of parts when the strip of paper is folded 11, 12, 13, and 14 times (see Figure 5). This modification was coherent with the teacher's strategy because it was not possible to propose to calculate the number of parts when the strip of paper is fold 100 times for example without the "expert" strategy.

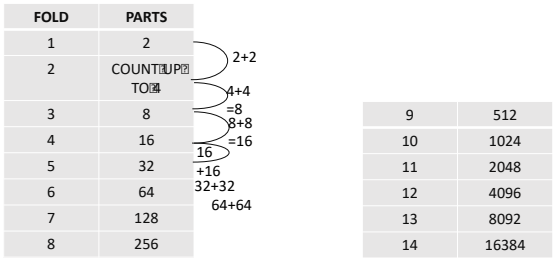

Figure 5: Reconstitution of the blackboard

In the blackboard, Océane wrote only additions to fill in the table, but nor multiplication neither "double of a number". To fill in the second line of the table, she wrote two strategies without linking: count up to 4 and 2+2.

This modification illustrated the focus of the lesson on calculation of double (with additions) and nor on modelling the problem, neither on the explanation of strategies and the links between the different strategies.

Representation of the *prescribed* task

Océane represented the *prescribed* task in according to her mathematical analyses. Before teaching, she prepared her lesson and realised mathematical analysis. The issue of the problem (modelling) took over by the teacher. In her analysis, the mathematical knowledge at stake in the problem are multiplication and doubling of a number. In the teacher's handbook, the aim is to represent, to model a problem, to develop reasoning capacities and research strategies. Her analysis was in contradiction with the teacher's handbook. Thus, she took freedom in relation to institutional constraints of the Teacher's textbook.

Redefinition of the *represented* task

Océane anticipated the two-column table to fill in, so she anticipated to take over the representation of the problem and his modelling before this lesson. During this lesson, she taught vocabulary, "double of", and she focused only on calculations. In her redefinition of the task, she modified the problem in a problem of calculation when the strip of paper is folded 2 until 14 times.

In her redefinition of the task, she modified the problem according to her mathematical analysis and her representation of the task.

Process of modification between the *prescribed* and *conducted* tasks

The process of modification between the *prescribed* and *conducted* tasks had its origins in her representation of the *prescribed* task for this lesson. Océane took into account the students' activity for the first time she taught this problem (last year). Then, she adapted her teaching when she taught this problem for the second time (for this lesson after LS): she took over the modelling and imposed

a two-column table to fill in. She did not take into account students' activity during this lesson but by anticipation.

Conclusion

This case study proposed an analysis of particular teacher's practices during a lesson after a LS process. After the LS process, this teacher has self-confidence over teaching problem solving. In teaching problem solving, it should be able to identify mathematics at stake in the problem. Mathematics at stake should be given by the mathematical textbooks, but it was not the case. In the French part of Switzerland, official textbooks lack mathematical analysis for the teacher to use while planning lessons. For this lesson, the representation of the *prescribed* task relied on the Océane's mathematical analysis which were not sufficient. Thus, her representation and her redefinition of the *prescribed* task did not allow to reach the mathematical learning intended by this problem. To conclude, the sources of the process of modifications for this lesson were her representation of the *prescribed* task and her mathematical analysis. This case study highlighted a gap between the *prescribed* and *conducted* tasks due to the teacher's representation and mathematical analysis.

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Developing student teachers' professional knowledge of low attainments' support by "learning-teaching-laboratories"

Ralf Benölken

University of Münster, Germany; rben@wwu.de

This paper presents a study that is part of a project named MaKosi ("Mathematische Kompetenzen sichern"). It aims at the conception and evaluation of a program in which primary student teachers and children who are low attaining in arithmetic work together. The organization refers to a specific form of project seminars called "learning-teaching-laboratories". The study investigates how knowledge of identifying and supporting low attaining children develops by participating in such a program. Qualitative data were generated by learning maps in a pre-post-design and analyzed by a reconstructive pedagogic-iconological image interpretation. The results indicate a sustainable positive development of student teachers' knowledge.

Keywords: Professional development, learning-teaching-laboratories, low attaining students.

Introduction

Concepts of teachers' professionalization are an important focus of current research in mathematics education (e.g., DZLM, 2015). Regarding a specific professional knowledge of teachers, the development of pedagogical content knowledge (PCK) including, in particular, the development of abilities to analyze children's thinking and learning is reputed to be one of the main goals (Sowder, 2007). Moreover, beyond cognitive aspects, some recent approaches on teachers' knowledge consider affective components like beliefs (Kuntze, 2012). Questions that arise from this are, e.g., how student teachers' education on analyzing children's learning trajectories can be realized, and as a result how cognitive and co-cognitive components of student teachers' knowledge develop. In this paper, the attention will be given to the aspects mentioned above by a synthesis of different approaches within a qualitative study: With regard to analyses of children's thinking and learning, the development of primary student teachers' knowledge about the identification and support of low attainments (ISLA) is focused on, since analyzing mistakes is assumed to be a valuable resource in this context (e.g., Ribeiro, Mellone, & Jakobsen, 2013). As to a suitable organization of a professional development program, the approach of "learning-teaching-laboratories" (LTL) is applied, which reflects an important part of current discussions on student teachers' education in Germany (Roth, Lengnink, & Brüning, 2016). Summarized, LTL provide project seminars intertwining student teachers' theoretical and practical education by working with children, i.e., via learning by teaching. The following questions will be investigated: *How can a development of student teachers' knowledge about analyses of children's thinking and learning in mathematics be organized by a connection of ISLA- and LTL-concepts? How does their knowledge by taking part in an ISLA-LTL develop?* First, brief overviews of the theoretical frameworks will be given. On these bases, a LTL-concept will be outlined. Finally, the study's design and results will be subsumed and discussed.

Theoretical frameworks – brief overviews

As to *teachers' professional knowledge*, the classical concept distinguishes between subject matter knowledge, PCK and curricular knowledge. Summarized, PCK refers to knowledge of possibilities regarding teaching subject matters (Shulman, 1986). Independent of certain approaches, there seems to be a consensus on the fact that PCK bridges subject matter knowledge and teaching, and it designates a specific-distinctive manner of teachers' professional knowledge (Brown & Borko, 1992). According to Ball, Thames and Phelps (2008), PCK covers knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. In particular, the first mentioned aspects provide facets that are connected to analyzing children's thinking and learning as well as to providing an adequate support (Sztajn, Confrey, Wilson, & Edgington, 2012). Because drawing an exact distinction between cognitive aspects regarded by knowledge and affective aspects like beliefs is felt to be difficult, some current research combines both aspects to describe teachers' professional knowledge; inter alia, pedagogical content beliefs (PCB) are described as an equivalent to PCK: Convictions about handling specific instructional situations (Kuntze, 2012) like ISLA. Recent competence frameworks of professional development programs are in line with such approaches. Beyond mathematics-related beliefs, self-oriented ones are considered including components like self-efficacy (e.g., DZLM, 2015), which produces a more holistic view. The study's framework refers to Kuntze (2012). Thus, a combination of cognitive (PCK) and affective aspects (PCB) is assumed. Additionally, self-oriented beliefs are considered in the context of PCB, since in this way a holistic base to describe changes of knowledge by taking part in a LTL is given.

Research on individual *problems in learning mathematics* covers a large range: Beyond approaches that describe such problems as a social construct, or approaches focusing on learning difficulties or disabilities in a narrower sense (for a survey: Scherer, Beswick, DeBlois, Healy, & Moser Opitz, 2016), different approaches focus on previously low achievements (e.g., Watson & De Geest, 2012). The perspective mentioned last mostly concentrates on arithmetic and in this context on typical phenomena such as rigidified counting (and a unilateral ordinal understanding of numbers) and an insufficient understanding of mathematical operations or the place value system (for surveys: Benölken, 2016; Denvir & Brown 1986). Mostly, a group of children is addressed which can be supported within a school's infrastructure, i.e., which does not show learning difficulties in the outlined narrower sense. As to identification or support, recent research independent of certain approaches demands a holistic view considering both cognitive and co-cognitive parameters (e.g., Nolte, 2009). Against the background of student teachers' education, the theoretical framework of both the LTL and, thus, the study corresponds to different aspects of the above outlined approaches: As to problems in learning mathematics, low attainments in arithmetic are focused on considering both typical phenomena and a holistic view in the identification and support procedures.

As to the development of PCK, practical situations that demand, e.g., scaffolding skills are assumed to be adequate opportunities of extending knowledge (Prediger, 2010). Existing findings indicate that one-to-one-interactions of a student teacher and a child might be a promising organizational form (e.g., Kilic, 2015). Against this background, *"learning-teaching-laboratories"* aim at a mutual growth and practical application of knowledge by a specific form of academic studies combining three dimensions: First, the support of children regarding a certain topic; then, the education of student teachers in this context, e.g., as to diagnostics and support; third, research aims like theory

building in the content focused on (Roth, Lengnink, & Brüning, 2016). Recent research mainly concentrates on a clarification of LTL-types and on an interdisciplinary consensus about defining the term of LTL. An example is given by the following definition:

> LTL define a specific form of organization as to student teachers' academic studies combining children's learning with student teachers' professional development in a holistic way. In contrast to, e.g., standard lectures, seminars or practice lessons, LTL offer student teachers opportunities to develop, to enhance and to apply iteratively various skills of diagnostics, support and, thus, both teachers' professional acting and knowledge with regard to specific focuses in authentic, but complexity-reduced learning situations. (Brüning, 2016, p. 1274; translated by the author)

Hence, LTL include aspects and influences that are considered as most important by approaches on teachers' professional growth like the individuality of their learning in mutual reflection and enactment processes (e.g., Clarke & Hollingsworth, 2002). Even if LTL are conducted at more and more German universities in different "STEM"-disciplines, ongoing studies still focus on their evaluation. First impressions indicate that LTL are highly valued by student teachers and they are suited to ensure a sustainable growth of their knowledge about the respective topic (e.g., Brüning, 2016). The study's framework refers to Brüning's definition. Its cornerstones are transferred to the context of ISLA. The demanded complexity-reduction is realized by one-to-one-interactions.

Survey of a LTL-concept in the context of ISLA

The presented study is part of the long-term project "MaKosi" that focuses on the conception and evaluation of a professional development program connecting ISLA- and LTL-approaches (for details: Benölken, 2016). Summarized, the aims are the support of children low attaining in arithmetic and the development of student teachers' knowledge of ISLA. The student teachers' education is organized as a combination of a theoretical course and a project seminar with children. The theoretical course covers information about approaches in the field of problems in learning mathematics as well as concepts of diagnostics and support. While the theoretical course is a regular seminar at university, the project seminar takes place at a primary school once a week about 15 times per semester. Against the outlined framework of ISLA, diagnostics triangulate different tools: In a first step, teachers are given information about the framework and they elect children providing a justification in written form. Then, parents have to fill in a declaration of consent. In a second step, children, student teachers and scientists come together to get to know each other in a playful first session. In a third step, process-diagnostics follow considering both cognitive and co-cognitive parameters; mostly, non-standardized tools such as observations on children's task solving using rating sheets or guided interviews with children, teachers or parents are applied. Every project seminar session is divided into three parts: First, a preparing workshop where student teachers and scientists come together for 15 minutes in order to highlight specific aspects of observation or other determining factors; second, a 90-minute-children-session; finally, a reflecting 75-minute-workshop, in which each child's problems and possibilities as to an appropriate support are discussed. Within this schedule, the children's session is divided into three stages: In the beginning, a playful problem task is offered avoiding arithmetic contents to provide an adequate imagination of mathematics or to support both a positive self-perception of mathematical abilities and joy of problem solving (for example, the problem of "a ferryman, a wolf, a sheep and a head of cabbage").

At this stage, children can organize themselves considering ideas of a natural differentiation. Subsequently, one student teacher and one child turn into one-to-one-interactions of diagnostics and support in established teams for 60 minutes. Thus, the student teachers can develop, e.g., scaffolding abilities in a complexity-reduced situation. Tasks and activities applied in this context are taken from well-proven examples of literature (for examples see: Benölken, 2016), which the student teachers got to know in the theoretical course. They develop suggestions on both their compilation and detailed planning which are discussed during the reflecting workshop with all participating student teachers and the supervising scientists in order to ensure sustainable conducts. Each session closes with a game to support the children's joy of participating in the LTL.

The study

The study focuses on the *question* how student teachers' knowledge of ISLA develops by taking part in an ISLA-LTL. The *participants* were 25 primary student teachers; 11 (10 females, 1 male) took part in the winter semester 2015/2016, and 14 (only females) in the summer semester 2016. Mostly, they were in their third year of undergraduate studies. The study's character is explorative, i.e., generalizations were not intended, but existential propositions (Lamnek, 2010) about possible developments of knowledge by participating in the LTL. Thus, a qualitative *design* was advisable. As to the *method*, qualitative data were generated according to Rott (2017) by applying learning maps in a pre-post-comparison which were anonymized by codes to ensure unbiased interpretations. In the head, the student teachers were given the impulse to craft their way between their current status and their future work at schools: "Dealing with low attainments will be a challenge as to your work as a teacher, especially due to the knowledge of identification and support: What does this mean to you personally? Which way have you covered or which way will you have to cover in the future? Please lay out your way." (translated from German) All participants designed the maps for the first time, and they had to do it before taking part in the LTL at the beginning of a semester, and, again, at its end. As to the *analysis*, the pre- and post-maps were compared by a reconstructive pedagogic-iconological image interpretation, which becomes more and more accepted in different scientific disciplines. Its characteristic steps were observed: (1) Discussion of previous history and selection of key images, (2) image description and analysis (with regard to the factual, expressive and form-related sense), (3) context analysis, and (4) comparative analysis (Schulze, 2013). Data were interpreted within two meetings at the end of the summer semester 2016: The 14 participants analyzed in groups of two or three at least one, but for the most part two pairs of maps. Afterwards, the results were presented, and major observations were discussed in a plenary session.

Results

As to *key images*, their *description* and *analysis*, three types were identified: (1) An interrupted path, (2) a continuous path, and (3) a system of paths. The first type was found only within the pre-, the third one only within the post-, but the second type within both the pre- and post-drawings. Subsequently, we focus on the reconstructions of the examples shown by the Figures 1 and 2, which were conducted in the group meetings mentioned above and which reflect typical main features.

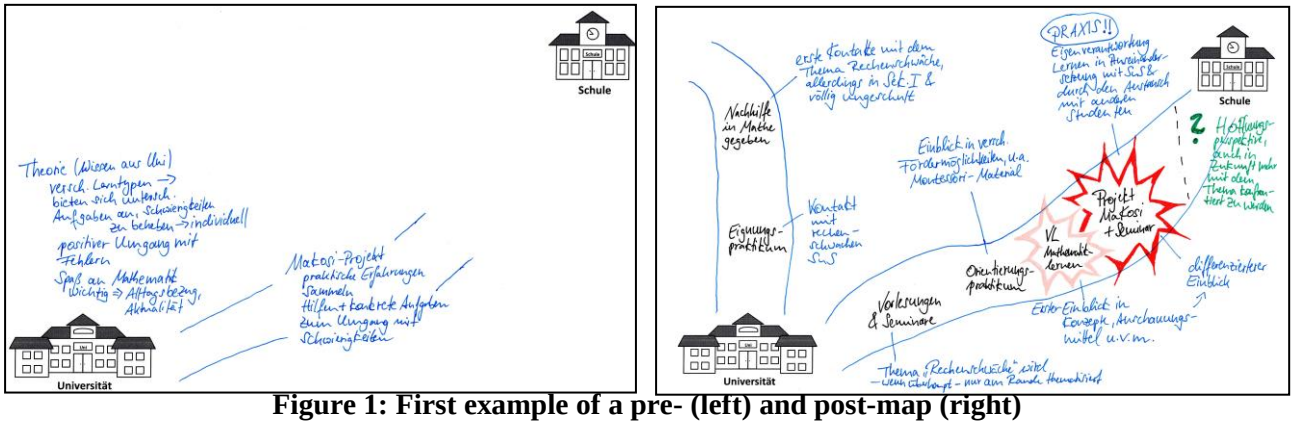

Figure 1: First example of a pre- (left) and post-map (right)

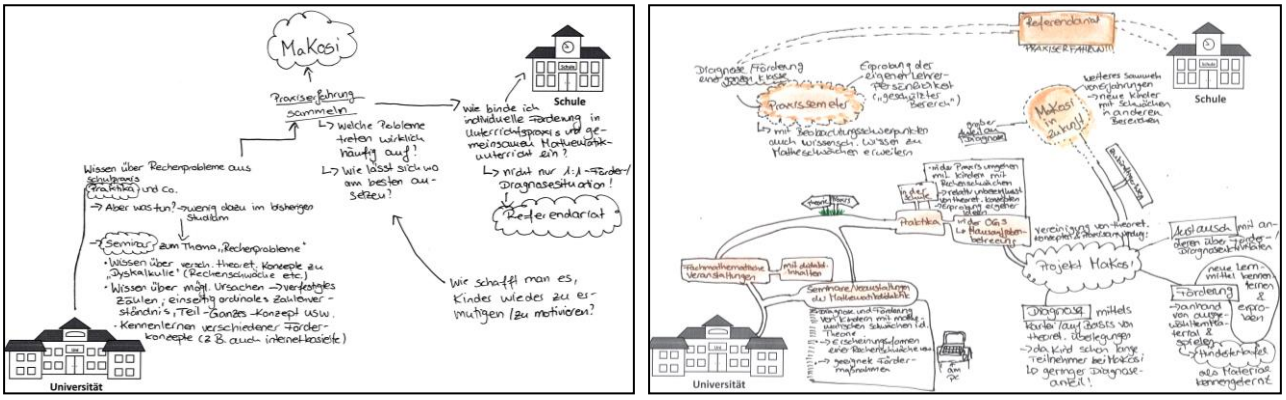

Figure 2: Second example of a pre- (left) and post-map (right)

As to the factual sense of the first example's pre-map (Figure 1, left), a lack of details is obvious, which might reflect that the creator is unfamiliar with the context. The expressive sense is characterized by monotony as to, e.g., colors, which might indicate the creator's uncertainty. Regarding the sense of form, the interrupted way seems to reflect that the creator cannot (yet) imagine how to achieve the purpose. As to the factual sense, the post-map (Figure 1, right) contains more details: Different remarks are phrased; thus, the creator seems to connect many thoughts to the path. Merged stars seem to reflect interdependent experiences that influenced the creator positively, but a question mark seems to indicate obscurity about future requirements. The expressive sense is characterized by a use of different colors highlighting the significance of the experiences' connection, for instance. As to the sense of form, the continuous path obviously reflects that the creator now perceives a way to achieve the purpose, even if it is flanked by the question mark. The path precedes the current status; thus, the creator seems to have developed a more holistic view on the way he or she passed. As to the factual sense of the second example's pre-map (Figure 2, left), a main feature is a wide range of remarks, which seems to reflect that the creator already connects several aspects to the path. An important detail is the remarks' phrasing in the form of questions in most of the cases; thus, the creator rather seems to ascribe uncertainty or just a small level of recognition to him- or herself. Moreover, clouds seem to emphasize particular past and future experiences. The expressive sense is rather monotonous, e.g., as to the coloring, which might indicate that the creator refers to a matter-of-fact way. Regarding the sense of form, the continuous path is drawn as a stairway; thus, the creator obviously distinguishes different steps of his or her knowledge's complexity. Arrows emphasize secondary objects, which might reflect that the creator at least considers different complex patterns. As to the factual sense of the post-map (Figure 2, right), a wide range of remarks still can be observed, but now they are put forward in the form of declarative sentences. Signposts seem to describe possible intentions and their connections. Clouds

and boxes seem to highlight important (mostly past) theoretical and (especially future) practical experiences. A computer seems to indicate an intertwining of the regarded focus with other domains. Finally, sections which the creator already had passed are characterized by continuous, but future sections by dotted lines. The expressive sense is characterized by different colors which underline the significance of main experiences; overall, a great confidence seems to be reflected. Aspects related to the sense of form confirm this impression: In contrast to the circle drawn in the first section of the map, the path continues afterwards directly, but winding to school, flanked by some concrete imaginations. The final part of the path system is drawn slightly broader compared to previous sections which might reflect that the creator will feel well prepared to enter school. In the creator's view, practical work seems to bridge impressions of running in a circle within theoretical studies, which are not connected directly to work at school, and achieving the objective, which is supposed to be a complex, but positive and manageable challenge.

As to a *context analysis* of the first example, the interrupted way of the pre-map (Figure 1, left) seems to reflect that knowledge on ISLA is assessed to be nonexistent or at least superficial by the creator (PCK), and he or she cannot imagine how to cope with ISLA at school (PCB). Comparing the post-map (Figure 1, right) indicates that the creator now reflects to have knowledge on ISLA (PCK), and that he or she perceives ways of handling ISLA at school (PCB). This impression is confirmed by concrete aspects and intentions given by the detailed remarks (e.g., top right, "self-responsibility", "perspective of hope to face the topic in the future"), which were missing in the pre-map, and by elements like stars or colors, which underline the significance of taking part in the LTL as to developing knowledge of ISLA and, thus, of bridging theoretical knowledge and practical work. Regarding the second example, the continuous path shown by the pre-map (Figure 2, left) indicates that the creator already knows about some aspects of ISLA, even if this knowledge seems to be rather superficial (PCK) and he or she seems to be rather uncertain (PCB). Comparing this with the post-map (Figure 2, right) suggests that the complexity of the system of paths reflects an increase of knowledge (PCK), and the creator can well imagine to cope with ISLA at school (PCB). Questions posed in the pre-map (e.g., on the first step, "What has to be done?") turned to concrete intentions and planning steps (e.g., top left, an intention as to the practical semester of academic studies "Enriching scientific knowledge by specific focuses of observation."); particular attention is given to emphasize the significance of practical experiences, like taking part in the LTL, as to developing knowledge of ISLA and as to intertwining theoretical knowledge and practice.

Based on these examples, a *comparative analysis* of all pre- and post-maps suggests that both the student teachers' ways and their location on the ways changed. Their PCK of ISLA developed to more profound patterns, and their PCB to more confident characteristics. Put more precisely, the comparisons indicate mainly the following typical changes: As to cognitive aspects, before taking part in the LTL the student teachers' knowledge about ISLA seemed to be rather fragmentary and superficial for the most part. Moreover, most of them seemed to equate problems in learning mathematics unilaterally with learning difficulties in a narrower sense. In contrast, after participating in the LTL, the student teachers developed a complex knowledge of the entire field: They distinguished different approaches and considered phenomenology-related issues. Regarding affective aspects, before taking part in the LTL, the student teachers rather seemed to express uncertainty as to dealing with ISLA, which seems to reflect disadvantageous characteristics of self-efficacy. In contrast, after participating in the LTL, the maps indicate more proactive views: The

student teachers declared more complex perceptions of problems in learning mathematics, in particular as to a child's individuality, and they proposed precise plans to develop their knowledge. Moreover, they seemed to connect ISLA closer to teachers' responsibilities, and emphasized the significance of practice as to the development of aspects such as self-efficacy. Finally, as indicated by the discussed examples, there seem to be different types: A first one representing an "optimistic novice" (Figure 1), and a second one representing an "expectant expert" (Figure 2).

Discussion

The results indicate that participating in an ISLA-LTL and, therefore, an intertwining of theory and practice contributes sustainably to a positive development of both student teacher's knowledge about ISLA and their abilities to analyze children's thinking and learning in mathematics. This observation is in line with reports emphasizing the benefits of practical work with children as to the development of knowledge, and one main reason for this might be seen in the student teachers' constructivist learning (Sowder, 2007). Thus, beyond the context of ISLA, the results suggest the hypothesis that LTL provide an appropriate professional development program for student teachers to develop their abilities in analyses of children's thinking and learning. Of course, the study's character is explorative, and it has obvious limitations; for instance, the reconstructions as to interpreting the use of colors (or, e.g., their absence) were conducted within a group, but it remains uncertain, if a consensus view is the right one (Lamnek, 2010). Subsequent research might focus on a deeper clarification as to evaluations of ISLA-LTL, and as to the benefits of LTL in general; in particular, different LTL-focuses like support of mathematics interest should be taken into account.

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Didactic assessment over a final work of a master for mathematics teachers in service

Adriana Breda[1], Luis R. Pino-Fan[1], Vicenç Font[2], Valderez Marina do Rosário Lima[3]

[1]Universidad de Los Lagos, Chile; adriana.brea@ulagos.cl; luis.pino@ulagos.cl

[2]Universitat de Barcelona, Spain; vfont@ub.edu

[3]Pontifícia Universidade Católica do Rio Grande do Sul, Brazil; valderez.lima@pucrs.br

The aim of this paper is to present what are the criteria used by a teacher when performing the didactic analysis in its final master thesis. For that matter a case study was performed, whose object of study is the master thesis conducted by a Math teacher in service. The analysis was based on didactical suitability criteria proposed by the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino, Batanero & Font, 2007). As a result of the analysis it was possible to notice that the teacher, in addition to using all didactical suitability criteria proposed by the OSA implicitly, highlights the importance of finding a balance among the suitability criteria to achieve the learning of the didactic proposal carried out by him.

Keywords: Didactic assessment, suitability criteria, masters thesis.

Introduction

The tendency to achieve an international convergence in the planning of College programs and, particularly, those related to the Professional Master education centered on the education of teachers, has fostered a series of reforms in different countries, so that there is a model organized by a sort of refinement and evolution around professional skills. In the Brazilian context, in an attempt to train Mathematics teachers who were currently working in the area, the Professional Master Program in Mathematics in the National Network (PROFMAT) was launched in 2010 by recommendation of the *Conselho Técnico-Científico da Educação Superior da Capes*. The program is an on-site and long-distance program, offered throughout the Brazilian territory, coordinated by the *Sociedade Brasileira de Matemática* (SBM), which has a main objective to support Mathematics teachers who work in the primary education level, especially in public schools. It is important to highlight that, although PROFMAT's main objective is to foster teaching of mathematics at all levels (Brasil, 2013) it is configured as a program composed almost entirely of mathematical disciplines. In addition, at the end of the course, the students must present a final work (ETM) consisting of design a sequence of tasks, not assigning the mandatory implementation.

The work presented in this document is part of a larger research (Breda & Lima, 2016; Breda, Font & Lima, 2016; Breda, Pino-Fan & Font, 2016; Breda, Pino-Fan & Font, *in press*), in which, through the analysis of 29 Master's Thesis Works (EMTs) of PROFMAT (Breda, 2016) and was concluded that the teachers who implemented the design of their sequence of tasks performed a much more refined and balanced didactic analysis compared to the teachers who did not implement the proposal. In addition, it was evidenced that when the teachers' opinions were clearly evaluative, they were organized implicitly using some characteristics of the components of the didactical suitability criteria proposed by the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino, Batanero & Font, 2007).

So, the objective of this paper is to present a case study that analyses what are the criteria used by a teacher in his reflection process (explained in his EMT of PROFMAT), who will be addressed as Mr. Lopes, in order to improve the design and implementation of new contents related to the Riemman integral in Elementary School.

Theoretical framework

In the field of Mathematics Education there is no consensus on the notion of "quality" and, in particular, there is no consensus on the "methods for assessing and improving the teaching and learning of mathematics". There are basically two ways to address this problem, from a positivist perspective or from a consensual perspective (Font & Godino, 2011). From the first, the scientific research in the area of Mathematics Education tell us what are the causes to be modified to achieve the effects as objectives to be achieved, or at least tell us what are the conditions and restrictions that must be taken into account to achieve them. From the consensual perspective, that tells us how to guide the improvement process of mathematics instruction, which must come from the argumentative discourse of the scientific community, when it is aimed at achieving a consensus on "what can be considered as the best".

The notion of didactical suitability criteria proposed by the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (OSA, from now onwards) (Godino, Batanero & Font, 2007) is positioned in the consensual perspective. Such notion is a partial answer to the following problematic: What criteria should be used to design a sequence of tasks to assess and develop mathematical competence of students and what changes should be made in its redesign to improve the development of this competition? Suitability criteria can first serve to guide the teaching and learning of mathematics and, second, to assess their implementation. Suitability criteria are rules of useful correction in two stages of the processes of mathematical study. A priori, the suitability criteria are the principles that guide "how things should be done". In hindsight, the criteria used to assess the study process effectively implemented. According to these authors, the didactical suitability criteria are: 1) Epistemic suitability, to evaluate if the Mathematics being taught are "good Mathematics"; 2) Cognitive suitability, to evaluate, prior to the beginning of the instruction process, if what is intended to be taught is at a reasonable distance from what the students already know, and after the process, if the knowledge acquired is any close to what was intended to teach; 3) Interactional suitability, to evaluate if the interactions contribute to clear doubts and difficulties students encounter; 4) Mediational suitability, to evaluate the adaptation of material and time-related resources used in the process of instruction; 5) Emotional suitability, to evaluate the implications (interests, motivations,…) of students during the process of instruction; 6) Ecological suitability, to evaluate the adaptation of the instruction process to the educational project of the school, the curricular guidelines, the social and professional environmental conditions (Font, Planas & Godino, 2010, p. 101).

Methodological aspects

We chose to conduct a case study (Ponte, 1994) where the didactic analysis performed by a mathematics teacher in service, as part of his master degree, is investigated. To analyse our case, we used the *indicators of didactical suitability* proposed by the OSA (Godino, Batanero & Font, 2007; Godino, 2011; Breda, Font & Lima, 2015), as theoretical model to analyse the reflections

performed by the teachers regarding ways to improve their teaching practices, related to the implementation of the didactical activities proposed as part of their EMT.

Research context

According to the guiding document of the PROFMAT, the EMT should, preferably, consist of a project with direct application to the mathematics classroom in Basic Education, thus contributing to the enrichment of the teaching of said discipline.

In this work we proceed on the assumption that the End of Master's Thesis (EMT) is a task that involves, implicitly, a didactical analysis exercise, since in their EMTs teachers must explain a didactical proposal and justify why it represents an improvement in teaching. In this sense, the reason for choosing the case of Professor Lopes is that, in addition to having applied the sequence of tasks with the students, he presents in his reflection aspects that "did not work" or that should be improved in future implementations.

Professor Lopes' didactical proposal

Professor Lopes' EMT (2014), entitled "A review of the introduction of Riemann's sums into High School Education", presents the design and the implementation of a didactic proposal for a group of third-year high school students (students aged 16 to 17) in order to intuitively introduce the integral calculus through the study of the areas of 2D geometric forms. Lopes (2014) explains that it is possible to introduce methods and notions of the integral calculus in High School Education intuitively, starting with area-calculation problems for curvilinear shapes. That is, the aim is to broaden the calculation of areas habitually studied in Elementary Education through the study of area-calculation of curvilinear shapes, using both Archimedes' and Riemann's methods.

To be specific, Lopes' EMT (2014) is organized into four chapters; in the first, the professor presents, using literature reviews, the argument: "Should integral calculus be used in Elementary School?" On the basis of this question, the professor seeks to justify- through the study of literature- the use of two methods: Archimedes' method (used to calculate the area of a circle) and Riemann's method (used to calculate the area of three curvilinear shapes: circles, eclipses and polynomial shapes with an x axis). In the second chapter, Lopes (2014) explains the didactic unit which was implemented with a group of third-year students from a state secondary/high school in Brazil. The group was formed of 41 students but at the beginning of the year, only 36 students attended the classes and participated willingly in the project. In this second chapter, the professor also explains in detail the initial self-evaluation he performed with the students and, in particular, he explains the method for evaluation previous knowledge on certain geometry topics, on mathematical software knowledge and also on the expectations of the project.

In the third chapter, professor Lopes describe the implementation he carried out. This section is a sequential report in which the author explains what happened during the implementation of the didactic sequence, placing emphasis on the set tasks, what the students learnt and the interactions made during the implementation. We are looking at a review written from the perspective of the professor but, in his very review, the professor ensures he presents evidence of the statements he makes. In the fourth and last chapter, the professor presents his reflections and conclusions on the implementation he carried out. In this way, it can be said that Lopes' proposal (2014) covers the four phases of didactic design (preliminary study, design, implementation and evaluation), which

other models of mathematics teachers' knowledge also cover, in order to answer the most fundamental question: "What knowledge should a mathematics teacher have to be able to appropriately manage their students' learning?" (Pino-Fan, Assis & Castro, 2015).

Professor Lopes' analysis on his own implementation project

When teachers have to reflect on a didactic proposal that implies a change to or an innovation in their own practices, they implicitly employ some of the *didactical suitability criteria*. Lopes' EMT (2014) has also allowed us to deduce the use of some of these criterions in the justification and reflection on the suggested proposal. In the following subsections, we show the extent to which the author considered- implicitly and explicitly- the suitability criteria put forward by OSA in attempt to defend his didactic proposal as improvement for mathematics teaching.

Epistemical suitability

Lopes (2014) justifies the 'innovative and creative' nature of his proposal by pointing out that it encourages students to perform relevant mathematical processes, in particular that of mathematical modelling. In his own words, he explains: "*In this way, the application process, divided in three stages, aims to build knowledge through the use of mathematical models. Starting with the first construction, on the basis that the topic is studied in depth and new elements arise, other models are built based on the previous ones* […]". (Lopes, 2014, p. 22)

The professor also considered that his innovative proposal allows students to perform other relevant mathematical processes such as connections, meaningful constructions, problem-solving, etc. "*In this sense, the aim is to (…) awaken the student's creativity and enthusiasm to learn geometry, to create geometrical models with the students, making connections with reality, and to provide situational problems with a geometric focus…*" (Lopes, 2014, p. 21).

It is evident in his review that some of the processes mentioned were in fact developed during the implementation of his proposal. In his thesis, the professor generally presents explicit reflections on the fact that his didactic proposal for teaching area-calculation is more representative (since it thoroughly explores the area-calculation of curvilinear figures) than the proposals that are commonly implemented at high school level.

Cognitive suitability

In Lopes' work (2014) there are comments, reflections, etc., that allow concluding that the author takes into account, in an implicit way most of the times, the indicator of cognitive suitability.

Background knowledge. The teacher carries out an initial evaluation in order to find out if the students had the necessary background knowledge for the study of the intended content. Furthermore, he makes sure that the students have such background knowledge, and specifically, he dedicates part of the time intended for the implementation, to revise the calculation of the area of triangles and quadrilaterals, and the study of trigonometric ratios. On the other hand, the learning objectives were attained by the students, "and there is confirmation that the Archimedes and Riemann methods are in the students' zone of proximal development" (Lopes, 2014, p. 19).

Curricular adaptation to individual differences. With the narration of the teacher it is not possible to conclude if he considers at some point complementary or reinforcement activities. However, when he assesses the learning related to the Riemann method, he concludes that many students will

not achieve such learning and adds: "…it would be necessary to have a more extensive study period, to be able to ask the students (…) to interpret results more thoroughly, considering that each student is unique and as such, needs a shorter or longer time to learn" (Lopes, 2014, p. 92).

Regarding the *learning* intended, the teacher states in a very clear way that he has to carry out evaluations to verify that his innovative proposal helps the students to achieve the learning objectives. Therefore, apart from the initial evaluation, the teacher carries out three formative evaluations that show the acquisition of the competences/learning implemented. With these evaluations, the teacher concludes that the learning related to the calculation of areas of quadrilaterals and triangles, and the Archimedes method was acquired, but the same cannot be said about the learning of the Riemann method, which he justifies with lack of time.

High cognitive demand. The author considers that his proposal requires a high cognitive demand from the students, since it activates relevant cognitive processes.

Interactional suitability

Teacher-Student interaction. The teacher describes a "teacher-large group interaction", through a dynamic of questions asked by the teacher and answers given by the students, which, according to him "facilitates comprehension among students" (Lopes, 2014, p. 32). He also presents some examples of how this type of interaction helps to clarify doubts that the students might have.

Interaction among students. In his narrative, the teacher also mentions that the students worked in small groups and although he did not comment if such dynamic has solved the student's semiotics conflicts, he concludes that this organization allowed some students that hardly participated in the classroom to express themselves in a larger group.

Autonomy. It is possible to conclude that there were moments in which the autonomy of students was encouraged. On the one hand, "the students had to do homework" (Lopes, 2014, p. 67); on the other hand, there were some moments in which it was possible to observe that the responsibility to study (exploration, formulation and validation) was assumed by the students.

Formative evaluation. As mentioned in the cognitive suitability section, the teacher carried out a formative evaluation that allowed a systematic observation of the cognitive process of the students.

Mediational suitability

It was possible to observe the use of *material resources* such as calculators and computers. The teacher explains that he used the *GeoGebra* software and the calculator during the teaching process. Regarding *GeoGebra*, he presents some implicit evaluative comments about the advantages of including this software of dynamic geometry in the teaching process.

Number of students, Schedule and classroom conditions. Regarding this aspect, the teacher makes several comments. In a relevant way, he explains that the number of students and the conditions of the classroom (both the physical space as well as the computer laboratory) somehow determined the use of *GeoGebra*. Thus, the software was mainly used by the teacher to illustrate and show mathematical practices (e.g., the calculation of the areas of quadrilaterals and triangles).

Regarding the *time – of group teaching and learning –*, the teacher makes comments and assessments about three indicators of this component: the adaptation of intended meanings in the

available time, the time spent in the most important and relevant contents, and the time spent in the contents that were more difficult for the students. In connection to the first indicator, the teacher states very clearly that he could not adapt the intended meanings in the time that was available. Particularly, he states that he did not have enough time to finish explaining all he had planned regarding the Riemann method. For the second indicator, the teacher states that it took him a lot of time to ensure the required background knowledge, and that, on the other hand, he did not have time to solve the initial problem that was contextualized in order to later introduce the Archimedes and Riemann methods. Finally, regarding the third indicator, it is possible to infer from the teacher's comments that it was impossible to carry out the whole study due to lack of time (e.g., there was not enough time to explain the Riemann method in depth).

Emotional suitability

In connection to this suitability indicator, no comments regarding the *interests and needs of the students* were found in Lopes' EMT (2014). No comments about *the attitudes* of the students were found neither. Regarding *emotions*, the teacher states that the implementation he carried out promotes the students' self-esteem.

Ecological suitability

According to the criteria and objectives that the teachers had to consider for the elaboration of their projects, professor Lopes adds that his proposal is a *didactical innovation* that adapts to the *Elementary school* curriculum and, according to his students, contributes to social and professional integration (*social and labour utility*) and that presents an intra-mathematical *connection* to higher level Mathematics (*intra and interdisciplinary connections*).

Final reflections

The analysis of the EMT of Professor Lopes shows how the indicators of *didactical suitability* proposed by OSA are -implicitly- present in his reflection processes on their own practice. An important aspect to highlight is that this EMT clearly demonstrates the issue of finding a balance between each of the suitability criteria. On one hand, the author plans an innovation with high epistemic suitability and he demonstrates in his review that he also made a substantial effort to achieve high cognitive suitability. On the other hand, however, he also demonstrates that he was obliged to neglect part of the content he had planned; in particular he could not solve the initial problem which was the very motive of his didactic proposal and the learning was not complete (in particular, the Riemann method) due to the fact that the suitability of means was not adequate; to be precise, there was not enough time.

We could say that, in terms of the suitability criteria, Lopes concludes that if in future implementations, cognitive and epistemic suitability are not to be neglected, and then it would be necessary to allow more time. One aspect, which is difficult to explain, is the reason why the criteria of didactic suitability function as implicit patterns in the Lopes' discourse, when he has to evaluate instruction processes without specific training on the use of this analysis tool. One possible explanation is that the training that he has received in the PROFMAT allows him to do, implicitly, this type of analysis. However, Caldatto's research (2015) and Caldatto Pavanello and Fiorentini (2016), leads us to believe that the characteristics of the PROFMAT do not encourage this kind of

reflection. Now, other answer to this very question is related to the origins of this construct. In the OSA, the didactic suitability criteria, its components and characteristics were constructed on the basis that they should be constructs which rely on a certain amount of consensus within the Mathematics Education community, albeit the local one. Therefore, one of the plausible explanations that the suitability criteria can be considered as teachers' reflections patterns is related to the extensive consensus that they themselves generate amongst persons involved in Mathematics Education. Therefore, another possible explanation, for Lopes case, could be based on his previous training and his experience, which would lead him to participate in this consensus.

The analysis of the EMT of Professor Lopes shows that, when the teachers' opinions were clearly evaluative, they were organized implicitly using some characteristics of the components of the didactical suitability criteria proposed by the OSA. This result has been evidenced in other investigations (Breda, 2016; Seckel, 2016; Breda, Pino-Fan, Font, *in press*) in which it is also suggested that the suitability indicators can be taught as powerful methodological tools to organize the teacher reflection –as it has already been done in different processes of teacher training in Spain, Ecuador, Chile and Argentina (Giménez, et al., 2012; Valls & Vanegas, 2015; Posadas, 2015; Pochulu, Font & Rodriguez, 2016; Seckel, 2016) –, that aim at the fostering of the "meta" dimension of didactical-mathematical knowledge (DMK) of Mathematics teachers (Pino-Fan, Assis & Castro, 2015; Pino-Fan, Godino & Font, 2016).

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Describing a secondary mathematics teacher's specialised knowledge of functions

Dinazar Escudero-Avila[1], Eric Flores-Medrano[1,] and José Carrillo[2]

[1]BUAP, Puebla, México; eadinazar@hotmail.com; ericfm_0@hotmail.com

[2]University of Huelva, Huelva, Spain; carrillo@uhu.es

Taking a case study into the specialised knowledge of linear functions demonstrated by a secondary teacher, we aim to demonstrate, and to raise discussion about, a set of methodological tools for organising and representing mathematics teachers' knowledge. The results are art of an ongoing study which employs the model of professional knowledge known as Mathematics Teacher's Specialised Knowledge. We also believe that the use of these tools help to demonstrate the effectiveness of the division of this knowledge into subdomains, and affirms the relevance of the descriptors defining their boundaries.

Keywords: Secondary, mathematics teacher's specialised knowledge, functions.

Introduction

Since the CERME 8 conference in Turkey, the University of Huelva research group has presented a series of papers demonstrating an analytical model that is being developed to aid the study of mathematics teachers' specialised knowledge (e.g. Montes, Aguilar, Carrillo & Muñoz-Catalán, 2013; Vasco, Climent, Escudero-Ávila & Flores-Medrano, 2015; Flores, Escudero & Carrillo, 2013), we outlined the model currently under development, known as Mathematics Teacher's Specialised Knowledge (MTSK). In this model, each subdomain is described in terms of mathematical concerns, an aspect which represents a significant move away from other models (Carrillo, Climent, Contreras & Muñoz-Catalán, 2013).

Continuing the group's work on this model, this paper illustrates the application of various methodological tools to studying and understanding mathematics teachers' knowledge, as brought into play in the classroom. At the same time, we consider developments in the theoretical foundations of the model, and present adjustments that have been made in the light of empirical studies and fruitful discussion in various academic platforms. We illustrate the use of these tools by means of an analytical study of the knowledge employed by a secondary school mathematics teacher in the process of planning classroom activities.

Mathematics Teacher's Specialised Knowledge (MTSK)

The distinguishing feature of the MTSK model is that it incorporates elements of knowledge which, considered as a unit, are uniquely relevant to mathematics teachers. It contemplates two chief knowledge domains – Pedagogical Content Knowledge and Mathematical Knowledge, each divided into three subdomains.

For its part, Pedagogical Content Knowledge is composed of Knowledge of Features of Learning Mathematics (KFLM - the teacher's knowledge of the processes involved in the students' assimilation of mathematical content), Knowledge of Mathematics Teaching (KMT - the teacher's stock of

resources and strategies for teaching such as examples, tasks, analogies and so on), and Knowledge of Mathematics Learning Standards (KMLS - the teacher's knowledge of the performance targets set for different educational stages). Meanwhile, Mathematical Knowledge comprises Knowledge of Topics (KoT - knowledge of the mathematical content pertaining to any particular course, along with the associated foundations, properties, definitions, phenomenological associations and so on), Knowledge of the Structure of Mathematics (KSM - the teacher's knowledge of mathematical connections between concepts), and Knowledge of Practices in Mathematics (KPM - knowledge of the syntax of mathematics and the procedural logic at the heart of the discipline) (Escudero-Ávila, Carrillo, Flores-Medrano, Climent, Contreras & Montes, 2015).

In addition to the division into subdomains, the MTSK model also places emphasis on beliefs about mathematics and about mathematics teaching and learning, as the relationship between beliefs and knowledge is a significant consideration, influencing, for example, how a teacher might deploy his or her knowledge. This provides a base to interpret knowledge evidences that we found and gives us sensitivity to interpret some elements that allow to contextualize David's teaching decisions and the exchange of opinions with his colleagues.

Methodological tools for studying and understanding the specialised knowledge of mathematics teachers

In this section we present an example of an analysis using MTSK, which we hope will demonstrate a deeper understanding of the nature of each of the subdomains involved and illustrate some interesting methodological tools.

We analyse the case of David (fictitious name), a secondary school mathematics teacher working in Colombia, who, at the time of the data collection, was studying an online Master's degree. His first degree was in Chemical Engineering, and he had been involved in education for over 25 years as a teacher of Mathematics, Physics and Chemistry. He had also participated in several teacher training courses and similar programmes. The bulk of his experience was in upper secondary (14-18 year-olds), but for the previous eight years he had worked in lower secondary (12-14 year olds), teaching Mathematics.

Our analysis focuses on David's participations in forums and his written productions as part of the Master's course. David plans a lesson for teaching linear and quadratic functions. The task required the teacher to devise an exercise for use with the class, and then to discuss aspects of it with the group tutor and the other course participants in various sessions of an online forum, with the aim of evaluating the task and making any modifications deemed necessary as a result.

David came up with two exercises, one dealing with linear functions, the other with quadratic, both aimed at pupils in eighth grade. In this paper, we focus on the specialised knowledge we were able to identify relating to the exercise concerning linear functions. It should be noted here that although David was given several opportunities to incorporate modifications to his planned activity, he ultimately decided to stick with the same plan through to the end of the course.

The Master's programme which forms the background to our analysis encouraged teachers to explore aspects of their day-to-day classroom practice. This required them to articulate their pedagogical and mathematical knowledge in order to explain to the other teachers and the tutor the decisions that had led to their particular activities.

Our system for classifying and organising the knowledge displayed by David was based on units of analysis drawn from his written discourse in forums and tasks over the duration of the course. These were assigned to a particular category within the subdomains making up MTSK. In order to establish these categories and refine the analysis, we used Grbich's *top-down* and *bottom-up* methodology (Grbich, 2013). This approach enables researchers to move from theory to data (top-down) by opening provisional categories suggested by the literature review, and from data to theory (bottom-up) by eliminating, merging and opening new categories consistent with the results obtained from empirical study.

With respect to the potential of this methodological approach, we propose the following example. In Escudero (2015), it is mentioned that the KFLM and KMT subdomains contain a category that reflects the knowledge that teachers have about learning and teaching mathematics, respectively. With this information we analyze the data collected from David to find elements related to teaching/learning theories. When David mentions Socio-Epistemological Theory, it is clear that he has knowledge about the theories we are looking for (top-down). On the other hand, when David refers to Socioepistemology, we could notice some differences with the "formal" constructs, therefore we think of the possibility to explore knowledge of formal theories and personal knowledge base in formal theories whose nature corresponds with the knowledge of teaching/learning theories (bottom-up).

As a result of this procedure, we developed a range of categories for each of the subdomains, enabling specific aspects of each one to be considered, and giving a fuller description of the MTSK model. These categories are illustrated alongside the model in Figure 1.

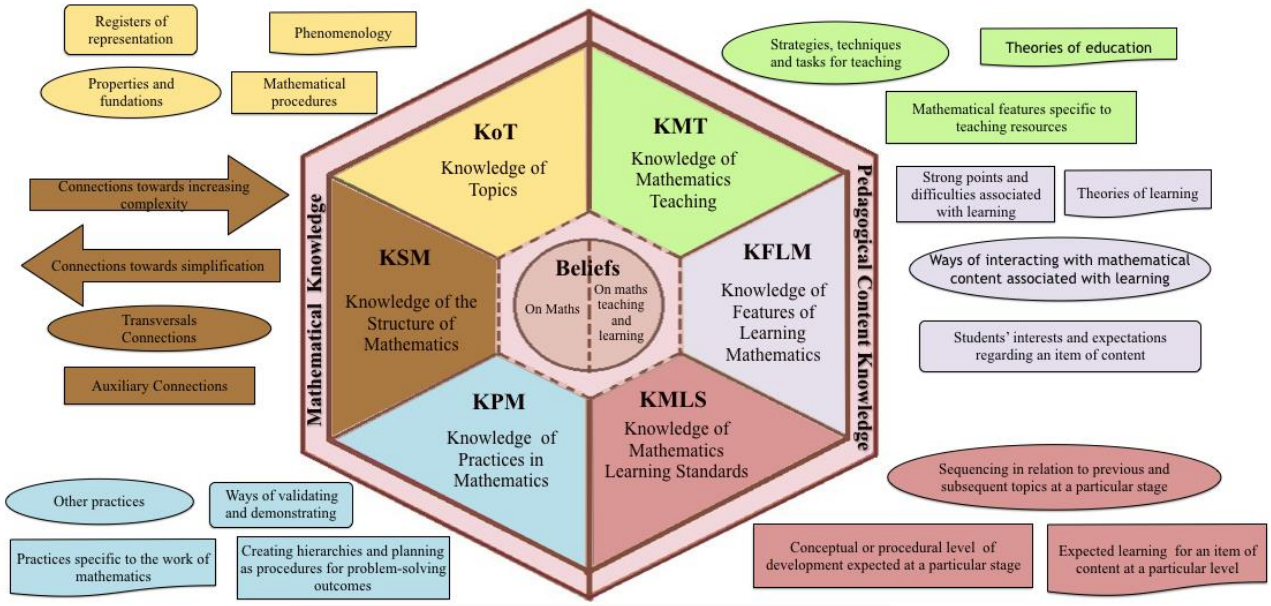


Figure 1: Mathematics Teacher's Specialised Knowledge

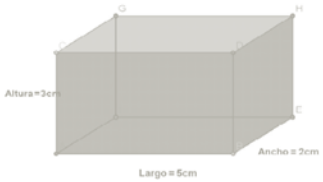
Once units of David's knowledge had been identified from his contributions during the course, we wanted to be able to describe and understand the specialized knowledge deployed in each instance. To do so required a tool that would allow us to determine the type of knowledge, the connections between different subdomains, and the nature of this knowledge in terms of the different categories within each subdomain. At the same time, we wanted to keep in sight the holistic character of this

kind of knowledge, and to maintain an awareness that the placing of boundaries around discrete areas of knowledge is a (necessarily) artificial procedure serving analytical purposes. To this end, we devised a coding system using shapes and colours, whereby subdomains were represented by colours and categories by shape. In addition, associations between different types of knowledge were schematized by a series of arrows joining the two elements. The result can be seen in Figure 2 below.

David's specialised knowledge of Mathematics

The task which we will analyse below concerns filling a tank, in the shape of a rectangular prism, with water.

David: There is a rectangular tank with a volume as shown in the figure, which is to be filled with water at the rate of 1 cm^3 per second.



Draw a table in which the variables are height and time.

If the inflow varies, what happens to the graph, if it increases and if it decreases?

What type of function do you get if you model the two situations?

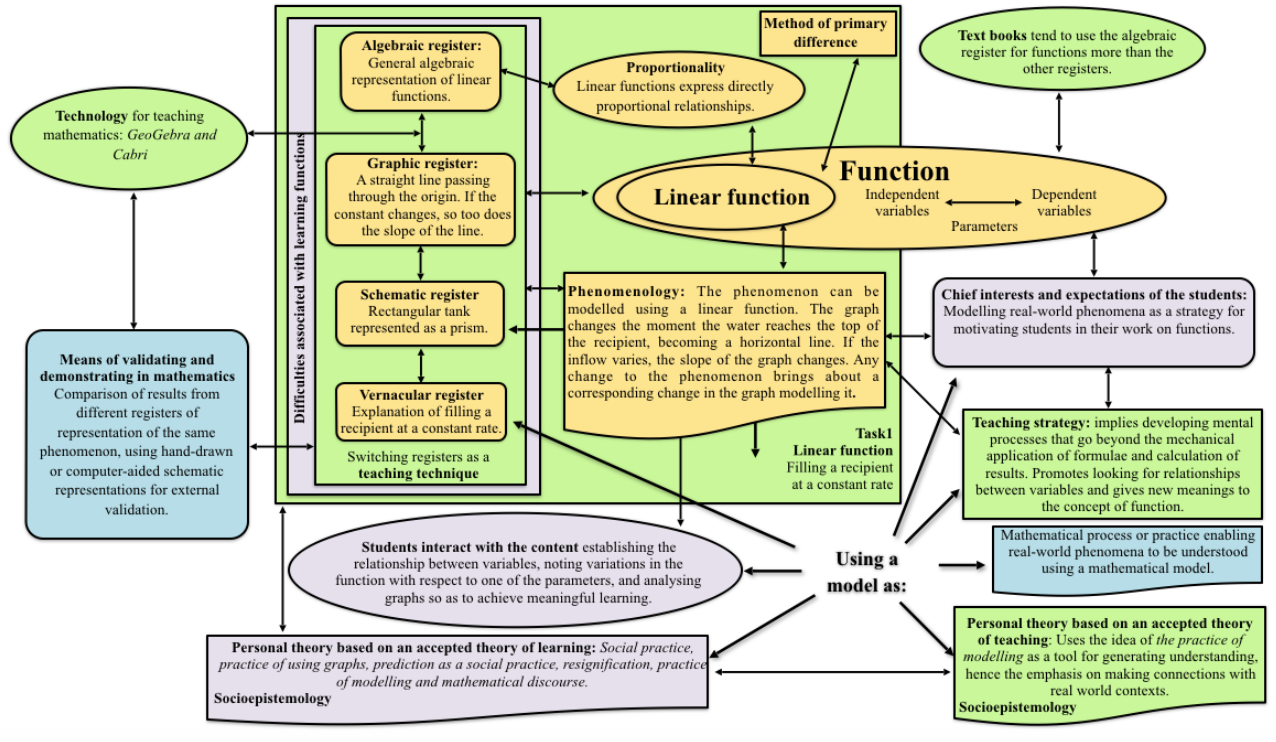


Figure 2: A representation of David's specialised knowledge regarding the concept of functions

An interpretation of Figure 2 is presented below. The aim is to reach an understanding of David's specialised knowledge, as identified in his contributions during the course, and of the connections between the different subdomains and categories which can be detected.

David goes on to specify the objectives of the task, underlining the importance of designing appropriate activities for the topic of functions, and gives a list of the resources to be used with suggestions for conducting the activity. These include a series of exercises aimed at guiding the students towards analysing the problem, establishing the relationship between the variables and modelling the process in order to arrive at a function modelling the behavior. In the final section, he makes various observations about the use of the teaching resources.

Although David does not write explicitly about his **knowledge of the properties** of linear functions (yellow ovals), the design of the exercise and his comments about it provide indications of the elements he considers fundamental to them, such as the parameters and concept of dependent and independent variables, and noting that directly proportional magnitudes are also linear functions.

Nor does David explicitly solve the activity, although he does talk about different ways the problem can be solved. One method he mentions is that of primary difference for calculating the parameters of the functions modelling the phenomenon in question. This indicates **knowledge of mathematical procedures associated with the mathematical content** (yellow rectangle), which represents evidence that he knows a specific mathematical process associated with the concept of linear functions.

> David: […] using differences to find out the parameters of the functions; the primary difference for the linear, and the secondary difference for the quadratic; that is for when I need to find out the parameters using the given data.

The students are encouraged by the task to employ different semiotic registers of representation (Duval, 1995) for the function, that is, the pictorial, numerical, graphical and algebraic. This provides us with evidence that David has knowledge of different registers of representation (yellow rectangles with rounded corners) associated with the concept of functions.

Regarding **knowledge of the mathematical characteristics specific to teaching resources** (green rectangles) used by David in his design, and which are located in KMT, we identified that he knows a specific mathematical task for dealing with the concept of linear functions, such as filling a receptacle at a constant rate, and also a specific technique for teaching the concept – the transition between registers of representation. Questioned about the exercise, David demonstrates a connection between understanding registers in terms of mathematics and understanding them in terms of teaching, when he considers the transfer from one register to another as a teaching strategy, thus making a connection between KoT and KMT:

> David: Generally, we, teachers, prefer to work in the *algebraic register*; but to a large extent we can also recognise the different variations a function can undergo using *tables* and *graphs*, by giving the dependent variable different values to the ones originally set.
>
> […] By moving from one register to the other [from the algebraic to the graphic], I hope that, by making graphs, [the students] establish the relations between the

> dependent and independent variables, and can see how this resource, that is the use of graphs, could be put to use in modelling.

A connection to KFLM is also made that contemplates potential errors, obstacles and difficulties to learning, as well as, conversely, areas that might offer an advantage when David considers **potential difficulties in learning the concept** of function (mauve rectangles):

> David: [I acknowledge] a high degree of complexity in learning the concept of function, because of the variety of its representations in different contexts, and its algorithmic nature.

Awareness of these potential difficulties in representing functions is inherent to the concept of function itself.

Elsewhere, another connection between KoT and KFLM can be seen when David shows that he is aware of how modelling real world phenomena can be a motivating factor in work on functions as the students can establish the relationship between dependent and independent variables through making graphs, thus providing a meaning linked to a tangible context for the linear and quadratic functions. It is precisely this kind of knowledge – understanding how drawing up graphs can help students learn about functions – which we consider **knowledge of the students' interests and expectations** regarding the teacher's approach to the concept in question (mauve rectangle with rounded corners).

As might be expected, David's KoT is the foundation on which the task is constructed, providing him with the background knowledge for sequencing activities and setting his goals. One important aspect is his knowledge of phenomena which can be modelled by linear functions, such as the rate at which a rectangular-shaped vessel fills with a constant inflow, which we have denominated **knowledge of the phenomenology associated with the concept** (yellow labels), on the basis that understanding the mathematical features of the phenomenon and the effect of its variation on the different modes of representation, enables David to make connections between the variables involved, and to model the phenomenon via the transition from one register to another, using modelling as a teaching strategy for generating "new" meanings of the concept of functions.

> David: [I recognise] the importance of learning this concept [function], and its significance as a tool for modelling different phenomena in mathematics, physics, chemistry and economics amongst others.

As can be seen in Figure 2, modelling plays an important role in the task design. Deconstructing David's understanding of modelling, we can see that he imbues it with different meanings, and we can duly recognise the different ways that David understands and recognizes modelling. Hence, on the one hand, we can locate David's knowledge of modelling as a **teaching strategy** (green rectangles) within KMT. On the other, within KPM, we can identify David's knowledge of modelling as a **mathematical practice or process** (blue labels), directly associated with the concept of functions which enables real world phenomena to be interpreted. Finally, within KFLM, David identifies the concept as a **means of getting students to interact with mathematical content for educational purposes** (mauve oval).

At the same time, KFLM includes **knowledge about theories of mathematics learning, whether formal or personal** (mauve label). In David's case, he was able to draw on various theoretical constructs at the design stage of his materials so as to supply a solid foundation for his work. Some of these constructs derive from Sociopistemological theory (Cantoral & Farfán, 2003). In the excerpt below, David makes reference to some socioepistemological constructs (marked in bold):

> David: One of our aims is to achieve the concept of function **resignification** using the **modelling practice**.

Asked about the origin of terminology, David replied:

> David: **Socioepistemology** allows me not only to view mathematics education as a practice in which we convey knowledge, express postulates, solve problems and do demonstrations, but also to see beyond the concepts to what lies behind them, to transform them and to transpose them to other contexts and so get closer to real life […] From the perspective of linguistics, we can note significant learning and **resignification.** Both attempt to modify the meanings of a concept, but Significant Learning is the process followed to achieve learning of value, and where our behaviour changes, while **resignification** is the means by which I achieve this learning.

Given the lack of rigorous definition of the constructs (it could even be said that some are not totally accurate, in that modelling and making graphs are not actually defined as social practices) we can say that David has a certain personal knowledge of this theory, which allows him to take a position in terms of how the concept of function is to be learnt (re-signified). That is to say, drawing on his knowledge of formal theory, David has developed a personal theory of the relevance of learning about functions based on the "practice of modelling". His personal theory extends to regarding education as a tool for creating specific meanings about functions by enabling connections to real world situations to be made.

Regarding KPM, as can be seen in Figure 2, David's **knowledge of the processes of validation, reasoning, and checking** via software (blue rectangles with rounded corners) is directly connected to teaching technique chosen for his lesson plan. It also links to his knowledge of technology specifically designed for teaching Mathematics (green ovals), namely GeoGebra and Cabri.

> David: The use of the software is to show the students that their work is not only checked by their classmates, but also by a tool which in addition to helping them show their results, is like an external agency which can tell them whether what they have done is right or not.

Final reflections

We can conclude that David's knowledge of various teaching resources, such as software and teaching strategies, and the use to which he puts this knowledge could be connected to his understanding of the ways of proceeding and producing in Mathematics, possibly as a result of the tendency of KPM to organize mathematical work, as discussed above.

The main connections that this analysis has brought to light are those between KoT, KMT and KFLM. As underlined above, these connections illustrate the interdependence between the different

categories within the subdomains, and highlight the holistic and indivisible nature of teachers' knowledge of the teaching and learning process. What a teacher knows about interacting with content, or about learning theories and possible difficulties that might arise in a specific topic, has a significant influence on how the teacher plans the lesson. However, it is important to distinguish between the influence that this knowledge brings to bear on the planning process and the teacher's knowledge of planning itself a teaching resource.

We hope that this study serves to open a debate within the group about the relevance, potential and shortcomings of using categories to analyse Mathematics teachers specialized knowledge, and likewise the use of colour-coding and shapes as a means of aiding the assimilation of the information and facilitating the analysis of the connections between subdomains and categories.

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Use of analogies in teaching the concept of function: Relation between knowledge of topics and knowledge of mathematics teaching

Gonzalo Espinoza-Vásquez, Diana Zakaryan[1] and José Carrillo Yañez[2]

[1]Pontificia Universidad Católica de Valparaíso, Chile; gonzalo.espinoza.v@gmail.com

[1]Pontificia Universidad Católica de Valparaíso, Chile; diana.zakaryan@pucv.cl

[2]Universidad de Huelva, Spain; carrillo@uhu.es

This paper addresses the knowledge about the function concept and the knowledge on how to teach this concept using an analogy. The analysis of an episode of one lesson in which the analogy between a washing machine and the concept of a function is shown allows identifying specialised knowledge about the function concept and teaching strategies. The study findings reveal links between knowledge of the topic and knowledge of mathematics teaching, permitting identification of potentialities and limitations of the analogy used.

Keywords: Function concept, mathematics teachers' specialised knowledge, analogies.

Introduction

Function is one of the most important concepts in mathematics, fundamental for development of mathematical analysis and mathematics in general (Ponte, 1992). This concept is not only present in many areas of mathematics, but is prevalent in the school curriculum and is, of course, studied as a part of the mathematics teacher training programs. Given its importance and great complexity (Dubinsky & Harel, 1992), it is essential that the mathematics teacher's knowledge considers both the discipline-specific knowledge of the function concept, as well as knowledge about how it is taught and learned.

Teachers' knowledge has been widely studied from different perspectives, using a wide range of theoretical models (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008; Carrillo et al., 2014). The Mathematics Teacher's Specialised Knowledge (MTSK) model is presented both as a conceptualisation for mathematics teachers' knowledge and as an analytical tool for acquiring this knowledge (Flores, Escudero, & Aguilar, 2013). Studying the subdomains and categories proposed by the MTSK model, and their relationships, allows us to advance the understanding and analysis of teachers' knowledge (Sosa, Flores-Medrano, & Carrillo, 2015). Part of this knowledge is related to the depth of the teacher's understanding of a concept (in this case the function) as a mathematical concept and as an object of teaching. Several studies conducted to date focused on the understanding of the function concept on students, pre-service teachers, and practicing teachers (e.g., Even, 1990; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Other studies highlight the importance of fully understanding the concept (e.g., Sierpinska, 1992), identification of its representations (e.g., Even, 1990), as well as difficulties in learning (e.g., Dubinsky & Harel, 1992) and relating their different representations (e.g. Ponte, 1992; Figueiredo, Contreras, & Blanco, 2015). The focus of the present investigation is on the relationship between teachers' knowledge about the function concept and the knowledge about its teaching from the perspective of the MTSK, particularly when a teacher uses an analogy to make the function concept understandable. In this study we ask what knowledge about

functions and their teaching can be inferred from the use of an analogy? How are these knowledge types related?

Analogies

According to Treagust, Duit, Joslin, and Lindauer (1992), analogy is achieved through a comparison of structures in two distinct domains, one of which is familiar (source or analogue), while the other is unfamiliar (target).

> An analogy refers to comparisons of structures between domains. An analogy is a relation between parts of the structures of two conceptual domains and may be viewed as a comparison statement on the grounds that these structures bear some resemblance to one another. (Treagust et al., 1992, p. 413)

The use of analogies in teaching, particularly as a didactic strategy in the teaching-learning process, has been extensively studied (e.g., Duit, 1991; Treagust et al., 1992). For example, Curtis and Reigeluth (1984) analysed instructional text and provided a classification of the ways in which the relationship between the source and the target domain is established. According to the authors, the relationship can be (1) *structural*, referring to physical similarity or similar construction; (2) *functional*, referring to the way of functioning of both structures; and (3) *structural-functional*, formed by combining the previous two. They add that analogies occur in two forms: verbal and pictorial-verbal, whereby the former is achieved solely via the use of words, while in the latter words are complemented by an image.

Teachers tend to produce analogies automatically when answering questions or explaining the concepts they are teaching (Ünver, 2009). According to Figueiredo et al. (2015), function as a machine is an example of such analogies and this comparison will show only some aspect of the concept. Function can be descripting operationally as a computational process or structurally as a set of ordered pairs (Sfard, 1991). To present the function trough this analogy conducts to understand the concept as an input-output process in an operational way.

Mathematics Teacher's Specialised Knowledge

The MTSK model (Figure 1), in the spirit of contributions by Shulman (1986) and Ball et al. (2008), proposes, within the teacher's knowledge, a discipline-specific component (MK, mathematical knowledge) and a didactic component (PCK, pedagogical content knowledge). A further component related to beliefs about mathematics and about teaching and learning it is introduced in the middle of the model.

According to Carrillo et al. (2014), MK corresponds to knowledge specific to the discipline being taught, and comprises of three subdomains: *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KSM), and *Knowledge of the Practice of Mathematics* (KPM). KoT considers the phenomenology, definitions, properties, procedures, and foundations of the topic, as well as the ways of recording and representing it. On the other hand, KSM pertains to the conceptual connections among mathematical concepts, relating a concept to prior contents (simplification), later contents (adding complexity), or to contents with a common property (transverse connections), and the auxiliary connections among objects. Finally, KPM is related to knowledge about the characteristics

of mathematical work, namely how to proceed and create knowledge in mathematics, practices linked to mathematics in general, and practices linked to a specific topic.

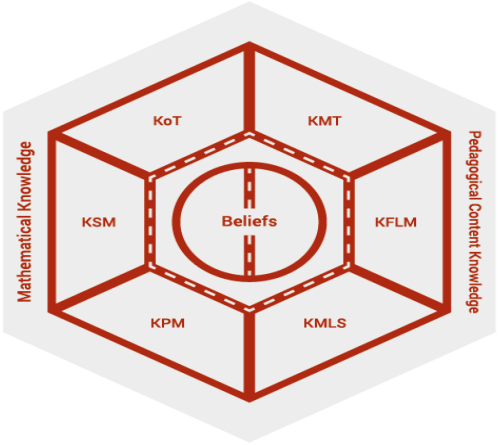


Figure 1: Sub-domains of the MTSK model (Carrillo et al., 2014)

PCK corresponds to didactic knowledge specific to teaching work in the process of teaching and learning mathematics. Once again, it comprises of three subdomains, namely *Knowledge of Features of Learning Mathematics* (KFLM), *Knowledge of Mathematics Teaching* (KMT) and *Knowledge of Mathematics Learning Standards* (KMLS). KFLM considers teachers' knowledge of their students' learning styles, strengths and difficulties associated with learning, way of interacting with mathematical content, students' conceptions of mathematics, and personal or institutional theories of mathematics learning. KMT is knowledge about mathematical content conditioned by its teaching, including knowledge about personal or institutional teaching theories, physical and virtual resources, and strategies, activities, examples, and help. Finally, KMLS pertains to knowledge about required mathematical concepts to be taught, knowledge about the level of conceptual and procedural development expected, and the sequencing of the various topics.

The aforementioned division into subdomains allows us to deepen our understanding of the elements of knowledge that are utilised in an integrated and interconnected manner. The MTSK model is a suitable analytical tool for meeting the objective of the present study because, in addition to highlighting mathematics, its categories, and subdomains, it allows focusing on teachers' knowledge about the function concept and its teaching (for example, the definition, its properties, representation, and the strategies used when teaching).

Methods

This research is grounded in an interpretative paradigm and is based on the instrumental case study design (Stake, 2007). The aim is to investigate from the perspective of MTSK the knowledge manifested by a high-school teacher when teaching the concept of function. The teacher that is in the focus of the study, henceforth referred to as Arturo, has ten years of teaching experience, teaching classes from fifth to twelfth grade. He is also a university teacher in first-year classes for engineering students and for pre-service teachers. He has also worked as a teacher in continuing education courses for primary-school teachers and has taken courses connected to university teaching, curricular updates in geometry, and curricular reform. At the time of this study, Arturo was teaching ninth-grade classes,

where he had planned to introduce the function concept. According to the information provided by Arturo, his group of students is familiar with the use of algebraic language, equation solving techniques, and Cartesian plane, among others.

To collect the data, classes in which Arturo planned to introduce the function concept were observed and video-recorded. The videos were transcribed and the transcripts served as the principal source of information. The resulting data was subjected to content analysis (Bardin, 1996), whereby class episodes were determined according to the tacit or explicit goals of the teacher. The units of analysis correspond to the teacher's interventions and responses provided by his students. In addition, only those that present evidence of teacher's specialised knowledge have been considered (Flores et al., 2013). In the first class, Arturo introduces the concept of function and its definition, and provides some examples of functions. An episode was selected from this class, in which the teacher uses an analogy to promote students' understanding of the definition of function. The episode was analysed in relation to the KoT and KMT sub-categories of the MTSK.

Results and discussion

Arturo defined function as *a correspondence between elements of two sets in which each element of the input set corresponds to a single element in the output set*. Knowledge of this definition is part of his KoT. To make this definition understandable, Arturo introduces a washing machine as an analogy for a function, alluding to a family context for his students. To know this context is not part of the MTSK, but it allows us to reflect on the scope and applicability of the analogy according to the type of students to whom they are presented. In the following extract, Arturo presents the analogy between function and washing machine:

Arturo: Before giving you more names, the function works like a kind of machine. An example could be a washing machine. A washing machine carries out a function. What is its function?

Student: Washing!

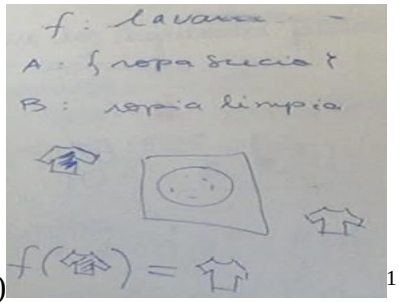
Arturo: What do you do? You take an article of clothing. It's dirty. You put it in the washing machine. How does it come out?

Student: Clean.

Arturo: Did the washing machine fulfil its function? Yes. The dirty article of clothing would be a member of the input set, and the clean article of clothing would be a member of the output set. This is what the function does. Here [he points to a diagram] we would have the dirty article of clothing. The function does what it does, depending on the machine, and arrives at the other side. In the case of a washing machine, it arrives clean.

Knowledge of analogies, as elements that enhance the teaching of a concept, is part of the teacher's specialised knowledge (Carrillo et al., 2014). The use of analogy shows teacher's knowledge about when to give any specific help to his students (KMT). The use of analogy favours the understanding and visualisation of abstract objects in students, besides being a motivation for a new theme. This analogy presents the function as a *process* and allows the students to better understand this concept (Sfard, 1991; Figueiredo et al., 2015). Moreover, different components can be identified, namely

domain, co-domain, pre-image, and image explaining the connection between the source and the target domain.

a) 

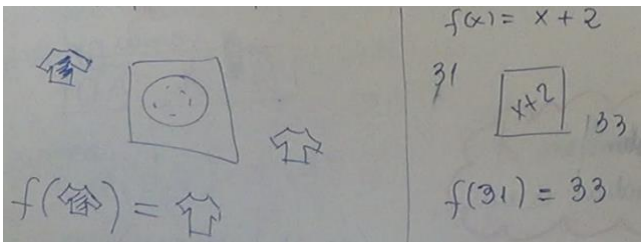
[1] b) 

Figure 2a: Presentation of the analogy. 2b: Relationship between the source and the target domain.

The analogy is presented in two formats: in the intervention described (verbal) and when the teacher draws a washing machine on the whiteboard (pictorial). This illustration (Figure 2a) shows the function as an input-output process, in which the object that enters is modified (in this case, a dirty article of clothing comes out clean). In this example, the objects are the same at entry and exit (T-shirts in both cases). Other analogies for this concept can relate objects of different nature. For example, function can be represented as a coffee dispensing machine into which money is entered in order to obtain a cup of coffee, highlighting the arbitrariness of related sets (Even, 1990). In this sense, the analogy of the washing machine impedes the association of arbitrary sets, which may result in students gaining a partial understanding of the concept. Likewise, the washing machine does not show other conceptions for the function as, for example, the co-variation of magnitudes. Similarly, it does not facilitate representation of more complex functions or the complexity of the concept itself (function algebra, composition).

Arturo takes advantage of this relation between input and output to clarify the definition of the concepts of image and pre-image, which are parts of his KoT. Similarly, evidence of his knowledge about the domain and co-domain of the function appears in the analogy as "dirty clothes" and "clean clothes." This knowledge and the exposed relations between the source and target domains account for the use of analogy as a strategy for teaching the function, evidencing a relationship between its KoT and its KMT. Figure 2a shows Arturo's knowledge of the notation $f(x) = y$ (knowledge about representations as a part of his KoT) that allows him to show the relationship between two domains of the analogy (as knowledge of strategies - KMT) and to introduce new ways to represent the function.

In the following excerpt, Arturo explains the relationship between the linked domains, source and target, using the analogy, in which we interpret the *structural* and *functional* character of the analogy presented (Curtis & Reigeluth, 1984).

Arturo: In our context, our function was the washing machine, washing. Set A would be dirty clothes and Set B clean clothes. If this is our washing machine, and it carries out its function; dirty clothes go in, and how do they come out?

[1] f: washing machine A: dirty clothes B: clean clothes

Student: Clean!

Arturo: The same as what we did here. The function was applied to this kind of T-shirt that I drew that was dirty. What will it equal?

Student: Clean, clean clothes.

Arturo: The same T-shirt, but clean. These two elements also have names. This element here is called the "image" of what I sent in. [...] And these elements here are called "pre image." What is the clean T-shirt?

Student: Image.

Arturo: The image of what?

Student: Of the dirty T-shirt.

The *structural* characteristic is shown in establishing the correspondences of the Set A with the dirty clothes, the Set B with the clean clothes, and the washing machine with the arithmetic process carried out by the function. That is, the structure of the laundry process is analogous to the evaluation process in the function. When Arturo presents the analogy "function as a washing machine," he also refers to its functional character, as he establishes a comparison between the operation of the machine and the function. After this intervention, Arturo represented verbally and as an algebraic expression (representations in his KoT) an example of function (Figure 2b), thus deepening the analogy between machine and function.

Arturo: With numbers, the function isn't going to do the washing. It's going to add two to whatever comes in [he writes "f(x) = x+2"]. Whatever comes into the function, to the machine, I add two to it. If this is my machine that adds two to whatever comes in, if a one enters, how does it come out? [Student: Three.]

It should be noted that, when Arturo teaches "with numbers," he aims to work in the target domain of functions. Consequently, the situation created becomes an example (knowledge of examples - KMT) that will allow him to present the *functional* characteristic of the analogy and propose a two-way process for understanding the relationship between a function and a washing machine (Figure 2b).

The relationship between an algebraic expression (function) and a washing machine presents functions as an input-output process, meaning of the concept that we consider part of his KoT. In the same way, the articulation and selection of representations for the function (the analogy, f(x) = y, algebraic expression and natural language) accounts for Arturo's KoT, relating to his KMT.

In this last intervention, by associating a function with a machine, Arturo highlights the process role of a function (part of his KoT). We do not have evidence of Arturo highlighting the object role of a function supported by the analogy studied here, although the structural character of the analogy may be the first approach to this conception.

Conclusion

According to the results yielded by the analyses presented above, the relations between the teacher's KoT and KMT are demonstrated in the articulation between knowledge of representations of the function and the choice and use of these representations as examples and analogies for teaching the

concept. In addition, the choice of analogies and examples given by Arturo reflect his KMT, which is nurtured and influenced by his KoT. The analogy shows the function as a correspondence in coherence with the definition given, while also permitting articulation of different representations: sagittal and algebraic diagram (Figure 2b). They also allow the teacher to produce other representations as a Cartesian graph or a table of values, thereby expanding the concept image for the students. Likewise, the analogy allows the students to appreciate the univalence character of the function (Even, 1990) and to extend the range of situations in which the concept of function is present (phenomenology - KoT).

The analogy utilised in this case is suitable for teaching and learning given that it is connected to the students' prior everyday experience. Moreover, its functional and structural character allows them to understand the concept from different perspectives and using different representations. Establishing a bidirectional relationship between the function and the washing machine can allow students to utilise the concept of a function they have learned to identify the concept in other areas of mathematics and other areas of knowledge. It should be noted that, in spite of the benefits presented by this analogy, students may have a partial notion of function if the conception of function as a machine is maintained. Thus, it is essential that other aspects of the concept be highlighted, such as the arbitrariness of the sets involved (Even, 1990). It is also important to relate it to other representations (Figueiredo et al., 2015). Lack of articulation between representations can cause a limitation in the development of the conception of function from the *structural* perspective (Sfard, 1991; Breidenbach et al., 1992). On the other hand, this presentation of the concept may constitute an obstacle to learning, for example, the algebra of functions or its composition (How do I sum two machines?). Likewise, students' direct experience with the laundry process can be detrimental in understanding the function: "the function did not fulfil its function." However, it is not the purpose of the study to evaluate the methodological proposal or Arturo's knowledge, but rather to approach it with the intention of improving our understanding of such knowledge, that of subdomain relationships in particular. In that sense, the analysis using the MTSK model is useful for advancing this understanding.

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Pre-service teachers using the Knowledge Quartet as a tool to analyse and reflect on their own teaching

Sindre Syversen Flesvig[1], Tim Rowland[2] and Elisabeta Eriksen[3]

[1]Inland Norway University of Applied Sciences - INN University, Norway; sindre.flesvig@inn.no; [2]University of Cambridge, UK; tr202@cam.ac.uk; [3]Oslo and Akershus University College of Applied Sciences, Norway; elisabeta.eriksen@hioa.no

This paper reports a qualitative study of the post-lesson reflections of two pre-service teachers in Norway. During their third school placement, Nora and Mia volunteered to use the Knowledge Quartet to analyse and reflect on their own mathematics teaching. Comparing the nature of their reflections at the start and at the end of the placement, we find that Nora and Mia exhibit some development, focusing more on mathematical content at the end of the study than in the beginning. Factors that can influence their reflections are discussed: their own experience of mathematics and their beliefs about mathematics seemed to play an important role in how they interpreted and made use of the framework.

Keywords: Mathematics teacher education, teacher background, elementary school mathematics, teacher practicum placement, Knowledge Quartet.

Introduction

The apparent disconnect between teacher education and the practice of teaching is of great concern to teacher educators (e.g., Solomon, Eriksen, Smestad, Rodal, & Bjerke, 2015). Systematic reflection on teaching might reduce this fragmentation, providing an educational experience based on genuine classroom experiences. However, teacher educators face the challenge of encouraging pre-service teachers to engage with classroom data in a meaningful way. In mathematics, in particular, research efforts have been made to find ways of focusing attention on mathematics as opposed to general pedagogy, with the ultimate goal of helping mathematics teachers (both pre- and in-service) to develop their teaching. The Knowledge Quartet (KQ) is an example of a research-based theory that resulted from this research effort (Rowland, Huckstep, & Thwaites, 2005). Through two case studies (Flesvig, 2016), this paper explains and exemplifies the situated challenge of using the KQ to reflect on mathematics teaching.

The research questions are: “To what extent does using the KQ as an analytical tool influence what pre-service teachers' (PSTs') attend to in the analysis of their own mathematics teaching? How do PSTs describe their experiences of using the KQ for lesson analysis?”

Literature review

Teacher education programmes prioritize increasingly the ‘core practices’ of teaching. The debate as to what these might be and what it means to focus on these in teacher education is ongoing (McDonald, Kazemi, & Kavanagh, 2013). We support the view that analyzing teaching is one such core practice:

> [it] involves learning to decompose instructional practice, to attend to particular events and interactions that are considered consequential for student learning, and to interpret the meaning behind those events to make informed teaching decisions. (Sun & van Es, 2015, p. 201)

The underlying assumptions are that engaging in analysis of teaching, and focusing on the details of the mathematical aspects involved and on students' mathematical thinking, will result in development of mathematical knowledge for teaching, as well as in more responsive teaching. There is evidence that the first assumption holds true both for PSTs (Turner, 2012) and for in-service teachers (Llinares & Krainer, 2006). While the second assumption is yet not well documented, Sun & van Es (2015) confirm it in their study of secondary PSTs exposed to a course with a focus on analysis of video recordings of the participants' own mathematics teaching.

Providing PSTs with opportunities to analyse teaching is not enough. PSTs need tools to direct their attention to salient aspects of teaching episodes. While focusing their reflections on the taught content (mathematics) is not a given either for in-service teachers or for PSTs, research provides examples to show that this is a trainable skill (Turner, 2012; Star & Strickland, 2008; Sun & van Es, 2015). Examples of 'tools' that support the process include frameworks for analysis of teaching (Rowland et al., 2005, Star & Strickland, 2008), routines for discussion of videos (Sun & van Es, 2015), and experienced mentor support to direct post-lesson review to focus on mathematics (Nilssen, 2010).

In theory, school 'practicum' placements should provide PSTs with excellent opportunities to reflect on the details of teaching, under the supervision of experienced teacher mentors. However, research has shown that there are significant differences in the experience of school placement of individual PSTs and that school placement is mostly about managing and doing the teaching, less about learning systematically from it (Solomon et al., 2015). This makes it all the more interesting for teacher educators to explore ways of supporting, with minimal involvement, PSTs' structured reflections on mathematics teaching in their school placements.

Theoretical underpinnings of the study

The nature of reflections on mathematics lessons

While we argued for the value (and difficulty) of attending to mathematics content in PSTs' lesson analysis, we recognise other salient aspects are likely to feature. To capture these aspects in PSTs reflections, we turn to the five-category framework of Star & Strickland (2008): classroom environment (class size and level, room layout, equipment, etc.), classroom management (classroom events and procedures), tasks (worksheets, presentations, homework, etc.), mathematical content (the topic, representations, examples, problems) and communication (questions asked, suggestion offered). The framework has been used as an instructional tool in two separate studies based on analysing video, leading to improved skills in observing classroom environment and communications (Star & Stickland, 2008; Star, Lynch & Perova, 2011). However, attention to the categories 'tasks' and 'mathematical content' seems harder to promote, and did not improve in the second study. For this reason, we chose another instructional tool for our study.

The Knowledge Quartet

The Knowledge Quartet (KQ) is a framework that classified the situations in which mathematics teachers' knowledge comes into play, in four broad categories: foundation, transformation, connection and contingency. The framework is empirically grounded in classroom observations, and the four categories encompass in total 20 different codes (Rowland, 2014). For example, foundation includes codes such as overt display of subject knowledge, adherence to textbook, concentration on procedures. Transformation encompasses ways of making the mathematics accessible to learners, such as choice of examples and choice of representation. Connection includes, for instance, both connections between concepts and sequencing within a lesson. Contingency is the dimension capturing unexpected events in the lesson, for instance in responding to students' ideas.

The KQ is used to analyse mathematics teaching with a focus on teacher knowledge, and is an appropriate tool to analyse and develop mathematics teaching when used in cooperation by PSTs, teacher mentors and teacher educators (Rowland, Huckstep & Thwaites, 2005). It has been successfully used as "an analytical framework to identifying mathematical content knowledge revealed through observations of practice" in a study with in-service teachers (Turner, 2012, p. 256). The participants, who collaborated closely with the researcher and were given considerable support in using the framework, saw the KQ as a tool to support them in reflecting more critically on their own teaching (Turner, 2012). This focus on the mathematics stands in contrast with the general pedagogical and organisational features of the lesson typically addressed in post-lesson review sessions between teacher mentors and PSTs (Solomon et al., 2015). The KQ is a means "to support focused reflection on the mathematical content of teaching" (Turner, 2012, p.253).

Methodology and methods

This paper reports on case studies of two PSTs' reflections on their mathematics teaching in school *practicum* placement. At the time of the study, the participants, called here Mia and Nora, were in their second year of a four-year Norwegian teacher education programme for grades 5-10 (age 10-15), specialising in mathematics. They were in the third school placement of their programme, and were based in the same grade 5 class.

Prior to the school placement, Mia and Nora attended a training session with the first author. This included a presentation of the KQ, and a joint analysis of a video from a Norwegian classroom. Nora and Mia were invited to use the KQ to analyse each mathematics lesson in their school placement. Since Mia and Nora were aware of the design of the study when they volunteered to participate, we expect that they attempted to use the framework as faithfully as possible.

Data collection included observations of two mathematics lessons for each participant, the first and the last of those taught in that third school placement (two weeks apart), followed immediately by audio-recorded semi-structured interviews. In the observed lesson, Mia and Nora taught statistics, and in the second they taught decimal numbers. This paper considers data from the interviews, since it is the PST's *reflections* on teaching, rather than the teaching itself, that will be analysed. However, the lessons were videotaped for stimulated recall during the interviews, and to allow recall of episodes discussed in the interviews.

The interview guides for the two interviews had a common core, and some additional questions that differed (regarding participants' background in the first interview, and regarding their experiences of using the KQ in the last interview). The core was structured around the dimensions of the KQ ("Last time we talked you mentioned being concerned with how tasks are sequenced. What about this lesson?"), but also included more open questions about the lesson observed ("Tell me about an episode you remember from this lesson. Why did this episode catch your attention?"). The interviews were transcribed and analysed in the original language (Norwegian), by the first author. The excerpts included in this paper were translated into English by the authors.

Given the design of the interview guide, with some open questions and some directly connecting to the KQ, this framework is not sufficient as the analytical tool. In this paper, our analysis draws on the framework of Star & Strickland (2008). This framework gives insight into the nature of the participants' reflections on their mathematics teaching, and their development during the school placement during which the study took place.

Participants

At the time of the study Mia and Nora were in their third school placement (lasting 13 days), both based in the same class (grade 5, age 10) under the supervision of the same teacher mentor. Both Mia and Nora had some experience working as (unqualified) substitute teachers.

While confident in her mathematics knowledge, Mia wanted more in terms of mathematics pedagogy and this was her motivation for participating in this study. She had enrolled in her current grades 5-10 teacher education programme after dropping out from a programme for mathematics teachers for grades 8-13 (age 13-18) in disappointment with the courses: "It was all about computations… there was nothing about putting it [the maths] across".

Nora found mathematics "fun, at least in grades 1-7", but to gain admission to the teacher education programme, she had to retake the final mathematics exam (grade 12). In teacher education, Nora experienced a "steep transition from upper secondary, quite a few notches over that". In the first interview she described mathematics as her favourite subject to teach, but was dissatisfied with the course: "A lot of what we learn is not what we will teach, and there is no use for it in our professional lives, while at the same time I miss something on how to teach the very basic stuff".

Findings

We consider Mia and Nora's reflections on their teaching of the two lessons, and their thoughts on The Knowledge Quartet. Some data from the videos is included by way of context for the interviews.

Interview 1 - Mia's reflection on her teaching

In the post-lesson interview following Mia's first lesson, some questions were directed towards specific dimensions of The Knowledge Quartet, such as transformation. Mia was asked how she selected tasks for her class. She mentioned that she does look at the textbook first, but she supplements the materials with additional problems that she finds online and selects carefully:

> I make sure they target the age group, fifth grade. That one [task on the handout] was actually a challenge for fourth grade, I found it online [...]. But it was about inserting, rather than drawing

> the chart, and there are no such tasks in the textbook. I always look for tasks that fit the topic and the age group and that complement the textbook, otherwise there is no point in it.

Interview 2 - Mia's reflection on her teaching and on using The Knowledge Quartet

The last lesson, like the first, had a traditional structure, with Mia showing some examples, then the students worked individually until the lesson ended, without any summary or discussion. Mia was invited to mention something she noticed during the lesson:

> I remember best and I was most surprised by how well the students remember from [...] the first lesson about decimal numbers. In that lesson I felt they got something out of it, but not everything, because it was hard. But now I suddenly felt that there were very many who were eager and who knew something about it [decimal numbers].

The interview included questions on the dimensions of The Knowledge Quartet, related to specific situations from the lesson observed. In terms of transformation, Mia commented on the role of the textbook and the choice of tasks and examples:

> I only use it [the textbook] to see what it says, given that the students will solve problems from there, so my teaching shouldn't deviate too much from it. But I don't really use it when I teach as such, then I use examples and tasks I prepared myself, that are suitable for the children. And these are […] examples I choose carefully so that I know them well if I get questions.

At the end of the interview, Mia was asked about her thoughts on the KQ:

> I had one lesson that I was really unhappy with, while my mentor thought it hadn't been so bad. But I was really irritated […] so I used it [the KQ], because I was really angry. I went carefully through all the codes and categories. I'm thinking this should be done when the lesson goes well, too, because it really helped me when it went poorly [...] I discovered that - here is something positive, and here as well. It wasn't all negative, although it felt that way to begin with.

Interview 1 - Nora's reflection on her teaching

Nora's first lesson was in statistics. The lesson had a traditional structure, starting with recalling and writing down definitions, solving a few problems on the interactive board, and then individual work.

Asked about the transformation dimension, about her choice of tasks and their sequencing, Nora explained:

> I asked first for the definition of the mode and the median, since they'd learned that earlier. [I asked them] to check if they remembered what they'd been told earlier. [...] They have a rulebook where it's good to write down things like this, so we started there, because I thought at least they have it there.

The interviewer asked her to explain her choice of tasks, why she considered them good, and why they were selected for the session on the interactive board:

> Because there was a bit of variation. But after a while … Well, there were [in the online resource of the textbook] ten levels [of difficulty]. That's a bit much, so I stopped a bit earlier. [...] It would have been too much of the same, but six-seven is okay, a chance to drill.

Interview 2 - Nora's reflection on her teaching and on using the Knowledge Quartet

Nora continued the lesson on decimal numbers from where Mia left off, continuing with individual work and then the whole class worked on exercises on the interactive board.

In the interview, she was invited to mention something she remembered from the lesson:

> The students worked individually for a long time, so I had to find some additional tasks [from the textbook] since they solved them much faster than I thought. So I just let them know [...] that they can carry on to the next page.

Invited to use the KQ to analyse the lesson itself, Nora recalls a contingent moment:

> One girl asked […] if the distance between 0.7 and 1.1 on the number line is 4. Then I answered that she has to think of the whole number line: here's 0 and here's 1, there is a whole between them. Do you think there are four between [0.7 and 1.1]? No, so then it's 0.4.

In this final interview, Nora was asked about her experience using KQ so far and if she thinks she might continue using it. She admitted that it can be helpful in reassessing a situation ("might not think of it without all these points") that might otherwise be overlooked ("so much happens during a lesson") and this will help to revise the teacher's approach next time. However, the traditional post-lesson review session appeals to her:

> I think it's helpful to talk about what happened in the lesson anyway, and we [Nora, Mia and the mentor] talked a lot. Then you get some insight in what is good and what could have been different, and so on.

In her experience, the KQ has "an awful lot of codes and dimensions" and using it resulted in:

> ... talking more about the lessons. And more about the examples. And sequencing, maybe. But not much otherwise.

Discussion

Mia

With reference to Star, Lynch & Perova's (2011) framework, the categories *tasks* and *classroom management* are especially prominent in Mia's first analysis/interview, both in response to the open questions, and when directed to use the KQ. A turn towards *mathematical content* occurs with more targeted questions about specific dimensions of the KQ, as in the case of transformation. By contrast, in her second analysis/interview Mia observed and reflected with *mathematical content* in mind, barely touching upon *classroom management*. Answering open questions, she refers to *tasks* and *communication*, without using any of the terms from the KQ. However, this changes when she goes deeper into her lesson using the KQ framework, with Mia using the terminology of the KQ, with attention to *mathematical content*, as well as *tasks* and *classroom management* in general. Questions on the dimensions of the KQ, such as the transformation dimension, direct Mia's focus to the fine grain of *mathematical content*.

Mia is convinced that the KQ supports a focus on the details of mathematics lesson. She pinpoints a specific situation in which breaking down the events of a lesson with the KQ helped her see

strengths, not only weaknesses in her teaching, thus regaining her confidence as a mathematics teacher, at a time when her mentor's more general feedback was not helping her.

Nora

Throughout her first teaching analysis/interview, Nora's reflections focused most on *tasks*, *classroom management* and to some extent *communication*, only superficially touching on the *mathematical content*. Even when the questions directed her to the KQ, she never actively used the terminology of the framework. In the second analysis/interview, Nora focused mostly on *mathematical content*. Although initially focusing on *tasks*, there was a clear change in emphasis towards *mathematical content* when she is asked to use the KQ framework, and even more so when the questions are specific to the KQ dimensions.

There is a tension in Nora's statements about using the KQ for teaching analysis. While she recognizes that the KQ creates an opportunity for development by making visible the specifics of a mathematics lesson, Nora prefers the unstructured form of traditional review sessions, explaining this in terms of the burden of the number of codes.

Concluding comment

Comparing the first and the last interview, we see that both Mia and Nora's reflections exhibited an increasing focus on mathematics. A number of factors could play a part in this, including the use of the KQ for analysis of the lessons, the experience of the school placement, and the mentor's guidance in the post-lesson reviews. The data indicate that the KQ does mediate this change, since even in the last interview we see that the more the questions are anchored in the KQ, the more marked the focus on mathematics was.

Both Mia and Nora described the KQ as a means to explore the details of a mathematics lesson, and an opportunity to improve their teaching. However, we see differences in the degree to which they embrace the use of the KQ, with Nora leaving the door open to use it for troubleshooting, and Mia positive to continuing using it both when lessons go well and when they do not. We recognize that there are differences in the mathematical knowledge and mathematics teaching confidence of the two PSTs, and this might play a part in these differences. For instance, analysing in such detail a lesson that ‘went well’ (as is often said in unfocused post-lesson reviews) is likely to reveal details that were problematic, an insight causing some emotional discomfort. In that case, Nora might have benefitted from receiving more support when using the KQ, to help her cope. Or, perhaps Nora interprets the KQ as an algorithm that requires her to go through the all 20 codes for every lesson, and finds the time commitment too much, in which case she would benefit from more in-depth training in using the KQ in a more holistic and efficient way, perhaps limiting the framework to low-inference codes (e.g. ‘choice of examples’) to begin with.

In conclusion, the study indicates that, even with minimal support, the KQ can contribute in some cases to focus pre-service teachers' post-lesson reflections on mathematics. Individual differences between the voluntary participants' willingness to continue using KQ after the end of the study suggest that teacher educators need to be mindful of factors that could deter PSTs from using the framework.

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Subject professional association activity: What can it offer teachers of mathematics and their students?

Jennie Golding

University College London Institute of Education, UK; j.golding@ucl.ac.uk

Professional association activity is commonly regarded as a professional 'good', yet there remains little systematic evidence of its impact. This paper reports on a small study that asked English teachers of mathematics participating in such activity what contribution they believe it makes to the development of their knowledge, skills and affect, and how that then impacts on their students. Participants claimed a range of significant and pervasive benefits, many of which are distinctive to this form of professional development. These include a renewed commitment to their role as teachers of mathematics, refreshment and inspiration, and a deep and lasting impact on both their own learning and that of their students.

Keywords: Professional association, professional development, mathematics, affect.

Background

Within mathematics education in the UK, there are four national 'classroom-facing' PAs, funded entirely from membership and each attracting up to several thousand members: The Mathematical Association (MA), the Association of Teachers of Mathematics (ATM), the National Association for Numeracy and Mathematics in Colleges (NANAMIC), and NAMA, the National Association for Mathematics Advisers. All have a core purpose of supporting the teaching and learning of pre-university mathematics through working with teachers and others.

Their annual conferences, residential for most, offer sessions that might focus on mathematics pedagogic knowledge or skills, learner enrichment or teachers' own mathematics enhancement and/or enjoyment, or mathematics curriculum, leadership or assessment issues. There is time for networking and also for social activities. Conferences are usually held in teachers' holidays and teachers who attend often fund themselves, so they clearly value what such activity offers. Between them the PAs also offer a range of day conferences, bespoke courses, professional periodicals, local groups, policy debate, and a variety of social media opportunities, so building professional communities of up to national scale. Additionally, the larger ones operate working days and weekends when resources are developed for publication, and Hodgen (2003) suggests this in itself can develop teacher reflection, knowledge and hence practice. Teachers can therefore personalize the extent and type of their involvement and the professional development (PD) targeted.

I adopt Cobb and Bowers' (1999) conceptualization of teacher PD as any planned experience intended to develop teachers' professional functioning (for the ultimate benefit of their students' learning) – and that process as both enculturation and construction. We know something of what makes PD effective, for example that it has a content focus, is coherent with teacher's prior learning and needs, active, sustained, features collective participation (e.g. Desimone, 2009). Golding (2017) suggests that for sustained development it should develop positive work-related identity and affect, as in Hodgen (2003) and Hannula (2011) respectively, including self-efficacy, resilience, enthusiasm and feeling valued. The importance of effective PD, job satisfaction and recognition to teacher retention

is also well evidenced (Lynch et al, 2016): critical when, as in England, there is a shortage of effective teachers of mathematics at all levels (Ofsted, 2012).

There are, though, some sizeable gaps in the PD literature, e.g. there is limited evidence of the impact on students' learning (Joubert and Sutherland, 2008), and the affordances of online activity continue to change. Further, I can find no systematic study of the contribution professional association (PA) activity can make to teachers' development, though Chetwin (2010) suggests there are likely to be gains from networking, growth of knowledge and/or skills, and taking responsibility for one's development. This small study therefore investigated the contribution of PA activity as perceived by participant English teachers of mathematics (n=185). It asked

- How is PA activity aligned with what is known about effective PD?
- What contribution can PA activity make to teachers' professional development, and to the development of their students?
- Is any of that distinctive?

The study

I conducted semi-structured individual interviews with a purposive sample of four anonymized participants from each of the four 2016 PA annual conferences (Table 1), drawing on a variety of phase/experience/PA background from those currently, or recently, active in the profession. Questions (Figure 1) offered opportunity for development of a grounded account (Charmaz, 2006).

All 2016 conference participants were invited to complete a (usually online) questionnaire designed in part to validate interview responses with a wider sample. The interview questions were complemented by Likert-style questions (Table 2) designed to elicit the value attributed to aspects of conference activity known to be particular to this form of PD. Additionally, I scrutinised documents and publications available on PA websites or at conferences. Data consisted of questionnaire responses (n=185), transcriptions of recorded interviews (n=16), and my notes from documents scrutinized. There is no claim to generalisation from this highly selective sample, though the study shows clearly the breadth and depth of perceived impact on some teachers.

Grounded analysis of all qualitative data was by open, axial and selective coding (Charmaz, 2006). Documentary evidence was used to validate participant claims, and interviewees validated all written interpretations of their talk; additionally, a colleague acted as a 'critical friend' - particularly important given I have a history of involvement in PA activity myself.

Throughout, though, the study is framed by its reliance on teacher accounts. The status of such claims has been contested: do teacher narratives represent warranted true belief, and if not, to what extent can they be represented as 'truth'? This issue is addressed in the literature, though Desimone (2009) argues concerns can be over-stated in relation to accounts of PD; I adopt here Doyle's (1997) position applied to teacher development, that the study aims to develop understanding of a highly contextualised and personal phenomenon, through access to participants' stories of intentions, motives, purposes and perceptions of effectiveness, rather than a universally knowable phenomenon susceptible to legislation through policy.

Table 1: Interviewees

Source	*Pseudonym and teaching context*	*PA activity(years)*
ATM	Alice (11-18, special education), Lara (11-18), Terry (11-18), Billy (11-18, Higher Education, Adviser)	6/8/ 2/'many'
MA	Jackie (5-11, 11-18, Higher Education), Janet (5-11, Adviser), Kim (11-18), Rachel (5-11)	41/35/3/10
NAMA	Charles (11-18, Higher Education), Gail (5-11, Higher Education), Graham (11-18, Adviser), Kathy (11-18)	16/20/16/'many'
NANAMIC	Sally (11-18, 16+), David (11-18, adults), Angela (16+), Susan (11-18, 16+)	'Many'/0/'12/10

1. Tell me about your professional background… and your history of involvement in (*the PA*).
2. What aspects of the conference are you finding/did you find particularly helpful (why)?
3. What are the limitations of a conference like this in terms of your PD – what aspects of your PD are better provided elsewhere? (*prompt:* institution or local opportunities, online affordances)
4. And how do all these different opportunities impact on your students? (*prompt*: and their learning? How do you know? Any conference-specific impact, or not? if not mentioned)
5. So if someone asked you how you stay up to date, maintain your skills, and develop further as a teacher, what would you say?... and has that changed over your career?
6. Any other comments you'd like to make about your PD and its relation to PA activity? Thanks.

Figure 1: Interview structure

Findings

Coding exposed themes around identity and values, specific gains for teachers and/or students, strengths and limitations of professional association activity, including some apparently distinctive benefits, and threats or disincentives to that. I consider each of those in turn. Whilst it was not possible to enact either questionnaires or interviews in precisely comparable ways across the four associations, I argue that differences did not fundamentally influence the nature of responses.

Identity and values

Interviewees were keen to talk about impact on their professional identity or values, often in terms of affirmation, empowerment and meeting with like-minded people who support and challenge them professionally. For 11 of 16 this was their first focus in response, often centred around face to face participation – a sense of community, sometimes built up over years, and talk about refreshment and renewal, often contrasting that with the draining nature of teaching. Claims were often extravagant: "It's been a life-changer, it builds me up as a maths teacher so I can do a better job in the classroom" (Kim), and for four teachers this was specifically linked with retention:

> PA activity helps me analyse and then be proactive about developing what I value. Hugely empowering, and … that keeps me in the profession despite the grinding demands. (Lara)

> Here is about personal PD, affirmation, values – challenge too, but support for…your long-term growth and enablement, that enables you to go back refreshed and keeps you committed to what can be a …very draining profession – I just couldn't stay in my job long-term without that injection of positivity and recharging. (Billy)

For interviewees with a background in Further or Adult Education, this identity work was talked about in even more fundamental terms:

> In FE, very often you don't even see other teachers of maths... So NANAMIC gives you that identity – there are other people out there struggling with you, valuing some of the same things as you do – otherwise you're just functioning in isolation, far too often. (Sally)

Specific gains for teachers and/or students

Interviewees commonly (10 of 16) talked about the high quality of PA publications, sessions and resources in terms of direct benefit to themselves and to their students:

> Support – inspiration - resources: I return brimming over with ideas and enthusiasm, with knowledge about innovations across the country, catholic ideas and approaches that have worked in different circumstances. The resources are creative and engaging, they really probe deep understanding and the students love them. (Terry)

Often the benefit was claimed to spread beyond the interviewee concerned:

> I've worked with teachers using these materials and boy are they effective. If they can make the right selection and the right tweaks, and we work on that, then they see real and immediate impact on learning. (Graham)

Many respondents (25 open questionnaire responses and 7 interviewees) greatly valued informal networking opportunities, claiming explicit benefits also to their colleagues and students – both immediate and also for sustained learning and positive disposition towards mathematics:

> The specific numeracy ideas, I took them straight back to my classroom and my students are already showing the benefits, in a couple of weeks – to confidence as well as skill. There are also 'seed' ideas, things that ...will come to fruition over a longer timescale. (Susan)

For some (seven interviewees and over 25 open questionnaire responses), the opportunity to be better informed about, and contribute to, national policy debate is valued; for others (in 6 interviews and some 15 questionnaires), the chance to engage with cutting-edge research relevant to their practice and reflect on its application is important. Teachers who engage in local branch meetings claim similar, but less extravagant, benefits. Questionnaire open responses were generally consistent with these interview response strands.

Strengths and limitations of PA activity

As well as the specific benefits to professional skills and knowledge, and to professional affect and identity, teachers identified the eclectic nature of professional association activity, and the fact that they can easily personalise it to their own professional needs, as underpinning its effectiveness. Many described it as 'uniformly high quality – the best professional development I get' or similar. It was often reported as having long-term benefits for both teachers and their students, sometimes in contrast to other courses which "focus on short-term skill or particular knowledge." (Janet).

Five interviewees talked about the benefits of being physically removed from their work environment and the luxury of sustained unhindered time committed to their professional growth. Several teachers described the desirability of also participating in institution-based development alongside colleagues, with access to familiar resources, and with whom they could contextualise new ideas. In three interviews they extolled the particular advantages of engaging in branch or conference activity with at least one colleague. On the other hand, teachers said they found distance learning can be effective and efficient for pure dissemination of information. Four identified PA activity as often limited by a 'light touch' for more substantial knowledge or skill development, particularly where there was a need for substantial subject or subject pedagogical knowledge, perhaps better provided in a series of inputs interspersed by classroom embedding. This was true in particular for two of the one-day NANAMIC conference participants. One commented:

> And of course then you go back into college and there's no-one to share it with…so unless you're really committed, those interesting ideas and good intentions might well get lost. (Angela)

However, all of those interviewed and virtually all of those completing questionnaires identified face to face PA activity as a central and rich component of their effective impact on students. These comments were echoed in questionnaire responses, though typically in less depth. Questionnaire responses added no significantly different responses.

Likert scale items in questionnaires largely concentrated on features of conferences. Table 2 shows mean response, on a scale of 1 (of little importance to me) to 5 (very important to me), together with standard deviations s. As quantitative data it is of limited robustness but gives some indication of the ranking of different aspects, similar but not identical for the different conferences. For the range of participants, working with others from a variety of roles and experiences is highly valued, as are opportunities to engage with new ideas or mathematics. These teachers also value opportunities to construct a programme that meets their individual needs.

Table 2: How important are the following aspects of the conference to you?

Association (responses from active teachers or those actively working with teachers)	*NANAMIC (n=10)*	*NAMA (n=29)*	*MA (n=56)*	*ATM (n=90)*	***Overall (185)***
Meeting people in comparable roles	4.2	3.7	4.3	4.2	4.2 (s=0.7)
Face to face rather than at a distance	4.4	4.5	4.3	4.3	4.4 (s=0.7)
Meeting people from other phases in education or with different roles or from different areas of the country	4.2	3.9	4.7	4.7	4.6 (s=0.6)
A mix of beginners and experienced colleagues	4.4	4.0	4.5	4.6	4.5 (s=0.7)
Sessions that are grounded in the classroom	4.1	3.4	4.4	4.2	4.1 (s=0.5)
Social activities	-	2.3	3.3	3.9	3.5 (s=1.1)
Immersion – it's residential	-	2.3	4.5	4.3	4.0 (s=0.7)

Opportunity to do mathematics or engage with new ideas, irrespective of whether I'll use them directly in the classroom	4.3	4.5	4.5	4.6	4.5 (s=0.8)
Being able to choose sessions which fit my needs/preferences	4.6	3.7	4.6	4.7	4.5 (s=0.8)

Threats or disincentives to such activity

There are, though, some clear threats to participation, of which funding was mentioned by nearly all interviewees. Although "cheaper for several days of exceptionally high quality development than many mediocre commercial courses" (Rachel), teachers routinely talked of schools and colleges prioritising performance-framed one-off courses for funding, and leadership teams not valuing the "deeper, wider learning that is supported by face to face PA opportunities" (Janet). Others said that colleagues "thought they were mad to spend holiday time at a …conference when there are so many pressures during term time you just want to curl up and die when you get to a holiday" (Kathy). Four interviewees claimed their schools/colleges would not pay for conference attendance because a better-informed teacher was more likely to move. Funding was a particular issue for those working in FE, with a majority of those respondents reporting little or no employer support for subject-specific development, so no choice but to fund such development themselves.

Four interviewees suggested that Primary or FE teachers without a strong mathematical background or a specific mathematics responsibility were unlikely to prioritise, or be confident to participate in, subject specific and self-funded professional development. They suggested incentives for Primary teachers to 'bring a local friend' might increase both confidence and impact, and identified day conferences as a good first step "where it's often desperately needed" (Janet).

Discussion

It is important to note that there is no claim to generalisability here: these are teachers in an English education culture who choose to attend these conferences in their own time, and often self-funded. They are therefore highly committed to their own development as teachers, but also claim that they gain motivation and energy for their work from PA activity, often in contrast to other opportunities available to their context. It is striking that almost all interviewees privilege talk about values, affirmation of professional identity, and improved self-efficacy in their accounts, together with deep, wide and reflective mathematical (subject and subject pedagogical) learning for the long-term effective exercise of their professional role. They commonly contrast that with much external PD and often generic local provision. With both preservice and inservice education in England increasingly adopting generic rather than subject-specific approaches, Joubert and Sutherland (2008) show such subject-specific opportunities are central to the development of a deeply effective teaching profession.

The benefits described align well with Desimone's (2009) and Golding's (2017) criteria for effective PD: showing a clear content focus, active and coherent with teacher's prior learning needs, featuring collective participation, and contributing very positively to teachers' affective and identity needs. PA activity can be sustained (sometimes over years) in the sense of offering longitudinal stimulus interspersed with everyday teaching, but not usually in the sense of a critical mass of hours focused on particular knowledge or skills, for which other avenues would appear to be more effective. The

benefits claimed for teachers, and for their students, are significant, deep, wide-reaching and long-lasting, including a renewed commitment to retention in the profession. There is no *a priori* reason why such benefits should not be experienced by far greater numbers, and it is important that perceived threats to participation are addressed. These are not just about funding, but, as in Lynch et al (2016), about the value teachers perceive management to give to PA activity and to teachers' PD beyond the short term specific needs of their school/college.

All teachers in this study were teachers of mathematics, but in the English context there are comparable professional associations in other curriculum areas, and an obvious question is whether the benefits cited here, particularly in relation to subject-related identity, apply also to them.

Benefits distinctive to professional association activity

Some distinctive benefits of PA activity appear to emerge. First, there is the opportunity to mould a PD programme to one's own professional needs, whether in terms of reading, resources or face to face development. Many teachers also referred explicitly to working with mathematics – for its own sake as well as for possible classroom benefits. In England, in contrast to many other jurisdictions, this is unusual after initial qualification, yet this aspect of PA activity was commonly highly valued.

There appears to be a great deal of professional affirmation, networking and identity work taking place. Some teachers particularly value the access to recent research offered by the PAs, and/or the informed policy work facilitated by PAs and based on professional discussion, and I would argue it is healthy for policy systems to be informed by such knowledge and discussion. Finally, through the PAs it is relatively easy for teachers to be able to offer something back to the PD of others, whether through writing about their professional work, giving a session at a conference, disseminating, receiving critique and discussing in a varied but informed peer group, producing resources or developing courses, or engaging in writing work that responds to deep desires to improve mathematics learning. This is itself is recognized to be developmental:

> Writing for journals has been very developmental, and the support you get for doing that. Very high quality writing sessions: a richness of ideas from people whose ideas have since shaped my practice, that I admire and aspire to. (Charles)

There are, though, other providers of different subsets of the cited activities, and it is an open question how the affordances and constraints of such provisions compare with those of PA activity.

Conclusions

These English mathematics teachers claim a wide range of benefits from PA activity (especially face to face events), some of which are perceived as either exclusive to such activity or most effectively provided by it. They say it gives them deep and wide professional learning which impacts on their students both through specific pedagogical tools and approaches, and through teacher refreshment and re-commitment. They claim an affirmation of their professional identity through sharing goals and values with others, and increased self-efficacy through peer validation and personalisation of development. Teachers value the subject-specific nature of PA activity, that in the English context contrasts with much school-based, typically generic, development. They appreciate the range of ideas and ways of thinking that far exceed what is available within one school/college or group of institutions. For many, these claimed benefits have been developed and sustained over years.

The study offers evidence to management and policy makers about the value these teachers of mathematics place on subject-specific development that affirms their professional identity and their values, recharges and renews their commitment and enthusiasm, and engages them actively in deep and reflective subject and subject pedagogic learning. With current performance pressures and limited budgets, it is not surprising English schools and colleges often privilege perceived immediate curriculum needs, but the cost of PA activity outside teaching time is small when compared with the costs to students of a stale and drained teacher – or no teacher at all. There is a need to identify politically-acceptable ways to invest in teachers' longer term subject-specific development, so that more teachers are encouraged to participate in such activity.

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Subject matter knowledge and pedagogical content knowledge in the learning diaries of prospective mathematics teachers

Jani Hannula

University of Helsinki, Finland; jani.hannula@helsinki.fi

In this study, 26 learning diaries by prospective mathematics teachers were analysed in order to describe the characteristics of mathematical and pedagogical knowledge discussed therein and to evaluate the potential and limitations of the learning diary in mathematics teacher education. Conceptualisations of teacher knowledge are typically discussed in terms of subject matter knowledge, pedagogical knowledge and pedagogical content knowledge. A central goal of mathematics teacher education is to strengthen all of these areas of competency. The results of this study indicate that, although the learning diary is a potential learning tool, prospective mathematics teachers tend to emphasise pedagogical content knowledge, placing less stress on subject matter knowledge. Consequently, more structured learning diary tasks could be used to support all the components of mathematical knowledge for teaching.

Keywords: Teacher knowledge, mathematics teacher education, learning diaries, subject matter knowledge, pedagogical content knowledge.

Introduction

Finnish mathematics teacher education consists of three somewhat distinct parts: subject matter studies, educational studies and practical teacher training at schools. Subject matter studies at mathematics departments form a major part of the mathematics teacher studies. Finnish mathematics teachers, however, report that university-level subject studies in mathematics lack a clear connection to the mathematics taught at school (Koponen, Asikainen, Viholainen, & Hirvonen, 2016). The transition to university-level mathematics requires a major change in mathematical thinking (e.g., Tall, 1992). At university, mathematics courses typically emphasise formal reasoning, meaning reasoning based on axioms, definitions and proven theorems (Viholainen, 2008). Informal reasoning is based on visual or physical interpretations of mathematical concepts (Viholainen, 2008). Some empirical studies (e.g., Chin, 2013; Viholainen, 2008) have shown that prospective teachers may have difficulties connecting formal and informal reasoning.

On the other hand, prospective mathematics teachers may emphasise the importance of a teacher's personal characteristics and pedagogical knowledge, while diminishing the importance of subject matter knowledge (e.g., Hoffkamp & Warmuth, 2015). Subject matter knowledge nevertheless plays a significant role in a teacher's professional knowledge. Firstly, subject matter knowledge is typically seen as theoretically necessary for developing pedagogical content knowledge (Baumert et al., 2010). The quality of prospective teachers' subject matter knowledge also affects their pedagogical choices when participating in practical training (Even, Tirosh, & Markovits, 1996). In addition, subject matter knowledge along with pedagogical content knowledge can be seen as a foundation for effective teaching, as a teacher's professional knowledge affects student achievement (e.g., Baumert et al., 2010).

This study is a part of a design-based research and development project that has been carried out at the University of Helsinki. The aim of the research is to develop instructional practices in order for

prospective teachers to both strengthen their subject matter knowledge and build up their pedagogical content knowledge. The research also aims to give insight into prospective mathematics teachers' conceptions of the relationship between school and university mathematics.

In this study, the specific focus is on learning diaries written by prospective mathematics during a six-week seminar. The seminar focused on finding connections between the mathematics studied both at university and at school and on discussions of mathematical content from the teacher's point of view. That is, the aim of the seminar was for prospective teachers to, first, strengthen their (structural) knowledge of mathematical topics (such as derivative) and, second, enhance their pedagogical content knowledge with relation to these topics. The aim of this study was to examine the potential and limitations of learning diaries as a learning tool in this context and to conceptualise the kinds of knowledge these prospective mathematics teachers discussed in their diaries.

Theoretical background

Theories used in design-based research can be divided into grand theories, orienting frameworks, frameworks for action and domain-specific instructional theories (DiSessa & Cobb, 2004). In this study, the idea of constructive alignment (Biggs & Tang, 2011), which provided the instructional design of the research setting, is used as a framework for action. The data analysis for this study is based on domain-specific conceptualisation of teacher knowledge (Ball, Thames, & Phelps, 2008). In the next two subsections, these frameworks will be discussed in more detail.

Teacher Knowledge

The distinctions between content knowledge (or subject matter knowledge), pedagogical knowledge and pedagogical content knowledge (Shulman, 1987) are an established starting point for conceptualisations of teacher knowledge (Scheiner, 2015). Especially the distinction between subject matter knowledge and pedagogical content knowledge has gained significant attention and generated a great amount of research and further development of the conceptualisations of teacher knowledge. According to the Mathematical Knowledge for Teaching (MKT) model (Ball et al., 2008), a teacher's content knowledge consists of common content knowledge (CCK), specialised content knowledge (SCK) and horizon content knowledge (HCK). In the MKT model, pedagogical content knowledge, in contrast, is divided into knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum.

The components of MKT model have been shown as important for effective teaching (Jakobsen, Thames, & Ribeiro, 2013). Hence, the model is valuable for this study, which aims to offer insight into prospective mathematics teachers' discussions of their learning diaries and to use this information for further development of instructional practices in teacher education.

Constructive alignment and learning diaries

Present research and development of instructional practices in higher education is typically based on the constructivist view of learning and concepts, such as learner approaches to learning, self-regulation and reflection. Constructive alignment is based on the constructivist view of learning and suggests that the intended learning outcomes, implementation of teaching and assessment should be carefully aligned and support active learning. Biggs and Tang (2011) suggest that by using more

active ways of learning (such as problem-based learning) even 'less academic' students can achieve more advanced levels of learning, such as applying and theorising.

In this study, the seminar was designed in the spirit of constructive alignment (e.g., the students worked in groups and specified their own study/discussion topics). The learning diary task was one of the ways to promote active learning and reflection among students. Typically, learning diaries are seen as texts that include both the central arguments of a course or a seminar and the writer's own interpretation of and reflection on these themes. That is, learning diaries are not supposed to promote *knowledge telling writing* (Bereiter & Scardamalia, 1987), which is understood as writing based on memorised facts. Instead, learning diaries promote *knowledge transforming writing* (Bereiter & Scardamalia, 1987), which is based on problem analysis and reflection.

Research questions

In Finnish higher education, learning diaries have been used successfully in subjects such as research methodology (Kyttälä, 2012). Journaling has also been found to be useful in studying university-level mathematics (Meel, 1999). There is, however, a lack of research evaluating the use of learning diaries in mathematics teacher education from the point of view of the teacher's knowledge. Additionally, more insight into prospective mathematics teachers' mathematical and pedagogical thinking is needed for further development of instructional practices in mathematics teacher education. Thus, the following research questions were formed.

1. Can learning diaries be used to promote knowledge transforming writing in mathematics teacher education?
2. What kinds of professional knowledge do the prospective mathematics teachers discuss in their learning diaries?

The first research question was posed in order to evaluate whether learning diaries have potential as a reflective learning method in mathematics teacher education. The second research question was posed in order to characterise the prospective mathematics teachers' discussions on teacher knowledge. The question of whether some/certain aspects of teacher knowledge would be emphasised in the diaries was also considered, as prior research has shown that prospective teachers may emphasise pedagogical knowledge and diminish the importance of subject matter knowledge (e.g., Hoffkamp & Warmuth, 2015).

Method

The data was collected during a seminar held in autumn 2014. The students (prospective mathematics teachers) attending the seminar formed small groups of 4–5 members. All groups prepared an introduction to a topic (such as dot product), so that both mathematical and pedagogical ideas were covered. These introductions led to group discussion and, as homework, the students reflected on their ideas by writing a learning diary. In their diaries, the students were asked to discuss 1) *What was discussed and how do the topics of discussion relate to other contexts?*; 2) *What did I learn and what was its meaning for me?*; 3) *Was something missing or unclear*?

Participants were mainly mathematics students at the end of their studies. Three students were studying another subject (such as physics) with minor studies in mathematics. Also, six students were second- or third-year students and, thus, not yet at the end of their five-year studies. The participants

were studying in a subject teacher programme that qualifies them to work as a teacher in the last years of comprehensive school (with students aged 13 to 16 years) and upper secondary school (with students aged 16 to 19 years).

Student learning diaries (N=26) were analysed using content analysis (Elo & Kyngäs, 2008). The units of observation were first placed in categories from the MKT model using deductive content analysis. Subcategories were then formed using inductive content analysis. In addition, the individual diary entries were classified either as *knowledge telling writing* or *knowledge transforming writing* in order to classify the entire diary either as *knowledge telling* or *knowledge transforming*.

The author of the present article created the coding. As it was not possible to use two independent coders, during the process, the author reread the diaries and the coding to ensure that his thinking remained constant during the coding process. The components of the MKT model may be difficult to distinguish from one another and this boundary problem has been highlighted in the research literature. This poses a challenge for coding, as two researchers may create different categorisations. The most problematic category seems to be HCK. In this study, HCK was understood, as defined by Jacobsen et al. (2013), as 'an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory'.

The coding of *knowledge telling writing* and *knowledge transforming writing* was based on a prior study by Kyttälä (2012). When coding each diary entry as either *knowledge telling* or *knowledge transforming*, the former was used if the entry included only repetition of the information discussed in the seminar and the latter code was used if the entry included personal reflection. *Knowledge telling writing* included excerpts such as 'This week we discussed linear algebra. Firstly, we discussed vectors in $\mathbf{R}^2$', whereas *knowledge transforming writing* included personal reflection such as 'I soon realised that I didn't remember much about dot product. I remembered that it had something to do with lengths and the perpendicularity of vectors.'

If at least half of the entries were labelled as *knowledge transforming* the entire diary was labelled accordingly. This methodology was chosen to ease the comparison of the results of this study to prior studies in the Finnish higher education context.

Results

The results of the study are presented in three parts. First, writing strategies (*knowledge telling* vs. *knowledge transforming*) are discussed. Then, in the following two subsections, the subject matter knowledge and pedagogical content knowledge observed in the diaries are discussed.

Knowledge telling writing vs. knowledge transforming writing

23 of the 26 diaries featured *knowledge transforming writing*. This seems to indicate that learning diaries can be used to promote reflective learning in mathematics teacher education as they have in other educational contexts, as Kyttälä (2012) has suggested. However, while most of the diaries were categorised as *knowledge transforming*, the content discussed in the diaries varied significantly. In some of the diaries, both mathematical and pedagogical topics/issues were discussed comprehensively, whereas in others, the mathematical content was discussed only cursorily and the

pedagogical issues were discussed in depth. The coding of subject matter knowledge and pedagogical content knowledge aimed to highlight this variation in greater detail.

Subject matter knowledge

The distinguished subcategories of subject matter knowledge are presented in Table 1. The frequency of each category is indicated in brackets. In the main categories, percentages are also given. Discussing representations of mathematical content was labelled as SCK, as according to Ball et al. (2008), knowledge of 'how to choose, make, and use mathematical representations' belongs to SCK. The category 'nature of mathematics' included utterances such as 'In mathematics you don't prove absolute truths. Instead, the proofs are based on chains where assumptions lead to something.' These can be also seen in connection to HCK, but as these comments were general, they were coded as CCK. Additionally, some discussion of the curriculum, such as 'Does knowing probability require knowing set theory? I suppose so. It would be good if there would be more of that in secondary school', were categorised as SCK instead of KCC. These comments also seemed connected to HCK, but were more focused on rethinking school mathematics and were consequently categorised as SCK.

Common content knowledge (66; 25 %)	**Specialised content knowledge (158; 59 %)**	**Horizon content knowledge (46; 17 %)**
• Giving a list of concepts (20) • Giving a definition (11) • Giving a theorem (9) • Explaining a property of a mathematical entity (8) • Giving a solution strategy (8) • Giving alternative definitions (7) • Discussing the nature of mathematical knowledge (6) • Giving a mathematical example (2)	• Discussing representations of mathematical content given in textbooks (72) • Discussing alternative representations of mathematical content (65) • Discussing relationship between mathematical knowledge and curriculum (11) • Going through some history of mathematics (6) • Giving and discussing matriculation examination tasks (5) • Reflecting on a mathematical example (2) • Modifying an example (1)	• Discussing the hierarchical relationship of mathematical concepts (28) • Giving an application of a mathematical entity or method (18)

Table 1: Subcategories of subject matter knowledge distinguished in the diaries

Common content knowledge was mainly discussed in terms of giving a list of concepts (related to the subject), giving a definition of a concept (such as limit) or giving a theorem (such as 'If function *f* is derivative, then function *f* is continuous'). This discussion was typically limited to telling the facts and no explanations or proofs were given. The specialised content knowledge typically focused on discussing the representations of mathematical content. Only one student adapted an example so that different versions of a problem were considered. The least discussed aspect of subject matter knowledge was horizon content knowledge; only 46 units of observation included discussion of the

hierarchical relationship of mathematical concepts or an application of a mathematical entity or method. Overall, discussion of subject matter knowledge was somewhat focused on SCK. More specifically, discussing the different representations of mathematical content was common in many diaries.

Pedagogical content knowledge

The distinguished subcategories of pedagogical content knowledge are given in Table 2.

Knowledge of content and curriculum (106; 27 %)	**Knowledge of content and students (133; 34 %)**	**Knowledge of content and teaching (148; 38 %)**
• Upper secondary school curriculum (57) • University curriculum (51) • Comprehensive school curriculum (17) • University of applied sciences curriculum (6) • Vocational school curriculum (2)	• Difficult content for students (45) • Student competence (28) • Student knowledge (23) • Learning process (18) • Misconceptions (10) • Affect (7) • Solving strategies (3)	• Ways to approach mathematical content in teaching (87) • Teaching methods (54) • Encouraging students (5) • Differentiation (2) • Answering student questions (1) • Correcting misconceptions (1) • Mathematical language and notation (1)

Table 2: Subcategories of pedagogical content knowledge distinguished in the diaries

In many diaries, the secondary school and university curricula were compared. Students discussed, for instance, the content of secondary school calculus courses and university analysis courses. The *knowledge of content and students* sections mainly focused on difficulties or misconceptions that school students may have. For example, affect and learning (e.g., emotions) were little discussed and the cognitive studies in mathematics education were mainly used as references. The *knowledge of content and teaching* focused on discussing different means of approaching mathematical content in teaching and different teaching methods. For example, no imaginary learning situations were introduced and only one student pondered the answering of school students' questions.

Overall, PCK was discussed more than subject matter knowledge. However, the PCK typically discussed in the diaries can be described as content-driven, as it was mainly placed in subcategories such as 'Ways to approach the mathematical content in teaching' or 'Difficult content for students'.

Discussion and conclusion

It is worth noticing that the results of this study cannot be generalised to whole student populations or other contexts. This study contributes only case-specific information, which can, however, be used in further development of the specific learning environment. In addition, the reliability of this study could be enhanced by using two independent researchers in the data analysis phase. Nevertheless, this study found that in this specific context, many prospective mathematics teachers adopted a *knowledge transforming writing* strategy in their learning diaries. The knowledge discussed in the diaries was somewhat focused on SCK and PCK. More specifically, the most discussed topics were representations of mathematical content, curricula, student knowledge and teaching methods.

Some of the learning diaries discussed both subject matter knowledge and pedagogical content knowledge. Some of the diaries, however, were more focused on pedagogical content knowledge. This seems to indicate that some of the prospective teachers emphasised pedagogical topics, while other prospective teachers discussed teacher knowledge more comprehensively. Further research would be needed to discuss this variation in detail and, especially, to compare students who are at different stages of their studies. In addition, it is notable that horizon content knowledge was rarely discussed in the diaries. This was somewhat surprising as the aim of the seminar was to connect the content of university-level mathematics and school mathematics. If HCK is understood as Jakobsen et al. (2013) have presented it, connecting mathematics as a discipline to school mathematics means discussing horizon content knowledge. In addition, some aspects of SMK and PCK (such as modifying tasks) also received little attention. This implies that learning diaries may lead to reflections that are not fully aligned with the intended learning outcomes. Further research is needed to determine whether more structured learning diary tasks would help students to better discuss desired sides of mathematical knowledge for teaching.

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The role of definitions on classification of solids including (non)prototype examples: The case of cylinder and prism

Mine Işıksal-Bostan[1] and Seçil Yemen-Karpuzcu[2]

[1]Middle East Technical University[1], Turkey; misiksal@metu.edu.tr

[2]Middle East Technical University[2], Turkey; ysecil@metu.edu.tr

The purpose of this study is to investigate prospective middle school mathematics teachers' (PMTs) formal definitions regarding cylinder and prism and the role of these definitions on their classifications of solids involving (non)prototype examples. Data were collected through questionnaire and interview protocol from three PMTs who were 3rd year students in one of the teacher education programs in Turkey. Data analysis revealed that PMTs mostly used inappropriate terminology in defining cylinder and prism. Despite their inappropriate definitions, one PMT could classify cylinders and non-cylinders among the groups of solids correctly. PMTs who could not classify cylinders correctly made incorrect inclusions regarding the cylinders and prisms.

Keywords: Definitions, solids, prospective mathematics teachers, middle school mathematics teachers.

Introduction

Definitions constitute the indispensable part of mathematical instruction. National Council of Teachers of Mathematics [NCTM] (2000) emphasized the importance of students' understanding of the role of mathematical definitions and students' use of definitions in mathematical practices. Researchers argue that definitions should have some critical attributes. According to Van Dormolen and Zaslavsky (2003), a good mathematical definition should have following necessary characteristics: criterion of hierarchy, criterion of existence, criterion of equivalence, and criterion of acclimatization. In addition to the necessary characteristics, criterion of minimality, criterion of elegance, and criterion for degenerations are classified as the preferred features for the good mathematical definitions (Van Dormolen & Zaslavsky, 2003). Additionally, Leikin and Winicki-Landman (2000) stated that definitions should have necessary and sufficient conditions for the given concept and they should have conditions that are characterized as minimal.

Beginning at early school years, students are encountered with geometric shapes and solid figures. Students' concept images that flourish during that years may support the concept definitions that will be introduced in later years. However, as Vinner (2011) argues, concept definitions and concept images are not always parallel and mismatch may arise between them. Keeping this in mind, educators should stimulate the development of concept images as early as possible that support the concept definitions (Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015). In other words, appropriate guidance should be supplied before those intuitions are rooted and difficult to alter (Tsamir et al., 2015). Hence, it is a well-known fact that teachers play important roles during the development of those concept structures. Shulman (1986) stated:

> Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted, why it is worth

> knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice. (p. 9)

To state differently, subject matter knowledge (SMK) was related to the structure of mathematics involving definitions, axioms, proofs, theorems and relationships among them (Shulman, 1986). Thus, teachers' knowledge on definitions, classified as one of the components of SMK (Ball, Thames, & Phelps, 2008) plays an important role during the development of students' conceptions of definitions. In addition, teachers' knowledge of mathematical definitions has an influence on didactic approaches in mathematics classes (Leikin & Zazkis, 2010; Zazkis & Leikin, 2008). In other words, teachers' understanding of the definition of concepts in addition to their concept images impact the teachers' way of representing the concept to the students, explanations given during the instructions, way of orchestrating the classroom discussion and their fluency and flexibility in teaching topics (Leikin & Zazkis, 2010). Thus, teachers' knowledge on definitions and to what extent they use these definitions in classifying solid figures deserves further attention for giving evidence for teachers' practices regarding the examples used in mathematics classrooms.

Moreover, an assertion was made that most of the time students do not know the definitions of mathematical terms that they need to use, and thus, they tend to memorize the given definitions without understanding (Edwards & Ward, 2008). Indeed, since stating the correct concept definition is not an indication for the understanding of the concept, critical understanding of definition is needed to apply the definition successfully. With this idea in mind, in this study as a continuation of our early work, we investigate PMTs' knowledge of definitions regarding the cylinder and prism and how their definitions influence their choice of classifications of solids involving prototype and non-prototype examples. Thus, the aim of this study was to answer the following research question:

How do PMTs define cylinder and prism, and how do their definitions influence their classifications of solids involving (non)prototype examples?

Framework of the study

In this research study, we used Leikin and Zazkis's framework developed for the analysis of teacher-generated examples of mathematical concepts (Leikin & Zazkis, 2010). According to the framework teacher-generated examples of definitions of the concepts were analyzed according to four components: 1) Accessibility, 2) Correctness, 3) Richness, and 4) Generality/Concreteness. Accessibility refers to the generation of examples with or without a prompt, whether they are generated easily or with difficulty. By means of generality, Leikin and Zaskis emphasize the use of specific properties pertaining the given concept apart from the general descriptions that belongs to the other class of concepts. In addition, correctness refers to the appropriateness of the logical statement generated for the mathematical concepts. While analyzing correctness of the examples, they make a distinction between appropriate and inappropriate examples. In other words, they classify examples of definitions as appropriately rigorous if the answers consist of necessary and sufficient conditions for defining the given concept and minimal use of correct mathematical terminology and symbols. On the other hand, the answers are classified as appropriate but not rigorous when there are some deficiencies in the definitions or imprecise language was used. When the given examples has deficiency in either satisfying necessary or sufficient conditions, they are categorized under the heading of inappropriate examples. For instance, while defining cylinder as a solid, one of the

necessary condition for the cylindrical surface is that it is generated by a straight line (generatix) which moves so as constantly to pass through a given curve (directix) remaining parallel to its original position (Beman & Smith, 1900). In addition, to have sufficient conditions, the curve should be a closed shape which designates the bases of the cylinder that are two parallel transverse sections (Beman & Smith, 1900). Therefore, cylinder is a portion of an enclosed space between two parallel bases created by a cylindrical surface. On the other hand, the definition of cylinder will be inappropriate unless the surface is composed of parallel lines. In addition, having a closed surface is a general description that belongs to other 3D solids and base should be a circle is not a necessary condition for a cylinder.

In the framework, richness refers to the number of different examples for a concept that are generated correctly by the participants. In the present study, we focus on cylinder and prism which are geometrical concepts and their examples have figural aspects. Thus, for analyzing richness of the examples, we considered Fujita and Jones's (2007) perspective on prototype examples since prototype images of geometrical objects and definitions of these geometrical concepts are related. Prototype example(s) of a concept are the examples that come first in persons' mind, and they also exist in the participants' concept images. To understand a geometrical object, the associated figure of this certain object would be active in mind. Thus, when a participant has only prototype examples of an object, she can give specific examples of an object with limited images (Fujita & Jones, 2007). On the other hand, when one has non-prototype examples of an object, she can generate different examples for an object. Therefore, in this research we accepted non-prototype examples as indicator of richness for the given definition. Considering the context of Turkish middle school mathematics curriculum (MoNE, 2013), hexagonal prism and rectangular prism are considered as prototypes of prism and circular cylinder is considered as prototype of cylinder. On the other hand, cylinder with non-circular base could be a non-prototype example. However, non-prototype examples are not given a place for exemplifying or defining the solids in Turkish curricula.

Method

In this study multiple case study approach (Yin, 2003) was used in which data were collected from three volunteer PMTs enrolled in a teacher education program in a public university in Ankara, Turkey. The rationales for selecting these PMTs are as follows: They took the methods of teaching mathematics-2 course, and they were already completed all the geometry courses offered in the program. Thus, they were supposed to give rich data about the definition of geometric shapes.

The data were collected by using a questionnaire and an interview protocol. Therefore, examples of the objects could be analyzed both in oral and written settings. The questionnaire involved items that asked participants to define cylinder and prism in their own words and to show the relationship between cylinder and prism, if any. After the implementation of questionnaire, semi-structured interviews were conducted with the participants using think-aloud method in order to get more information regarding the richness, correctness and generality of the participants' definitions. The semi-structured interview protocol involves questions on definitions of solids and classification of groups of solids as given in Table 1. More specifically, during the interviews participants were given back their questionnaires and they were asked to elaborate on their answers (E.g. Here is your definition for the cylinder. Do you want to change any wordings of your definition? Or will you keep the definition same?). Then, they were asked to name the classification of objects given in Table 1

and to express their reasoning behind this classification. In the Table 1, solids taken from Van de Walle, Karen, and Bay-Williams (2013) are numbered in each group. Group A and Group B involve cylinders with both prototype and non-prototype examples. Further, Group C and Group D involve prisms with prototype and non-prototype examples. Lastly, Group E involves general 3-D objects which are neither cylinders nor prisms.

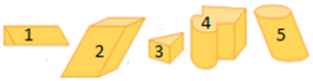
Cylinders	**Prisms**	**Not cylinders**
Group A 	Group C 	Group E 
Group B 	Group D 	

Table 1: Groups of solids used in the interview protocol (Van de Walle et al., 2013, p. 413)

The data obtained from questionnaire and interviews were analyzed using Leikin and Zazkis's (2010) framework and Fujita and Jones's (2007) definition of non-prototype examples. More specifically, to analyze generated definitions regarding the cylinder and prism, correctness, richness and generality dimensions of the Leikin and Zazkis's framework were used. While coding the data in terms of correctness criterion, we focus on whether participants' definitions of cylinder and prism satisfy the necessary and sufficient conditions and involve proper terminology. In the analysis of the generality of the definitions, if a participant's exemplification of definition corresponded only to cylinder/prism (e.g. circular cylinder, rectangular prism), it was named as specific. If the definition corresponded to any 3-D solid without critical attributes of cylinder/prism, it was named as general. In addition, in order to evaluate the richness of examples, Fujita and Jones's (2007) definition of non-prototype examples was used. In other words, richness of the examples are determined according to the non-prototype examples expressed/drawn by participants. Researchers analyzed the data until reaching a consensus on the categories of the definitions. Considering that definitions are conventional and contextual, the analysis of the present study was made based on Beman and Smith's (1900) definitions of cylinder and prism.

Findings

Analysis of PMTs' definitions and their classifications for the cylinder and prism are presented in Table 2. Participants' definitions of cylinder and prism were analyzed according to the correctness, richness and generality criteria. The analysis of PMTs' knowledge of definitions revealed that their definitions were categorized as inappropriate, which satisfy necessary but not sufficient conditions or neither necessary nor sufficient conditions. The dimension of richness included both prototype and non-prototype examples. Analysis regarding generality showed that participants' definitions included specific, partially specific, and general statements. Lastly, the classification of solids are given in Table 2. The symbols '+' and '-' showed that a PMT identified the classification of a group as cylinders, prisms or not cylinders correctly and incorrectly, respectively.

	Correctness		Richness		Generality		Classification		
	Cyl	Pr	Cyl	Pr	Cyl	Pr	Cyl	Pr	None
P1	Inapp. Not N not S cond.	Inapp. Not N not S cond.	Non-pro	Proto.	General	Specific	+	+	Non-cyl
P2	Inapp. With N but not S cond.	Inapp. With N but not S cond.	Non-pro.	Proto.	General	Partially Specific	-	+	Non-pr
P3	Inapp. Not N not S cond.	Inapp. Not N not S cond.	Non-pro.	Proto.	General	Partially Specific	-	+	Non-pr

Note: Cyl: Cylinder, Pr: Prism. P1: Participant 1, P2: Participant 2, P3: Participant 3, N: Necessary, S: Sufficient. Symbol +: participant correctly identified solids. Symbol-: participant do not correctly identified solids

Table 2: Correctness, richness, and generality in participants' definitions and their classification

Data analysis showed that P1 defined cylinder as "a 3D shape formed by union of two parallel surfaces with another surface that cover these two surfaces' surroundings" and then she expressed that "the surfaces do not have to be a polygon and one under the other". In her statements, the definition would be correct if "parallel surfaces" are taken to mean "parallel transverse sections enclosed by closed curves" and "another surface" is taken to mean "surface composed of parallel line segments". In addition, her use of "surfaces" terminology is not appropriate and lead misinterpretation. This is why her definition was categorized under the example of inappropriate definition with neither necessary nor sufficient conditions. In addition, P1's definition not only include improper terminology but also satisfies general conditions that belong to general 3D objects whose surface is not restricted with a cylindrical surface (e.g. vase). However, analysis of data showed that her drawing of cylinder included a non-prototype example, a cylinder with non-circular closed curve bases. While classifying the group of objects in Table 1, P1 correctly identified the examples of cylinders (group A and group B) which contain both prototype and non-prototype examples. While identifying them, she referred to the attributes that she mentioned while defining cylinder. In addition, she correctly identified Group E as examples of non-cylinders referencing inappropriate examples. For instance, by showing the object #6 she stated that "there is a space here [she showed the surface of the object #6] thus this group cannot be categorized as cylinder". However, she did not address this critical attribute in her definition of cylinder.

As for the prism, P1 defined it as "a 3D figure formed by combination of two polygon surfaces in two different planes with a surface that covers surroundings of that polygons". In her definition she did not mention critical attributes like bases and faces. This definition was also considered under the category of neither necessary not sufficient conditions. While classifying, she correctly identified the groups of prisms through referring visual characteristics and making inclusion as "they are prisms, a special case of cylinders since their bases are polygons".

P2 defined cylinder as "it is a 3D figure whose top and bottom bases are parallel and congruent closed curves" and then she expressed that "cylinder is formed by combining two identical, parallel and closed curves from corresponding points". In her statements, although she mentioned about some necessary conditions for the cylinder, they were not sufficient. For instance, she did not mention any information about cylindrical surface. Thus, those characteristics which are general and not specific to the cylinder can be valid for other 3D objects. In addition, the curve that bounds the base should

be planar. Therefore, definition was categorized under the examples of inappropriate definition with necessary but not sufficient conditions. Similar to P1, her drawing of cylinder consisted of non-prototype example which is a cylinder with non-circular bases. However, while classifying, she incorrectly identified the groups of cylinders. She identified objects (2, 3, 4, and 5) in group A as prisms based on their attributes. She also incorrectly identified #1 as pyramid. She stated that "Bases are congruent polygons and parallel. If I take the bases of #5 as polygon, I can say that it is also a prism". This incorrect identification was also related to her conception of "cylinder is [a special case of] prism". Besides, she identified group E as non-prisms, based on the assertion that they lack the critical attribute of prism as "their bases are not congruent", but did not refer to the cylinders.

P2 defined prism as "a 3D object that has top and bottom bases which are congruent and parallel and whose surface area is polygon". In her definition, while she mentioned some necessary characteristics of bases, she missed that the bases are polygons and overgeneralized faces to polygons rather than parallelograms. Therefore, the definition was categorized under the example of inappropriate definition with necessary but not sufficient conditions. While classifying, she referred to the critical attribute "prisms have top and bottom bases that are congruent and parallel polygons" and correctly identified prisms. Although she confused for a moment about whether object #4 in group D is a prism or not, she identified it as a prism by referring the concavity as "I can consider both concave and convex polygons. Can I think them as polygonal region? Ok, then, I considered polygon bases as both convex and concave and decide that they are prisms".

P3 defined cylinder as "It has non-polygon bases. It consists of two parallel bases which can be regular or not and combined to each other with infinite parallel lines". In her statements, she did not mention any information about the congruency of bases. Thus, those properties could be attributed to general properties of 3D solids that are not specific to cylinder. She unconventionally used the term "lines" rather than "line segments". In addition, she stated that the bases should be non-polygon. Therefore, her statements were examples of inappropriate definition with neither necessary nor sufficient conditions. In addition, she drew a non-prototype example of a cylinder with non-circular bases. While classifying the groups, she incorrectly identified solids in Group A and Group B as prisms. In this process, she considered cylinders as a special case of prisms and identified the objects according to their attributes. In other words, for Group A objects, she stated that "these bases are parallel and congruent. There are parallel lines between them. Thus, these shapes can be classified as prism" and for Group B objects she stated that "these (#1 and #3) are regular cylinders. Can we say whether #2 is non-regular cylinder? But we could not name them as cylinder unless bases are circles. But, we can admitted them as prisms". In addition, she classified objects in group B as both prism and cylinder since cylinder was included in prisms in her images. Moreover, she identified objects in group E as "non-classified solids and non-prisms". While explaining her answer, she referred to the critical attribute of prism that "bases should be congruent" and classified them as non-prisms.

P3 defined prism as "a shape formed by infinite parallel lines that pass through two parallel and congruent polygonal bases". Similar with P3's cylinder definition "lines" is not appropriate terminology and the definition is insufficient since critical characteristics of bases and faces are not stressed. Therefore, this example was considered under the category of inappropriate definition with neither necessary nor sufficient conditions. In addition, as an example of prism she drew a prototype

example that is a right hexagonal prism. While classifying, she identified prisms correctly by just referring to visual form as "they are all prisms" and not mentioning any critical attributes.

Discussion and conclusion

In this study, we first examined PMTs' knowledge of definitions of cylinder and prism through questionnaire and interview data. Then, we analyzed their classification of cylinders and prisms while identifying the 3D objects and how their identification of classes of objects was influenced by their informal definitions. The definition analysis showed that PMTs have difficulty in defining cylinder (Ertekin, Yazıcı, & Delice, 2014; Tsamir et al., 2015) and prism (Gökbulut & Ubuz, 2013) where they mostly used inappropriate terminology that refers to more general attributes of solids rather than attributes specific to cylinder and prism. More specifically, we deduced that PMTs have inadequacy in defining the (sur)faces of the cylinder and prism. This might be the result of their construction of concept definition that depends on the bases rather than cylindrical or prismatic surfaces.

The other finding was that even a PMT did not make a correct definition of cylinder, she identified group of cylinders and non-cylinders correctly. In other words, while PMT did not express some critical attributes that are specific for cylinders in her definition, she correctly identified the group of non-cylinders based on her correct inclusion of cylinder and prism. However, two participants identified cylinder as inclusion of prisms and thus classified non-cylinders (group E) as non-prisms. In other words, participants' misconception of "cylinder is a special case of prism" arouse from the prototype image of circular cylinder let them classify the solids incorrectly. Thus, PMTs' prototype images of cylinders might be an obstacle that influence their inclusion relations of cylinder and prism. On the other hand, PMTs could identified the group of prisms correctly by referring to the critical attribute of polygon bases in their definitions and making visual judgements. Thus, PMTs easily identified prisms compare to the cylinder. However, we could deduce that PMTs' preferences toward pedagogical considerations over a rigor definition inhibit their formal definitions and correct classifications of solid figures.

Based on the findings, some implications could be suggested to the teacher educators. The content of courses like methods of mathematics teaching and school practice offered in teacher education programs should be enriched with activities that demands identifying, explaining, defining, and classifying mathematical concepts from both pedagogical and mathematical perspective. By this way, PMTs could have the opportunity to analyze the critical attributes and non-prototype examples of the given concepts and establish the relationships among the concepts through examining general and specific characteristics. In addition, as mentioned above definitions are conventional. Thus, depending on the theoretical perspective 3D solids could be defined differently in different nations. Thus, further research studies could be conducted to compare and contrast definitions and their instructional practices in different nations.

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Preparing teachers for teaching: Does initial teacher education improve mathematical knowledge for teaching?

Dun Nkhoma Kasoka[1], Arne Jakobsen[2] and Mercy Kazima[3]

[1]University of Stavanger, Norway; dun.n.kasoka@uis.no

[2]University of Stavanger, Norway; arne.jakobsen@uis.no

[3]University of Malawi, Malawi; mkazima@cc.ac.mw

In this study, we examined whether the Malawian Initial Primary Teacher Education Programme develops teachers' mathematical knowledge for teaching [MKT]. We administered measures adapted from the Learning Mathematics for Teaching project to more than 1,700 pre-service teachers from eight colleges to measure their knowledge for teaching. Paired samples t-test using 725 pre-service teachers whose pre-test (M = .049, SD = .960) and post-test (M = .084, SD = 1.043) scores we have paired, showed no significant gains in knowledge overall (t(724) = -.808, p=.419), and in seven colleges individually. However, mathematical knowledge for teaching for pre-service teachers from college C2 had increased significantly (t(165) = -2.062, p = .041). While the results showed a significant correlation between pre-test and post-test (r = 0.544, p < 0.001), we fail to conclude based on the results that the initial teacher education improves mathematical knowledge for teaching.

Keywords: Primary teacher education, mathematical knowledge for teaching, number concepts.

Introduction

Primary school education sector in Malawi experienced a critical shortage of teachers due to increased enrolments following the introduction of free primary education in 1994 (Ministry of Education Science and Technology [MoEST], 2011). The increased enrolments in primary schools resulted into an exceptional demand for teachers. The government of Malawi responded by recruiting unqualified teachers and recalling retired qualified teachers as a temporary measure. In order to expedite training of the unqualified teachers, the MoEST developed a new teacher education programme called the Malawi Integrated In-service Teacher Education Programme [MIITEP] which begun in 1996/7 and remained until 2005/6 when it got replaced by the current Initial Primary Teacher Education [IPTE] programme. Unlike MIITEP, IPTE is a two year pre-service certificate teacher education programme designed to respond to reforms in primary school curriculum (MIE, 2008). During the first year, pre-service teachers are full time learning in teacher colleges with minimal supervised peer and micro-teaching. In the second year, the pre-service teachers are attached to primary schools for teaching practice with assistance of experienced mentors. To enrol into the programme, prospective students must possess a Malawi School Certificate of Education (MSCE) with a credit pass in English and a pass in Mathematics and one other science subject.

Despite the reforms that have occurred in teacher education since 1997 and curriculum shift to align it with the primary curriculum, pupils' performance and achievement in primary mathematics remain poor. For instance, Malawian pupils have fared poorly on surveys by the Southern and Eastern African Consortium for Monitoring Educational Quality (SACMEQ). In SACMEQ II, Malawi was ranked 13th in mathematics out of 14 Southern African countries (SACMEQ, 2010). The trend in pupils' achievements did not change when SACMEQ III was implemented, with Malawian pupils

being ranked 14th out of 15 participating African countries (SACMEQ, 2010). Efforts to explain the underperformance of Malawian pupils in regional surveys have ruled out use of English and increased enrolments as factors responsible for the trend because these factors were not unique to Malawi (World Bank, 2010). Following analysis of the SACMEQ results, the MoEST observed that most teachers did not have sufficient training and/or experience (MoEST, 2011). The analysis therefore recommended strengthening of pre-service and in-service teacher education.

In order to strengthen the current teacher education programme, we need to understand how it operates and identify weaknesses. Therefore, the current study as part of a larger study contributes to efforts to improve the IPTE programme by assessing the extent to which it develops pre-service teachers' mathematical knowledge for teaching. We therefore attempted to address the following null hypothesis: There is no significant change in pre-service teachers' mathematical knowledge for teaching after undergoing initial primary teacher education in Malawi.

Related literature

Issues surrounding teacher preparation are of growing concern in Malawi. In its evaluation report of the IPTE programme, MIE (2008) highlights the importance of basic mathematics and numeracy for teaching of primary mathematics. Through this evaluation and further analyses, MIE identified the following issues relating to IPTE and mathematics teacher education: (1) lack of motivation among trainees to learn mathematics and to become mathematics teachers, (2) mistakes and wrong information in the teaching and learning materials, (3) mismatch between syllabus and content in handbooks, (4) conflict between curriculum and assessment reforms, and foundation studies, and (5) missing illustrations for number operations in student handbooks. The MIE report concludes that these "…will eventually make the student teachers fail to understand what they are supposed to do and even fail to do the activities they are being asked to perform" (p. 54). We consequently observe that, if the goals of the reforms in teacher education are to be met, we must approach the assessment of the current (mathematics) teacher education in a multifaceted and holistic manner. This study is therefore part of that holistic approach.

Although several factors affect a teacher's effectiveness, Hill, Rowan and Ball (2005) argue that teacher's knowledge is one of the biggest influences on teaching and students' attainment. As a matter of fact, they have put forward evidence that show that teachers with sufficient knowledge produce significantly positive changes in their students' learning and attainment after controlling other variables that are believed to influence student achievement. Furthermore, Ball, Thames and Phelps (2008) state that there is nothing more foundational to teacher competency than knowledge of their subject matter. Their assertion concurs with Ma (1999) who contends that a deep understanding of fundamental mathematics affords a crucial base for effective and successful mathematics teaching. We agree with Ball et al. (2008) that the quality of mathematics teaching depends, largely, on teachers' mathematical knowledge. Research in teacher education has considerably changed the way in which mathematical knowledge for teaching development is understood. Traditionally, it can be thought that knowledge for teaching develops during and after formal teacher education. However, Grossman (1990) and Hill, Rowan and Ball (2005) suggest that knowledge for teaching develops from pre-teacher education experiences, teacher education experiences and teaching experiences. Hence our interest in pre-service teachers' mathematical knowledge for teaching before and after their initial teacher education.

Vigorous enthusiasm in knowledge for teaching research followed Shulman's (1986) work to identify, classify and define elements of knowledge for teaching. In mathematics education, the Learning Mathematics for Teaching [LMT] Project at the University of Michigan has made substantial contribution towards identifying and defining the type of knowledge necessary for teaching mathematics. This knowledge is today referred to as MKT (e.g. Ball, 2003; Ball et al., 2005; Ball et al., 2008; Hill et al., 2005). The LMT Project has shown that general mathematics ability does not entirely adjudge the knowledge and skills for effective teaching of mathematics. Ball and colleagues have also defined a special type of knowledge needed by mathematics teachers that is specific, distinct from pedagogy and knowledge of students, and not needed in other professional settings (Ball, et al., 2008; Hill et al., 2005). This is because the tasks of teaching mathematics require knowledge beyond ability to confidently perform algorithms (Ball, 2003; Ball et al., 2005). They argue that teaching mathematics demands of teachers to be able to, apart from thinking pedagogically, fragment mathematical reasoning which is not needed by other professions that use mathematics (Ball et al., 2008). This mathematical knowledge and skill unique to teaching is, in their view, specialized content knowledge (SCK). Conversely, common content knowledge (CCK) is the knowledge which enables an individual to succeed mathematically in terms of "being able to do particular calculations, knowing the definition of a concept, or making a simple representation" (Thames & Ball, 2010, p. 223) and is the knowledge also needed in other professions.

It is our considered view therefore that for the IPTE programme to prepare pre-service teachers for quality mathematic teaching, the programme must offer the teachers opportunities to develop their mathematical knowledge for teaching, among other things, because "improving the mathematics learning of every child depends on making central the learning opportunities of our teachers," (Ball, 2003, p. 9). The programme must embrace deliberate implementation approaches that develop SCK, in addition to CCK, of pre-service teachers within the teacher education cycle.

Design and methodology

The study examined whether pre-service teachers' knowledge for teaching number concepts and operations improved after completing two terms (about 6 months) of their initial teacher education in a pre-test–post-test design. Mathematical knowledge for teaching was examined using measures that were adapted from the LMT measures. The measures were piloted using a different group of pre-service teachers a year prior to the current study (Kasoka, Kazima, & Jakobsen, 2016).

Sample

Pre-service primary teachers were sampled from all the eight public teacher education colleges in Malawi. Only one of the colleges is a single-sex institution. The recruitment and posting of the student teachers to the colleges is centrally done by the MoEST. For the purpose of this study, we identify the colleges as C1 to C8. The pre-service teachers enrolled into the IPTE programme in September 2015 and had covered number concepts and operations in term one that runs from September to December of an academic year. The pre-service teachers had learnt how to teach number concepts and operations by the end of term one. Table 1 shows the composition of the sample.

Age/Gender	Teacher college								Total
	C1	C2	C3	C4	C5	C6	C7	C8	
<21 years	49	65	36	26	28	42	18	20	284
21 – 25	75	81	53	16	34	52	20	16	347
26 – 30	13	12	15	3	6	11	2	3	65
>30 years	9	8	0	0	3	3	4	2	29
Female	84	94	22	45	25	57	17	19	363
Male	62	72	82	0	46	51	27	22	362

Table 1: Pre-service teachers numbers by college

The instrument

For this study, we used adapted MKT measures for number concepts and operations from the LMT measures (see Kasoka et al., 2016). We used the previously validated measures to design an instrument comprising of two forms, namely Form A and Form B. The forms that we designed were intended to specifically measure the pre-service teachers' CCK and SCK of number concepts and operations. The final version of the instrument we administered had 67 items distributed between Form A and Form B. Form A had 38 items while Form B had 35 items. The two forms had six anchoring items.

Data collection and scoring

The first set of data was collected from the pre-service teachers in the third week (September 28 – October 2, 2015) of term one. The forms were re-administered during the second week (May 16 – 20, 2016) of the third term. Although number concepts and operations are covered in the first two terms, we deliberately collected post-test data at the beginning of term three so as to avoid the pre-test directly affected the post-test. We also noted that there are instances where first term material spills to term two due to some unforeseen circumstances in the colleges. To minimize the test-retest effect further, we swap the forms among the colleges such that the pre-service teachers that took Form A on pre-test, took Form B on post-test and vice versa.

The administration of the forms took place on different days of the same weeks as it was not possible for us to cover all the eight colleges on a single day. Each of the two forms was administered in four colleges. Permission to administer the forms was sort from college management prior to travelling to each college. The pre-service teachers completed the forms during class times without any personal incentives. However, they were briefed about the study and its objectives hence they participated willingly. The participants were allowed to work on the forms for a maximum period of 90 minutes. We had more than 1,700 preservice teachers at each administration. However due to other logistical hiccups, we have only been able to pair pre and post-test scores for 725 pre-service teachers. Hence the sample size, $n = 725$. It must be noted that some participants who took the pre-test dropped out of college before the post-test and some participants who took the post-test reported to college late. These missed the pre-test. Table 1 shows the number of pre-service teachers whose scores we have paired so far.

The two tests were scored simultaneously using an item response theory (IRT) software, BILOG-MG. IRT was chosen because of its robustness in analysis of item level data to measure inter-individual variation. For Hambleton and Swaminathan (1985), rich item level information extracted through IRT offers many advantages over classical test theory (CTT). Each of the pre-service teachers got a pair of IRT scores placing them along a standardized ability scale with mean of 0 and standard deviation of 1. The IRT scores were then entered into IBM-SPSS for further analysis.

Statistical analysis

Pre-test scores were used to calculate Cronbach's alphas to test internal reliability of the test. Form A appeared to have a good internal consistence, $\alpha = 0.733$ (Pallant, 2007). All the 38 items seemed worthy retaining since the greatest increase in alpha was only .002 after deleting either item FA8 or FA13. Form B had a lower reliability, $\alpha = 0.656$ and none of the item deletion increased the alpha value significant. To examine the change in mathematical knowledge for teaching between pre-test and post-test, a dependent samples *t*-test was used to compare means of the tests for all the colleges, and within each college. We also compared knowledge growth among the pre-service teacher in the eight colleges using one way analysis of variance (ANOVA). To achieve this, a new score we called knowledge growth was obtained for each participating teacher by calculating the differences between post-test and pre-test scores. ANOVA of the new scores was then carried out. Levene's test confirmed the assumption of homogenous variances, $F(7, 717) = 2.273$, $p = .049$.

	Teacher college							
	C1	C2	C3	C4	C5	C6	C7	C8
T	-.085	-2.062	.951	1.586	-.453	-.873	-1.705	1.590
p-value	.932	.041	.343	.120	.652	.385	.950	.120
Df	145	165	219	44	70	107	43	40

Table 2: Summary of *t*-test results

Results and discussion

A dependent samples *t*-test was carried out to test the hypothesis that the pre-IPTE mean ($M = .048$, $SD = .833$) and post-IPTE mean ($M = .960$, $SD = 1.044$) of mathematical knowledge for teaching are the same. Before conducting the test, the assumption of normality for the distribution was ascertain using Q-Q plots. We also observed that there was a significant correlation between pre and post-test scores ($r = .544$, $p < .001$). This suggests that a pre-service teacher with a high mathematical knowledge for teaching at pre-IPTE was more likely to have a high mathematical knowledge for teaching at post-IPTE. The two assumptions support our use of dependent samples t-test. Dependent samples t-test returned $t(724) = -.808$, $p = .419$ showing that the test was not significant. We therefore failed to reject the null hypothesis and concluded that the teachers' knowledge means before and after teacher education were not significantly different. We also carried dependent samples t-tests to test the null hypothesis using knowledge scores for each college. The results are summarized in Table 2.

Teachers' understanding of number concepts and operations is critical to quality teaching of primary mathematics (Hill et al., 2005; Ma, 1999). The poor achievement in primary mathematics and misconceptions are attributed to misunderstanding surrounding number concepts and operations, including counting, ordering, order of operations, associativity, commutativity, and fractions (e.g. Brombacher, 2011). It is critical that pre-service teachers understand and competently use basic number concepts and operations properties for them to effectively teach mathematics (MIE, 2008). The items used in this study were purposively selected to address specific aspects of number concepts and operations that can be considered prerequisite for the learning of school mathematics beyond literacy level. The items examined pre-service teachers' knowledge of whole number operations, subtraction of integers, representation and operations of fractions, decimal representations, prime numbers, and the order of operations. The results of data analysis show that pre-service teachers' knowledge for teaching these essential aspects of mathematics did not improve as a consequence of the participants undergoing teacher education. Similar results are observed from analysis of data from seven individual colleges. These results are surprising and not encouraging since the IPTE programme was developed on the premise of setting the foundations for formal schooling. However, at college level, we observe that mathematical knowledge for teaching number concepts and operations significantly improved among pre-service teachers from college C2.

We noted that the IPTE programme has some effect on pre-service teachers' mathematical knowledge for teaching. However, the effect varied among the colleges. ANOVA test was conducted to understand how the IPTE programme affected mathematical knowledge for teaching growth of pre-service teachers across the colleges. The ANOVA analysis yielded insignificant results ($F(7) = 1.808$, $p = .085$). This result shows that the means of knowledge growth were not statistically different among the eight colleges.

Implications

The IPTE programme should include aspects related to mathematical knowledge for teaching. These may include use of non-traditional lesson design, improved quality of activities, and classroom engagements of pre-service teachers during their college based education. This study suggests that mathematics teacher educators in teacher education colleges in Malawi ought to understand what mathematical knowledge for teaching is, why it is important, and how they can enhance its development among pre-service teachers. These results encourage teacher educators to network among themselves to learn from each other. The study found that one college had improved MKT. The other colleges can therefore learn from this college.

The significant growth of MKT for college C2 supports the notion that MKT can be developed and enhanced during pre-service teacher education. Mathematics educators in education colleges are encouraged to blend content, pedagogy and practice to develop pre-service teachers CCK and SCK simultaneously. However, the determination of what exactly went on at college C2 to improve MKT is beyond the scope of the current study but a possible direction for future research.

The result also point to pre-service teachers' very poor understanding of basic mathematics and hence their mathematics background. This may have explained the low levels of mathematical knowledge students had on pre-test and the inability for the knowledge to grow in six months. IPTE's minimum

entry requirements for mathematics teachers may need to be revised. Currently, all pre-services primary teachers study mathematics in college and are expected to teach mathematics on completion.

More research is required to identify, understand, and replicate variables affecting growth of mathematical knowledge for teaching in the Malawi context. Although this study is unique in that it implemented MKT measures on multiple colleges and compared knowledge growth, it was limited to MKT growth. There is need to clearly understand how this knowledge is developed and can be enhanced in Malawi. As we continue to make further attempts of pairing pre-test and post-test scores so as to increase our sample and understand how the current results would be affected, the study has been able show that growth in MKT is possible within the settings of the IPTE.

Conclusion

As more and more research reports highlight poor achievement in primary mathematics in Malawi (Brombacher, 2011; SACMEQ, 2010; World Bank, 2010), it is necessary to assess how teachers are prepared under the IPTE programme to teach mathematics. Research suggests that for student teachers to succeed in their profession, they must have sufficient mathematical knowledge for teaching mathematics. Research has also illustrated that student learning and achievement can be affected by teachers' knowledge for teaching and that this knowledge can be assessed (Hill et al., 2005). According to the framework suggested by the LMT Project, mathematical knowledge for teaching consists of mathematical content knowledge specific to the needs of task of teaching (SCK) over and above what is considered common for other professionals (CCK) (Ball et al., 2008; Thames & Ball, 2010). Therefore, our teacher education programme can meaningfully prepare teachers by ensuring that the pre-service teachers are provided with opportunities to develop both CCK and SCK. We can therefore argue that by assessing student teachers' knowledge for teaching mathematics, we can identify possible gaps that need to be addressed as a contribution towards the provision of quality mathematics teacher education in Malawi.

The current study found that pre-service primary school teachers' MKT did not improve after undergoing IPTE programme. We expected that students' MKT to positively change post-IPTE. However, mathematical knowledge for teaching for students from one college, C2, improved significantly relatively to its pre-test average but not to other colleges.

While the study findings are discouraging, they suggest the need for a review of the IPTE curriculum and its implementation so that it improves pre-service teachers' Mathematical knowledge for teaching

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A preservice secondary school teacher's pedagogical content knowledge for teaching algebra

Florence Mamba[1], Reidar Mosvold[1], and Raymond Bjuland[1]

[1]University of Stavanger, Norway; flothomo@gmail.com

This study investigated a preservice secondary school teacher's pedagogical content knowledge (PCK) for teaching algebra. Data were generated using video-recorded interviews and analysed using thematic analysis. Findings indicate that content knowledge was influential in the preservice teacher's PCK. Secondly, the preservice teacher, who was one of the best students in his class, displayed some knowledge of analysing students' errors and anticipating their possible misconceptions. He appeared to be familiar with some methods of handling students' errors and misconceptions. However, it appears that he was not prepared to apply such methods in his teaching of algebra.

Keywords: Pedagogical content knowledge, preservice teacher, secondary school, algebra.

Introduction

A body of research indicates that teacher knowledge influences the quality of their teaching and student learning (Hoover, Mosvold, Ball, & Lai, 2016). Although there appears to be general consensus that mathematics teachers need to know the content in ways that surpass the knowledge of educated people outside the teaching profession (Ball, Thames & Phelps, 2008), more research is needed in order to investigate the types of knowledge needed for teaching particular mathematical topics at particular levels (Hoover et al., 2016). From her review of literature on teaching and learning of algebra, Kieran (2007) suggests that researchers have barely begun to investigate the knowledge needed for teaching algebra. Some studies have contributed to this area of research. For instance, Bair and Rich (2011) investigated the development of specialised content knowledge for teaching algebra among primary teachers. In the present study, we contribute to the field by investigating pedagogical content knowledge (PCK) for teaching algebra in secondary school.

Our study was carried out in Malawi – a country in southern Africa that experiences severe challenges in the education system. Secondary school students' performance in national examinations continues to be poor with only around 50 percent of students passing end-of-cycle examinations (Ministry of Education Science and technology, 2008), and students' performance in algebra is poor (Malawi National Examinations Board, 2008-2013). While explanations have been proposed concerning system failure, (Ministry of Education Science and Technology, 2008), we suggest that further investigations of teacher knowledge as a potentially relevant factor of influence is necessary. As such, the aim of this study was to investigate a Malawian preservice secondary school teacher's PCK for teaching algebra. Possible implications are discussed.

Theoretical framework

Two constructs guide the theoretical framework for this study: mathematical knowledge for teaching (Ball et al., 2008) and algebraic thinking (Kriegler, 2007). Ball et al. (2008) distinguish between three sub–categories of subject-matter knowledge and thus extend Shulman's (1986) original category. *Common content knowledge* refers to a kind of mathematical knowledge and skill

that is used in settings other than teaching (Ball et al., 2008). *Specialised content knowledge*, on the other hand, refers to mathematical knowledge and skill that is unique to teaching. In addition, they present *horizon content knowledge* as a third category of subject-matter knowledge. For sake of simplicity, we focus broadly on content knowledge in this study. Ball et al. (2008) also distinguish between three sub-categories of PCK. *Knowledge of content and students* (KCS) is knowledge that combines knowing about students and knowing about mathematics. *Knowledge of content and teaching* (KCT) combines knowledge about teaching and knowledge about mathematics. Finally, Ball et al. (2008) present *knowledge of content and curriculum* as a third sub-category of PCK. In the present study, we focus on content knowledge and two sub-categories of PCK: KCS and KCT.

The second construct of the theoretical framework draws upon Kriegler's (2007) work on mathematical thinking tools of the algebraic thinking framework. Kriegler asserts that mathematical thinking tools are analytical habits of mind. They are organised around three topics: problem solving skills, representation skills, and quantitative reasoning skills. Teachers should be able to solve algebra problems using multiple approaches and problem solving strategies. They should be able to translate among representations and solve problems inductively and deductively.

Method

The study reported here is part of a larger qualitative study that investigates four preservice secondary school teachers' mathematical knowledge for teaching (see Mamba, 2016a; Mamba, 2016b). In previous publications, the first author reported on results from a task based interview that she conducted with one preservice secondary school mathematics teacher and video lesson observation for one lesson. In the current paper, we explore PCK for teaching algebra, considering the case of Dinga (pseudonym). Dinga was a diploma in education student in a three-year programme at a college of education in Malawi. When data for this study were generated, he was in the final year of study. Being a high-achieving student in mathematics in his class, he was considered an "information-rich" case for in-depth study (Yin, 2014). Video-recorded semi-structured interview was used to generate the data, allowing for multiple, in-depth rounds of analysis of the data (Girden & Kabacoff, 2011). Interview tasks reported here were adapted from a Malawi secondary school mathematics textbook (Gunsaru & Macrae, 2001) and were piloted before the main study.

The first author conducted the interview and transcribed the video recordings. These transcripts were analysed using a combination of inductive and deductive thematic analysis (Yin, 2014). Analysis of the interview transcripts involved identification of the PCK that Dinga displayed as he answered the interview questions. The themes that guided analysis were developed *a priori* from the theoretical framework. Themes for content knowledge were problem solving skills, representation skills, quantitative reasoning skills and justifying. Themes for KCS included predicting students' errors and misconceptions, understanding reasons for errors and misconceptions, and asking questions to reveal or understand students' reasoning and misconceptions. KCT was coded into methods of handling students' misconceptions and knowledge of instructional tasks to be used to enhance conceptual understanding and choosing instructional strategies. Some themes that were not in the theoretical framework were developed *a posteriori* from the data. The themes were further grouped into three categories: CK, KCS and KCT. To achieve credibility of the results, another researcher analysed the data. In all cases we got at least

90% agreement, with no discussion between the researchers. Furthermore, the findings were read and critiqued by other researchers.

Results and discussion

In the following, we present illustrative examples of results from the analysis of data from the video-recorded interview.

Content knowledge

During the interview, Dinga was asked to solve the following equation: $x^2 = 2x + 8$. He solved the equation using two approaches: factorisation method and the quadratic formula. When asked about other methods apart from the two he used, Dinga explained that he had forgotten the other method. His explanation showed that he had knowledge of completing the square, but he had forgotten how to use this method for solving quadratic equations. Dinga's solution processes revealed that he had procedural knowledge, since he did not display knowledge of the conceptual foundations of quadratic equations. For instance, when he solved the equation $x^2 = 2x + 8$ by factor method and quadratic formula, his difficulties of explaining the procedures indicated that he knew the "what" of the procedure but not the "why". For mathematics teachers, knowing both the "what" and the "why" of a procedure is important (Shulman, 1986). Dinga's knowledge of content therefore seemed limited in depth and breadth. Inability to solve the equation, using the other approaches, like completing the square and graphing, also indicated limitations in problem solving skills and representation skills. When solving the equation using the two methods he remembered, however, Dinga used rules of logic to come up with next steps in the procedures he used – thus indicating skills in deductive and quantitative reasoning.

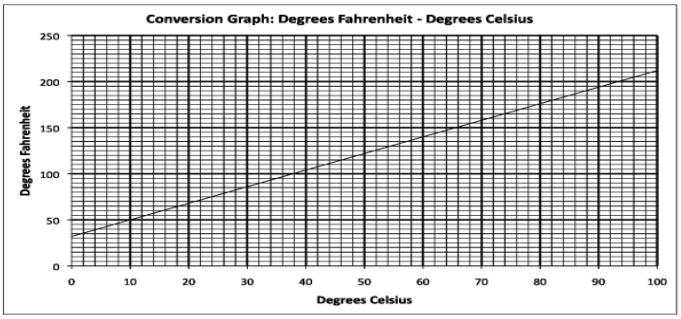
When answering questions in Task 2 (Figure 1), he did not attempt to interpret the graph first, although the interviewer asked him to do so. This also indicates limitations in content knowledge and algebraic thinking skills.

Knowledge of content and students

Dinga displayed knowledge of predicting methods that students may find easy or difficult, predicting students' errors and misconceptions, understanding reasons for misconceptions, and asking questions to reveal or understand students' reasoning and misconceptions. For instance, the interviewer asked Dinga to assume that he gave this equation $x^2 = 2x + 8$ to his algebra students to solve. When asked to explain what methods he thought his students would use, Dinga explained that his students would use factorisation because, to him, factorisation method is easy and the other methods are difficult. Our interpretation of Dinga's response is that he based his prediction of what students would find difficult on his own experienced difficulties.

The second task in the interview involved interpretation of a conversion graph between °C and °F (see figure 1 below).

Task 2: Suppose you give your students the following problem:
Below is a graph of an equation $Tf = \frac{9}{5}(T_C) + 32$ where $Tc =$ temperature in degrees Celsius and $Tf =$ Temperature in degrees Fahrenheit. Use the graph to convert the following temperatures:
(a) 10°C, 40°C and 77°C respectively; (b) 50°F, 100°F and 170°F respectively.



i. What error(s) may students exhibit as they answer this question?
ii. What underlying mathematical misconception(s) or misunderstanding(s) might lead the student to the error presented in this item? How might the student have developed the misconception(s)?
iii. What further question(s) might you ask to understand the student's misconception(s)?
iv. What instructional strategies and/or tasks would you use during the next mathematics lesson to address the student's misconception(s)? Why?

Figure 1: Conversion task

During the discussion extracted in the below excerpt from the transcripts, Dinga was first asked to read, solve and understand task 2. Then, the interviewer asked him to explain the errors and misconceptions that students might display as they attempt to answer this question.

Dinga: (...) They may misread the scale, or they may not understand the scale and they may come up with different values.

FM: Ummmm! Hummm!

Dinga: Some students don't know what horizontal axis is and what vertical axis is. So in this kind of problem, they can easily do the reverse.

FM: What misconceptions might lead to the errors you presented in this item?

Dinga: Sometimes, they put forward the belief that mathematics is difficult. So they may think that they may not manage.

FM: What else apart from the belief that mathematics is difficult?

Dinga: Teachers also contribute. If a teacher does not understand a topic, he/she does not teach it. He/she teaches a topic that is easy.

By explaining that students will misread scale and come up with different values from the expected values due to misunderstanding of scale, and that students may misallocate a point on the coordinate system, Dinga displayed ability to predict errors students may exhibit as they interpret that graph. However, when Dinga said, "if students do not understand scale, they would find some difficulties about how to come up with a conversion graph", he changed the purpose of the task from interpreting the graph to drawing the graph. Secondly, although Dinga suggested that graph interpretation is easier for Malawian students than drawing the graph, he did not interpret the graph himself, hence what he said contradicted with what he did during the interview. By explaining that graph interpretation is easier than drawing graphs, Dinga displayed limited knowledge of levels of

graph interpretation. According to Cursio (1987), graph interpretation is a cognitive task involving three levels of understanding, namely reading the data, reading between the data and reading beyond the data. Dinga's understanding of these levels may influence students' development of graph interpretation abilities. Thus, Dinga displayed limited ability to predict students' difficulties about interpreting a linear graph – possibly because he struggled to interpret the graph himself. Failure to interpret the graph was also an indication of limitation in representation skills. He was unable to interpret information within a representation. We noted that while algebra students often create graphs from equations, they rarely practice creating equations from graphs and interpreting the graphs – hence Dinga's difficulty.

When asked to explain the misconceptions that may lead to the errors mentioned, Dinga pointed out beliefs and teachers. By explaining that teachers also contribute to errors and misconceptions, Dinga proposes that some misconceptions originate from experiences in school – students' interaction with teachers being one of the experiences. However, his responses indicate that he did not know which causes the other. He identified the errors but was not in a position to understand the misconceptions that could be possible causes of such errors. Instead, Dinga explained the causes of misconceptions. This confusion might have resulted from lack of understanding of errors, misconceptions and their causes. Understanding of each is important for the teacher, because, as with the weeds, the roots must be tackled if the weeds are to disappear. Similarly, teachers must deal with the causes of errors and misconceptions to help students overcome them.

To understand or reveal students' reasoning and misconceptions, Dinga explained that he would ask the students about the meaning and interpretation of scale, and he would identify the misconceptions from their responses. He also explained that he would ask the students to tell him what the horizontal and the vertical axes represent, because some students do not know what horizontal and vertical axes represent. By explaining that students confuse between horizontal and vertical axes, Dinga displayed understanding of students' confusion between independent and dependent variables. Asking him to give examples of probing questions that he said he would ask, we expected that he would mention questions like "explain your answer", "why do you think so?", "How did you get that?" Dinga did not suggest such questions. We interpret this as an indication of limited knowledge of what questions to ask in order to identify misconceptions.

Knowledge of content and teaching

The KCT that Dinga displayed included predicting strategies for teaching how to solve the quadratic equation $x^2 = 2x + 8$, and strategies for handling students' errors and misconceptions. When asked how he would teach solving quadratic equations using the equation $x^2 = 2x + 8$, Dinga explained that he would give them steps to follow for them to come up with the roots of the equation. The procedural knowledge he displayed when solving the equation might have influenced this response. Dinga also explained that he would ask volunteers to solve the equation on the chalkboard in any way they want in order to understand the way students understood the problem. In this case, Dinga decided to use a hybrid of teacher centred and student centred teaching methods. Although, this might be an improved version of teacher centred strategies, Dinga revealed knowledge of teaching the "what" but not "why". Thus, Dinga displayed lack of an important aspect of subject matter knowledge: "knowing why". Even and Tirosh (1995) argue that "knowing that" is not enough, and they suggest that "knowing why" enables the teacher to make better

pedagogical decisions. It could thus be expected that Dinga lacks the knowledge necessary for teaching equations effectively to his students. In the closing statements of the interview, Dinga explained his limitations with lack of teaching experience.

In order to handle students' errors and misconceptions, teachers need to use appropriate strategies to create moments of cognitive conflict and help students resolve the conflict (Sayce, 2009). Dinga pointed out that he would use discussion and group work. The transcript below illustrates this.

FM: What instructional strategies would you use to address the misconceptions you have mentioned?

Dinga: I will use discussion, grouping, … (silence).

FM: How could these methods work in addressing these misconceptions?

Dinga: If I put my students in groups of 5 or 10, those who understood may assist others.

FM: Um! Humm!

Dinga: I will also give a summary of the topic so that students can easily correct their mistakes.

FM: How else can you address the misconceptions?

Dinga: Maybe giving my students a lot of exercises to do on the topic which they have problems so that the students understand the concept.

The results in the transcript above concur with Sayce (2009) who asserts that, to induce cognitive conflict, teachers should encourage collaborative working, especially in mixed ability groups. However, Dinga did not reflect on any features of group work that may facilitate this. He also did not reflect on the possibility that cognitive conflicts might be introduced by peers, but he seemed to suggest that the less able students could be passive recipients of the other students' explanations. We also argue that since errors and misconceptions are deeply rooted erroneous conceptions, summarising content covered in a lesson or giving students more exercises might neither be the root to cognitive conflict nor the way out of the conflict. The possible cause for Dinga's limitations in methods of handling students' errors and misconceptions might be that he did not solve the mathematical task (task 2) – possibly because he seemed not to know what it takes to interpret graphs (Cursio, 1987). A teacher needs to be able to solve the mathematical task before presenting it for the students in the classroom. Solving the task enables the teacher to anticipate students' solution methods, errors, misconceptions and questions to ask the students.

Conclusion

In this study, we investigated the PCK for teaching algebra displayed by one preservice secondary school teacher – Dinga. The findings reveal that content knowledge was the overriding determinant of Dinga's KCS and KCT. For instance, Dinga explained that he would teach solving the equation $x^2 = 2x + 8$ by giving the students a procedure to follow to solve the equation, probably because he lacked the "why" of the procedure and he solved the equation likewise. He also had some difficulties predicting students' errors and misconceptions because he was unable to interpret the graph. These findings support the fact that PCK involves knowledge and skills that are highly interrelated to each other (Even & Tirosh, 1995). Thus, teachers should possess in-depth content

knowledge, have a rich repertoire of teaching strategies to promote students' understanding of a particular topic and to understand and handle students' errors and misconceptions (Kilic, 2011). The findings also support the fact that preservice teachers possess limited PCK (Kilic, 2011). Although Dinga appeared to be familiar with some methods that can be used to handle students' errors and misconceptions, he seemed to lack conceptual approaches to students' errors and misconceptions because of his limitations in content knowledge and the "why" of his ideas.

It is also worth noting that Dinga's responses to the interview questions might be influenced by several factors. Firstly, it might be that Dinga might be able to solve equations but might not be able to explain his ideas. The fact that the interview tasks were adapted from a textbook might also limit Dinga's explanations and responses during the interview if the textbook was not familiar to him. But I gave the preservice teachers the textbooks a week before the commencement of the data collection for them to study in preparation for the tests and interviews. Furthermore, Dinga's responses might be influenced by the secondary school mathematics curriculum he went through as a mathematics student. The mathematics curriculum did not give students opportunities to explain their reasoning during mathematical problem solving. The way mathematical problem solving is handled in preservice teacher education in Malawi might also influence the results from the interview with Dinga. For instance, there is less practical work on mathematical problem solving in teacher education, yet the preservice teachers are expected to teach their students how to use the approach during teaching practice.

The results from this study cannot be generalized to all preservice teachers at Dinga's institution. Further research with a larger sample needs to be carried out to find out whether results from this case study are generalisable at a large scale. In addition, the measurement of Dinga's PCK was somewhat constrained due to the limitation of not having access to classroom students. In his responses, Dinga had to 'work' within a hypothetical situation. While his limitations in PCK were more evident during the task-based interview, it is possible that additional evidence of PCK could be obtained through a variety of problems and video lesson observations. These results, however, help us to learn that developing PCK and problem-solving skills among preservice secondary teachers by "providing teachers with opportunities to learn mathematics that is intertwined with teaching" (Hoover et al., 2016, p. 12) is crucial for mathematics teaching and learning.

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Analysis of mathematics standardized tests: Examples of tasks for teachers

Francesca Martignone

Università del Piemonte Orientale, Italy; francesca.martignone@uniupo.it

This paper discusses examples of tasks for teachers proposed during some laboratory activities developed in two Italian teacher education courses for middle school mathematics teachers. Following a common script, the teachers carried out and shared an a priori analysis of some items selected from the Italian National Standardized Assessment. The tasks fostered the reflection on epistemological, cognitive and educational aspects: the teachers reflected on the mathematical contents involved, the link with the National Guidelines, and the possible students' answers. The specific tasks and the laboratory approach make these activities well suited for showing and improving some aspects of teacher knowledge and skills.

Keywords: Teachers' knowledge and skills, teacher education, tasks for teachers, laboratory activities.

Introduction

For some years now, the research on Teacher Education has been concerned with tasks for teachers. Drawing from specific research about the knowledge used by teachers in their work (Shulman, 1986; Ball, Thames, & Phelps., 2008) and about the features that make a task well suited for developing that knowledge (Suzuka et al, 2009), this study presents examples of *meta-didactical praxeologies* (Aldon et al., 2013), which show the ability of the teachers to intertwine the different knowledge that can characterize them as mathematics teachers. The paper discusses a laboratory activity carried out during two teacher education programs for middle school teachers, in which we can identify tasks that aim at bringing out teachers' knowledge and skills and at improving their ability to dynamically relate these knowledge. Teachers carried out *a priori* analysis of items on number line, selected from Italian mathematics standardized tests. These items are assessment tasks but they were chosen and analyzed individually as mathematical problems that could be also become part of learning activities. The fact that they were selected from national tests gave also the opportunity to take into account the statistical results on the national sample. Teachers reflected on the mathematical contents, aims, and students' possible solution strategies, errors and misconceptions. In order to do that, they used different mathematical content knowledge. These activities, by means of shared analyses and collective discussions, put into light some professional skills of the teachers: in particular those related to the use of their *Subject Matter Knowledge* and the *Pedagogical Content Knowledge* (Shulman, 1986; Ball et al., 2008) and to the management of the links between these knowledge.

After a brief presentation of the background and the theoretical framework, the paper discusses examples of activities, tasks, and teachers' reflections developed in two teacher education programs, which involved in-service and pre-service middle school teachers.

Background and theoretical framework

Making the cultural background explicit is fundamental to understand the goals of a teacher education course (Bartolini Bussi & Martignone, 2013). Therefore, it is important to present briefly the background of the activities analyzed in this paper. In fact, the choices made in the design of the activities are due to factors that are also linked to the Italian context: e.g. the choice of analyzing items taken from the Italian National Standardized tests (http://www.invalsi.it/invalsi/index.php). The teachers are interested in the discussion about these items, because all their students faced or will face these tests. For each item the teachers could refer to the quantitative results coming from the surveys on the national sample (these results are annually reported and they are public as well as the test items). Another factor that justifies the choice of analyzing these items is that the teacher educator was involved in a research on Italian national standardized mathematics tests (Branchetti et al., 2015; Lemmo, Branchetti, Ferretti, Maffia, & Martignone, in press) and she wanted to discuss the results of this study with the teachers. The teachers and the teacher educator shared specific *meta-didactical praxeologies* (Aldon et al., 2013) about the analysis of standardized test items. The term "meta-didactical" denotes that the praxeologies shared during the courses deal with the actions and the reflections of teachers about the educational activities. The reflective actions can be fostered by a particular *praxis* that includes different kinds of tasks (in this case the task for teachers about the *a priori* analysis) as well as *techniques* available to face them (e.g. the development and sharing of a common script that takes into account institutional aspects, the mathematical contents involved, the link with the National Guidelines, and the possible students' answers and mistakes). The *meta-didactical praxeologies* change over time because of the dialectical interactions between the researchers and the teachers communities (Martignone, 2015). This work would lay the foundations of the growth of a *community of inquiry* (Jaworski, 2003) in which the teachers and the teacher educator can share and develop their knowledge. There are aspects of teacher knowledge that many researchers agree on as been characteristic of the teacher knowledge, such as the *Pedagogical Content Knowledge*: "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9). There is a wide literature that started from the Shulman idea about the knowledge for teaching. In particular, the studies on the *Mathematical Knowledge for Teaching* (MKT - Ball et al., 2008) propose a refinement of Shulman's classification of *Content Knowledge*. Ball and colleagues try to define a so called *Specialized Content Knowledge*: "the mathematical knowledge and skills unique to teaching" (Ball et al, 2008, p. 400). As the authors write (Ball et al, 2008, p. 403), often it is difficult to discriminate *Specialized Content Knowledge* (SCK) from the *Common Content Knowledge* (CCK) and from the *Pedagogical Content Knowledge* (PCK), but there are examples in which we can see their different aspects.

> "[…] for example, consider what is involved in selecting a numerical example to investigate students' understanding of decimal numbers. The shifts that occur across the four domains, for example, ordering a list of decimals (CCK), generating a list to be ordered that would reveal key mathematical issues (SCK), recognizing which decimals would cause students the most difficulty (KCS[1]), and deciding what to do about their difficulties (KCT[2]), are important yet subtle" (Ball et al., 2008; p.404)

[1] Knowledge of Content and Students.

By means of specific tasks and methodology, some aspects of teachers knowledge and skills can emerge during teacher education programs. Therefore, the tasks for teachers are an important part of teacher learning: they can include "the mathematical prompts, many of which may be classroom tasks, that are used as part of teaching learning" (Watson & Sullivan, 2008; p.109). With regard to the tasks used to develop the MKT, Suzuka and colleagues (Suzuka et al, 2009) make a list of features that make a task well suited for developing MKT:

- "Creates opportunities to unpack, make explicit, and develop a flexible understanding of mathematical ideas that are central to the school curriculum
- Provokes a stumble due to a superficial "understanding" of an idea
- Opens opportunities to build connections among mathematical ideas
- Lends itself to alternative/multiple representations and solution methods
- Provides opportunities to engage in mathematical practices central to teaching (e.g., explaining, representing, using mathematical language, analyzing equivalences, proving, analyzing proofs, posing questions)" (Suzuka et al., 2009; pp.12−13)

These features are linked to the analysis of some examples of tasks (Suzuka et al, 2009). There are shareable critics (Flores, Escudero, & Carrillo, 2013) about the fact that the tasks described can also be faced by individuals who know the topic and, therefore, these knowledge and skills are not identifiable as specialist of the teachers: according to Flores and colleagues there is not sufficient evidence to guarantee that the knowledge, labelled SCK, is exclusive to mathematics teachers. Even if I agree with the critics on these tasks, the list of features about the MKT tasks presented by Suzuka and colleagues seems to well identify some aspects of tasks that characterize the work of a teacher.

This paper will present some examples of tasks for teachers that can be suited for showing and improving the mathematical knowledge for teaching. The methodology is also important in raising the teachers knowledge: in the activities that will be presented, the teachers are involved in laboratory sessions in which they work in small groups, discuss and share their reflections and analysis.

The teacher education program

This paragraph discusses some examples of educational activities carried out in the "Didactics of Mathematics" courses for middle school teachers involved in two Italian post-degree programs for the achievement of the teaching credential: the Active Internship (in Italian "Tirocinio Formativo Attivo" -TFA) and the Special Teaching Certificate Course (in Italian "Percorso Abilitante Speciale" -PAS). These programs follow the indications of the Ministry of Education and they are established by universities. The first one is attended by pre-service teachers and the second one by in-service teachers. The author of this paper is the teacher educator/didactician. During the teacher education programs, the teacher educator had the opportunity to work with teachers and to discuss with them some theoretical tools to analyze tasks from the Italian National Mathematics Standardized Tests. These tests are administered at the end of the school year in grades 2-5-8-10 (grade 6 was involved from 2010 to 2013). Only for grade 8 (the end of middle school in Italy), the

[2] Knowledge of Content and Teaching.

test is part of the national final examination, so it contributes to the final assessment of the students. The items aim at assessing students knowledge and skills identified in the Italian National Guidelines.

The teacher education courses analyzed involved 72 in-service teachers and 25 pre-service teachers. The activities followed a laboratory approach in a Vygotskian perspective (Bartolini Bussi & Mariotti, 2008): the teachers learn by doing, seeing, imitating, and communicating with each other within the course community. In these courses, the teachers and the teacher educator shared, discussed and reflected on their *a priori* analysis of mathematics problems, focusing on the institutional, epistemological, cognitive and didactical aspects.

The laboratory activity consists of five phases. In this paper we discuss the first three phases that deal with the *a priori* analysis of the items and that are the same for pre-service and in-service teachers. The last two phases involved only the in-service teachers, because these activities consist in the implementation and analysis of classroom activities carried out with their own students.

At the beginning of the activity, the teacher educator shows an *a priori* analysis of some Italian National Standardized Test items according to a common script (Martignone, 2016). It starts with the analysis of what is necessary for the students to know and how to do it. Therefore, the focus is on the epistemological and institutional aspects. The question is: what is significant from the point of view of the teaching and learning of mathematics? This starting point is necessary, but it is not an innovative task in a teacher education program. It is an activity that all teachers commonly develop. The next step consists in foreseeing the students' possible solution strategies and mistakes. This type of fine grain analysis is common in the educational studies in mathematics, while the teachers more often discuss the student performance after an activity. Afterwards, the teacher educator presents the national sample results about these items, adding also some examples of students' answers collected in the study carried out by her research group (Branchetti et al., 2015). By means of the analysis of these data, during the subsequent collective discussion, the hypothesis shared in the *a priori* analysis are confirmed or rejected. In this first part, the *meta-didactical praxeologies*, related to the analysis of standardized test items, begin to be shared.

The activities of the teacher education course

In the second phase of the course, the teachers, divided into small groups, carry out on their own an *a priori* analysis of other items selected from the Italian National Assessment. During these laboratory sessions, teachers compare and discuss their works and the teacher educator provides other information about possible students behavior, by also quoting some educational research results collected in the literature.

The tasks for teachers

This is the task proposed to the teachers: "Carry out an *a priori* analysis of the item, following the common guidelines shared: (i) the mathematical contents, the necessary skills to face the item, and the links with the Italian National Guidelines; (ii) the possible students solution strategies; (iii) the possible mistakes and difficulties; (iv) the strengths of the task; (v) the critical aspects of the task; and (vi) some proposals for changes".

Figure 1 shows an example of an item analyzed. This item was administered to grade 6 students in 2011, but there were many similar items also in grade 8 tests.

Place the following numbers on the line:

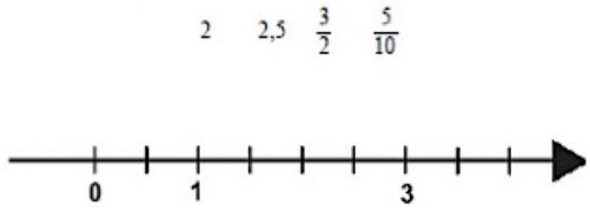

Figure 1: Item D8 administered in grade 6 (2011)

The in-service teachers have much more experience in predicting students' possible behaviors, but also pre-service teachers manage to foresee the most common mistakes. The written analyses (word documents or Power Point presentations) were shared on the Course Moodle platform.

Some excerpts from the teachers' works

This paragraph presents some excerpts (author's translations) from the *a priori* analysis carried out by teachers on item D8 presented above. The first excerpt shows an example of how the teachers wonder about what the item actually assesses.

> "The item assesses simultaneously two different skills: to know how to transform numbers from one representation to another, and to place them on the number line. Because a student may be able to carry out only one of these two actions, he/she gives a wrong answer. Therefore, it is not possible to understand (from the final result) in which part of the process the student made the mistake".

The following are examples of students possible difficulties foreseen by the teachers.

> "If students consider only the numerators of the fractions (3 and 5), the way in which the numbers appear in the stem might suggests that they are already written in order"
>
> "The students do not manage the symbol of fraction: they consider only the value of the numerator (or denominator); they put the fraction close to the value of numerator (or of the denominator)"
>
> "The student pinpoints 5/10 counting five hash marks for two possible reasons: he/she makes the straight line longer to obtain ten hash marks and then he/she considers five of these; or he/she considers only the value of the numerator and counts five hash marks".

The teachers have to read the colleagues' works in order to be ready for the next collective discussion, that is orchestrated by the teacher educator.

Collective discussion

All the hypotheses about the possible students behavior are collectively discussed. The student behaviors in these types of items are amply documented in the literature. The teacher educator refers to the studies summarized in the Encyclopedia of Mathematics Education (2014) and to the theoretical framework of the study on the Italian National Mathematical Standardized Tests carried out by her research group (Lemmo et al., in press). All the research papers quoted (or translations of

parts of them) are shared on the platform of the course. Because the items analyzed are taken from Italian national standardized tests, also the quantitative results could be taken into account. The national sample results concerning item D8, administered in 2011 in grade 6, are: 85% of the students gave the wrong answers, 11% the correct one and the remaining 4% either did not answer nor the answers made no sense. The questions faced by the teachers are: which could be the reason for this low percentage of right answers? When all the numbers are well positioned, the item is considered correct; il instead the students make mistakes, then how can the teachers identify the students' difficulties? As a matter of fact, based on the statistical data, we cannot know which mistakes the students made, but only the percentage of wrong answers. For this reason, during the discussion the teacher educator shows some examples of students' answers collected and analyzed in her research project: she wants to intertwine quantitative and qualitative analyses. As we can see in the following examples (Figures 2-3-4), we find the difficulties and the mistakes foreseen by the teachers (showed in the previous paragraph). The teacher educator supports the teachers' interpretations, by quoting some results from educational studies in Mathematics concerning the placing of rational numbers on the number line (Lemmo et al., in press). In fact, many mistakes can be generated by the interlacement of misconceptions about rational numbers and number line management: e.g. some students write the numbers in each hash mark without considering the unit of measure and place the fractions considering only the numerator (Figure 2). In this specific case we are not able to know if the students only copy the numbers as they are presented in the stem.

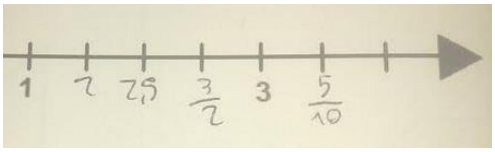

Figure 2

Other students show difficulties in placing fractions on the number line: they consider 3/2 equivalent to 3+½, or to 3.2. The same mistake could justify 5/10 put over the end of the line (Figures 3-4).

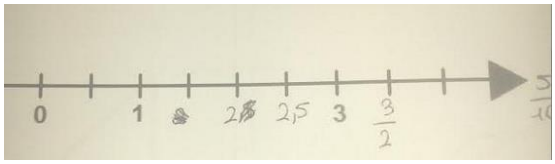

Figure 3

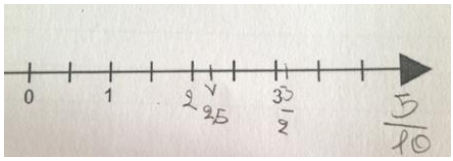

Figure 4

At last, the teachers design new items in order to investigate specific questions raised from the collective discussion. They ask themselves: what would happen if we change/add some numbers? Which ones? Why? What would happen if we change the unit of length or the numbers already written on the number line? Some examples of the proposals for modifications are: to change the order of the numbers in the stem (e.g. to alternate fractions and decimals numbers and do not write 2.5 after 2); to make the number line longer to see where, the students who make a mistake, would place 5/10; to add more numbers (e.g. ½) , etc.

Conclusions

This paper discusses some tasks for teachers carried out during laboratory activities for middle school teachers. The teachers and the teacher educator share praxeologies concerning the analysis of some items selected from the Italian National Assessment. During the laboratory activities, the teachers analyze and discuss the mathematical contents involved, the link with the National Guidelines, the possible students answers and mistakes, and propose changes in the tasks. The quantitative data of the national sample are taken into account and the teacher educator also provides results from different educational studies collected in the literature. The *shared praxeologies* (Aldon et al., 2013) are made by the intersection of the didactician's knowledge of theory, research and systems and the teachers' knowledge about subject matter, students and school: elements of mathematics, didactics and pedagogy are intertwined. The *meta-didactical praxeologies* (Aldon et al., 2013) presented highlight how some aspects of the MKT can emerge in the analysis of mathematical problems carried out by teachers. The actions that characterized the teachers' activity and their knowledge and skills are: to imagine many possible different solution strategies; to interpret different representations and students' solutions; and to analyze the mistakes thinking about the possible causes. The paper shows how these knowledge and skills can emerge and be improved during a teacher education program by means of specific tasks (e.g. *a priori* analysis of math problems) and methodologies (e.g. laboratory activities that involved teachers and researchers). The tasks presented aim at developing an interlacement among the teachers' *Subject Matter Knowledge* and *Pedagogical Content Knowledge* (Shulman, 1986; Ball et al., 2008): i.e. they highlight the different aspects related to the knowledge of the subject matter to be taught, to the curriculum, and to the possible behavior of students and their difficulties related to specific mathematical content. The ability of the teachers to dynamically relate and intertwine the different content knowledge (among that there are fuzzy boundaries) characterizes their professional skills and therefore it can be identified as typical of the work of teachers. A further development of this research will be the analysis of the actual teachers' actions in the following design and implementation of educational activities (phases 4 - 5 of the teacher education program).

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An exploration of challenges of engaging students in generative interaction with diagrams during geometric proving

Lisnet Mwadzaangati

University of Malawi, Malawi; University of Stavanger, Norway; mwadzaangatilisnet@yahoo.ie

This paper reports from a single qualitative case study which investigated challenges that might arise in a geometric proof lesson involving generative interaction with diagrams. Data consists of video recordings from a 120 minute lesson, taught with a focus on geometric proof in a Malawian grade 11 secondary school classroom. A post-lesson interview was also conducted with the teacher soon after the lesson. The findings indicate that the teacher faced challenge of lack of time, hence he did not complete some planned activities. The students faced the challenge of lack of understanding of the problem. As a result, they were unable to devise a correct plan during their initial involvement in the generative activity. The findings suggest that both the teacher's and students' challenges would have been avoided if problem solving was conducted appropriately.

Keywords: Geometric proving, generative interaction with diagrams, empirical activity, geometric diagrams, problem solving.

Introduction

Malawi National Examinations Board (MANEB) chief examiners' reports show that students fail to construct geometric proofs because they do not interact with diagrams successfully (MANEB, 2013). The report attributes students' failure to construct geometric proofs to lack of teacher knowledge for engaging students in successful proving activities. The reason suggested by MANEB supports the argument that teachers are responsible for engaging students in activities that involve interaction with diagrams during proof construction (Herbst, 2004). In geometry, to prove means to construct a sequence of argumentation from X (hypothesis) to Y (conclusion) with supportive reasons (Cheng & Lin, 2009). Herbst (2002) describes the work of geometric proving as a didactical contract between a teacher and his/her students. In this contract, the teacher's responsibility is to provide the problem statement and a diagram which contains the givens and the unknown. The students' responsibility is to develop logically connected statements from diagrams by making appropriate geometric interpretations and relationships. Herbst (2004) proposed four distinctive ways of thinking about how students interact with geometric diagrams during proving. These are empirical, representational, descriptive, and generative modes of interaction. Empirical interaction supports hands on geometry in the sense that the student is free to make a variety of operations on the diagram (measuring, looking at, and drawing in the diagram). Representational interaction supports abstract geometry, in the sense that the student is restricted by prescribed rules when making operations on the diagram. In descriptive interaction, the diagram contains features like marks and labels. Students use the features to complete a proof. Herbst (2004) argues that the availability of features in a descriptive diagram reduces students' responsibility for producing the proof and portrays situations of doing proof as only learning of good logic rather than discovering of new mathematics. The author therefore proposes that teachers should engage students in generative activities by using generative mode of interaction with diagrams during geometric proving. In this mode of interaction with diagrams, students are authorised to anticipate operations and results on

the diagram. Depending on the given information and the anticipated result, students can add features like drawing in lines and labels into the diagram. This means that generative interaction with diagrams is supposed to involve exploratory teaching strategies. Despite acknowledging that it is not easy to involve students in generative activities, the studies by Herbst have not focused on clarifying the challenges that might arise during such activities. Likewise, to my knowledge, no study has been done in Malawi to examine challenges of involving students in different modes of interaction with diagrams during geometric proving. Therefore this study has both a local relevance to teaching of geometric proving in Malawi and general relevance to the field of teaching of geometric proofs. The study addresses the following research question: What are the challenges of engaging secondary school students in generative interaction with diagrams during geometric proving?

Theoretical framework

Several investigations have been conducted with an aim of addressing challenges of teaching and learning geometric proof construction. Studies by Jones and his colleagues aimed at developing strategies for teaching proof construction with focus on helping students to understand the proof and appreciate its discovery function (Ding & Jones, 2009). The focus on value of geometric proofs arose because the authors found that there were some students who were able to construct proofs but could not appreciate its discovery function in mathematics (Jones et al., 2009). The authors therefore propose for a shift to exploratory pedagogical strategies. One of these teaching strategies is problem solving approach which involves four stages; understanding the problem, devising the plan, carrying out the plan, and looking back (Polya, 1945). For a geometric proof problem, understanding of the problem involves understanding of the hypothesis and the conclusion. Hypothesis is the given information and conclusion is the statement to be proved. Polya (1945) suggests that when a proof problem is connected to a figure, the stage of understanding the problem must involve helping the students to draw the figure, to introduce suitable notations, and to label in the diagram the hypothesis and conclusion. Devising the plan involves finding the connection between hypothesis and conclusion, deciding on the theorem to use, and making decisions whether to introduce auxiliary elements into the diagram to enable proving (Polya, 1945). Carrying out the plan involves writing of the proving statements logically, each statement accompanied by a valid reason. Looking back stage includes reviewing the solution and checking if the arguments can be derived using a different approach. This study agrees that problem solving strategy can provide opportunities for students to appreciate the value of geometric proving. In addition, this study argues that students' opportunities to involve in generative interaction with diagrams can be enhanced through exploratory activities. This means that teachers require knowledge of problem solving strategy to engage students successfully in generative interaction with diagrams. This study was therefore guided by Polya's (1945) problem solving framework in analysing the data.

Methodology

The study was conducted using qualitative case study design because the goal was to expand understanding of social issues in their context (Yin, 2009). The study was conducted on one Malawian secondary school teacher and one lesson. The teacher, Kim (pseudonym) is regarded as one of the best teachers due to his long teaching experience and because his students perform well

in mathematics during national examinations. Kim was selected for the study on assumption that conducting research on an experienced teacher could offer an opportunity to study the issue in depth. The lesson episode analysed for this study is part of the video data that was collected for a larger project which aims at studying knowledge for teaching geometric proofs. The lesson episode was considered for analysis because it involved moments of generative mode of interaction with diagrams as well as problem solving teaching strategy. Post-lesson interview was conducted and audio recorded in the teacher's office soon after recording the lesson. The teacher was mainly asked to explain his views about the lesson in terms of what went well and what did not go well during the lesson. Although the data for the study is from one lesson, it is considered to be sufficient for illustrative purposes because it was generated from a real-life context (Yin, 2009). The empirical material, both from the video recording of the lesson and the audio recording of post-lesson interview were transcribed and further analysed separately by using thematic analysis. The aim of thematic analysis was to capture and interpret sense and substantive meanings in the data (Ritchie, Spencer & O'Connor, 2004). Polya's (1945) stages of problem solving were used as a priori themes for analysing the data. The transcribed data was read several times to understand it and to identify moments that were related to a particular stage of problem solving. The findings from the two types of data are discussed under each theme for purposes of comparison. Due to space limitations, this paper has mainly discussed findings related to the first two stages of problem solving; understanding the problem and devising the plan.

Findings and discussions

Kim started the lesson by writing a geometric statement and making a drawing of a diagram on the chalkboard. He told his students that the aim of the lesson was to prove that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference. After explaining the lesson aim, Kim asked his students to go into their usual small groups to draw a similar diagram and discuss how to prove the theorem. There were six groups in the class and each group contained five to eight students. Table 1 shows the diagram that was drawn and the statement that written by Kim on the chalkboard.

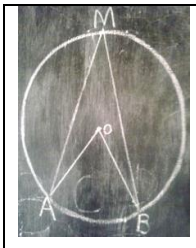	Given: a circle with centre O, with arc AB subtending angle AOB at the centre and angle AMB at the circumference. Prove that the angle at the centre is twice the angle at the circumference.

Table 1: Diagram and statement given to students for proof construction

It can be argued that the problem statement in table 1 is ambiguous because the teacher has not specified the two angles whose relationship is to be proved. There is one angle at the circumference referred by the statement, but there are two angles at the centre (reflex AOB and obtuse AOB). Furthermore, the diagram in table 1 did not contain any features to indicate the required angles. This means that during group discussion, students were challenged to decide whether to relate the angle

at the circumference to the reflex angle or the obtuse angle at the centre. As such, the diagram required generative mode of interaction. Hence the discussion activity involves problem solving.

After about 10 minutes, Kim moved around to check what students were discussing and doing in the groups. The following lesson segment 1 presents a dialogue between Kim and students in group 6.

Kim: Okay so what are you going to do, have you discussed?

Student 1: Yes, we will join MO and prove that these two triangles (pointing at triangle AMO and BMO are congruent). Then relate the corresponding angles.

Kim: Can you show me how you will relate the angles.

Student 1: First, AO = BO (radii), OM is common, and AM= BM (third side) AOM is congruent to BOM. Then angle MAO = MBO, the two angle here are also equal (pointing at the reflex angle at AOB) and the two angles here are equal (pointing at M). (The student is silent).

Kim: Go ahead.

Student 2: Then we add angles here (pointing at the reflex angle at AOB) and angles here (pointing at M) uhhh…. (silence 4 seconds).

Kim: Yes go ahead what about the other group members, how do you proceed from here to the theorem? (silence for 4seconds) how do you relate the two angles? (silence 6 seconds), how do you arrive at the question that you have been asked using that theorem? (silence 4 seconds). Do you know the angle at the centre referred in the theorem?

Student 3: Yes this one (pointing at the reflex angle at AOB).

Student 4: No this one (pointing at the obtuse angle at AOB).

Kim: Can you try to measure the angles and see if it is the upper or lower angle which is twice the angle at M? After that think of another way, this one might not work.

Then Kim went to check students in other groups and asked questions. When he noticed that most of the groups were not focusing on a correct angle at the centre, Kim interrupted the generative activity and asked all groups to measure the three angles and relate their values to find out the correct angles. The activity took about 15 minutes, students measured the angles and made comparisons. The following lesson segment (segment 2) is a continuation of a conversation between Kim and group 6.

Student 3: This angle (pointing at angle AMB) was 52^0 while this one (pointing at obtuse angle AOB) was 104^0. So this (pointing at obtuse angle AOB) is twice this (pointing at angle AMB).

Kim: Okay so how are you going to prove the theorem?

Student 5: We tried similarity but we found that it was going to be difficult as well because it was not saying anything about this angle (pointing at obtuse angle at O) it was only saying about this one (pointing at the reflex angle at O). So since this angle is outside these two triangles, we agreed to use the property of exterior angle of

> triangle. So we extended MO to N to create exterior angles here (pointing at obtuse angle at the centre).

Understanding the problem

Lesson segment 1 shows that the students agreed to add a feature (auxiliary line) into the diagram by joining MO to form two triangles (AOM and BOM). The students also agreed to construct the proof for the theorem by firstly constructing an in-between proof of congruency of triangles AOM and BOM. This means that in lesson segment 1, the students were devising a plan for the proof. Depending on geometry background at their level, there are two possible approaches that the students could use in developing their plan. The first approach involves joining of MO, then forming two equations using properties of isosceles triangles, sum of interior angles of a triangle and sum of angles at a point, and finally making substitutions to reach the conclusion. The second approach involves joining MO and extending it to some point within the circle to form exterior angles of triangles AOM and BOM, and then use properties of isosceles triangles and exterior angle of a triangle to form two equations, and finally make substitutions. This shows that congruency theorem is not appropriate for both approaches. As such, the students' decision to join MO and then use congruency theorem as an in-between proof was not correct. It can be argued that the students' decision was based on lack of understanding of the problem. This argument is based on several observations. Firstly, in segment1the students were unable to explain how they would use the congruency theorem to connect to the conclusion. But segment 2 indicates that after realising that the angle at the centre is the obtuse angle, the students changed their plan, and they were able to justify their decision to use property of exterior angle of a triangle. Secondly, the argument is based on students' disagreements regarding angle at the centre in segment 1. Some students pointed at the reflex angle while others pointed at the obtuse angle when Kim asked them to identify the angle at the centre referred by the theorem. This means that the students were not sure of the correct angle at the centre. The disagreement among the students regarding the angle at the centre provided an opportunity for Kim to shift students' focus from the stage of devising the plan to the stage of understanding the problem. The findings from the lesson segments are supported by the following extract in which Kim expresses the challenges that students faced when constructing the proof:

> It was very difficult for the students to come up with relevant constructions when proving. Because I think the main trick in the proof was to know what type of construction and in-between theorem to use for the proof. But after doing the measurements, the students were able to make correct construction and to know the theorem to use. Their reasoning and their work showed that they now understood the statement they were asked to prove.

The extract shows that Kim realised the difference in the students' ability to come up with a correct plan before and after the empirical activity. Kim mentions two challenges that the students faced before the empirical activity. The first challenge involved failure to decide on features to be added to the diagram. The second challenge involved failure to use the diagram to generate a correct in-between theorem and proof. Both challenges involved generative interaction with diagrams as they required students to make explorations with the diagram (Herbst, 2004). The extract also indicates that the students were able to devise a correct plan upon understanding the problem. Cheng & Lin (2009) call the proving of an in-between theorem as construction of intermediary condition (IC).

These authors argue that students can only construct correct ICs if they understand the hypothesis and the conclusion. Polya (1945) advises that solving of any mathematical problem should not be started unless the problem is well understood. Probably, the challenge of understanding the problem could have been avoided if Kim carried out the phases of problem solving before going to the class (Polya, 945). Through this preparation activity, Kim could have anticipated that students were more likely to regard the reflex angle as angle at the centre due to its location. Thus the reflex angle is close to and in the same quadrilateral (AMBO) with the angle at the circumference. In so doing, Kim could avoid the mistake by helping the students to understand the theorem (which also implies to understand the problem) before asking them to discuss how to develop its proof.

Devising the plan

Both lesson segments show that students devised the plan for the proof in their groups through exploration. Lesson segment 1 shows that the first attempt to devise the plan was not successful due to lack of understanding of the problem. Lesson segment 2, shows that after the empirical activity, students came up with a proper construction and correct IC for linking angle at the centre and angle at the circumference. The students' suggestions to introduce a line into the diagram confirm that they were involved in generative interaction with the diagram. Lesson segment 2 also shows that the students were able to evaluate their ideas by considering the questions asked by Kim in lesson segment 1. The questions are, "how do you relate the two angles? How do you arrive at the question that you have been asked using that theorem?" The questions helped the students to focus on identifying a construction and theorem that could help them to connect the given information to the conclusion. This is observed in utterance by student 5 who explained that they tried to use similarity theorem but they realised that it was not appropriate because it could not help them to link angle at the circumference and the obtuse angle at the centre. The technique of asking questions that probe students' thinking is regarded as one of the strategies for helping students to devise a plan (Polya, 1945). The findings from analysis of students' explanations in lesson segment 2 show that the empirical activity which focused on understanding angle at the centre helped the students to devise a correct plan for the proof. This observation is also confirmed in Kim's explanation regarding what went well during the lesson. Kim explained that the lesson was generally successful because the students were able to construct the proof independently. Kim explained that his lesson objective was to help the students to understand the theorem and construct its proof on their own. He explained as follows:

> If you just start proving without engaging students in an activity like measuring or discussions on how to prove, they just memorise the proof. So to avoid memorisation, I involved the students in discussions. When I found that they were referring to a wrong angle at the centre, I did not tell them the angle, I wanted them to find out on their own by measuring the angles.

The extract shows that during the first activity Kim regarded the students' inability to construct a correct proof as an opportunity for them to understand the theorem which is also the statement problem in this case. Kim's idea of giving students opportunity to do explorations on the problem to be proved is supported by Ding & Jones (2009). But Kim was not supposed to wait until students got stuck in order to suggest the empirical activity. According to Polya (1945) the teacher is

supposed to prevent students from answering a question that is not clear to them, and from working for an end that they do not desire. The author argues that students might be frustrated if they either get stuck or come up with undesirable solution to a problem. As such, teachers are supposed to avoid making students' frustrated by ensuring that they understand the problem before beginning to devise its plan. This suggests that Kim was supposed to first of all engage the students in an exploration activity that could help them to understand the problem before involving them in activity of discussing how to prove the theorem. By doing so, Kim could have avoided the challenge of lack of time that he pointed out when explaining what did not go well during the lesson. Kim explained that he had planned to discuss three examples of how to apply the theorem in solving different geometric problems, but he only managed to discuss one example because some time was spent on the unplanned activity of measuring angles. Apart from examples, Kim also explained that he had planned to give the students an exercise which he wanted to start marking during the lesson, but he turned it into homework due to lack of time. Kim seemed to have planned many activities for the lesson which involved problem solving. But Polya (1945) explains that problem solving might be time consuming because students explore different ways when devising a plan for finding solution of the problem. This means that a teacher is supposed to plan few activities for a problem solving lesson to ensure that students have ample time for explorations. The planning of many activities and the sequence of activities during the lesson indicates that although Kim used problem solving strategy, he was not aware of some of its skills. This agrees with Herbst's (2004) caution that engaging students in generative mode of interaction with diagrams is challenging.

Conclusion

This study has found that both the teacher and students faced some challenges during the lesson of geometric proving which involved generative interaction with diagrams. The teacher faced the challenge of lack of time hence he did not complete some of the planned activities. The students faced a challenge of lack of understanding of the problem during their first attempt to devise a plan for the proof. Due to this challenge, the students came up with wrong construction and IC for the proof. As a result the students were unable to complete the plan for the proof during their first attempt to devise the plan. However the findings show that the students were able to devise correct plan for the proof after doing an empirical activity which focused on understanding the problem. The study implies that the challenges faced by both the teacher and the students could have been avoided if proper problem solving skills were followed. Thus the first activity could have focused on understanding the problem and the second activity could have focused on devising the plan for the proof. The findings suggest that successful involvement of students in generative interaction with diagrams require knowledge of several problem solving skills including proper planning and sequencing of activities. Lastly, the findings show that empirical interaction with diagrams enhanced students understanding of the problem. Further study is needed to explore whether successful involvement of students in generative interaction with diagrams require a combination with other modes of interaction with diagrams.

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An investigation of middle school mathematics teachers' knowledge for teaching algebra

Nurbanu Yılmaz[1] and Ayhan Kürşat Erbaş[2]

[1]Bülent Ecevit University, Faculty of Education, Zonguldak, Turkey; nurbanuyilmaz@beun.edu.tr

[2]Middle East Technical University, Faculty of Education, Ankara, Turkey; erbas@metu.edu.tr

The purpose of this study was to investigate middle school mathematics teachers' knowledge for teaching algebra. The participants of the study were 48 mathematics teachers from various middle schools. A questionnaire was conducted in order to collect data about the teachers' knowledge related to teaching of algebra. The results showed that the participating teachers were competent in making transitions among different algebraic representations. However, they had difficulties in explaining the conceptual bases of some of the algebraic concepts and procedures. In addition, results indicated that some of the teachers had difficulties and misconceptions similar to those of students as depicted in the scenarios provided in the questionnaire.

Keywords: Pedagogical content knowledge, algebra, middle school mathematics teachers.

Introduction

Teachers' knowledge is considered as one of the most important predictor of student achievement (Hill, Rowan, & Ball, 2005). In recent years, therefore, researchers have focused on the professional knowledge of teachers (Ball, Thames, & Phelps, 2008; Grossman, 1990; Shulman, 1986, 1987). As Knowles, Plake, Robinson, and Mitschell (2001) stated, what teachers should know and be able to do are issues which continuously change and develop as values of the society come up with the changes. Therefore, teachers need different types of knowledge in order to fulfill those expectations. Subject matter knowledge (SMK), pedagogical knowledge, and pedagogical content knowledge (PCK) are among the major components of teacher knowledge referred frequently in the literature (Ball et al., 2008; Cochran, DeRuiter, & King, 1993; Magnusson, Krajcik, & Borko, 1999; Shulman, 1987). Among these, having a strong subject matter knowledge is often considered the central component of teacher competency (Krauss et al., 2008). However, merely having strong mathematics knowledge does not guarantee effective teaching (Ball et al., 2008; Kind 2009).

Effective teaching requires making the content accessible to students, interpreting the questions and productions of students, and being able to explain or represent ideas and procedures in multiple ways (Hill, Sleep, Jewis, & Ball, 2007). In this context, teacher competency is the cognitive ability in order to develop solutions for problems concerning teaching profession and applying these solutions in various situations successfully (Weinert, 2001). Providing meaningful and effective activities for students' learning is considered essential for teacher competency (Knowles et al., 2001). Therefore, pedagogical content knowledge, defined as "the most useful ways of representing and formulating the subject that makes it comprehensible to others" (Shulman, 1986, p. 9), is seen as a core component of teacher competency and an indispensable part of teacher knowledge base. A study conducted by Kind (2009) in which several PCK models were investigated revealed that representations and instructional strategies and students' subject specific learning difficulties were considered as two core dimensions of PCK in most of the studies (e.g., Grossman, 1990; Magnusson, Krajcik & Borko, 1999;

Shulman, 1987). Therefore, among others, the components that are knowledge of students' learning of mathematics and knowledge of teaching mathematics were chosen as the focus in this study.

In the literature, there have been several studies on the pedagogical content knowledge of mathematics teachers concerning knowledge of algebra and teaching of algebraic concepts (e.g., see Doerr, 2004; Güler, 2014; McCrory, Floden, Ferrini-Mundy, Rackase, & Senk, 2012). As the Mathematics Study Panel (2003) indicates, proficiency in algebra is important for students' mathematical thinking and understanding. Moreover, teaching of algebra in middle school is particularly crucial since the algebra learnt there constitutes a basis for the high school and university level mathematics (Mathematics Study Panel, 2003). Thus, teachers' knowledge of algebra and teaching of algebra in middle schools are worth studying as teachers' professional knowledge is one of the most important predictors of student achievement (Hill, Rowan, & Ball, 2005). Thus, the assessment of teacher knowledge is an important step to understand the competency of teachers or the quality of teacher education programs. Although there are several studies based on the algebra knowledge of students, there is limited research on algebra instruction (Güler, 2014; Kieran, 2007; Ladele, Ormond, & Hackling, 2014). Therefore, which knowledge component is required by the teachers and how this knowledge could be developed need to be investigated in order to improve algebra instruction (Kieran, 2007). For this reason, there is a need for theory building on what teachers need to know related to teaching of algebra and how this knowledge could be developed by teachers. In this study, SMK and PCK were considered as separate dimensions of teacher knowledge and PCK of middle school mathematics teachers was investigated for teaching algebra. As for the main theoretical framework, we used "algebraic knowledge for teaching" which was adapted by Güler (2014) from Ferrini-Mundy, Floden, McCrory, Burrill, and Sandow (2005). The framework is a three-dimensional model consisting of three main components: algebra content, algebra knowledge for teaching, and domains of mathematical knowledge. Algebra content consists of two main categories: algebraic expressions, equalities, and inequalities and linear and non-linear functions and their properties. Algebra knowledge for teaching includes advanced algebra, knowledge about learning of students, and knowledge about representations of the content. Domains of mathematical knowledge include basic concepts and procedures, representations, applications, and reasoning and proof. In this study, algebraic knowledge for teaching dimension of the model was focused on. The only difference between the adapted framework (Güler, 2014) and the original one (Ferrini-Mundy et al., 2005) was in algebraic knowledge for teaching dimension which includes school algebra, advanced algebra, and teaching knowledge components. The knowledge about learning of students, and knowledge about representation of the content segments of this dimension were investigated within the study in relation with algebra content and mathematical knowledge context dimensions.

The purpose of this study was to investigate the knowledge of middle school mathematics teachers in relation with teaching of algebra. In this scope, two components of pedagogical content knowledge of middle school mathematics teachers were investigated; knowledge of learning of students and knowledge of teaching mathematics. Moreover, the difficulties and strengths of middle school mathematics teachers were investigated in relation with the knowledge of teaching algebraic concepts. Therefore, the research questions that guided this study were:

- What are the difficulties and challenges that middle school mathematics teachers face in teaching of algebraic expressions, equations, inequalities, and linear and non-linear functions?

- What pedagogical content knowledge do middle school mathematics teachers have about algebraic expressions, equations, inequalities, and linear and non-linear functions?

Methodology

Participants of the study were 48 middle school mathematics teachers from different public and private schools in Turkey who voluntarily participated in the study. Their teaching experiences (in years) ranged from 1 year to 26 years ($\overline{X}$ = 8, *SD* = 5.3). In Turkey, middle school mathematics teachers are responsible for teaching mathematics in grades 5, 6, 7, and 8. All of the participants in this study had at least a bachelor's degree in elementary mathematics education. Based on the current mathematics curriculum in Turkey (MoNE, 2013), formal teaching of algebra starts in 6th grade with the introduction of algebraic expressions and the concept of variable. Then, the concepts of equality, equation and linear equation are introduced in 7th grade. In 8th grade, algebraic expressions and identities, linear equations, equation systems, and inequalities are further dealt.

Developed by Güler (2014) and used with his permission, the instrument used in this study was a 20-item questionnaire comprising of multiple-choice and open-ended items intended to assess teachers' knowledge for teaching algebra at the middle school level. For the instrument, the scores were averaged across 20 items to control the reliability of the instrument ($\overline{X}$ = 17.8, *SD* = 5.9). Also, Cronbach's alpha was estimated as 0.81 for person reliability and 0.94 for item reliability. Moreover, in order to ensure validity of the instrument, item analyses were carried out and expert opinion was taken in order to show that the content of the instrument coincides with the conceptual framework. In the questionnaire, the participants were confronted with situations or scenarios related to the work or practice of teaching middle school level algebraic concepts (see Figure 1 for an example). The algebra content tested in the instrument was in line with the mathematics curriculum for the middle school level (grades 5 to 8) in Turkey (MoNE, 2013). The instrument consists of two components in relation with PCK: knowledge of students' learning and knowledge about representation of the content. Moreover, the content of the instrument is constructed under two domains: mathematical knowledge content and algebra content. Mathematics content includes basic concepts and procedures, representations, applications, and reasoning and proof. Algebra content includes algebraic expressions, equality, and inequality, linear and non-linear functions and their properties.

The use of a questionnaire with a survey type design was preferred in order to collect data from a large sample of teachers. There was no time limitation for completing the questionnaire. Descriptive analyses and item based in-depth analyses were carried out in order to have a general overview on algebra related PCK of teachers. Based on the rubric prepared by Güler (2014), analyses were conducted for each item separately. The frequencies and percentages of correct, particularly correct, or incorrect answers were calculated in order to investigate the performances of all teachers for each item. To illustrate, the answers to the 10th item (see Figure 1) were categorized as correct if the teachers explained why the inequality sign changed direction by using algebraic expressions or trying out particular values to make generalization, partially correct if the teacher explained it by using particular values of *x*, and incorrect if the answer was wrong, invalid, or missing.

Mr. Reyhan gave the inequality $-x < 7$ to the students and got them find the solution set of the inequality. One of the students divided both sides of the inequality with -1 and got the solution as $x > -7$. Then, another student asked why the direction of the inequality sign was changed when both sides of it were divided by -1. What would you tell to the student if you were the teacher of the classroom?

Figure 1: The Item 10 in the questionnaire

In general, answers were coded as correct when the teacher provided correct answers/results to the questions by using algebraic expressions or trying out particular values in order to make generalization in explaining an algebraic topic conceptually, and explained why the answer of the student was wrong in the scenario by providing underlying reasons. The answers coded as partially correct included those in which the teachers gave inadequate explanations and used particular values to show the validity of an algebraic procedure. The answers coded as incorrect were the wrong, invalid, or missing ones. To ensure the reliability of coding, two independent scorers coded all of the open-ended items in the questionnaire for half of the participants (i.e., 24 teachers). The interrater agreement across both scorers was high (*percent agreement* = 92.80%).

Findings

As the findings suggested, most of the teachers in this study were able to make the transition among the rhetoric, symbolic, and geometric representations of the algebraic expressions. Below we present the findings in relation to the dimensions of knowledge for teaching algebra framework (Güler, 2014), namely algebra content, algebra knowledge for teaching, and domains of mathematical knowledge. In some items, most of the teachers gave correct responses in the questionnaire. The Item 18 ("The difference between an equation and (algebraic) identity") was one of the items that most of the teachers answered correctly (see Table 1). The item 18 was related to three dimensions of the algebra knowledge for teaching model, namely algebra content, algebra knowledge for teaching, and domains of mathematical knowledge.

	Correct		Partially correct		Incorrect	
	f	%	*f*	%	*f*	%
Item 6	3	6.3	17	35.4	28	58.3
Item 10	0	0.0	27	56.3	21	43.8
Item 18	33	68.8	12	25.0	3	6.3

Table 1: Results of the analyses of three items in the questionnaire

The answer of a participant (P_{18}) for the Item 18 item was categorized as correct since it presented the difference of the two concepts effectively.

> P_{18}: Equation is an algebraic expression which holds for particular real number(s) while (algebraic) identity is an algebraic expression which holds for all real numbers.

The answer of another participant (P_{31}) for the Item 18 was categorized as partially correct since the teacher considered all equations as if they were just first degree equations.

> P_{31}: Equations hold for just one value (i.e., If $3x + 5 = 8$, then $x = 1$). However, identity holds for all values of the unknown.

Moreover, some of the items got partially correct or incorrect answers by most of the teachers. To illustrate, the results for Item 6 (see Figure 2), which was related to the algebra content and knowledge for teaching algebra dimensions and basic concepts and procedures segment of the model, are presented in Table 1. On the other hand, the results revealed that the teachers were incompetent in some of the areas such as finding and correctly expressing the solution set of equations and identification and correction of students' incorrect ideas and misconceptions.

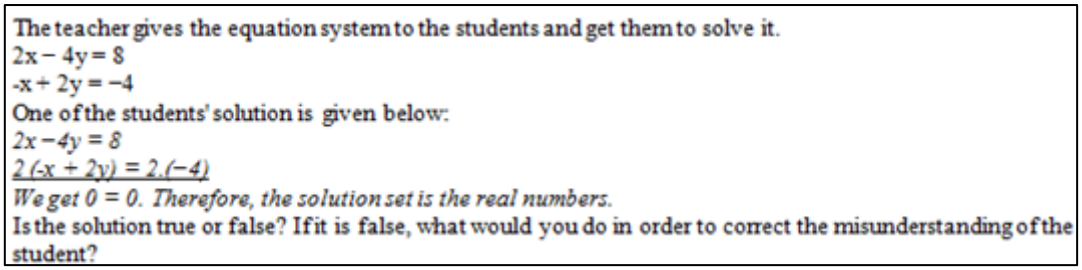
The teacher gives the equation system to the students and get them to solve it.
$2x - 4y = 8$
$-x + 2y = -4$
One of the students' solution is given below:
$2x - 4y = 8$
$2(-x + 2y) = 2.(-4)$
We get 0 = 0. Therefore, the solution set is the real numbers.
Is the solution true or false? If it is false, what would you do in order to correct the misunderstanding of the student?

Figure 2: The Item 6 in the questionnaire

The answer provided by one of the participants (P_{34}) was categorized as correct since the teacher stated that the solution set could not be real numbers and gave a suggestion on how to show it to the students.

> P_{34}: The solution set cannot be real numbers since the equations do not hold for each (x, y) when x and y are real numbers. Therefore, this could be shown to students by substituting some x and y values which are real numbers but do not satisfy the equations.

The answer of another participant (P_{32}) was categorized as partially correct as it stated that the equation could not be solved.

> P_{32}: The solution is false. The solution set of the equation system should be in the form of (a, b). It cannot be real numbers. Those are not two different equations. The first equation is double of the second equation. The two are the same equations. That is, it cannot be solved.

Some of the answers for the Item 6 were categorized as incorrect if the teachers stated that the solution was correct without any explanation or if they gave invalid/missing explanations. The Item 10 where the teachers were asked "Why the direction of the inequality sign is changed when both sides of the inequality $-x < 7$ are divided by a negative number?" (see Figure 1) was also answered incorrectly by most of the teachers in this study (see Table 1). This item was related to the algebraic expressions, equations, and inequalities, basic concepts and procedures, knowledge about learning of students segments of the model. None of the teachers used algebraic expressions for the solution. Rather, the teachers mostly used particular values for x in order to show the change of the direction of sign when both sides were divided by a negative number. The answer of one of the participant (P_{35}) is categorized as particularly correct since the teacher used particular values for the solution without making generalization.

> P_{35}: I would give particular values. For example, if we multiply or divide both sides of $2 < 5$ with -1, the inequality will be $-2 > -5$. By considering the values of the numbers, we can see that the inequality sign should be changed when both sides are multiplied or divided by -1.

Also, the answer of one of the participants (P_{44}) is categorized as incorrect since the teacher stated that it was just a rule to be memorized. In addition, the invalid/missing answers were also categorized as incorrect.

> P_{44}: I would say that it was a rule to be memorized. Thus, the inequality sign changes when both sides are divided by a negative number. Then, I would give examples with particular values.

Discussion and conclusion

Results revealed that middle school mathematics teachers presented strength and weaknesses in terms of knowledge for teaching algebra. Most of the teachers in this study show indications of ability and knowledge to make transition among different representations of algebraic expressions. Moreover, most of them were successful at items which required knowledge about particular algebraic concepts such as the difference between an equation and (algebraic) identity or the properties of nonlinear functions. Although some studies concluded that the concept of equation and the concept of identity were frequently confused by pre-service middle school mathematics teachers (Altun, 2006; Güler, 2014), nearly all of the teachers in this study were able to differentiate between these concepts. It might be safe to say that, in general, teachers in this study performed well in items related to algebra content dimension and basic procedures and representations segment of the domains of mathematical knowledge dimension of the model (Güler, 2014).

Analyses of the answers illustrated that some of the teachers had errors and misconceptions similar to those of students, as also observed by Güler (2014). Moreover, most of the teachers presented not only signs of misconceptions beyond the ones given in the scenarios but also difficulties in explaining students' reasoning and proposing an appropriate teaching method in order to guide them in the right direction. One of the basic reasons for teachers' inadequacy to propose an appropriate teaching method might be the lack of knowledge of algebraic concepts. Since the teachers could not provide a valid explanation for students' thinking and reasoning in most cases, they could not provide a suggestion for the teaching of related algebraic concepts (Güler, 2014). In some of the items, teachers were required to give conceptual explanations for the algebraic situations. For example, regarding "How would you explain to your students why 2^0 is equal to 1?"), only about half of the teachers were able to provide conceptually sound explanations why 2^0 is equal to 1. However, other teachers could not give a conceptual answer for that question or stated that "2^0 is equal to 1" is just a rule. This indicates that such lack of knowledge lead teachers to teach algebra as a collection of rules to be memorized. Another situation supporting this was Item 10 which required explaining why the direction of the inequality sign changes when both sides of the equation were divided by a negative number. There were nearly no conceptually based explanations provided by the teachers for it. Some of the teachers explained that they were just rules to be memorized like any other mathematical rules, like the multiplication of two negative numbers' being equal to a positive number. Conversely, some teachers explained it by showing that it holds for many different values. However, none of them provided an explanation by using and considering algebra as a tool for generalization (Bednarz, Kieran, & Lee, 1996). Thus, it might be concluded that the teachers in this study had lack of strong knowledge based on representation of the content segment of knowledge for teaching algebra dimension in general. Also, teachers had deficiencies related to the mathematical knowledge dimension of the model.

Results obtained in this study supports that middle school mathematics teachers might be incompetent about the conceptual bases of some algebraic concepts and in identifying and explaining students' errors and misconceptions in order to provide appropriate representations (Ball, 1990; Güler, 2014; Tirosh, 2000). Since teacher knowledge is considered as the first step to provide an effective teaching, it should be investigated and developed. As Cochran, DeRuiter, and King (1993) stated, teacher knowledge has a dynamic form and it develops continuously. Therefore, the results of the current study should simply be taken as a call for teacher educators to strengthen and develop teachers' conceptual knowledge of algebra and its teaching. In this context, for example, a balance should be sought between requiring several advanced mathematics/algebra courses and methods of teaching mathematics/algebra courses in mathematics teacher education programs in order to equip them with the knowledge and pedagogy needed to teach algebra effectively. Furthermore, courses and professional development efforts for learning to teach mathematics/algebra should focus on concepts and big ideas with an emphasis on why to use a particular procedure or why a certain idea/procedure works in some particular contexts.

On the other hand, as assessing in-service teachers' knowledge for teaching (algebra) would require a more carefully constructed instrument and research design, the questionnaire used in this study itself and limiting the research design to survey type might be considered as methodological limitations of this study. Even though the purpose was to collect data from as many teachers as possible, further studies should at least consider collecting data through interviews (with selected participants at least) and require more elaboration on the answers given in the questionnaire and conducting classroom observations when possible. Furthermore, as much as it is challenging, more studies should focus on developing instruments for assessing teachers' knowledge for teaching algebra.

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Is teacher knowledge affecting students' transition from primary to second-level mathematics education?

Niamh O'Meara[1], Mark Prendergast[2], Lorraine Harbison[3] and Clare O'Hara[4]

[1]EPI•STEM, University of Limerick, Limerick, Ireland; niamh.omeara@ul.ie

[2]School of Education, The University of Dublin, Trinity College, Ireland; mark.prendergast@tcd.ie

[3]CASTeL, Dublin City University, Dublin, Ireland; lorraine.harbison@dcu.ie

[4]Central Statistics Office, Dublin, Ireland; clare.ohara@cso.ie

Similar to counties such as the U.K. and the U.S.A, the Irish education system is divided into four key stages; pre-school education, primary level education, second level education and tertiary education. Transition between each of these phases has its own set of challenges but many believe the most challenging of all is the transition from primary to second level education. This quantitative, national study investigates the transition from primary to second level mathematics education from the perspective of teachers. It investigates sixth class teachers' knowledge of the mathematics curriculum and teaching strategies employed at second level and first year mathematics teachers' knowledge of the mathematics curriculum and teaching strategies favored in primary school. The results of the study highlight low levels of knowledge in these domains amongst both sixth class and first year mathematics teachers. The ramifications of this gap in teacher knowledge are also discussed in detail.

Keywords: Primary school mathematics education, second-level mathematics education, transition, teacher knowledge, continuity.

Background to the study

As is the case in Australia, the United States and the United Kingdom, Ireland's education system is divided into four phases; pre-school education, primary level education, second level education and tertiary education. The transition from primary mathematics education to second level mathematics education is one of the greatest challenges that young people experience during their school years. According to Bicknell, Burgess and Hunter (2009) the challenge presented by this transition is multifaceted and involves challenges from social, academic and systematic perspectives. As such, this is a pertinent research area and one which has been looked at in depth in recent years.

The overarching finding to emerge from the research carried out to date was that the transition from primary to second level mathematics education resulted in a decline in students' attitudes, academic performance and confidence (Attard, 2010; Economic and Social Research Institute [ESRI], 2007). Furthermore, Bicknell et al. (2009) found that the gap between high achieving and low achieving students widened significantly during this transition period. Due to the serious nature of these consequences, researchers, such as Green (1997) and Attard (2010), have sought to investigate what constitutes effective transition and what are the main factors that contribute to an educationally poor transition for students.

In her study on students' experiences of the transition from primary to second level mathematics education in Australia, Attard (2010) listed curriculum, pedagogy, assessment strategies, social interactions and students' relationships with others, as key factors that dictate the success of transition. Likewise, Barber (1999) describes the transition as a set of five hurdles all of which must

be overcome at once. The hurdles to be overcome to ensure a smooth transition, as listed in this study, are bureaucratic, social and emotional, curriculum, pedagogy, and management of learning. In addition to this, Evangelou et al. (2008, p. 2) stated that a successful transition for children entailed:

> …developing new friendships and improving their self-esteem and confidence; having settled so well in school life that they caused no concerns to their parents; showing an increasing interest in school and school work; getting used to their new routines and school organisation with great ease [and] experiencing curriculum continuity.

All research conducted into what constitutes effective transition make some reference to curriculum and pedagogical continuity. Likewise, research conducted in the area of problematic transitions all point to a lack of continuity in this regard. For example, Elkins (1989), Green (1997) and Tilleczek (2007) all found that the attainment and motivational losses that students often experience when moving from primary to second level mathematics education can, in no small way, be attributed to a lack of continuity in terms of both curriculum and pedagogical approaches.

However, in order to ensure continuity between both curriculum and pedagogical approaches it is critical that teachers who are teaching students that are about to enter or have just completed the transition process have an in-depth Mathematical Knowledge for Teaching (MKT). Such knowledge encompasses knowledge of the mathematical content previously studied and that which they will study in subsequent years (Ball, Thames & Phelps, 2008). Ernest (1989) reiterates that a teacher's MKT is not limited to knowledge of curriculum, but also knowledge of students, in order to enable them to teach mathematics effectively. The authors further ascertain that teachers, especially those involved in the transition process, must have a comprehensive MKT comprising of the curricula, students and teaching methodologies utilised before and after the transition process. Teachers who do not possess such knowledge have yet to develop the full range of knowledge domains proposed by both Ball, Thames & Phelps (2008) and Ernest (1989) and as such, their knowledge could be considered inadequate for teaching. It is this belief, in conjunction with existing research, which led the authors to investigate the following research questions.

> 1. How familiar are sixth class primary school teachers with the second level mathematics syllabus and the teaching methodologies being promoted at second level and vice versa?
>
> 2. What are the consequences of these levels of MKT in terms of (a) the fluidity of the transition between primary and secondary mathematics education and (b) the teaching approach adopted by second level teachers when teaching mathematics to first year students?

Methodology

The research design for this quantitative study involved the distribution of questionnaires to a representative sample of two groups of stakeholders involved in the transition process; namely sixth class teachers in primary schools and first year mathematics teachers in second level schools[1]. For the purpose of the study two advisory groups, one involving primary teachers and another involving second level mathematics teachers, were established. Their role was to help with the development

[1] In Ireland 6th class is the final year of primary education which 1st year is the name given to the first year of second level education.

and piloting of the questionnaires and to help the authors in relation to sampling issues. To allow for comparison of responses from primary and second level teachers the questionnaires were of a similar nature and both were based on the framework for transition developed by the authors from the work of Anderson, Jacobs, Schramm and Splittgerber (2000) and the models of knowledge proposed for primary teachers by Ball, Thames & Phelps (2008) and for second level teachers by Ernest (1989). This theoretical framework is outlined in Figure 1.

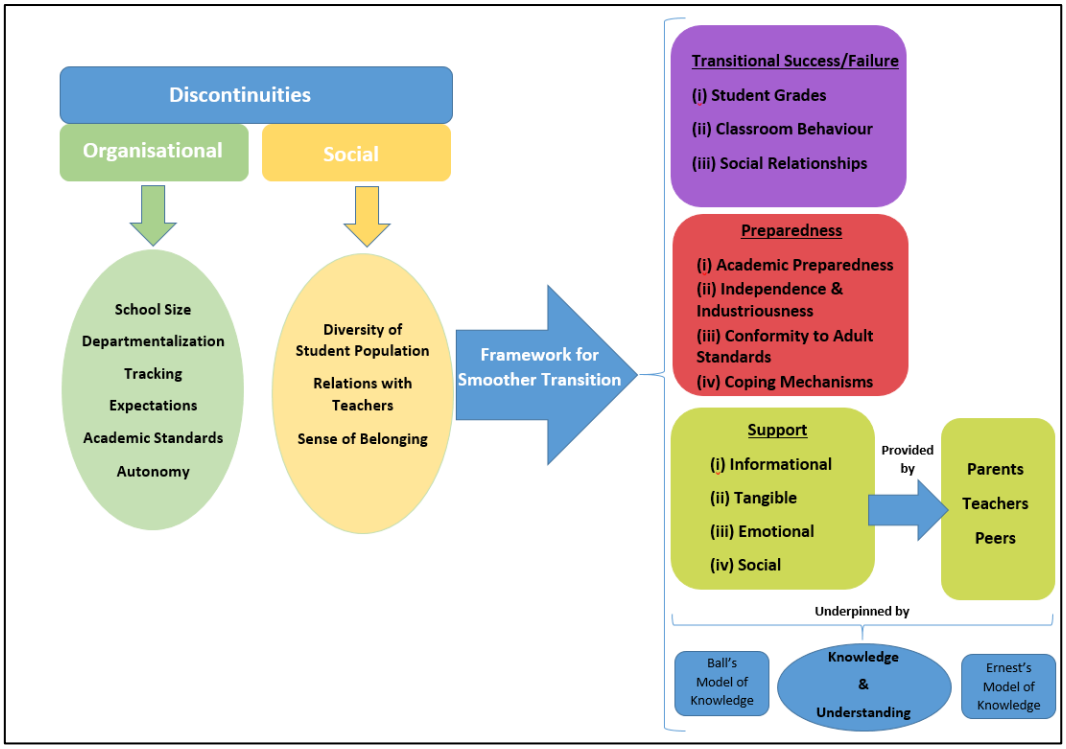


Figure 1: Theoretical framework

This study was unique in that it looked solely at the issue from the perspective of teachers. As such, only some dimensions of this model were relevant to this study namely the discontinuity pillar, the support pillar and the teacher knowledge pillar. This particular paper has an even narrower focus and looks solely at the pillar of teacher knowledge.

The sampling frame for this study was a list of all 3,300 primary schools and 723 second level schools in Ireland (DES website February 2016). The targeted sample was 700 sixth class teachers and 400 first year mathematics teachers. By consulting the primary school advisory groups, the authors established that on average, there is one sixth class teacher in each primary school in Ireland. As a result, a simple random sample of 700 primary schools was selected. Overall, the sample included 21.2% of all primary schools. Having consulted with the second level advisory group it was established that on average, there are two mathematics teachers teaching first year mathematics in each school in Ireland. Hence using this estimate, a stratified random sample of 200 second level schools around Ireland was selected. This sampling technique ensured that an accurate representation of each type of school (secondary, vocational, community and comprehensive) in Ireland was included in the sample. Overall, the sample included 27.7% of all second level schools in Ireland.

The questionnaires were distributed to the 700 primary schools and 200 second level schools in April 2015. The primary school questionnaires were sent to the principal of each school and they were

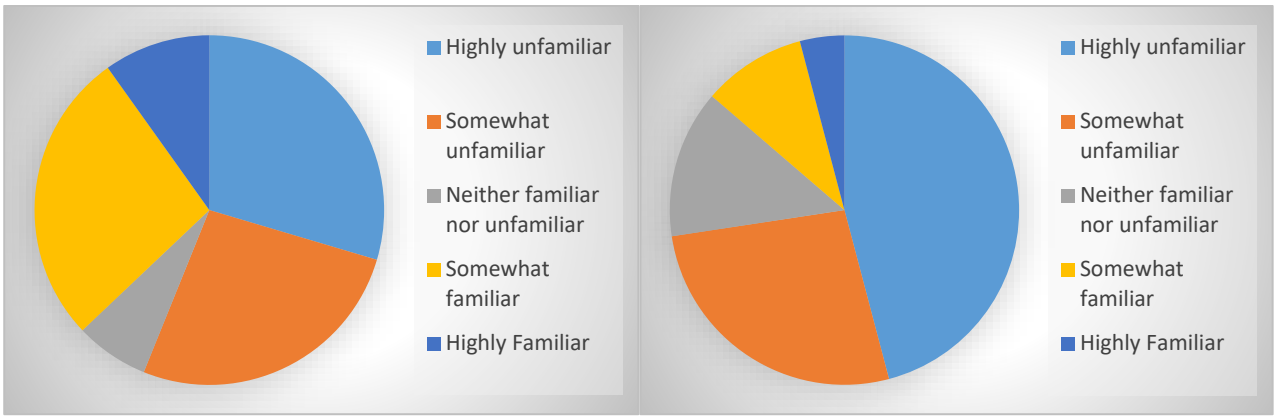
asked to distribute these questionnaires to the sixth class teacher. The pack sent to each of the 700 principals included an information sheet for the principal, a teacher information sheet along with the questionnaire and a stamped address enveloped for the questionnaire to be returned in. The second level questionnaires were sent to the Head of Mathematics in each of the 200 second level schools and they were asked to distribute the questionnaires to the first year mathematics teachers in their school. The pack sent to each department head included an information sheet for their perusal, an information sheet for first year mathematics teachers along with two questionnaires and two stamped address envelopes in which the questionnaires could be returned. At both primary and second level, each stamped addressed envelope included was given a number corresponding to the school selected so the researchers could identify the schools that had not returned the completed questionnaires. Two weeks after sending the questionnaires, follow-up telephone calls to each of these schools were undertaken with the aim to increase the response rate.

Upon receipt of the completed questionnaires the quantitative data was inputted and saved into the computer programme SPSS. Descriptive analysis examined primary teachers' knowledge of the mathematics curriculum and teaching strategies employed in secondary school and second level mathematics teachers' knowledge of the mathematics curriculum and teaching strategies employed at primary level. Descriptive analysis also allowed the authors to determine how these levels of knowledge affected the approach adopted by second level teachers when teaching first year students and also to determine if the transition from primary to secondary was educationally successful from the teachers' perspective. The authors will now present the results of this analysis in an attempt to address the aforementioned research questions.

Findings

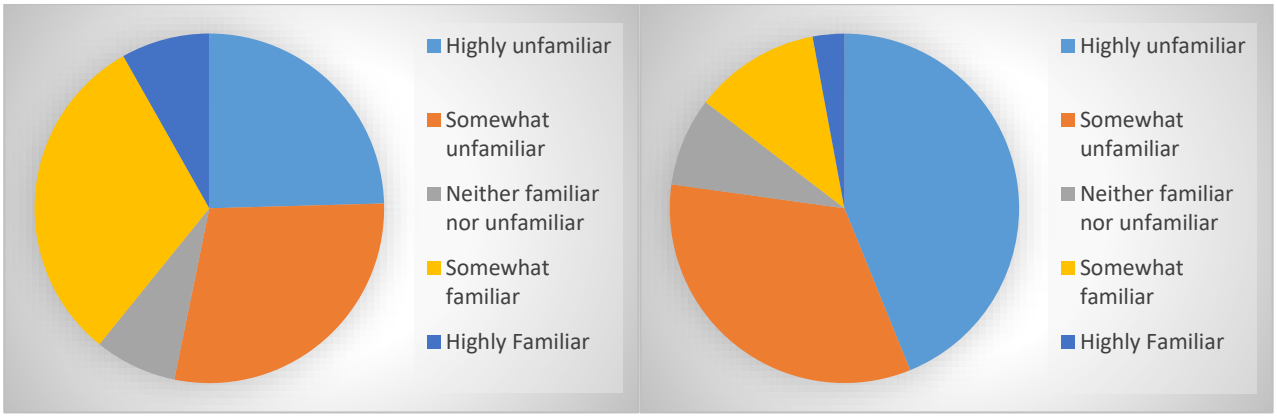
Based on the population size it was determined that, to allow for a 5% margin of error, the study would require 263 responses from sixth class teachers and 133 responses from first year mathematics teachers. The actual response rate was 296 primary school teachers (approx. 42%) and 171 second level teachers (approx. 43%). The primary teachers who responded were distributed across 271 schools (38.7% of schools surveyed) while the second level teachers who responded were distributed across 101 schools (50.5% of schools surveyed).

The first research question sought to ascertain sixth class teachers' knowledge of the first year mathematics curricula and the teaching strategies employed by first year mathematics teachers as well as first year teachers' knowledge of the sixth class curriculum and the teaching strategies adopted by sixth class teachers. The findings related to this research question are presented in Figure 2 and Figure 3.



(a) **(b)**

Figure 2: Primary teachers' responses when asked (a) *How familiar are you with the first year mathematics syllabus?* and (b) *How familiar are you with the recommended teaching methods for first year mathematics?*



(a) **(b)**

Figure 3: Second Level teachers' responses when asked (a) *How familiar are you with the sixth class mathematics syllabus?* and (b) *How familiar are you with the recommended teaching methods for sixth class mathematics?*

These findings demonstrate that teachers, at both levels, have a deficient understanding of the syllabus and teaching strategies that their students were/will be exposed to in their previous/next year of schooling. Over half of sixth class teachers (56%) reported that the first year mathematics syllabus was either highly unfamiliar or slightly unfamiliar to them. The corresponding figure for second level teachers was 49%. The responses in relation to knowledge of teaching strategies were even more pronounced. Almost three-quarters of sixth class teachers (73%) stated that they were highly unfamiliar or slightly unfamiliar with the teaching approaches used in mathematics classrooms at second level. Likewise, 77% of first year mathematics teachers stated that they were highly unfamiliar or slightly unfamiliar with the pedagogical approaches employed by sixth class teachers. Furthermore, only 13% of sixth class teachers claimed to be in any way familiar with the teaching approaches used by first year mathematics teachers while the corresponding figure for the first year teachers who responded was 15%.

The second research question was two-folded and sought to analyse the knock on effect of the gaps in teacher knowledge discussed previously. In order to address this research question both groups of teachers were first asked to rate their agreement with the statement *"There is a fluid transition*

between primary and secondary mathematics". The second level teachers were then further probed on these knock on effects when they were asked to describe the approach they adopt when teaching first year mathematics students upon their entry to second level. For this, they were asked to pick from a pre-determined list of four options, which included "Other". The responses received are provided in Figure 4 and Figure 5.

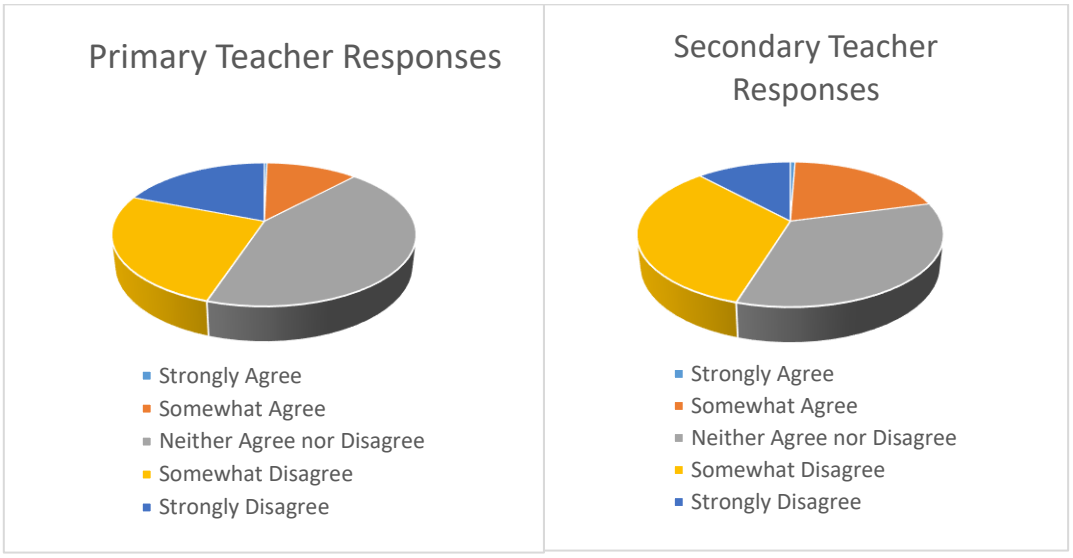


Figure 4: Teachers' responses when asked their level of agreement with the statement *"There is a fluid transition between the primary and secondary school mathematics curricula."*

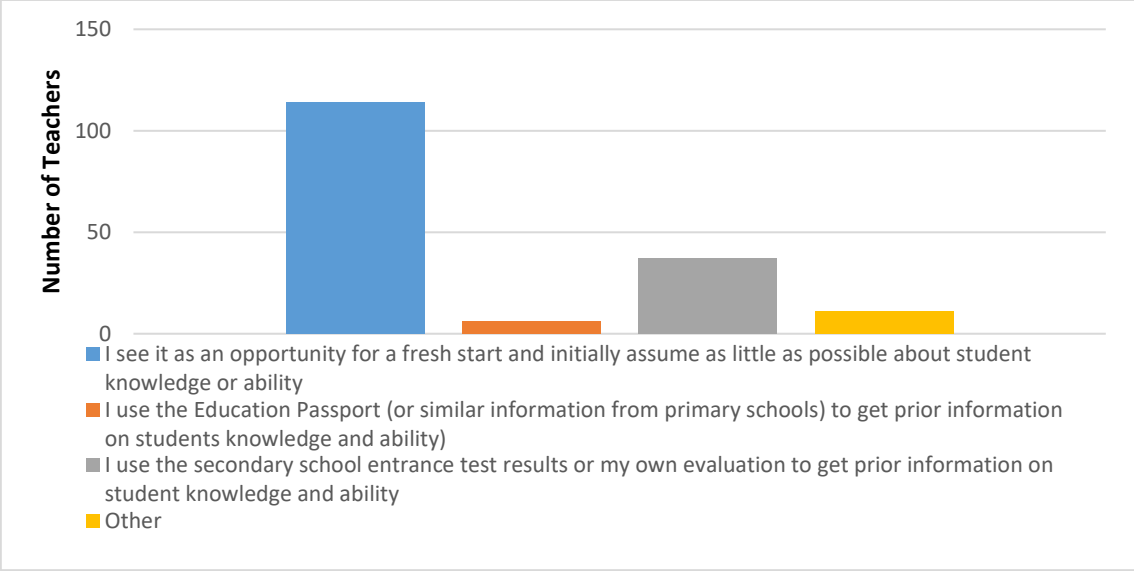


Figure 5: Second level teachers' responses when asked which of the 4 strategies outlined best describe their approach to teaching first year mathematics

Figure 4 shows that a large proportion of both groups of teachers believe that the transition from primary school mathematics to second level mathematics is not smooth. For example, 44.6% of sixth class teachers believe this to be the case compared with 44.4% of first year mathematics teachers. Only one teacher in both groups strongly agreed that there was a fluid transition between primary school mathematics and second level mathematics with a further 34 in each group agreeing with the sentiment. The lack of fluidity or continuity is elaborated upon further when secondary teachers were

asked to describe the approach that they adopt when teaching first year mathematics students. Of the 168 teachers who responded to this question 67.9% stated that they "See it as an opportunity for a fresh start and initially assume as little as possible about student knowledge or ability". Despite mechanisms, such as the Education Passport[2] being introduced in recent years, this study shows that teachers, most probably due to their own lack of knowledge of the primary school curriculum, continue to adopt a "fresh start" approach. This will undoubtedly lead to a disjointed and fractured transition from primary to secondary mathematics education.

Discussion and conclusion

"If a teacher is largely ignorant or uninformed he can do much harm" (Conant, 1963: 93)

This research study has demonstrated that sixth class teachers have gaps in their knowledge in relation to the syllabus and pedagogical approaches being adopted at second level while the same can also be said about second level mathematics teachers in relation to the primary school mathematics syllabus and favoured pedagogical approaches. Teachers in both of these sectors do not have the full repertoire of knowledge prescribed by Ball, Thames & Phelps (2008) and Ernest (1989). As Conant (1963) points out, such gaps can be detrimental to students' progress and prove a hindrance in their academic progression. This gap in teacher knowledge can prevent teachers from adequately preparing students for the transition process or providing them with a sense of continuity when they make the transition. For example, in a study carried out by Bicknell et al. (2009) teachers expressed concerns that gaps in their own knowledge meant that they were not equipped to prepare students for the mathematics they would face at second level. Likewise, students in a study carried out by Green (1997) reported that the lack of continuity between primary and second level mathematics education, which stemmed from the lack of understanding of the mathematics syllabi and teaching strategies being employed in the years either side of the transition on the part of teachers, meant that they did not face new challenges on entry to second level and as a result their motivation and attitudes declined. Hence, internationally it has been shown that these gaps in teacher knowledge can play a role in the declining attainment levels and attitudes of students during the transition. As a result, it is critical that steps are taken to improve teachers' knowledge in this regard in order to improve students' experience of transition.

In addition to the ramifications already discussed in international literature, this study found that the knowledge levels reported by teachers had other consequences, namely the lack of fluidity in transition and the approach adopted by teachers when students enter first year. The lack of fluidity in transition reported by teachers in this study is unsurprising, as without an in-depth understanding of the previous or subsequent syllabi and teaching approaches it is difficult for teachers to ensure curriculum or pedagogical continuity. Such continuity is critical in order to allow for a educationally successful or fluid transition from primary to second level mathematics education (Evangelou et al., 2008). Teachers who do not possess the knowledge domains outlined in the work of Ball, Thames & Phelps (2008) and Ernest (1989) struggle to provide curriculum and pedagogical continuity and are

[2] The Education Passport was an initiative introduced in 2014. It requires primary schools to pass documentation onto second level schools which details a rounded picture of the child's progress and achievement at primary school as well as signalling to second level schools what support a child may need. The overall purpose of the Education Passport is to help the child progress and experience continuity as they move from primary education to second level education.

forced to adopt a "fresh approach" with their first year students. This is the only option available to teachers who are not informed about the syllabus and/or pedagogical practices that students were exposed to in their previous year of schooling. It is not surprising that this deficiency in the area of MKT among teachers has resulted in pedagogical approaches that are not well received by students, and thus lead to boredom, lack of motivation and a consequential decline in students' attainment levels (Bicknell et al, 2009). Due to such concerns, the authors believe it is of paramount importance that teachers are given the opportunity to develop knowledge of the sixth class and first year mathematics syllabi; of students in both these years; and of the teaching strategies in place across both levels. Only when such opportunities are available will teachers be in a position to develop the range of knowledge domains required for teaching and the hurdle of discontinuity will be overcome.

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Intertwining noticing and knowledge in video analysis of self practice: The case of Carla

[1]Miguel Ribeiro, [2]Edelmira Badillo, [3]Glória Sánchez-Matamoros, [4]Miguel Montes, [2]Genaro de Gamboa

[1]Research Centre for Spatial and Organizational Dynamics, University of Algarve (Portugal); cmribas78@gmail.com, [2]University Autònoma of Barcelona (Spain); edelmira.badillo@uab.cat, Genaro.DeGamboa@uab.cat [3]University of Sevilla (Spain); gsanchezmatamoros@us.es [4]University of Huelva; miguelmontesnavarro@gmail.com

This paper discusses a primary prospective teacher analysis of practice from a video episode she selected from her own practice – aimed at exploring the use of non-standard length measurement units. The analysis focuses on the revealed knowledge (MKT) while in practice, on the reflection when justifying the choice of the episode, and on her analysis of such episode. The results reveal aspects of PT knowledge associated with anticipating students' difficulties, but also her difficulties in interpreting and give meaning to student use of non-standard measurement units in a non-standard way. From the analysis, the need for an improvement in the use of video-based tasks in teachers' education is discussed, as well as the importance and impact of analyzing and discussing the analysis made by PT in and for educators' professional development.

Keywords: Mathematic teachers' knowledge, measurement, video analysis.

Introduction

Mathematics teachers' knowledge and professional development has acquired an important relevance for research, enhancing its complexity. Particularly, the research focusing on mathematics prospective teachers' (PTs) education brings to the fore the role and importance of the relationships between teachers' mathematical knowledge and their knowledge of the content and students (e.g., Ball, Thames, & Phelps, 2008). Following such focuses, new trends (conceptualization and implementation of interventions) for accessing, understanding, and developing teachers' and PTs knowledge are being developed (e.g., Ribeiro, Mellone, & Jakobsen, 2013; Santagata & Bray, 2015; Sherin, Linsenmeier, & van Es, 2009).

More recently, analyzing video episodes has been both a focus of attention and a source for teacher education (e.g., Llinares & Valls, 2010; Sherin & van Es, 2002). In that sense, the reflection and discussion upon one's own practice through video analysis is perceived as a powerful path for the development of a teacher's knowledge and awareness—focusing essentially on the mathematically critical features, both for teachers and students (e.g., mathematical content, competencies, interactions). One such critical aspect for students concerns measurement (e.g., Sarama, Clements, Barrett, Van Dine, & McDonel, 2011), particularly with regard to length. Considering the core role of teacher knowledge in student learning a focus on such knowledge is essential to better understand students' difficulties.

This paper aims at contributing to a broader and deeper understanding of the hows and whys sustaining the intertwining of teachers' knowledge (in the sense of Mathematical Knowledge for Teaching—MKT; Ball et al., 2008) and professional competency of noticing (in Mason's 2002 sense). Therefore, the research question in this study is what kind of knowledge is mobilized by the teacher when analyzing students' interactions with the mathematical content in a videotaped episode

of measurement. For doing so, a video clip of a primary PT practice on length measurement is discussed, as well as her analysis of such video. The results reveal powerful trends concerning both the video analysis process as well as PT knowledge concerning length measurement.

Theoretical framework

As in other mathematical topics (e.g., adding or dividing fractions), the understanding of the mathematical whys of the measurement process is not straightforward, the understanding of such process being much more complex than the process itself. Piaget (1972) mentions that acquiring the notion of magnitude requires going through different stages, from the use of words to express the magnitude (correctly) until one has the knowledge about the measurement of such magnitude. Going through such stages is not a straightforward path, and developing a broader and deeper understanding of the concept image and definition involved (in the sense of Tall, 1988) is a core aspect of such development.

Two aspects in measuring a length are crucial: the dimension and the distance. The dimension is connected with the use of physical resources, and the distance concerns the space between two points/objects. Although both notions are perceived in an intertwined and inseparable way, due to its nature, one can't approach them in a single identical manner with students (Clements & Stephan, 2004). Such a measuring process requires the choice of the unit to use and perceive the quantity of units (or unit parts) are needed (how many) to go from one point to the other. Ultimately, one would need to combine both processes in order to get a more approximated value for the considered magnitude. In that sense, measuring is linked with two core ideas: the inverse relationship between the size of the unit used/number of units needed and the need for using the same unit in the same process of measuring (e.g., Clements & Stephan, 2004), applying it using a certain algorithm. Measuring the length thus requires knowing the standard measure(s) used, as well as the differences from using different measurement units (e.g., hands, foot, fingers) and the possibility of using these non-standard measurement units in a non-standard way.

Teachers' MKT on measurement is essential for developing students' knowledge and awareness of the topic. Among the MKT subdomains, and due to the aim and context of the work reported here, we consider for discussion the Common and Specialized Content Knowledge (CCK and SCK) and the Knowledge of Content and Students (KCS) subdomains. For doing so, examples linked with length measurement are used. The Common Content Knowledge (CCK) is associated with the mathematical knowledge required by teachers, including being able to use the instruments to correctly perform a certain measurement, knowing that no empty space must be left, as well as that there shouldn't also exist any overlays (instrumental knowledge). In doing so, knowing what instruments to use (in the sense of standard instruments/measurement) for measuring different entities and the differences between the entities they measure is involved. It corresponds to knowledge of how to perform, assuming a user's perspective (knowing how to measure). Complementary to this CCK, following the MKT conceptualization, teachers are required to be in possession of mathematical knowledge specifically linked with the tasks of teaching (Ball et al., 2008). Such Specialized Content Knowledge (SCK) includes knowing the mathematical whys justifying the different measuring processes (considering all the stages mentioned by Piaget, 1972). In addition to requiring knowledge of the different ways of measuring different entities, from one side, teachers' knowledge should also include knowing the whys of using such different forms of measuring (and the associated units).

Complementarily, it should include knowing possible different units for measuring the same entity and the ways of doing so (e.g., length), as well as the whys associated with the use of (non)standard units. In that sense, such knowledge does not include only knowing the whys associated with the procedures, but also includes the concepts (both image and definition in Tall's 1988 sense) and the whys associated with such concepts (e.g., the inverse relationship between the size of the unit and the number of iterations needed). Thus, it corresponds to a core aspect of the knowledge that allows teachers to give meaning and interpret students' solutions and comments (part of the interpretative knowledge, e.g., Ribeiro et al., 2013) while in practice.

Both CCK and SCK (and HCK, although it is not discussed here) give support for teachers' developing their practice—conceptualizing and implementing the tasks, hearing students' comments, and interpreting them in order to decide the path to follow at each moment. Those decisions are also informed by the knowledge teachers have of their students' difficulties or what they consider easier.

Intertwining knowledge of the content and knowledge of the students' learning processes concerns the KCS. It includes knowing that one of students' difficulties concern the measuring process (e.g., the need for using units with different natures), which is related to the complexity of understanding the measuring process (e.g., Clement & Stephan, 2004). In such subdomain of knowledge, one can also include the knowledge allowing teachers to anticipate students difficulties in differentiating the measurement instrument (non-standard unit, e.g., the hand) and the measuring unit, or on ways of using non-standard measuring units (e.g., using the finger length or width).

Recent research has shown the need for designing instruments/resources for teacher education allowing them (and the researcher) to characterize the knowledge in action when analyzing their own practice (e.g., Kersting, Givvin, Sotelo, & Stigler, 2010). One such resource is video analysis (of classroom episodes), which allows a focus on how and which knowledge (prospective) teachers bring to front when interpreting, analyzing, and reflecting upon the recorded interactions (e.g., Kersting et al., 2010; Van Es & Sherin 2002). When focusing on selecting and characterizing a video episode, Sherin et al. (2009) consider three dimensions: (i) window, (ii) depth and (iii) clarity. In the particular case we address here, focusing on length, windows dimension is related to evidences of students' different levels of comprehension of the measurement of length; depth dimension is related to evidences of interactions in which students participate in the decision making process about choosing the measuring unit, the instruments and the measurement procedure and clarity is related to evidence of student's arguments that transparently show their comprehension of measurement of length.

Reflecting upon what seems to be happening, and discussing grounded on the analysis elaborated, is perceived as a pathway for developing teachers' knowledge and professional noticing (Sherin et al. 2009). Such professional noticing includes identifying what is important in a teaching situation, using what one knows about the context to reason about a situation, and making connections between specific events and broader principles of teaching and learning (Van Es & Sherin, 2002). Such noticing is thus linked with the ability for examining practice pinpointing the significant aspects in order to be better informed at the time of making their pedagogical decisions.

The context of the study

Aiming at identifying and deepening the understanding of the content of teacher knowledge and how it intertwines with teacher actions and beliefs several case studies have been developed – in different

contexts and involving different school levels, mathematical topics, and competencies, contributing also to conceptualizing tasks of different natures focusing on developing teachers' knowledge (e.g., Ribeiro et al., 2013).

Method

We first present the participant of this study, and afterward, the specific context and analysis stages and process. Here we focus on the knowledge and awareness about the topic of length measurement of a primary PT (Carla), who was part of a bigger research project in which her professional development through video analysis and reflection was the aim. Carla was in the last year of the teachers' training program at University Autònoma of Barcelona, and she was teaching the field practice to grade 2. Previously, she had some training concerning classroom analysis, identifying and interpreting relevant events in a video from a novice teacher. As part of the field practice, PTs have to record the 10 classes they teach (one hour each) and choose one episode to analyze and reflect upon – which has to be transcribed. The selection of the episode needs to be justified, mentioning the whys associated with its mathematical richness and referring explicitly to the mathematical goal pursued. For selecting and analyzing such an episode, Sherin et al.'s (2009) criteria (windows, depth and clarity) should be used (which have been previously explored with the PTs). Therefore, the analysis should focus on the mathematical content approached, student-teacher interaction, and students' understanding.

For selecting the episodes, PTs' were advised to look for situations they perceived as involving a "high level" of students mathematical knowledge and discussions/argumentations. Carla's choice was an episode aimed at introducing "the measurement of the length and width of objects using non-standard units" (e.g., the length of the classroom using the foot; the length of a glue package using the finger). An example of the proposed tasks is: using an unconventional length measure (finger), determine the length of the glue package. Although Carla chose a 15 minutes long video associated with the goal "review the content," she only considered it important to analyze and reflect upon the last six minutes – sustaining such choice on her perceived richness, in terms of Sherin et al.'s criteria (2009). In the video, one can perceive how the students assign the corresponding number (amount of units) to the length or width of the objects as a result of comparing such amount with the unitary measurement they consider as a reference. In the discussion, the inverse relationship between the length to be measured and the unit used is also explored. Complementary to the video and Carla's analysis, Carla (as all the other PT) were also asked to justify (in a written form) their actions in the video they selected.

The analyses were made in different stages and each of them followed the same structure. First, each researcher focused on the divergent aspects, then there was a joint discussion focusing on them. Such joint discussions contributed to a refinement of the analysis and increased the researchers' own mathematical awareness and "interpretative knowledge." Here we expand the notion attributed to this expression by Ribeiro et al. (2013) by including the researcher's ability to read, hear, and understand the interactions and knowledge in action. The first stage focused on analyzing Carla's revealed knowledge in the part of the video she considered important to analyze and reflect upon—the last six minutes. In the second stage, the focus of attention was Carla's analysis of her own practice using the same criteria PTs use for doing the analysis (Sherin et al.'s 2009 criteria). The third stage focused on the mathematical aspects associated with what happened in the first part of the video—which Carla

chose not to include in her analysis—and the possible whys concerning her knowledge and awareness leading to such choice.

Analysis and discussion

When analyzing Carla's practice (the 15' video) with a focus on the CCK, SCK, and KCS, different aspects of her revealed knowledge are identified that sustain her awareness and professional competency of noticing. She reveals knowing the importance of measuring with different non-standard units (CCK), leading to the inverse relationship between the number of units and the size of such units (CCK):

> Carla: *I gave an example, showing to the students that is not the same measuring with bigger o smaller hands, so I put my hand in the sheet they had to measure and asked a student to put his hand next to mine, to compare the different measures of both hands, telling them: "Do you see that his hand is smaller? Then my measure will be smaller than his".*

Also, the students' difficulties in measuring without leaving empty spaces between units is anticipated (KCS) when interacting with students:

> Carla: *Of course, another thing is how we put our hands. If some of you put them like this (partially opened) and some of you put them as Isaac did (completely opened), Isaac will get less . . . . But it doesn't mean that it is wrong; it simply indicates that we have different hands and we have measured differently.*

However, we can see a potential conflict identifying the unit of measurement, arising at least four different possible unit combinations: kin's and teacher's hand, both in a close or open position. In spite of Carla's ability to identify the previous situation as an interesting topic to be discussed in the class (noticing), it is noteworthy that there is a lack of awareness of the necessity of exhaustiveness when re-covering the measurable object (Clements & Stephan, 2004).

While teaching, Carla indistinctly uses dimension and distance—the two aspects of length (Dickson, Brown, & Gibson, 1991)—revealing aspects of the content of teachers' knowledge that need to be a specific focus of attention in training. Such knowledge would sustain the conceptualization and implementation of tasks aimed explicitly at exploring both concepts and their complexity. Not overcoming such difficulties, and thereby enriching teachers' SCK on measurement, would contribute to a low level of professional competence of noticing, particularly concerning the students' difficulties in distinguishing perimeter and area (and volume)—KCS.

Focusing on her analysis of the students' reasoning (only the last six minutes of the video), the fact that she can differentiate various aspects of understanding from different students reveals Carla's (advanced) level of professional competency of noticing according to Sherin et al.'s (2009) criteria. Such can be linked with her KCS as well as her interpretative knowledge (Ribeiro et al., 2013). In that sense, she takes into consideration (CCK and SCK) the dimension, the absence of empty spaces between the use of the measurement units, the use of anthropomorphic units, and the inverse relationship between the size of the unit and the number of iterations – on the written justification Carla wrote: *Hugo and Daniel notice that the different results of the measurement depends on the size of the hands; Miguel, on the other hand, concludes that the result depends on the ways the hands are placed*).

When justifying the choice of the episode to analyze, Carla's reflection reveals an awareness of the relationship between the SCK on the content, her decisions, and her ability to anticipate the students' difficulties and understanding of the topic (KCS).

> Carla: *I chose this particular part of the class because it reveals the moment when the pupils become aware that when measuring with non-standard instruments and units, they get different results…Through the students' reflection and reasoning, they get that the size of the hands matter when measuring the length (thus getting different results) and that not only the size of the hand matters, but also the ways one uses it. I also have to note that after the mentioned reasoning emerged from students, I registered the different values on the blackboard, and I could then use it to explain to the whole class. During this period, I also use different teaching strategies, including the registration on the board (with the number and the unit used), giving time for students to present their results, and promoting reflection concerning the measurement technique used, supported by visual examples, as they could have difficulties in the content concerning both the measuring process and finding the relationship between length and units used.*

However, in the remainder time of the episode Carla didn't chose to analyze (the first nine minutes), something curious and mathematically important happened. Analyzing it can shed some light on teacher decisions and associated specialized knowledge as well as concerning the need for a change in Sherin et al.'s (2009) criteria for selecting and analyzing he video episodes.

During the first part of the episode, one student (Miguel) provides a completely different answer to the posed questions (e.g., length of the class in steps, when the rest of the class provided answers around 30, his answer was 14). Carla interpreted it as a misunderstanding of the measurement process, showing him in each case "how to do it correctly." But the consistency of Miguel's answers reveals a high level of understanding (concept image and definition, Tall, 1988) of measurement (Clements & Stephan, 2004), using non-standard units. When Carla discussed the question "How many fingers are needed to measure the glue package?" Miguel again disagrees with the other students and with Carla's validation of a length of eight fingers. With the glue package on the top of the table, on the vertical, the following dialogue occurs:

> Teacher: *No? How many fingers did you get?*
>
> Miguel (putting the finger vertically along the glue package): *One!*
>
> Teacher: *One? Like this? (The teacher repeats the measurement process using the indicator finger horizontally)*
>
> Miguel: *No, two…*
>
> Teacher: *Two? With two you can cover all the distance?*
>
> Miguel: *No…ah…four…*
>
> Teacher: *Don't know what you are measuring…*
>
> Miguel: *Ah, four, four…*
>
> Teacher: *No! It can't be…you should get eight, you are doing it wrong.*

Although the goal of the class was "*to use non-standard length measures*," such measures have been used and explored by Carla only in a standard way (the one with which she was used to – taking into consideration the cultural aspects). Such exclusive use, linked with the content of SCK, makes it difficult to anticipate and understand (KCS) Miguel's answers—use of non-standard measurement units in a non-standard way. The fact that when reflecting upon her practice and analyzing the video she did not point to this aspect as problematic sustains the need for a change in the way the video analysis task has been conceptualized. And, in an intertwined way, makes us wonder about Sherin et al.'s (2009) criteria for professional competency, as when focusing only on her own analysis (of the last six minutes), Carla could have been considered to have an advanced level of such competence; when looking at the part of the video she did not analyze, a different conclusion could be drawn. In that sense, there is also evidence of the interdependent nature of such competence and teachers' knowledge.

Carla seems to not be aware of the inexplicit use of a one-dimensional measurement unit on a three-dimensional object, leading thus to some contingency moments. To overcome such moments, she opted (grounded in her own revealed knowledge of measurement) to tell the student "how to do it". Although aspects of professional awareness and competencies of noticing are present both in her practice and in her analysis of the video, her knowledge shaped such aspect, leading to a partial view of the students' understanding of the measurement length and units. This supports the need for a complementary discussion and reflection upon the video analyses PTs make in order to focus also on the mathematically critical aspects they are not aware of, and which are a barrier to completely achieving the advanced level of professional competency of noticing expected when using Sherin et al.'s (2009) criteria.

Final comments

From the work focusing on the video analysis, three aspects can be enhanced: teachers' knowledge and professional noticing abilities; reflection and awareness capabilities; video-based task design and its potentialities. Although of different natures, these aspects have in common the educator's responsibility in changing the training process. Through this case study, one can better understand aspects of the content of the different subdomains of MKT, both revealed in practice as well as expressed when reflecting upon such practice when analyzing the video. Other aspects concern the impact of the interactions and the prevalence of non-standard length measurements in standard ways, and that revisiting the practice didn't become a prompt for awareness and noticing development, obviously linked with previous experiences and MKT. This calls our attention to the need for an exterior element to pinpoint these critical features, leading them to become a starting point for developing such knowledge and awareness.

Although the video analysis performed by Carla with the provided instrument did not accomplish completely the defined aim, our analysis of her practice and of her own analysis of the video was a prompt for inquiring our own practice as educators in terms of interpretative knowledge (Ribeiro et al., 2013) and of awareness of the educators professional noticing abilities. One future research path concerns thus the develop of a complementary instrument for video analysis that would allow trainees

to dig deeper into their awareness of the mathematical whys sustaining what happens and to focus also on the educators specialized knowledge, awareness and noticing abilities.

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Investigating Lesson Study as a practice-based approach to study the development of mathematics teachers' professional practice

Anita Tyskerud, Janne Fauskanger, Reidar Mosvold, & Raymond Bjuland

University of Stavanger, Faculty of Arts and Education, Norway; anita.tyskerud@uis.no

The study, whose methodological approach is the focus of attention in this paper, is a qualitative, single longitudinal case study. The object of study is Lesson Study (LS), and the unit of analysis is two LS cycles. What teachers learn about teaching practice and student learning in mathematics from participating in the two cycles is investigated. LS and teaching practice are in the study regarded as object-oriented activities. It is claimed that indications of what the teachers learn during LS processes can be uncovered by the use of discourse analysis because learning is considered as a change in discourse.

Keywords: Lesson Study, mathematics teachers, professional development, teaching practice.

Introduction

Recent studies have focused on the potential of practice-based approaches for developing mathematics teachers' knowledge and practice. In this paper, we investigate a methodological approach to study how Lesson Study (hereafter LS) as a particular practice-based approach to professional development can contribute to teachers' development (Thames & Van Zoest, 2013). In Japan, LS has been used for professional development of teachers for more than a century (Ronda, 2013). Since Stigler and Hiebert (1999) wrote "The Teaching Gap", researchers from other countries have become interested in LS as a structured approach to teachers' professional development (e.g., Fernandez, 2002). In Norway, the Ministry of Education and Research calls for more school-development projects, and LS is mentioned specifically in a recent strategy document (KD, 2014).

Cohen, Raudenbush and Ball (2003) suggest that teaching can be regarded as instructional interactions among teachers and students around a certain content. Increased student learning thereby requires a change in these instructional interactions. Thames and Van Zoest (2013) call for research to focus more directly on these instructional dynamics. We suggest that LS provides a great venue for studying the development of teachers' interactions about teaching practice and student learning.

We focus on issues related to research design and methods in a project where LS is used to study indications of what the teachers learn during LS processes. To frame this discussion, the paper presents an ongoing research project in a Norwegian lower secondary school, where teachers learn and develop their professional practice from participating in two LS cycles. The aim is to highlight and discuss some methodological issues that occur when coordinating two sociocultural theories in a study of teachers' learning about their own practice and student learning. Although this is a theoretical rather than an empirical paper – some would suggest that it sits at the border in between – we provide a brief empirical example from the study to illuminate our approach.

Context of the study

A group of mathematics teachers is observed in two LS cycles with an overall focus on what the teachers learn about their own teaching practice and student learning from participating in these two cycles. A sociocultural stance is used to investigate teachers' learning and to understand the

participants' perspectives and interactions in the LS group. Knowledge is regarded as shared and collective rather than individual and develops through social negotiation (Radford, 2008). The role of verbal interaction in the learning process is essential, because new knowledge is considered to develop through talk in social interaction (Dudley, 2013). The theoretical and analytical frameworks used in this study coordinate two sociocultural theories: activity theory (Leontiev, 1978) and the commognitive theory (Sfard, 2008). Research on human development and learning thus becomes the study of development of discourse.

Tabach and Nachlieli (2016) propose a combination of activity theory with communicational theories to study mathematics teaching, and our coordinated theoretical framework adheres to this proposal. Activity theory is used as a grand theory, and LS and teaching practice are seen as activities in the way Leontiev (1978) thought of activity. To Leontiev, all human activities are oriented towards an object with a certain motive. The activities consist of three components at dynamic levels: object-motive, actions-goals and operations. In coordination with this theoretical perspective, and to identify what the teachers have learned on a discourse level, the commognitive theory (Sfard, 2008) is used as a local theory. This theory defines learning in terms of discourse, and it presents certain characteristics of a mathematical discourse: word use, visual mediators, routines and endorsed narratives. In this study, the development of teachers' mathematical discourse about teaching practice and student learning in the goal-oriented actions and operations in the LS activity is studied.

The main data sources for the ongoing research project are video-recorded observations from two LS cycles and focus group interviews (FGI). One LS cycle lasts about three months. The two cycles took place in spring (first cycle) and autumn (second cycle) 2016. The school implemented LS as their school-development project in January 2016, and this was the teachers' first experience with LS. The LS group consists of four mathematics teachers, one participant from the school administration (the group leader) and one external expert (the first author of this paper). All the teachers' meetings are video- and audio-recorded. In addition, all documents produced by the teachers during the whole process, and some of the students' written works, are collected. Since conversation and communication are crucial in the study, FGIs before and after each LS cycle are conducted. The purpose of a FGI is to get a variety of perspectives on a given subject (Kvale, 2007). In the first FGI, a discussion about the current teachers' teaching practice, including making plans, teaching, evaluation and the teachers' thoughts about student learning is facilitated. In the second FGI, it is important to let the participants reflect on what they have learned about their own teaching practice and student learning. In the third FGI, the focus is on the LS process and what can be done differently in the next cycle. In the last FGI, the most crucial topic relates to teaching practice and student learning, and contains the same focus as the second FGI.

A coordinated theoretical framework to investigate teacher development

Describing the landscape of using more than one theory, Birkner-Ahsbahs and Prediger (2014) refer to networking strategies or connecting strategies. They distinguish these strategies by the degree of integration, from *ignoring other theories* on the one hand, to *global unification* on the other. Figure 1 gives an overview of the approaches in between. Networking strategies are useful to analyse the same empirical phenomena using different approaches (Birkner-Ahsbahs & Prediger, 2014).

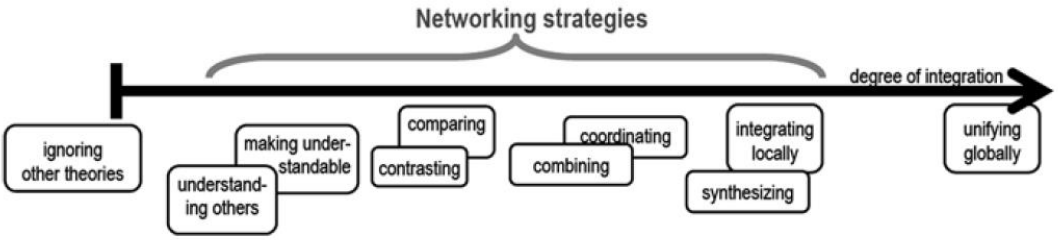

Figure 1. Networking strategies (Bikner-Ahsbahs & Prediger, 2014, p. 119).

The strategies of coordinating and combining are mostly used for a networked understanding of an empirical phenomenon or a piece of data (Bikner-Ahsbahs & Prediger, 2014, pp. 119–120). The difference between coordinating and combining theories depends on how elements from the two theories are well fitting or not. While coordinating can only be possible if the theories have compatible cores, theories with conflicting basic assumptions can be combined. In this study, activity theory and the commognitive theory are coordinated, because they have similar ontological and epistemological perspectives. LS and teaching practice are seen as object-oriented activities. Tables 1 and 2 give an overview of these activities. LS and teaching practice as object-oriented activities have a common motive: to promote student learning. They have however, different objects and goal-oriented actions. Another main difference is the teaching in LS (the research lesson). Because of the participating observers and the teachers' research question(s) – they are researching their own teaching practice – the teaching is planned in order to promote teachers' learning and development as well as students' learning. The focus on instructional interactions between teacher and students around content (cf. Cohen et al., 2003; Thames & Van Zoest, 2013) is naturally embedded in LS. In order to study your own teaching practice to increase student learning, teachers are conducting goal-oriented actions. These actions represent each step of the LS cycle; formulate goals and plan the lesson, teach the lesson, observe students and reflect on/evaluate the research lesson. Each action has its own goal – prepare teaching, facilitate students' learning, gather data to answer research questions and learn from the lesson. To constitute the goal-oriented actions, there are different operations, listed in Table 1.

To describe the meaning of teachers' professional teaching practice, we draw upon the work of Ball and Forzani (2009). They consider mathematics teaching as professional work and this work of teaching mathematics does not come natural. It has to be learned through deliberate training. Teaching practice is all about designing activities that increase student learning, but the work of teaching mathematics can also be decomposed into several core components. For instance, a teacher must present mathematical ideas, respond to students' mathematical questions, find examples that illustrate certain mathematical points and so on – all examples of what Ball and colleagues refer to as "the mathematical tasks of teaching" (Ball & Forzani, 2009; Ball, Thames, & Phelps, 2008). This is another aspect of what they mean by referring to teaching as professional practice. In order to carry out the tasks of teaching mathematics, a specific knowledge is required that is connected with the work of teaching. This constitutes a particular knowledge base that is shared within the teaching profession.

Activity	**Lesson Study**														
Objects/ motive	Investigate own teaching (object) to improve students' learning (motive)														
Actions	Plan the lesson				Teach the lesson			Observe the lesson				Evaluate the lesson			
Goals	Prepare teaching				Facilitate students' learning			Gather data to answer research questions				Improve the lesson			
Operations	Study other textbooks, curriculum materials, teacher journals, and previous research	Formulate lessons goals, research question	Select artefacts, design worksheets, group work, differentiation	Prediction on students responses and plan observation	Introduce and exemplify	Present problems	Support students' problem solving	Observe students	Observe case students	Documenting learning indicators	Interview case students	Discuss achievement	Discuss the strengths and weaknesses of teaching	Discuss observation	Reflect on improvement and make a new plan	Document and present results

Table 1: LS and Activity Theory (translated from Mosvold & Bjuland, 2016, p. 188)

Considering teaching or "the work of teaching" as an object-oriented activity, the object/motive is teaching in a way that leads to student learning. The goal-oriented actions are the tasks of teaching, and the operations are when the teachers actually conduct the tasks of teaching (see Table 2). These operations require mathematical knowledge for teaching (Ball et al., 2008). Developing teaching practice also includes developing teachers' knowledge for teaching (Lerman, 2013). In this study, Ball et al.'s (2008) knowledge component: "Knowledge of content and students" is used when studying what teachers learn about student learning.

Activity	**The work of teaching mathematics**
Objects/motive	Help students learn
Actions	Mathematical tasks of teaching
Goals	Student learning of specific mathematical content
Operations	Conduct the mathematical tasks of teaching, depending on teachers' knowledge for teaching

Table 2. Teaching practice and Activity Theory

The following example illustrates how the local theory is applied in coordination with the grand theory. In the activity of LS, one of the goal-oriented actions is evaluation. Within this action, an

operation is to "discuss observations" (see Table 1). In the discussions of observations from the second research lesson, the teachers discuss how the students had worked on a task of finding the shape of a sandpit that can fit 500 litres of sand. They recall how one boy responded when challenged by the teacher to try another figure than a rectangle – like a triangle. The boy responded, "Yes, then we just double, because that is half, then…" Another teacher comments that the boy found another solution. Although the example is limited, it displays some characteristics of a mathematical discourse. In this action of discussing their observations, the teachers use mathematical words like shape, rectangle and triangle. Their restatement of a student's response illustrates a mathematical routine that appears to involve the area of a triangle. When analysing the teachers' discourse in the actions of LS over time, the local theory may help us identify changes in discourse – which is how Sfard (2008) defines learning – on an object level or meta level. Introduction of new words are examples of object level learning, whereas changes in the metarules of the discourse constitute learning or development on meta level.

Considering the role of the researcher

In the described study – like in numerous similar LS research projects – the researcher acts as participant observer. Being a participant observer in research – as the first author of this paper – might lead to both advantages and challenges. Connelly and Clandinin (1990) underline one advantage when they focus on the relationship between the researcher and the participants in the context of research in education. They stress the importance of all parties' equality, which gives rise to better collaboration. The first author is a participant observer in the way Bryman (2012) defines as being an "overt full member" (p. 441). This means that the researcher is completely involved in the group's work. Bryman distinguishes between "covert full member" and "overt full member". The differences being if the members of the group are aware of the researcher's status as a researcher or not. In the present study, the participants are aware of the first author's role as a researcher. Bryman (2012) claims that there are some challenges associated with the "overt full member" role. As an active participant, you may forget your role as a researcher. He refers to this as "going native" (Bryman, 2012, p. 445). To avoid this, it is important to be aware of the different roles you have as a participating observer. In the group meetings, the researcher switches between a conversation role and a member role. In the research lesson, the researcher does not teach the lesson, but participates in activities as an observer. Wadel (1991) refers to this role as the role of the apprentice. In addition, another essential aspect of the researcher role in this study is "the knowledgeable other" in the LS group, the role of observer-spectator (Wadel, 1991). The most important part of this role is to guide the group through the LS cycle and help the teachers to keep focus on their own research. Previous research has shown that without an external expert, teachers easily forget the research question (e.g., Takahashi, 2013) and collaborate without actually doing LS.

Staying long in the field increases the stability of observations and dependability in a qualitative project (Cohen, Manion, & Morrison, 2007). In this project, data collection spans over a calendar year. The time span is particularly important when the researcher acts as participant observer in a LS group, in order to reduce potential reactivity effects (Cohen et al., 2007). Another element that supports the dependability in the study is the teachers' reflections on the outcomes of their own learning. This is useful for the analysis, because we can then compare findings (related to observed change in discourse) with the teachers' own reflections. The participants' opportunity to agree with

the descriptions and interpretations the researcher makes during the LS cycles underpin the confirmability in this research. Since one researcher is participating in all the conversations when the teachers talk about their own reflections on a meta-level, this researcher's voice – repeating their different opinions – enables the participants to confirm or disconfirm. This can only happen because one researcher is a participant observer.

In the final step of a LS cycle, the teachers have to think through what they have learned during the whole process. Based on interpretations of the data material, the researcher attempts to make thick description of teachers' learning through LS. In the process of creating such thick descriptions, we follow Stake (2010) who emphasizes the connection to theory in addition to providing rich descriptions and interpretations of data – thus supporting the transferability of the research.

Concluding discussion

In this paper, we have referred to a study of teacher learning in LS as a starting point for discussing some theoretical and methodological issues that can be involved when studying what teachers learn about teaching practice and student learning. In their call for more practice-based approaches to study the development of mathematics teachers' knowledge and professional practice, Thames and Van Zoest (2013) argued that such efforts required "work on conceptualizing practice, formulating questions about practice, and developing methods for studying it" (pp. 592–593). We suggest that LS provides a useful venue for such studies, but we agree with these researchers that further work – conceptual and methodological – is necessary. A possible approach is to use our proposed coordination (Bikner-Ahsbahs & Prediger, 2014) of activity theory and the commognitive theory to study mathematics teachers' learning in the context of LS. This might be useful in multiple ways. In the following, we highlight two potential benefits of such a coordinated theoretical framework.

First, the application of Leontiev's (1978) activity theory provides a useful framing for a reconceptualization of the work of teaching mathematics. Ball and Forzani (2009) propose that the work of teaching mathematics is constituted by the recurrent tasks of teaching that teachers encounter when carrying out this work. Their conceptualization fits within the idea of teaching as professional practice. In the TeachingWorks (2015) project, they develop these ideas further and identify a number of core practices that are particularly important in the work of teaching. A challenge with these and other efforts to conceptualize the work of teaching is that the components of practice – for instance the mathematical tasks of teaching – sometimes appear to be on different levels, and the issue of purpose often appears absent. Using Leontiev's (1978) idea of distinguishing between object-oriented activity, goal-oriented actions and operations in a reconceptualization of the work of teaching mathematics, may solve both of these potential challenges while at the same time preserving the obvious strengths of previous conceptualizations. Such a theory-based reconceptualization enables new questions to be posed and may support the development of a theory of mathematics teaching that communicates with existing theories of learning and development.

Second, the application of Sfard's (2008) commognitive theory enables the development of more operational definitions of teaching and teacher learning about teaching practice and student learning. When applying a definition of teaching that coordinates perspectives from activity theory with Sfard's theory, the issues of motives and purpose are embedded. The proposed definition of Tabach and Nachlieli (2016, p. 303) is a good candidate: "teaching can be defined as the communicational activity

the motive of which is to bring the learner's discourse closer to a canonical discourse." This definition draws upon Sfard's (2008) definition of learning as an observable change in discourse, and the application of such a theory makes teaching and learning more easily observable.

In interpretative research, the goal is to understand and interpret the meanings of human behaviour such as teachers' talk, and it is important for the researcher to understand motives, meanings, reasons and other subjective experiences rather than to predict causes and effects (Hudson & Ozanne, 1988). This paper highlights and discusses some methodological issues that may arise when investigating development of mathematical knowledge for teaching in LS from a participationist (rather than acquisitionist) perspective (Sfard, 2008), focusing on teachers' participation in object-oriented LS activities and analysing their learning in terms of discourse as two different grain sizes. The two levels occur because the theories look at learning differently; activity theory is focusing on acting humans, whereas discourse theory is focusing on humans who communicate. Both perspectives are arguably embedded when mathematics teachers' professional practice is developed through LS, and an application of such a coordinated theoretical perspective might represent another step towards the efforts to understand what teachers learn about teaching practice and student learning (cf. Thames & Van Zoest, 2013).

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First year student teachers dealing with non-routine questions in the context of the entrance examination to a degree in Primary Education[1]

Aitor Villarreal, Lluís Albarracín[2] and Núria Gorgorió

Universitat Autònoma de Barcelona, Facultat de Ciències de l'Educació, Spain;
aitor.villarreal@uab.cat; lluis.albarracin@uab.cat; nuria.gorgorio@uab.cat

We understand Fundamental Mathematical Knowledge (FMK) as the initial mathematical background we expect students to have at the start of their education to become Primary School teachers. In this paper, we focus on the answers given by 241 first-year student teachers to two non-routine questions that were part of an entrance examination. Non-routine questions are those for which students do not have a straightforward algorithm available to reach the answer and/or the result obtained by using it needs to be interpreted within the problem's context. The results show that non-routine questions are a powerful tool to assess the solidity and stability of their initial mathematical knowledge but also a powerful tool to find out whether the students have appropriated the essential ways of thinking and working in mathematics. Our findings also suggest that the mathematical knowledge of first-year student teachers is far from the desired FMK.

Keywords: Fundamental Mathematical Knowledge, non-routine questions, teacher education.

Introduction

Within the framework of a program that aims to improve initial teacher training in Catalonia – *Programa per a la Millora de la Formació Inicial de Mestres, MIF*– an entrance assessment has been created. Its purpose is to regulate the access to primary teaching degrees and will be applied for the first time in June 2017. The assessment includes a mathematics test intended to ensure the mathematical knowledge of future students of teaching degrees. In this context, in an attempt to find evidence of the distance between the student candidates' initial mathematical knowledge and the desirable mathematical knowledge, our study focuses on characterizing the knowledge demonstrated by a group of 241 students in their first year of training as primary teachers. In this paper we focus on the analysis of their responses to 2 non-routine questions.

Fundamental Mathematical Knowledge (FMK)

In Castro, Mengual, Prat, Albarracín and Gorgorió (2014) we introduced Fundamental Mathematical Knowledge (FMK) as the disciplinary mathematical knowledge that students need to benefit from courses in mathematics and mathematics teaching during their education to become teachers, considering the requirements of the professional practice and the competences to be developed by children in primary education. FMK is the disciplinary knowledge on which to build throughout teacher training to attain the mathematical and pedagogical content knowledge required

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[2] Lluís Albarracín is a Serra Húnter Fellow at the Universitat Autònoma de Barcelona.

for professional practice. As teacher educators, we regard FMK as the mathematical knowledge starting point for our courses, which should be based on a thorough knowledge of elementary mathematics, being the foundation that would support a structurally robust training.

As we have defined FMK, it is not explicitly part of the different models that characterize teacher knowledge. However, it relates to them. Ma (1999) develops Shulmans' proposal (1986) and brings up the notion of profound understanding of emergent mathematics. Our idea of FMK is related to Ma's proposal, but focuses on the initial knowledge required for teacher training, as opposed to this author who is interested in the educated teacher. Ball, Thames and Phelps (2008) propose a characterization of mathematical knowledge for teaching from what teachers do when they teach mathematics and from the knowledge and skills they need to achieve for students to learn.

Considering that we are focusing on young adults that want to become teachers, FMK is close to their notion of Common Content Knowledge (CCK), understood as the subject-specific knowledge needed to recognize and solve mathematical problems that any educated adult should have. However, since FMK is defined in terms of requirements to enter a teaching degree, FMK would be mathematical knowledge essentially linked to the school-subject "mathematics". FMK would be the knowledge upon which our students would construct their Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK) during their education to become teachers. Rowland (2008), based on observation in the classroom to characterize situations in which the teachers' mathematical content knowledge is visible, proposes the reference framework called Knowledge Quartet Framework. The Knowledge Quartet has 4 dimensions: foundation, transformation, connection and contingency. Again, the idea of FMK can be related to the Foundation dimension, even though the latter refers to expert knowledge

There is extensive research aimed at developing theoretical models of different types of knowledge required to teach mathematics. However, much less attention has been paid to research focused on establishing what students' mathematical knowledge is (or should be) when entering teacher training programs (Linsell and Anakin, 2012). According to Linsell and Anakin (2013), the theory developed so far around knowledge for mathematics teaching shows limitations when analysing students' knowledge on entrance to the faculties, since it is based on what teachers do in practice.

FMK focuses on the knowledge of core concepts and on the ability to solve exercises, problems and situations applied to different fields – numbering and arithmetic, relations and change, space and shape, measurement, statistics, and randomness – together with the ability to assess the adequacy and reasonableness of the response in each case. We understand mathematical competence as the ability to use mathematical knowledge encompassing both mathematical and non-mathematical situations. Mathematical competence is based on factual knowledge and concrete skills to carry out mathematical activities and it includes the ability to ask and answer questions in and with mathematics, and the ability to deal with mathematical language and tools (Niss and Højgaard, 2011).

Mathematical competence goes beyond knowledge of procedures and it manifests itself in the use of conceptual knowledge in different situations. It requires the knowledge of rules, definitions and connections and domain structure, and knowing why certain procedures work for certain problems, what the purpose of each step of a procedure is, and making the connections between these steps

and their conceptual foundations. Non-routine questions cannot be approached in an automated way and solving them requires a deep understanding of the concepts and procedures involved in them. This is why we want to know whether or not non-routine questions are a suitable tool to evaluate the initial mathematical knowledge of student candidates in relation to FMK in terms of competences.

Method

In the process of construction and refinement of the mathematics test of the entrance assessment, we examined different groups of students who had just started their training as teachers but had not yet had any courses related to mathematics or didactics of mathematics. The main data used in this study comes from the test answers of the 241 first-year students of the Primary Education Degree at the Universitat Autònoma de Barcelona during the school year 2015-16. For this cohort of students, the test was not yet a requirement for admission. The test students had to pass had 4 different versions around the same mathematical ideas, and was made up of 25 questions. Students were given 90 minutes to complete the test not being allowed to use any type of calculator. They were randomly handed out one of the four versions, and we afterwards verified that the four groups of students were statistically equivalent in terms of their entrance characteristics.

The questions in the test were related to the core content of the curriculum for compulsory education. At least half of them were non-routine questions in the sense that there was no straightforward algorithm available to the student to reach the answer. In the cases in which such an algorithm was available, the result obtained by using it had to be interpreted within the problem's context. When there was a simple approach to solve a question, its application would however be a cumbersome and time-consuming task. Therefore, given the limited time, these questions called for the development of efficient approaches to solve them.

Results

Ruler question

Questions 1 and 2, given in Figure 1, were question 15 in versions 1 and 4, and versions 2 and 3 of the test respectively. In the students' test, the centimetre in the images corresponded to exactly one centimetre.

This question involves the measurement of a length with a tool, which is part of the content of compulsory education and is similar to questions that can be found in tools aimed to test primary children's mathematics. It's a non-routine question, since it cannot be answered by a direct reading of the image because the segment to be measured does not start at point 0 and the subdivisions of the ruler do not correspond to decimal units. The question requires knowledge of the number line and representing fractions on it.

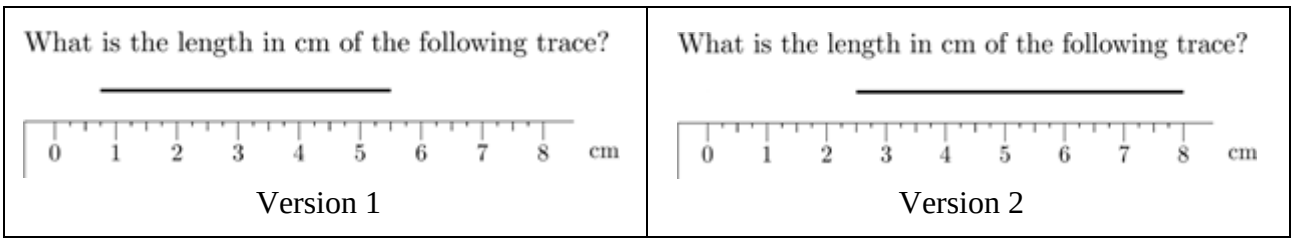


Figure 1: Question 15 of versions 1 and 2

The students' answers are summarized in Table 1. Since all the students answered this question, both in version 1 and 2, we interpret that they believed to know the correct answer. For the question in version 2, where the fraction involved is 1/2, 78.22% of the students gave a correct answer, while in version 1, which additionally requires a measurement using the ¾ fraction, the percentage of success is 38.46%.

Version	Correct Answers	Errors	Blank	Total
1	45 (38.46%)	72	0	117
2	97 (78.22%)	27	0	124
Total	142 (58.92%)	99	0	241

Table 1: Answers to the *Ruler question*

We generally observe a tendency to induct, i.e. the students answer the question from what they would do to approach a similar routine question. The correct answer is 4.75 cm, and little more than a third of the students answered correctly, and the answer 4.3 arises from considering that each subdivision of the unit corresponds to 0.1 cm and calculating the total length counting subdivisions. This answer contains an additional error, since units are ignored when expressing the measurement.

Similarly, the answers 4.7 cm, 5.3 cm and 5.7 cm stem from automatically considering that each subdivision equals 0.1 cm. The three answers consider a reading of the graphical information contained in whole units. Thus, $4.7 = 5 – 3 \times 0.1$. In the 5.3 answer, there is an additional error since subtraction would be required $5.3 = 5 + 3 \times 0.1$. The 5.7 answer implies an incorrect reading of the graphical information when supposing that the segment contains 6 whole units $5.7 = 6 – 3 \times 0.1$.

The error of 19 cm also answers to an automated process, since it stems from considering each subdivision as 0.1 cm. The incorrect answer 9.5 cm arises from considering that every two divisions equals 1 cm and then acting routinely. In addition, those who make this mistake do not stop to think whether the segment may be longer than the ruler being used to measure. We also noted another type of errors resulting of applying a routine process. Here the students gave the result of the measurement by rounding amounts, such as in 4.8 cm or 5.8 cm, from rounding 4.75 and 5.75, respectively. In version 2, the correct answer is 5.5 and the errors detected similarly stem from applying automated processes – such as acting inductively or rounding – without questioning the meaning of the procedure they are using or the answer obtained.

Cubes question

Question number 16 – shown in Figure 2 – was slightly different in the four versions of the test, but was always related to the concept of volume. It approached the idea of measuring the volume of a solid directly by counting units, again a concept present in compulsory education. However, all these were non-routine questions in the sense that they differed from typical textbook exercises that require the calculation of a volume by applying a formula. This question additionally required the interpretation of figural information, even though the figures were rather simple.

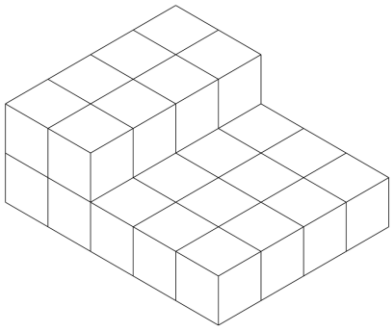
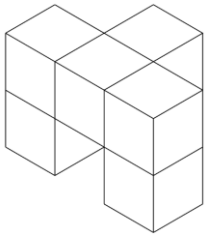
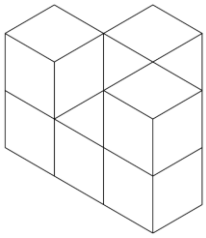
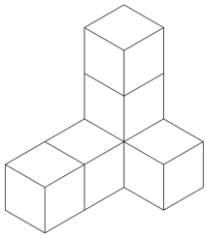
If the edge of each cube measures 4 cm, what is the volume of this object?  Version 1	If the edge of each cube measures 4 cm, what is the volume of this object knowing that we can see all the cubes that integrate the object?  Version 2
If the edge of each cube measures 4 cm and the volume of the object is 512 cm^3, how many hidden cubes are there?  Version 3	The volume of the object is 384 cm^3. All the cubes that integrate it are identical. What is the length of the edge of the cubes?  Version 4

Figure 2: Versions 1, 2, 3 and 4 of the Cubes question

In versions 1 and 2, the question required interpreting figural information, mastering the notion of unit of measurement, being able to calculate the volume of a cube with a 4 cm edge and, particularly, having a way to quickly figure out how much 4 x 4 x 4 is. In both cases, students should be able to express their answers using the appropriate units. Version 2 was more complex than version 1, since it explicitly brought up the idea of hidden cubes. The questions in versions 3 and 4 also required interpreting figural information in a representation that also included the idea of hidden cubes, but this time providing the volume of the global object.

The question in version 3 required determining the number of hidden cubes, therefore calculating the volume of a cube with an edge of 4 cm, using the multiplication algorithm or having a way to quickly calculate 4 x 4 x 4. Then it required being able to find how many times 512 cm^3 could contain 64 cm^3, by either using the traditional division algorithm or any other efficient procedure. The question in version 4 was more complex, since once the number of small cubes was determined, the students had to find the volume of each of them, and after that finding the length of the edge. With a given length, finding the volume is certainly a routine question; however, the opposite is clearly non-routine.

Table 2 shows the number of correct, erroneous and blank answers for each version of question 16. In contrast to what happened with the *Ruler question*, in all versions of question 16 there was a

significant percentage of blank answers, which indicates how the students perceived the difficulty of the different versions.

Version	Correct	Erroneous	Blank	Total students
1	23.21%	46,43%	30,36%	56
2	20.69%	43,10%	36,21%	58
3	9.09%	59,09%	31,81%	66
4	8.20%)	39,34%	52,46%	61
				241

Table 2: Answers to the Cubes question

If we pay attention to the percentage of corrects answers, we see that the success in versions 1 and 2 is similar and much higher than that of versions 3 and 4, which also have a similar success rate. We interpret this difference as due to the fact that questions in versions 3 and 4 are much further away from textbook questions than the questions in versions 1 and 2, and all of them are far from how they have dealt with volume in their previous schooling.

In version 1, the correct answer is 1792 cm^3, since there are 28 cubes with a volume of 64 cm^3 each. Among the 56 students, only 13 reached the correct answer and 18 gave no answer at all. Among those that gave a wrong answer, the most common errors lie in not knowing how to find the volume of the small cube or in mixing the idea of volume with that of surface area. Another common error lies in not identifying the right number of cubes in the figure: only 8 of the wrong answers are attained using the correct number of cubes, the others that make sense deal with the question considering there are 20 cubes.

In version 2, the correct answer is 384 cm^3, since there are 6 cubes with a volume of 64 cm^3 each. Among the 58 students dealing with these questions, only 12 reached the correct answer and 21 gave no answer at all. The percentage of blank answers being higher than in version 1 would confirm the idea that students considered this task to be more difficult than task 1, most likely because it brings up the idea of hidden cubes. Calculations are simpler than in version 1, since the number of cubes is a one digit number. However, students also have problems in knowing how to find the volume of the small cube or mix the idea of volume with surface area.

In version 3, the correct answer is 2 hidden cubes, since the object is made up of 8 cubes and 6 of them are visible. The total number of cubes in the object can be found by dividing 512 by 64, since the volume of each cube is 64 cm^3. Among the 66 students, only 6 students reached the correct answer and 21 gave no answer at all. The percentage of blank answers being close to the one in versions 1 and 2 would suggest that students do not consider this question to be more difficult than the others. As in version 2, calculations are simple, since the number of cubes is a one digit number. One common error is to not answer to the number of hidden cubes, but to answer the number of cubes that make up the object. Among the incorrect answers, 7 of them clearly show that the students knew they had to divide the volume of the object by the volume of the small cube, but again encountered problems in relating the length of the side of the cube to the volume of the cube, and directly used the side length or the surface area of one the sides of the cube. It is interesting to

note that, among the wrong answers, there is at least one that suggests that the student attempted to directly read the figural information given.

To solve the question in version 4 the student had to divide the volume of the object, 384 cm^3, by 6 which is the number of the cubes, obtaining 64 cm^3. Version 4 was given to 61 students, 5 of them solved it properly, 24 obtained wrong answers and 32 gave no answers, a much higher percentage than in version 3. Among the 24 wrong answers, we found three where the student divided 384 by 6 but went no further, giving as a result 64 cm^3, 64 cm or 64. Among the wrong answers, we also find those of 11 students that divided the volume by 3 to calculate the length of the side and, in some cases, there are added errors like using the wrong units or rounding off the result.

We wish to note that, in all versions, miscalculations were often present. Once again, as in the *Ruler question*, what is most striking about the students' answers is that, too often, it seems that they did not try to make sense of what they were doing. We might think that the students would be content with providing a numerical answer without paying attention to the meaning of either the question or the answer they are giving.

Discussion and conclusions

In this paper, we have analysed how a group of 241 first year student teachers deal with two non-routine questions in the context of a test intended, in a near future, to select those students that will be allowed to access a degree in Primary Education. Our results show that the percentage of correct answers relates to how far apart the questions posed are from the type of questions the students may have encountered in their previous schooling. The difficulties encountered by the students show that their initial mathematical knowledge is not solid enough to be considered a sufficient basis on which to build up their mathematical content knowledge and didactical knowledge. The results also show a significant offset between the initial mathematical knowledge of the student tested and the FMK understood as the desirable starting-point knowledge even though they may consider they have the knowledge required to address these questions, given that the number of blank questions is generally small. Moreover, we have little evidence that they master mathematical modes of thought or that they have problem tackling competence. Even to find a reasonable answer – one that makes sense – does not seem to be part of their way of doing mathematics.

Non-routine questions in the examination were designed to assess the students' understanding of basic algorithms and core concepts and their ability to move from specific to general thinking. Therefore, even if the questions essentially dealt with ideas taught in compulsory education, they challenged the student to reason and to use mathematical structures, and required that the student thoroughly understood the knowledge, skills and problem approaches he was using.

Non-routine questions seem to be a powerful tool not only to test basic concepts, but also to check the existence of the relationships among them, and to verify whether the students grasp the essence of doing mathematics and the thought processes involved. It is in this way that we consider that non-routine questions were useful to assess the students' initial mathematical knowledge. Our results also seem to suggest that when non-routine questions are simple, as in the *Ruler question*, students tend to solve them by using a procedure that would solve a similar standard question, without critically analysing what they are doing or how they are doing it. In the same way, when the question is non-routine due to its complexity, the evidences we obtained show gaps in their

knowledge of concepts and relationships between concepts. However, these are hypotheses that would need further research to be tested.

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Anticipating students' thinking through Lesson Study: A case study with three prospective middle school mathematics teachers

Nadide Yılmaz[1], İ. Elif Yetkin Özdemir[1] and Bülent Çetinkaya[2]

[1]Hacettepe University, Faculty of Education, Ankara, Turkey; nadideylmz@hacettepe.edu.tr

[1]Hacettepe University, Faculty of Education, Ankara, Turkey; ozdemiry@hacettepe.edu.tr

[2]Middle East Technical University, Faculty of Education, Ankara, Turkey; bcetinka@metu.edu.tr

This study aimed to examine prospective teachers' anticipations of students' thinking on the measure of the arc and measure of the central angle, circumference, and area of the circle and to explore the changes in their anticipations as they conduct three lesson study. For this purpose, case study method was used. Three prospective middle school mathematics teachers participated in the study and completed three lesson study. The data were analyzed in terms of three components: the prospective teachers' anticipations of (1) how students' would think; (2) what difficulties they would have; and (3) what powerful ideas they would have. The results showed that lesson study cycles with real classroom experience provided opportunities for the prospective teachers to develop anticipations of students' thinking.

Keywords: Lesson Study, prospective teacher knowledge, anticipating student thinking, middle school mathematics.

Introduction

Recently, teachers' knowledge of mathematics has become an object of concern. Ball and colleagues have expanded teacher knowledge proposed by Shulman (1986) by defining "the mathematical knowledge for teaching" (Ball, Thames, & Phelps, 2008). Pedagogical content knowledge (PCK) is part of the mathematical knowledge for teaching, and it focuses not only on content knowledge and pedagogical knowledge, but also on integration and transformation of content and pedagogy (Ball et al., 2008). According to Ball and others (2008), the components of PCK are knowledge of content and teaching (KCT), knowledge of content and students (KCS), and knowledge of content and curriculum (KCC). This study focuses on knowledge of content and students (KCS). Hill and others (2008) defined "KCS as content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (p. 375). Ni Shuilleabhain (2015) added that KCS includes students' understanding of content, student developmental sequences, typical student errors, anticipation of what students are likely to think or find confusing and common student computational strategies. Teachers should consider students' needs and interests when they plan for lessons. They must anticipate students' typical thoughts, ideas, and difficulties when selecting teaching materials and making decisions about the implementation of the lessons. Teachers must also attend to and interpret students' emerging and incomplete ideas during the instruction. Each of these tasks requires knowledge and skills developed through the interaction between mathematical understanding and knowledge of student thinking (Ball et al., 2008). Prospective teachers who can recognize and appreciate students' thinking and cognitive development could design and implement learning activities to meet students' needs and interest (Ball et Al., 2008; Llinares, Fernandez, & Sanchez-Matamoros, 2016).

However, their experiences and knowledge of students' thinking are very limited (Peterson & Leatham, 2009). Hence, teacher education programs should be designed to help prospective teachers improve students' mathematical knowledge and skills related to KCS by anticipating student thinking, among other abilities.

Lesson study is one of the models that helps teacher candidates develop their knowledge related to student thinking. It is a professional development program in which teachers collaboratively work on teaching. In this program, teachers first determine learning goals of the lesson and plan the lesson. Subsequently, one of the teachers in the group teaches a lesson and other group members observe the teaching process. Finally, they evaluate and revise the lesson plan so that it can be implemented second time (Lewis, Perry, & Hurd, 2009; Murata, Boffering, Pothen, Taylor, & Wischnia, 2012). In planning a lesson, the teachers are usually guided to focus on expected learning activities, expected student reactions or answers, teacher's responses to student reactions, and possible evaluation activities. This professional development model may provide prospective teachers with opportunities to plan, implement, criticize, and reflect on lessons collaboratively (Carrier, 2011).

Although research studies emphasize the importance of teachers' knowledge of student thinking, we know little about the development of prospective teachers' knowledge and skills in anticipating student thinking (Webb, 2006). Lesson study practices are found helpful in developing teachers' competence to anticipate students' mathematical thinking (Lewis et al., 2009; Tepylo & Moss, 2011). In this regard, the present study aimed to investigate prospective teachers' knowledge of student thinking of the concept of circle as a two-dimensional figure (disk) and as a one dimensional curve and to explore any changes in their anticipations as they work on three lesson study cycle.

Conceptual framework

The theoretical basis of the lesson study model acknowledges that cognition is social and that learning takes place in enriched learning environments in a cooperative way (Fernandez, 2005). In this regard, lesson study is a professional development program that encompasses constructive learning and creates learning opportunities (Lieberman, 2009). As lesson study is based on planning before the practice, observations during the practice, and cooperation and reflection throughout the practice, it contributes to making 'learning' a cultural activity, which ultimately makes this model more significant (Dudley, 2013). According to Davies and Dunnill (2008), this model differs from other cooperative models in that the cooperation adopted in lesson study continues before, during, and after the practice. All these characteristics of lesson study model give teachers different perspectives and show them how an effective mathematics education should be offered (Erarslan, 2008). Researchers emphasize that integrating lesson study model into teacher education programs can help prospective teachers develop knowledge and competence by learning from practice (Rasmussen 2016; Sims & Walsh 2009). Lesson study enables prospective teachers to work on lesson plans collaboratively and to conduct careful observations of learning and teaching activities. The model also allows prospective teachers to discuss and reflect on their own practice (Fernandez 2010). It mainly helps prospective teachers gain curriculum knowledge and pedagogical content knowledge, including knowledge of common student mistakes. Observing lessons gives prospective teachers the opportunity to notice things about the classroom environment, and most importantly, to see the situation from the perspective of students (Lewis, 2002). Thus, they can become familiar

with what students actually know, how they think, and what they can do as well as with areas in which they may have difficulty (National Research Council, 2001, Takahashi, 2005).

Methodology

This study focuses on investigating the prospective teachers' anticipations of students' thinking of the measure of the arc and angle and circumference and area of the circle. Furthermore, it explores how such anticipations change during the lesson study. For this purpose, case study design was adopted. This design provides an in-depth analysis of single or multiple cases by means of various data collection tools (Creswell, Hanson, Clark, & Morales, 2007).

Participants and context

The participants of this study were three prospective middle school mathematics teachers enrolled in a state university in Ankara. They were in their last year of the program and were willing and motivated to participate in the study. They were 22 years old, and their GPAs were between 3.29-3.47 out of 4.00. The teacher education program generally focuses on content knowledge (i.e., mathematics) in the first and second years. On the other hand, during the third and fourth years, most of the courses are related to mathematics education. Prospective teachers who graduate from the program can work as mathematics teachers in middle schools (grades 5 to 8).

Data collection

Within the scope of the three-week study, the prospective teachers worked as a group and planned, implemented, and revised three lessons (each of which lasted about 80-120 mins) for each learning objective on the topic of circle that is a part of the 7th grade math curriculum (MoNE, 2013). The prospective teachers used a lesson plan template to design the lessons. It consisted of four columns listing the (1) learning activities and key questions (and time allocation), (2) expected student reactions or responses, (3) teachers' responses to student reactions, and (4) goals and method(s) of evaluation (see column 2 of Table I).

Steps of the lesson: learning activities and key questions (and time allocation)	Student activities/ expected student reactions or responses	Teacher's response to student reactions / Things to remember	Goals and Method(s) of evaluation

Table 1: Lesson plan template

Afterwards, one of the group members implemented each lesson plan in real classrooms. The implementation process was video recorded. Regarding the last stage, the first author and mentor teacher evaluated and provided feedback to group members. Subsequently, the group revised the lesson plan (see Figure 1). Moreover, prospective teachers were expected to write a diary about this process.

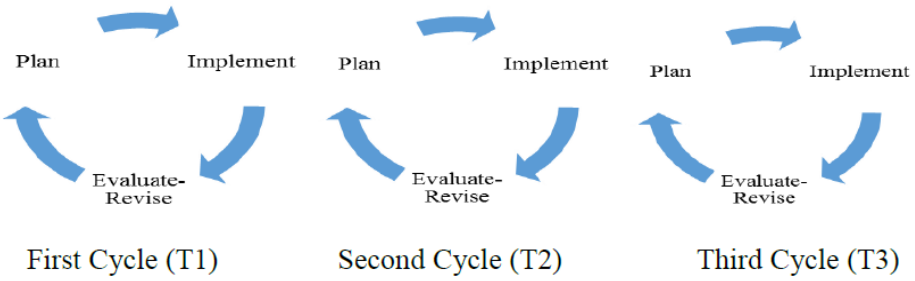

Figure 1: Lesson study cycles

The same cycle was repeated for each lesson plan. Prospective teacher 1 (T1) focused on the following learning objective: "Identifying central angles, their intercepted arcs, and the relationship between the measure of the arc and measure of the angle." T2 focused on "Calculating the circumference of a circle and a segment of a circle." T3 focused on "Calculating the area of a circle and a segment of a circle" (MoNE, 2013). The initial and revised lesson plans designed by the prospective teachers, the video recordings of the lesson study meetings, the video recordings of the lessons conducted, debriefing meetings after each lesson, and the observation notes and diaries taken by the prospective teachers, researchers, and training teacher during the implementation of the lesson provided the data for this study.

Data analysis

The data were coded based on the themes from the relevant literature (Ball, et. al., 2008; Fernandez & Chokshi, 2002; Hill, et. al, 2008; Schoenfeld, 1994) as well as from the participants' responses. For this purpose, researchers examined the prospective teachers' articulations and behaviors during study lessons and searched for the incidents showing their anticipations of (a) student thinking, (b) difficulties that students would have, and (c) students' powerful ideas. Anticipations of how students would think involve predicting how students think in general, what deductions they can make, and what types of connections they can make. Anticipations of students' difficulties involve predicting the challenges, mistakes, and misconceptions of students regarding the concept of circle. Lastly, anticipations of the powerful ideas that students might have involve expecting the key ideas about the relevant concepts. The data were examined based on this analytical framework for each lesson study cycle, and the findings were compared across three lesson study to examine any changes observed throughout the study.

Findings

This section presents the prospective teachers' anticipations of (1) how students would think, (2) the difficulties that students would have, and (3) the powerful ideas that students would have, as teachers design and implement three study lessons on the concepts of the measure of the arc, measure of the angle, and circumference and area of the circle. In Table 2, frequencies represent prospective teachers' anticipations related to each component of student thinking observed for each study lesson.

Table 2 shows that the prospective teachers pointed out that 26 different thoughts could be considered as typical for students (i.e., component 1) in the study lesson. For instance, they expected

that students could explain central angles as the midpoint angle of the circle, or they anticipated that students would know how to draw a circle using different tools, such as a coin or a compass.

	Types of plan	**How students' would think typically**	**What difficulties they would have**	**What powerful ideas they would have**
First Cycle	Lesson plan	10	2	3
	Revised lesson plan	10 (0 new anticipation)	3 (2+ 1 new anticipation)	3 (3+ 0 new anticipation)
Second Cycle	Lesson plan	8	-	5
	Revised lesson plan	9(8+ 1 new anticipation)	1 (0+ 1 new anticipation)	6 (5+ 1 new anticipation)
Third Cycle	Lesson plan	7	5	6
	Revised lesson plan	7 (7+0 new anticipation)	5 (5+ 0 new anticipation)	8 (6 + 2 new anticipation)

Table 2: Frequency of prospective teachers' thoughts related to the components of anticipating student thinking

Among these 26 different anticipations, only one new anticipation was included in the revised plan after its implementation in a real classroom. More specifically, in the second study lesson, they thought that the students would quickly give the correct answer when they were asked to spot the circumference of the circle. However, since the students pointed the region inside the circle as the circumference of the circle, the prospective teachers included such typical student thinking in the revised lesson plan and made some revisions, as illustrated in the following dialogue:

T2: Some students conceived the circumference of the circle as inside of the circle. We had never considered this.

T1: Yes, I didn't know what to say when they gave this answer.

T3: So, we need to include some questions in the lesson plan. For example, shall we say 'What do you think of when we say 'circumference of the school ground?' Then we can tell them to walk around the school ground.

T2: Let's decide a starting point and tell them to start walking from there and walk around the school ground until they reach the same point.

T1: Yes, then students would realize that circumference is not actually the same as inside [of the circle]. Then we can ask them to think about the circumference of a circle.

According to Table 2, the prospective teachers reported 9 different student thoughts could be considered difficult (i.e., component 2) in the study lesson. For example, they thought that students might confuse the concepts of circle and sphere. They also anticipated that students might consider meter as a unit of arc. Among these 9 different anticipations, two were included in the revised plans after its implementation in real classroom. In the first cycle, during the implementation of the lesson, the prospective teachers realized that students would confuse the central angle with inscribed angle. Thus, in the revised lesson, they decided to remind the students about the difference between these two types of angles. The following dialogue shows how they considered this idea as they were revising the lesson plan:

T1: Did you see that students confused central angle with inscribed angle?

T2: Yes, it seems that they didn't know the difference. We never thought about it. So, what shall we do? Let's draw a figure and ask them if it's a central angle. If they say 'yes', we'll use a material that shows an angle as a combination of two rays (half lines). Here, we can make a straight angle and ask the students what type of an angle it is.

T1: Then, students would say it's a straight angle based on what they learnt it in the previous lesson. Let's make it a 360-degree angle and draw a circle taking the origin of the angle as the center and ask what type of an angle it is.

T2: The students would say "It's a central angle because the origin is at the center of the circle."

T3: Yes, then we can show the example with an inscribed angle and ask if it is a central angle. I think students will say it's not because the origin does not cross the center.

T1: This way, they can differentiate between a central angle and an inscribed angle.

As seen in Table 2, the prospective teachers pointed out 17 different student thoughts could be considered as powerful ideas (i.e., component 3) in the study lessons. For instance, they expected that students could explain differences between circle and disk, and they anticipated that students would know what pi is. Among these 17 different anticipations, three new anticipations were included in the revised plans after their implementations in real classroom. In the third cycle, the prospective teachers asked students to find the area of a circle by using the area of a parallelogram. However, following the lesson, they realized that the instructions in the activity sheet were not clear enough to guide the students. This experience led them to contemplate about their expectations concerning students' powerful ideas (i.e., making connections between the area of a circle and of a parallelogram) based on which they made some changes to the instructions. The following dialogue illustrates how their expectations changed:

T1: The activity sheet was not very clear, so students did not understand the relationship between the area of a disk and the area of a parallelogram. We wanted to make it easier using what they learnt in quadrilaterals, but it didn't work. What shall we do?

T3: I think we should say that circle doesn't look like a quadrilateral. Then they can notice it. Later, we can ask what part of a disk and the height of a parallelogram look similar. They might say 'the radius of disk'. But, what's important here is whether they can tell, which part of a disk has a relationship with the base length of a parallelogram.

T2: Yes, so let's include this in the lesson plan.

To sum up, the data analysis showed that the prospective teachers identified various anticipations related to the three aspects of student thinking. They thought about how students develop ideas related to the fundamental elements of a circle as well as the circumference and area of a circle. They also recognized the importance of designing the lesson in the light of powerful ideas. In addition, they successfully anticipated different student difficulties and mistakes. The study lesson also help the prospective teachers produce new anticipations of student thinking after implementing and reflecting on the lessons. The prospective teachers might not have thought about these issues if they had not had a chance to implement and revise study lessons.

Conclusion and results

The findings showed that prospective teachers' anticipations of student thinking for each study lesson varied. The study lesson provided the prospective teachers an opportunity to consider student thinking as an essential part of the planning. They were expected to work collaboratively and think deeply about students' thinking (typical student responses, misconception, powerful ideas, prior knowledge, and understandings, etc.) as they plan their lessons. The lesson plan template guided prospective teachers to document their ideas about student thinking, making them explicit and the object of discussion. In this way, study lessons created a context for discussion about student thinking.

Furthermore, the lesson study allowed them to develop knowledge and skills related to student thinking through powerful experiences in real classrooms. Such opportunities allowed them to get to know students well and analyze the learning process from students' perspective. Some previous studies reported similar results (e.g., Ni Shuilleabhain, 2015; Webb, 2006). In this regard, lesson study could be used as a model in undergraduate programs to help the prospective teachers develop knowledge and skills related to student learning.

Even though prospective teachers' anticipations of student thinking were observed for each study lesson, a clear development through the three cycle was not observed. In this study, each cycle focused on learning objectives involving different concepts and skills (e.g., identifying central angles and calculating circumference and area of a circle). Such differences might have influenced prospective teachers' anticipations. Another reason for not being able to detect a clear development through three cycles might be related to the analytical framework used in the study. To reveal different aspects of development of teachers' anticipations, a more structured and detailed analytical framework could be used.

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Mathematics teachers' knowledge and professional development in a Lesson Study context

Vanessa Crecci, Miguel Ribeiro, Dario Fiorentini, UNICAMP, Brazil; vancrecci@gmail.com, cmribas78@gmail.com, dariofiore@terra.com.br

Keywords: Teachers' professional development, interpretative knowledge, mathematics teachers' specialized knowledge, Lesson Study.

Some preliminary words

The Saturday Group [Grupo de Sábado] (GdS), which emerged in 1999, is a collaborative group that brings together teachers from schools and academics (prospective teachers, master and PhD students, teacher educators) interested in researching the teaching of mathematics in a collaborative environment. Although gatherings are held on the university landscape, there are no formal academic regulations controlling participation. Recently the participants have focused their interest in improving their practices by deepening their mathematical and pedagogical knowledge. In that context, in order to frame the work to be developed, a Lesson Study (LS) has been devised, involving teachers from primary, lower and upper secondary, prospective teachers and researchers. For the implementation of LS, a structure with three subgroups has been established (one for each school level – primary, lower secondary and upper secondary). Complementary to the subgroups meetings (discussion and reflection upon tasks conceptualization and implementation), there will be meetings involving all the GdS elements. In those large group meetings (following the work already being done) the work goes around discussing and reflecting upon the situations emerged from teachers' practice the participants consider problematic. In particular, such discussions are aimed at contributing for developing teachers' knowledge, professional competency of noticing (in Mason's 2002 sense) and professional awareness. For the development of this project we consider what we call a hybrid methodology, where goals of two different natures are pursued. On the one hand, there is an interest in studying the development of a Lesson Study as teacher education strategy in the Brazilian context. On the other hand, the research goal concerns the teachers' professional development process, considering such development addressed through the lens of the Mathematics Teachers' Specialized Knowledge – MTSK (Carrillo et al., 2013) conceptualization, intertwined with what we term of interpretative knowledge (Jakobsen, Ribeiro, & Mellone, 2014). When discussing teachers' knowledge, besides the analysis of teachers' classroom practices and interviews, also a focus on the dynamics emerged in the subgroups and on the GdS will be analysed, as well as teachers' narratives grounded on their own experience (Connelly & Clandinin, 1988).

Lesson Study as a teacher education strategy

Considering the GdS context and the goals of the project in which this hybrid Lesson Study will be developed, six steps are considered. We have to recall that these steps have been emerged from the work with participants – by the nature of the GdS (a collaborative working group). The environment in which these steps are going to be developed are perceived as the context in which data for the research dimension will be gathered.

1) Teachers identify a critical aspect from their own mathematical practice which they aim to discuss, reflect upon and improve. Such critical feature will be firstly discussed in each of the

subgroups, taking into consideration the specificities of the contexts, and afterwards socialized and discussed in the large group (GdS) with all the participants;

2) After the identification of the critical features, all the subgroup elements will discuss their own previous experiences, also grounded on some readings on documents (e.g., papers, books) where such problematic (or similar) is discussed. Such discussion aims at allowing deepening the participants MTSK, understanding and awareness on the problem at hand;

3) Grounded on the discussions and reflections focusing on the problematic, teachers will conceptualize a task or sequence of tasks aimed at contributing for minimize the identified problem. Such tasks will be firstly prepared and discussed in each of the subgroups;

4) The following considered stage concerns the implementation of each task prepared on the subgroups in the GdS, in order to deepen the levels of discussion both on the nature of the tasks, its mathematical goals and ways of implementation. We have to recall that in such discussion are involved teachers from different school levels which is perceived as an opportunity to intertwine different aspects of the MTSK and teachers' awareness;

5) The next stage concerns the participants analysis of the implementation (all the participants will analyse the task, the implementation process and the teachers knowledge involved and required). Such analysis will occur firstly on the subgroups and afterwards some episodes will be discussed in the GdS;

6) The last stage (of each cycle), concerns the writing and discussion of narratives focusing on the lived experiences.

The development of LS as a research project

Understanding LS as a proficient way to engage mathematics teachers in a professional development landscape, we focus on discussing the MTSK mobilized, involved and recognized by teachers when preparing, implementing and discussing mathematical tasks. In an intertwined manner the dimensions of the interpretative knowledge will be focus of attention at all the previously considered moments. Data collection will be taken from the video recordings of the subgroups meetings and the GdS meetings; classroom practices; interviews to teachers; interviews to students after the implementation of the tasks and teachers narratives. With the analysis we intend to contribute for a broader understanding of teachers' professional development and on the MTSK and interpretative knowledge developmental processes.

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Combining two frameworks: A new perspective on mathematics teacher knowledge

Knud Thorbjørn Sloth[1] and Ingi Heinesen Højsted[2]

[1]University College Capital, Denmark; thorbjrn@yandex.com

[2]University College Capital, Denmark; ingi.heinesen.hoejsted@skulin.fo

Keywords: Mathematical knowledge for teaching (MKT), competencies and mathematical learning (KOM), combining frameworks.

For many years, there has been an interest in the role of the teacher in mathematics education research. During this time, efforts have been made to conceptualize the professional knowledge required to be a mathematics teacher. However, yet no consensus has been reached on how to describe the knowledge and ability, which is special to mathematics teachers. One framework, "Mathematical Knowledge for Teaching" (MKT) (Ball, Thames & Phelps, 2008), developed in Michigan includes subject matter knowledge, pedagogical content knowledge and tasks of teaching, and is widely used, in praxis and theory. The Danish competency-based framework "Competencies and Mathematical Learning" (KOM) (Niss & Jensen, 2011), which is described in terms of possessing eight fields of mathematical competency and six competencies related to the teaching of mathematics, is less used. The frameworks have not yet been used together. This is what we propose to do using the framework for networking of theories as formulated by Bikner-Ahsbahs and Prediger (2010), which describes strategies for connecting theoretical frameworks.

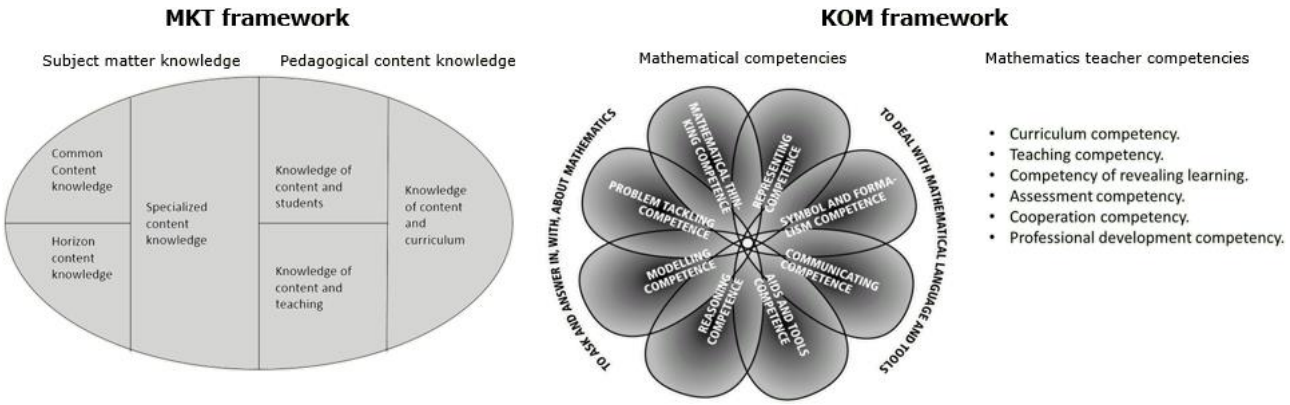

Figure 1: The MKT and KOM frameworks

Method

Based on a case study of the development of mathematics teacher knowledge among students in the Danish preservice mathematics teacher education program (Sloth & Højsted, 2016), which aims to qualify prospective teachers for work in primary and lower secondary schools, we investigate how and to what extent MKT and KOM can capture what the preservice teachers learn. Finally, referring to the model for networking of theories proposed by Bikner-Ahsbahs & Prediger (2010) we compare the two frameworks.

Results

We find that MKT and KOM can be used to describe most of our findings regarding the development of mathematical teacher knowledge in our case study, but the manner they describe them is different and do not always overlap. For example, when preservice teachers learn about different subtraction algorithms, which pupils might employ, we find the MKT framework can give a nuanced description of the unique mathematical knowledge and skills involved in teaching through its description of "Specialized content knowledge". The KOM framework does not address the issues of teaching at this level of detail, but one could say that the teaching of subtraction algorithms requires different mathematical competencies, like representation competency and symbol and formalism competency as well as teaching competency. Another example is when preservice teachers learn how to perform mathematical modelling and how to analyze the mathematical models of others. We find that KOM can capture this with modelling competency, whereas MKT lacks a way to describe mathematical processes like modelling, problem solving and reasoning skills. Meanwhile the details of the didactical aspects of the development of modelling competency are not elaborated in KOM. For example, typical difficulties pupils encounter when working with modelling are not described. This could call for the development of what corresponds to knowledge of content and students and specialized content knowledge, but with regards to competencies.

Conclusion

We conclude that MKT and KOM give different perspectives on mathematics teacher knowledge, that there are overlaps and differences when applied to practical situations, but also that the frameworks themselves may benefit from the perspective of each other. Using both frameworks on our case, we find that they can complement each other and describe a greater range of mathematics teacher knowledge. Furthermore, we suggest that using or combining concepts from both frameworks can result in a new understanding of the knowledge and ability needed by mathematics teachers.

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Teachable moments at elementary school mathematics level

Amanjot Toor[1] and Joyce Mgombelo[2]

[1]Brock University, Faculty of Education, St. Catharines, Canada; amantoor115@hotmail.com

[2]Brock University, Faculty of Education, St. Catharines, Canada; jmgombelo@brocku.ca

Keywords: Mathematics teaching, teachable moments, phenomenology.

Introduction and purpose of the study

Teaching is a complex phenomenon that encompasses ever-changing situations, where teachers are challenged to positively deal with unanticipated circumstances such as teachable moments. This poster draws from an ongoing doctoral research study, a hermeneutic phenomenological study, which seeks to identify and explore teachable moments corresponding to intermediate mathematics teachers' lived experiences. The research is guided by the following questions: What does it mean to teach intermediate mathematics? What does it mean to recognize and utilize teachable moments at an intermediate mathematics level? At this stage in the research, the term *teachable moment* is generally defined as an opportunity that arises when connections have been made to advance learning by a learner and/or an educator. More specifically, the term *moment* is distinctly defined as an expected or unexpected occurrence that allows learners and/or educators to deepen their understanding. The poster focuses on the background and context of the study, by highlighting the rationale, framework and methodology of the study.

Background and context

Teachers often question the underlying purpose of teaching and its subjective meaning across three general categories: teaching subject content; teaching based on certain pedagogical tools; or teaching the child. Researchers in mathematics education have asked: What does it means to teach mathematics? How can one become an effective mathematics teacher? On one hand, researchers tend to focus on teachers' content knowledge and/or pedagogical knowledge, or an integration of the two. However, having in-depth context knowledge in mathematics does not necessarily equate to teaching it effectively (Muri, 2008). Also other researchers question how the two categories of knowledge—mathematics (subject matter) and teaching (pedagogy)—can be integrated. On the other hand, mathematics researchers such as Doxiadis (2003) suggest that teaching mathematics effectively means humanizing mathematics for learners, and that "education is—should be, at its best—a process involving the complete human being" (p. 2). In this sense, humanizing mathematics means teaching it in a way that involves students as participants in mathematics, which is beyond mere content delivery or teaching certain skill sets. This poster draws from a study that aligns with this second view.

Theoretical framework

The epistemology of phenomenology centers on didactic meaning as opposed to arguing or developing abstract theory. In their discussion of the theoretical and conceptual framework for a phenomenological study, Savin-Baden and Howell Major (2012) stated that "the essence experience

is so central and is to be uncovered before it is categorized, researchers do not tend to use a theoretical or conceptual framework… [because] doing so could impose presuppositions on the meaning of the experiences" (p. 221). The objective of this study is not to make broad generalizations about experiences of all mathematics teachers, but instead to examine individual teachers' personal experiences associated with a very specific phenomenon of teachable moments.

Methodology

In order to understand what teachable moments means to mathematics teachers, and how teachers use them in their day-to-day, moment-to-moment teaching, it is necessary to first gain insight into teachers' lived experiences by exploring their reasoning, beliefs, and intentions for teachable moments. As a consequence, to this underpinning, the study employs e a hermeneutic phenomenological research design founded on an epistemology of interpretivism. The study uses qualitative methods to collect data. These include semi-structured interviews, field notes and researcher journal. Participants comprise a purposeful sample of intermediate teachers with experience of teaching mathematics in Ontario, who have had lived experiences related to the phenomenon of teachable moments.

Conclusion

The "teachable moment" is viewed as a somewhat intangible pedagogical prize; a teacher might know what it feels like, yet may not identify its characteristics. Educators such as math teachers seek moments of openness and creativity with their students so they can personally experience "the psychic rewards" of teaching (Lortie, 1975). Too often, such moments happen suddenly and slip away just as quickly, leading some teachers to conclude that such occurrences are a matter of chance. This study therefore seeks to shed some light on the characteristics of teachable moments in mathematics education to deepen teachers' understanding and use of such teachable moments in their respective practices.

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TWG21: Assessment in mathematics education

Introduction to the papers of TWG21: Assessment in mathematics education

Paola Iannone[1], Michal Ayalon[2], Jeremy Hodgen[3] and Michiel Veldhuis[4]

[1]Mathematics Education Center, Loughborough University, UK; p.iannone@lboro.ac.uk

[2] Department of Mathematics Education, University of Haifa, Israel, mayalon@edu.haifa.ac.il

[3] School of Education, Nottingham University, UK, ttzjh@exmail.nottingham.ac.uk

[4] Freudenthal Group - FSW, Utrecht University, Netherlands, m.veldhuis@uu.nl

Keywords: Summative assessment, formative assessment, validity, feedback, assessment methods.

Introduction

Given the prominence that the general education research ascribes to the impact of assessment on teaching and learning it is surprising that there had been so far no TWG dedicated to assessment of mathematics at CERME. TWG21, which met for the first time at CERME10 in Dublin, aimed to fill this gap. Given that this was a new group we decided to focus on assessment of mathematics considered broadly in order to gauge where the interest of the mathematics education community lies in this field, which encompasses very many different aspects. Although traditionally assessment has been discussed across many TWGs at CERME, TWG21 aimed to bring researchers together who have an interest in this topic and can, for the lack of a common forum, at times feel isolated. To reflect the landscape in the general literature we called for papers investigating the nature of assessment and its effects on student learning making use of a wide range of methodologies, from large quantitative and mixed methods study to small investigative qualitative studies. We were delighted to have 24 papers and one poster discussed at the conference. In what follows, we have grouped the papers in thematic clusters to reflect the variety of submissions regarding both focus and methodology. We conclude with some reflections on the working of the group and some suggestions for the directions this group can take in future CERME conferences.

Thematic clusters

We identified six overarching themes that could serve as an organizing tool for the papers submitted to TWG21. Below, we describe each of these themes in turn.

Different approaches to assessment: Papers in this theme considered the affordances, drawbacks and validity of innovative assessment, both for students and for teachers. Davies proposes comparative judgment at university level as a new way of assessing students. In his paper, he investigates issues connected to the validity of this method for assessing conceptual understanding in mathematics. Lemmo and Mariotti investigate the issues connected with transitions of tasks from a paper and pencil form to an electronic form. They challenge the view that students employ similar solving strategies in both environments and find that indeed students solve the task differently in the two modalities. Teledhal investigates the validity of narrative accounts as an assessment tool for problem solving and concludes that those accounts do not offer enough details of the problem-solving process to be a valid tool for assessment. Dahl describes the perceptions of a group of science students (engineers, mathematicians, and other sciences) for group oral assessment. She finds that students

across disciplines agreed that a group exam gives less differentiation of grades compared to an individual exam. Finally, Reit discusses whether the validity of teachers' intuitive assessment practices is supported by empirical findings and shows that a sequential consideration of thought structures in a solution approach leads to reasonable results and may justify its application in school due to its straightforward implementation, especially when assessing modelling tasks.

In service and pre-service teachers' views: A second important theme that emerged from the submissions to TWG21 was related to teachers' views, beliefs, and use of assessment methods, both during their training and in their professional practice. Hofmann and Roth report on a study aimed at fostering preservice teachers' diagnostic skills with a focus on students' abilities, problems and misconceptions with graphs of functions. They explore the affordances of two tools for promoting diagnostic skills: video analysis and task analysis. Pratt and Alderton analyse English mathematics teachers' assessment approaches in the context of the current changes in assessment policy in the UK. To this end they use a Foucauldian analysis of teachers' discourse to sketch the power structures involved. They find that the official removal of the levels only superficially affected teachers' practices and teachers still relate these to the 'old' language of attainment levels. Kaplan and Haser investigate 27 preservice middle school mathematics teachers' purposes in planning the assessment and their views and suggestions about the assessment part of a lesson plan. Findings of the study indicate that purposes underlined by preservice teachers in preparing the assessment part of the lesson are similar across the sample and they all related to the teacher actions.

Professional development: Papers in this cluster addressed the role of professional development in fostering teachers' (both in service and pre-service) competences in assessing student understanding. Grapin and Sayac investigate the use of external (e.g. researcher-created) assessment tasks by primary school mathematics teachers and teachers' practice by using and an activity theory perspective. They find that teachers design tests with low levels of complexity and did not invest much in assessment as a professional activity. Pilet and Horoks present analytical tools to characterize assessment activities as part of teachers' practice in algebra. The authors exemplify why high school teachers came to consider assessment as a potential lever to enhance both the students' learning in mathematics and the teachers' development. Initial results indicate that the teachers developed better indicators to select the students' productions that they will use for the discussion after a task, but that they use they make of these products hasn't improved. In her theoretical paper, Andersson argues that the addition of the dimension Teacher Instruction (ATI) as a key strategy to the five key strategies proposed in Wiliam and Thompson's (2007) framework of formative assessment could facilitate the analysis of teachers' use of formative assessment activities and improve the guidance and support of teachers' implementation of high quality formative assessment practice. Finally, in this group Santos and Domingos investigate portfolio assessment in geometry for pre-service teachers through the lenses of activity theory and procepts. They find students engage in qualitative different pathways when solving these problems.

Formative assessment/feedback: We received many papers discussing formative feedback and the submissions in this group spanned from primary to upper secondary school with focuses both on teachers' use of formative assessment and students' engagement with such assessment. Chanudet investigates the assessment of problem solving by using a grid of criteria. The paper focuses on the use that teachers make of such tool to facilitate formative assessment and offers the example of the

practice of one teacher where she analyses instances of formative feedback occurring in this classroom. Zhao, Van den Heuvel-Panhuizen, and Veldhuis investigated the effects on student achievement of supporting Chinese primary mathematics teachers' use of classroom assessment techniques. In this experimental study, the intervention consisted of teachers participating in workshops on the use of these techniques and using them in their classrooms. Results indicate that the students of teachers that gained more insight about their students from using the techniques, improved their mathematics achievement scores more than other students. Gurhy focuses on Irish students' perspectives on the use of assessment for learning in primary school. Findings indicated that students were positive about the feedback in, and practices of, assessment for learning, became more confident and expressed a feeling of enjoyment related to this. Two related papers reported findings from FaSMEd, a European project on the use of technology for formative assessment. In the first paper, Cusi, Morselli and Sabena analyse a teacher's strategies to provide feedback during class discussion. They identify five strategies: revoicing, rephrasing, rephrasing with scaffolding, relaunching, and contrasting. In the second paper, the authors describe how materials were designed to facilitate technology-enhanced formative assessment practices. They then show how the design framework can be used to analyse the implementation of technology-enhanced materials. They argue that materials designed in this way, combined with the functionality of technology, enhance a teacher's capacity to activate Wiliam & Thompson (2007) formative assessment strategies.

Task design: Three papers were dedicated to this theme. O'Brien and Ní Ríordáin describe the development, design, and theoretical underpinning of a diagnostic test for algebra. The test is aimed at lower secondary students in Ireland and is intended to help teachers identify the causes of students' errors. The authors discuss their reasons for adopting this approach. Beck investigates students' written solutions from CAS-allowed exams. Based on the analysis of students' solutions a descriptive model for assessing these solutions is set up. The paper also discusses how formative assessment could help students develop their competencies in communicating mathematics. Moomaw investigates the validation of a constructivist game- and story-based measure (Teddy Bear Picnic) for pre-school mathematics. In this measure, pre-school pupils are assessed while playing several interactive games. Psychometric tests show that the test appears to be a valid and reliable measure of pupils' level of mathematical development.

Large-scale/standardized tests: Finally, we received several papers addressing issues related to the use and design of large nationwide standardized tests. Garuti, Lasorsa, and Pozio describe the development of items for national assessment in Italy. They show how both quantitative and qualitative analysis can be used to improve the psychometric properties of items, whilst also improving their validity in terms of appropriate and relevant mathematical content. Ferretti and Gambini investigate the persistence of certain misconceptions in the transition between school and university. They focus on properties of powers and analyse two Italian nationwide databases to find that indeed certain misconceptions persist across this transition. Drüke-Noe and Kühn analyse characteristics of statewide exams in eight countries through task analysis and find that that the cognitive demands of most competences needed to solve these tasks are rather low with the only the competence 'working technically' being often assessed. Cunningham, Shiel, and Close investigate the relation between the current Junior Certificate mathematics examination in Ireland for Grade 9 to the PISA and TIMSS frameworks. Their findings show that the Junior Certificate examination is moving closer in the direction of the PISA approach, but this is also motivated by the comprehensive

reform in mathematics in this country. Finally, Olande investigates how Grade 9 students solve an item involving the interpretation of graphs. Using student responses to an item from the national test in Sweden, his analysis shows that only a very small proportion of students use graphical reasoning in their solutions.

Conclusions

In the process of preparing for this new group at CERME10 we were impressed not only by the variety of work we received but also by the methodological variety of the papers that spread from small qualitative case studies to large statistical surveys. The theoretical frameworks employed were also varied, from Activity Theory to Foucauldian analysis. We believe this variety to be sign of a growing interest in mathematics education for assessment; not only in the sense of validation of large scale tests, but also in terms of the effect that assessment has on teachers' actions in the classroom and as such on student learning. This variety, however, can also be sign of a field which has yet to find its unifying themes: the presence of a forum for discussion like TWG21 can therefore help define these emerging unifying themes. Validity of assessment for example – although ubiquitous in many papers – was hardly explicitly addressed. Indeed, in the final session of our group which was dedicated to reflecting on the group experience with an eye to future meetings, we observed some issues which at times have hindered communication. One of those was the lack of uniformity in definitions of recurring terms or sometimes the lack of clear definitions at all. It was felt that agreement on definitions of basic terms is important for communication and collaboration, and the lack of this clarity of definitions can be again a manifestation of a developing and growing field. We also noticed the absence of papers discussing the impact of assessment methods on student learning, a theme which is very much present in the assessment literature. The final reflection of the group concerned the presence of *mathematics* in the research presented. The group felt that in a topic such as assessment it may be easy to lose the focus on the mathematics assessed and instead discuss generic assessment research. While assessment research in general education is obviously very important to the work of this group, all participants felt that the focus should be on the mathematics assessed, and that indeed it may be a difficult balancing act not to replicate research and constructs that are already used in the general assessment literature and keep the focus on the fact that we aim to use these findings and constructs to investigate the assessment of mathematics. Although this balancing act might make for a difficult enterprise, we are confident that in the coming CERMEs we will be able to continue discussing general assessment issues such as validity, but always with a clear focus on the mathematics to be assessed and its didactics.

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Formative assessment - and the component of adjusted teacher instruction

Catarina Andersson

Department of Science and Mathematics Education, Umeå University, Sweden;
catarina.andersson@umu.se

This theoretical paper is based on an empirical study where the framework of formative assessment by Wiliam and Thompson was used to analyze teachers' use of formative assessment in their mathematics classroom practice. The paper argues for treating a component named Adjusted Teacher Instruction (ATI) as a key strategy in complement to the five key strategies in the original framework. ATI is a significant component in formative assessment, but also particularly challenging for teachers to implement in their classroom practice. Treating ATI as a key strategy could facilitate the analysis of teachers' use of formative assessment activities and enhance the understandings about what kind of ATIs are most useful for whom under what conditions. Extended understandings about effective formative assessment activities are important in decisions about what formative assessment to include in teacher education and in-service training for teachers.

Keywords: Formative assessment, assessment for learning, mathematics, instruction.

Introduction

The encouraging potential noticed in formative assessment has motivated scholars to further engage in both empirical studies and theoretical work in the research area of formative assessment. The theoretical understandings of formative assessment have evolved during a long time, often affected by empirical studies in which researchers has been responsive to teacher practice. This paper focuses on a component of formative assessment that is regarded particularly significant, but also difficult in carrying out formative assessment. In this paper this component is called Adjusted Teacher Instruction (ATI) and it is argued for treating ATI as a key strategy in parallel to teacher feedback.

Black and Wiliam (1998) in their research review demonstrated that large student achievement gains are possible when formative assessment is employed in classroom practice. This review received widespread attention and caused a discussion about the need for and role of an extended assessment culture and practice. Since then, implementation of formative assessment has been on school policy agendas in many countries (Tierney, 2006), but this implementation has often proven to be challenging (Birenbaum et al., 2015). Several attempts have been unsuccessful in accomplishing a substantially developed formative assessment practice (James & McCormick, 2009; Schneider & Randel, 2010) and misunderstandings and distortions of essential features of formative assessment are detected in policy and practice (Swaffield, 2011). Some factors facilitating and hindering the implementation regarding the teacher, student, assessment and context are identified (Heitink, Van der Kleij, Veldkamp, Schildkamp & Kippers, 2016), but still a strong research base supporting how to effectively help regular teachers to implement a high quality formative assessment practice is lacking (Schneider & Randel, 2010; Wiliam, 2010). Such a research base needs to include both how to design effective professional development programs for teachers and what kind of formative assessment to include in such programs.

This paper is related to the latter issue. The discussion in the paper is theoretical, but originates from an intervention study in which a group of mathematics teachers learned about formative assessment

(see Andersson, 2015). In this study, formative assessment was conceptualized as one big idea and five key strategies (see Wiliam & Thompson, 2008) in a framework that was used in the professional development program and for structuring the data collection and data analysis.

Background

Black and Wiliam defined formative assessment as "encompassing all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (Black & Wiliam, 1998, pp. 7–8); a definition that provide several possible focus in carrying out formative assessment. Consequently, Black and Wiliam's review included studies investigating different strategies for carrying out formative assessment, using the term formative assessment in different meanings or using alternative terms such as feedback, self-regulated learning or peer-assisted learning. As Bennett (2011) points out, without a consensus about the term formative assessment, the effects will be unclear. A common and clear terminology and definition of formative assessment is also desired to eliminate misunderstandings and distortions in policy and practice.

To maximize instructional benefits, we need to know more about what constitutes effective formative assessment (Wiliam & Thomphson, 2008; Wiliam, 2007). To gain valuable insights about best practices it is important to be clear about the way formative assessment is conceptualized in for example studies of implementations of formative assessment that are empirically linked to student achievement. The formative assessment practice needs to be carefully analyzed and described to provide information about specifics of such practices as well as how these specific characteristics may have functioned as part of an enhanced learning process.

Black and Wiliam's review included studies showing the potential instructional benefits of different strategies for carrying out formative assessment. It can be expected that a classroom practice that integrate such key strategies to a unity would open up extended opportunities for learning and thus offer higher potential for improving student achievement. The empirical study motivating this paper is one of few studies investigating the impact of such an integrated practice on students' achievement. Following sections of the text outline Wiliam and colleagues' conceptualization of such a practice, followed by a description of the operationalization of that framework.

One big idea and five key strategies

A more recent definition of formative assessment by Black and Wiliam is more detailed:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about next steps in instruction that are likely to be better, or be better founded, than the decisions they would have taken in the absence of evidence that was elicited. (Black & Wiliam, 2009, p. 9)

This definition clearly demands every formative strategy to fulfill the big idea of using evidence of student learning to adjust instruction to better meet students' learning needs. The conceptualization of formative assessment as one big idea and five key strategies (Wiliam & Thompson, 2008; Black & Wiliam, 2009) is visualized in Figure 1. The matrix visualizes how three processes (horizontally) and three agents in the classroom (vertically) construct five key strategies (KS) in formative assessment. The three processes constitute the defining characteristics of formative assessment

inherent in the definition above and are central for the big idea of using evidence of student learning in decisions about how to proceed in the instruction. The three agents who are responsible for the learning in the classroom are defined as the teacher, the learner and the peers.

<table>
<tr><th></th><th>Where the learner is going</th><th>Where the learner is right now</th><th>How to get there</th></tr>
<tr><td>Teacher</td><td>KS 1 Clarifying learning intentions and criteria for success</td><td>KS 2 Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding</td><td>KS 3 Providing feedback that moves learners forward</td></tr>
<tr><td>Peer</td><td>Understanding and sharing learning intentions and criteria for success</td><td colspan="2">KS 4 Activating students as instructional resources for one another</td></tr>
<tr><td>Learner</td><td>Understanding learning intentions and criteria for success</td><td colspan="2">KS 5 Activating students as the owners of their own learning</td></tr>
</table>

Figure 1: ***The relationship between key strategies (KS), instructional processes and agents in the classroom (After a figure in Black & Wiliam, 2009, p. 8)***

The different key strategies in formative assessment are connected and sometimes dependent on each other's existence and performance. For example, clear learning intentions guide the teacher to chose questions/tasks that elicit relevant information about students' learning and help the teacher to provide goal directed feedback. In addition, learning intentions clear to the students enhance their opportunities to be engaged and involved in the learning process (their own or their peers').

Using the big idea and five key strategies to analyze classroom practice

The author of this paper participated in a research group responsible for a study about professional development in formative assessment for a group of randomly selected mathematics teachers. The framework above was used in the professional development program, in the data collection and in analysis of formative assessment used in the teachers' mathematics classroom practices.

Two rounds of data collection and analysis were made, before and one year after the professional development program. Both times, observations were completed in each teacher's mathematics classroom practice and all teachers were interviewed. The big idea and the five key strategies structured observation schemes and interview guides, with a focus on what formative assessment activities the teachers used in their mathematics classroom practice.

Each formative assessment activity was supposed to be classified in relation to one of the five key strategies or the big idea. From a teacher perspective the big idea pertains to Key strategies 1–3. Accordingly, activities aimed at clarifying where the learner is going could be classified as belonging to KS 1 and activities aimed at eliciting where the learner is right know as belonging to KS 2. Teacher

feedback aiming at moving student learning forward was classified as belonging to KS 3. This process led to one remaining group of teacher activities concerning teachers' use of information about learning needs (the big idea) that did not fit to any of the other key strategies. Therefore, to clarify our data and for consistency reasons, we decided to include the new category Adjusted Teacher Instruction (ATI) as a new 'strategy' in parallel to feedback. Both strategies (ATI and feedback) aim at taking learning forward. Consequently, no activities were classified as belonging to the big idea. The formative activities classified as ATI were the activities aiming at taking learning forward that did not concern teachers' oral or written feedback.

Before the professional development program (PDP) the most common ATI activity was to use results from a diagnosis in the textbook to choose a group of tasks (regular or advanced) for each student's individual work with the chapter in the textbook. Other ATI activities used by a smaller group of teachers were, for example: individualized tasks for a student; adapted materials for example work sheets, homework or tactile materials; extra or modified lecture for the class, a group of students or for individual students; and adaption of time set aside for a chapter in the textbook. After the PDP individual teachers extended their repertoire of ATI activities (from the same type of activities as identified before the PDP), for example lectures for group of students became more common. The teachers' use of ATI activities was also affected by new activities connected to Key strategy 2. Many teachers had started to make use of students' misunderstandings, which were often identified by using mini-whiteboards as an all-response system. In general, the teachers received information about student learning more often and in various ways and could consequently make adjustments of instruction more often and with more precision.

In our analysis of formative assessment activities in teacher's mathematics classroom we decided that teacher activities connected to the use of evidence of learning needs could either be classified as a feedback activity or as an Adjusted Teacher Instruction activity. Thus, feedback and Adjusted Teacher Instruction would have a shared position within the third teaching and learning process (How to get there) for the teacher's actions (see Figure 1). This proposed shared position will be discussed below.

Discussion

In this paper, Adjusted Teacher Instruction (ATI) is suggested as a component of particular significance in formative assessment to be treated as a key strategy, a proposal that could improve the use of the formative assessment framework by Wiliam and Thompson (2008) in research, policy and practice. The advantage is twofold: (1) the ATI component will get a more prominent place and (2) the framework will be more coherent. These advantages, but also some concerns, will be discussed below.

One advantage of treating ATI as a key strategy would be that teacher activities within this strategy could be defined and further studied in the same way as for activities belonging to other strategies. The few studies using frameworks unifying several formative assessment strategies in analysis of teacher classroom practices do not always provide specifics about teachers' adjustment of instruction (e.g. Wylie & Lyon, 2015; Randel et al., 2011). Such specifics are desirable because the instructional decisions and actions taken to better meet student needs are crucial for students' continued learning opportunities (Wiliam, 2007) and because using evidence of learning to inform next instructional

steps has been experienced as a challenging aspect of formative assessment (Cowie & Bell, 1999; Heritage, Kim, Vendlinski, & Herman, 2009; Oláh, Lawrence, & Riggan, 2010).

Research addressing this crucial and difficult component in formative assessment is desirable to enhance the understandings about what ATIs are effective under different circumstances, but this area of research needs improvement in terms of becoming more prominent, the definition of the area and the number of empirical studies conducted (Bellert, 2015). Research about feedback has resulted in guiding models for what type feedback is more or less effective, for different students and under different conditions (e.g. Hattie & Timperley, 2007; Shute, 2008). Similar knowledge about ATI is important because a main aspect of formative assessment is that planning of instruction is decision driven. To secure that the information from the assessment will be useful, a feasible way is to plan instruction backwards with a clear decision in mind and searching for relevant evidence to make decisions in a smarter way (Wiliam, 2007). Skilled teachers can design teachable moments into their lesson because they have already thought of alternative instructional decisions before the information was collected (ibid., p. 1089).

If we know more about what instructional adjustments are likely to be most effective in different situations this would be helpful guidance for policy and practice, for example in teacher education and in-service training for teachers. The knowledge base can be extended from conducting studies that empirically link different types of ATI to student achievement and by careful analysis, descriptions and conclusions about ATI characteristics and their function as part of an enhanced learning process. One example of a study contributing to this knowledgebase is a study by Ruiz-Primo, Kroog and Sands (2015). This study of science and mathematics teachers was restricted to *informal* formative assessment, in which the interaction between teacher and students is central. Studied within this interaction, the characteristics of the teacher's response were classified according to type of oral feedback or type of *instructional move*. Additionally, the type of teachers' actions observed in more and less expert teachers was studied. The results were separated for individual student work and whole class work. Using a two step cluster analysis, there were five variables included for teachers' instructional moves: (1) re-teaching (e.g., going over content again in the same or similar way as before); (2) solving problems with students (e.g., asking students for their input along the way while solving a problem); (3) solving problems without students (e.g., solving or modeling the solution to a problem without student input); (4) re-clarifying the task (e.g., reminding students of what they need to do); and (5) providing the correct answer (e.g., giving the answer without explanation) (Ruiz-Primo et al., 2015, p.17). This study does not only tell us that instructional adjustment where implemented, but also specifies these adjustments and compare the use of them in two kinds of work conditions and by two groups of teachers.

Another advantage of treating ATI as a key strategy in parallel with feedback is that the framework would be more coherent. In fact, also Wiliam includes instructional adjustment as a second aspect of feedback in the meaning that feedback is provided to the teacher so he or she can modify the instruction to be more effective (Wiliam, 2010, p. 33). However, the feedback *to* the teacher is comparable to elicited evidence of student learning in Key strategy 2, and this might generate confusion. The suggested shared position for feedback and Adjusted Teacher Instruction might clarify the framework to avoid misunderstandings and distortions in policy and practice and thus provide a better guidance. Such guidance might have affected the design of the professional development

program in the study behind this paper (see Andersson, 2015). In the program ATI was not treated as a key strategy and did not get the same focus and time set aside as the other key strategies did. This might have affected the limited extension of types of ATIs at group level in the results.

A concern about treating ATI as a key strategy regards the need and meaning of the big idea. Elaborating the big idea from a teacher perspective, Key strategy 1 is not indispensable for teachers' eliciting and using information of students' learning needs, only critical for ending up with evidence useful for formative assessment. Wiliam distinguishes between diagnostic assessment and assessment that is instructionally tractable, where the latter form not only indicates what needs attention but also what needs to be done to address the issue (Wiliam, 2007, p. 1063). Wiliam points to the need of the teacher to have a range of instructional alternatives beyond just repetition:

> For formative assessment to be instructionally tractable, the teacher must be clear about the range of alternative instructional moves that are possible, should then decide what kinds of evidence would be useful in choosing among the relevant alternatives, and only then elicit the evidence needed to make that decision. (Wiliam, 2010, p. 33)

While Key strategy 1 is ultimate, Key strategy 2 is a prerequisite for the implementation of the big idea. Key strategy 3 concerns the very foundation of the big idea about formative assessment. The big idea is important in the evaluation of the function of the implementation because it reflects the whole assessment cycle, which Wiliam suggests should be performed backwards (see above).

Another concern regards the distinction between feedback and Adjusted Teacher Instruction. The distinction we made (see Andersson, 2015) do not match the distinction made by Ruiz-Primo et al. (2015), probably caused by the different conceptualization of formative assessment. Even when using the same conceptualization of formative assessment, some activities will be a definite feedback *or* ATI activity, but other activities will have a more uncertain belonging. In our case, when teachers used the "thumb of role" giving feedback as *two stars and a wish* (showing the student two excellent aspect of their work and an idea for improvement), this is categorized as feedback. When a teacher decides on finishing work on algebra a week earlier than planned, this is ATI. A more uncertain activity would be when the teacher together with the student decides what tasks are most appropriate for the student to work with. We would classify this as an ATI activity, from the rationale of not being restricted to oral or written feedback from the teacher.

Conclusion

The advantage of treating ATI as a key strategy put forward in this paper is ultimately about improving the guidance and support of teachers' implementation of high quality formative assessment practice. ATI is experienced as difficult to implement by teachers. At the same time, the ATI component does not always receive much focus in analysis of teacher classroom practices. A more specified analysis and communication of research results about ATI could provide teachers with better guidance. In addition, a more coherent framework could be easier for teachers to understand. The big idea is an important guiding idea and the key strategies concretize this idea. The quality of any formative assessment activity is dependent on the extent the activity meet the aim of the key strategy as well as the big idea.

This paper argues for more studies conceptualizing formative assessment as a unity of different strategies of which ATI is one. Ultimately such studies examine the effect of implementation of

formative assessment on both teacher classroom practice and student achievement. The formative assessment practice needs to be carefully analyzed and described to provide information about specific characteristics and their function as part of an enhanced learning process. Such research might end up in models similar to those of feedback (Hattie & Timperley, 2007; Shute, 2008), showing more and less effective ATI, for different students and under different conditions.

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Written documentations in final exams with CAS

Johannes Beck

Julius-Maximilians-Universität Würzburg, Germany; beck@dmuw.de

This article focuses on students' written solutions from CAS-allowed exams. Based on the analysis of students' solutions a descriptive model is set up. It can be used for assessing students' solution as well as creating exemplary documentations. The purpose of these documentations is to help teachers reflect about their practise of writing down solutions and the norms they set by this for exams. This paper also shows how formative assessment could be a means to help students develop their competencies in communicating mathematics.

Keywords: Upper secondary education, exams, written solutions, Computer Algebra Systems, formative assessment.

Introduction: Challenges in exams with computer algebra systems

Digital technologies like graphical calculators and computer-algebra-systems (CAS) influence various aspects of mathematics education in general and the classroom-practise in particular (cf. Barzel ea. 2005). Most prominently, the way tasks can be solved changes drastically: where once it was necessary to differentiate a function with pen and paper, now every student can use a CAS and by pressing a short series of buttons the result appears on the screen. The new possibilities fuelled the hope that "[t]he new tool provided the chance to concentrate more on central competencies in mathematics education, concept formation, problem solving and modelling competencies, and to outsource algorithmic operations to the machine" (Drijvers & Weigand, 2010). While the shortcutting of the work of calculating with pen and paper enables classes to have more time for the central competencies, it also forces teachers and students to think about how to document the process of "working" on a task. This article focuses on the aspect of how communication in a CAS-environment could be shaped.

In this context, the study aims at identifying problems, difficulties and possibilities concerning students' solutions, categorize elements in those according to their function within the solving process and suggest a possible standard for written solutions with CAS. Against this background, the suggestions can also be interpreted as assessment criteria and, thus, a given students' solution can be assessed. Furthermore, it is a goal to provide a theoretical model that describes the development of students' documenting competency. The following research questions are

1. How do students write down their solutions in final exams? Which different forms of documentations do they use? What kinds of problems or difficulties (if any) are connected with these forms?
2. How can students' written solutions be described by means of a category system?
3. How could a developmental model look like that
    a. describes how students' competencies in documenting the solving process with CAS can be developed?
    b. offers learning strategies and exemplary solutions for shape this development?
    c. encompasses criteria for assessment in final exams?

The major motivation for this study is to create a sound set of suggestions for teachers, for application in the classroom and for preparing students for the final exams. Formative assessment plays a crucial role in developing an adequate documentation competence. This article gives an answer to the second question and presents an outline of ideas to research question (3).

Theoretical framework

The theoretical framework to tackle the questions above encompasses two important fields: written documentations as communicative texts, and formative assessment, which will be the basis for creating didactical material for teachers.

In exams the purpose of documentations is to enable others to understand how the solution has been gained and to evaluate to which degree the solution is correct or incorrect (cf. Ball & Stacey 2003). Thus, the communication of mathematical knowledge is the primary aspect. In terms of the communication model by Jakobson (1960) the communication situation can be described as follows: the learner is the ADDRESSER, the teacher is the ADDRESSEE and the written-down solution is the CONTACT (or channel) for the MESSAGE (1960, p. 353). The CODE in this communication situation can be considered as coming from three different areas: (1) the natural language, (2) the mathematical language, encompassing the symbolic language as well as the mathematical register (cf. Pimm 1987), and (3) the computer world with CAS-commands and also its own register (cf. Siller & Greefrath, 2010).

In exams the written documentation is – according to the communication model above – the only channel by which the message is sent from the addresser to the addressee. Naturally, in such a situation it is neither possible nor allowed for the corrector to inquire in case he or she does not understand a part of the solution. Busse speaks of all written communication as "reduced communication situation[s]" (2015, p 320, translation by the author), arguing that only the text itself and the recipient of the text are present in the situation. As a result, the understanding of texts can be reduced to the allocation of the recipient's knowledge to elements of the text.

The second part of the theoretical background is about formative assessment and how it might be used to develop the documentation competence of students over a longer period. "Assessment for learning", as formative assessment is sometimes called, can be outlined as "the process of seeking and interpreting evidence for use by learners and their teachers to decide where the learners are in their learning, where they need to go and how best to get there" (ARG 2002, p. 2). Black & Wiliam (2009) describe how five key strategies constitute formative assessment:

1. Clarifying and sharing learning intentions and criteria for success;

2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;

3. Providing feedback that moves learners forward;

4. Activating students as instructional resources for one another; and

5. Activating students as the owners of their own learning (Black & Wiliam 2009).

The crucial and most difficult point here is to have criteria for good written solutions. As Weigand

points out “there are no algorithmic rules or norms how to document a solution on paper” (Weigand 2013, p. 2772). He reports from a long-term project in Bavaria (a part of Germany) that students have “difficulties in using SC [scientific calculator] and (problem-)adequate representations especially, as well as the documentation of the solution with paper and pencil” (Weigand 2013, p. 2763). Therefore, teachers need to focus on the development of the competence to document adequately over a longer time. Students have to reflect about documentations and grow into the communication practices of the mathematics community. In Germany, the most important framework of mathematical competencies is the one by the KMK (cf. KMK 2012). The KMK distinguishes three requirement-levels to describe the requirements that can be addressed in tasks in relation to six central mathematical competencies and five central mathematical guiding ideas. This framework is not made for the development of the competence to document (which I see as only a part of the competence of communicating mathematically) but for describing and testing the competencies. Thus, a model was created that focuses more on the development and tries to reflect the difficulties of handling the CAS, too. The competence model by Dreyfus & Dreyfus (1991) is insofar an important reference work that it describes how the development of competencies from a novice stage to an expert stage happens in five steps. According to Dreyfus & Dreyfus, novices act according to rules very explicitly while experts have internalized the rules so much that the behaviour has become part of them. The model for the development of the documentation competence distinguishes only three stages. For teachers it is thus easier to think of the stages as of the three consecutive years (10th to 12th grade) when CAS-classes can be allowed permanently in special classes. The model tries to reflect some difficulties teachers reported in CAS-classes (cf. Beck 2015, Weigand 2013) by shifting the focus of each of the three stages:

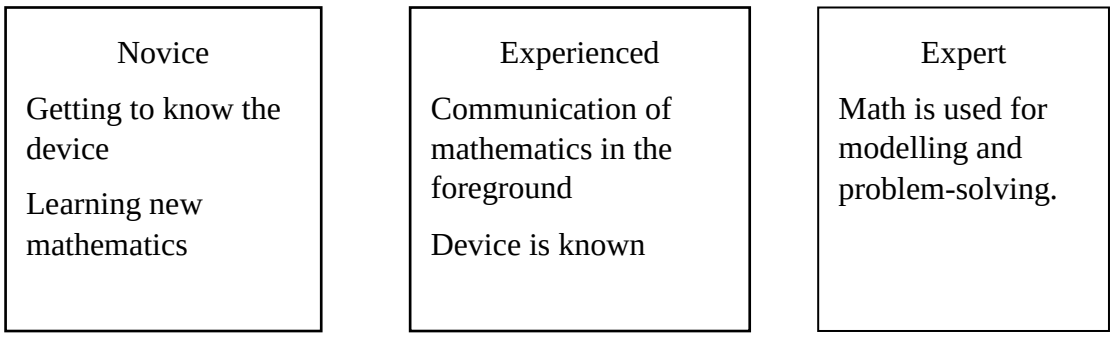

Figure 1 – Development model

1. Novice: Focus on technology-use: Learning to deal with already known and new mathematical content *with the new tool*, in order to get accustomed to it.
2. Experienced: Focus on communication: Reflecting about the use of the tool and the communication of mathematical content.
3. Expert: Focus on modelling and problem-solving: Applying the mathematics to modelling problems with the help of the CAS.

Although high-school CAS-classes only use one device (like the TI-nspire or the CASIO Class Pad II) it is not the aim to restrict the mathematical competence to just this one device. Therefore, also the competence to document should not be chained to the tool that is used but be applicable to all kinds

of tasks and problems. To achieve such a flexible competence teachers could use formative assessment as shown below.

It is the teacher's task to initiate a discussion about how the digital tool changes the nature of the solving process and, as a consequence, the written solution (strategy 2, see above). This is also a reflection of the benefits of a CAS. As already mentioned, it is not necessary anymore to write down every step in the solving process in order to perform it. A lot of algorithmic procedures can be outsourced to the CAS. This reflection might help students to progress further on their way from novices to experts.

Furthermore, as Black & Wiliam state "peer- and self-assessment are activities that might be used to pursue the fourth and fifth [key strategy] respectively" (p. 8). As a useful activity students (and teachers as well) could check and discuss whether their own documentation and the documentation of classmates meet the criteria. Students can discuss problems regarding the understandability of solutions amongst themselves and with their teacher. From a theoretical perspective, it seems to be most promising to apply these activities with experienced students (Fig. 1), after the students are used to work with the CAS but before complex (modelling-)problems are treated. Yet, this assumption has not been tested empirically. Regarding the documentation of solutions one aim is a high level of language use (most prominently: mathematical terminology). This is part of the competences communicating mathematically and mathematical reasoning (cf. KMK-Bildungsstandards, K6 and K1; Blum 2010).

Methodology

As a first step, a descriptive model has been developed from students' authentic solutions from high-stake final exams. These exams are the last time in the students' life and therefore these reflect (to a certain degree) the knowledge and the practise of the students. From a linguistic perspective, it is one aim to identify which elements students use in their documentations and which function is connected with each form. This is a typical pragmatic approach (cf. Meibauer 2008). The underlying question of this form-function-analysis can be formulated as follows: With which forms of representation do students document each step of the solving process? From a mathematical perspective, it is the aim to identify difficulties and problems in the students' solution. One problem is that traditional mathematical notation is mixed with computer language with the result that the created expressions do not fit the requirements of "the community of mathematicians".

Bavarian teachers of CAS-classes have been asked to send in nine written solutions each from the final exams. The students have three groups of tasks to solve (calculus, geometry, data & statistics) and have 180 minutes time. Three solutions came from students who have been average, three from students who have been above average, and three from students who have been below average in the preceding semester. Similar data has been collected every year (starting with 2014) for further evaluation and research. Four to five teachers answer this request every year.

The first research question is how students document their solving process in exams. So far, in Bavaria (Germany) only little official advice about documentations of solving processes is given. Normally, the Institute for School Quality and Educational Research (ISB) provides such material and official notes in addition to the curriculum. In order to develop such advice, it is a very valuable first step for researchers to analyse authentic documentations and to develop a descriptive model with which

problems and difficulties can be identified and categorized.

The representational dimension describes with which forms of representation students document. There might be expressions, which use some kind of formulaic symbols (traditional mathematical, computer-syntax, mixed-forms), verbalisations (both natural language and the special mathematical vernacular) and graphic representations. In the latter category, mixed forms (such as graphs, tables, sketches, etc.) are also counted.

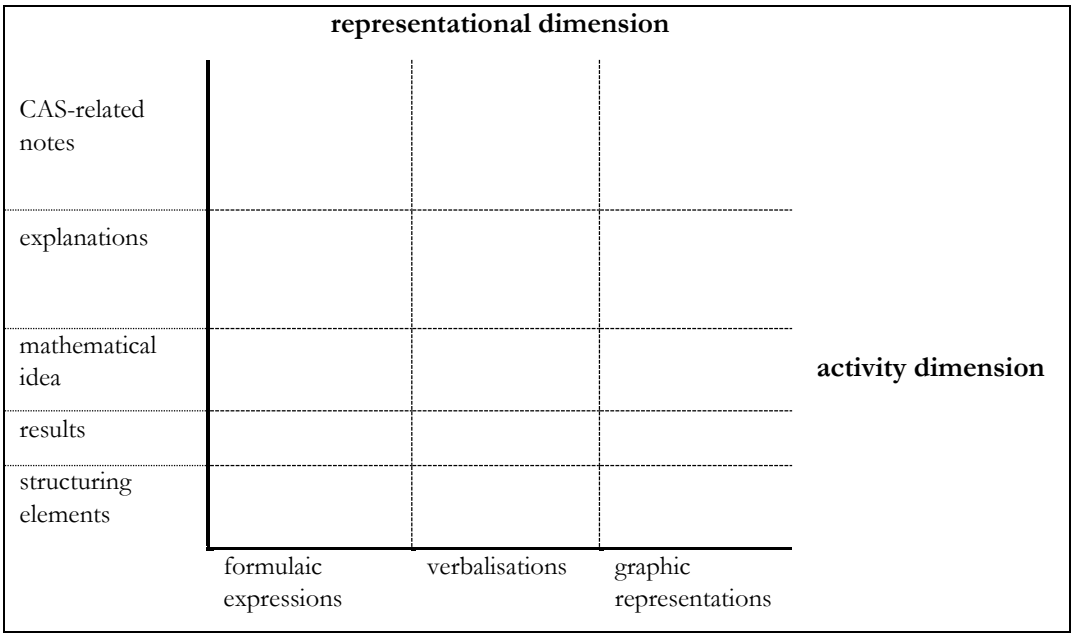

Figure 2 – Category system for description of students' solutions

The second, *activity dimension* describes which purpose an element has, that is what actually is documented with it and which step, or activity, in the solving process it is related to. Central categories are:

- *CAS-related notes* make the use of CAS explicit, either by stating the CAS command (input), by writing down the output (e.g. "false", which is odd in a German text), or by unspecifically writing – in short form – that the CAS was used (e.g. "CAS: …").
- According to Wagner and Wörn (2011) *explanations* comprise three different facets: concepts and ideas (what-explanation), algorithms and procedures (how-explanation), argumentations and logical connections (why-explanation). They often focus on:
    - *mathematizations*, which show that information given in the task-description is translated into mathematical notation or terminology;
    - *interpretations*, which are translations of computer-output and the construction of meaning in relation to the task.

  The terms mathematizations and interpretations are related to the respective activities in the extended modelling cycle by Siller & Greefrath (2010).
- Furthermore, there are elements which refer to the underlying *mathematical idea*, e.g. in order to find the maximum of a function *f* it is necessary to solve the equation *f'(x)=0*. From this element, the mathematical idea can be reconstructed.

- Every mathematical activity leads to some *result*. This might be the answer to a posed question or a step that takes one closer to the final result.
- *Structuring elements* are used to structure the text on the surface (the layout) and the way the information is presented. They can also be used to set up links between pieces of information such as single steps in the solving process and the chronological order in which they were performed.

Results

The first result is that the category system above (Fig. 2) is suitable to describe students solutions. It can be observed that in regard to the documentation of CAS-commands the style was very homogenous throughout each class. In one of the classes, CAS-commands have been documented. In the second class, the CAS-use was indicated by writing "CAS" either over an equation or at the beginning of a line. In the third and fourth class there were no CAS-commands at all. This phenomenon can be explained by the normative standards that the respective teacher had set in the preceding year. Secondly, students who had a correct solution always showed the necessary mathematical ideas. Fig. 3 shows two different ways how mathematical ideas can be presented: either in verbalized form (line 1) or encapsulated in a formulaic expression (line 3).

A further result of the analysis of the students' documents is that written solutions *without* verbalised explanations were often harder to understand and that the solving process could not be reconstructed that easily.

Authentic and examplary solutions

As shown above, elements of written solutions can have different functions. Among them explanations can contribute a lot to make students' documents easily understandable. According to Jörissen and Schmidt-Thieme explanations can be characterised as "primarily verbal statements" with the goal that the reader can understand connections (2015, p. 401, translation by the author).

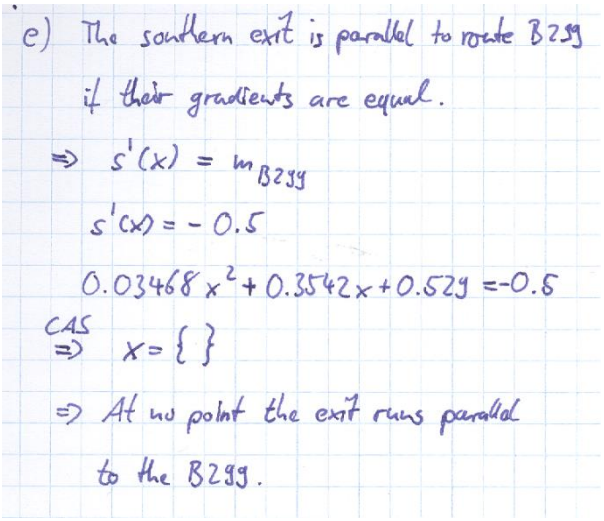

Figure 3 – Student's solution: original and translation

Furthermore, additional explanations extend the transmitted information with the possible consequence of redundancy. However, misunderstandings can possibly be prevented. As already mentioned Wagner & Wörn distinguish three different types of explanations: explain-what, explain-how and explain-why (2011). These sub-categories can be found – rudimentarily – in the students'

solutions, too. It is most important to notice here that students often explained verbally although it was not explicitly asked to do so in the formulation of the task.

The task of the example (Fig. 3) is to check whether there is a point at which the exit of a highway – modelled by a polynomic function *s* – runs parallel to another road – the route B299.

In the example (Fig. 3) we see that the student explains the mathematical idea of his solution verbally at the beginning. It is a rudimentary how-explanation. The verbal inaccuracy at this point is not that important because the information given in the text is supported by the mathematical formulaic expression, which is the equation. The output ("{}") follows a CAS-use which is documented *unspecifically* (see abve). The student confuses proper mathematical syntax with device-specific CAS-output and mixes both into an incorrect expression. As a concluding answer to the task a verbal interpretation of this output is written down.

The categories from Fig. 1 can be used to describe and explain students' solutions. But they can also be used to help teachers to reflect written solutions and their own practise of writing mathematical texts. Furthmore, on the basis of the categories exemplary solutions can be created, as shown below (Fig. 4).

Category	**Exemplary solution**
Explanation	The roads run parallel to each other when there is a point at which f has the same gradient as $g(x) = -0.5x$.
Mathematical idea	$f'(x) = -0.5$
Result	This equation has no solution, therefore,
Expanation	the roads do not run parallel to each other.

Figure 4 – Exemplary solution for teachers

It cannot be expected of students' solution to show such a degree of verbalisation. It is not the purpose of exemplary solutions to set minimal standards for students but to show teachers how solutions can be documented. The categories help to structure the text and to make the function of single elements more apparent.

Discussion of results and conclusion

Teachers may apply the categories in two ways: firstly, the categories can provide guidelines for documentations in a constructive way. For example, every documentation should make clear what the *mathematical idea* was that lead to the solution. Furthermore, students and teacher could agree in classroom discussions that verbalized *explanations* help to make clear the connections of different (symbolic) elements and, thus, allow the reader(s) to follow the solving process more easily. The notation of *CAS-commands* (if and how) could also be agreed upon in the class or amongst teacher even on a school level. Secondly, they then might be used for assessing students' solutions for formative purposes as well as summative purposes. The categories might help teachers and researches alike to see documentations more clearly as a set of elements with different functions that make up the whole text. The exemplary solutions illustrate different possibilities of documentations. They can

be used for reflection in pre-service professional development courses as well as in in-service professional development courses.

In conclusion, it is important that students develop the competence to assess by themselves if a solution is acceptable and to create good solutions on their own (formative assessment key strategies 4 and 5). The main determining factors are the purpose of the documentation and the intended addressee, both of which may make an additional verbal explanation necessary. To provide support for this development is a major challenge for modern mathematics education. The combination of "learning to document" with some elements of formative assessment is a promising way to meet this challenge.

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Teachers' formative assessment practices: The case of an IBME-centered course

Maud Chanudet

University of Geneva, Switzerland; maud.chanudet@unige.ch

In Geneva canton (Switzerland), a special course has been designed to deal with problem solving. Within one 45-minute period per week, teachers have to improve students' problem solving competencies and to assess them frequently. In order to assess students' problem solving competencies, most teachers use grids of criteria with a summative purpose. The global objective of our research is to find out if and how using such a tool can also foster formative assessment processes. In this paper, we present an exploratory study focused on teachers' formative assessment practices. In order to do so, we study the practices of a teacher giving this IBME-centered course and show how she uses formative assessment in her teaching practices.

Keywords: Formative assessment, summative assessment, problem solving, IBME, teachers' practices.

A course dealing with problem solving

In French-speaking Switzerland, the shared curriculum for compulsory education insists on the importance of problem solving in mathematics education in order to make students familiar with inquiry based mathematics education (IBME). The aim is to promote students' scientific processes of thought. But as Dorier and Maass say "inquiry based mathematics education remains quite marginal in day-to-day mathematics teaching" (2014, p. 303). That is why in Geneva canton, a special course called *mathematics development* has been created to focus on and develop students' problem solving competencies. Students aged 13-14 years old (grade 8) with a scientific profile are involved. Within one 45-minute period per week, teachers have to improve the students' problem solving competencies and at the same time, assess them frequently.

This course is subject to many constraints and raises two fundamental questions in mathematics education: how to foster and how to assess students' problem solving competencies? Thus it is necessary to identify what problem-solving competencies are and consequently what students are expected to learn and to know.

IBME and problem-solving competencies

In the French mathematics teaching tradition, problem solving has been seen for many years as a means to develop specific mathematical content and knowledge (Brousseau, 1998). For the past couple of years, however, many countries, and especially European countries, have been emphasizing problem solving in mathematics and inquiry-based mathematics and science education (IBMSE) as a learning goal for its sake. The European Rocard's report (Rocard et al., 2007) promotes a wider implementation of IBMSE in classrooms as a tool to make sciences and mathematics more attractive to students. Nevertheless, this increasing interest in IBMSE has not been followed by a concise and commonly shared definition (Dorier & Garcia, 2013). If we are to summarize, it

> refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, making conjectures and generalizations), interpret and evaluate their solutions and communicate and discuss their solutions effectively. (Dorier & Maass, 2014, p. 300)

Intended learning outcomes of IBME

The goal of IBME is to make students work in a way similar to the one of mathematicians and to make students familiar with a scientific approach to solve problems. For Hersant (2012), a scientific approach cannot be considered as a relevant learning goal, especially because it is unclear, non-unique and too ambitious. Thus the intended learning outcomes of IBME are not so easy to interpret and implementing IBME in classrooms remains a crucial issue. If we look at institutional instructions of the *mathematics development* class, teachers are invited to propose *open-ended problems* (Arsac, Germain, & Mante, 1991) to students, which is, in France and in French speaking Switzerland, a traditional way of introducing students to IBME. An *open-ended problem* is a problem which has a short text, has no obvious solution and method, deals with students' familiar conceptual domain and enables students to make the problem their own. Facing such a problem, students should learn different strategies. It aims more generally at both establishing scientific debate rules and developing a scientific approach following the pattern of *try - conjecture - test and prove*. But according to Hersant (2010), what gives this approach a scientific dimension is not only the existence of trials, conjecture and proof but the articulation among these. She also emphasizes that there is no unique scientific approach. The first goal is not so clear, neither is the second. Debate rules can indeed refer to logical rules (several examples don't prove a proposition, a counter example is sufficient to disprove a conjecture, etc.) or to social rules (listen to the others, etc.). Consequently, curriculum and instructions about *open-ended problems* do not seem to be sufficient to help teachers to define what is institutionally expected about students' problem solving competencies.

Identifying what we want students to learn and to know about problem solving is still a problematic issue. The identification of the intended learning outcomes from IBME is by no means obvious even for teachers, and the danger is that students might not be aware of what they are supposed to learn and to know. That is why IBME learning goals should be at the midst of specific discussions with students in class. Even though such discussions should also be encouraged when acquiring a more classical mathematical knowledge, it is all the more important in the case of IBME.

Problem solving narration activity

To assess students' problem solving competencies, teachers have to be able to access what students did in order to solve the problem and especially what solving strategies they used. That is why the *problem solving narration activity* (Bonafé et al., 2002) has been institutionally chosen as a means to assess students. It can be defined as a new contract between students and teachers in which students have to explain the best they can, how they solved, or tried to solve, the problem (including mistakes, wrong ways, dead-ends, help they received…) and teachers have to assess students on these and only these points and especially not take into account the fact whether students found the right answer or

not. With this activity, the fact that students have to explain all the strategies they tried and all the ideas they had to someone else, presupposes that they are capable to do so firstly to themselves. They have to reconstruct their reflection and make a synthesis of which strategies were effective, which one were wrong ways or led to dead-ends, etc. In that sense, it can emphasize students' reflection about what solving problems in mathematics means, about their own problem solving competencies and it can encourage the development of para and proto-mathematical knowledge. *Problem solving narration activity* as a scheme used principally for summative assessment can also foster students' problem solving competencies and assume a formative function. This last observation leads us to consider the assessment of problem solving competencies, not only with summative purpose, but also with a formative purpose.

Assessing students' problem-solving competencies

According to Allal (2008) assessment is summative as soon as a synthesis of the competencies and knowledge learnt by the student at the end of his curriculum is established. Thanks to the distinction made by Scriven (1967) and then by Bloom (1968) between summative and formative assessment, Black and Wiliam give the following definition:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited (Black & Wiliam, 2009, p. 9).

The notion of feedback is a key component of formative assessment. Formative assessment contains "all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities" (Black & William, 1998, pp. 7-8). Another key component of formative assessment is that students understand the target of their work and that they grasp what is expected (Harlen, 2013). But it means that "students need to have some understanding of the criteria to apply in assessing their work" (Harlen, 2013, p. 17). Once again, the necessity of specific discussions with students about assessment criteria and about what they are expected to learn is emphasized.

To classify classroom formative assessment, Shavelson et al. (2008) are using a continuum, that ranges from formal embedded assessment to informal, on the fly formative assessment. It means that formative assessment does not take a unique form but that it can be planned or not, it can refer to formal tools to collection of data or not, etc. Adopting this point of view, formative assessment can be considered as a practice integrated within the learning process (Lepareur, 2016). Referring to formative assessment about IBME is all the more relevant that

> the practice of formative assessment, through teachers and students collecting data about learning as it takes place and feeding back information to regulate the teaching and learning process, is clearly aligned with the goals and practice of inquiry-based learning. (Harlen, 2013, p. 20)

These definitions of summative and formative assessment enhance that identifying assessment according to *when* it occurs (after a phase of teaching *vs* within a teaching activity for instance) or *how* it occurs (paper-pencil test *vs* worksheet for instance) seems less relevant than distinguishing assessment according to its function. But it does not mean that these two principal functions of assessment (summative and formative) cannot coexist. Thus some researchers (Allal, 2011; Harlen,

2012; Shavelson et al., 2008) argue that they can coexist in what Earl (2003) calls *assessment for learning*. The same assessment activity can serve to summative and formative purpose. It means that data collected by the teacher can be used to give students a mark but also to improve learning and teaching. On the other hand, for Shavelson et al. "formative assessment could serve summative needs" (2008, p. 298). In our research, we deal with teachers' practices in the *mathematics development* class. Our objective is to find out if and how using an assessment tool as a grid of criteria, firstly with summative purpose, can also foster formative assessment processes. In this paper, we focus only on teacher's formative assessment practices. For that, we study the practices of a teacher giving this IBME-centered course.

Teachers' formative assessment practices

Context of the research

The teacher whose practice we are going to analyze, was a member of a one year commission, created in September 2015, and gathering another teacher and ourselves. The purpose of this commission was to give teachers of *mathematics development* classes a common tool to assess students' problem solving competencies with both summative and formative purpose, and consequently to ensure common expectations about IBME (from teachers, and more globally from schools). To do so, we have been working for one year to elaborate a grid of criteria aiming to assess students' *problem solving narration activity*. This development of the tool was mainly based on teachers' expertise. Indeed, teachers have implemented the grid in their class, and according to their experiences, we adjusted, removed and added some criteria. Nevertheless, we dealt with an existing tool elaborated by the Geneva team in the wake of the PRIMAS[1] project and were careful to take into account some research results (as those of Hersant (2010)). The grid in its final version summarizes criteria related to five dimensions of such an activity: presentation, narration, research, technique and modelling. These dimensions induce ways to look at the students' production according to expected qualities. For instance, the modelling dimension is characterized by two criteria: "Appropriation of the problem: rephrase the problem in French and/or express it with drawings, diagrams, tables" and "Use of pertinent mathematical tools and theories, strategies".

So, this teacher whose practice we are going to analyze has been reflecting on the intended learning outcomes of this IBME-centered course and on the summative assessment of problem solving competencies, for one year, thanks to the meetings with the other members of the commission. She used the grid elaborated by the commission, in her class, principally with a summative purpose. However, thanks to our hypothesis that summative and formative assessment can co-exist, we would like to see if she also referred to formative assessment, thanks to specific discussions with students about criteria, and feedback related to their production. That is why we analyzed her formative assessment practices.

[1] Available at http://www.primas-project.eu/fr/index.do

Theoretical framework

To characterize teachers‘ formative assessment practices, we referred to criteria elaborated by Lepareur **(2016)**. She defined five strategies[2] : eliciting goals and criteria (S1); managing discussions and activities which can produce some evidence of effective learning (S2); giving feedback to students which make them progress (S3); helping students to be responsible for their learning (S4); helping students to be a resource for their peers (S5). Even though her research dealt with science and mathematics teachers' formative assessment practice, in our case, we only study mathematics teachers' practices. In that sense, we made some adaptations about key words and sub-strategies she defined. The table 1 is the grid we used to characterize the mathematics teacher's formative assessment practices. The T is used for the teacher, S for the students.

Methodology

We video recorded two consecutive periods of *mathematics development* given by this teacher, member of the commission, at the end of the school year. The nine students of the class were working in four groups (3 groups of 2 students, 1 group of 3). They were working on two problems related to the introduction of algebra. At the end of the second period, students had to give a narration about the problem they were working on to the teacher. Consequently half of the second period was devoted to the narration and students were invited to use office software to write their research down. To interpret data and make it relevant with our theoretical framework, we transcribed all interactions occurred in class for both lessons (about 67 minutes) and classified interactions according to strategies and sub-strategies defined in the table 1.

Strategy	Key words	Code	Description
S1	Goals	S11	T explains the goals of the activity.
	Criteria	S12	T explains the intended learning outcomes, what will be assessed.
S2	Progress in activity	S21	T collects information about students' progress in the activity.
	Strategies	S22	T collects information about strategies used by S.
	Understanding	S23	T questionnes S about their understanding of the goals.
	Knowledge	S24	T takes information about previous S' knowledge.
	Self assessment	S25	T helps S to situate themselves in relation to assessment and success criteria.
S3	Feedback (what students have to do)	S31	T gives an information to make explicit what S have to do left.
	Feedback (how students can do it)	S32	T provides explicit information about how S have to do it, to move on.
S4	Responsabilisation	S4	T emphasizes S' ideas, gives them independancy to access resources.
S5	Interactions (group)	S51	T encourages S to discuss with others members of their group. Peers are seen as a resource.
	Interactions (class)	S52	T integrates S' propositions and encourages others to react on.

[2] Translated from Lepareur (2016).

Table 1: Grid of analysis of formative assessment practices, adapted from Lepareur (2016)

For instance, during the following interaction between the teacher and a student, we can see that the teacher tries to make the student explains his strategy.

Teacher:	Yes but how did you find this?
Student:	I made a lot of stuff.
Teacher:	But try… what did you do? It's interesting to know how you were thinking.
Student:	I made all of this but like everything in reverse.
Teacher:	Yes so the first step. What did you do at the first step?
Student:	24 minus 7.
Teacher:	Yes, you went backwards in your calculations. Yes. It's a good idea. Doing backward calculation is in fact a lead.

In that case, we identify a formative assessment practice, according to the strategy S2 "managing discussions and activity which can produce some evidence of effective learning" and more specially the strategy S22 "collecting information about strategies used by students".

Results and analysis

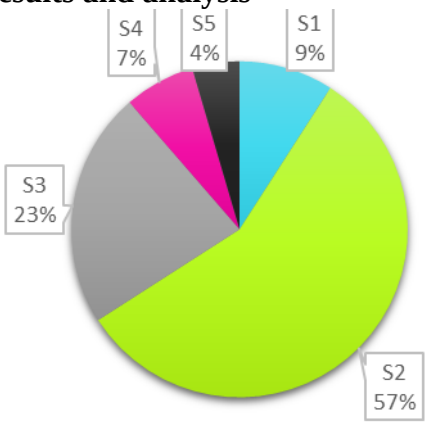

Figure 1: Strategies of formative assessment

We found out 44 episodes when formative assessment strategies occurred thus we can say that this mathematics teacher refers frequently to formative assessment in her practice. We identified 4 times the strategy S1; 25 times strategy S2; 10 times strategy S3; 3 times strategy S4 and 2 times strategy S5. The figure 1 illustrates the percentage of apparition of each strategy. To summarize, we can notice that the teacher refers to every strategy (S1, S2, S3, S4 and S5).

On top of that, she uses principally the strategy S2: "managing discussions and activities which can produce some evidence of effective learning". It represents more than half of the strategies of formative assessment used by this teacher. The strategy S3 which refers to "give feedback to students which makes them progress" is also used frequently, about one time out of four.

But if we look deeper, we can see that each sub-strategy is not used with the same frequency (figure 2).We can see that for the strategy S1 related to the goals and criteria, the teacher only explains the goals of the activity (S11) but not the criteria of assessment or the intended learning outcomes (S12). This lesson occurred at the end of the year so we can make the hypothesis that by then students knew well what they were expected to do.

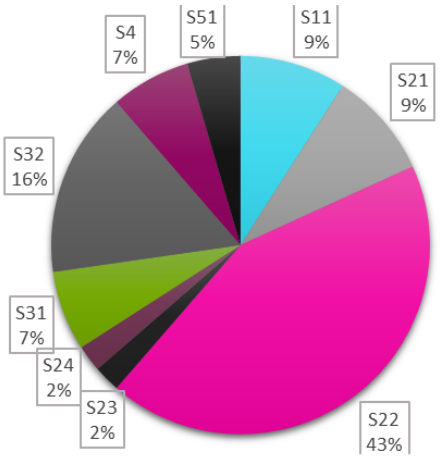

Figure 2: Strategies of formative assessment according to key words

The most represented sub-strategy related to "discussions and activities which can produce some evidence of effective learning which is used frequently" (S2) is "collecting information about strategies used by students" (S22). It appears 19 times. It is also the strategy that occurs the most, all categories taken into account. The only other significant strategy dealing with S2 used by the teacher is "taking information about students' progress in the activity" (S21). So the teacher focuses on where students are in the activity and what they have done to get there.

Then, for strategy S3 about feedback, the two ways; "provides information to make it explicit what is left for the students to be done" (S31) or "how they have to move on" (S32) are represented, but the second one more than the first one. The idea is that this teacher helps students both knowing what they have to do but even more how they can continue. For the last strategy (S5) dealing with interactions, the teacher focuses on discussions within the groups. The absence of strategy S52 about "the integration of students" can easily be explained by the fact that during these two lessons, students only worked in group, without any collective classroom discussion.

Conclusion

We can say that this mathematics teacher refers frequently to formative assessment in her practices (44 times during a 67 minute-lesson). She uses a very large set of formative assessment tactics; eliciting goals and criteria; managing discussions and activities which can produce some evidence of effective learning; giving feedback to students which make them progressing; helping students to be responsible for their learning; helping students to be a resource for their peers. But she uses principally formative assessment to manage discussions and activities which can produce some evidence of effective learning, and especially, she collects information about strategies students use. The feedback she provided to students is mainly about how they can continue, how they can do what they have to do to solve the problem. The only strategy which does not appear is this related to explanation of criteria and expected learning outcomes. We can make the hypotesis that it has been at the core of a discussion in the first part of the schoolyear. In that sense, it should be interesting to focus on what happens at the beginning of the schoolyear and to study how teachers explain and negociate the intended learning outcomes with students.

To conclude this paper, we can say that this exploratory study shows that formative assessment seems to be relevant for this teacher in order to foster her teaching practices in the case of an IBME-centered course. It is, nevertheless, necessary to enlarge the study, in order to compare and expend or not the results, and to have more information about how teachers refer to formative assessment. On top of that, we can imagine that working with teachers about these strategies could foster their formative assessment practices.

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Ireland's Junior Certificate mathematics examination through the lens of the PISA and TIMSS frameworks: Has Project Maths made any difference?

Rachel Cunningham[1], Seán Close[2] and Gerry Shiel[3]

[1]Educational Research Centre, Drumcondra, Dublin 9, Ireland; rachel.cunningham@erc.ie

[2]Educational Research Centre, Drumcondra, Dublin 9, Ireland; sean.close@erc.ie

[3]Educational Research Centre, Drumcondra, Dublin 9, Ireland; gerry.shiel@erc.ie

It has been argued that the PISA assessment has had a disproportionate impact on Project Maths, the new mathematics curriculum recently implemented in post-primary schools in Ireland, which seeks to emphasise deep conceptual understanding and problem solving in real-life contexts. This paper describes an analysis of the content, cognitive processes and contexts underpinning Junior Certificate mathematics examination questions set for students in Grade 9 in 2003 and 2015, using the frameworks underpinning the PISA and TIMSS studies. Despite a significantly increased reading load for students, the Junior Certificate mathematics examination continues to emphasise lower-order processes, at the expense of higher-level thinking, as defined by PISA and TIMSS, while there has been a small increase in the proportion of items presented in practical contexts. The need to examine the effects of Project Maths in classroom settings is highlighted.

Keywords: Mathematics, assessment, Project Maths, Junior Certificate.

Introduction

Since 2010, all students entering post-primary school (Grade 7) in Ireland have studied under Project Maths, an innovative mathematics curriculum introduced in an effort to increase the relevance of mathematics for students, and to improve teaching and learning. Since 2015, all aspects of Project Maths have been assessed in the Junior Certificate (JC) mathematics examination, a state examination taken by almost all students at the end of Grade 9 (Third Year). The purpose of this paper is to examine changes in how and what mathematics is assessed in the JC, and how these changes relate to the frameworks and approaches to assessment of mathematics underpinning the OECD Programme for International Student Assessment (PISA) (OECD, 2016), and the Trends in International Mathematics and Science Study (TIMSS) (Grønmo, Lindquist, Arora, & Mullis, 2013).

Ireland has participated in PISA since its inception in 2000. Although Ireland has been consistently among the highest-performing countries on reading literacy in PISA, performance on mathematical literacy has generally been at the OECD average, and, on one occasion (2009), significantly below it. Although average performance on mathematics in Ireland was above the OECD average for the first time in 2012, this reflected a decline in the OECD average rather than an increase in the performance of students in Ireland, compared with, for example, 2003 and 2006.[1] Students in Grade 8 in Ireland

[1] It should be noted that over three-quarters of students in PISA 2012 in Ireland had not studied under Project Maths, with students in Grades 9 and 10 studying the preceding curriculum, except those in a small number of Project Maths pilot schools.

did not participate in TIMSS between 2003 and 2011, but did take part in the most recent cycle, in 2015.

Concerns about standards in mathematics have increased in recent years and outcomes in PISA are just one factor contributing to this. There have also been concerns about declining performance among entrants to university mathematics courses (Gill, O'Donoghue, Faulkner & Hannigan, 2010).

In 2005, the National Council for Curriculum and Assessment began a process leading to the implementation of a revised mathematics curriculum for post-primary schools. This involved the commissioning of a research paper looking at international trends in mathematics education (see Conway & Sloane, 2006), and a consultation process involving interested parties (NCCA, 2005). Following two years of development work, the Project Maths curriculum was introduced into 24 pilot schools in 2008, and implementation began in all post-primary schools in 2010, with a phased introduction that was completed by 2015. The aims of Project Maths at Junior Cycle level (Grades 7-9) include developing mathematical knowledge, skills and understanding needed for continuing education, for life and for work; fostering a positive attitude to mathematics; and developing the skills of dealing with mathematical concepts in context and in applications, and in problem solving (DES, 2013). There are five inter-related strands in the curriculum: Statistics and Probability; Geometry and Trigonometry; Number; Algebra; and Functions. Learning outcomes are identified for students intending to take the JC mathematics examination at Higher and Ordinary levels, with no separate course for Foundation level.

Project Maths has received a mixed reception. The Irish Mathematics Teachers Association (IMTA) (2012) noted that insufficient detail was provided on aspects of course content, giving rise to uncertainty as to whether certain topics were included or not. Drawing on a survey of teachers in Project Maths pilot and non-pilot schools conducted as part of PISA 2012 in Ireland, Cosgrove et al. (2012) reported more frequent use of ICT in pilot schools, and more positive changes in learning and assessment, though teachers in pilot schools were less confident in their teaching. Concerns about readability were raised in the same study, with teachers in pilot schools arguing that the problems presented to students in classroom and assessment contexts contained more text and greater linguistic complexity than was the case prior to Project Maths, when teaching and learning mathematics were more formal and less contextualised. Similar concerns have been raised about the effects of readability on performance in the PISA mathematics assessment as a result of the complexity of contexts that are presented to students (Eivers, 2010).

A small number of studies have looked at the initial effects of Project Maths on students' performance. A study by the National Foundation for Educational Research in the UK (Jeffes et al., 2013) found no achievement differences between students in schools implementing Project Maths for longer or shorter time periods, or between those that had implemented more, compared with fewer, content areas. Although students in pilot schools in the PISA 2012 sample in Ireland achieved higher mean scores on each PISA mathematics content area, and on overall performance, differences were not statistically significant (Merriman, Shiel, Cosgrove & Perkins, 2014). Worryingly, students in pilot schools had significantly higher levels of anxiety about mathematics than their counterparts in non-pilot schools.

The lack of evidence for a significant change in mathematics performance since the implementation of Project Maths, as well as claims that PISA has had a disproportionate impact on the Project Maths curriculum (e.g., Kirwan, 2015; Grannell, Barry, Cronin, Holland & Hurley, 2011) points to the need for a critical look at changes to the JC mathematics examination. This paper looks at whether the contexts, content and processes underpinning the examination differed in 2003 and 2015 (when analysed through the lens of PISA and TIMSS) and whether examination papers have become more or less readable between the same two time points. The research questions addressed are as follows:

(i) What changes in mathematical content, cognitive processes, and item contexts can be identified from a comparison of the pre-Project Maths 2003 Junior Certificate examination with the post-Project Maths 2015 examination with reference to the TIMSS and PISA mathematical frameworks and tests?

(ii) What changes in readability can be identified from a comparison of the pre-Project Maths 2003 Junior Certificate examination with the post-Project Maths 2015 examination?

Methodology

Examination paper analysis – Context, content and process

This study focuses on the JC state examination papers in 2003 and 2015 as examples of papers before and after the introduction of the revised curriculum. The 2003 papers were chosen because a similar classification exercise was previously carried out on these papers by Close and Oldham (2005). The 2015 papers were chosen as a comparison as they were the first to include all Project Maths content areas for all JC students.

Junior Certificate mathematics is examined at three levels: Higher Level (HL), Ordinary Level (OL) and Foundation Level (FL). There are two papers each for HL and OL and one for FL. Consequently, the study included ten papers in total (five for each year). The aim was to classify the questions in each examination paper in terms of the main components of the TIMSS 2015 Grade 8 mathematics framework (Grønmo et al., 2013) and the PISA 2003 and 2015 frameworks (OECD, 2003; 2016). The relevant characteristics of these frameworks are outlined below. The PISA framework was chosen as it reflects recent trends internationally towards realistic mathematics and problem-based learning in rich contexts. The TIMSS framework was also used in this study as it reflects a more traditional, curriculum-based approach, with a focus on the mathematical concepts, skills and applications seen as necessary for further study of mathematics and for life. Using both frameworks allowed the analysis to capture more fully any changes in the JC examination between the selected time points.

TIMSS mathematics has four content domains: Number, Algebra, Geometry and Data and Chance. There are three cognitive domains: Knowing, Applying and Reasoning. The PISA mathematics framework has three dimensions: Context, Content and Competency[2]. PISA classifies each mathematics item in terms of its context – Personal, Occupational, Societal or Scientific. The PISA

[2] For the purposes of this study, the 2015 framework was used to classify items by Content and Context. However, the 2003 Competency Clusters were used as they are more suited to the JC examinations and are more consistent with the TIMSS Cognitive Domains. In addition, the 2003 Competencies were used in Close and Oldham (2005).

content categories are Change and Relationships, Space and Shape, Quantity, and Uncertainty and Data. The PISA 2003 framework also classifies items by groups of cognitive processes or 'Competency Clusters'. The three clusters are Reproduction, Connections, and Reflection.

The Reproduction cluster in PISA can be summarised as the 'reproduction of practised knowledge' (OECD, 2003), which covers the category of Knowing in TIMSS but also some of Applying. The Reflection cluster involves 'advanced' reasoning, abstraction and generalisation in novel contexts, which often requires a higher cognitive demand than some Reasoning items in TIMSS.

Within each JC paper, each part of a question (a i), ii), etc.) was treated as a separate item. Each item was classified by two of the authors (RC and SC) according to the dimensions outlined above. Classifications were carried out independently and all disagreements were recorded and discussed until a consensus was reached. Very few disagreements arose in relation to the content or context of the items. By comparison, more disagreements occurred where judgements were made about the processes involved in answering the items i.e. in identifying TIMSS Cognitive Domains and PISA Competency Clusters. This is not surprising, as determining the primary process required to answer a test item is, by nature, a more subjective exercise. Initial agreement rates for cognitive processes per examination ranged from 74 percent (TIMSS classifications for FL 2015) to 97 percent (PISA classifications for FL 2003). Most examinations had an agreement level above 80 per cent.

Reading load and readability analysis

An analysis of the reading load and readability of JC mathematics examination papers administered in 2003 and 2015 was also conducted. Reading load here refers only to the number of words to be read, while measures of readability aim to assess the overall difficulty of the text. Data were generated for word count, number of sentences, average number of words per sentence, and average number of complex words (words with three or more syllables). An overall measure of readability for each paper was obtained by taking the average results (in Grade level units) of eight readability measures including the Flesch-Kincaid Grade Level. Prior to applying these formulae, title information, general instructions and item numbers were removed. In addition, diagrams were deleted (though labels and numbers were retained), and functions were replaced with a placeholder, as readability formulae are not designed to assess the complexity of these elements.

Results

Classification of items by content, cognitive process and context[3]

The data in Table 1 show that, between 2003 and 2015, there was an increase in the number of JC examination items in Data and Chance (+10 percentage points), as defined by TIMSS, with a corresponding increase in Uncertainty and Data (+9 percentage points), as defined by PISA. The change in these content domains can be ascribed to the increased emphasis on Statistics and Probability in the revised curriculum. These increases were more or less counter-balanced by decreases in Number (-5 percentage points) and Geometry (-4 percentage points) on the TIMSS content dimension and in Quantity (-8 percentage points) in PISA.

[3] For brevity, the results of the item categorisation are collapsed across HL, OL and FL for each year.

TIMSS Content Domain	TIMSS % Items	JC 2003 % Items N = 187	JC 2015 % Items N = 208	PISA Content Domain	PISA % Items	JC 2003 % Items N = 187	JC 2015 % Items N = 208
Number	30 (30)[4]	21	16	Quantity	25	27	19
Algebra	30 (25)	24	27	Change & Relations	25	31	34
Geometry	20 (30)	30	26	Space & Shape	25	25	24
Data & Chance	20 (15)	11	21	Uncertainty	25	11	20
Not covered[5]		13	10	Not covered		6	3

Table 1: Percentages of items in each TIMSS and PISA content domain, and percentages of total JC 2003 and 2015 items, by TIMSS and PISA content domains

Table 2 presents the results for the cognitive process dimensions. Relatively few of the JC examination items for either 2003 or 2015 fell into the TIMSS Reasoning category, although there was an increase between 2003 and 2015 (+6 percentage points). Only two items in 2015 (1%) were categorised as PISA Reflection, with none in 2003. However, the proportion of Connections items was higher in 2015 than in 2003 (+8 percentage points), with a corresponding decrease in Reproduction items. Despite this, most of the JC items for both years were classified as TIMSS Knowing and Applying and PISA Reproduction.

TIMSS Cognitive Domain	TIMSS % Items	JC 2003 % Items N = 187	JC 2015 % Items N = 208	PISA Cognitive Domain	PISA % Items	JC 2003 % Items N = 187	JC 2015 % Items N = 208
Knowing	35	35	45	Reproduction	25	88	79
Applying	40	57	42	Connections	50	12	20
Reasoning	25	7	13	Reflection	25	0	1

Table 2: Percentages of items in each TIMSS and PISA cognitive (process) domain, and percentages of total JC 2003 and 2015 items, by TIMSS and PISA cognitive domains

Part of the intention of Project Maths reform was to place more emphasis on using mathematics to solve problems set in practical realistic contexts. Table 3 shows the results of classifying the items in the 2003 and 2015 JC exams into items with some sort of practical context and items which are purely mathematical or intra-mathematical. The results show that around half of the items in the 2015 examination papers had a practical context reflecting a small change since 2003 (up from 40%). These figures are similar to the TIMSS percentages for mathematical and practical contexts, whereas all PISA items are placed in a practical context. It is important to note that the practical contexts of

[4] TIMSS 2003 Content Domain weightings in parentheses.

[5] Some JC topics are not included in the TIMSS and/or PISA frameworks e.g. sets, trigonometry and proofs of geometric theorems.

TIMSS items and JC 2003 and 2015 exam items are generally minimal compared with the more substantial and often realistic contexts of PISA items.

Context Category	TIMSS approx.% Items	PISA % Items	JC 2003 % Items, N = 187	JC 2015 % Items, N = 208
Mathematical	50	0	60	49
Practical	50	100	40	51

Table 3: Comparison of TIMSS and PISA test items and JC exam item percentages by context category

Reading Load and Readability[6]

Table 4 shows that, at JC HL, the number of words students were expected to read on Paper 1 increased from 765 in 2003 to 1335 in 2015 (a 75% increase). At OL, the word count increased from 662 to 1240 (an 87% increase), and at FL, the increase was from 530 to 1027 words (a 94% increase). However, the average difficulty of the text that students were expected to read remained about the same at HL (Grade 4) and OL (Grade 2). There was an increase at FL (from Grade 0 to Grade 2). The average number of words per sentence remained more or less the same (and even decreased by 3 on HL Paper 1 in 2015), while the proportions of complex words also remained about the same.

Examination Paper	No. of Item Parts	No. of words	No. of sent-ences	Avg. no. words per sentence	No. of complex words (Percent)	Flesch-Kincaid grade level	Average read-ability grade
2003 HL Paper 1	32	765	58	13	61 (8)	5.0	4
2015 HL Paper 1	45	1335	135	10	123 (9)	4.1	4
2003 OL Paper 1	40	662	76	8	35 (5)	2.6	2
2015 OL Paper 1	41	1240	181	7	64 (5)	2.2	2
2003 FL	32	530	73	7	23 (4)	1.6	0
2015 FL	43	1027	140	7	57 (6)	2.3	2

Table 4: Readability measures for JC examination papers in 2003 and 2015

Conclusion

The impetus for the Project Maths curricular reform arose, in part, from Ireland's performance in PISA mathematics (NCCA, 2012). However, the current analysis does not indicate that the assessment of JC mathematics has been unduly influenced by the PISA approach. The distribution of items by content area on the JC examination in 2015 was similar to its predecessor in 2003, when viewed through the lens of TIMSS and PISA, though more Data and Chance items were included in

[6] Again, for brevity, results for this section are reported for Paper 1 only for HL and OL. There is only one paper for FL for each year.

2015. This is in line with the increased emphasis on Statistics and Probability in the new JC curriculum. While there were proportionally more Reasoning items (as defined by TIMSS) in the JC examination in 2015 than 2013, there were no Reflection items (as defined by PISA) in 2003 and only 2 in 2015. However, there was an increase in the proportion of PISA Connections items in the JC examination, and a reduction in the proportion of Reproduction items. One objective of Project Maths is to place more emphasis on higher-order cognitive processes, such as problem solving and reasoning in the mathematics curriculum and examinations. The analysis above suggests some movement in this direction in JC state examinations, but not to the extent that might be expected from such a comprehensive reform. The analysis also indicates that the 2015 JC examination is more similar to TIMSS than to PISA in terms of content and processes. This is reinforced by the finding that the proportions of items described as being presented in practical (rather than purely mathematical) contexts were similar in the JC 2003 and 2015 exams, despite large increases in the amount of text that students had to read. Remarkably, however, the readability (overall difficulty) of the text in the JC examinations was broadly similar in 2003 and 2015. A study by King and Burge (2015) found that readability levels for clusters of PISA 2012 items ranged from US grade levels 7.5 to 10.9 (UK reading ages 12.3 to 15.5 years). Hence, the linguistic complexity of PISA items is greater than that required for JC mathematics. The analysis here focuses only on state examinations of JC mathematics, and not on the implementation of Project Maths in the classroom. Given that Project Maths has now been fully rolled out, it would seem timely for an exploration of the extent to which teaching and learning has changed.

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Designing and analysing the role of digital resources in supporting formative assessment processes in the classroom: The helping worksheets

Annalisa Cusi[1], Francesca Morselli[2] and Cristina Sabena[3]

[1]University of Turin, Italy; annalisa.cusi@unito.it

[2]University of Genoa, Italy; morselli@dima.unige.it

[3]University of Turin, Italy; cristina.sabena@unito.it

In this paper we present the design of specific digital resources and related methodology, conceived with the aim of exploiting connected classroom technology to carry out formative assessment processes in the mathematics classroom. The digital resources have been created and experimented within the European Project FaSMEd. By using a multi-dimensional theoretical frame developed within FaSMEd, we offer elements of validation for the design, focusing in particular on the activation of formative assessment strategies through the use of "helping worksheets". The elements of validation will be illustrated through an example from a case study.

Keywords: Formative assessment, technology, digital resources, task design.

Introduction

This contribution stems from the European Project FaSMEd ("Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education"), aimed at investigating the role of technologically enhanced formative assessment (FA) methods in raising students' attainment levels.

Within FaSMEd, FA is conceived as a method of teaching where evidence from learning is used to adapt both teaching and learning. Wiliam and Thompson (2007, in Black & Wiliam, 2009) focus on *three central processes* in learning and teaching, which represent the aims related to the collection, interpretation and exploitation of these learning evidence: (a) *Establishing where learners are in their learning*; (b) *Establishing where learners are going*; (c) *Establishing how to get there*.

In the model developed by Wiliam and Thompson these three *central processes* are connected to the *three main agents* that intervene (the teacher, the student, the peers) and to the *FA key-strategies* that could be activated: (A) Clarifying and sharing learning intentions and criteria for success; (B) Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (C) Providing feedback that moves learners forward; (D) Activating students as instructional resources for one another; (E) Activating students as the owners of their own learning.

A theoretical model to analyse the use of technology in FA practices has been elaborated within FaSMEd (Aldon et al., in print, and Cusi, Morselli & Sabena, 2016). The model extends Wiliam and Thompson's model, taking into account three main dimensions: (1) *the five FA key-strategies* described by Wiliam and Thompson (ibid.); (2) *the three main agents* that intervene; (3) the *functionalities of technology*. The third dimension - the *functionalities* of technology – was added within FaSMEd to focus on the ways in which technology can support the three agents in developing the FA strategies: (a) *Sending and sharing*, that is the ways in which technology

supports the communication among the agents of FA processes; (b) *Processing and analysing*, that is the ways in which technology supports the processing and the analysis of the data collected during the lessons; (c) *Providing an interactive environment*, that is when technology enables to create environments in which students can interact to work individually/in group on a task or to explore mathematical/scientific contents.

We argue that the FaSMEd three-dimensional framework may represent a useful tool for:

- *designing* digital materials for technology-enhanced FA practices and the corresponding methodology, and
- *analysing* how these materials are implemented in the classroom.

In this paper we will refer to the study carried out in Italy, where we added the fundamental assumption that, in order to raise students' achievement, FA has to focus not only on cognitive, but also on metacognitive factors (Schoenfeld, 1992). For this reason, our design is aimed at i) fostering students' ongoing reflections on the teaching-learning processes, and ii) focusing on making thinking visible (Collins, Brown & Newmann, 1989) through students' sharing of their reasoning with the teacher and the classmates, by means of argumentative processes.

In the following we present the design of digital materials and of the methodology for their implementation in the classroom. We will then focus on the specific case of "helping worksheets" and analyse its implementation in a case study. The analysis will be based on the FaSMEd three-dimensional framework. The analysis of FA strategy C (Providing feedback that moves learners forward) will be deepened with reference to the four major levels of feedback introduced by Hattie and Temperley (2007): (1) *feedback about the task*; (2) *feedback about the processing of the task*; (3) *feedback about self-regulation*; (4) *feedback about the self as a person*.

Design of the digital materials and of the methodology for their implementation

The research developed within FaSMEd has been built on the model of design-based research (Cobb et al, 2003), so it is based on successive cycles of design, observation, analysis and redesign of classroom sequences. The design experiments we carried out in Italy were characterised by three subsequent cycles of design. The first two cycles were carried out in March-May 2015 and in September-December 2015. The third cycle started in May 2016 and has not been completed yet. The results we present in this paper refer to the first two cycles of the design.

In tune with the theoretical assumptions presented in the previous paragraph, we chose to use a technology that supports the students in sharing, discussing and comparing both their written productions and the strategies developed to carry out the different tasks. Specifically, we explored the use of a connected classroom technology (CCT), which creates a network between the students' tablets and the teachers' laptop, allowing the students to share their productions, and the teacher to easily collect the students' opinions and reflections: IDM-TClass.

The design experiments involved 25 classes (from grade 4 to grade 7) from three different clusters of schools located in the North-West of Italy. Each school was provided with tablets for the students and computers for the teachers, linked to IWB or data projector. In order to foster collaboration and sharing of ideas, students were asked to work in pairs or in small groups on the same tablet.

During the first two cycles of design, we carried out about 450 hours of lessons. The researcher was in the class as both an observer and a participant (to support the teacher in the use of the technology and in the implementation of the digital resources). In some cases, also Master students were present as observers. The corpus of data is constituted by video-recordings of the lessons, written transcripts, field notes taken by the observers, teachers' interviews after sequences of lessons, students' written questionnaires and groups of students' interviews during a Q-sorting activity (questionnaires and Q-sorting data were collected at the end of the design experiments).

The use of IDM-TClass was integrated within a set of activities on relations and functions, and their different representations (symbolic representations, tables, graphs). These activities, in line with the aims of the FaSMEd Project, where adapted starting from existing research-informed materials.

For each activity, we have prepared a sequence of different worksheets, to be sent to the students' tablets or to be displayed on the IWB (or through the data projector). The worksheets were designed according to *four main categories*: (1) worksheets introducing a problem and asking one or more questions (*problem worksheets*); (2) *helping worksheets*, aimed at supporting students, who meet difficulties with the problem worksheets, through specific suggestions (e.g. guiding questions); (3) worksheets prompting a poll between proposed options (*poll worksheets*); (4) worksheets prompting a focused discussion (*discussion worksheets*).

Usually the activity starts with a problem worksheet, sent from the teacher's laptop to the students' tablets. Students work in pairs or small groups of three. After facing the task and answering the questions, the pairs/groups send back to the teacher their written productions. The teacher can decide to send helping worksheets to some groups, or the groups can ask for them.

After all groups have sent back their answers, the teacher sets up a classroom discussion in which the students' written productions are shown and feedbacks are given by the teacher and by classmates. The discussion is engineered starting from the teacher's selection of some of the received written answers, to be shown on the IWB, and aims at highlighting: (a) typical mistakes; (b) effective ways of processing the tasks; (c) the comparison between the different ways of justifying. During the part of the discussion focused on these aspects, therefore, the criteria for success could be clarified through the analysis and comparison of the different written productions.

The teacher can also display the discussion worksheets or poll worksheets, if she realises that some specific aspects were neglected, to support the class discussion during different parts of the lessons. It is also possible to create polls on the spot to check students' understanding, or their awareness about what has been developed during the activity, or their attitudes toward the activity.

In the next paragraph we illustrate the design of the helping worksheets and the corresponding implementation, and present the analysis of an example from a case study. The example has been chosen because it is a paradigmatic one, which enables to highlight how the implementation of helping worksheets, through the support of CCT, fosters the activation of FA strategies and the dynamics between them.

The design and implementation of the *helping worksheets*: Analysis of an episode

Helping worksheets are conceived to support students in facing the tasks posed through the problem worksheets and are sent to selected students during the problem-solving phase, when: (a) they ask to receive a help; (b) the teacher realises that they are stuck; (b) the answers they send to the teacher highlight mistakes or difficulties. Moreover, helping worksheets may be sent to all groups, after they sent their answers to the teacher, as a checking tool for their work.

Usually we design sets of differentiated worksheets, according to the possible difficulties students could meet when facing a problem worksheet. Since our activities are adaptations of existing research-informed materials, the hypothesis about students' difficulties and the corresponding feedback that could be provided are drawn also from these materials.

We focus on helping worksheet 1A (see figure 1), which is matched to problem worksheet 1 within an articulated activity on time-distance graphs. The activity, which is our adaptation of some materials from the Mathematics Assessment Program, developed at the University of Nottingham (http://map.mathshell.org/materials/lessons.php), starts with the interpretation of a given time-distance graph and develops through the matching between graphs and stories and the construction of graphs associated to specific stories. We adapted the tasks in order to propose them to students from grade 5 to 7. To ground the time-distance graph on a meaningful activity, we designed an introductory activity on the use of a motion sensor (a device connected to a graphic calculator, showing, in real time, the Cartesian representation of a produced motion).

Worksheet 1 introduces a task on the interpretation of a time-distance graph representing the journey of a student, Tommaso, from home to the bus-stop. The worksheets' sequence connected to this task was conceived to gradually lead students in the interpretation of the graph, focusing their attention on the meaning of ascending, descending and horizontal traits of the graphs. Students are also asked to focus on the reasons supporting the correct interpretation of a time-distance graph, with the aim of making them, on one side, reflect on their thinking processes and share these processes with their classmates and, on the other side, consolidate their competencies in justifying and analysing their answers. The question on worksheet 1 (within the white box, see fig.1) requires students to interpret the meaning of a descending line within the graph. Students have to highlight that in the period of time from 50s to 70s the distance from home decreases, so Tommaso is going back for a while. Helping worksheet 1A (fig. 1) first of all makes students focus on the word "straight" to help them to abandon the idea that the graph could represent the drawing of the road. Moreover, it aims at fostering a correct interpretation of the descending line in the graph, making students look at two specific points within the graphs, that is (50, 100) and (70,40), to highlight that the distance from home is decreasing.

Sending a helping worksheet to a specific group of students is a way to activate FA strategy C, because students are provided with feedback about the task (if they receive this kind of worksheets, they realise that their answers should be completed and/or corrected) and feedback about the processing of the task (the suggestions and the guiding questions on the helping worksheets are aimed at supporting the students in facing the problem). Moreover, giving feedback represents a way of making students activate themselves as owners of their learning (FA strategy E).

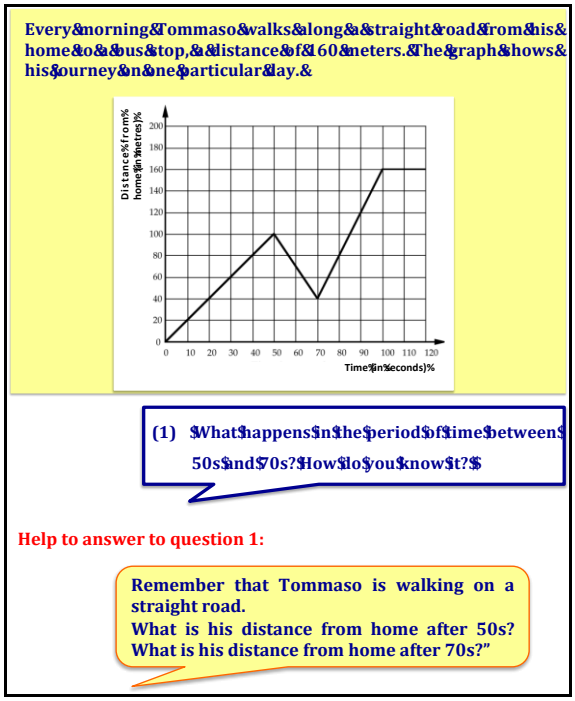

Fig. 1: The helping worksheet

During our design experiments, some students (in particular, low-achieving students) face difficulties also in interpreting the *purpose* of the provided helping worksheets, as supports to face the tasks. This is a manifestation of their lack of metacognitive control. For this reason, after the first cycle of design experiments, we introduced the displaying and collective meta-level analysis of helping worksheets as a fundamental characteristic of the methodology for their implementation.

As an example of this specific implementation of helping worksheets, we present and analyse an excerpt from a discussion on helping worksheet 1A, which was carried out in a 5th grade class. The discussion was aimed at making students aware of the goal of the helping worksheet and at pointing out specific mathematical aspects related to the task, namely to make them: (1) look at points within time-distance graphs as bearers of two linked information (the distance from home and the time spent); (2) interpret the variation of the distance in terms of moving away/approaching home; (3) avoid the typical mistake of interpreting the graph as a drawing. We remind that the researcher takes part in the discussion as both an observer and a participant.

220 Researcher: The first ones who are going to speak are those who did *not* receive this helping worksheet. Let's read the help that is given and try to say why, in your opinion, it is an help…what it helps you to do… The main question to be answered is still this one (*she indicates question 1, presented in Worksheet 1*). The help says (*reading*) "Remember that Tommaso is walking on a straight road. What is his distance from home after 50s? What is his distance from home after 70s?"

221 Teacher: Why do the suggestions focus on this?

222 Researcher: What do these questions help to do?

Several students raise their hands.

223 Carlo: Because they help you to understand the distance in the period between 50s and 70s. Because, at 70s, he is nearer…

224 Researcher: So you are saying that it enables to look at the distance, aren't you?

This discussion was planned with the objective, on one side, of eliciting evidence of students' understanding at a metacognitive level (strategy B), and, on the other side, of activating some students as resources for their classmates (strategy D). In fact, the researcher (lines 220, 222) and the teacher (line 221) are fostering a meta-reflection, involving the students that did not receive the help in clarifying the reasons why the questions posed on helping worksheet 1A could provide help in answering questions 1. Their aim is, therefore, to activate FA strategy C at the peer's level. Specifically, Carlo's intervention (line 223) represents a feedback about self-regulation because he highlights that the questions in worksheet 1A enable to focus on the change in Tommaso's distance from home, during that period of time. Through this comparison with their classmates, students can therefore become aware of the kind of support that helping worksheets could give and can also develop new tools to face similar activities in an effective way.

Then, the teacher focuses students' attention on a part of the helping worksheet that was *not* mentioned by Carlo:

225 Teacher: And why does it [the help] suggest that Tommaso is moving on a straight road?

226 Carlo: Because it wants to make us reason on the fact that he is going back.

227 Researcher: What mistake couldn't be done if I remember that the road is straight? … (*Silence*) If I don't know that the road is straight, what could I think?

Anna mimes a curvy road with her hands.

228 Arturo: I could think that the sensor initially indicates a direction, then he goes on the right… (*Arturo is referring to the introductory activity with the motion sensor*)

229 Teacher: So a change in the direction.

230 Researcher: That we are zigzagging, in a strange way.

231 Teacher: It is the reason why it remembers us that the road is straight. You recalled, with your memory, what we experimented last time. If we hadn't worked with the sensor, you, maybe, would have proposed different answers.

Again the teacher and the researcher focus students' attention on the suggestions contained in helping worksheet 1A to make them become aware of its role in supporting the resolution of the task (strategy C). The teacher (line 225) focuses on the first suggestion given in worksheet 1A (*Remember that Tommaso is walking on a straight road*) and the researcher (lines 227) aims at making students reflect on the possible misinterpretations that this suggestion wants to prevent. Students are, in this way, provided with both feedback about the processing of the task and feedback about self-regulation, because they can become aware of the possible mistakes that could be done in the interpretation of this kind of graphs, learning how to monitor their work. Also the teacher (line 231) provides a feedback about self-regulation because she is making the students notice how the previous experience has influenced their answer to the current question. Carlo (line 226) and Arturo (line 228) are activated as instructional resources for their classmates (strategy D).

Discussion

In this paper we referred to the theoretical lenses provided by the FaSMEd framework to present and discuss the design of digital resources and the corresponding method of implementation, with a special focus on the helping worksheets. The analysis we developed, on one side, shows that the

FaSMEd three-dimensional framework represents a useful tool for both designing digital materials for technology-enhanced FA practices and the corresponding methodology, and analysing how these materials are implemented in the classroom. On the other side, this analysis provides a validation of the design and implementation because the FaSMEd framework offers some important criteria according to which the digital worksheets and the methodology for their implementation can be evaluated as effective tools to foster FA processes: (1) the activation of different FA strategies; (2) the involvement of all the agents; (3) the evolution of the FA strategies (in particular toward strategy E, which should constitute a constant objective of the activities); (4) the different levels of feedback provided; (5) the support provided to the three fundamental FA processes.

At the same time, the analysis of the design and implementation of the helping worksheets in the chosen episode enabled us to highlight a pattern that characterises the evolution of FA strategies when helping worksheets are implemented. In fact, the use of the helping worksheet, combined with the sending and displaying functionality of technology, turned into the activation of several FA processes, with the involvement of all the agents. During the group-work phase, by sending the helping worksheet to the students, the teacher is activating FA strategy C with the aim of activating also strategy E. After the group-work phase, a meta-level discussion devoted to the sharing and analysis of helping worksheet is planned by the teacher (strategy B). As a result of the design based process, two different ways of fostering students' meta-level reflections have been identified: initially, the students who did *not* receive the worksheets are asked to reflect on the possible role played by the provided help (becoming instructional resources for their classmates, strategy D); then, the students who *did* receive the help (mostly low achieving students) are asked to discuss on the ways in which they used it, making their reasoning explicit and being activated as the owners of their own learning (strategy E). During the discussion, all the students receive feedback from the teacher and their classmates (strategy C) and are provided with the opportunity to clarify the learning intentions associated to the worksheet (strategy A).

We think that this pattern, since it is recurring throughout our corpus of data, represents an important validation of the design of helping worksheets, because it highlights the effectiveness of these resources and their implementation in fostering the development of FA strategies and the fruitful involvement of all the agents.

Other elements of validation can be highlighted if we interpret the results of the activities carried out through the helping worksheets in terms of the three fundamental FA processes that are supported: (a) the students are supported in establishing *where they are in their learning* when they use the help as a feedback to assess their own answer; (b) the teacher is supported in helping students clarify *where they are asked to go* when, during the class discussion, the characteristics of given answers are analysed and discussed; and (c) the teacher and the students are supported in establishing *what needs to be done to get there* when, during the class discussion, the helping worksheets are analysed to highlight in what ways they could help and what kind of suggestions they give.

We are now developing a similar analysis to highlight, referring to these criteria, how the other categories of worksheets are used to foster the activation of FA strategies through the support provided by technology. This will enable us to identify the connections and mutual support between

the different worksheets and the methodologies through which they are implemented during the lessons.

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Enhancing formative assessment in mathematical class discussion: a matter of feedback

Annalisa Cusi[1], Francesca Morselli[2] and Cristina Sabena[3]

[1]University of Turin, Italy; annalisa.cusi@unito.it

[2]University of Genoa, Italy; morselli@dima.unige.it

[3]University of Turin, Italy; cristina.sabena@unito.it

This contribution is framed in a European project on the use of technology to foster formative assessment strategies (FaSMEd project) and addresses the crucial issue of feedback therein. The theoretical framework refers to formative assessment, with specific focus on different levels of feedback. By analyzing data from teaching experiments in grades 5 and 7, we identify strategies employed by the teacher to provide feedback during class discussion and investigate the effect of such strategies on the enactment of formative assessment.

Keywords: formative assessment, technology, feedback, teacher, peers.

Introduction and theoretical background

Formative assessment and feedback

This contribution is framed within the European Project FaSMEd ("Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education"), aimed at investigating the use of technology to promote formative assessment (FA) practices in the mathematics and science classroom. FA is conceived as a method of teaching where

> "[...] evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited" (Black & Wiliam, 2009, p. 7).

Wiliam and Thompson (2007) describe five key FA strategies: (A) Clarifying and sharing learning intentions and criteria for success; (B) Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (C) Providing feedback that moves learners forward; (D) Activating students as instructional resources for one another; (E) Activating students as the owners of their own learning.

Feedback is a crucial issue in FA. Hattie and Temperley (2007) define feedback *as* "*information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding*" (p.81) and identify four *major levels of feedback*: (1) feedback *about the task* (concerning how well a task is being accomplished or performed); (2) feedback *about the processing of the task* (concerning the processes underlying tasks or relating and extending tasks); (3) feedback *about self-regulation* (concerning the way students monitor, direct, and regulate actions toward the learning goal); (4) feedback *about the self as a person* (consisting in positive (and sometimes negative) evaluations and affect about the student). Hattie and Temperley also point

out that feedback is a consequence of specific actions and that *can also be sought by students, peers, and so on, and detected by a learner without it being intentionally sought*." (p.82).

Enhancing formative assessment: technology and tasks

Specific theoretical and methodological assumptions of the Italian team (the three authors) within FaSMEd concern the importance of fostering students' development of ongoing reflections on the teaching-learning processes, and helping students to make their thinking visible (Collins, Brown and Newmann, 1989), sharing their ideas with the teacher and the classmates. These basic assumptions entail specific choices concerning the technology and the tasks.

Concerning technology, each class is equipped with a Connected Classroom Technology (CCT) through which the students' tablets and the teacher's laptop are connected. In order to foster collaboration and sharing of ideas, students are asked to work in pairs or in small groups on the same tablet. By means of the CCT equipment, students are able to receive worksheets from the teacher, send back their written answers, and answer to instant polls proposed by the teacher; the teacher can easily collect the students' opinions and reflections during or at the end of an activity, as well as the written answers, and receive the statistics concerning the answers to the polls. The teacher's computer is connected to an Interactive White Board (IWB) or a projector, so that it is possible to select and display written productions and the results of polls.

Concerning the tasks, students are asked to work on sequences of activities with a strong argumentative component: they are required to provide their answer and explain it in a written text. In this way, they are encouraged to make their thinking visible and to provide the teacher and the peers with a written text that will be shared and analysed during mathematical discussions (Bartolini Bussi, 1998). The mathematical content at issue is relationships and functions, and their different representations (symbolic, tabular, graphic). Activities are adapted from the ArAl project (Cusi, Malara & Navarra 2011) and The Mathematics Assessment Program (http://map.mathshell.org).

Summing up, the typical lesson starts with a peer activity on one worksheet. After having collected all the students' written answers, the teacher promotes a class discussion, starting from the analysis of some written answers (selected and displayed on the IWB). The discussion concerns the task level (correct answers and typical mistakes), the task processing level (effective ways of approaching the task) and the communicative level (effective ways of communicating the answer and the explanation). Comparison between different solutions is especially promoted. For further details on the organization of the lessons, see (Cusi, Morselli and Sabena, 2016).

Previous results and the current research questions

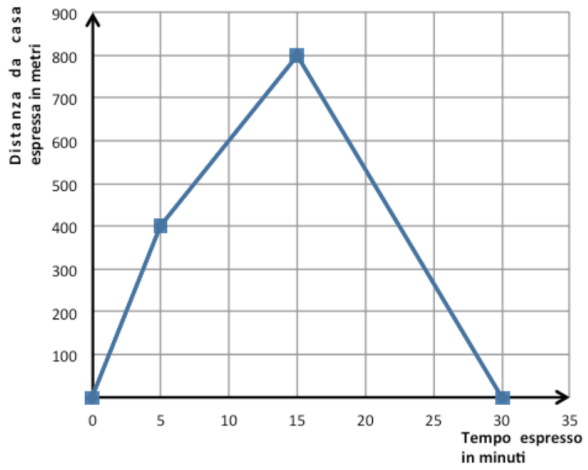
In former studies (Cusi, Morselli and Sabena, 2016) we analysed classroom discussions performed within the FaSMEd teaching experiments, and highlighted that CCT may support the activation of FA strategies by the teacher, by the peers (peer assessment), and by the student himself (self-assessment). In this contribution we focus on *strategy C* (*providing feedback that moves learners forward*) and we investigate: what are teacher's strategies that may foster FA strategy C; which level of feedback is provided; what are the effects of this strategy (in terms of activation of other strategies such as D and E).

The context

In Italy the FaSMEd project involved 20 teachers, from three different clusters of schools located in the North-West of Italy (from grade 4 to grade 7). During all the teaching experiments, one of the authors was present in the classes with the teachers, acting as a participant observer. The analysis is based on video-recordings of the classroom discussions, with the help of written transcripts and field notes by the participant observer (one of the authors).

Analysis

By analysing several episodes from the teaching experiments, we came to a first characterization of teacher's feedback strategies, that is the ways in which she gives feedback to students. In the subsequent part, we provide a short example for each kind of feedback strategy, highlighting the level of feedback provided. Moreover, we will discuss the effects of each feedback strategy in terms of activation of FA strategies. All the examples come from the third lesson on time-distance graphs, performed in grade 7. The lesson sequence on time-distance graphs (about 20 hours, 8 worksheets) was adapted from the Mathematics Assessment Program (http://map.mathshell.org) and was introduced by an experience with a motion sensor, which provided instantaneous graphical representation of a linear motion performed by the students. This lesson was chosen because it contains all the typical teacher's feedback strategies that recurred in different classes and grades in our teaching experiments. Here we refer to worksheet 6, where a graph and three possible stories are presented:



Story A: Tommaso took his dog for a walk to the park. He set off slowly and then increased his pace. At the park Tommaso turned around and walked slowly back home.

Story B: Tom rode his bike east from his home up a steep hill. After a while the slope eased off. At the top he raced down the other side.

Story C: Tommaso went for a jog. At the end of his road he bumped into a friend and his pace slowed. When Tommaso left his friend he walked quickly back home.

The students are asked to answer to the following question: "*What is the story that this graph represents*? *Justify you answer.*" Students work in pairs and send their written answers to the teacher's computer, as soon as they feel ready. The teacher, together with the participant observer, reads the answers as they arrive at her laptop and selects some of them for the discussion. The first selected answer is the one by the group of Mil and Pon:

"For us the answer is B for two reasons:

1. You cannot do 1600 meters by foot in half an hour

2. The graph represents precisely the information given by the story. Then Tommaso climbs the hills, the first trait is the climb, the second is still a climb but less steep. When he comes to the top, then Tommaso climbs down and goes back home".

We may observe that Mil and Pon highlight two reasons for the choice of story B: the first one is based on *everyday life experience* (they draw from the graph the information that 800+800 meters are walked, and they point out that it is not possible to walk for 1600 meters in half an hour; since it is actually possible to walk 1600 meters in half an hour, this argument is wrong), the second one is based on a *wrong interpretation of the graph*: they interpret the graph as a picture of the hill, that Tommaso first climbs up and then descends down. For the teacher, the discussion of students' production is the occasion for *giving feedback on two levels: about the task* (clarifying that the graph *represents* the relation between distance from home and time, and is not a picture of the hill, so it does not share with it any resemblance, in principle) and *about the way of processing the task* (pointing out that the justification must be based on a careful analysis of the information provided by the text and the graph). To this aim, the teacher promotes a discussion (*strategy B*). Mario is asked to read the production of Mil and Pon, then the discussion starts.

Transcript	**Analysis**
217. Teacher: Then, answer B for two reasons. Ok, Lollo?	The teacher encourages the students to activate themselves as resources for Mil and Pon (strategy D).
218. Lollo: We did, because… we did the experience with the motion sensor… that if the line was more oblique the… the line, if it was more oblique, it meant that he (Tommaso) went faster, it did not mean that the road was steeper, because if the road is steeper you go slower…	Lollo gives a feedback about the task (strategy C), suggesting that the different inclination of the segments should be interpreted in terms of different speed. To warrant his statement, he refers to the experience with the sensors. He activates himself as resource for Mil and Pon (strategy D). He also adds that, when the road is steeper, usually one goes slower, and not faster, referring to everyday experience.
219. Teacher: Rob?	
220. Rob: This is a graph, it is not the drawing of the hill.	Rob makes explicit that the graph does not represent the drawing of the hill, giving a feedback about the task to Mil and Pon. He activates himself as instructional resource for his classmates (strategy D), providing feedback about the task (strategy C).
221. Teacher: It is not the drawing of the hill, it is the graph that represents what?	The teacher encourages Rob to make explicit his comment to Mil and Pon's answer. This intervention is a **relaunching**: she poses another question, linked to Rob's intervention, with the aim of deepening the analysis. Relaunching Rob's intervention the teacher implicitly gives a feedback (strategy C) to Rob himself, suggesting that his intervention is worthwhile.

222. Rob: The… the journey of one boy, and anyway they told that it is not possible to do 1600 meters in half an hour, we already said it last time [*he refers to the lesson with motion sensors*], it is a graph, it doesn't have to be really real… really near to reality.	Rob gives a feedback (strategy C) about the processing of the task, pointing out that the justification must not rely on empirical arguments but on the interpretation of the task. The teacher's **relaunching** is efficient in turning Rob's former intervention, which provided a feedback about the task, into a meaningful feedback about the processing of the task.
223. Observer: Do you understand what he is saying?	
224. Mario: For me you can do it easily, you can even do 2 or 3 kilometers…	Mario challenges Mil and Pon's justification A, on the basis of empirical experience. Mario is giving a feedback on the task (strategy C): the first answer relies on a wrong argument.
225. Rob: For me yes…	
226. Teacher: Then, the fact of 1600 meters in half an hour, your classmate says that actually you can do it in half an hour, then that is not a good motivation. Somebody else was talking about the second motivation, motivation B, the fact that the graph explains us that Tommaso climbs the hill and so on. Lollo said: "No, because when we did the experience with the sensor we went on a oblique line, but the path we were doing was not on a hill, it was not steep".	The teacher synthetizes the interventions of Lollo, Mario and Rob, stressing that the justification 1 is not correct. Then she shifts the focus on justification 2, focusing on the correct interpretation of oblique lines within a time-distance graph. In this way, she activates strategy C, giving Mil and Pon a feedback about the task (it is a mistake to interpret the task as the picture of a hill) and the processing of the task (focusing on the ways in which the time-distance graphs should be interpreted). Here we may see instances of both **rephrasing** (the teacher reformulates some arguments so as to make them more intelligible to the mates) and **revoicing** (the teacher revoices some of the students' interventions, so as to draw the attention on specific effective parts of the given arguments).
227. Ur: Teacher, but I agree with what Lollo said. I thought that if it is steep you walk slowly, while after, when it becomes less steep, Tommaso goes faster.	Ur intervenes, referring to Lollo's first intervention (218). Ur activates herself as owner of her own learning (strategy E). This intervention confirms that Lollo became a resource for his mates.
228. Teacher: But the fact that… you say: "the fact that the road is more or less steep can give us information on the reasons why he goes faster or slower"…	The teacher gives a quick feedback to Ur, reformulating her sentence, so that other students can intervene. This is again an example of **rephrasing**.
229. Mark: Teacher, moreover we told that with the sensor if we went faster… the segment went more	Mark intervenes making reference to the experience with sensors (thus linking the inclination to the speed)

vertically, but here … they say that he is climbing and he goes too much… he goes fast, and then when it [the segment] becomes less steep he goes less fast. I don't know, in the descent he goes really faster than on the other two traits, but if they say that he climbs up in the first trait, he goes fast, and then when it starts being plane he goes less fast. […]	and pointing out that something doesn't work in what Mil and Pon wrote: in their interpretation of the graph as a picture the first trait is the steeper part of the hill, but in the interpretation of the graph in terms of speed (as in the previous experience with the motion sensors) the segment is steeper when the speed increases. Mark expresses his own doubts about the two contrasting interpretations: in reference to everyday experience, it is not so common to walk faster in the steeper trait of a hill. Mark's intervention is an instance of strategy E, but his intervention could also act as feedback for Mil and Pon (strategies C, D).
234. Teacher: But I… this answer really tells that the first segment, the first two parts of segment that go up describe the hill, the steep climb, the less steep climb, the top and after the descent…	The teacher goes back to Mil and Pon's written answer, so as to foster the comparison between their answer and the intervention of Mark. By **contrasting** in this way the two answers, the teacher is implicitly giving a feedback to Mil and Pon (strategy C) and turning Mark as instructional resource for them (strategy D).
235. Student: That is wrong.	This intervention confirms that the contrasting was efficient in fostering the comparison between the different positions of Mark and Mil and Pon.
236. Teacher: Then the idea that the segments, as Rob said… "the graph is different from the drawing of a hill", or Lollo said "when we did it with the sensors we saw this kind of segments but we were not climbing, it meant that we changed the speed"… Let's remember always that the y axis describes what? The distance from home in meters.	The teacher intervenes with a **rephrasing**: she teacher reformulates and synthetizes the interventions of the students, so as to give a feedback to Mil and Pon. The activated strategy is C (providing feedback). In this way she is efficient in turning the feedback about the task into a feedback about the processing of the task (she draws the attention on the meaning of the two axes). We call this kind of intervention a **rephrasing with scaffolding**, since the teacher, besides rephrasing, adds some elements to guide the work on the graph.

Results and discussion

Within the FaSMEd project, we performed several teaching experiments in grades 5 to 7, setting up task sequences and proposing them in a CCT environment. As a first result (Cusi, Morselli and Sabena, 2016), we showed how technology may support the activation of several FA strategies. In the current paper we focused on FA strategy C (providing feedback) and explored the ways in which the teachers may intentionally provide feedback during class discussions, the kind of feedback that is provided and the possible links with FA strategies.

The analysis of several class discussions performed during the teaching experiments led us to identify typical strategies employed by the teacher to provide feedback. Such strategies are exemplified in this paper through the analysis of a class discussion in grade 7. Here we summarize the strategies and discuss further developments of our study. The first strategy is **revoicing**, that occurs when the teacher mirrors one student's intervention so as to draw the attention on it. Often, during the revoicing, the teacher, stresses with voice intonation some crucial words of the sentence she is mirroring. **Rephrasing** takes place when the teacher reformulates the intervention of one student, with the double aim of drawing the attention of the class and making the intervention more intelligible to everybody. Rephrasing is applied when the teacher feels that the intervention could be useful but needs to be communicated in a better way so as to become a resource for the others. We also found special instances of rephrasing, when the teacher, besides rephrasing, adds some elements to guide the students' work. Drawing from Wood, Bruner & Ross (1976) the term "scaffolding", we call this special strategy a **rephrasing with scaffolding.** The revoicing and rephrasing strategies are used to activate strategy D, since they turn one student (the author of the intervention) into a resource for the class. Moreover, we observed that often revoicing and rephrasing (and rephrasing with scaffolding) are efficient in promoting the evolution of the kind of feedback, for instance (as in the reported example) from a feedback on the task to a feedback on the processing of the task. **Relaunching** occurs when the teacher reacts to a student's intervention, which (s)he considers interesting for the class, not giving a direct feedback, but posing a connected question. In this way, by relaunching the teacher provides an implicit feedback (strategy C) on the student's intervention, suggesting that the issue is interesting and worth to be deepened or, conversely, has some problematic points and should be reworked on. **Contrasting** takes place when the teacher draws the attention on two or more interventions, representing two different positions, so as to promote a comparison. By contrasting, FA strategy D and E are activated (the authors of the two positions may be resource for the class as well as responsible of their own learning).

The aforementioned strategies, besides being efficient ways to boost the discussion, are powerful formative assessment tools, since they foster the activation of formative assessment strategies. When addressing one student's statement, the teacher gives an implicit feedback on it (strategy C), suggesting the intervention deserves further attention. Moreover, in this way strategies D and E are activated and the feedback may evolve from feedback on the task to feedback on the processing of the task. We deem that this kind of classification may shed light into the crucial role of the teacher in enhancing FA within class discussions. All the documented strategies seem to be intentionally applied by the teacher. Anyway, the given feedback is implicit, since the teacher does not address directly the correctness of the student's intervention. As a consequence, the feedback is not always sought by the students. We are aware of the fact that we were able to single out and discuss only some effects of a given feedback, namely when a student explicitly refers to a previous intervention or changes his mind immediately after an intervention by a peer or by the teacher. Other effects of a given feedback are less visible during a class discussion: in order to study them, it will be necessary to analyse further activities of the students or collect a-posteriori interviews.

For the moment we focused on class discussions around the analysis and comparison of students' written productions. In the future we plan to go on with our analysis, focusing on other crucial

moments of the teaching experiments, such as the discussion after a poll, or the discussion on specific helping worksheets. As a further development, we plan to compare the strategies we outlined with Bartolini Bussi (1998)'s classification of teacher's interventions during a mathematical discussion. Moreover, we aim at complementing the present study, concerning the *way* feedback is given (feedback strategies), with a study on the *content* of feedback. To this aim, we plan to deepen the categorization of levels of feedback provided by Hattie & Temperley (2007), so as to take into account the specific features of the proposed mathematical tasks.

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Mathematics students' attitudes to group-based project exams compared to students in science and engineering

Bettina Dahl[1]

[1] Aalborg University, Denmark; bdahls@plan.aau.dk

At Aalborg University, science, engineering, and mathematics students spent half the time each semester working in groups on projects within a problem based learning (PBL) curriculum. They are assessed through group exams. A survey showed that overall, the students are positive towards the group exam but there are significant differences between engineering, science, and mathematics students. Within this collectivistic student culture, some engineering students are very positive towards group exams, while mathematics, science, and other engineering students are less positive. In terms of the opportunity to obtain a fair grade in a group exam, the mathematics students are moderate positive, with different engineering students being more or less positive. All students agree that a group exam gives less differentiation of grades compared to an individual exam.

Keywords: attitudes; project exam; group exam

Introduction

At Aalborg University (AAU) in Denmark, science, engineering, and mathematics students work half the time each semester in groups of four to eight students on a project in a problem based learning (PBL) curriculum. PBL is student-centred and self-directed learning in teams with problem analysis and problem solving (Barge, 2010). The project is documented as a joint written report with an oral group exam lasting around four hours. This exam has traditionally consisted of two phases: First the group presents the project, then the external examiner and the supervisor examine the group. Each student is awarded a grade that may not be the same as the others. From 2006–2012 the Danish government banned group exams, but during this time AAU students still worked in PBL groups but the exam became an individual oral exam of around half an hour per student. This situation led to research on assessment methods in PBL, and Kolmos and Holgaard (2007) concluded that the students, the academic staff, and the external examiners preferred the group exam. One argument was that the students were not able to interact with each other during an individual exam, hence it was not possible to test PBL process competencies such as collaboration and teamwork. Since Danish law states that a grade solely depends on the student's performance at the exam (they cannot earn partial credits during the semester), this created a misalignment between PBL teaching and the assessment methods. In 2013 the group exam was reintroduced in Denmark and at AAU. The Faculty of Engineering and Science (FES) now added an individual phase into the previous group exam, where each student is questioned without interference from the group. Dahl and Kolmos (2015) found that the students overall were in favour of the reintroduction of the group-based project exam, but that students from two different engineering programmes were not equally positive, partly owing to their previous experience with the individual project exam and partly owing to professional cultures of individualism or collaboration influencing their attitudes. Maull and Berry (2000) and Bingolbali et al. (2007) showed that mathematics and engineering students are different in terms of their learning of mathematics, so one might ask whether a similar division

between engineering and mathematics students is seen in relation to which type of exam is perceived appropriate. This paper therefore compares students from eight programmes in science, engineering, and mathematics in relation to how they perceive the new group exam, the new individual phase as well as the opportunity to obtain a fair grade.

Theoretical background

Alignment and exams

Biggs and Tang (2011) argue that in order for students to learn the intended learning outcomes (ILOs), teaching should be constructively aligned with the ILOs and the exam. This theory fits other studies stating that an upcoming exam is a key factor for students' motivation and learning (Boud & Falchikov, 2006); i.e. the 'backwash effect' of exams. Hence, one can argue that in a PBL curriculum, the exam method should be aligned with the team-based and collaborative teaching method and the ILOs on PBL process competencies. Romberg (1995) argues that a group exam is able to test "reflection on one's own thinking, reasoning and reflection, communication, production, cooperation, arguing, negotiating" (p. 165). One can thus argue that a group exam assesses PBL competencies of communication and cooperation. However, each programme also prepares the students for a professional life after the university so the problems that the students address vary and AAU's PBL model is developed "on the basis of both professional and educational argumentation" (Kolmos et al., 2004, p. 9). One might anticipate that professional culture influences the students' views of the group exam, particularly master students.

Cultural differences in engineering, science, and mathematics

Murzi et al. (2015) studied how students perceived their discipline culture using Hofstede's dimensions. One dimension measures individualism versus collectivism. Overall, students had a high individualistic score. Mathematics, computer engineering, and electronic engineering students were among the *less individualist* students. This fits the study by Burton (2004) where a majority of professional mathematicians worked co-operatively. Murzi et al. (2015) further argued that they had expected industrial design students to be more collectivistic as they rely on collective work in team projects but the results were opposite. Architect students' scores fell between mathematics and industrial design. Dahl and Kolmos (2015) also found significant differences between the engineering programmes Architecture and Design (AD) and Software Engineering (SE) at AAU. SE students were significantly more positive toward the group exam than those of AD. AD combines architecture with civil engineering and students here expect a more individual-oriented programme whereas SE is a system-oriented approach and a collaborative profession.

Research questions

How do the students from the eight programmes view the group-based project exam compared to the individual project exam and the individual phase of the new group exam? How do they experience the grading? What does this tell us about mathematics assessment in PBL?

Methodology

The questionnaire was piloted after the January 2013 exams and the revised questionnaire consisted of 20 questions of which most had several sub-questions. After the June 2013 exams, all 4,588 FES students received a link to this questionnaire and 1,136 responded. The response rate was relatively low (25%), which unfortunately is not uncommon for online surveys, but the level is still reasonable (Nulty, 2008). The response rate for each study programme cannot be determined separately but the number of student responses were as follows: Computer Science (CS: 40), Energy (EnE: 50), Mechanics and Production (MP: 39), Physics and Nano science (PN: 27), Architecture and Design (AD: 79), Mathematics (M: 28), Software (SE: 51), and Electronic (ElE: 48). In this paper, all questions are translated from Danish by the author. The programmes compared all had a relative large number of students who responded. The engineering programmes are civil engineering.

Results

Views of individual versus group-based project exam

Of all FES students, 34% preferred the individual exam and 57% the group exam, but students in different programmes were not equally positive towards the group exam (see Figure 1).

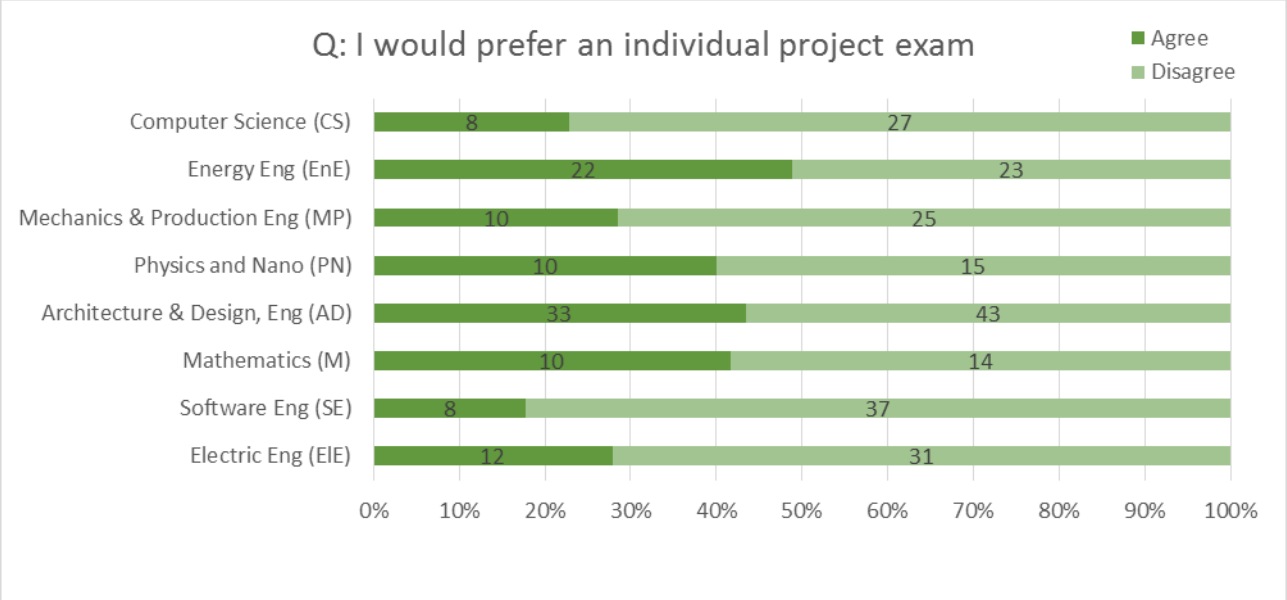

Figure 1: "I would prefer an individual project exam"

EnE students were the least positive towards the group exam while SE students were the most positive. Table 1 shows the programmes that were significantly different.

	EnE	AD	PN	M
CS	.047	.037		
ElE	.043			
SE	.002	.004	.041	.031

Table 1: Significant differences in answers to the question if they preferred the individual exam

SE and EnE are at opposite ends of the group-individual preference and they are each significantly different from many programmes. SE, CS and ElE are the most collective while EnE, AD, PN, M are the more individualistic, although M appears to be more moderately individual. MP are not significantly different from any. Master students are significantly more positive towards the individual exam than bachelor students (p = .001). Almost half the master students preferred the individual exam while only a third of the bachelor students did. Comparing bachelor and master students in each programme, there is only a significant difference for ElE (p = .002).

The survey asked a related question: "The time spent on the individual part is better spent on a longer joint part?" Of all FES students, 34% agreed, 50% disagreed, and 15% did not know. The students do not differ except MP, which is significantly different from AD (p = .005) and M (p = .030) (see Figure 2). Overall the students are positive towards the individual phase, particularly M. There is not significant difference (p = .114) between bachelor and master students for all FES. When comparing students in each programme, MP bachelor and master students are significantly different (p = .024) but here the master students are the more positive toward the group exam.

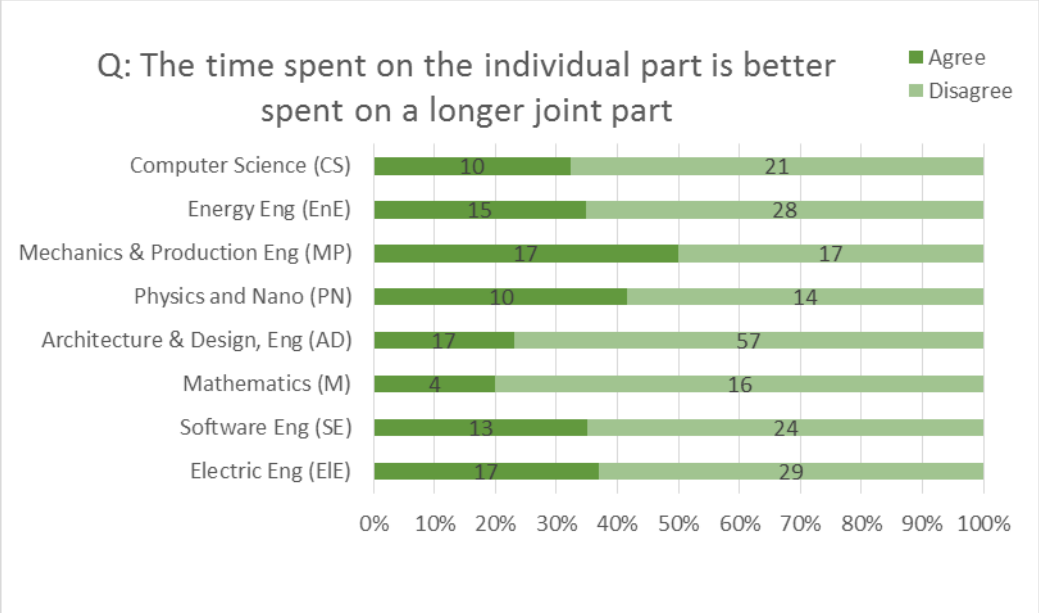

Figure 2: "The time spent on the individual part is better spent on a longer joint part"

How did the students experience the opportunity to obtain a fair grade?

A question asked if the students were satisfied with their own grade. Overall, 83% agreed and there was not a significant difference between the eight programmes. Another question asked whether they found that all their group members had received a fair grade. Here only 66% agreed. In all groups, a majority of students had agreed but EnE (61%) and AD (59%) agreed significantly less than SE (76%) and PN (85%). EnE was different from SE (p = .046) and PN (p = .037) while AD was (p = .019) different from SE and PN. When comparing each programme, most had given significantly different answers to the two questions except ElE (p = .064), PN (p = .324), and SE (p = .760).

The survey also asked the students who had tried the individual exam whether the new group exam gave a better or a worse opportunity to obtain a fair grade. The views differ greatly (see Figure 3).

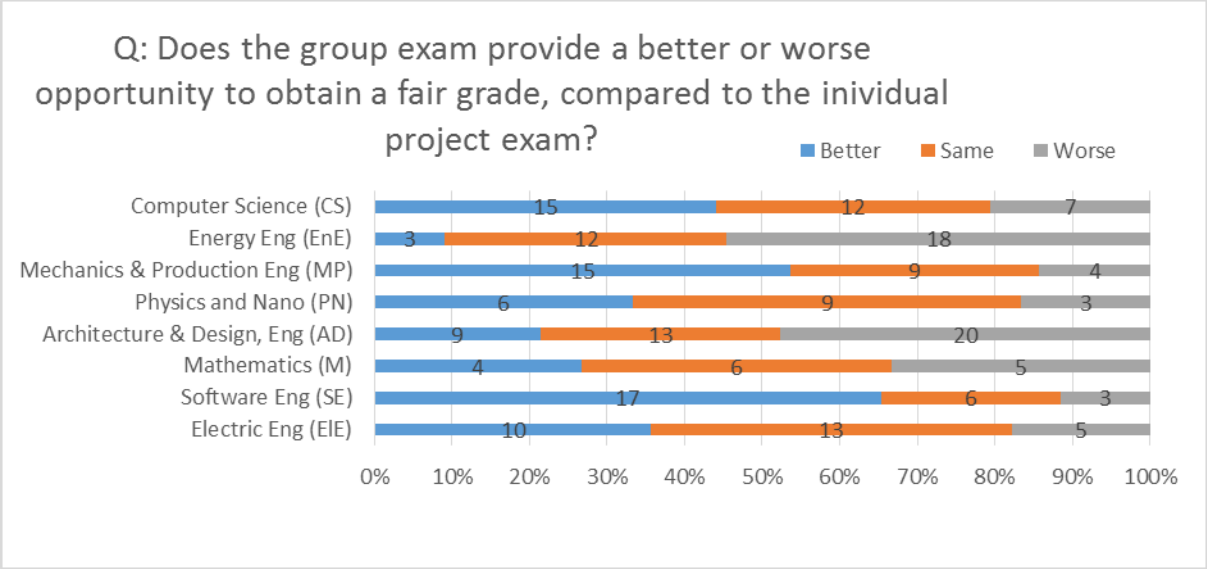

Figure 3: "Does the group exam provide a better or worse opportunity to obtain a fair grade…?"

When comparing the programmes, one sees that only M is not significantly different from any of the others while students from several of the engineering programmes answer significantly differently from each other (see Table 2). M therefore appears to be quite moderate in their views of whether or not the group exam gives a better or worse opportunity to obtain a fair grade. CS, ElE, SE, MP and to some extent PN are more positive towards the group exam as giving them a fair grade, while AD and EnE in general state that it provides them a worse opportunity to obtain a fair grade, compared with the individual exam. The students answering this question were all from the second year or older, as these were the only ones who had experienced both types of exams.

	CS	ElE	SE	MP	PN
EnE	.001	.004	< .001	< .001	.015
AD	.030	.039	< .001	.005	

Table 2: Significant differences in answers to the question if the group exam provided a better or worse opportunity to obtain a fair grade

The survey also asked the students whether there was a larger differentiation of grades in the group exam compared to the individual exam. The Danish grade scale has five passing grades (2, 4, 7, 10, 12) and two failing grades (-3, 0). This question relates to an internal discussion at both AAU and in Denmark debating if the group exam uses less of the grade scale as it is harder to give a precise individual grade and weaker students can hide and good students are not rewarded. Only a minority of the students confirmed that the group exam resulted in more differentiated grades. The lowest was ElE, where 7% said that the group exam resulted in more differentiated grades to some or to a higher extent, while the highest was M with 29%. This difference was not significant (p = .069).

Conclusions

Individual versus group exam and the individual part of a group exam

The students were in general very positive towards the group exam but there are significant differences. Murzi et al. (2015) found that mathematics, computer engineering, and electronic engineering were among the *least* individualistic within a very individualistic student culture. One might argue that in general the AAU students are used to working in PBL and thus have a more collectivistic culture since, overall, the majority of students were in favour of the group exam when asked to compare it to an individual exam. This study of AAU students shows that in agreement with Murzi et al. (2015), CS and ElE were among the most collectivistic students as they were among the most positive towards the group exams. However, this is stated within the frames of a collectivistic AAU culture. Murzi et al. (2015) found mathematics students to also be among the least individualist, however at AAU, M appeared to be among the more individualistic students, even though AD and EnE appeared to be even more individualistic. Architect students in Murzi et al. (2015) were not among the individualist groups. In relation to their attitudes to the time spent on the individual part of the group exam, M were the most positive toward the individual part. M was again close to AD but quite different from MP. PN students were closer to the M students.

Master students were generally more positive toward an individual exam than bachelor students, especially ElE. However MP master students appeared very positive toward having a longer joint part in the new group exam. One might argue that this is related to the fact that the master students have been used to the individual exam prior to 2013, or perhaps to how they perceive their future professional life (wrongly or rightly) might have an impact.

Fairness of grades

The students were overall satisfied with their own grade but relatively less satisfied with the grades given to their peers. Given the relatively low response rate, it could make sense that students answer this question significantly differently. The question in the questionnaire did not explicitly ask if their peers were over/under-graded, but it appears that seeing how their peers behave at the exam, perhaps with reference to their work during the semester and then experiencing what grade they received, often left the other students feeling some degree of unfairness. More research is needed here in order to determine why. The students also differed when they compared the group and the individual exam in relation to the opportunity to receive a fair grade. Here, M was more or less in the middle, not being significantly different from any of the other programmes. In general one sees that the same programmes as above show ‘collectivist’ preferences (CS, ElE, SE, MP) and ‘individualistic’ preferences (AD, EnE), which to some extent validate the results shown above and show that the students are consistent in their answers. However, one also needs to discuss to what extent their experience of receiving a fair grade is correct. Do students always know which grade they deserve? Furthermore, the perception – rightly or wrongly – of not being awarded a fair grade, might negatively influence their view of the exam. Students are occupied by fairness in grading and their perception of justice is significantly affected by the assessment method (Burger, 2016).

Students appear to obtain more similar grades when they are assessed as a group than if they are assessed individually. The question is then – which is the right grade? One might argue that in a group exam of up to eight students, it might be difficult to make a distinction between each group member, which to some extent might explain the different opinions about own grade and the grade of the group members, and the same question can only be asked once. On the other hand, one might also argue that since a group exam to some extent is able to test PBL process competencies, which an individual exam cannot, the grades given in a group exam are the more accurate.

Summing up and impact for mathematics assessment

It appears that mathematics students are not distinct from engineering or science students on the issue of preference for individual or group exams. The eight groups were mixed; ergo mathematics students were more similar to some engineering students but different from others. This is different from what is known from how the learning of mathematics takes place when comparing mathematics and engineering students. For instance Bingolbali et al. (2007) found that engineering students see mathematics as a tool and therefore wish to see the application side. In science, engineering, and mathematics PBL projects, mathematics is applied to solve problems, but in a mathematics project, the body of mathematical theorems used to solve a problem usually takes up a considerable part of the project work and the report. Thus, the role of mathematics is different in the PBL projects in each programme. Assuming that the group exam is the best fit to PBL, it is unexpected that the students are not more in agreement with each other about the group exam. An obvious answer is that the group exam does not fit each programme equally well. The more moderate views of the mathematics students could indicate that it is a reasonably good fit when there is an individual phase as this is a way to serve both the individual and collaborative aspects of mathematics.

The question of individualistic or collective attitudes may also depend on the overall student culture. Murzi et al. (2015) found that mathematics students were among the *least individualistic* within an individualistic culture, while this study found the mathematics students to be among the *most individualist* students within a collectivist culture. With caution, one might argue that the most individual student cultures were AD and EnE, with M and PN also being individualistic but not as much. The most collectivist student cultures were SE and CS, with ElE and MP also being collectivistic but not as much.

However, the above conclusions should be treated with caution as the students base their attitudes about the group exam on the ‘learnt’ curriculum (Bauersfeld, 1979), which is not necessarily the same as the intended curriculum. The group exams were intended to be the same throughout the FES and prior to the reimplementation, workshops had prepared the supervisors for this type of exam. It is, however, unlikely that all group exams were identical as students, supervisors, and external examiners were different. One should therefore hesitate to draw too strong conclusions, particularly also taking into account the relatively small response rate. The results are *indications* of how students in different programmes at a Danish PBL university perceive a group exam and therefore which attitudes curriculum planners might expect from students if other universities wish to implement a type of project work or group exams. Curriculum planners need to consider what is the general ‘culture’ of collaboration both at the university and in the future profession, they need to

consider how the exam has a backwash effect on how the students work, and that bachelor and master students might not perceive such an exam in the same way. The group exam might also be a better fit for some groups than others, but in terms of mathematics, it neither appears to be an obvious fit, since mathematics students might lean more towards the individualistic culture, nor does it appear to be a bad choice since the students in general are positive. This might reflect the mathematics culture as being both individual and collectivistic. In terms of grading, curriculum planners need to consider that group exams might result in a smaller distribution of grades.

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A case for a new approach to establishing the validity of comparative judgement as an assessment tool for mathematics

Ben Davies

Loughborough University, United Kingdom; b.m.j.davies@lboro.ac.uk

Traditional, criterion-based assessments have recently been criticised for rewarding the procedural over the conceptual, limiting exam writers' ability to focus on authentic mathematics and necessitating complex mark schemes that are difficult and time consuming to implement (Jones, Swan & Pollitt, 2015; Bisson, Gilmore, Inglis & Jones, 2016). This paper discusses an assessment innovation, comparative judgement, *that avoids the above criticisms by using the innate human capacity for comparative, over criterion-based, judgement. After a review of the reliability and validity literature in this area, a theoretical examination of validity research on comparative judgement results in the proposition that a new approach is necessary. The final section of the paper suggests that investigation of* predictive validity, *as defined by Trochim (2006), may address the concerns raised regarding previous research on the validity of comparative judgement.*

Keywords: Assessment, comparative judgement, conceptual understanding, reliability, validity.

If presented with two large bags of gold and invited to keep one, most people would have a strong preference. While unlikely to know the exact quantity in either bag, it is usually not difficult to identify which is heavier by comparison. *Comparative judgement* is an assessment tool that exploits this innate capacity for comparison (Thurstone, 1927, "The Law of Comparative Judgment") rather than isolated evaluation against specified criteria. This well-established principle of psychology forms the basis for an innovative assessment that offers an alternative to the criterion-based status quo. This paper is a theoretical discussion of investigations into the reliability and validity of this approach to mathematics assessment. This is followed by a detailed theoretical examination of validity, resulting in the suggestion that further work and a new approach is necessary to establish validity in a more substantive manner. While making few definitive claims regarding the nature of the new approach, a case study in assessing conceptual understanding is used to illustrate the potential of two approaches based on what Trochim (2006) refers to as predictive validity.

What is comparative judgement?

Comparative judgement relies on expert judges making a series of comparisons between two responses to a given question. Without any specific criteria, judges are simply asked to 'choose the better response'. Using an automated system to generate pairs of students' responses for comparison, comparative judgement generates a ranked order from 'best' to 'worst'. This ranked order, produced after each response has been compared to others several times, is based on the Terry Bradley model (Pollitt, 2012) that generates a standardised parameter estimate (*z*-score) representative of the relative quality of each response. In the absence of specific marking criteria, the rankings produced are "instead grounded in the collective expertise of the judges" (Bisson et al., 2016, p. 143), avoiding the necessity to explicitly define the target of the assessment.

A summary of the literature on Comparative Judgement

Before examining the literature on reliability and validity, this section begins with a discussion of two key benefits claimed in the comparative judgement literature.

Benefits, motivations and intentions

Comparative judgement facilitates the targeting of specific aspects of mathematics without having to embed a precise definition into a marking schedule (Pollitt, 2012). For example, Bisson et al. (2016) used comparative judgement to investigate *conceptual understanding* in undergraduate mathematics. This "important but nebulous construct" (Bisson et al. 2016, p. 143) has been the topic of much debate. Bisson et al. (2016) claim experts are capable of recognising examples of conceptual understanding, but find it difficult to generate comprehensive, reliable scoring rubrics. As suggested by Bisson et al. (2016), by grounding the definition of an area of assessment solely in the collective understanding of judges, this problem is at least partially avoided. Assessment of conceptual understanding will be returned to in a later section of this work.

Jones, Swan and Pollitt (2015) also argue for the need to "free assessment designers from restricting tasks to those that may be easily scored by conventional means" (Jones et al. p. 170). They claim this would allow for the introduction of components that more adequately reflect the assessments purported purpose. A study involving four experienced GCSE (General Certificate of Secondary Education for England, Wales and Northern Ireland) exam writers producing an exam explicitly for comparative judgement marking found that exam writers experienced greater freedom to write more open-ended questions providing students with the freedom to experience more genuine problem solving situations (Jones & Inglis, 2015). Moreover, this freedom was successfully used, as evidenced by the survey responses of 23 teachers who evaluated the suitability of this exam script. Examinations of this type have been long sought-after but have seldom materialised due to the need for a demonstrably reliable and valid tool for assessment (Jones & Inglis, 2015).

These benefits, particularly that of eliminating assessment criteria, raise questions of reliability and validity. An assessment based on such apparently subjective foundations might initially attract substantial skepticism. These questions are the focus of the following two sections.

Reliability

In the context of comparative judgement, reliability has universally been used to refer to the notion of repeatability across different judges and judging communities. This is often referred to as *inter-rater reliability.*

There is a growing collection of case studies demonstrating high inter-rater reliability in the use of comparative judgement. Bisson et al. (2016) demonstrated high inter-rater reliability in the contexts of conceptual understanding of entry-level undergraduate statistics (*p*-values), undergraduate pure mathematics (derivatives) and elementary school-level algebra (using letters in mathematics). Using a split-half technique in each of the three studies, judges were divided into two groups of equal size. As a result of the computerised method by which judges make comparisons, this split-half comparison was repeated 20 times in each study without requiring any additional comparisons from the judges. Pearson's correlation coefficients were computed for each iteration of each study, the results of which are summarised in Table 1.

	Participants	Range in *r* over 20 trials	Mean *r*	Median *r*	Standard Deviation
***P*-values**	10 judges, 20 students	.664 - .855	.749	.762	.06
Derivatives	30 judges, 42 students	.826 - .907	.869	.872	.02
School Algebra	10 judges, 46 students	.678 - .837	.745	.742	.04

Table 1: Summary of comparative judgement reliability results from Bisson et al. (2016).

Jones, Swan and Pollitt (2015) produced results consistent with the above in a problem solving context. When applied to both traditional GCSE examination scripts and purpose-written open-ended examination scripts, comparative judgement performed well when implemented by a diverse group of 23 judges, consisting of ten GCSE examiners, one non-GCSE examiner, seven mathematics education lecturers, two researchers, one research student and two advisors. When separated into two groups (of 12 and 13), the Rasch sample separation reliabilities, often considered similar to Cronbach's α (e.g. Wright & Masters, 1982) were .80 and .93, indicating acceptably high internal inter-rater reliability for each group. Jones et al. (2015) also report a Pearson product-moment correlation coefficient of .87, providing further evidence for the inter-rater reliability of comparative judgement when assessing problem solving.

In every reviewed study on the reliability of comparative judgement to-date, the results have positively indicated that comparative judgement performed reliably across judging communities.

Validity

Consistent with the majority of studies in the comparative judgement literature this paper adopts the broad definition that a valid assessment measures that which it purports to measure (Koretz, 2008). Most investigations of the validity of comparative judgement have focused on what Trochim (2006) refer to as convergent validity, based on comparison with traditional measures of mathematical achievement. Bisson et al. (2016) has taken a different approach, by investigating content validity, based on comparisons with other psychometrically validated instruments.

A measure is said to have *convergent validity* if it produces results similar to others that should theoretically be similar (Trochim, 2006). On investigating assessment of problem solving, McMahon and Jones (2015) reported "comparative judgement outcomes correlated as expected both with test marks and with existing student achievement data, supporting the [convergent] validity of the [comparative judgement] approach" (p. 368). Jones and Inglis (2015) reported a correlation of .86 between students' GCSE marks and their parameter estimated *z*-score based on a comparative judgement assessment, providing further evidence in this direction.

Finally, Bisson et al. (2016) investigated both convergent validity and content validity across all three studies on which they report; see Table 2. The former was evaluated with methods similar to those

above, analysing correlations between traditional student achievement data and comparative judgement results. Content validity, a comparison of the measure against the relevant content domain (Trochim, 2006), was based on correlation analysis between comparative judgement results and other psychometrically validated measures of conceptual understanding; see column two of Table 2. It should be noted that the choice of topics (*p*-values, derivatives and algebra) in Bisson et al. (2016) were "driven by the existence of validated instruments to measure conceptual understanding of those topics" (p. 144). As is discussed later, the absence of such instruments for most topics in mathematics appears to serve as a significant barrier for this line of research. For the list of topics available Bisson et al. (2016) conclude that comparative judgement is a valid measure of conceptual understanding.

	Traditional measure	**Instruments**
***P*-values**	.555	.457
Derivatives	.365	.093
School algebra	.349	.428

Table 2: Summary of comparative judgement validity results from Bisson et al. (2016). The 'traditional measure' line refers to the correlation between achievement data and comparative judgement *z*-scores. The 'instruments' column refers to the correlation between comparative judgement *z*-scores and scores on an existing psychometrically validated measure (RPASS-7, CCI and 'Concepts in Algebra', respectively).

Jones and Inglis (2015) also investigated content validity of exam scripts. They asked experienced experts to analyse the content of the exam scripts produced when writers were freed from the constraints of traditional assessment criteria. Their qualitative analysis resulted in the assertion, consistent with their hypothesis, that these exam scripts placed more emphasis on conceptual thinking than their GCSE counterparts. It should be noted that this aspect of the study was focused on the exam scripts themselves, not on students' responses.

The evidence in this section suggests that comparative judgement is likely to perform at least as well as established tools for mathematics assessment in many or even most contexts. Moreover, Bisson et al. (2016) provide evidence of content validity in three specific mathematical domains. As discussed in the final section, the scope for expansion of the argument for content validity is limited, given the absence of other psychometrically validated measures. The following section focuses on the necessity for a complementary approach to establishing the validity of comparative judgement.

The necessity for a different approach to validity

This section has two aims. First, this section establishes the necessity for an approach other than the convergent validity-based studies discussed above. Second, this section discusses the limitations of the content validity approach proposed by Jones et al. (2015) as a suitable answer to the shortcomings of convergent validity discussed above.

Assessing the validity of comparative judgement only by comparison with traditional assessment appears to significantly limit the argument for comparative judgement. While arguments regarding

efficiency (Bisson et al. 2016) and the freedom to write examinations with a different mathematical focus (McMahon & Jones, 2015) still stand, proponents of comparative judgement claim it to have the potential to assess something fundamentally different from that assessed by traditional approaches. Bisson et al. (2016), for example, argue that comparative judgement can be used to assess 'conceptual understanding', an aspect of mathematics known to be difficult to assess using traditional assessment. However, in establishing the validity of comparative judgement in this area, they refer to comparisons with traditional assessment data. This approach to validity appears problematic from the outset as it limits a successful measure of validity to something not worse than traditional assessment; an approach claimed to be ineffective in assessing their domain of interest. Jones et al. (2015) note similar reservations pointing out that traditional student achievement data, often compared with comparative judgement, can be criticised for its overly-procedural focus. Jones et al. (2015) go on to assert the need for a different approach, pointing to comparisons with 'other psychometrically validated measures' as a satisfactory solution.

A further analysis of the limitations of convergent validity

A key aspect of the argument for comparative judgement is that it facilitates reliable assessment of areas previously difficult to assess. Thus it seems logical to believe that comparative judgement, when explicitly focused on assessing those areas (e.g. conceptual understanding) should reward different abilities/strengths than those rewarded by traditional assessment.

Taking this reasoning to its logical conclusion results in the assumption that some students should perform better under comparative judgement than under traditional assessment. Thus it seems at best limited to evaluate a new assessment tool by its ability to reproduce the results of traditional assessment. Consider a hypothetical student with a particular aptitude for memorising procedures but difficulty with the conceptual understanding underlying and drawing together important ideas. Based on previous critiques of traditional criterion-based approaches, one might assume this student would perform well on such an assessment. However, one might hope that this student would score poorly on comparative judgement assessment given the claims of Bisson et al. (2016). Hence, arguing for the validity of comparative judgement by pointing to its comparability with traditional assessment seems inherently limited.

Establishing convergent validity has made a significant progress toward gaining legitimacy and the credibility to justify further, more detailed work. However, it seems that a complementary direction is necessary in establishing other aspects of validity, thereby further developing the argument for the adoption of comparative judgement.

The limitations of content validity

As noted by Jones et al. (2015), making comparisons with alternative psychometrically-validated instruments appears to be an obvious and fruitful way forward in establishing the validity of comparative judgement. However, the absence of such instruments in most domains serves as a substantial barrier to progress. Returning to conceptual understanding, the absence of such instruments in this realm is well-documented and is a significant reason Bisson et al. limited their study to just three content domains.

It is theoretically possible to develop more such instruments. However, the number of instruments will always be far fewer than the number of mathematical domains in need of assessment. At present,

the content validity approach relies on inherently inductive reasoning. Evidence supporting comparative judgement in a small number of domains has little relevance for the new and previously untested. If comparative judgement is to be established as a viable assessment tool for a greater range of domains, I further claim that yet another approach is necessary.

The final section below seeks to identify a way forward that is cognizant of the issues raised above.

Predictive validity through the case of conceptual understanding

Having argued the necessity for something new, this final section offers a plausible approach attempting to avoid at least some of the above criticisms. This new approach, *predictive validity*, is illustrated here by further considering the case of conceptual understanding. It should be noted that this is just one such possible solution that, like previous sections, is intended to raise more questions than it answers. It too is not without its flaws.

As defined by Trochim (2006), *predictive validity* refers to a measure's ability to predict something it should theoretically be able to predict. If one is to accept conceptual understanding as fundamental to mathematical development (NCTM, 2000; Ofsted, 2008; Rittle-Johnson, Siegler, & Alibali, 2001), a valid measure of conceptual understanding should serve as a predictor of other mathematical success. In the absence of suitable comparative measures for evaluating content or convergent validity, predictive validity appears a logical alternative. This section addresses two possible such approaches that may aid in establishing the validity of comparative judgement.

First, comparative judgement could be evaluated as a predictor of performance on traditional assessment. Allowing for the passage of time, moving away from an immediate comparison with traditional assessment (see convergent validity) creates the opportunity for a student's general mathematical maturity and understanding to progress. This argument hinges on the assumption that conceptual understanding is so fundamental to mathematical development that it should act as a future predictor of mathematical performance in general, even on assessment for which conceptual understanding is not an explicit focus. If one is to accept this assumption regarding conceptual understanding then those that perform well on a comparative judgement assessment of conceptual understanding at time 1, should perform well on any assessment at time 2.

This approach may be criticised by those who argue that if an attempt to avoid immediate comparison with traditional assessment results in a delayed comparison with the same traditional assessment, then nothing has been gained by analysing predictive over convergent validity.

Another approach to predictive validity is to seek other measures of mathematics success not beholden to traditional, criterion-based assessment. Such measures of mathematical success may appear hard to come by. However, metrics such as success at postgraduate level, or success on relevant career pathways may provide an interesting solution. While arguably somewhat crude, these measures may provide useful sources of comparison. Using such measures, one could compare comparative judgement *z*-scores from prior years with success on these future measures to evaluate the predictive validity of comparative judgement here. For example, in an attempt to evaluate validity in the tertiary context, it would be possible to consider success in tertiary mathematics. By using comparative judgement to assess a series of tertiary exam scripts, it may be possible to evaluate predictive validity by comparison with future success in postgraduate studies or related mathematical

careers. A significant correlation between comparative judgement rankings and future success would stand as strong evidence for the predictive validity of comparative judgement.

This second approach to predictive validity benefits from avoiding any comparison with traditional assessment. However, unsurprisingly raises many questions of its own, particularly regarding the quality and availability of data related to such future measures of success. Moreover, predictive validity still does not grapple with the ways that scores and interpretation of scores are used. Kane (2013) offers yet another approach, drawing on the general principles of construct validity (i.e., how well an assessment reflects the true theoretical measurement of a concept) to grapple with the assumptions underpinning the interpretation and use (IUA) of assessment scores, rather than the scores themselves. The implications of this 'argument-based approach' approach to validation of comparative judgment are unclear, but his detailed discussion of IUAs draws attention beyond the process of assessment in of itself to the use and interpretation of the assessment findings in the real world. For example, the evaluation of comparative judgment results could usefully require an evaluation of the consequences of the assessments' use, and negative consequences in some could render the use of the results unacceptable in some circumstances.

I do not claim to have covered the full range of criticisms that may be levied at the new approaches. Nor do I claim predictive validity is the only response to the limitations of existing research discussed earlier. The intention of this section is to initiate a discussion regarding a new approach to validation of comparative judgement that might facilitate an extension of the argument for comparative judgement beyond the mathematical domains for which we have other tests.

Conclusions

This paper has been a discussion of an exciting and relatively new approach to assessment. In particular, the focus has been on the necessity for a new approach to evaluating the validity of comparative judgement. It should be noted this work does not advocate for the abandonment of one approach in favour of another. Rather, it advocates for a wider range of complementary approaches that together will form a comprehensive body of research.

This paper has argued that by moving beyond convergent validity, researchers will have the ability to investigate the validity of comparative judgement in a manner less limited by comparisons with the status quo. Moreover, it has been argued that the recent content validity solution, suggested by Jones et al. (2015) is insufficient in scope and is inevitably limited by the non-existence of psychometrically validated measures for the vast majority of mathematical content.

Finally, this paper proposed two new approaches, based on predictive validity, that avoid some of the discussed limitations. A research programme including investigation of predictive validity and the interpretation and use of assessments has the potential to extend the discussion about comparative judgement, and ensure that further uptake of this exciting new development in mathematics assessment is evidence-based and beneficial to the students for whom it is designed.

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Cognitive demand of mathematics tasks set in European statewide exit exams – are some competences more demanding than others?

Christina Drüke-Noe[1] and Svenja Mareike Kühn[2]

[1]Pädagogische Hochschule Weingarten, Weingarten, Germany, druekenoe@ph-weingarten.de

[2]Universität Koblenz-Landau, Koblenz, Germany, kuehn@uni-koblenz.de

Statewide exit exams play an important role in education as they tend to define what is considered important in a subject. This paper examines characteristics of Mathematics tasks set in such exams in eight European countries at the end of lower secondary education. The main result of this descriptive study is that the cognitive demand of most competences needed to solve these tasks is rather low. The only exception is 'working technically'. So far, these results can neither be explained with the public impact these exams have nor with relevant exam regulations.

Keywords: Exit exams, mathematics, competences, routines and facts, cognitive demand.

Introduction

Key competences and statewide exit exams

A changing need for competences in a knowledge-based society has caused literally all European countries to agree to equip all learners in compulsory schooling with certain 'key competences' "which represent a combination of knowledge, skills and attitudes that are considered necessary for personal fulfilment and development; active citizenship; social inclusion; and employment" (Eurydice 2012, p. 7). 'Mathematical competence and basic competences in science and technology' is one of the eight key competences defined. As these are promoted either by a national strategy or by applying large-scales initiatives instead (Eurydice, 2012), competences are relevant in different forms of assessment, too.

Central written exams at the end of a particular educational stage are one form of assessment in the majority of European countries. In secondary education such exams are frequently taken at the end of upper secondary education (ISCED level 3) and in some countries additionally at the end of lower secondary education (ISCED level 2), which marks the end of compulsory schooling. Subjects tested are usually at least Mathematics and the language of instruction.

Exit exams are mostly compulsory either for all students, regardless of the type of school attended, or just for those in public-sector schools (Eurydice, 2009). Even if these exams are "optional, as in the case of the 'national qualifications examination' in the United Kingdom (Scotland), or the Dutch CITO test, nearly all pupils take them in practice" (Eurydice, 2009, p. 23) which hints at the relevance of these exams. However, despite having the same name and sharing the purpose of summarizing the achievement of individual students at the end of an educational stage, statewide exit exams of any two countries differ in several respects such as the time awarded or the institutions which set them (cf. Klein, Kühn, van Ackeren & Block, 2009) or the ways the results are used or published (cf. Eurydice, 2009).

Research on statewide exit exams

International research on statewide exit exams reveals various perspectives. Considering governance and accountability Klein & van Ackeren (2011) identify considerable national differences, e.g. as regards the role of the governing bodies, the degree of standardization of these exams, the use of the results or varying intentions ranging from control to support of schools. Other research focuses on setting standards and promoting innovation by means of such exams (e.g. Bishop, 1998) but also draws attention to contrary positions. Wößmann (2008) even gives evidence that such exams enhance students' achievement. Further research addresses the impact of such exams on the selection of tasks for teaching (e.g. Cheng & Curtis, 2012; Neill, 2003). In this context a backwash-effect of exams on teaching and learning is also frequently discussed (e.g. Biggs & Tang, 2011; Boud & Falchikov, 2006). Especially such summative forms of assessment have this effect and exert a considerable influence on both teachers and students because exit exams set standards as to what is to be achieved or considered important. As a consequence, exams ought to be carefully designed, even more so if governing bodies use them as a strategic instrument to influence teaching. However, there is only scant research on characteristics of tasks set in exit exams, and some of these studies are somewhat dated (see overview by Krüger (2015) for various subjects).

With respect to tasks particularly set at the end of lower secondary education in Europe, this lack of research might be due to the fact that only few countries like Northern Ireland and Scotland have a longstanding tradition of such exams, whereas most other countries only introduced them in the last one or two decades (Eurydice, 2009). However, given the broadly conceded relevance of tasks in Mathematics (e.g. Arbaugh & Brown, 2005; Neubrand, Jordan, Krauss, Blum & Löwen, 2013), it is astounding that only very few studies examine tasks set in exit exams in this subject. One study by Kühn and Drüke-Noe (2013) identifies a low level of cognitive demand of the tasks set in the 15 German states (*Länder*), and it reveals that applying routines and using facts (working technically) is a dominant competence when solving the tasks, whereas more complex competences, such as problem solving or argumentation, are far less needed. Another study by Vos (2013) focusses on specific modeling aspects of tasks set in the Netherlands and reveals that concerns about test reliability seem to limit the range of modeling aspects tested.

Methodology

Based on the discussion in the section above, the following four research questions are addressed: Which mathematical competences are needed to solve the tasks set in statewide exit exams and which level of cognitive demand is realized? Are the competences of similar importance? Which national exam characteristics can be identified which might be related to their public impact in each country? Can national exam characteristics be explained on basis of national exam regulations?

To address these questions a classification scheme by Kühn (2011) is used, which was developed in a German comparative study on mathematics tasks. The scheme's subject specific categories, which are based on educational standards in mathematics, are used to identify and understand the structure as well as content- and process-related characteristics of tasks (cf. Kühn & Drüke-Noe, 2013). In the scheme the various categories and their subcategories are described in detail and illustrated by examples. Some of these categories are: mathematical content (subcategories: arithmetic, algebra, geometry, stochastics), types of mathematical activities and mathematical competences. In the

course of developing this classification scheme, country-specific requirements for the design of exam tasks (e.g. *Specifications* in England, *Notes des Service* in France, or *Examen Programma* and *Constructie opdracht* in the Netherlands) have been examined if they were publicly available. Thus, the classification scheme includes general (instead of country-specific) categories for task analysis so that it can be used for a transnational comparison of task characteristics.

Based on this classification scheme, various task characteristics, such as competences necessary to solve a task and the level of cognitive demand of each competence, can clearly be identified. Each task is assigned to one or more of six mathematical competences (argumentation, problem solving, modeling, use of representations, working technically, communication), and for each competence one of four levels of cognitive demand is marked by a score (0: not needed, 1: low level, 2: intermediate level, 3: high level).

Analyses of the impact of these exams rely on documents published by the Eurydice network.

The sample comprises 655 tasks set in eight European countries (Ireland (IE), France (FR), Italy (IT), the Netherlands (NL), Norway (NO), Portugal (PT), England (UK-ENG) and Scotland (UK-SCT)). In both years 2008 and 2011, only these countries set written statewide exit exams in Mathematics at the end of lower secondary education (ISCED level 2), i.e. when students are aged 16. All tasks were translated into German and then categorized by a student who was excellent in mathematics and its didactics. The student was trained in several steps: tasks were classified and reasons for classifications had to be articulated. By means of this communicative validation agreement was gradually achieved. Additionally, the student could query classifications at any time.

Results

Quantitative findings

The descriptive results presented here are aggregated for both years 2008 and 2011. For each of the eight countries the findings provide insight into mathematical competences needed to solve the exam tasks. Table 1 informs on the number of tasks (N) set in each country and it provides information on the level of cognitive demand realized by giving mean scores (M) and standard deviations (SD) for each of the six competences.

		argumentation		problem solving		modeling		use of representations		working technically		communication	
Country	**N**	**M**	**SD**	**M**	**SD**	**M**	**SD**	**M**	**SD**	**M**	**SD**	**M**	**SD**
IE	96	.01	.102	.28	.452	.46	.501	.15	.355	1.41	.535	.02	.144
FR	60	.40	.807	.77	.647	.67	.629	.63	.712	1.13	.724	.28	.585
IT	63	.17	.423	.37	.485	.79	.652	.67	.762	.90	.560	.37	.517
NL	50	.32	.713	.72	.497	.78	.648	1.22	.932	1.32	.768	.74	.527
NO	124	.10	.400	.40	.582	.52	.618	.54	.562	.94	.621	.29	.522
PT	39	.26	.637	.54	.643	.51	.506	.82	.885	1.18	.756	.77	.742
UK-ENG	152	.26	.581	.37	.512	.50	.575	.67	.726	1.16	.656	.45	.639
UK-SCT	71	.08	.327	1.00	.378	.65	.657	.56	.626	1.39	.547	.59	.523

Table 1: Mean scores of cognitive demand of the six competences

Table 2 reveals if exam results are used to award certificates and if and how exam results are published. These categories are used to judge the exams' impact (cf. Eurydice 2009, 2012).

Country	IE	FR	IT	NL	NO	PT	UK-ENG	UK-SCT
Award of certificates	x		x			x		x
No impact on students' progression		x		x	x		x	
Publication of results organized or required of schools, by central/local government				x			x	x
Publication of results at the discretion of schools			x					
No publication of results	x	x			x	x		

Table 2: Ways in which results of statewide exit exams are used and published

With respect to the first two research questions, the results in table 1 indicate an overall very low level of cognitive demand of all six competences for all countries. Since a maximum score of 3 can be reached, which stands for a high level of cognitive demand of a competence, the means realized indicate that most competences are only rarely needed to solve the tasks. This means that if a specific competence is needed at all its cognitive demand is mostly low. As mean scores of about 1.00 or more are almost exclusively limited to 'working technically', it can be concluded that this competence considerably determines the overall cognitive demand of the tasks set. This shows the relative importance of this competence, as more complex ones, such as argumentation, problem solving or modeling, are only rarely needed in any of these countries. This fact, however, is not fully in accordance with national exam regulations, such as those of France, England and the Netherlands, which explicitly require that these competences are tested, too. All in all, these results hint at the relative importance of applying routines and using facts to solve the tasks set.

Finally, based on the results given in both tables, for none of these countries a systematic relation can be identified between the overall mean levels of cognitive demand realized and the impact of the exams. For example in Scotland the impact can be considered high (certificates awarded, results published) and the mean levels of cognitive demand of all competences are relatively high, too. On the contrary, in France the impact can be considered low (no impact on students' progression, no results published) and the mean levels of cognitive demand are fairly high, too.

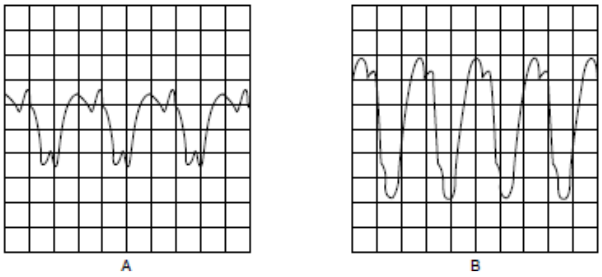
Examples of tasks

The following two examples of tasks and their analyses are to illustrate the quantitative findings given in table 1. The first example serves to illustrate the relative importance of working technically. In many tasks this is either the only competence needed or it is the only one which is cognitively more demanding. The task shown in figure 1 is typical in the sense that it only requires to work technically. To factorize the given sum of products, several steps need to be carried out (working technically, intermediate level). In all countries, however, most tasks which require this competence are of an even lower cognitive demand.

Factorise: $x^2 + 7x + 12$

Figure 1: Working technically at an intermediate level (source: Ireland, 2011)

Sounds are vibrations in the air. A sound spreads through the air. We then speak about sound waves. Sound can be visualized using a device that turns sound vibration into an electric vibration. Below you see what this device shows for two different sounds. Both figures show a number of vibrations within a certain period of time (for example, 1 millisecond).



The number of vibrations per second is called the sound frequency.

◊ Which figure shows a higher frequency? Explain how you get to your answer.

Figure 2: Relatively high cognitive demand of a task (source: the Netherlands, 2011)

The second example given in figure 2 shows one of the few tasks in the entire sample which are not only relatively long and complex but also require a number of rather complex competences.

To understand both diagrams (figures A and B) the introductory text needs to be read first (communication, intermediate level). Then a given model needs to be applied to the context (modeling, low level). The comparison of both complex diagrams (use of representations, high level) is supported by problem solving strategies (intermediate level). The final decision on the appropriate diagram requires an argumentation that needs to be written down (communication and argumentation, both intermediate level). In contrast to the majority of tasks in the sample this one does not require to work technically.

Influence of national exam regulations on task characteristics

To explain the different levels of cognitive demand of tasks set in different countries (cf. table 1) national exam regulations are considered, too. As not all regulations are publicly accessible, the following analyses are only based on the ones applicable in England, in France and in the Netherlands. The document analyses are to show if and in which way individual competences are mentioned. Furthermore, the analyses are to reveal if the relevance of any of the competences is stressed in relation to other ones.

The analyses of these documents firstly reveal that exam regulations in these three countries demand explicitly that competences such as argumentation, modeling and communication are tested. While in France and in England these competences are merely mentioned, in the Netherlands the cognitive demand of individual competences is described in some more detail. For example with respect to argumentation and communication regulations in the Netherlands explicitly ask for “reasoning strategies” and “communication by means of adequate mathematical language” (cf. CEVO, 2009). In addition, the requirements concerning modeling are even more specific as the entire modeling cycle including its individual steps is mentioned.

Contrary to expectation, though, the way competences are specified does not directly correspond with realized mean levels of cognitive demand. One example is the level of cognitive demand of modeling in France (brief specifications) which is similar to the one in the Netherlands (more detailed specifications). A second example is the considerably different mean levels of cognitive demand of argumentation in France and in England despite the fact that both countries have rather brief and similar specifications of competences.

The document analyses secondly reveal that none of the exam regulations studied state in any way how intensively individual competences are to be considered in the exams. There is also no statement as to the relative importance of working technically. Thus, on the basis of the exam regulations it is not possible either to explain the relatively high mean levels of cognitive demand of this competence in all countries.

All in all, the findings show that exam regulations do not correspond systematically with task characteristics. Furthermore, the more detailed analysis of only three countries shows that further research is necessary to explain how exam regulations are put into practice. Considering theories of educational governance, it must be assumed that such regulations do exert an influence on the selection or the development of exam tasks but so far it is not entirely clear in which way this impact works in practice. As a consequence and with special regard to competences, more needs to be found out on the application of exam regulations by those who either select and develop exam tasks or design entire exam papers by assembling tasks. For these reasons it is necessary to find out which other elements directly influence characteristics of exam tasks in different countries.

Conclusion and implications

Although all eight countries examined supposedly promote key competences including mathematical ones, the task characteristics raise the question how thoroughly these competences have been implemented, since cognitive demand is primarily determined by working with routines. As exam tasks influence the selection of tasks for teaching and as the task characteristics identified in this descriptive study can neither yet be explained by the use of the exams' results nor be explained by exam regulations, further research is necessary that addresses both the influence of national curricula and that of existing exam design regulations as well as their application. Beyond this, further research should also address reforms actually taken to promote competences in these European countries to explain the results.

More detailed research of the tasks set in two different years is of interest, too. This also has to do with the influence such tasks have on teaching (cf. Black & Wiliam, 1998). More detailed research of this kind could possibly better help to understand the so far unsystematic relation of the exams' impact (see table 2) and the characteristics of the tasks set.

The results presented in this study also raise further interest in more recent developments on exam design regulations and the ways in which they influence actual exams. Especially with respect to exams taken at the end of lower secondary education not much is known about this relation yet. Research on further development in the direction of a broader and more explicit notion of competences which includes explicit research on the cognitive demand of individual competences in these exams is desirable, too.

From a process-oriented perspective, a number of interesting research questions arise that focus on factors which influence the development of tasks set in central exams. There are two important factors of interest: One of them is longstanding traditions of tasks in these countries which might help to understand the task characteristics presented in this study. A second factor is the professional knowledge of those people who design the exam tasks. As the findings presented cannot fully be explained by exam regulations, attention is shifted towards the role of those people who develop tasks set in the statewide exit exams. It is well worth finding out if these people are aware of the different competences and – even more and on a more detailed level – of the various facets which these competences have. Modeling can be taken as an example here, as this competence comprises more than just mathematising (cf. exam regulations in the Netherlands). Furthermore, it is of interest to see if people who design the tasks are aware of different levels of cognitive approach of individual competences.

Finally, more recent theories on educational governance as well as findings from research on implementation support the assumption that the design of exam tasks is not exclusively influenced by relevant exam regulations. There is evidence for a considerable and multi-factorial influence of any people who are involved in the design of exams. More traditional attempts of research, which explain directions of influence, are based on a model of governance which is hierarchically structured and works in a linear way. More recent approaches, however, that try to explain these directions of influence, even identify a systematic discrepancy between formal exam regulations set by both political and administrative bodies and the way they are applied in practice. There is evidence that this seems to be caused by a necessary re-contextualization and transformation of existing regulations by the individuals involved (cf. Altrichter & Maag Merki, 2010). Research both on innovation and implementation shows similar findings and reveals that administrative regulations and their implementation in practice can even differ widely (cf. Gräsel & Parchmann, 2004). To sum up, far more research is necessary to understand how exam design regulations are put into practice and how task characteristics can be explained.

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A vertical analysis of difficulties in mathematics by secondary school to level: Some evidence stems from standardized assessment

Federica Ferretti[1] and Alessandro Gambini[2]

[1]Alma Mater Studiorum, University of Bologna, Department of Mathematics, Bologna, Italy;

federica.ferretti5@gmail.com

[2]INVALSI, Italy; a.gambini@unibo.it

A longitudinal analysis of the recurring mistakes at different school levels in national standardized assessment tests is presented. The analysis of the outcomes highlights some difficulties common across different school grades. Subsequently, we extend our research to university students: we investigate the results of tasks solved by students at the end of high school and at the beginning of university in an e-learning environment called AlmaMathematica. We examine whether there are commonalities between errors that lead to wrong answers at school level and university level. Results show that university students share the same difficulties of high school students when faced with similar tasks.

Keywords: Mathematics test, summative evaluation, semiotics.

Introduction

Our research is carried out selecting a set of tasks sharing common features among the national standardized assessment tests INVALSI (National Assessment Institute for the School System) and the results of the AlmaMathematica tests administered to students at the end of high secondary school and at the beginning of university. As we can see in the next section, the INVALSI gives back the results at national level to each item of each test. Thus, we have, each year, results for grade 2 and 5 (in Italy, second and fifth years of Primary School), grade 8 (third year of low secondary school) and grade 10 (second year of high secondary school). Most of the research based on these data concerns vertical and common features arising at different test level outcomes (Branchetti et al, 2015). In this research perspective, we investigate common difficulties that emerged from the vertical analysis of INVALSI secondary schools test results. In the second phase of the project we further develop the research expanding to university level students. In order to do this, we analyzed the results of mathematics tests in the AlmaMathematica project, an e-learning environment in which many students of the fifth (and last) year of high secondary school and university freshmen perform mathematics tasks. Hence, we asked the following research question: Is it possible to identify some common student behavior when facing mathematics tests from secondary school to university? For this we analyse two tasks of INVALSI mathematics tests of grade 8 and grade 10 giving an example of analysis of linked tasks in which different approaches are implemented. The comparison between the two tasks allows us to interpret some difficulties encountered by the students. The analysis of a related task in AlmaMathematica suggests some possible ways to interpret students' behaviors at university level. All three tasks have the same structure and the same mathematical content, but the likely difficulty is represented by the switch from a representation register to another one. The same skills are required for all three tasks and the purpose of our research is to investigate if there are common aspects in the task solutions.

National Evaluation Service and AlmaMathematica Project

INVALSI is the Italian national institute for the educational assessment of instruction and every year, through National Evaluation Service (SNV), it carries out periodic and systematic national assessment to check student knowledge and skills in Mathematics and Italian language. Every year, INVALSI performs assessment tests in a census to school grades 2 and 5 (Primary School), 8 (low secondary school), and 10 (high secondary school) and it returns the results of the sample for each item of all tests administered. As part of our research we consider in detail the results of two tasks in mathematics standardized tests carried out by grade 8 students in a.y. 2010/11 and by grade 10 students in a.y. 2011/12. To conduct this research, it has been crucial the use of a research tool of INVALSI tests, the *GestInv Database* that we will describe in detail later. To investigate difficulties at university level, we examine the results of the performance of students using AlmaMathematica. AlmaMathematica (almamathematica.unibo.it) is a project of the University of Bologna designed specifically to create links between Secondary School and University. It is aimed at students who wish to enroll in undergraduate courses at the University of Bologna and it provides basic courses in mathematics, statistics and probability. The environment that has been created refers to the tasks of the Entrance Tests to a restricted number of Curriculum courses (TOLC), and to the evaluation of basic knowledge tests. Access to the statistical data allows us to investigate the percentage of correct answers for each question.

Theoretical lenses

Our main hypothesis is that a longitudinal analysis involving a large number of students can give relevant information about difficulties existing at secondary school level and can allow us to infer whether these difficulties remain at the beginning of university and in what fashion. Our research stems from evidence arising from the analysis of Large Scale Assessment (LSA) tests. We do not consider LSA merely as a way to provide a ranking or scores for benchmarks or as a search for correlations between variables of context, but we assume that its results can provide information on the teaching/learning process. In accordance with many researches (i.e. Looney, 2006) we consider the analysis of the results of standardized assessment through the lens of formative assessment. The information given back by LSA contains not only global scores (measured by statistical models), but it also highlights the specific phenomenon observed individually. Among the results of standardized assessments many significant macro-phenomena are visible that can be explored and interpreted through some of the lenses of mathematics education as the most frequent difficulties described in literature are also reflected in the students' responses. We conjecture that this kind of longitudinal analysis carried out through the comparison between the data sets from different years and levels could be useful to better interpret the difficulties that arise in secondary school and remain until university. For this reason, we need some criteria to link tasks from different grades: we have chosen the tasks that had the lowest correct response rate in the standardized tests and for which the topic is present both at secondary school and university. In particular, our research focuses on powers and manipulation of exponents and the difficulties with these topics have been widely reported in the literature (Pitta-Pantazi et al., 2007, Cangelosi et al., 2013). Indeed some studies have already reported common difficulties with management of exponentials between university students and high school students (Cangelosi et al., 2013). This led us to think that some misconceptions regarding exponential expressions are persistent over time. The students' mental

constructions and the way in which they develop a meaningful understanding of exponentials has been the subject of other studies (i.e. Pitta-Pattanzi et al., 2007). In this is a vertical analysis made among secondary school students the authors found that, independently from the age of the students, there is an issue in the treatment of the exponentials that led them to provide wrong arguments in comparing powers with the same base or with the same exponents. Starting from these evidence, we conducted our research concerning expressions with exponentials, in particular the manipulation of different representations of exponentials. We are interested in understanding if the phenomenon showed from the quantitative analysis also implies that the difficulties encountered by students at different school levels are the same. To investigate this phenomenon, we need a further qualitative analysis, which is on-going, and its results will not be included in this paper. As we can see below, we detect that the main common difficulties are related to the semiotic representations management and, in order to interpret it, we use the semiotic approach proposed by Duval (1993, 2006). According to Duval, for each object there is more than one possible semiotic representation and one of the highest processes of mathematics is precisely the management of different representations of the same object. Our analysis will show that recurrent errors made by students, in all investigated levels, can be reduced to the difficulties concerning the management of different semiotic representation of the same object and the transformation of representations within different registers. We can then identify the main difficulties in the conversion (Duval, 1993). Duval (2006) suggests that the switch from natural language (verbal register) to algebra (symbolic register) requires a high level of complexity. Furthermore, according to Duval, it is possible to classify the different representations of a mathematical object in different registers, which are a set of signs and rules that can be manipulated. Such registers may themselves be classified as discursive (natural language written or spoken, mathematical symbols) or non-discursive (diagrams and figures). Still, it is possible to distinguish within each category those that are multifunctional registers, i.e. suitable to explain processes that cannot be put in algorithmic form, from those mono-functional, i.e. especially dedicated to algorithmic processes. In the first category there is natural language, in the second arithmetic and algebraic symbols. In the mono-functional register treatments can take the form of an algorithm, while in multi-functional ones, this is not possible. This fact will be crucial in the analysis of the behaviour of the students in our research.

Methodology and data analysis

Our research is a mixed method sequential research (Johnson & Onwuegbuzie, 2004), with design QUAN → QUAL → QUAN. The first quantitative phase consists in an analysis of statistical results of the standardized items. Then, among the selected standardized items, we search for the ones with a topic common between secondary school and university levels. Subsequently, we look for the ones that highlight the same educational phenomena. Finally, we search on AlmaMathematica for the ones with the same features. We conduct a research looking at tasks at low secondary school level, high secondary school level and the initial stage of university undergraduate level. We needed a common topic to start with and we chose powers. In Italy, this is a topic used in the final national examination at the end of low secondary school, which is then elaborated in the second year of high secondary school and it is considered an “entry requirement” (and therefore investigated) for all university courses that require mathematics knowledge and skills. To search for tasks concerning powers among all the ones of the standardized assessment INVALSI test from 2008, which has had a low rate of correct answers, the INVALSI Database Gestinv is used. This database is an online

tool of research (www.gestinv.it) that contains more than 1,400 items administered in the Italian national standardized tests and it is used in professional development programs implemented by schools and in research in mathematics education. Inside the database, there is a PDF of all tests administered in Italy from 2008, in which each item of these tests is accompanied by detailed results, statistical classifications, and data split into different categories. In respect of each item there is the image of the question, the goal of the content, the process, the reference to the National Guidelines for Curricula, some keywords characterizing the content, the text of the question, the correct answer or the image of the correct answer, the percentage of national response, the characteristic curves, and the item information. The Gestinv database can be used in many ways: when entering the section of Mathematics it is possible to search by National Guidelines for Curricula, Keywords, Full Text, and to do a Guided Search: a cross-search - with connectors and/or - of all parameters in respect of each item and all its features, such as the percentage of national response. Through the tool Guided Search we searched for all of the secondary school tasks of INVALSI Tests of Mathematics, referring to the keyword "powers" which had percentages of correct national responses below 50%. The research displayed about ten tasks with these features, and we looked for those whose analyses represent the same didactic phenomenon. As we see in the next section, we studied two tasks that had "common errors" displayed at national level: one by low secondary student (13 years old) and one by high secondary school students (15 years old).

Once such tasks are identified, we check if the analysis of the tasks about powers in AlmaMathematica Project shows the same type of errors. Students who performed exercises and problems in this e-learning environment are 18-20 years old; this allows us to investigate whether the same phenomenon persists with students of different age and how it occurs in various levels.

The analysis of tasks

The following task was administered in a census at grade 8 Italian students in a.y. 2010/11.

E11. A tenth of 10^{20} is:

A. 10^{10}

B. 1^{20}

C. 100

D. 10^{19}

Figure 1: Question D11, Grade 08, INVALSI Test, a.y. 2010/11

Figure 2 shows the national percentage of correct, mistaken, missed, and invalid answers and the percentage of choice for each option.

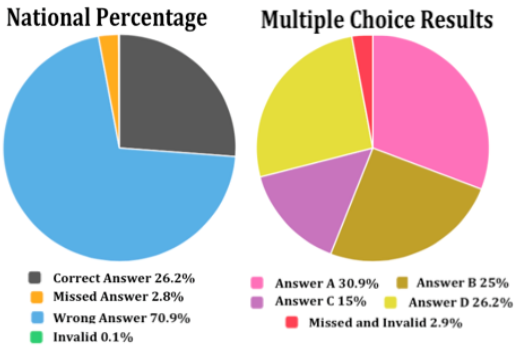

Figure 2: Data of Question D10, Grade 08, INVALSI Test a.y. 2010/11

The question was administered in a.y. 2010/11 to a population of approximately 600,000 grade 8 students, the sample (from which the statistical data are calculated) was composed by about 27,000 students. To give the correct answer a correct management from natural language to algebra is necessary. As we notice, the percentage of correct answers is low (the correct answer is option D, and it has been chosen by 26.2% of students). Option A and B have been chosen almost by the same percentage of students. Students that answered A worked incorrectly on the exponents (they probably halved the exponents or subtracted ten from the exponents). Students who choose option B had divided the base by ten. A similar situation appears also in the following question, administered at grade 10 students in the INVALSI N of the a.y. 2011/12.

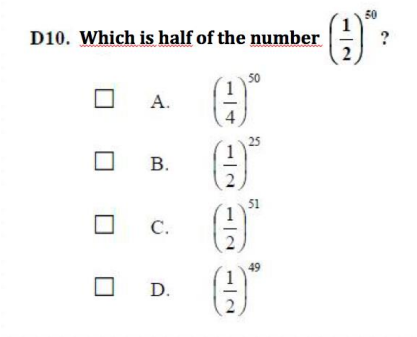

Figure 3: Question D10, Grade 10, INVALSI Test, a.y. 2011/12

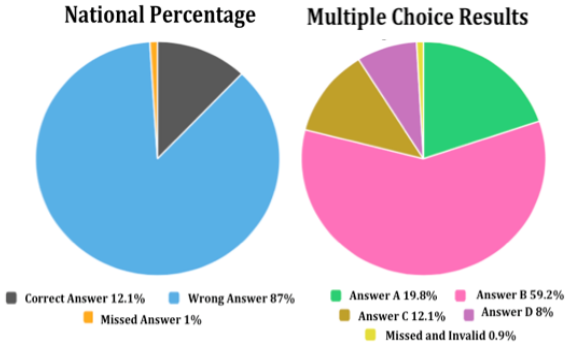

Figure 4: Data of Question D10, Grade 10, INVALSI Test, a.y. 2011/12

As shown in Figure 4 the correct answer was chosen only by 12.1% of students. This question was administered in a census to 530,000 grade 10 students, and about 42,000 students composed the sample. The most common option is B: students who have chosen this option have halved the

exponents. This situation has some features in common with the previous one: the task structure is the same and the construction of the distractor is similar. In the 8th grade task, the numbers involved are integers and in the 10th grade task are fractions, but the solution of both tasks require the manipulation of powers, and the conversion from different registers. Another similar situation occurred in the analysis of the results of the exercise referring to powers in the AlmaMathematica Project. Entering the e-learning environment AlmaMathematica there are 5 sections and one of them is about algebra. Inside this section there are 7 subsections including "Powers and Roots". Students enter the online environment and perform the exercises; each student can perform the exercises more than once. When a student makes a mistake, it is only reported that the given answer is incorrect but the display does not show the right one. The percentages of answers, shown in Table 1, are related to the first attempts given by students. When we extrapolated the data, there were 1625 registered and 773 students attempted the exercise. By analysing the data we can see that one of the tasks has some characteristics in common with the previous ones; the task is the following one (Figure 5).

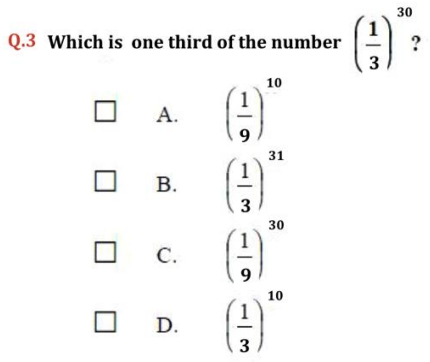

Figure 5: Question Q.3, Section "Powers and Roots", AlmaMathematica Project

Answer A	Answer B	Answer C	Answer D
13.7%	25.9%	12.3 %	48.1%

Table 1: Data of Question Q.3, Section "Powers and Roots", AlmaMathematica Project

Also in this case we notice that the structure of the task has several elements in common with the previous two. Indeed, the situation is similar to the situations that occurred in the INVALSI tests: the solution of the task requires interpreting a verbal delivery and employing working with powers. Also the distractors are similar to the distractors in the INVALSI task grade 10th. Specifically, the number presented in option C, in which the exponent is divided by a third, is obtained by an incorrect manipulation of the exponent exactly as the number present in option B in the previous task. As we can see in Table 1, the percentage of correct answer is slightly higher and almost half of the students chose option D, just like the majority of students of grade 10 chose option B. The exercises in AlmaMathematica were performed by students at the end of the secondary school and at the beginning of university but almost all users are university students. Thus, we can observe that among all analysed levels (from low secondary school to university) the conversion from natural language to symbolic representation about power manipulation is an issue. Specifically, observing the results obtained in high secondary school and university tests, we show that the students who

made the same mistake: they manipulated in the wrong way the exponent of the powers leading back to the exponent the "verbal indication" provided in the stimulus. Indeed, in both tasks they chose the option in which there is an incorrect manipulation of the exponent.

Conclusions and further directions

Our main hypothesis is that a longitudinal analysis, performed with many students, can give relevant information on the directions to link (and, then interpret) longitudinal shared difficulties from low and high secondary school to university. We study students' behaviour when solving mathematics exercises in which the management of representation of powers in different registers is required. As shown by Cangelosi et al. (2013) certain errors when working with exponential expressions persist as the students progress through their mathematical studies. Many students memorize algebraic rules with little or no conceptual understanding of their meaning because the rules of algebra and its terminology seem distant from their way of thinking. It follows that these students have trouble keeping track and applying the rules appropriately (Kieran, 2007). The description of the difficulties of students in algebra, particularly in the interpretation of mathematical symbols, was also addressed by Carraher and Schliemann (2007). Kieran (2007) noted furthermore that a main issue is the ways in which students work with variables and algebraic expressions, discussing in depth the development of algebraic thinking in middle and high school. In our case the problem is the management of verbal representation of power and algebraic representation. The processes put in place to manage these different representations are well framed in Duval (2006). In all tasks analysed, a switch from natural language (verbal register) to the algebra (symbolic register) is necessary to give the correct answer, and this presents a high level of complexity (Duval, 2006). Indeed, we studied the difficulties of students in conversion from two different registers, from natural language to symbolic representation. Particularly, we analysed the difficulties to convert from one multi-functional register to one mono-functional register, and despite this represents a difficulty, it is impossible to avoid this situation in the teaching/learning processes. Results show that students make common errors in managing different representations of an object. For a better interpretation of the phenomena that we observed, we shall need a further qualitative analysis and for this reason we are conducting some interviews with school and university students. Research in mathematics education regarding the transition from secondary to tertiary education highlights that students' difficulties are related to a multiplicity of factors – cognitive and meta-cognitive – and it is still more problematic when accessing university education (Gueudet, 2008). These difficulties highlight that one of the causes is the gap in the prerequisite knowledge, specifically in the manipulation of different objects representations. In conclusion, information acquired by LSA and by the e-learning environment has brought to light some recurrent mistakes. The analysis of this data allows us to interpret a didactic phenomenon, and it is also in this perspective that we consider standardized assessment as a tool for formative assessment.

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The Italian national education assessment system: Building mathematics items

Rossella Garuti[1], Cristina Lasorsa[1] and Stefania Pozio[1]

[1]INVALSI, Rome, Italy, rossella.garuti@libero.it

[1]INVALSI, Rome, Italy, cristina.lasorsa@invalsi.it

[1]INVALSI, Rome, Italy, stefania.pozio@invalsi.it

In this paper we describe how mathematical items are constructed for the Italian National Education Assessment System (SNV). After a brief description of the structure of the Italian SNV, we describe – through an example – how mathematical items are analyzed and finalized before being submitted to Italian students of different school grades. Each item is analyzed from two different perspectives: first the mathematical content and its relevance in the teaching and learning of mathematics is considered and then the statistical analysis of the field trial results are examined. The challenge we face is to maintain an appropriate balance between these two different aspects.

Keywords: Assessment, mathematics tests.

Introduction

Italian National Assessment System (SNV[1]) started its work in 2008 through annual surveys conducted by the National Institute for the Evaluation of the Education System (INVALSI) at different school levels. The INVALSI develops standardized national tests to assess students' reading comprehension, grammatical knowledge and mathematical competence, and administers them to the whole population of primary school students (Grades 2 and 5), middle school students (Grades 6 and 8), and high school students (Grade 10), see Figure 1. From 2008, the SNV test on reading and mathematical competence is part of the 8th Grade national final examination. Therefore it contributes to the final evaluation of the students (Garuti & Martignone, 2016).

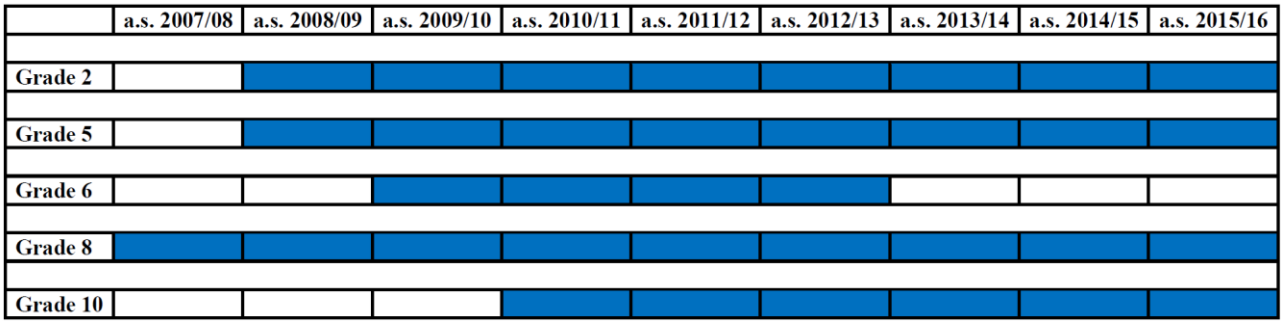


Figure 1: The design of INVALSI surveys from 2008 to present

SNV surveys aim at taking a snapshot of schooling as a whole: in other words, it is an evaluation of the effectiveness of education provided by Italian schools. The results of a national sample are

[1] https://invalsi-areaprove.cineca.it/.

annually reported[2] stratified by regions and disaggregated by gender, citizenship and regularity of schooling. These results are public, as well as the tests and the scoring guides. However, the results of each school are sent confidentially to the principal.

The mathematics items are connected to the National Guidelines for the Curriculum and to some teaching practices that have consolidated over the years. Another important reference is the UMI-CIIM curriculum "Mathematics for the citizen" (Anichini et al., 2004), which is based on results of mathematics education research and has deeply influenced the last formulation of the national curriculum. The SNV Framework defines what type of mathematics is assessed by the SNV tests and how it is evaluated. It identifies two dimensions along which the questions are built: the mathematical content, divided into four major areas (Numbers, Space and Figures, Relationships and Functions, Data and Forecasts); and the mathematical processes involved in solving the questions (Knowing, Problem solving, Arguing and proving). These dimensions are closely and explicitly related to the National Guidelines. The framework adopted by SNV assessment includes aspects of mathematical modelling as in PISA survey (Niss, 2015), and aspects of mathematics as a body of knowledge logically consistent and systematically structured, characterized by a strong cultural unity (Arzarello et al., 2015).

The SNV tests differ from PISA or TIMSS surveys not only for its frequency (annual vs. triennial), for the type of tested population (census vs. sample), and for the target population (grade-based vs. age-based students for PISA), but mainly for its goals: as a matter of fact the SNV tests results aim at providing to the Ministry a national benchmark for the assessment of the Italian students at different grades taking into account the national curriculum. In addition, each school and each class receives its own results according to the dimensions described above, compared with the national results and the results of 200 schools or classes with the same socio-economic background. Results are returned to the schools according to the two dimensions of the framework: content and mathematical processes as well as for each item (including the response rates for each option in the case of multiple-choice item).

The goals of the study

The aim of the present study is to illustrate the two approaches used to select mathematics items: mathematical content and its relevance in the teaching and learning of mathematics and the statistical analysis of the field trial results.

The construction and analysis of mathematics items

The preparation of the SNV items takes place in two steps. A first set of items is prepared by in-service teachers of all levels, subsequently, the SNV National Working Group builds the test by selecting items so that the test is balanced both from the point of view of contents and processes.

Since 2013, the item development has been carried out during a summer school held in July, involving about 200 in-service teachers. All item developers are teachers who teach at the school level for which they prepare the items. All items are classified according to the SNV framework

[2] https://invalsi-areaprove.cineca.it/index.php?form=precedenti_risultati.

(mathematical content, question intent, mathematical process involved and specific links with the National Guidelines).

Towards the end of the year in which the summer school is held, the items are revised and assembled in booklets. Five different working groups (group-level), one for each grade involved, prepare the field trial. The group's work consists of the analysis of the items and the selection of those deemed most suitable to be included in the booklets. Each group-level constructs at least two booklets, making sure that they are balanced from the point of view of both content and processes. Once the items have been tested in the field trial, the results are analyzed by the SNV National Working Group in order to prepare the final booklet for the main study. This National Working Group is formed by the coordinators of the level-groups, generally experienced teachers, researchers in mathematics education and statisticians. The revision of the booklet is supported by psychometric analysis, conducted according to CTT model (classical test theory) and the IRT model (item response theory – 1p Model) (Rasch, 1960, Hambleton, 1991).

If one item does not fit the parameters, we try to figure out what doesn't work. Sometimes the problem can be that the text is not entirely clear from a linguistic point of view, that some distractors are too attractive or not attractive enough, or the percentage of correct answers is too low, especially when the question requires an open constructed response. In this case, if the item is interesting from an educational point of view and it tests important skills, it is modified according to the analysis done and included in the next field trail. In some cases it is difficult to change the item while maintaining its significance from a mathematical point of view, so if the psychometric properties of the item are too weak we prefer to exclude the item from the final booklet.

Our research questions are:

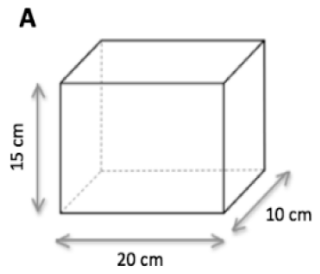
a. In which way item analysis, that is the statistical analysis of the psychometric properties of an item within a test, may support the progressive refinement of the item formulation?
b. How to balance the tension between producing a test with good psychometric properties and guaranteeing at the same time that it has appropriate or relevant mathematical content?

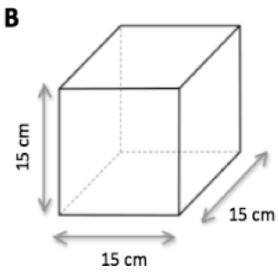
Qualitative and quantitative analysis: one example

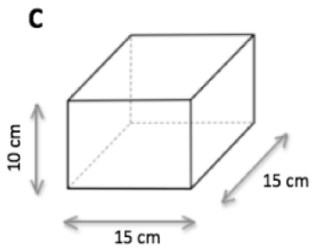
The example below shows the analysis performed on each mathematical item. Each item is analyzed from two different perspectives: the mathematical content and its relevance in the teaching and learning of mathematics and the statistical analysis of field trial results. In fact, an item may be robust in terms of statistical analysis, but not significant from the point of view of mathematical competence to be measured and vice versa an item could be very interesting from the aspect of mathematics education, but not suited to a standardized test. The challenge that we face is how to ensure the right balance between these two aspects.

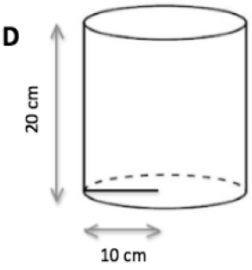
From the mathematical educational point of view, the item that we propose as an example in Figure 1, belongs to Space and Figures content. It doesn't require the calculation of the volume of a solid, which is a typical and traditional request at this educational level (8th grade), but the understanding of the relationship between a given quantity (1 litre) and the shape and dimensions of a container. The four figures have different sizes and shapes. Each response option is accompanied by a plausible justification.

One litre of water is poured in each of the container showed below. In which container will the water reach the maximum height?









A. ☐ In container A, because the base area is smaller

B. ☐ In container B, because its sides are all equal

C. ☐ In container C, because its volume is smaller

D. ☐ In container D, because its height is higher

Figure 1: Original item (before the field trial)

The item was administered to a random sample of 389 students, representative at the national level (see Table 1). The high number of respondents ensures the robustness of the results of the test. Analyses are conducted by considering the overall functioning of the item as well as the functioning of each option (4).

item:37 (A25-M8SF601)
Cases for this item 389 Item-Rest Cor. 0.11 Weighted MNSQ 1.09
Item Delta(s): 1.30

Label	Score	Count	% of tot	Pt Bis	t (p)	PV1Avg:1	PV1 SD:1
A	1.00	38	9.77	0.11	2.08(.038)	-0.94	1.11
B	0.00	44	11.31	-0.07	-1.46(.145)	-1.39	0.81
C	0.00	136	34.96	0.14	2.77(.006)	-1.13	0.78
D	0.00	144	37.02	-0.11	-2.27(.024)	-1.40	0.86
7	0.00	1	0.26	-0.06	-1.14(.254)	-1.92	0.00
9	0.00	26	6.68	-0.06	-1.27(.206)	-1.43	1.13

Table 1: The original item: Results from field trial

The item has some problems from the point of view of both the overall functioning and the functioning of individual distractors. A, B, C, D labels represent the different options, the code 7 represents invalid answers and the code 9 represents missing answers. Taking into account the correlation coefficient (in table 1, Item-Rest COR.= 0.11), we notice that the value is positive, but below what we consider the reference value (0.20). This means that the item, as was proposed in the field trial phase, is not able to discriminate students with different skill levels. The correct option (A) is chosen by a small number of respondents, (9.77%). The question, therefore, seems to have a high level of difficulty but, at the same time, it is not so discriminant. In fact, if we consider the correlation coefficient for the correct answer (A) the correlation is low (Pt Bis 0.11). The three distractors have problems too: the weakest is the third (C): besides being very attractive (34.96%) it

has a positive biserial-correlation index, even slightly higher than the correct option (Pt Bis = 0.14). This means that the students who choose this option have a higher level of ability, as measured by the whole test, than the students having selected the correct answer. This is confirmed by the IRT analysis shown on the plot of characteristic curves by category (see Figure 3).

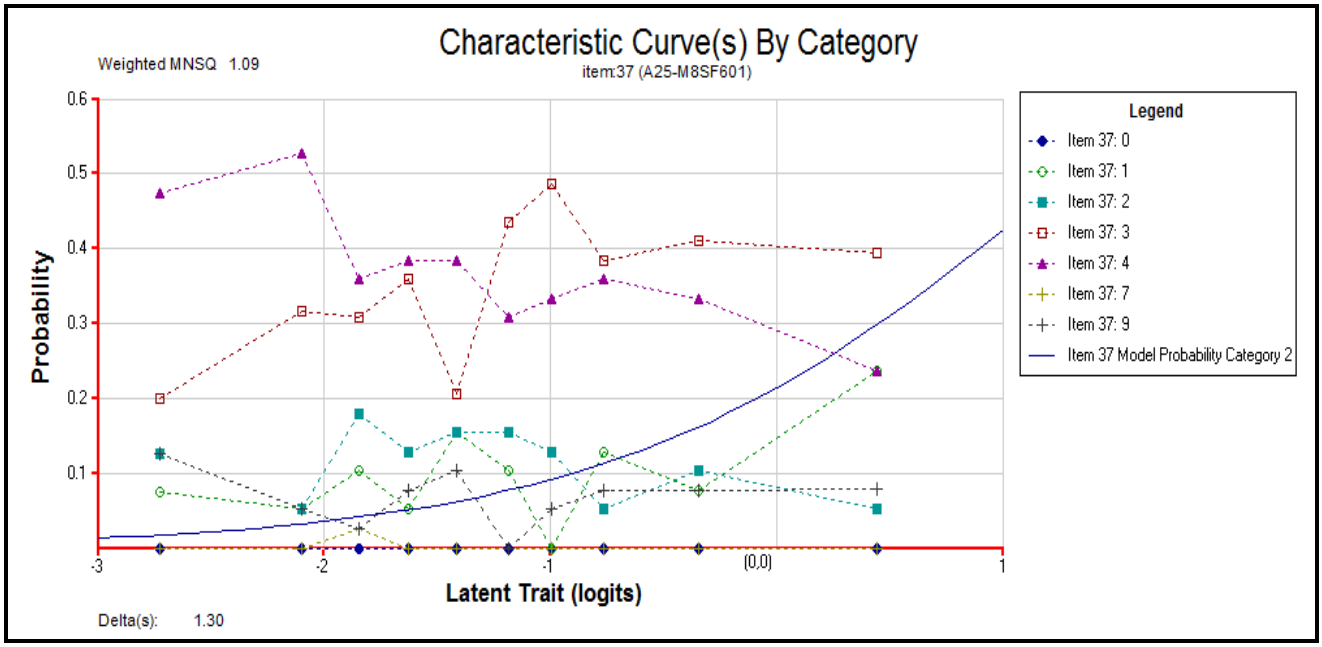


Figure 3: Characteristic curves by category

The item difficulty, estimated by Rasch scale, is 1.30 (Figure 3, Delta(s): 1.30). The IRT model confirms the difficulty of the item. The chart shows on the x-axis the estimated ability of respondents (in logit units) and on the y-axis the probability of answering correctly to the item. Therefore each curve represents the evolution of a response option, in terms of skill of students and chance to choose a certain option. The curves relative to observed data are then compared with theoretical trends that question should have, depending on the Rasch model (continue blue curve). This comparison reveals that observed behaviour of correct option (in legend, Item 37:1) is not fitting with the theoretical model. From the plot, we can see that the probability to choose the correct answer does not increase with the increase of the skill level. The distractors trends have some problems as well. In particular, the distractor C (Figure 3, Item 37:3) has a probability to be chosen which is higher than that to choose the correct answer, even for the highest level of ability.

The field trial item analysis results help us to make hypothesis about which mathematical aspects of this item don't work. The two more often selected options (C and D) contain both terms that probably confuse students (volume and height), and elements linked to students' cognitive difficulties related to the concept of volume, as shown by the relevant research (Vergnaud, 1983).

In option C, the word "volume" probably attracts high level students as they know that the item is about the volume. Probably they think that the lower the volume, the greater the height, because they do not grasp the right relationship between the measures of the base area and the supplied quantity (1 litre). Option D is attractive probably because the question is about the "maximum height" and this option is the only one in which the term "height" is mentioned. This item is interesting from the point of view of mathematical skills and it is closely related to the Mathematics

National Guidelines, but it does not seem to be clear for students. Probably its question intent is not well focused either. In fact, given the formulation of the item, we cannot understand the reason why students are unable to answer correctly: does it depend on the fact that students are not able to identify the right container or on the fact that they are not able to identify the right justification?

The question has therefore been modified for the Main Study test (2015) in order to remove its problematic elements (Figure 4).

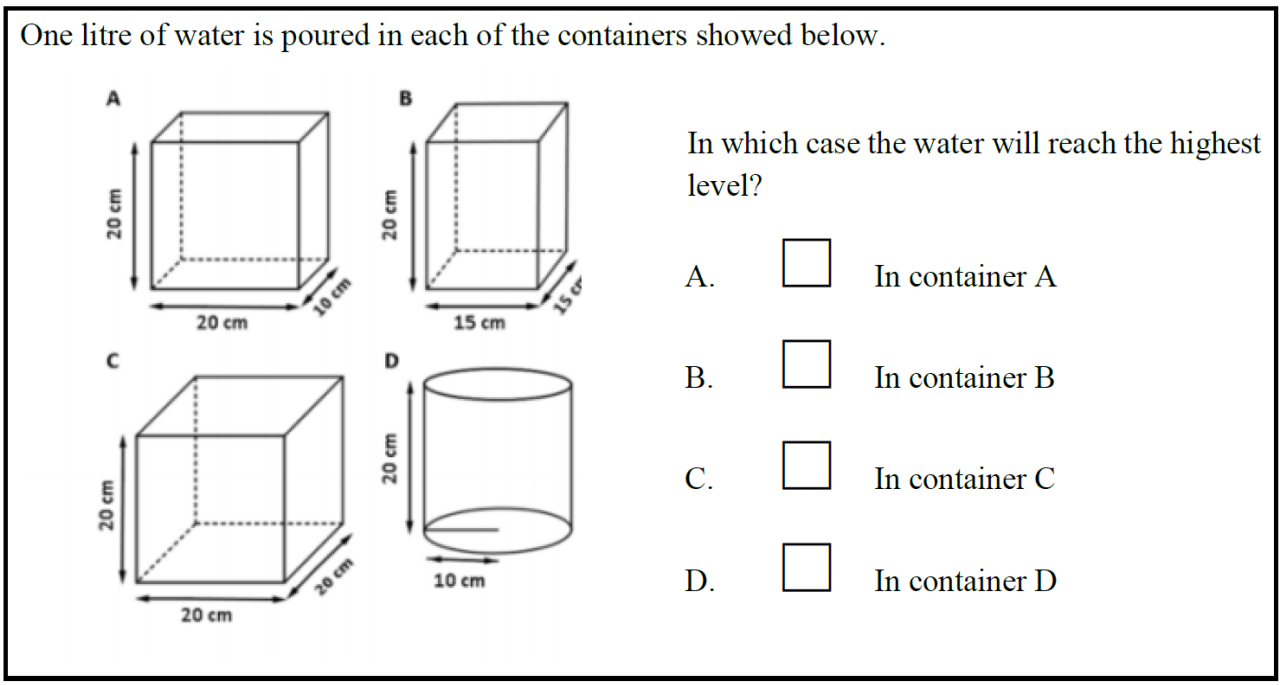


Figure 4: Modified item (after field trial)

In the new version of the item "the maximum height" is no more mentioned, because the question is about "the highest level". Moreover, all figures have the same height (20 cm), in order to help students to focus their attention on the shape of the containers and on the dimensions of the base.

Finally, the justification has been removed from the response options. The item question intent was partly modified, but it is more focused.

The fact that the figures have all the same height allows students to focus more on the size of the base of the containers rather than on the meaning of volume or on the relationship between volume (1 L) and the shape of the containers.

From the point of view of mathematics education the first version of the question was probably more interesting because it was more stimulating (the containers have different sizes and each option has different justifications). This type of question would be very suitable for a class discussion. For example, as part of work with the class, it would be interesting to isolate the crucial variable through a discussion in order to highlight the different ideas of the concept of volume. However, in a standardized national test, where the item is administered to all the Italian 8th grade students, it should be able to discriminate between skill levels, without losing in mathematical relevance.

Figure 5 show the characteristic curves of the modified item administered to the national sample (about 28500 students, with the presence in the classrooms of external observers).

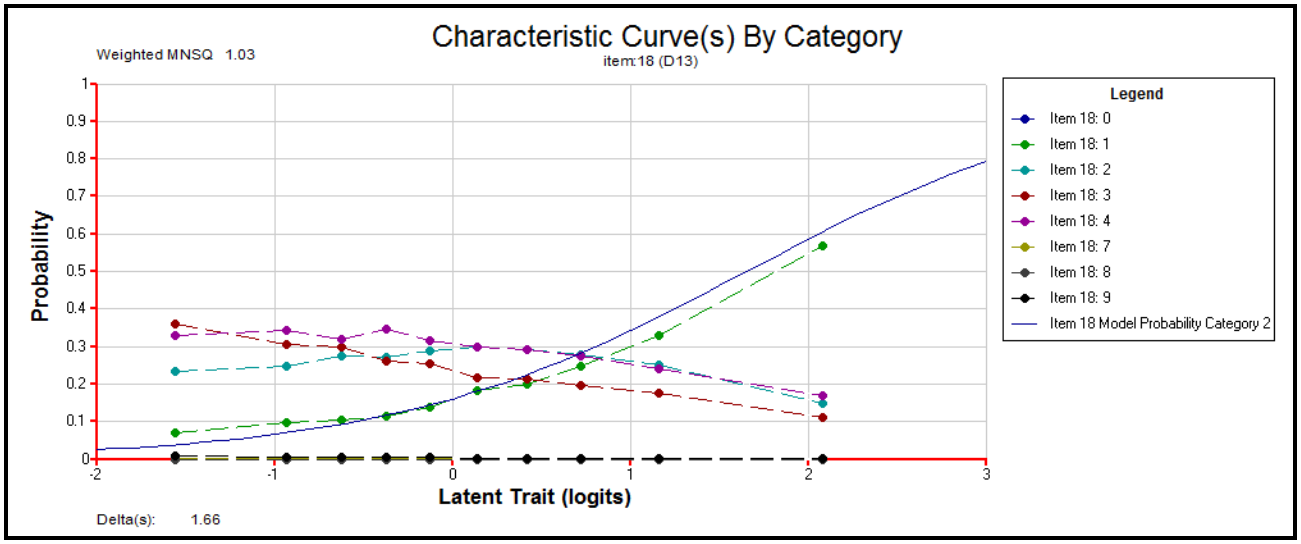


Figure 5: Characteristic curves by category

The item is still rather difficult (Item Delta = 1.66, and the percentage of correct answers 19.58%), but its discrimination has improved (Item-Rest Cor. = 0.25) as well as the index of FIT (Weighted MNSQ = 1.03). Only the option B seems to have some problems: its correlation coefficient, in fact, have a positive value (Table 2, Pt Bis= 0) even if this correlation is not significant (in table, sig. p = 0,952). The real change is observed on the curves of each response option (Figure 5). In the previous version of the item, the trend of the curves highlighted a number of issues about the different response options. Now we can see that the correct option (Label A; Item 18:1) follows the theoretical model (continuous blue curve) and the curves of the incorrect options drop with increasing student skills. Option B still remains problematic: you can notice from its curve on the graph that this option (in legend, Item 18: 2) has lower probability to be chosen, in comparison with the other distractors, by students of any skill level. Analysing the item from the mathematical content point of view you can see that option B involves an estimate of the base area which is slightly more complex (15x15) than the other options. On the other hand, the other two options C e D, (Figure 5, 18:3 e 18:4) work well: in fact they have higher probability to be chosen by students with low abilities and this probability decreases according to the increase of skill. As intended, the item so modified results to be better focused on a specific question intent.

Discussion

In this paper, we have described how we construct mathematics items for the Italian Assessment System and which aspects are taken into account in the selection of items for the main study. The debate between statisticians and researchers in mathematics education (as well as experienced teachers) of the National Working Group for the construction of the mathematics national test is always very lively. The challenge is to be able to maintain the right balance between these aspects, with the awareness that through a standardized test many mathematical skills cannot be measured. The choice is never between pretty or ugly questions, but between suitable or unsuitable questions in a standardized test aimed at all the students of the country. In this example we showed how statistical analysis has allowed us to improve the item from the linguistic point of view and better focus on the purpose of the item. In this way the mathematical aim of the item is saved while its

psychometric properties are improved. Over the past few years, in a few cases we have submitted items that were very interesting from a mathematical point of view, but unsuitable for a test of this kind as they didn't provide information about students' ability. However some of these questions would be more appropriate to be discussed in a classroom setting, proceeding by trial and error until arriving at a shared solution, rather than being included in a standardised test.

The challenge that we try to face is to be able to build interesting questions on math skills that are important to the teaching and learning of mathematics, but that provide a solid measure of students' mathematics skill levels, with the aim of providing useful information for teachers to enrich their work in the classroom.

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Using external assessments for improving assessment practice of primary school teachers: A first study and some methodological questions

Nadine Grapin[1] and Nathalie Sayac[2]

[1] Université Paris Est Créteil, Laboratoire de Didactique André Revuz (Universités Paris Diderot, Paris-Est Créteil, Artois, Cergy Pontoise et Rouen). France; nadine.grapin@u-pec.fr

[2] Université Paris Est Créteil, Laboratoire de Didactique André Revuz (Universités Paris Diderot, Paris-Est Créteil, Artois, Cergy Pontoise et Rouen). France; nathalie.sayac@u-pec.fr

A recent research (Sayac, 2016) has shown that assessments proposed by primary school teachers are mainly summative and not used to increase student knowledge. To further this work, we have decided to study teachers' assessment practices in mathematics, but also to improve them in the context of a collaborative environment. In such a project, as researchers and teacher educators, we share the same goal with the teachers involved: developing assessment for learning and helping students to learn better in mathematics. For studying this environment and for analysing the professional development of any actor, we use the Activity Theory framework (Engeström, 2001) but also the notions of "evaluative episode" and "professional judgement in assessment", developed by Sayac (in progress) for defining the didactic paradigm of assessment. In our paper, we present our methodology and focus on one aspect of this research: the use of external assessments as tools for improving assessment practices.

Keywords: Assessment practices, external and internal assessment, validity in assessment.

In France, recent modifications in curricula and institutional directions encourage teachers to assess competencies (and not only knowledge), but also to develop, in their classes, "assessment for learning": teachers should consider that assessment is a part of the didactic process and use assessment information for adjusting their teaching strategies. A recent study by Sayac (2016) with some primary school teachers, has showed that assessments in mathematics are mainly summative and that teachers use the results principally for the end of term report. Moreover, assessment tasks are not complex and previously studied before taking the test.

Furthermore, few assessment tools are made available to primary teachers. Until 2012, national diagnostic assessments were organized at the beginning of Grades 3 and 6 and teachers could use them for a diagnostic purpose, but these tests do not exist anymore. In the same time, the number of large scale assessments in primary school has increased in France, but items are not free and only results are published (Brun & Pastor, 2009, Dalibard & Pastor, 2014; Lescure & Pastor, 2008,). So, teachers cannot use items or results directly for their classes and educators can only exploit general results for providing an overview of trends in students' mathematical knowledge or difficulties.

Since 2016, a bank of exercises (with scoring procedure, explanation of the difficulties and propositions of teaching strategies) has been created for helping teachers to assess students at the beginning of Grade 3. It is a commendable initiative, but it raises a lot of questions about these exercises and their use: even if each task is relevant with regard to its assessment aim, teacher must select many of them to elaborate a complete test. How is this selection done and which competencies are finally assessed? Is scoring guide used or not by teachers? How? Etc.

Like other external assessment tools, this could improve teaching and assessment practices, but merely delivering assessment tasks seems not to be sufficient: for designing tests and using results for a better regulation in classes, we think that, in addition to assessment tools, teachers must have knowledge about mathematical notions, teaching them.

Finally, these observations about the bank of exercises have led us to conduct a study to analyze and improve teachers' assessment practice; we suppose that training teachers in assessment will also impact their teaching as a whole. We have chosen to conduct our research in a specific mathematical domain (whole numbers) at the beginning of the elementary school (Grade 1 to 3), in a collaborative environment. After specifying our research and training aims, we explain our theorical frameworks and methodology for studying this environment and the professional development of any actor and conclude with perspectives.

Research aims and context

This research aims to continuing to explore the assessment practices primary school teachers in mathematics but also to improve them as part of a collaborative research-training. It is undertake in a special network "AeDeP" at FIE (standing for Associated educational Design-experiment Places at French Institute for Education); this type of project is initially based on an educational question shared by different actors (teachers, researchers, school directors, local authorities) and is built for sharing experiences and designing common tools. Our two key issues are:

1. How do primary school teachers assess their students? We focus on test content (which type of tasks do they propose? What techniques are necessary to solve them? Etc.), but also on how teachers design their assessments (what kind of resources they use? What do they do with the results? Etc.)

2. How can such a collaborative research-training improve teachers' assessment practice, and more generally mathematics teaching?

We present in this paper one part of our three-year project, called "EvalNumC2" and we focus on the development of assessment practices in mathematics at primary school. For this part of the project, regular meetings are planned (one per month) with all the actors (ten primary school teachers and us, two educators/researchers) and with different aims depending on the timing of the research. At the beginning, researchers will only collect the tests produced by teachers (without training) and information about teachers' practice; after, external assessments and didactic tools designed by researchers will be introduced.

Theoretical framework

Activity theory for studying professional development in a collaborative environment

We consider that the Activity Theory expanded by Engeström (2001) is a good framework to study this kind of collaborative project of research and training through the objects and the tools used by the subjects, i.e. the teachers involved and us, as mathematics teacher educators/researchers (MTE-Rs). In this framework, the activity of teachers in which we are interested is their assessment practice and the activity of MTE-Rs is to explore and develop these assessment practices within the collaborative environment. We look at the activity of the different subjects in order to produce

results concerning professional development of each one, promoted through this collaborative device (Jaworski, 2006).

Among the tools used in the AeDeP, we have chosen to focus on two specific ones: a list of criterions for studying the validity of test items and tests designed by researchers. In the following we explain why.

Didactic paradigm of assessment

To study the assessment practices of teachers, we adopt the didactic paradigm of assessment developed by Sayac (in progress). In this framework, teachers' assessment practices are studied through the evaluative episodes they propose during the learning process, but also from the evaluative logic of teachers that becomes apparent in the design of the episodes (resources, method, nature of the tests provided), through their professional judgment in assessment and their grading practices.

A number of researchers draw on the notion of professional judgment when considering learning assessment by teachers (Klenowski & Gunn, 2010; Laveault, 2008; Wyatt-Smith, Morgan & Watson, 2002). For them, professional judgment includes both cognitive process and social practice (Mottier Lopez & Allal, 2008), which is not same as a "mechanical gesture of measurement" (Wyatt-Smith et al., 2010), but must be considered as a "flexible dynamic process comprised of middle and final judgments" (Tourmen, 2009). The professional judgment of teachers could be viewed as an act of discernment and as the ability to build intelligibility of the phenomenon of assessment, while taking into account the epistemic, technical, social, ethical and paradigmatic dimensions of classroom assessment practices (Tessaro, 2013). In the didactic paradigm of assessment, the professional judgment is considered as a kind of "didactic vigilance" (Pézard, 2010) specifically applied to the assessment activity of teachers. This allows them to on the one hand give a valid verdict (Chevallard, 1989) about students' mathematical knowledge, individually and collectively, from data collected during the different evaluative episodes. On the other hand this allows them to mutually articulate the different moments of the learning process (especially to connect evaluative episodes to the other moments of the learning process), based on data collected during the different evaluative episodes. This professional judgment in assessment is related to teachers' mathematical and didactical knowledge and assessment skills. It also depends on individual factors as beliefs on learning and assessment as well as professional and personal experiences on assessment (Brady & Bowd, 2006; Di Martino & Zan, 2011; Jong & Hodges, 2015).

Validity of tests

Researchers and teachers, and more generally, all assessment designers, have a same preoccupation about the test: they want to be sure that their tests assess what they should assess and only that. In previous work, we have described a methodology and listed didactic criterions for analysing the validity of an external assessment in mathematics (Grapin, 2015; 2016). We transfer and adapt these principles with two different aims: as researchers, for analyzing the content of internal tests designed by teachers (classroom assessment), and, as educators, for helping teachers to construct their own assessments.

For studying the validity of a test of a mathematical domain, we consider two levels: locally (exercise by exercise) and globally (the test as a whole). From a didactical point of view, the *a priori* analysis of each item is crucial because it gives indicators to guarantee that a task is relevant for achieving its assessment aim. For each item, we realize such an analysis specifying the tool or object aspect, the registers implicated with their possible congruence (Duval, 2006), the types of tasks involved in the resolution, the different techniques (adequate and inadequate through curricula) for solving the problem, the arithmetic problem classes (Vergnaud, 1996), the complexity levels (Sayac & Grapin, 2015). We also take into account the techniques involved in the resolution; if an item can be solved with a technique or a strategy different from the ones expected relatively to his assessment' aims, we consider the item as inconsistent.

For studying and ranking items according to their complexity, we have developed a tool (Sayac & Grapin, 2015) which takes into account three factors. In the first one, the wording and the task context are considered (what is difficult to understand the question?), in the second one, the mathematical knowledge involved in the solving process is studied, and finally in the third one, concerns the level of competency (is the task usual or not, does the student have to take initiative?). For each of these factors, we attribute a degree of complexity between 1 (simple) and 3 (complex). We also observed that discussions between teachers arise during the use of this tool because they do not have the same ideas about the complexity, depending on their teaching or their representations of mathematical notions (Sayac & Grapin 2013). So, this tool seems particularly appropriate to use with the teachers in our project.

On a global level, we study whether the items are representative for the curriculum: have all types of tasks been represented? What are the complexity levels (defined *a priori*) of the items? Are they different or similar? Which registers of representation are involved? When a same type of task is represented by different items, are the effective techniques similar? Etc.

Methodological elements and preliminary results

The Engeström triangle (Engeström, 2001) allows identifying, for each subject, objects that will evolve during the collaborative project of the research and training, through the mediation proposed via the tools, the rules and the division of labor and the communities.

For the teachers involved in the project, the main object is to assess their students. It comes to study specifically, from the collaborative environment, tests designed by researcher as one of the tools used. We will study how these tests could:

1- Enhance assessment tasks proposed by the teachers in terms of diversity, complexity and coverage of the mathematical domain.

2- Develop teacher's professional judgment through the study of students' answers to these tests and the confrontation between all the teachers during the meetings.

3- Work on the coding of students' answers and therefore, on grading.

Studying or using external tests could foster the teachers' professional development, because they will be validly designed from the epistemological and didactical point of view (Grapin & Grugeon,

2015). The evidence of validity will allow us to show how these tests could be relevant for the three points above.

We consider as Johnson, Severance, Penuel and Leary (2016) that:

> Professional development organized around the analysis of mathematical tasks has potential to prepare teachers for standards implementation by helping them develop common understandings of standards and how to help students meet ambitious new learning goals. (p. 173)

Therefore, we believe that the contribution of assessment tasks from external tests, in a collaborative context, could develop teachers' skills on assessment tasks design and contribute to enhance their professional judgment in evaluation (Gueudet, Pépin & Trouche, 2013).

Methodology

At the start of the school year, we collected tests designed by teachers involved in our project in order to analyze the assessment tasks with the tool developed in previous research (Sayac & Grapin 2015). We will also collect new tests designed by the teachers at the end of the school year, after the collaborative group work in the meetings. Each assessment will be analyzed in terms of its validity with the criterions listed above; we will principally observe the change in content between the beginning and the end of the project (variety of type of tasks, complexity of tasks and the coding of students' answers).

Each teacher will also fill in a questionnaire about his or her assessment practices (How does he or she design tests? Which resources does he or she use? What are the periods of assessment in his/her classes? How does he or she use the results?). The results of these questionnaires will be used to compare teachers but also as an element for analysing the evolution of their practice.

Lastly, for relying assessment and teaching in classes, each teacher involved in the project will have to keep a "daily book" (journal) in which he or she explains briefly the aim and the content of each mathematical course (in the numerical domain). He or she will also have to identify, according to his/her own representations, the evaluative episodes and describe these more specifically. We will analyze the content of the tasks proposed in tests and during teaching to study their correlations and their evolution during the project. As observed by Grugeon and Bedja (2016), we suppose that training teachers in assessment will also improve teaching: teachers should propose a wider variety of types of tasks, but they also should be able to have a better interpretation of students' errors and propose adapted instruction to upgrade students' level of understanding.

Preliminary results

At the time of writing, we cannot present full results because this project started in September 2016 and we are in the process of collecting first data; we show however two example of tasks, extracted from the same test, one of the teachers in our study used in Grade 3.

In a first task (Figure 1), the five questions are similar and aim to assess the decomposition of written numbers in canonical expressions. In all examples, the underlying structure is regular. We observe that such a task is not complex (we quote level 1 on each factor of complexity) and assesses five times the same knowledge: only the positional aspect of numeration (and not the decimal aspect).

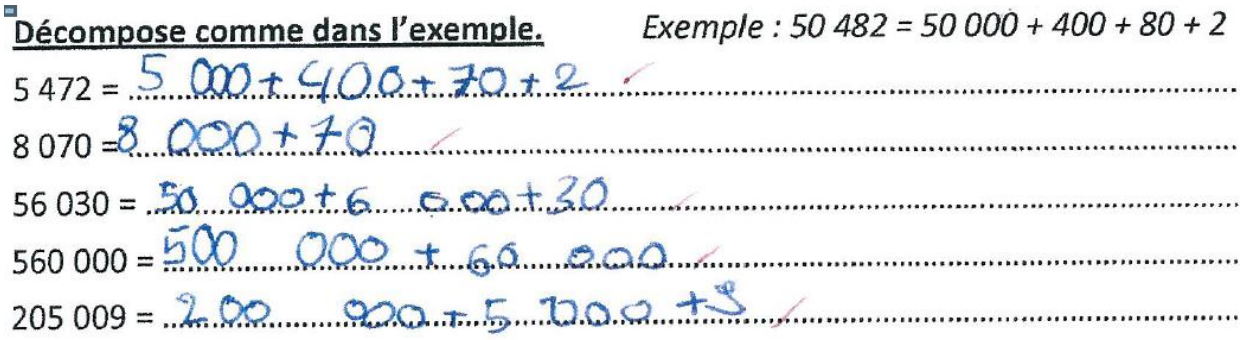

Figure 1: First exercise and students' answer extracted from a classroom assessment designed by a Grade 3 teacher.

The second exercise of the test (Figure 2) looks like the first one and assesses the same type of knowledge.

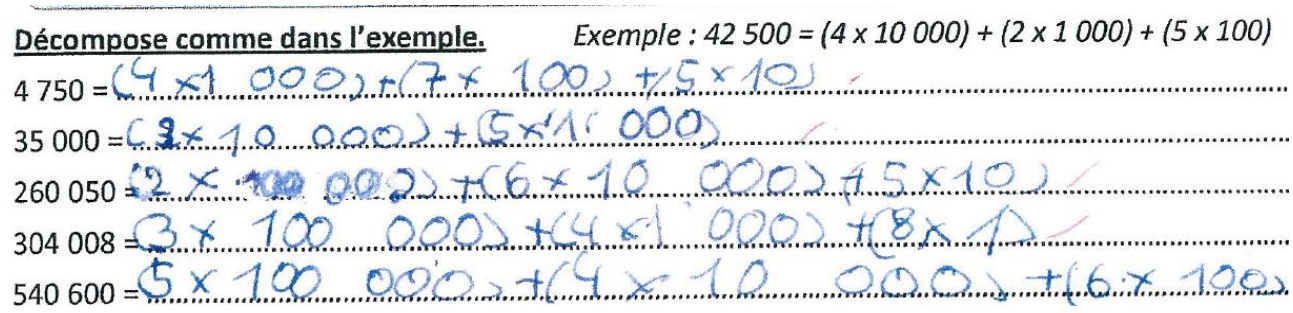

Figure 2: Second exercise and students' answer extracted from a classroom assessment designed by a Grade 3 teachers.

Throughout the full test, there isn't any exercise for assessing the decimal aspect of numeration. We conclude that such a test is not valid and in the project we are going to elaborate other questions with the teachers to fill this lack.

We have not yet achieved the analysis of all classroom assessments designed by teachers involved in the project, but it seems that, as we can observe in the two previous examples, tasks are repetitive, having low complexity and for the numeration, assess principally positional aspect of the numeration. Such observations led us to propose external assessments with other types of tasks: exercises designed to assess numerical aspect of the numeration but also complex situations intended to develop students' abilities.

Conclusion and perspectives

We have focused in the paper on the original and theorical notion of "didactic paradigm" designed recently by Sayac (in progress). What we can tell currently, from the first data collected (tests, questionnaires, interviews) is that the teachers, participating in our research, design tests with low levels of complexity and have invested very little in assessment as a professional gesture. They assess their students as they can, with very subjective practices. So, it seems that our research, with its training dimension, will make possible a real professional development concerning the assessment tasks proposed in mathematics by these teachers and their professional judgment in assessment.

So, besides studying and providing primary school assessment practice in mathematics and designing assessment tools, our research aims also to develop these theorical elements. At the end of the project, it will be possible to show the impact on the teachers' practice and interests, and the limitations of this methodology, according to our theorical framework. It would then be possible to extend such studies in other mathematical domains (as geometry) or in other levels, for example at secondary school.

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Using assessment for learning to enhance mathematics education in the primary school: Irish students' perspectives.

Ann Marie Gurhy

Dublin City University, Dublin, Ireland; amgurhy@eircom.net

Previous research suggests that assessment for learning (AfL), when used effectively, can greatly enhance student learning and achievement. However, students' views regarding AfL are often overlooked. This paper reports on one part of a broader study that investigated the interplay between lesson study, continuing professional development in AfL and mathematics teaching and learning. The paper explores the views of students in one Irish Primary School regarding their use of AfL practices during mathematics lessons over the course of one academic year. Evidence suggests the use of AfL strategies and techniques enhanced children's mathematical confidence, and improved their engagement with, and attitudes to, mathematics. By the end of the intervention, children readily used the language of AfL, engaged in AfL practices and played a more active role in their own learning and assessment, identifying the use of self-and peer-assessment as a highlight.

Keywords: Assessment for learning, mathematics, student voice, student engagement.

Background and focus of the project

Assessment, as argued by Gardner (2012), is a "hot topic" across the entire education spectrum and rarely out of the limelight (p.103). In particular, assessment for learning, with its emphasis on learning as opposed to measurement has in recent years, according to Chappuis (2014), "garnered the lion's share of assessment attention and established a pretty good name for itself" (p.21). This paper is about assessment for learning (AfL) in mathematics which is conceptualised using the following *second generation* definition generated by the Third International Conference on Assessment for Learning in New Zealand in 2009 which states:

> Assessment for Learning is part of everyday practice by students, teachers and peers that seeks, reflects upon and responds to information from dialogue, demonstration and observation in ways that enhance ongoing learning. (Klenowski, p.264)

This definition clearly captures the key tenets of AfL, foregrounds classroom practices, highlights the notion of AfL as a bridge between teaching and learning (Wiliam, 2011), and views teachers and students as the primary agents of educational change (Lysaght & O'Leary, 2013). Research evidence suggests that focusing on the use of day-to-day AfL is one of the most powerful ways of improving learning in mathematics classrooms and can result in significant learning gains (Wiliam, 2007). Additionally, various studies have also linked AfL to increased student motivation and self-esteem (e.g., Clarke, 2008), enhanced self-regulated learning and metacognitive abilities (e.g., Andrade, 2013), and better student-teacher relationships (e.g., Clarke, 2014). In the Irish context, government policy emphasises the centrality of AfL in teaching and learning although few teachers have received assessment-related continuing professional development (CPD). The Department of Education and Skills (DES, 2011a) has highlighted that AfL is not used sufficiently widely in Irish schools and

concerns have also been raised about teacher assessment literacy. Regarding mathematics, data from the 2009 National Assessments of Mathematics and English Reading (DES, 2010b), school inspections (DES, 2010a) and international reports (PISA, 2009) have suggested Irish students are underperforming. Indeed, Hislop (2013), chief inspector with the DES, argues that it was Ireland's poor performance in PISA 2009 that precipitated publication by the Irish government of a strategy aimed at improving standards of literacy and numeracy in Ireland: *Literacy and Numeracy for Learning and Life; The National Strategy to Improve Literacy and Numeracy for Children and Young People 2011-2020* (DES, 2011a). This strategy is one of the most significant documents pertaining to education in the Irish context in recent years, and it is especially pertinent to this research since it has particular implications for numeracy, assessment and CPD. The *Literacy and Numeracy Strategy* increased the amount of time allocated to literacy and numeracy at all class levels, and set out *ambitious* improvement targets in English and mathematics as measured on standardised tests to be achieved by 2020. Compulsory standardised testing in English and mathematics changed from two to three points in the primary cycle (second, fourth and sixth classes), with mandatory annual reporting of aggregated results to the DES to facilitate collation of a national picture of achievement. Additionally, schools must use these results as part of "robust self-evaluation" (p.40) and to prepare three-year improvement plans for the promotion and improvement of numeracy and literacy. Results also have to be given to Boards of Management and parents. Nevertheless, while various scholars (e.g., Leahy & Wiliam, 2012) agree that AfL, when used effectively, is a warranted strategy that can improve student learning and achievement, and Irish government policy emphasises the importance of using AfL in teaching and learning, apart from some small-scale studies (e.g., Lysaght, 2009), research into AfL in the Irish context remains sparse, particularly in the area of mathematics. Furthermore, there has been little or no research investigating what is going on in the hearts and minds of learners while engaging in AfL practices. This study seeks to address this gap in the field by specifically investigating the following research question presented as a hypothesis:

> *The use of AfL strategies and techniques, and the adoption of AfL principles, would enhance children's mathematical confidence, and improve their engagement with, and attitudes to, mathematics.*

Methodology

This is a practitioner action research mixed methods explorative case study, which operated within the pragmatic paradigm. Over the course of one academic year, it investigated the impact of AfL practices on the teaching and learning of mathematics at fourth-class level in one primary school in the Republic of Ireland. Specifically, it explored how the use of AfL principles, strategies and techniques affected students' attainment on standardised mathematics tests and their dispositions towards mathematics. Additionally, the research investigated the potential of lesson study (LS) as a vehicle of collaborative professional learning in AfL and considered the impact engaging in LS had on teachers' skills, knowledge, and use of AfL, and their beliefs towards AfL as a form of assessment. A key part of this study was the use of peer-to-peer learning as a vehicle of continuing professional development (CPD). Meeting after school on twenty-three occasions over the course of the intervention the teachers learned about AfL strategies and techniques on a phased basis before implementing them in their mathematics lessons prior to the next meeting. The AfL strategies used were as follows: learning intentions and success criteria; questioning and classroom discussion;

feedback; self- and peer-assessment. Additionally the teachers learned about and implemented more than twenty AfL techniques, for example, rubrics, think-pair-share, two stars and a wish, fist-to-five, learning logs, ABCD cards and comment-only marking (For further details see Wiliam, 2011).

Site selection and research participants

This research project took place exclusively in the school where I teach, Scoil na nAingeal (pseudonym), from September 2012 to June 2013. It is a vertical, urban, girls-only Primary School in the Republic of Ireland with an enrolment of 438 students and an all-female staff. The profile participants included all students enrolled in fourth class for the academic year 2012-2013, 51 girls, the average age of whom was ten years in September 2012. Classes were pre-formed, intact groups and so, according to Creswell (2009), random sampling was not considered appropriate. Three teachers participated in the project, the two fourth-class teachers and one member of the Special Educational Needs (SEN) team.

Data collection and analysis

Both quantitative and qualitative data were collected to decide whether to accept or reject the above research hypothesis. This aided triangulation, enhanced the study's findings and enabled better understanding of the research problem. The quantitative data were collected using one scale of an instrument developed by the researcher called the Attitude to Mathematics Questionnaire (ATMQ). This scale was labeled the ATMQ-TIMSS since it used statements 8a-8h of the Trends in International Mathematics and Science Study (TIMSS) 2007 Grade Four Student questionnaire verbatim. These statements examined "students' general attitudes towards mathematics" and "their self-confidence in learning mathematics" (Mullis, Martin & Foy, 2008, p.173). Leaving the TIMSS questionnaire unchanged facilitated comparative analysis with national and international data, and ensured the reliability of this scale (median reliability coefficients across all TIMSS countries at fourth grade was 0.83, Mullis et al., 2008, p.401). Fifty children completed the ATMQ-TIMSS pre- and post-intervention. Scoring was done using a four-point Likert scale, with response options ranging from '*Agree a lot*' to '*Disagree a lot'*. Raw data from the ATMQ-TIMSS were coded or categorised, recorded and prepared in Microsoft Excel and then imported into the software package, SPSS 21, where dependent (paired) samples t-tests were conducted in order to compare the scores of the same participants at Time 1 and Time 2, to ascertain whether or not the intervention had an impact (Pallant, 2013).

The qualitative data comprised transcripts and video from focus groups (FG) interviews, teachers' learning logs (TLL), students' learning logs (LL), and the researcher's journal. In addition to aiding triangulation, these data supplemented the quantitative data by facilitating a more in-depth analysis of students' views about using AfL strategies and techniques in their learning of mathematics. Braun and Clarke's (2006) six-step approach to thematic analysis was used to guide analysis of each data item individually and subsequently the complete qualitative data set.

Results and discussion

Quantitative data

Results from analysis of the ATMQ-TIMSS indicated that there was a statistically significant difference between pre-test (M = 2.03, SD = 0.73) and post-test scores (M = 1.56, SD = 0.42; t (49)=

5.09, $p < .001$), i.e. post-test scores indicated more positive attitudes towards mathematics (mean values in post-test scores were closer to 1 = *Agree a lot*). The magnitude of the difference between the pre- and post-test means can be interpreted as being large (eta squared = .35). Table 1 presents combined '*agree'* percentages for pre- and post-results for each statement in the ATMQ-TIMSS scale.

	Statement	Subscale	n	Combined % Agreeing PRE-TEST	Combined % Agreeing POST-TEST
a	I usually do well in Maths	SCM	50	76	96
b	I would like to do more Maths in school	*	50	60	70
c	*Maths is harder for me than for most other students in my class*	SCM	50	28	14
d	I enjoy learning Maths	PATM	50	80	98
e	*I am not good at Maths*	SCM	50	26	6
f	I learn things quickly in Maths	SCM	50	58	70
g	*Maths is boring*	PATM	50	26	6
h	I like Maths	PATM	50	80	100

Note. % Agreeing = Agree a lot +Agree a little

Italicised text highlights statements that were recoded.

For discussion purposes, percentage scores for the three PATM (Positive attitude towards mathematics) statements and the four SCM (Self-confidence in learning mathematics) statements were amalgamated to provide a composite percentage score for these scales which could then be compared with TIMSS data.

*Regarding statement b, Clerkin (personal communciation, April 15, 2015) suggests it was probably originally intended to be part of the PATM scale but following factor analysis was found not to represent positive affect in the same way as the other scale items and so was excluded.

Table 1: ATMQ-TIMSS Scale

A more detailed exploration of the eight statements from the scale revealed that following the intervention the combined '*agree'* percentages scores for the five positively-worded statements (a, b, d, f, h) had increased by between 12 and 20 percentage points while the percentage of students agreeing with the three negatively-worded statements (c, e, g) had decreased by between 14 and 20 percentage points. This suggests that children believed the intervention had positively impacted their general attitudes towards mathematics and their self-confidence in learning mathematics. Specifically, regarding the three statements which measured students' general affect towards mathematics (d, g, h), results indicated that following the intervention almost 100% of the participants agreed a little or a lot with these three statements (Table 1).

Qualitative data

Five main themes were identified following the qualitative analysis but due to space limitation these are discussed only briefly here.

Enjoying the AfL journey

When describing their experience of using AfL practices in mathematics, the children regularly used words such as 'fun' and 'enjoyment'. Their enthusiasm regarding using AfL can be succinctly summarised by Maria's comment: "I love doing AfL and I would like to continue doing it" (LL, 13/02/2013).

Growing positivity and self-confidence in mathematics

It is clear from the data that at the end of the study students reported a growing positivity towards and increased self-confidence in mathematics that they attributed to their use of AfL. Some believed that using AfL practices made mathematics easier and increased their liking of mathematics: "It makes maths so much more fun, 'cause like in third class I used to hate maths and then like now that there's all these different strategies, it just makes maths so much easier" (Sophie, LL, n.d.).

A changed classroom dynamic

Scholars suggest (e.g., Hayward, 2012), engaging in AfL practices in the *spirit* in which it is meant impacts learning and teaching and acts as a leverage for change in classroom practices, and roles and relations. Children in this study, while recognising the teacher as overall guide and arbiter in the classroom, also identified teachers as learners. Additionally, they were beginning to monitor their own learning and to evaluate their progress. For example, Chloe commented, "I think it's helped me, that it's not letting the teacher correct all your work, that you, you kind of have to check it, and you have to, because there's some silly mistakes that you could make" (FG1). Others displayed an increased confidence in their ability to assess, both themselves and others: "I think that when you correct it yourself, or for your friend, you know what you're correcting, you know why you're correcting, what you did wrong, so you know what you're doing" (Emma, FG1). This suggests that through the process of engaging in AfL strategies and techniques over the course of one academic year, these students were becoming more independent learners, accepting the responsibility this brought, and were moving towards self-regulated learning.

Peer- and self-assessment: - a highlight for children

Using peer- and self-assessment was undoubtedly a highlight for children participating in this study as this comment illustrates: "I love self-assessment and peer-assessment" (Hollie, LL, n.d.). The following comment from Ruby demonstrates that by the end of the intervention the children had developed a good understanding and appreciation of peer- and self-assessment:

> I thought the self-assessment was excellent because we were judging ourselves and could learn from our mistakes. Peer-assessment was brilliant for your partner or pair could judge your work and spot mistakes that you might not have spotted yourself. (Ruby, LL, 12/06/2014)

The children believed that self- and peer-assessment not only enhanced their learning but that it was also fun. Similar to research by Topping (2010), trusting your peers was mentioned by a number of participants as integral to good peer-assessment practice. For example, Hollie remarked: "I think peer-

assessment is the best because you get to like trust your friends more, so they'll be more honest with you in the future" (FG1).

Unexpected insights

The qualitative data also revealed some unexpected insights about students' perspectives regarding AfL. The children revealed they used the learning intentions and success criteria from their mathematics lessons to help them when doing their mathematics homework. Additionally, they described how they appropriated the AfL strategies and techniques to help them learn in other subject areas. Many children reported they liked using rubrics since they scaffolded the assessment process. Maria wrote: "My favourite thing about the AfL was using the rubric" (LL, 12/06/2013) and later explained why:

> I like using the rubric because when we were first going to do peer-assessment I was like 'oh God, what will I say was wrong?' and 'I don't know what to do here', but then you showed us the rubric, and I was like 'oh, it's ok', ... with the rubric, it tells you what you're supposed to assess, (FG2)

Conclusions

While acknowledging the limitations of this study, such as the fact that it takes place in a single school and employs a case study strategy with convenience rather than non-probability sampling, it nevertheless contributes new insights regarding AfL and mathematics, especially from students' perspectives. It demonstrates that children in primary school are capable of engaging in AfL practices, including peer- and self-assessment, and want to do so. Findings suggest that using AfL practices enhanced children's mathematical confidence, and improved their engagement with, and attitudes to, mathematics.

This study is important since, to date, little empirical research has been done into the effects of AfL practices on students' mathematics learning in the Irish context at primary level. Additionally, this study can supplement research done by academics regarding AfL since it provides a practitioner researcher's perspective of the field, thus "inside-outside" (Cochran-Smith & Lytle, 1993). Notwithstanding, perhaps the most significant contribution made by this research is that it provides a unique opportunity to listen to, and contemplate, the voice of young learners as they discuss their experiences of using AfL practices in their mathematics learning.

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Assessment of students' thinking when working with graphs of functions – Promoting pre-service teachers' diagnostic competence

Rita Hofmann[1] and Jürgen Roth[2]

[1]University of Koblenz-Landau, Campus Landau, Germany; hofmannr@uni-landau.de

[2] University of Koblenz-Landau, Campus Landau, Germany; roth@uni-landau.de

Diagnostic competences are essential for teacher actions; however pre-service teachers often do not have the opportunity to train these skills at university. Thus, there is a need to find out the best way to promote diagnostic competences in teacher training. During the last decades, several projects introduced videos as a tool for the training of diagnostic skills, but there is no evidence that pre-service teachers really acquire diagnostic skills better by analysing videos than by analysing tasks. The present study contributes to this growing area of research by exploring which one of these two methods promotes diagnostic skills better. Video analysis and task analysis are compared as training methods in an intervention study with a pretest-posttest design. Fostering preservice teachers' diagnostic skills with focus on students' abilities, problems and misconceptions with graphs of functions, is the specific objective of our study.

Keywords: Pre-service teacher training, diagnostic competence, formative assessment, graphs of functions, video vignettes.

Theoretical background

Good lessons require a lot of competencies on the teacher's side. Diagnostic skills, in particular, are an important part of teachers' professional knowledge and competence (e.g. Baumert & Kunter, 2006). Weinert (2000) regards diagnostic competence as one out of four basic and essential competences of teachers. Having good diagnostic skills enables a teacher to differentiate and individualise amongst learners – an ability that becomes increasingly important in today´s classrooms, "[…] because lessons can no longer be planned completely in advance, and teachers have to make many decisions in the midst of instruction about how to proceed" (van Es & Sherin, 2002, p. 574). Good lessons require teacher actions that are adapted to the students' needs and abilities and, therefore, are based on diagnostic information (Klug et al., 2013; Schrader & Helmke, 2001). For the adaptation of teacher actions to pupils' needs during a lesson, relevant information needs to be obtained during the students' whole learning process. Getting an insight into the students´ abilities only through the results of a final exam, is often too late to work on the students' problems and misconceptions. For this reason, our focus lies on diagnostics which take place in the learning process of the students where the teacher is still able to guide and influence the learners and their learning process. In the following, an overview of different aspects concerning diagnostics and assessment will be given.

Diagnostic competence

The term "diagnostic competence" is often used in the literature, but there is no agreement on a definition of this expression. A wide-spread definition would be that diagnostic competence involves all the abilities of an evaluator enabling him to correctly asses other people (Schrader, 2010). Artelt and Gräsel (2009) understand diagnostic competence as the teachers' competence to

evaluate the characteristic traits of their students in an adequate way and to suitably assess the demands of learning and of the tasks. Except of these two, various other definitions are used. Diagnostic competence is often described as "accuracy" in teachers' judgements – mostly in correlation with standardized tests – and therefore concerns the students' achievement in tests (Klug et al., 2013). Other definitions refer to the learning process of the students itself. In this regard Weinert (2000) defines diagnostic competences as:

> […] an amount of abilities to continuously asses during lesson the state of knowledge, the learning progresses and the performance issues of the individual students as well as the difficulties of different learning-tasks, so that the teaching actions can be based on diagnostic insights. (Weinert, 2000, p.16, own translation)

All of these definitions have in common that diagnostic skills are presented as the tool allowing the teachers to gain information about the learners. This information can be used for different pedagogical decisions like grading and lesson planning (makroadaptations), but also for short-termed interventions during lessons (mikroadaptations) (Schrader, 2013). As our study does not focus on achievements in tests but on the learning process of the students, we refer to the definition of Weinert. Moreover, there are different facets of diagnostic competences (e.g. Praetorius, Lipowsky, & Karst, 2012), so that we prefer to use the term diagnostic skills, as we focus on specific parts of it: the analysis of tasks and the analysis of video sequences – both with regard to abilities, problems and misconceptions of students working on tasks with the content functional relationships.

Formative assessment vs. summative assessment

The terms formative and summative assessment are quite similar to the foregoing described diagnostic competence. Again, there is no common and widely accepted definition although they are widespread in the international literature (Black & Wiliam, 1998). While summative assessment corresponds to the evaluation of students' academic achievements, formative assessment can be equated to diagnoses during learning processes. According to Bell and Cowie (2001b, p. 538), such diagnoses during learning processes "could include continuous summative assessment", which is why the authors "explored formative assessment as classroom assessment to improve learning (and teaching) during the learning". Bell and Cowie (ibid.) distinguish between planned formative assessment and interactive formative assessment. The former describes an assessment activity which is planned in advance, the latter includes assessments that arise out of learning activities during the lesson (Bell & Cowie, 2001a). The purpose of interactive formative assessment is to help the students by accompanying the learning process (Bell & Cowie, 2001a). According to Bell and Cowie (2001a, p. 86) this process involves three parts: noticing, recognizing and responding. Noticing in this context means to gather information about the patterns of thought and actions of the students. This information is gathered while the pupils are working or talking. Thus, this interpretation differs from the term "noticing" described by van Es and Sherin (2002). In contrast to the meaning of "noticing" characterized by van Es and Sherin, which already includes the identification of important aspects of a teaching scenario, Bell and Cowie (2001a) regard the recognition of relevant interactions and moments as a second step. "Recognising may be differentiated from noticing in that it is possible to observe and note what a student does without appreciating its significance" (Bell & Cowie, 2001a, p. 88). Consideration of "responding" as one of

the stages of interactive formative assessment shows, that the noticing or assessment should not stand alone – the following action of the teacher is indispensable.

In this sense, formative assessment involves diagnostic as well as didactical competencies - action competence, respectively. Diagnosis/ noticing and the action which follows up the diagnosis are both parts of formative assessment. To sum up, the subject of our study is diagnostic competence according to Weinert (2000) and the following teacher action. Thus, the regarded skills manifest themselves in the three stages of interactive formative assessment: noticing, recognizing, and responding (Bell and Cowie, 2001a).

Graphs of functions

The focus of the diagnosis in our project is on the students' learning processes while working with graphs of functions. The interpretation and construction of graphs of functions are essential skills - not only in mathematics education. The ability to use different (external) representations is an important issue here. It is one of the six mathematical competences mentioned in the German educational standards for mathematics and also influences two of the remaining standards (KMK, 2004). Moreover, the use of graphs of functions is essential for the topic "functional relationships", being one of five central topics of mathematics education (KMK, 2004). In addition to that, the abundance of graphs of functions in our everyday life (e.g. functional relationships or graphical representations of data) makes them indispensable in teaching and learning. Nevertheless, previous research has shown that dealing with graphs of functions can be difficult and easily leads to misconceptions. In the literature one can find a lot of those mistakes and misconceptions (e.g. Nitsch, 2015; Leinhardt et al., 1990; Clement, 1985; Bell & Janvier, 1981), like the graph-as-picture misconception, the slope-height confusion or the interval-point confusion. Moreover, what the students think a function is or how a graph of a function should look like (concept image) does not always correspond to the definition of a function, the students have in mind (concept definition) (Tall & Vinner, 1981).

However, not all of these mistakes and misconceptions are visible on the surface but they need to be uncovered in time. Otherwise, there is the danger of a consolidation of wrong thinking making it very hard to work against them (Nitsch, 2015). In this case, wrong conceptions might still be present when students leave school or even when they enrol at university. Teachers need to be able to diagnose students´ misconceptions and difficulties in time in order to foster their correct use of graphs of functions.

Giving effective feedback is a crucial aspect of teacher-learner interactions (Hattie, 20120), but often there is a lack of time for reflection and decisions on necessary actions to be taken (Black & William, 2009). The perception and processing of crucial situations often takes place intuitively – "on the fly" – when the teacher is monitoring the classroom and listening to student conversations while students are working with their partners or in groups. This is a highly demanding situation for teachers (William & Thompson, 2007). Consequently, in the beginning of teaching, teachers can experience an overloading by the wealth of information. Thus, the skills to notice, recognize and respond should already be fostered during preservice teacher training. A common way to train diagnostic skills is the analysis of tasks as it can easily be embedded in university teacher training. Thereby the university students reflect the skills which are needed to solve a task as well as

problems which can occur with the task. This method focuses on skills which are primarily necessary in lesson planning. No influence of task analysis on teachers´ diagnostic skills could be found yet. It could be assumed, that a good analysis of tasks helps a person to notice things – which are expected through the analysis – in reality. Nonetheless the analysis of gestures is not part of this method and can still be a difficulty for beginning teachers. Furthermore, noticing in a situation is more complex and can be cognitive overwhelming. Therefore, another approach to train such diagnostic skills is the use of videos as part of the training of diagnostic competences, as videos are very close to reality (compare Janík et al., 2009).

Up to now, several studies have shown that pre-service teachers often do not have the opportunity to train their diagnostic skills so that these competences are only poorly developed (Ostermann et al., 2015; Praetorius, Lipowsky, & Karst, 2012). For this reason, we want to foster these skills already during the university teacher training.

Research Question

The goal of our research is to enhance pre-service teachers´ diagnostic skills through experimental settings at university. As mentioned before, there are different aspects of diagnostic skills, all important for professional teaching. On the one hand, a teacher should be able to identify possible difficulties of a task and be aware of the skills needed for solving the task. On the other hand, the teacher needs to be able to identify the concrete difficulties and misconceptions an individual student has and to react appropriately. The analysing of tasks is one common way to train diagnostic skills of pre-service teachers. During the last decades videos were introduced as training tool for diagnostic skills as well. Looking at the two approaches to the training of diagnostic skills, several questions arise that need to be answered:

1. How does the training of task analysis influence the skills for analysing learning situations?
2. How does the training of analysing videos influence task-analytical skills?

Furthermore, as the diagnoses should be the basis for teacher action, the impact of both trainings with regard to this issue is another interesting part of the investigation:

3. Which intervention results in a noticeable improvement of the actions following the diagnoses?

Method

In order to verify the effects of the different trainings on the preservice teachers' diagnostic skills we will conduct an intervention study using a pre-posttest design. Thus, it will be a setting with two experimental groups: Experimental group one (EG1) will practice diagnostic skills by analysing videos, experimental group two (EG2) by analysing the tasks the students work on (Figure 1). The preliminary study will be conducted in winter term 2016/2017. The participating pre-service mathematic teachers (approximately 60 persons) are currently attending the same lecture in mathematics education (didactic of algebra) and will be randomly distributed into the two experimental groups. The participants of both groups receive the same content input during the lecture. The information given in the lecture will be on functional relationships and particularly focus on the representation graph of functions. Furthermore, the skills which the learners shall

acquire as well as possible student mistakes and misconceptions which can occur during learning, are of special interest.

In the intervention the participants of EG1 are asked to analyse video-vignettes. The participants of EG2 have to analyse tasks which contain the construction or the interpretation of graphs of functions. The focus of both analyses lies on diagnosis of errors regarding problems and misconceptions as well as the skills the students already have or need. The video-sequences used for the intervention can be watched multiple times, stopped at any point and the participant can jump to any point in the video that is of interest to him. This circumstance is meant to help the pre-service teachers as well as possible while they are analysing the learning process of the pupils. The tasks which are given to the participants of EG2 are the same tasks used for the video vignettes. Therefore, differences between experimental groups are limited to the characteristics of the learning resources. The analysis – both of the videos-vignettes and the tasks – happens at each individuals' home, not during the university lecture. In contrast to the test situations, there will be no time constraint during the intervention in order to foster the development of diagnostic skills.

During the pretest, additional data will be collected: teaching experience, attended university lectures in education (other subjects included), differentiating between those already attended, and those happening in the meantime of the intervention. Knowing these influences gives us the opportunity to consider them as covariates for the computation and the results.

The pre- and posttest will be conducted to measure the diagnostic skills of the pre-service teachers at the beginning and the end to be able to see the changes of these skills between before and after the intervention. The tests inquires diagnostic skills which are important for the preparation of lessons as well as those needed to be able to notice situations relevant for successful learning in class. Furthermore reactions based on the participants' diagnoses will be part of the inquiry. The test for diagnostic competencies asks participants to first analyse tasks. Then, a three-minute video will be presented, showing pupils working on the tasks previously analysed. The video can only be watched once and doesn't provide the possibility to pause. This way, we are trying to create a test-situation which is as close to reality as possible. The test includes both open and closed questions asking the participant to communicate what they have noticed and to reason about their findings.

By testing both types of analysis, we want to investigate whether different diagnostic skills have influences on each other. Moreover we expect the test to resolve, whether one method is superior to the other one. This would be the case if for one training method superior gains in both types of diagnostic skills could be observed.

Both settings of the intervention and the tests for diagnostic competence are embedded in the learning environment ViviAn (see Figure 2) developed by Bartel and Roth (2015). This learning environment provides a combination of video vignettes and further material and thereby further approximates the information available in real-life teaching situations. Hence, the user gets information about the students (type of school, grade, sex), the content and the learning goals of the entire lesson, and the materials the students use such as the given task and the materials (for example a big sheet of paper with a graph of a function on it). The students' protocols (products) are only available to the participants of EG1 who are analysing videos. As the participants of EG2 analyse the task in more general they shall not be influenced by the solution of the pupils.



Figure 2: The learning environment ViviAn (Bartel & Roth, 2015)

The data will be analysed with mixed methods. The approach of qualitative content analysis (Mayring, 2008) will be applied to create a coding guideline. Thereby the answers of the participants will be compared to experts' diagnoses. As experts serve mathematics teachers and academic staff working in the field of didactics of mathematics. These experts' diagnoses will be used as a criterion norm for the measurement of diagnostic skills by using the resulting criteria to rate the participants' answers. To resolve group differences descriptive statistics as well as inferential statistics with variance analysis will be considered.

Expected results

The preliminary study was conducted in winter term 2016/2017. It will reveal potential problems concerning our approach, the used material and tasks. Based on these findings we will be able to improve our approach and the used material. Moreover, the preliminary study contributes to the investigation of differences between the diagnoses of tasks and videos. Prospectively, with the results of the main study, we will then be able to point out, whether trainee teachers better acquire diagnostic skills by analysing videos than by analysing tasks. Furthermore it will provide insight into whether different aspects of diagnostic skills have an influence on each other.

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Preservice mathematics teachers' views about planning the assessment

Gözde Kaplan[1] and Çiğdem Haser[2]

[1]Middle East Technical University, Ankara, Turkey, gkaplan@metu.edu.tr

[2]Middle East Technical University, Ankara, Turkey, chaser@metu.edu.tr

The study attempted to investigate senior preservice middle school mathematics teachers' purposes in preparing the assessment, and their views and suggestions about the assessment part of a lesson plan through employing basic qualitative method. First, the "Incomplete and Improper Lesson Plan Task" prepared by the researchers was administered to the participants (N=27). Then, one-to-one interviews were conducted (N=11). Findings of the study indicated that preservice teachers underlined similar purposes in preparing the assessment part of the lesson plan, all of which related to the teacher actions. They mainly emphasized the kind of feedback gathered by the teachers rather than the students. They also considered the assessment part in the task weak since there were insufficient number and diversity of the questions.

Keywords: Formative assessment, feedback, preservice mathematics teachers.

Introduction

Formative assessment or *assessment for learning* is utilized deliberately *for learning* (Laud & Patel, 2013). It includes all activities that provide feedback for adjusting teaching activities and instruction (Black & Wiliam, 1998). Any assessment is formative if it is utilized to collect evidence about students' learning progress and current level of understanding the concept (Heritage, 2007), and to make instructional adjustments in line with their needs (Wiliam, 2007).

Feedback plays a crucial role in formative assessment (Sadler, 1989). Ramaprasad (1983, p.4) defined feedback as "the information about the gap between the actual level and the reference level of a system parameter which is used to alter the gap in some way." Feedback cannot be differentiated from the instruction and it is formative when the information provided is utilized to enhance learners' performance (Wiliam, 2007). Information gathered from feedback can be used by both teacher and students (Sadler, 1989). Teachers use it to specify students' needs and give decisions about the adjustments for further instructions (Wiliam, 2007). Students use it to realize their strengths and weaknesses (Moss & Brookhart, 2009) and to learn how to modify and improve their performances to reach the reference level. Therefore, it affects students' learning positively.

Wiliam and Thompson (2008) focused on three instructional processes; *where students are in their learning*, *where they are going*, and *how to get there* which are emphasized in Ramaprasad's (1983) definition of feedback and they suggested a formative assessment framework shown in Table 1. According to the framework, formative assessment is composed of five key strategies and one *big idea* that the outcome of students' learning processes can be utilized to make necessary changes in the instruction with respect to students' needs (Wiliam & Thompson, 2008).

	Where the Learner Is Going	Where the Learner Is Right Now	How to Get There
Teacher	(1) Clarifying and sharing learning intentions and criteria for success	(2) Engineering effective classroom discussions and tasks that elicit evidence of learning	(3) Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	(4) Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	(5) Activating students as the owners of their own learning	

Table 1: Framework relating strategies of formative assessment to instructional processes (Wiliam & Thompson, 2008, p.63)

The five strategies can be explained in the following way: Teachers are responsible for "engineering" effective learning environment since their role is only to scaffold learning. They provide this environment by generating productive discussion setting, asking deep questions and monitoring the learning process (O'Connor, 2002). Learners' active participation is associated with challenging tasks and providing feedback for these tasks which assist students' learning (Black & Wiliam, 1998). Since students need to understand the learning intentions and standards for which they will be assessed, clarifying and sharing learning intentions and success criteria with the students is also very important (Wiliam, 2007). Additionally, activating learners as instructional resources for one another and for themselves is essential for any assessment approach. In this way, learners can improve the ability to judge and decide about what to do next (Berry, 2005).

Formative assessment is a significant process that can be followed in order to have information about students' progress. Hence, each step of the assessment needs to be decided and planned continuously (Heritage, 2007). Since, as future teachers, preservice teachers need to be qualified in planning and implementing formative assessment practices, teacher education programs play a crucial role in raising the awareness of preservice teachers about the significance of formative assessment and in teaching them how to plan and use it in their classes effectively.

The current study aimed to investigate preservice mathematics teachers' (PST) purposes in preparing the assessment part, and their views and suggestions about the assessment part of the lesson plan. Aforementioned framework guided the researchers in the process of both preparation of the task used for as the data collection instrument and analysis the participants' responses.

Methodology

In this study, basic qualitative research method was employed in order to reveal PSTs' views about planning the assessment part of the given lesson plan.

Context and participants

The study was conducted in a four-year middle grades mathematics teacher education program (MTE). The program offers mainly mathematics and introductory education courses in the first two years and mathematics teaching courses in third and fourth years. A total of 27 senior PSTs who were

enrolled in MTE program participated in the study. They were selected among PSTs who completed the Methods of Mathematics Teaching and Measurement and Assessment courses, and who were taking the Practice Teaching course. The first data collection instrument, Incomplete and Improper Lesson Plan Task (LPT) was implemented to 27 PSTs. Then, 11 PSTs were selected for the interviews with respect to their diverse answers to the questions in the given task.

Instruments and data collection

Data were collected by LPT and a semi-structured interview protocol. LPT consisted of incomplete and improper lesson plan with three 6th grade objectives on equivalent fractions and a case where PSTs assumed to be in-service mathematics teachers implementing this lesson plan. A basic lesson plan template which addressed the lesson in beginning, middle, end, and assessment parts was used because participants were familiar with the template in the MTE program courses. The plan was incomplete because there were not any expression implying any formative assessment strategy. PSTs were expected to realize the nonexistence of these strategies and integrate one or more strategies in the given lesson plan. The plan was improper also because the first objective was unmeasurable and unobservable, there were inconsistencies between objectives and questions in the assessment part, there was no rubric for fair scoring, and questions had a weak structure in the assessment part. Certain multiple choice questions were selected because PSTs were expected to realize that feedback gathered through these questions about students' learning was limited. PSTs were anticipated to notice and eliminate these reasons of improperness of the lesson plan. Figure 1 presents the lesson objectives and 4 yes-no questions in the assessment part of the lesson plan.

Objectives:

- Students should be able to develop a conceptual understanding of equivalent fractions.
- Students should be able to explore the same quantity can have different fractional names.
- Students should be able to look for patterns in equivalent fraction.

Assessment:

For each equivalence please write T in the given blank if it is correct; write F if it is false.

A) $\frac{3}{4}=\frac{6}{8}$ (__) B) $\frac{46}{14}=\frac{23}{6}$ (__) C) $\frac{5}{12}=\frac{20}{36}$ (__) D) $\frac{37}{74}=\frac{1}{2}$ (__)

Figure 1: Objectives and assessment part of the incomplete and improper lesson plan

LPT asked PSTs to write the strengths and weaknesses of the assessment part, and give suggestions about how to improve it with regard to weaknesses they found. During the interview, PSTs commented on how they would have designed the beginning, middle, end, and assessment parts if they had prepared the given lesson plan. They responded the questions by assuming to be in a hypothetical classroom environment; they did not implement the lesson plan in their teaching. The current research report, a part of a broader study which addressed all indicated features, focused only on PSTs' views about the assessment part of the given lesson plan.

LPT was implemented in a course where PSTs attended after necessary permissions were granted. PSTs were asked if they would like to volunteer to participate in the study and those who volunteered completed the task in about 50 minutes. After one-month of data analysis period, 11 PSTs were

selected to be interviewed in one-on-one setting. They all participated in the interviews voluntarily. Interviews took 30-80 minutes and were audio-recorded with the permission of the participants. PSTs' answers to the task were reminded with the purpose of eliciting whether they wrote the responses for formative assessment purpose or not when they did not advert to the same issues during the interview. Data were analysed through content analysis. PSTs' expressions which imply the action of gathering evidence about students' current knowledge and their own competence in teaching were grouped under the "providing feedback that moves learners forward" subdomain of the formative assessment framework (Wiliam & Thompson, 2008). PSTs' purposes in preparing this part were examined under the categories of "feedback for teacher" and "feedback for students" (Sadler, 1989). One researcher, other than the authors, examined the data and they agreed that the categorization was conceivable regarding to the data. Preservice teachers' intended further actions, views and suggestions about the assessment part were also reported.

Findings

Feedback for teacher and students

Findings of the interviews indicated that all interview participants agreed that the assessment part provided feedback for the teacher. PSTs mainly stated that they would perform this part to gather feedback about students' level of knowledge and their own competence in teaching. PST4 emphasized both aspects as in the following conversation:

> Researcher (R): What is your purpose in preparing the assessment part of the lesson plan?
>
> PST4: In order to learn about whether I could teach the concept or not. Did I have students achieve the objectives? There can be some points that the students did not get. I prepare [the assessment part] in order to determine these points [as well].

PST13 also underlined the necessity of providing feedback about students' learning as follows:

> I think the [assessment] part is necessary in order to provide feedback about what the students have learnt or have not learnt… I think using exit card is very beneficial in order to understand whether the students have learnt the concept or not.

Only one participant mentioned that this part also provided feedback to the students about their learning:

> PST23: Definitely I will not grade students' work. Here, grading is so ridiculous. I check whether they understood the concept or not. I think it should provide me feedback.
>
> R: Why do you not grade their works?
>
> PST23: We implement it in last five minutes [of the lesson]. The students have learnt the concept in that lesson. Their knowledge is so fresh. I think there is no need to grade the exit card if the students have a perception that the assessment part serves for testing themselves. The teacher will use it to learn about what the students understand (and) also the students will realize whether they understand the concept or not through the assessment part.

Remaining ten interview participants also preferred not to grade students' responses to the questions in the assessment part since they thought that this part would be used to check students' current level

of understanding of the concept or to have them comprehend the concept better.

All interview participants expressed that feedback they obtained by means of the assessment part would affect their further instructional plans. They indicated that they would make some instructional changes according to the feedback about students' needs. They mostly preferred changing the next class's activities or teach the lesson again. However, some of the participants claimed that they probably would not have time to repeat the lesson since they need to keep the pace of the national curriculum:

> I would decide what to do according to the answers of the majority of the class. If the students made major errors, I would think that I was responsible for their mistakes. Maybe, I would repeat the lesson or I would probably teach another lesson in which I could emphasize the points that the students misunderstood. However, I do not know whether I have time to do it when I would be a teacher because there is a curriculum [need to follow]. These plans are only utopia. (PST15)

PSTs' views and suggestions about the assessment part of the lesson plan

In both task implementation and the interview, PSTs expressed similar ideas with different frequencies. In the LPT implementation, more than half of the PSTs commented on the strength of the assessment part and mainly emphasized the consistency of the questions in the assessment part with the lesson content (n=5). PST12 expressed that "it is a good activity [since] the students can implement what they have learnt into the assessment part." Two PSTs claimed that the assessment part was strong since "it is efficient in assessing whether the students understand the second objective" (PST8) and "the indicated questions can measure easily whether the students understand the relationship between two equivalent fractions." (PST15). PST11 and PST27 reflected on different features of the questions. They indicated that this part was useful since there were questions related to both enlargement and simplification of the fractions. On the other hand, there was not any coherence within other PSTs' expressions. For instance, PST9 stated that she liked the questions in the assessment part because "they have uncontroversial and single answer" whereas according to PST23, "the questions do not have specific answer and they prompt students to think."

Only four interview participants mentioned the strength of the assessment part by commenting on only the specific options. Two participants stated that they would keep the option c since it could assist to detect some misconceptions or errors. PST17 expressed that she liked the questions in the assessment part since they included numbers such as 37 and 46 which were not much used.

Regarding to the weaknesses of assessment part, in both LPT implementation and interviews, PSTs addressed the questions in the assessment part as insufficient in number and diversity. In LPT implementation, 6 among 27 PSTs commented on and suggested ways to improve such weaknesses:

> There is only one type of question. There should be (questions) which are supported by the shapes. Not only true-false questions, but also some interpretation questions and the questions that the students can write equivalence of the indicated fractions should be added. (PST21)

According to five participants, assessment part was weak since it included questions that students had fifty per cent chance to answer them correctly. They recommended to add different types of questions to reduce students' guessing. For instance, PST19 suggested to "add some open ended questions in order to see how much the students understand the concept in an easy and reliable way." Adding verbal and daily life questions, and questions with shapes was also recommended.

During the interview, PSTs mostly underlined similar weaknesses of the assessment part and proposed suggestions to deal with them as they did in the task implementation:

> I think, whether the students understand the concept or not is not assessed exactly here because there is fifty per cent chance. If I write all of them true, I will answer one or two of them correctly. (PST18)

To overcome this weakness, PST18 also recommended to add different type of question as:

> If I were… I would give them and want them write an equivalent fraction to this one rather than asking true-false [questions]. Even, I can ask the same questions with the one that I asked in the beginning part of the lesson. I change its numbers. "If such a number of pieces of cake were eaten, how many pieces were eaten?" and "Is there another fraction which represents the (same) amount?"

On the other hand, none of the participants mentioned the inclusion of the rubric for fair scoring which was one of the reasons for the improperness of the given lesson plan.

Discussion and conclusion

Findings of the study showed that all PSTs planned to prepare assessment part of the lesson plan in order to gain feedback about students' current knowledge level and their own competence in teaching, all of which related to teacher actions. Only one interview participant stated that assessment part would also provide feedback to the students about their own learning. PSTs mainly emphasized the kind of feedback gathered by the teachers although they were expected to comment on that students can also obtain feedback about their own learning and needs through the assessment. This tendency towards teacher-centred assessment was the action which provided feedback only to the teacher in order to determine the problematic areas that required more emphasis and practice (Antoniou & James, 2014). It might be due to PSTs' views that the students may not benefit from the assessment part to determine their needs and take necessary actions to meet these needs.

PSTs were against grading the assessment part of the lesson plan since their purpose in the preparation of the assessment part was only to check students' learning or to have them understand the concept well. This finding was congruent with the idea that the main purpose of utilizing formative assessment was to facilitate and improve students' learning instead of simply assigning a grade (Marshall & Drummond, 2006). When it is considered that the formative assessment serves its purpose when the teacher avoids grading students' performance (Elawar & Corno, 1985), it might be deduced that the PSTs planned to use this part for formative purpose.

PSTs indicated that feedback gathered from the assessment part assisted them to adjust their further instructional plans according to students' needs. This finding might indicate that PSTs used the assessment part of the given lesson plan formatively because they would make some changes in the next class's instruction and planned further instructional steps (O'Connor, 2002). PSTs generally preferred to repeat the previous lesson or teach another lesson by changing the existing activity in case the students had difficulty in learning the content. However, some PSTs indicated that they probably would not have such time since they needed to keep track of the national curriculum. Similar types of adjustments and lack of time issues were also reported previously (Antoniou & James, 2014). Although all PSTs talked about teachers' further actions, they did not mention students' possible further actions to enhance their own learning. The reason might be PSTs' disposition towards teacher-

centred assessment. They might not consider students' further actions because they disregarded the fact that the students could also obtain feedback through assessment part to monitor their progress and enhance their learning.

In both task implementation and interview, majority of the PTSs were able to detect the improperness of the assessment part resulted from the structure of the questions. They emphasized that they could not know whether students had learned the concept or not through these questions since students had fifty per cent chance to answer them correctly. PSTs generally recommended adding open-ended questions to eliminate this weakness. Being able to detect improperness might be due to their awareness of the requirement of the alignment between objectives and the questions in the assessment part. They might have suggested adding different questions to eliminate the inconsistency between the objectives and questions. PSTs also suggested increasing the number of the questions in the assessment part. The reason for this might be attributed to the demand of introducing students with wide range of questions to prepare them for high-stake national examinations, as reported for beginning Turkish middle grades mathematics teachers (Haser, 2006).

Due to the fact that the measurement and assessment course stresses mainly the assessment tools rather than focusing on the whole picture of the lesson plan with regard to the utilization of the formative assessment strategies, PSTs might have had difficulty in examining and integrating the intended formative assessment strategies in the lesson plan. Therefore, courses on assessment can be offered with the methods of mathematics teaching courses or lesson contents of these courses can be associated with each other so that PSTs can integrate what they have learnt about assessment into the lesson plans they prepared in the methods courses. Hence, they can have a chance to look at the whole picture of the lesson plans in terms of employing the formative assessment practices.

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From paper and pencil- to computer-based assessment: Some issues raised in the comparison

Alice Lemmo[1], Maria Alessandra Mariotti[2]

[1]University of Palermo, Italy; alice.lemmo@unipa.it

[2]University of Siena, Italy; mariotti21@unisi.it

Comparative studies on pen-and-paper and computer-based test principally focus on statistical analysis of students' performances. In educational assessment, comparing students' performance (in terms of right or wrong results) does not imply a comparison between the solving processes followed by students. In this paper we present an example of task analysis that allows to highlight how students' solving processes could change in switching from paper to computer format and how these changes could be affected by the use of one environment rather than another. The aim of our study lies in identifying possible consequences that specific changes in task formulation have, in terms of students' solution processes.

Keywords: Computer based assessment, comparative study, task analysis.

Introduction

Computer-based assessment is an actual issue. The increasing use of tests administered in the digital environment allows research in mathematics education to develop new fields of study. On the one hand research in computer based tests concerns the validity of these tests, on the other it focuses on their comparability with existing paper tests. In these two perspectives, large-scale surveys were conducted; they involve students from different educational levels, from primary to secondary instruction (Drasgow, 2015; Way, Davis, & Fitzpatrick, 2005).

Computer-based tests mainly involve institutions in large scale assessment (OECD-PISA, OECD-PIAAC, NAEP, …); one of the major interests of these institutions is to anchor every new test with ones from the previous years rather the diachronic study of students' performances in the different surveys. For this reason, some studies focus on the test-mode effect comparing performances of students taking on computer and paper-based tests.

Literature on these topics shows very mixed results; there is empirical evidence that paper-based and computer-based tests will not return the same results. On the one hand, some studies show equivalence in students' performances; on the other hand, different researches highlight a significant discrepancy on scores. For example, Kim and Huynh (2007), as many other researchers (e.g. Kapoor & Welch, 2011; Lottridge et al., 2008), show that there is no statistical evidence suggesting that the administration modality changes the coherence and consistency of computer-based tests. On the contrary Clariana and Wallace (2002) point out to empirical evidence suggesting that students involved in paper-based and digital-based tests will not obtain the same results. At a more general level, in a meta-analysis of computer versus paper-based cognitive ability tests, Mead and Drasgow (1993) found that on average, paper-based test scores were very slightly higher than computer-based test scores.

The main characteristic of all these studies, which involve comparative analysis of outcomes using quantitative and statistical methods, is that they show the comparability between tests administered

on paper and pencil and in a computer environment. Such comparison is developed contrasting students' performances and it is grounded in the implicit assumption that students who achieve the same performance implement the same solution processes. Threlfall et al. (2007) propose a more accurate analysis; they focus on students' solution processes and explore the effect on students' attitudes when they are involved in paper and pencil tests migrated into a digital environment. As shown by Threlfall et al., in some cases changing to a different environment seems to make little differences in the solution process. However, for some particular tasks the computer environment deeply affects how students approach the tasks. An important issue arises: task comparability cannot be measured only in terms of students' outcomes but it is also established by the comparison between the solving strategies that they use.

These diversified results suggest that task comparability needs a deeper analysis. In particular, the comparison between students implies the problem of how and when two tasks could be considered equivalent. Ripley (2009) proposes a possible solution to this question. He distinguishes two main approaches to the use of digital devices in order to enhance assessment: *migratory* and *transformative* approach. He defines the migratory approach to be the use of technological support as a tool of administration; it consists in a transition in digital format of tasks conceived for paper format. The transformative approach involves the transformation of original paper tests integrating new technological devices which support interactive tools (graphs, applets, …) that enhance new affordances. There are no specific studies comparing these two approaches; a possible reasonable hypothesis is that migratory could be a suitable approach to construct what in the literature is called an equivalent task. By definition, the migratory approach has the aim to maintain most of the task features unvaried in the translation process but this transition to a new environment cannot be completely unbiased. The migration from an environment to another one is not neutral because it depends on intrinsic properties of the environments. The adoption of migratory approaches is undervalued; the assumption that the translation process causes few changes on the task formulation and that these changes do not cause significant alterations on the solution processes is not to be neglected.

The purpose of this study is to examine whether the migratory approach may have effects on students' solving processes. In this perspective, the issue of test validity arises; in other words, does the use of a migratory approach maintain the validity of the original test? Below, we present part of a wider study that has the aim to analyse possible changes in students' solution procedures related to the migration from a pen and paper to a digital environment. In Ripley's words, we consider tasks that could be defined migratory or other authors could call equivalent tasks. In particular, we present one example of analysis that compares a task in his migration from paper to computer, highlighting the impact that the changes could have on students' solutions.

Word problem in a migration process

In many tests, especially in large scale assessment, knowledge and skills are assessed through units consisting of a stimulus (e.g. text, table, chart, figures, etc.) followed by a certain number of tasks associated with this common stimulus. These particular features connect these kinds of tasks with word problems. In a wide perspective, the term *mathematical word problem* refers to any mathematical task where significant background information is presented through a verbal text rather

than in mathematical notation. As word problems often involve a narrative of some sort, they are occasionally also referred to as *story problems* (Verschaffel , Greer, & De Corte, 2000).

Mathematical word problems have an important role in teaching; for many decades researchers in mathematics education have focused on the possible difficulties that students encounter when they solve them. Verschaffel et al. (2000) highlight the fact that many of the difficulties met by students lie in the preliminary phase of understanding the problem situation. Interpreting students' attitudes in solving word problems is complex because it involves multiple interacting factors, both cognitive and metacognitive: stereotypes of standard problems, implicit and explicit rules that regulate mathematical activity, students' beliefs, etc. (Verschaffel , Greer, & De Corte, 2000).

Considering word problem texts (in particular, its formulation features) introduces the important issue of *representation*. Goldin and Kaput (1996) describe two distinct meanings of the term *representation*. On the one hand, the *external representations* refer to "physically embodied, observable configurations such as words, graphs, pictures, equations, or computer microworlds" (ibid., 400); on the other hand, the *internal representations* concern "possible mental configurations of individuals, such as learners or problem solvers" (ibid., 399). In the case of word problems, the solver interacts with the *external representation* presented and produces a personal *internal representation* linked with the one that she already has. Obviously, being internal, such configurations are not directly observable but they could be inferred through the solution process that the solver employs. For this reason it is possible to confirm that a change in the *external representation* could influence the construction of the internal representation and so the adoption of the solving process. Therefore, it is possible to suppose that the formulation of mathematical word problem influences both cognitive and metacognitive factors that are involved in word problem solution. Goldin (1982) highlights that small differences in some features of word problems can deeply affect the process of solution. In particular, Mayer (1982) and later De Corte & Verschaffel (1985) observe that the difficulties noticed within problem solving activities may come from an inadequate interpretation of the text.

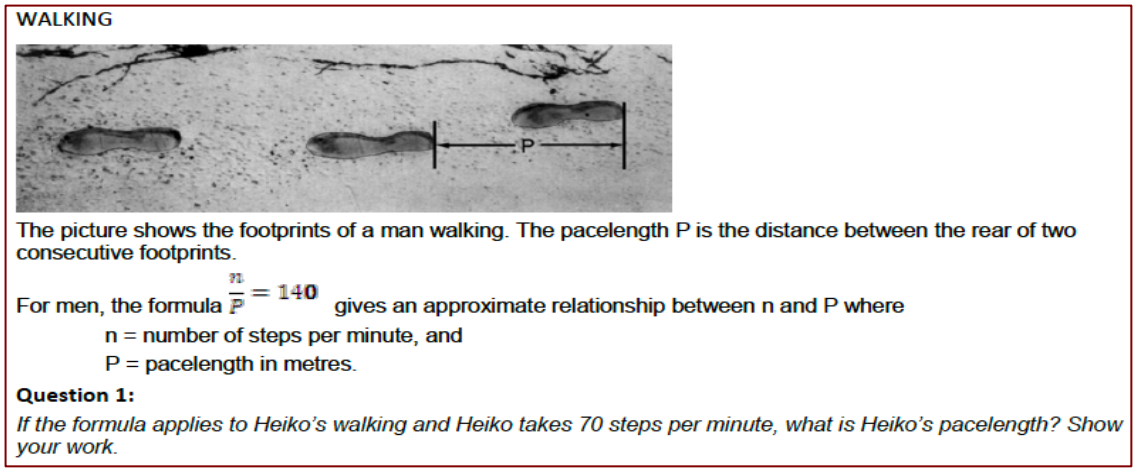
Thus, in the perspective of comparison, it is necessary to analyse the differences between tasks to determine the possible differences that occur in students' solution processes. Identifying possible changes in a mathematical word problem requires to consider many text features. For this reason, the task is simplified by dividing the word problem into simpler elements. Gerofsky (1996) describes word problems in terms of three main components: the *set-up component* which establishes the characters and location of the story; the *information component* which encompasses the information needed to solve the problem; and finally, the *question component* which expresses the request and focuses on goal and aim.

Our purpose is to analyse tasks through specific variables that might influence the behaviour of students in the solving process. Obviously, checking these differences is a general issue that could be presented whether or not there is a migration process in a new environment; possible changes could happen even just in the paper environment.

Analyses of a migrated word problem

We present the analysis of one of the items presented in the Draft 2015 PISA Mathematics framework (OECD, 2013). Figures 1 and 2 show the two versions of the famous task: "Walking", administered

in PISA 2003 survey. The text of the item has not changed; therefore, narrative or linguistic differences are not recognized in the *set-up* component.



WALKING

The picture shows the footprints of a man walking. The pacelength P is the distance between the rear of two consecutive footprints.

For men, the formula $\frac{n}{P} = 140$ gives an approximate relationship between n and P where

n = number of steps per minute, and

P = pacelength in metres.

Question 1:

If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pacelength? Show your work.

Figure 1: "Walking" paper version, administered in PISA 2003 survey

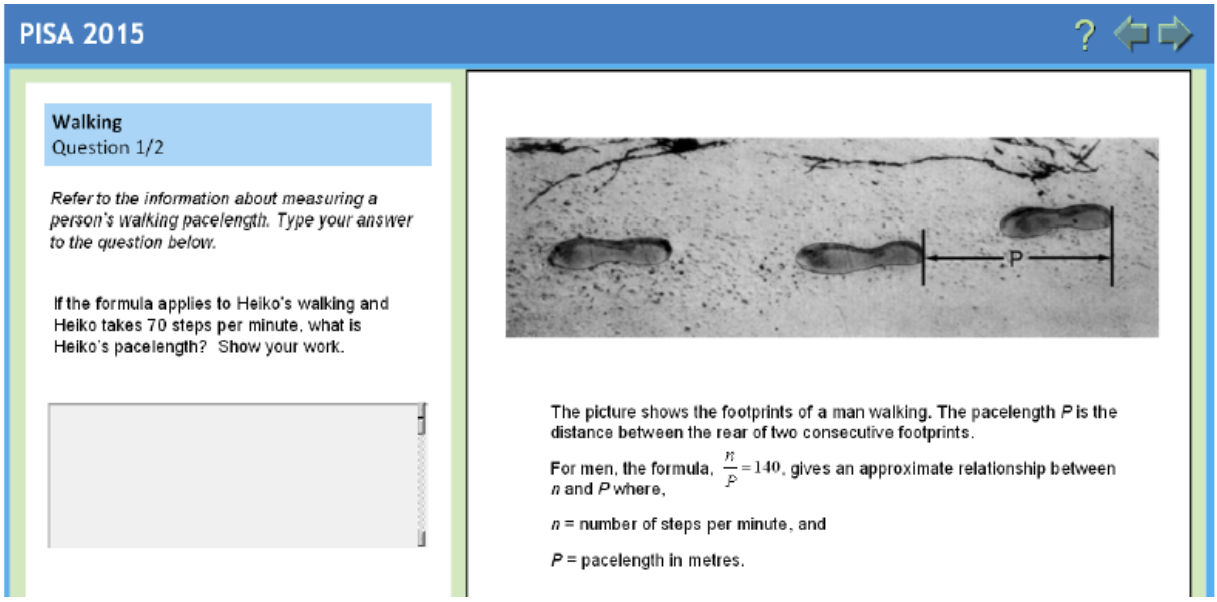


Figure 2: "Walking" computer version, shown in PISA 2015 Framework Draft

First of all, there is a difference in the editing of the text. In the paper version the task is presented in a compact way: *set-up* and *information* components are given in the same text and the *question* is presented under this text. In the digital format, the task is divided into two main sections. On the right there are the *set-up* and *information* components: they consist of an image and a description of the situation, both in words and algebraic formulas. On the left there is the question. The difference in editing seems to complicate the task; in the digital format, the text is presented in two separate columns. Therefore, the change of the question position could create variances in solution processes: the solver has to coordinate the interpretations of the different parts in which the text is divided. In other cases this change could affect the solver's comprehension. For instance, Thevenot et al. (2007) show that putting the question before a word problem (rather than classically presenting it at the end) conditions problem solution in young students and in particular it facilitates students in engaging a correct solution process. In Fig 2, the question is presented in the bottom-left of the screen; in this case the solver probably reads the question before reading the *set-up* and *information* components.

According to Thevenot et al, this fact suggests that in the digital format the interpretation of *set-up* and *information* components could be affected by the previous reading of the question.

Concerning the component *question*, there is another notable difference. In the digital format (Fig. 2), the first part of the *question* text shows the instructions for answering to the task ("Type ... below") and how to coordinate the information presented in the context ("Refer to … pacelenght"). This aspect enriches the *question* and the length of the text that students have to comprehend and interpret (in the paper version there is not any kind of instruction).

The test item format is changed; a text box in the digital version replaces the free space presented in the paper. The test item format has a strong impact on students' solution process. Kazemi (2001) investigates children's mathematical performance on test items focussing on the typology of the questions. In his study, Kazemi uses multiple-choice questions and juxtaposes them to other open-ended problems. He highlights that the typology of questions affects students' thinking in designing and interpreting problems. This impact is emphasized when there is a change of environment and so a change of tools available to the solver. Concerning computer and paper and pencil based tests, Russell and Haney (1997) describe a comparative study in terms of students' performances. They show that there are differences in performance related to the type of test item formats; substantial changes are not found in the case of tasks with multiple choice questions but there are relevant differences in the case of open response items. Moreover, assuming that the student is familiar with the writing tools available (for example the keyboard) it is reasonable to suppose that this change would not result in significant differences in the solution process. However, in using the free space in paper format, the solver has a different freedom of expression with respect to the case of the text box: in paper and pencil, the solver can produce sketches, calculate and write text both in natural and in symbolic language. These actions are not allowed in a simple text box in which one can only enter the characters on the keyboard or otherwise perform the actions allowed by the available writing tool, depending on the software used.

Finally, in both tasks the same picture is presented; nevertheless, it is possible to notice that in the digital version the picture is presented on the screen with all the strengths and limitations of the software that supports it. For example, it might be difficult (or impossible) to analyse the image through common and simple manipulation action such as turning the paper, complete the picture by drawing lines, highlight points, etc.; these actions are possible only in paper and pencil environment.

Conclusion

In the previous example, the migration process could at first appear accurate but a deep analysis shows the opposite. At a first glance, the highlighted little differences might appear superfluous; however they are crucial to analyse and interpret students' behaviour in the solving process. The literature described in the first part of the paper indicates that each difference observed in the example may affect the solver. For instance, the change in the task editing could simplify the text comprehension if it is presented in a linear way; on the contrary, the reading could be difficult if the verbal description is fragmented in several parts. Furthermore, the position of the *question* may encourage the solver to develop a correct solution process or it could complicate the *set-up* comprehension because the solver has to coordinate its interpretation with the information presented in the text components. These little differences hide important consequences for assessment, especially if the purpose of the migration process is to ensure continuity between the paper and the

computer administration. The comparison studies presented in the literature assume that the tasks administered in the two environments are equivalent. However, our analysis shows that this starting assumption should be changed. The equivalence between performances (in terms of right or wrong results) does not imply an equivalence between the processes adopted by the students. Therefore, the analysis of the results collected in the two environments probably is not equivalent in terms of educational assessment. The answers produced by students in the digital environment seem hardly comparable with what they do on paper. Thus, there is a substantial difference in terms of the assessment; it cannot be ignored especially by national or international large scale assessment.

In addition, the change in type of test item format is crucial because it strictly depends on the intrinsic feature of the environment and on the familiarity that the solver has with the tool available. We recall also that there are cases where it is impossible to translate a task from paper to digital format through the migratory approach; for example, the tasks that require the use of physical tools and measuring instruments as ruler, compass, or other. A special case is the one of items that have the goal of assessing students' drawing abilities. In this case, the item may request to draw a figure starting from a given one, or from given measurement, or from written verbal instructions. In these cases, it is possible to introduce an ad hoc software or applet that simulates the use of drawing tools. Moreover, the issue of students' familiarity with these software or applets arises (Bennett, Persky, Weiss, & Jenkins, 2010). In the case of lack of familiarity with the use of the instrument, the digital device could be largely useless; students that use digital tools may be disadvantaged comparing them to students that use paper and pencil and physical tools. This example highlights a very serious and complex issue. Further research is needed to define criteria of control that allow to check and to compare all the little differences that occur in the migration process.

In our wider study we define a specific instrument to monitor such differences. In particular, we identify specific variables that might influence the students' behaviour in solving a certain task. We organize such variables into a table that we call *comparison tool*. Such tool is constituted by a system of indices related to the structure of word problems described before. In particular, we identify five different indices that represent possible changes that may occur in the migration process:

- **Story** refers to the *narrative dimension* (Zan, 2011) of the task (for example: characters, background, narration, etc);
- **Linguistic form** that indicates also the number and the length of sentences (for example: syntactic, organization of the sentence, lexical, etc);
- **Type of item formats** concerns the types of possible responses (for example: the question has constructed-response, selected-response, etc);
- **Format and editing** refers to layout features and position of the different components (for example: paragraphs, font, underlines, spaces, etc);
- **Data representation** is related to semiotic register used for representing information (Duval, 1993) (for example: verbal register, iconic register, math register, etc).

Each index is related with studies concerning word problem formulation and its impact on students' solution process. For example, many authors show that the narrative dimension attached to mathematical tasks is relevant to students in terms of the their availability to solve the task (Sowder, 1989). Other studies draw attention to the importance of language in student performance on

assessments (Abedi, Lord, & Plummer, 1995). Moreover, many authors pay attention to the role of representation in the teaching and learning process (Duval, 1993).

Considering the example above, the comparison tool highlights differences related to three of the five indices. In particular, tasks are different in terms of *linguistic form*, *type of item format* and *format and editing*. In fact, in the digital version there are more sentences than in the paper task. Furthermore, even if both questions are open-ended, in the computer task there is the restriction caused by the text box where type. Finally, there is a strong difference in editing: in the two task versions, the components are presented in different parts of the page/screen. The comparison tool highlights a certain number of differences; such differences could confirm our hypothesis in terms of possible differences in students' behaviour.

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Teddy Bear Preschool Mathematics Assessment: Validation of a constructivist game- and story-based measure

Sally Moomaw[1]

[1]University of Cincinnati, Cincinnati, Ohio, USA; sally.moomaw@uc.edu

This research evaluated an interactive story- and game-based measure of the level of mathematical development in preschool children. Nine measurable learning trajectories identified by previous researchers—quantification, counting, set comparison, numerals, number line, positional terms, shapes, addition/subtraction, and patterning—were assessed through a series of games played by assessor and child to re-enact the story of teddy bears on a picnic. This format was selected because it engages young children, connects to constructivist curriculum and materials, and extends previous research by the author. Confirmatory factor analysis indicated that the nine developmental trajectories were strong and significant contributors to the latent construct of level of mathematical development. Results from test-retest and criterion comparisons indicated that the assessment tool was a valid and reliable measure of mathematical development for this sample.

Keywords: Curriculum based assessment, number concepts, preschool tests, child development, developmentally appropriate practices.

Purpose

The purpose of this research was to create and validate the Teddy Bear Preschool Mathematics Assessment (TBPMA), a play-based, constructivist measure, as a novel alternative to traditional quantitative assessments. The story- and game-based framework, in which the assessor participates with the child, was selected because it engages young children, connects to constructivist curriculum and materials frequently used in preschool classrooms (Moomaw & Hieronymus, 2011), and provides both formative and summative assessments. The hypothesis was that nine measurable variables based on seminal research—(1) quantification level, (2) counting, (3) set comparison, (4) numeral recognition and understanding, (5) movement along a number line, (6) emergent addition/subtraction, (7) understanding positional terms, (8) shape recognition, and (9) patterning—would make a significant contribution to the latent construct of mathematical development.

Theoretical framework

Assessment of preschool children has grown rapidly during the 21st century (Wortham, 2012). Assessment is used to monitor children's development, make educational decisions, and evaluate programs. However, valid and reliable assessment of young children is often difficult. If children are not interested in the assessment, they quickly lose attention and refuse to participate. Also, many standardized assessments do not provide information that directly relates to the curriculum. This is frustrating for teachers, who need formative assessments in order to guide their planning and interactions with children. Therefore, assessments that are aligned to curriculum and can inform planning and instructional decisions (Stecker, Fuchs, & Fuchs, 2005) are of particular interest.

Assessment of early math development is important; research has shown that children develop substantial math knowledge prior to first grade (Clements & Sarama, 2007). Early number sense

development is a strong predictor of later mathematics achievement (Duncan et al., 2007); however, a substantial achievement gap exists between low- and middle-income children that persists as they advance in school (Jordan, Kaplan, Olah, & Lucuniak, 2006).

The theoretical framework of this research is constructivist; children are active creators of knowledge rather than passive recipients (Piaget, 1952). An instructional corollary is that students require a context that allows them to formulate or discover important relationships (Geary, 2003). For young children, play is an important mode for learning that is considered essential for cognitive development (Frost, Wortham, & Reifel, 2008).

Learning trajectories in early mathematical development

Formation of content knowledge follows a developmental progression that reflects progressively higher levels of thinking (Piaget, 1952). Understanding of developmental progressions in the mathematical thinking of young children is essential for effective teaching because it guides teachers in the selection of appropriate curriculum and in effective modeling and dialogue with children. These documented developmental progressions are now referred to as learning trajectories, which serve as a bridge between theory and practice (Sarama & Clements, 2007).

The learning trajectory for quantification (Kamii, 1982; Piaget, 1952) shows a progression in young children from visual perception to more logical forms of reasoning when determining quantity and comparing sets of objects. The earliest level is referred to as *global*, in which children make a visual or tactile approximation of quantity, perhaps by taking a small or large handful of objects to represent a given amount. At the *one-to-one correspondence* level, children realize they can accurately represent a given quantity by taking one object for each item in the original group and aligning them. This important development indicates that children are now able to focus on the units in a set rather than just the global parameters. Eventually, children realize that they can *count* to determine the number of objects in a group or to create an equivalent set. Rather than simply counting because an adult tells them to, children at this level select a counting strategy because they understand that the last number they count represents the total.

Counting is an important mathematical tool for children. In their seminal research, Gelman and Gallistel (1978) developed five principles that children must understand in order to successfully use counting to quantify. Of these principles, the first three are designated as “how to count” and are likely to be developed by children in the 3- to 5-year age range. The *stable order* principle indicates that children understand that they must say the counting words in the correct order. Application of the *one-to-one* principle shows that children know they should count each object one and only one time. In practice, many (if not most) young children tend to recount objects or count some more than once, particularly as the number of objects increases. The *cardinality* principle means that children understand that when they count objects, the last number word they use refers to the total amount, not just one item. These principles do not necessarily develop in a prescribed order, and accuracy is variable throughout the early years. It is the ability to apply the principles with growing accuracy when counting increasingly large sets that constitutes a learning trajectory.

Both researchers and educators have documented that children can name numerals before they can use them to represent a specific quantity. In particular, Kato, Kamii, Ozaki, and Nagahiro (2002)

have shown a disconnect between children's ability to name numerals and their ability to use them to represent quantities.

Research by Ramani and Siegler (2008) suggests that linear number board games enhance young children's understanding of number. However, research by Moomaw (2015) indicates that there is a developmental trajectory in which children are first able to represent quantities on a grid, or bingo-type board, followed by moving a specific amount along a straight path, or incipient number line. Representing quantities by moving along a longer, curved path is more advanced.

Spatial reasoning is an important component of geometry. Bowerman (1996) has demonstrated that the order in which children learn spatial terms is consistent across languages, thus forming a learning trajectory. Terms that relate to an object that is in direct contact with another ("in," "on," and "under") are the first to develop, along with the movement terms "up" and "down." Words of proximity, such as "beside" and "next to," develop next. These are followed by terms related to position but not necessarily close proximity ("in front of," "in back of," or "behind"). Directional words, such as "right" and "left," are often not learned until first or second grade.

Seminal work by van Hiele (1999) suggests a progression, or learning trajectory, of four levels of geometric understanding, of which the first two relate directly to young children. At the first level, *visual*, children judge a figure according to its appearance. They often have a prototype in mind, so they may identify only equilateral triangles as "triangles." At the second level, *descriptive* or *analytic*, children begin to use language to describe properties of shapes. For example, they may indicate that a figure is a triangle because it has three sides.

Research has established a trajectory for early addition (Baroody & Tiilikainen, 2003). Children start by counting individual sets of objects. Next, they realize that they can count the objects in both sets all together. Eventually, children begin to count forward from the cardinal value of one of the sets; they also begin to remember particular combinations, such as doubles.

Patterning is considered a foundational component of early mathematical thinking by the National Council of Teachers of Mathematics (USA). However, there is insufficient research in this area to establish a developmental learning trajectory.

Development of the Teddy Bear Preschool Mathematics Assessment

Conception of the TBPMA evolved from the author's 25 years of experience teaching preschool and kindergarten children. During this time, considerable contributions to the understanding of how children develop mathematical reasoning were made by theorists and researchers. That information was shared with interested teachers by Professor Anne Dorsey, then Director of the Arlitt Child Development Center at the University of Cincinnati. With her input, teachers began to redesign their preschool math curriculum to focus, in part, on teacher-created math games. Children's responses to these games were carefully observed, and changes to the curriculum were made accordingly. Eventually, this game- and play-based curriculum, which extended to all areas of the classroom, was published in a series of books for teachers (Moomaw & Hieronymus, 2011).

For her doctoral research, the author developed and validated an assessment of number sense in preschool children that was essentially a quantification game played between assessor and child. It

showed that this interactive form of assessment could be reliably quantified. The TBPMA extends that research to more completely measure number and operations as well as geometry.

Methods

The TBPMA consists of a series of games that allow children to re-enact the story of teddy bears on a picnic. The game board depicts a park, including picnic blanket, wading pool, path, small climber, and merry-go-round (Figure 1). Assessor and child take turns drawing cards arranged in a prescribed order. Teddy bear counters are used throughout the game. Although the assessor participates in the game, assessor role and comments are clearly scripted to ensure procedural reliability.

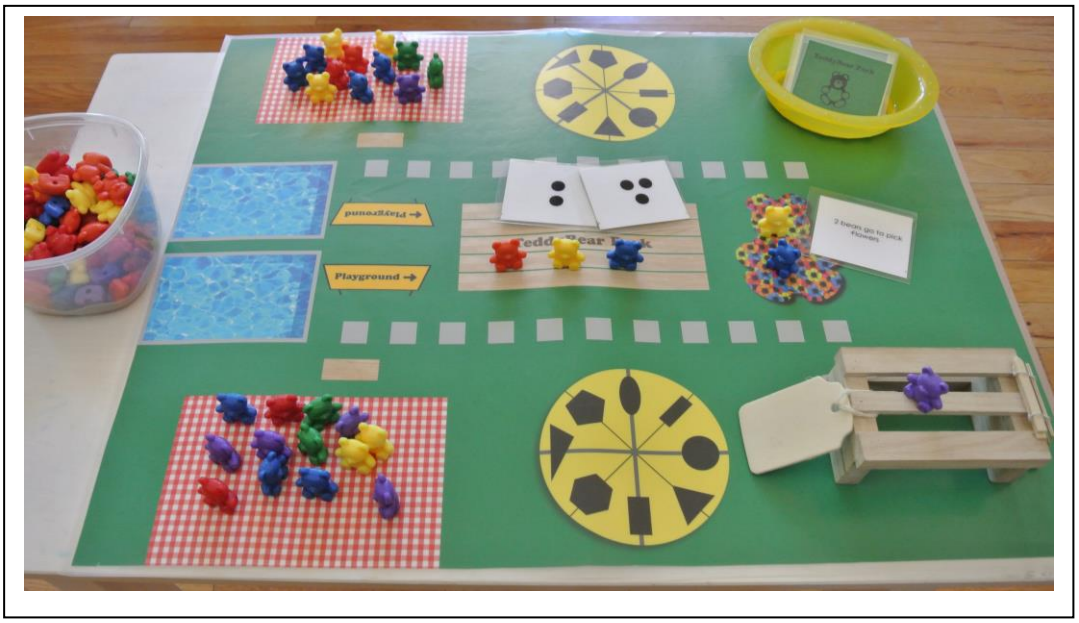

Figure 1: Game Board for the Teddy Bear Preschool Math Assessment

The scales quantification, counting, and set comparison are measured as assessor and child draw cards with 1–8 dots to determine how many bears to place on their picnic blankets. One point is awarded for a global strategy, 2 points for one-to-one correspondence, 3 points for counting, and 4 points for counting with accuracy. Following each turn, the assessor asks how many teddy bears are on the child's blanket and scores 1 point for application of each counting principle (stable order, one-to-one correspondence, and cardinality). Next, the assessor asks who has more bears, which elicits set comparison and is scored similarly to quantification. Set sizes increase over subsequent turns.

Recognition and understanding of numerals is assessed when the bears go swimming. Children are scored on whether they recognize numerals 1–8 and can put the corresponding number of bears in the pool. Movement along a number line is measured when children draw cards with dots and attempt to move a corresponding number of spaces along a path. Understanding of spatial terms is assessed as children draw cards that tell them where to place their teddy bear in relation to the climber, as in "Your bear wants to play under the climber." For assessment of shapes, the bears play on a merry-go-round, which has prototypical and non-prototypical shapes for seats. Scoring is based on shape recognition and the explanation the child gives for determining shape.

The addition/subtraction trajectory is assessed as children draw two cards with dots to determine how many bears to move to the picnic table. They then draw a card that indicates how many bears leave and asks how many are left. Finally, the assessor arranges the bears in an AB and an ABC color pattern for a dance and asks the child to complete the pattern.

Data sources

A total of 146 children, ages 3 to 5 years, enrolled in childcare centers in a Midwestern USA city, were assessed. Of the sample, 60 (41.1%) were poverty level (determined by voucher participation); 37 (25.3%) were African American or mixed race; 73 (50%) were female; 65 (44.5%) were 3 years old; 63 (43.2%) were age 4; and 18 (12.3%) were age 5. For test-retest comparison, 50 (34.2%) received a second assessment; 38 (26.0%) were administered a criterion measure. Assessors were required to demonstrate reliability by independently scoring assessments conducted by the lead researcher and achieving an inter-scorer reliability of 90% over three successive tests.

Results

The maximum possible total score on the TBPMA is 118. Among the 146 initial tests, scores ranged from 5 to 112 (M=62.44, SD=28.01). Assumption checks indicated the total scores were normally distributed. The correlation between child's age in months and total score was r=.572 (p=.000).

There were two specific questions for the analysis:

1. What is the contribution of each sub-construct to the general construct?
2. What is the technical adequacy of the TBPMA?

Regarding question 1, confirmatory factor analysis (AMOS 22) was used to evaluate the hypothesized model for the TBPMA. Results indicated good model fit: $\chi^2(23,N=146)=25.297$, p=.335; NFI=.974; CFI=.998; and RMSEA=.026. All standardized path coefficients were highly significant (p<.001), with robust standardized regression weights ranging from .57 to .90 (Figure 2), indicating all scales were strong contributors to the latent variable mathematical development.

For question 2, technical adequacy was assessed through Cronbach's alpha, test-retest reliability, and criterion validity. Cronbach's alpha was .922, indicating high internal consistency.

Test-retest reliability was assessed with paired t-tests on mean differences for the nine scales and total score (Table 1). Differences significant at the .05 level occurred with the addition/subtraction scale, in which the mean retest score was one point higher on the 16 point scale, and with the total TBPMA score, about 2.5 points higher on the 118 point scale. Given that the average time between the initial test and the retest was 24 days, these small increases do not appear to affect the reliability of the TBPMA. Reliability is further confirmed by the highly significant correlations between the initial and retest scores. For all scales and the total score, all correlations were strong, above .50.

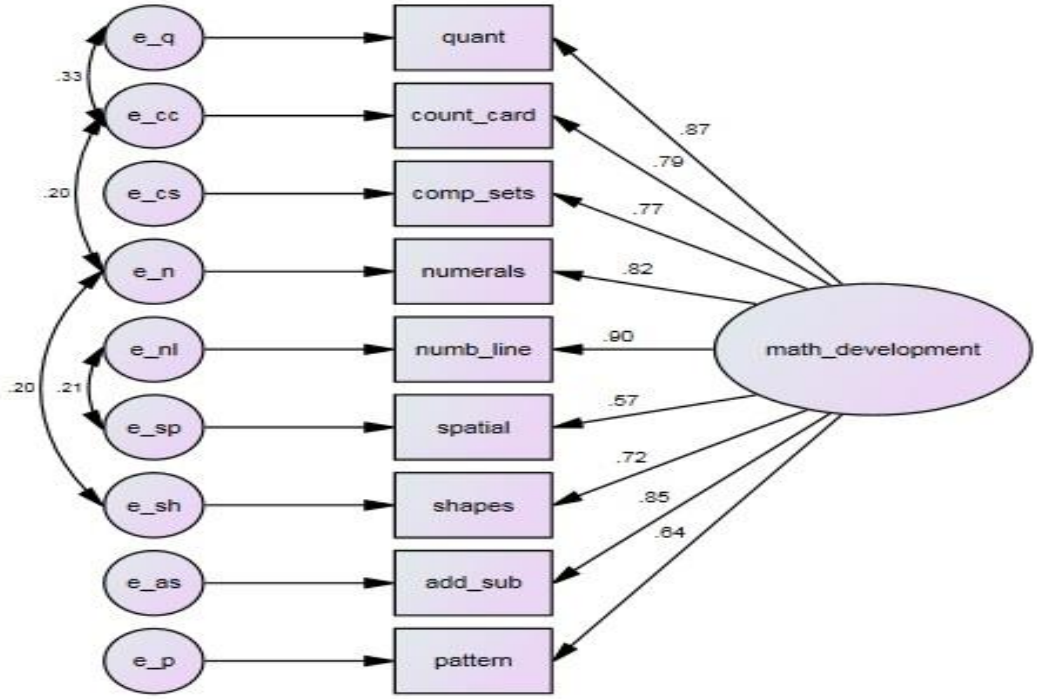

Figure 2: Standardized Path Coefficients for the Confirmatory Factor Analysis for the Teddy Bear Preschool Math Assessment

Scale	Max score poss-ible	Initial Test		Retest		Difference in means		*T* test on difference in means		Pearson correlations	
		Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*	*t*	*Sig.*	*r*	*Sig.*
Quantification	20	14.58	4.77	15.04	4.49	0.46	2.42	1.34	.186	.864	.000
Counting/Cardinality	20	12.52	4.84	13.32	4.47	0.80	3.30	1.72	.092	.752	.000
Comparison of Sets	16	7.32	4.30	7.48	4.02	0.16	2.97	0.38	.705	.747	.000
Numerals	9	5.92	3.43	5.90	3.39	-0.02	1.17	-0.12	.904	.941	.000
Number Line	12	7.22	4.86	7.38	5.06	0.16	2.61	0.43	.666	.863	.000
Spatial	6	5.02	0.96	5.18	1.17	0.16	1.02	1.11	.272	.560	.000
Shapes	12	5.96	2.76	5.52	2.87	-0.44	2.44	-1.27	.209	.623	.000
Addition/Subtraction	16	4.72	4.29	5.76	4.78	1.04	3.55	2.07	.043	.698	.000
Patterning	7	2.58	2.98	2.72	2.98	0.14	2.08	0.48	.636	.756	.000
Total	118	65.84	27.30	68.30	27.67	2.46	6.80	2.56	.014	.969	.000

Note: This table uses data from the 50 subjects who were assessed twice.

Table 1: Test/Retest Reliability: Means, Test of Differences between Means, and Correlations between Means of Initial Test and Retest

TEMA-3 (Ginsburg & Baroody, 2003), a recognized measure of early mathematical ability, was used to assess criterion validity. The correlation between the total TBPMA score and the TEMA-3 raw scores was high (r=.867, p=.000), demonstrating criterion validity.

These results indicate that the TBPMA was a valid and reliable measure of mathematical development for this sample.

Significance

Preschool educators and funding sources recognize the need for accountability and for monitoring children's development. Conversely, they understand that testing is often stressful for children, unreliable, and unrelated to the curriculum. The TBPMA offers a validated approach to quantifiable assessment that is compatible with young children's social and emotional development; all children assessed to date have enjoyed the experience.

The TBPMA is a novel alternative to traditional quantitative assessments. It can provide both formative data for teachers and summative data for program evaluation. Because it assesses multiple domains of mathematics, it can indicate areas of strength within a curriculum and areas that may require more attention. Based on acknowledged developmental trajectories, it informs teachers of the child's current level of thinking so that they can target their interactions to move the child forward. This is the essence of constructivist teaching.

It is natural for educators to focus on material from class or program assessments (i.e., teach to the test). The TBPMA reverses this process. It evaluates the types of experiences and interactions that are highly recommended for preschool classrooms and the areas of mathematics designated as focal points (NCTM, 2006).

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Examining difficulties in initial algebra: Prerequisite and algebra content areas for Irish post-primary students.

Aoife O'Brien[1] and Máire Ní Ríordáin[2]

[1]School of Education, National University of Ireland, Galway; a.obrien47@nuigalway.ie

[2]School of Education, University College Cork; maire.niriordain@ucc.ie

This research aims to investigate the algebraic under performance of second year post-primary students in Ireland (approximate age 14 years). To this end a diagnostic test for algebra has been developed to profile and identify students who are struggling with algebra. This paper examines the development of the test, which involved the identification of key mathematical content areas that are critical for success in algebra. Both prerequisite, and algebra content areas are key to a students' success in algebra and how each of these areas contribute to a students' progress with algebra is discussed in this theoretical paper. Test items have been selected and adapted from the literature which are aligned with both the key content areas and the Irish mathematics syllabus at junior cycle, the initial three years of post-primary education in Ireland.

Keywords: Algebra, diagnostic test, secondary education.

Introduction

This paper reports on the development of a diagnostic test for algebra, designed specifically for second year post-primary students in Ireland. It is noted in the literature that few adequate assessments are available to provide formative information on a students' progress with algebra, but they are essential to allow timely and informed instructional decisions for teachers (Ketterlin-Geller, Gifford, & Perry, 2015). A key outcome of this research will be a validated diagnostic test specifically designed to identify students' conceptual errors when working with algebra and aligned with the Irish post-primary mathematics syllabus. The test can be used by teachers in a normal class period (35-40 minutes as relevant to the Irish context) to help inform their instruction of algebra. This paper focuses on identifying key content areas which are required for success in algebra and accordingly developing a bank of questions to support the development of a diagnostic test.

Background and rationale

The mathematical deficiencies of students entering third level education in Ireland and internationally is widely reported and commonly referred to as the "Maths Problem" (Treacy & Faulkner, 2015). Over a decade ago it had become apparent that there were issues with the mathematics curriculum at post-primary level in Ireland resulting in radical reform of the syllabi, introduced as "Project Maths" (Treacy & Faulkner, 2015). The vast majority of students study mathematics throughout their time in post-primary education and the syllabus is delivered in five interwoven strands; 1. Probability and Statistics, 2. Trigonometry and Geometry, 3. Number, 4. Algebra and 5. Functions. Mathematics is offered at higher, ordinary and foundation levels throughout. Despite the changes made to the Irish syllabus and teaching methods, it is clear that problems with algebra persist. This is evidenced in the latest chief examiners report which states students need to "gain comfort and accuracy in the basic skills of computation, algebraic manipulation and calculus" (Department of Education and Skills, 2015 p. 29). Algebra serves as a gateway to higher mathematics and deficiencies in basic algebra result in overall mathematical deficiencies in students (Lawson, 1997). However, issues with the

teaching and learning of algebra remain internationally, and research into progress measuring instruments to inform the area continue (Ketterlin-Geller et al., 2015). Given the poor performance of Irish students in algebra a diagnostic test has been developed to give insight into how second year students perform in this area. It is noted that many tests exist to measure performance in algebra and these provide useful information about the content areas in which students need further assistance. However, few of these tests provide information about why students are struggling (Russell, O'Dwyer, & Miranda, 2009). This test may help inform part of why students are struggling by looking at the conceptual errors students make in other content areas. These content areas, which include fractions, equality and patterns for example, have been identified in the literature as essential for the understanding of algebra (Bush & Karp, 2013; Warren & Cooper, 2008).

Theoretical frame

The theoretical framework guiding this research is that the conceptual errors students make in working with algebra perpetuate the difficulties they encounter, subsequently interfering with their understanding of various algebraic concepts (Russell et al., 2009). The test is designed to identify the conceptual errors students make when working with a particular concept. The test results could inform the teacher who can then guide instruction aimed at alleviating these conceptual errors (ibid.:2009). It is possible that many teachers do not realise the essential connections that algebra has to numerous other mathematical content areas, and that how conceptual errors in one or more of these areas can hinder a students' progress, which will be discussed in more detail below (Bush & Karp, 2013). This test, in conjunction with an appropriate framework to interpret students' answers, could help inform instruction. The teaching and learning of algebra in Ireland aligns closely with Kieran's (2004) model, which outlines three activities that learners of school algebra must participate in: 1. Generational activities, 2. Transformational activities and 3. Global/Meta Level activities. It is noted that two aspects of algebra underlie all others, namely generality and abstraction (Department of Education and Skills, 2016 p. 26). These aspects have led to the definition of three types of algebraic activities that mathematics students in Ireland must engage in - representational, transformational and activities involving generalising and justifying. The test items have been designed to support the implementation of the new syllabus and in line with the approach to algebra being used in Ireland.

One potential criticism of the approach used in developing this test instrument is that it is a test of mathematics globally, given the broad range of content areas included. However, difficulties with algebra lie with both algebra itself and also with other areas of mathematics that students will have encountered, which are seen as prerequisites for the learning and understanding of algebra (Bush & Karp, 2013). Problems and conceptual errors in any one of the content areas can lead to problems in another area thus hindering a student's progress in algebra as a whole. Although these prerequisite areas support other areas of mathematics learning such as number and numerical operations, they are identified in the literature as core areas for the learning and understanding of algebra (Bush & Karp, 2013). The following section outlines the content areas identified throughout the literature as pertinent to success in algebra, on which test items have been based.

Prerequisite and algebra content areas

As stated the focus of this paper is the key content areas required for success in algebra. These content areas are outlined in Table 1 (Bush & Karp, 2013; Warren & Cooper, 2008).

Content Areas	Junior Cycle Syllabus
Ratios and proportional relationships	*3.1 Number Systems:* - consolidate their understanding of the relationship between ratio and proportion. 4.4 Examining algebraic relationships: – proportional relationships
Fractions	*3.1 Number Systems*: - Investigate models to think about operation on fractions. - Use the equivalence of fractions, decimals and percentages to compare proportions.
Decimals and Percentages	*3.1 Number Systems*: - Calculate percentages - Use the equivalence of fractions, decimals and percentages to compare proportions.
Integers	*3.1 Number Systems*: - Investigate models, such as the number line, to illustrate the operations on integers
Exponents	3.2 Indices
Order of operations	*3.1 Number Systems*: - Appreciate the order of operations, including use of brackets
Properties of numbers	*3.1 Number Systems*: - Investigate the properties of arithmetic and the relationships between them.
Compare and order numbers	*3.1 Number Systems*: - Use the number line to order natural numbers, integers and rational numbers. - Use the equivalence of fractions, decimals and percentages to compare proportions.
Equality	*3.1 Number Systems:* - Consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value.
Variables	4.6 Expressions: - Using letters to represent quantities that are variable.
Algebraic expressions	4.6 Expressions: - Arithmetic operations on expressions. - Transformational activities
Algebraic equations	4.7 Equations and inequalities: - Selecting and using suitable strategies for finding solutions to equations and inequalities.
Functions	*4.2 Representing situations with table diagrams and graphs:* - use tables, diagrams and graphs as a tool for analysing relations – present and interpret solutions, explaining and justifying methods, inferences and reasoning.
Patterns	*4.1 Generating arithmetic expressions from repeating patterns:* - use tables and diagrams to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence.

Table 1: Content Areas for Diagnostic Test and alignment with the Junior Cycle syllabus

Related content domains from the Irish syllabus were used as a framework to align the prerequisite and algebra content areas identified in the literature. The Number Strand (3) of the syllabus builds on primary school learning and facilitates the transition from arithmetic to Algebra (Strand 4). The Common Introductory Course (CIC) is the minimum course to be covered by all students at the beginning of the junior cycle, elements from the CIC are in italics within Table 1. The numbering

within Table 1 refers to the strand numbers, for instance 3.2 refers to section 2 of strand 3. The diagnostic test has been developed for use with students in second year and therefore it was important to align the test items specifically with the content of the CIC. Once the CIC is complete teachers use their own discretion to introduce their topics (Department of Education and Skills, 2016). There is no prescribed structure for following the syllabus however, it is desirable that students will have completed their basic algebra skills including equation solving by the end of first year (Project Maths, n.d.-a).

Prerequisite content areas

Difficulties with algebra lie with both algebra itself and also with other areas of mathematics that students will have encountered, which are seen as prerequisites for the learning and understanding of algebra. Proportional reasoning is a key aspect of numeracy and it leads to relational thinking which is important in the development of algebraic skills, it is highly conceptual and a skill that develops gradually. Equally, fractions are an integral part of algebra and can be found as coefficients, constants and solutions to equations, the slope of a line, and, in general, proportions are written in fraction form in algebra (Bush & Karp, 2013). Knowledge of decimals, their value and placement on a number line, computation with decimals, and the ability to convert between decimals, fractions and percentages is also important for success in algebra (Bush & Karp, 2013). Studies have been conducted to identify what element of decimal and fraction understanding best indicate a students' performance in algebra. It has been found that the relational understanding of the bipartite format of a fraction and unidimensional magnitude, measured with the placement of decimals on a number line are the best predictors (DeWolf, Bassok, & Holyoak, 2015).

In learning about fractions, decimals and percentages, with the use of number lines and graphs, students at junior cycle are expected be able to compare and order numbers. This provides students with the skills and knowledge to apply the rules correctly when working with variables. It also enables a student to assess if a solution to an equation or inequality is reasonable. However, if students do not understand a fraction, decimal or percent, they are unable to extend their understanding to which is greater than or less than or equivalent (Bush & Karp, 2013).

Furthermore, a solid understanding and procedural fluency with integers is required for success in algebra. Misconceptions about negative integers can impede progression, where, for example, a student may fail to accept a negative number as a solution to an equation. Research has suggested that the number line and graphs of functions can be used to help correct students' misunderstandings and conceptual errors with integers (Bush & Karp, 2013). Equally, an understanding of exponents is required for both the transformational skills, in dealing with expressions, and the generational and global/meta level skills, where knowledge of the shape of functions are required (Bush & Karp, 2013). Moreover, to succeed in the transformational rules of algebra it is essential to understand the order of operations. Some students believe that order of operations do not matter, that the same answer will result regardless. Others believe that the context of the problem determines the order of operations and in the absence of context operations should be performed from left to right. Research suggests that students should learn the hierarchy of operations more naturally by attending to more complicated operations first (Bush & Karp, 2013).

Finding equivalent expressions is frequently required in algebra, and this manipulation requires an underlying sense of the properties of numbers. Allowing students to investigate the properties of numbers will assist in learning, retaining knowledge and developing relational understanding, which in turn will create a strong foundation for algebra (Bush & Karp, 2013). Numerous studies have focused on development of the concept of the equal sign in the early stages of learning algebra (Bush & Karp, 2013). Students often misinterpret the meaning of the sign viewing it as an operational sign. Those who interpret the equal sign correctly and see it as a relational symbol have more flexibility when working with equations.

Algebra content areas

Kieran (1992) asserts that many misconceptions and common errors in algebra are generally rooted in the meaning of symbol or the letters used in algebra. Much research has been conducted into students' difficulties in working with algebraic variables and the misconceptions student's hold. These misconceptions include viewing variables as labels, the belief that the value of a variable has something to do with its position in the alphabet, and the belief that a variable is just a missing value rather than something which has varying values. These difficulties are then compounded when a student attempts to create and manipulate an algebraic expression (Bush & Karp, 2013).

The underlying misconceptions and difficulties students hold in relation to variables, expressions and indeed all the prerequisite content areas can then lead to difficulties in solving algebraic equations. The ability to solve equations is reliant on both procedural and conceptual understanding. Conceptual understanding is strongly related to student's equation solving performance, as without it students learn by rote a series of transformational rules for dealing with equations. A solid understanding of how to use variables to write algebraic expressions, form subsequent equations and solve when necessary is the essence of success in algebra at junior cycle level (Bush & Karp, 2013).

A function is defined as a correspondence between two sets (Kieran, 1992), and there are two general approaches to teaching and learning functional relationships mentioned in the literature; a correspondence approach and a covariation approach (Ayalon, Watson, & Lerman, 2015). The correspondence approach deals with an input-output model, whereby an output value y is calculated for a given input value x, often listed in a table of values or as couples. This approach allows for determining the rule which generates the y-value from the x-value and is in line with the approach to teaching functions in Ireland. The concept of a function is not simple when you consider that at least three representations are used to convey the notion of a function; a table, a graph and an equation. True procedural fluency and competency in working with functions is obtained when one can move between the different representations of a function with ease and this aligns with the multi-representations approach advocated in the Irish syllabus (Bush & Karp, 2013; Project Maths, n.d.-a)

Finally, algebra can be seen as the language used to describe patterns and relationships for the ultimate goal of problem solving and as a systematic way of expressing generality (Project Maths, n.d.-a). Students at junior cycle learn to identify the relationship which lies between the pattern and its position is a functional relationship meaning an expression or formula must be created using variables. In doing this a context for the use of variables is set for Irish students, assisting their understanding of a variable as a varying quantity rather than a specific unknown, laying down the

foundation for understanding expressions and solving equations in what is known overall as a functions based approach to algebra (Project Maths, n.d.-a).

The diagnostic test

Test items were taken from previous relevant studies pertinent to measuring ability in the core content areas required for algebra outlined in Table 1. The diagnostic test currently contains twenty one questions summarized in Table 2 where the source of each test question is detailed.

Content Areas	Test Question Number and Source
Ratio and proportion	1. Number Line/Decimal Magnitude from DeWolf et al. (2015) 7. Proportional reasoning from Hilton, Hilton, Dole, and Goos (2013)
Fractions	2. 4. 5. Fraction Knowledge from DeWolf et al. (2015) 3. Fraction Knowledge multiplication (Bush & Karp, 2013)
Decimals and Percentages	1. Number Line/Decimal Magnitude from DeWolf et al. (2015) 11. Comparing and Ordering Numbers, Project Maths (n.d.-b)
Integers	15. Integers and equations adapted from Vlassis (2008)
Exponents	6. and 8. adapted from discussion in Mok (2010)
Order of operations	9. Order of operations adapted from Linchevski and Livneh (1999)
Properties of numbers	10. Distributive property, adapted from discussion in Mok (2010)
Comparing and ordering numbers	11. Comparing and Ordering Numbers, Project Maths (n.d.-b)
Equality	12. 13. adapted from Stephens, Knuth, Blanton, Isler, Gardiner, and Marum (2013)
Variables	14. Variable as label adapted from Küchemann (1981)
Algebraic expressions	16. adapted from Hodgen, Kuchemann, Brown, and Coe (2009) 17. Simplifying expressions based on errors discussed in Kieran (1992)
Algebraic equations	20. adapted from Clement, Lochhead, and Monk (1981) 18. 19. Next step of solution adapted from Chung and Delacruz (2014) 21.3 Forming equations adapted from Ayalon et al. (2015)
Patterns	21 Interpreting from a geometric pattern from Ayalon et al. (2015)

Table 2: Summary of content and source of items on the diagnostic test

An example of a test item, which assesses relational fraction knowledge, together with understanding of a variable and algebraic expression is taken and adapted from (DeWolf et al., 2015) as follows;

n is a whole number greater than 0. If n continues to get bigger in value, please circle one of the following options A, B or C in the answer box for what happens to $\frac{1}{n}$.

Hint: Think about the following sequence of numbers $\frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \ldots$

Figure 1: Question 5 on the diagnostic test based on relational fraction knowledge

Students have space for workings and are then asked to circle the correct answer from the following options; A. $\frac{1}{n}$ gets very close to 1, B. $\frac{1}{n}$ gets very close to 0, or C. $\frac{1}{n}$ increases in value too. Adaptations from the original question include changing the word "integer" to "a whole number" and offering the "Hint", to ensure the question is more in line with the age profile of those being tested as informed by the pilot of this test and feedback from teachers. The above question was included as relational understanding of a fraction was an element identified in the DeWolf et al. (2015) study to predict performance in algebra. All test items have been developed with such a theoretical underpinning that is using multiple choice responses based on possible conceptual errors. In addition language was adjusted where necessary to make the test items more accessible for fourteen year old students.

Conclusion

There are fresh concerns in relation to student attainment in mathematics in Ireland, specifically algebra and for progression to third level education (Treacy & Faulkner, 2015). There is a clear need to intervene early in the effort to address the issues students are facing with learning and understanding algebra. The overall aim in using this test is to identify conceptual errors that students make in both algebra and the prerequisite content areas required for success in algebra, therefore assisting to identify possible root causes of the students' errors, and as a result, through appropriate intervention improve students' knowledge of algebra and therefore general mathematical ability. Ultimately, this will be a tool for teachers to use in the classroom allowing them to make informed decisions and to plan appropriate interventions (Russell et al., 2009).

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Analysing students' graphicacy from a national test

Oduor Olande

Linnéuniversitetet, Sweden; oduor.olande@lnu.se

The present study explores a task containing graphical artefacts from the Swedish national test (Nationella provet – NP) for a sub-sample of grade 9 students' solutions. The sub-sample comprising of 115 students' solutions to the task is closely analysed, using an analytical construct founded on "identification" as well as the "critical-analytical" approach to problem solving. Based on this construct it is observed that a sizable number of students' solutions follow a visual strategy with strong reliance on everyday forms of expression. Given the nature and purpose of NP we posit that students use the methods and tools that reflect general school practice. The analysis used in this study is perceived as problematising the assessment of competency from high stake tests, and the educational setting in general.

Keywords: Assessment, examinations, graphical artefacts, misleading diagram, competencies.

Introduction

National tests generally play a vital role in the evaluation of educational systems, informing educational reforms and policy implementations as well as serving as a local comparative instrument (Eurydice, 2009; cf Skolverket, 2016). This implies that results from national tests can be used to explore the effects of curriculum documents on teaching and learning. Some researchers, (e.g. Boesen 2006) have suggested that centrally administered tests can influence instructional practices, for instance by guiding the time set aside to teaching specific topic strands in the mathematics classroom. Given the orientation towards competencies in the mathematics classroom, it is also envisaged that the Swedish national test (Nationella provet – NP) might be a suitable indicator of teaching and learning of mathematics competencies. Thus, national tests can be perceived as having the potential to provide insight on students' fluency with mathematical concepts, as well as mathematics competencies (NCTM 2000; Niss & Højgaard, 2011; Skolverket, 2011; see also Sáenz, 2008).

The goals of the study

The aim of the present study is to gain insight into the strategies and approaches that some students at the end of the compulsory school in Sweden employ, as they interact with a mathematics task containing graphical artefacts. This study is based on written solutions from NP. The purpose is to explore the strategies and tools used to solve this task, as well as to determine the potential provided by a model focusing on mathematical tools and forms of expression as a means of exploring students' mathematics competency.

Background and theoretical construct

Several methods have been used to analyse students' responses to test items (e.g. Åberg-Bengtsson, 2005; Goodchild & Grevholm, 2007). In some of these studies the focus has been on general performance with microanalysis done at topics strand levels. While there are models proposed for examining the response to tasks (e.g. Gal, 2002, 1998; Ben-Zvi & Arcavi, 2001; Watson & Callingham, 2003), these seem to suggest general skills needed to solve the task with a bias towards statistical literacy. Friel, Curcio and Bright (2001) developed a three-tier model entirely devoted to

graphical artefacts: without loss of generality this can be collapsed into two tiers (cf. Gal, 1998; Bertin, 1983; Olande, 2013). While there are models for analysing students' test results, there is a dearth of studies exploring the use of subject specific tools and forms of expression in students' solutions.

In the present study an analytical construct (see Olande, 2013), focusing on the application of mathematics tools and forms of expression while solving test items, is employed. In this construct response to test items is perceived as being oriented towards *identification* and *critical-analytical approaches*. Items eliciting the identification approach are largely "self-evident*"* and as such, problem solvers might not need to unpack their mathematical skills entirely in order to solve the problem. On the other hand, items seen as demanding a critical-analytical approach require focused engagement from the problem solver. For example, this could be in the form of a critical analysis of underlying factors in the task, evaluation and selection of appropriate tools for interacting with the task, as well as reporting the solution with relevant subject specific forms of expression.

The construct guiding the present study borrows from a socio-semiotic paradigm where the emphasis is on artefacts as a means of coming to know. Radford (2008) posits that the investigation of students' interaction and use of semiotic means of objectification is a methodological way of accounting for learning (see also Radford, 2003; Vygotsky, 1978). It is recognised that a sign or symbol does not exist in isolation but is always bound with intentions, motives and the objects of action (Roth, 2008; Olande, 2014). Thus, in a critical-analytical approach, "being critical" encompasses more than visually interrogating a graphical artefact, but also includes the means and the tools used in the sense making process. The assumption about what can be perceived as a non-hierarchical path to cognition is significant for the analytical framework: i.e. it is access to tools and forms of expression that is perceived as largely determining a problem-solving trajectory.

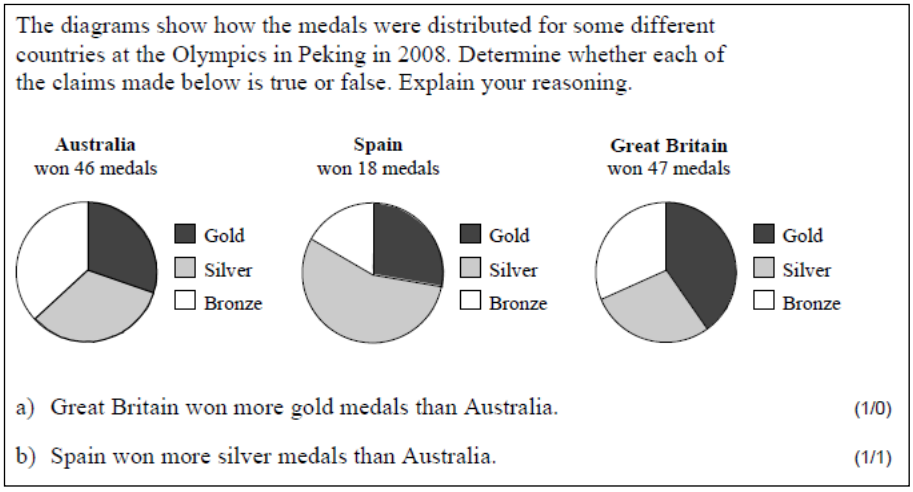
The research questions

Thus, the concern of the study is to outline and analyse tools and forms of expressions used by students as they interact with a task from NP. This task was picked from section C of the national test, a section that requires students to justify in one way or another how they arrive at their solutions. The task (figure 1) was selected for further analysis given that it provided a combination of *visual-identification* as well as *critical-analytical* components. This way of assessing the task is different from the marking process employed by graders while awarding credit to students' solutions. The marking scheme used by graders did not indicate assessment of diverse solutions provided by students but largely gave written statements as guidelines. Thus item *a)* scored full credit when the solution contained the expression TRUE with corresponding justification such as making a comparison based on the sizes of the shaded areas. For item *b*) partial credit was awarded where the solution contained the expression FALSE with corresponding justification indicating an understanding that the pie charts express different quantities e.g. "Australia has more medals than Spain". Full credit was awarded where the solution, in addition to the general statement FALSE, explicitly made comparisons based on mathematical forms of expression e.g. computation with fractions. A reliability check conducted by an independent entity (Skolinspektionen, 2010) from a representative sample of Swedish students for section C of the test, indicated that for items scoring grade G - (for the purpose of this study *partial credit*) in 51% of the cases the graders gave higher grades than the assessors. In 27% of the cases there was correspondence in the credit award. For items scoring VG - "*full credit*" in 37% of the cases

the graders' credit award was higher than that of the assessors, with 41% indicating correspondence in the credit award.

From a competency perspective (Skolverket, 2011), this task might be considered to focus on developing competencies in the following: using and analysing *mathematical concepts*, use of appropriate *mathematical methods* to solve problems, and the use the of mathematical forms of expression to discuss, *reason and give account* of questions, calculations and conclusions. However, given that the task explicitly called for a justification of the student's given solution, it was generally perceived as eliciting a critical-analytical approach. The success rate for item 9a was 78% for both girls and boys, while the success rate for item 9b was 47% and 55% for girls and boys respectively.

Figure 1: Task No. 9 from NP Sweden



The research questions are outlined as:

1. What range of tools and forms of expression are made manifest as students interact with the task containing graphical artefacts?
2. How does tool selection and use impact on the characteristics of the task solution provided?

Task analysis

Students' solutions to task No. 9 were closely analysed with respect to: *a)* forms of expression (see Table 1 for typical examples of student responses) mathematical, everyday, graphical or one-word and *b)* tool use/sign of tool accessibility – mathematical operators and symbols. With regard to tool use seven approaches to task solving were identified, namely: *i)* visual comparison of graphical artefact, *ii)* critical - questioning the production of the graph *iii)* fraction i*v)* proportion *v)* percent *vi)* division *vii)* multiplication and *vii*) other solutions.

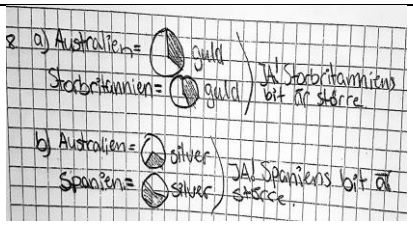
Mathematical	Everyday	Graphical	One-Word
Britain ≈ 5/12 ≈ 0,40 40% gold (18,8 medals) Australia ≈ 35/120 ≈ 0,30 30% gold (13,8 medals) Answer: true	True: Britain obtained most gold in its diagram and Australia more silver.	 Yes! Britain's bit is larger Yes Spain's bit is larger	False

Table 1: Forms of expression identified from test task

Based on aspects of *identification* and *critical-analytical approaches* (Olande, 2013) and aspects of Sfard's (2008; cf. Schleppegrell, 2007) categories of mathematical discourse, the author developed a coding scheme and analysed the students' solutions. The focus was on the forms of expression and *mathematical* tools – concepts used in working out a solution, rather than the correctness of students' solutions. The correctness or otherwise of the item solution was pegged on the test score awarded by test graders within the framework of the test situation. Significantly, within this coding scheme the perception of the forms of expression was as follows: *1*) everyday – the use of causal expressions wherein aspects of the *obviousness* of solution are embedded. In Table 1, while the type of language used in the solution is *everyday*, it is apparent that there are quantities being compared: *"Britain obtained* ***most*** *gold*" and *"Australia* ***more*** *silver" 2*) mathematical – the use of mathematical concepts and methods in the solution *3*) graphical – the use of illustrations in an attempt to amplify the visual aspects of the task. In the illustration given in Table 1 above while the solution for item *b)* did not receive credit award from the graders, the solution appears to be appealing to the visual faculties in a comparison exercise. Zooming in on gold or silver, the student seem to be posing the question *can't you see they are different?*

Results and analysis

The success rate for task No. 9 seems to indicate that students did not have as much difficulty with aspects of item 9*a* as compared to item 9*b*. This can be explained in part by the nature and the array of tools needed for effective interaction with the different items. Thus, the different forms of expression of the items were analysed.

	Everyday	Mathematics	One-word	Graphical
Form of expression	**0.72**	**0.15**	**0.11**	**0.02**
Correct responses	0.89	0.95	0.32	1.00
Incorrect responses	0.11	0.00	0.60	0.00
No response	0.00	0.05	0.08	0.00

Table 2: Forms of expression identified from item 9*a*

For this item, a majority (72%) of the students' solutions used everyday forms of expression, of these 89% provided successful solutions. While only 15% of the students used mathematical forms of expression, the success rate was relatively high at 95%. Students providing a one-word response to the task gave the majority of unsuccessful solutions. For this item it was observed that more than 50% of the students used visual comparison and/or the comparison of totals.

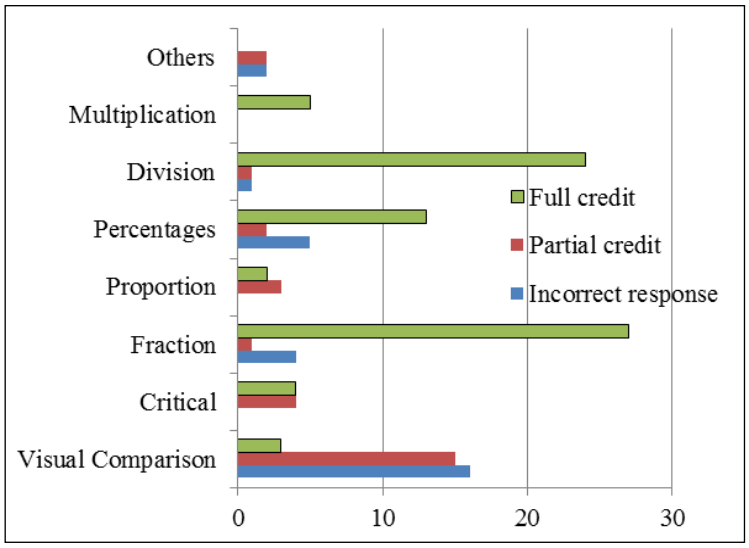
The general pattern for forms of expression for item 9*b* was no different for item 9*a* (see Table 2).

	Everyday	**Mathematics**	**One-word**	**Graphical**
Form of expression	**0.50**	**0.32**	**0.13**	**0.04**
Full credit	0.29	0.91	0.00	0.25
Partial credit	0.43	0.04	0.13	0.00
Incorrect responses	0.27	0.02	0.79	0.75
No response	0.00	0.05	0.08	0.00

Table 3: Forms of expression identified from item 9*b*

It is noteworthy that students providing one-word answers and those using graphical strategies recorded rather higher rates of incorrect responses. Evidently, students' results indicate that the use of forms of expressions reflects one way or another on the success rate of their solutions. Consequently, it is of particular interest to explore the subject specific tools and forms of expression used in solving the task, and how these might have impacted on the quality of the response provided. This analysis is conducted for item 9*b* (see figure 2)

Figure 2: Frequency of tool use and credit award



From figure 2 it is shown that most of the solutions awarded full credit included the application of mathematical symbols and calculation. For these solutions the predominant tools of manipulation/calculation were fraction, division and percent. The solutions providing incorrect responses seem to be largely based on visual strategies.

From the results it is apparent that students using mathematical tools and forms of expression had a relatively higher success rate. The predominant tools used in solving the task were fraction, division and percentage. The frequency with which fraction and division are used is not entirely unexpected since these tools are closely related; the same applies to some extent to percent. It could be suggested that the task in some way elicits the use of fraction: It is not uncommon for mathematics teachers to use pie charts in the teaching of, or in the introduction of, the topic strand of fraction. From this approach there is a natural connection to percentage; a circle divided into two gives two halves – this is often perceived as 50-50 (%). The results also indicate that some of the students using these tools (fraction, percentage) experienced some difficulty in application. It is worth noting that the students using multiplication obtained full credit. These students seem to employ a tool that might not be considered "self-evident" for the task. This necessitated using the tool in a creative manner, thus indicating a higher level of confidence and "procifiency" in using the tool.

For the students scoring full credit for the items, it is observed that most of them are proficient in the use of item specific tools namely, fraction, division and multiplication. From the students' solutions it seems that only a small number indicate some aspect of interrogating the purpose of the graphics. Indeed there seems to be a general scarcity of solutions communicated using subject specific forms of expression, that is, mathematical language.

Discussion

The present study sought to apply a construct focusing tools and forms expression in solving mathematics tasks containing graphical artefacts. The purpose was to outline the usage of tools and forms of expression, and the quality of the solution thereof. A deeper analysis of task No. 9 provided more insight into strategies and tools used in interacting with graphical artefacts. It is observed that it is in part the grasp of the tool in use that determines the quality of the solution given. Based on the level of confidence in tool use, it is possible for the test taker to interrogate the task from different perspectives. Based on what can be considered as overlapping tool use (see figure 2), the results are perceived as suggesting that "reading" a graphical artefact can be a complex undertaking that might involve *reading the graph – reading within the graph – reading beyond the graph* (cf Friel et. al., 2001). In the case of this task, there is indeed a different array of tools available to the students as they solve the task. However, it is communication using subject specific forms of expression that appears to be wanting – this might be an indicator that as much as the tools are available and "visible" to some of the students, the competency to apply and organize the same to produce a sound solution is a major challenge. This was observed in the case of students indicating knowledge of appropriate tools needed to solve the task, but apparently lacking the necessary skills to effectively apply the same in a problem-solving situation. Thus the observation made in the present study underscores the importance of having a solid foundation in the use of mathematical tools and forms of expression (concepts) in different settings. The importance of the use of subject specific forms of expression is also observed. For item *b)* there was higher correspondence in credit award between the graders and assessors as compared to item *a)* which did not elicit the use of the subject specific forms of expression as such. The analytical framework employed in the present study also helped to identify the strengths and weakness in students' written solutions, thus providing valuable indicators for developing classroom practice.

Given the interest in, and the focus on mathematics competencies in the Swedish mathematics classroom, the present study can also be perceived as drawing attention to the practice of assessment: if the concept of mathematical competencies implies imparting aspects of such skills as mathematicians use in the processes of mathematisation (cf Niss & Højgaard, 2011; Sfard, 2008), then assessment practice might need to be refocused to examine such use of tools and forms of expression that enable the learner to understand, and to participate in, activities within the mathematics community.

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Assessment in mathematics as a lever to promote students' learning and teachers' professional development

Julia Pilet[1] and Julie Horoks[2]

[1]Laboratoire de Didactique André Revuz, Université Paris Est Créteil, France; julia.pilet@u-pec.fr

[2]Laboratoire de Didactique André Revuz, Université Paris Est Créteil, France; julie.horoks@u-pec.fr

In this paper, we are presenting our analytical tools to characterize assessment activities as part of teachers' practice, on a specific mathematical content (algebra). We are also presenting the principles of our collaboration with high school teachers, inside a particular workgroup (LéA), to explain why we came to consider assessment as a potential lever to enhance both the students' learning in mathematics, and the teachers' development. We are presenting a few results on the effects of this collaborative work on teachers' practice when assessing students' learning, and on our means to analyze the students' results throughout the process.

Keywords: Assessment, algebra, teachers' practice, teachers' professional development, collaborative work.

In this paper we are presenting the analytical framework that we are building to characterize the practice of high school teachers in mathematics regarding student assessment. We study assessment throught one of its particular functions, promoting learning, with a didactical point of view, focussing on a particular content (algebra). We consider three inputs for assessment: assessment as a framework to characterize teachers' practice, assessment as a tool to enhance students' learning in mathematics, assessment as a lever for professional development. We will present some results on teachers' professional development, from the collaborative work we lead with high school teachers on assessment.

Assessment as a framework to characterize teachers' practice

In this first part, we are presenting the framework that we have built to analyze teachers' assessment practice, leaning on previous studies about teachers' practice, and on the teaching of algebra in particular.

Defining assessment practice

Assessment can be found in many aspects of the teachers' activity, and it would be easy to call assessment any interaction between the teacher and the students. To restrain our observations, we draw upon De Ketele's definition (1989), and call assessment any gathering of information by the teacher on the students' activity and knowledge, and the interpretation and use of this information.

By activity, we mean everything the students do, say, think, (or do not do). Of course, not everything is accessible to neither the researcher nor the teacher, but we consider that students' learning happens through their mathematical activities, at least partially (cf. Rogalski, 2013, about the use of Activity Theory as a framework for research in the Math Education field). These activities may consist on participating in a debate around a task or listening to a mathematical discourse in class. But in many occasion, they result from the tasks proposed by the teacher, and on the choices that the teacher makes

to manage the solving of the task, which are elements that we consider when analyzing the teachers' practice in class.

Assessment is easier to pinpoint for the researcher, when it is formal, for example a written summative test at the end of a teaching sequence, or short diagnostic tests happening at the beginning of every session. Informal assessment, on the other hand, is more difficult to identify, but can happen in many occasions during the class, through the interactions between the teacher and the students, giving information on the students state of knowledge, to the teacher or to the students themselves. To be able to characterize assessment practice in any case, we have drawn a list of criteria, whether the assessment is formal or not, and at any point in the teaching sequence.

Characterizing assessment practice

One of the elements that we take into consideration to characterize assessment practice, is the distance between each assessment task and the similar tasks previously given by the teachers on the same mathematical content (Horoks, 2006). For example, when a teacher is assessing the students' knowledge on a mathematical content through a final test, we can question the choices of tasks made by this teacher and their link with the tasks that were actually worked on before the test. A certain gap between the test's and previous tasks (or between tasks from a diagnostic test and all the possible prerequisite tasks) can be interpreted in different manners: it could be explained by the function given to this test, (rewarding or challenging the students for example) or maybe by a lack of pedagogical content knowledge (Shulman, 1986) for the teacher. The epistemological and didactical analyses of the mathematical contents are crucial here to make the comparison between the two sets of tasks. In the case of informal assessment, each new task given by the teacher has to be considered among a set of previous tasks, depending on the "study moments" (Chevallard, 1999; Barbé & al., 2005), related to the distance from the first encounter with the mathematical notion. The distance between the tasks allows us to measure the complexity for the students compared to the tasks that they have already worked on. More globally, the whole range of tasks proposed by the teacher on a mathematical content, with the absence of particular tasks in the related assessments, can tell us some of the intentions of the teacher for assessment, in relation with teaching.

Another element that we take into account, is the "depth" of the information: indeed, the process of "taking / interpreting / exploiting information on students' activity" in mathematics, can take its roots in the solution that the students produce (the result of a task), the way they solve the task (procedure to achieve a result), or the knowledge that is put into action to complete this procedure. Linking the activity to the student's knowledge requires, from the teacher, an understanding of the conceptualization of the mathematical contents behind the procedure, which is usually specific to the particular content. Regarding algebra in particular, it leads us to consider some specific elements, such as the form of writing for calculations or the type of reasoning. Indeed, some forms of writing for the calculation for example (cf. figure 1) can inform the teacher on the meaning of the equal sign for each student (computation, equivalence), or on the student's structural / procedural view of numerical or algebraic expressions.

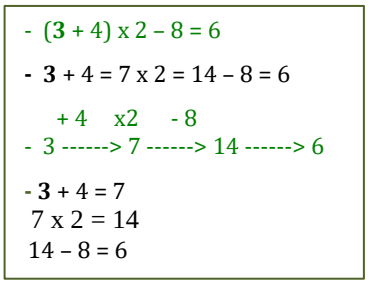

Figure 1: Different writings for the calculation, giving information on the students' knowledge

The interpretation of the information can also differ, depending on the reference taken for this interpretation: a comparison with what is expected by the institution (curricula, external assessment), or what could be expected by the teacher, considering all the previous work and the teacher's knowledge of the didactics of the mathematical notion at stake (errors, obstacles and breaches, steps in the conceptualization, etc). It can also be a comparison between the students' different procedures or a comparison in time for one student, to appraise his or her progress. These comparisons can be made explicit or not to the students. Here again, it can be linked to the various possible functions of the assessment.

Finally, the exploitation of the information differs depending on the moment when it occurs: whether it leads to immediate feedbacks related to students' result, procedure or knowledge or, in a more or less short term, when it influences the planning for the next activities.

Before giving an example of teachers' formal assessment practice, we will first describe our working context with a group of teachers.

Description of the collaborative work inside the *Léa*

A "Lieu d'Education Associé (*Léa*)" is an instance created by the French Institute for Education (IFE) to promote research with people who play an acting part in education. For 3 years, they are associated with a team of researchers to investigate questions about education and to build realistic resources for teachers or educators. Our *Léa* takes place, since May 2014, in a high school (students from 11 to 15 year old) in an Educational Priority Area, with 9 teachers (4 at the beginning) and 7 researchers, who meet every month to work together, to build teaching materials for algebra and to discuss assessment practice that could promote students' learning. The *Léa* can give us access to those teachers' evaluation practice in the long term.

An example of a comparison of teachers' formal assessment practice

We asked *Léa* teachers to design a diagnostic test at the beginning of the year for their 7th-grade students, to assess their numerical and pre-algebraic knowledge before introducing algebra. The tasks of the tests that they individually proposed were not covering all the range of the required knowledge for the introduction of algebra (Carraher & Schliemann, 2007, Kieran, 2007). The teachers justified their choices by giving institutional or social reasons, rather than epistemological or didactical ones.

Teachers	G	M
Variety of tasks	Repetitive task	Different tasks
Complexity of test's tasks / previously given in class	Similar to the previous ones in class	More complex than the previous ones in class
Information (declarative)	On the result	On the procedure
Feedback to the students (declarative)	Marks on the paper	Marks on the paper
Function of the formal test (declarative)	- To be able to give marks for the institution - To work on the basics	- To learn by adapting to a different situation - To adapt the teaching plan ahead

Table 1: Formal assessment (summative test)

We also conducted interviews with these teachers to find out about their views about assessment, after they had proposed their first summative test of the year. They were asked questions about their choices of tasks and the feedbacks they gave to the students afterwards (cf. table 1 for two of the teachers). Their answers showed a great variety in the tasks they proposed, regarding the distance with previous tasks, and probably resulting from different views on the functions given to formal assessment, despite the fact that these teachers often worked together. What was common to all the teachers on the other hand, is that they did not usually give many feedbacks to their students. Indeed, those teachers gave a mark without informing the students with the necessary elements to understand their mistakes and the limitations of their reasoning. Another comparison, related to informal assessment, for one teacher at different moments, will be made in the last part of this paper.

Assessment as a tool to enhance students' learning in mathematics

Definition of formative assessment

For Black & Wiliam (1998), an assessment can be formative when a teacher uses the information on the students to help them engage in the work on a task, or to help each of them auto-evaluate their knowledge:

> The term 'assessment refers to all those activities undertaken by teachers, and by their students in assessing themselves, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged. Such assessment becomes "formative assessment" when the evidence is actually used to adapt the teaching work to meet the needs. (page 2)

In terms of gathering/interpreting/exploiting information, formal assessment can play a more or less formative function, depending on the chosen tasks (if the tasks are way too complex or too simple; the students' productions might not reveal many useful information for the teacher). It depends also on the feedbacks made to the students. These facts can both be witnessed and analyzed by the researcher.

But when in comes to informal formative assessment in class, even if it is possible to see a teacher going around in the class when the students are working on a task, we can only witness the information actually gathered if the teacher is using it right away to guide the students' work. Deciding not to use the information right away, but reorganizing the plan of the next sessions, for the entire class or a particular student, could also be an exploitation of the information to promote the students' learning, but the researcher would then hardly acknowledge it. In any case, the research time allowed to the students to work on the task will probably have an influence on their mathematical

activity, and on the information that the teacher will be able to take on this activity, depending on the task.

A key moment: sharing the students' productions after letting them work on a task

The moment of pooling of students' procedures, after letting them work on a task, alone or in groups, seems to us like a good opportunity for informal formative assessment, where students could compare their solution with others' and know if they are close to what was expected. It depends of course on how the teachers choose to manage this moment of the session, and on the use they will make of the students' productions. This is why we will look more closely at those moments in the classrooms to analyze the use that the teachers make of the students' productions: how is the students' work taken into consideration?

What kind of productions do the teachers choose to share with the entire class? Is there a variety in these productions, regarding the result of the task or the possible procedures? Are there errors, typical or not, showed to the students? These elements are indicative of the information probably gathered by the teacher on the students' work while they were working, but depends also on an *a priori* analysis of the task, strongly linked to the mathematical contents to be mobilized, in order for the teacher to anticipate the possible outcomes.

We also analyze the exploitation of these productions. However, the interpretation of the information by the teachers remains mostly invisible to the researcher, except when the teachers explicitly mention the reference they use to compare (with what is expected at the end of the year, with what the students already did before, between students…). We note if the teachers organize a comparison of the results or of the procedures. Do they rank them to show the relevance and limits of each solution? How is organized the (in)validation of the solutions? Who is (in)validating them? With which arguments? And which conclusion? These elements can inform us on the role given to the students in the validation and institutionalization process and in the assessment process in general. We will give an example of this type of analysis for one teacher, in the last part of this paper.

We have hypotheses on the conditions that we consider more favorable towards student's learning, for example by making use of various students' procedures and errors and by implicating them in the validation, using mathematical arguments. We will confront these hypotheses with the results in algebra of a hundred of high school students, whose teachers' assessment practice was analyzed in this study,. In order to do that, we will analyze each task given by each of the teachers participating in the *Léa*, as part of the formal assessment process in algebra during 3 years, in terms of kinds of tasks (Chevallard, 1999) and adaptations (cf. Robert 2003), to determine their variety and complexity. We will collect the students' productions, analyze their answers and characterize them according to the different degrees of algebraic competencies defined by Grugeon & al. (2012) and Chenevotot-Quentin & al. (2015) for the design of a diagnostic assessment in algebra. This analysis is now in progress, as we are beginning the third year of our work inside the *Léa*.

Assessment as a lever for professional development

Our tools to analyze professional development

For our research, we already have collected a wide range of data inside our *Léa*, that we still have to fully process: to document the teachers' assessment practice, we gathered their personal documents

for the class and asked them to film themselves regularly in their own class; to measure the possible effect of this practice on the students' learning in algebra, we collected many students' productions; and at last, to try to estimate the effects of our collaborative work on the teachers' practice, we recorded the discussions during our meetings, and kept the reports of these meetings, when written by the teachers.

The analysis of the teachers' assessment practice takes into account, as explained before, the list of tasks proposed to the students in algebra and the management of the resolution of these tasks in class (informal assessment) or after a test (formal assessment). Their point of view about assessment is also visible through the interviews we conducted at the beginning of the project, or through the discourse of the teachers during the meetings of the *Léa*. More specifically, we are interested in the ways they argument their choices for their class, through the type of reasons they give for their choices of tasks or management (institutional, social, didactical, mathematical, etc). The moments when the teachers disagree with the researchers, or try to convince new teachers who joined the collaborative workgroup, are particularly interesting for us for that matter; to gather information on a possible evolution on the teachers' point of view on assessment, and more globally, on the teaching of algebra.

Some results about the changes in the teachers practice and arguments

To illustrate our analyses of the teachers' practice and their development, we are giving here an example of two sessions, for the same teacher, filmed one year apart. We analyzed the moment of sharing the students' productions, after working on similar tasks. These tasks both involve testing a calculation program with several numbers to notice an unchanging result or property and proving it with algebra. Our analyses are based on the indicators that we have already listed (the variety of the productions chosen to be displayed, the exploitation of students' errors, the initiatives in the validation, the arguments for justification).

Our analyses after the first year of collaboration (see table 2 "year 1") tend to find that teachers' assessment practice are very settled and stable. It seems that our didactical contributions about the teaching and the learning of algebra have helped teachers' practice to evolve (Horoks & Pilet, 2015): indeed they have better indicators to select the students' productions that they will use for the discussion after a task. But the exploitation that they make of these productions hasn't really changed after the first year: when sharing them with the class, the *Léa*'s teachers don't usually organize a comparison between the students' productions nor give a validation based on mathematical reasons. We also found that these teachers usually leave no initiative for students when working on the more complex algebraic tasks, which leads to the impossibility to rely on their production (see table 2, "year 1").

After the second year, where we decided to share some of our tools to analyze the sessions in class with the teachers, we can notice some evolution in the exploitation of the information (see table 2, "year 2"). Even though the second part of the task is more complex, this teacher still relies on the students' productions, even if they are not mathematically correct, to build, along with the students, the reasoning that will allow the class to invalidate the proposed solution.

However, even if the tasks are similar between year 1 and year 2, students have a higher grade in year 2 which may also explain the different choices made by the teacher. We should go on studying practice for a longer time, to identify its stability, and this is what we plan to do in the *Léa* project.

<table>
<tr><th></th><th></th><th>M(year 1)</th><th>M(year 2)</th></tr>
<tr><td rowspan="5">Testing with numbers</td><td>Duration of individual work</td><td>6.00</td><td rowspan="5">this numerical step is not part of the second task</td></tr>
<tr><td>Variety and comparison on results/procedures from the students' productions</td><td>3 procedures not compared</td></tr>
<tr><td>Presence of errors in the displayed productions</td><td>error in the procedure</td></tr>
<tr><td>Student's initiative in the validation</td><td>yes</td></tr>
<tr><td>Mathematical arguments of proof</td><td>no</td></tr>
<tr><td rowspan="5">Proving with algebra</td><td>Duration of individual work</td><td>2.30</td><td>6.30</td></tr>
<tr><td>Variety and comparison on results/procedures from the students' productions</td><td rowspan="4">this algebraic step is handled by the teacher without any support on the students' productions</td><td>4 procedures compared</td></tr>
<tr><td>Presence of errors in the displayed productions</td><td>yes</td></tr>
<tr><td>Student's initiative in the validation</td><td>yes</td></tr>
<tr><td>Mathematical arguments of proof</td><td>counter-example</td></tr>
</table>

Table 2: Evolution of informal assessment practice for teacher M

Both the cognitive (contents and tasks) and mediative (organization of the sessions in class) elements of the teacher activity play a part in the assessment process that we are trying to analyze here. But, as emphasized by Robert and Rogalski (2005) there are other constraints of this professional occupation to be taken into account: the social (type of school), the institutional (curricula) and personal (carrier and education) components, playing a significant part when interpreting teachers' practice, including for us their choices in terms of assessment. We analyze teacher's practice through all these components, at different levels: locally in the classroom or globally within all the teaching plan, and we believe that it can give us access more deeply into the teachers' consistency and explain their stability. Yet, after the second year, we noticed some changes in the arguments that the teachers are giving to justify their choices, shifting a little from social and institutional reasons to mathematical or didactical ones, that we would hope to link to our work together, that is still going on..

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It's different, it's difficult, it's unknown': Letting go of levels

Nick Pratt[1] and Julie Alderton[2]

[1]Plymouth University, UK nick.pratt@plymouth.ac.uk

[2]University of Cambridge, UK jha32@cam.ac.uk

This paper explores primary teachers' accounts of their responses to major changes in the curriculum and assessment system in England, which has recently re-designated expected standards of achievement and progress. Analysis is informed by Foucauldian poststructural understandings of power/knowledge and truth to examine how they reorganise their practices as mathematics teachers within a policy context which continues to compel schools to focus on performance. By means of a small-scale empirical study, we identify the tensions created when the 'rules of the game' change and how technological assessment tools require and enable teachers to reproduce levels and labels to categorise pupils. Our aim in undertaking this analysis is not to compare teachers' assessment practices to an ideal, beyond policy, but to illustrate how government-driven changes to assessment are insufficient to change underlying discourses of performativity which ultimately shape practice.

Keywords: Elementary school mathematics, assessment, Foucault, governmentality.

Introduction

> The examination [assessment] combines the techniques of an observing hierarchy and those of a normalizing judgement. It is a normalizing gaze, a surveillance that makes it possible to qualify, to classify and to punish. It establishes over individuals a visibility through which one differentiates them and judges them. (Foucault, 1977, p. 184)

The quote from Foucault begins to set out both our substantive interest and the theoretical stance, namely, an interest in mathematics assessment from a sociological perspective. Our focus is on assessment not simply as a technical activity to improve pupil outcomes, but as a mechanism through which teachers manage their professional selves; the way in which mathematics assessment is used as part of their ongoing professional identification and as the basis, and evidence, of their success. Our starting point is the claim that in English primary (5-11) schools, assessment, and the curriculum alongside which it takes place, plays a major – perhaps *the* major – role in influencing teachers' actions. There are many reasons why this is the case but, as Pratt (2016a) argues, in essence they revolve around the marketized and high-stakes, accountable nature of the English system and the 'performativity' (Ball, 2003) this manifests in teachers' work.

The changing context of English mathematics education

It is difficult in a short paper to describe fully the complex landscape of an education system and how it is changing and we refer the reader to Pratt (2016a) and Keddie (2016) for more detailed discussions of English primary schools. However, in summary the system is based in a neo-liberal, neo-conservative framework which affords an increasingly marketized, competitive and accountable approach to school improvement. This has led to a strong discourse of 'progress', since it is the change in pupils' levels of attainment across each year which has been the key measure against which schools, and individual teachers, have been judged. In turn this leads to a strong discourse of control, a belief that pupils' progress is predictable and controllable across time; and therefore of teachers' responsibility for learning outcomes obtained through their teaching (Pratt, 2016b).

However, over the last 18 months, both the curriculum for mathematics and the assessment system have been reformed. The new primary national curriculum (NC) (DfE, 2013) stipulated increased expectations in mathematics with more challenging national tests. Perhaps most importantly for teachers, previous NC 'attainment levels' have been superseded by an 'expected standard' set for 2016, at a higher level than in 2015 (DfE, 2016). The rationale for this change is described in the final report of the government's Commission on Assessment Without Levels (McIntosh, 2015, p. 5), as follows:

> Despite being intended only for use in statutory national assessments, too frequently levels also came to be used for in-school assessment between key stages in order to monitor whether pupils were on track to achieve expected levels at the end of key stages. This distorted the purpose of in-school assessment, particularly day-to-day formative assessment. The Commission believes that this has had a profoundly negative impact on teaching.
>
> Too often levels became viewed as thresholds and teaching became focused on getting pupils across the next threshold … Depth and breadth of understanding were sometimes sacrificed in favour of pace.

Guidance specifies that the majority of pupils should move through the programmes of study of the NC at broadly the same pace (DfE, 2013), crucially replacing previous advice to accelerate high attaining children through new content. At the classroom level, 'progress' *through* the curriculum has been replaced by 'progress' *within* it; and a new language of 'mastery' has sprung up to describe this, which "denotes a focus on achieving a deeper understanding of fewer topics, through problem-solving, questioning and encouraging deep mathematical thinking" (McIntosh, 2015, p. 17). Progress measures of pupils and schools across key stages are also calculated differently. Monitoring progress by levels and sub-levels has been replaced by a value-added measure. Pupils' results at the end of key stage 1 and key stage 2 (at ages 7 and 11) are compared to the achievements of other pupils with similar attainment nationally, and a new 'floor' standard requires that at least 65% of pupils meet the expected level in mathematics (and English), or that a school achieves sufficient progress scores (DfE, 2016). Schools not achieving the floor standard will be scrutinised through additional inspection and may have their freedom curtailed. Indeed, the Commission notes that "with freedom, however, comes responsibility" (McIntosh, 2015, p. 10) and "recognises that the transition to assessment without Attainment Targets and levels will be challenging, and that schools will have to develop and manage their assessment systems during a period of change" (p.16). However, it justifies this on the basis of "a much greater focus on high quality formative assessment as an integral part of teaching and learning"; the raising of "standards" in line with neoliberal policy. As we have previously pointed out, in a performativity culture such as this one, assessment has become a means by which teachers gain and maintain professional capital (after Bourdieu, see Pratt, 2016a).

Theoretical Framework

To understand the effect of changes to the 'rules of the game' of assessment, we draw on Foucault, particularly his notion of governmentality (Foucault, 1977) that surrounds English education (Ball, 2013; Llewellyn, 2016); the notion that dominant discourses become normalized to such an extent that (teacher) subjects consent to particular action and hence come to govern themselves. (Note, discourse here refers to "a group of rules proper to discursive practices … [which] define the

ordering of objects" (Foucault, 1972, p. 49) and is more than just language.) Our aim is to make visible the ways in which assessment discourses normalize certain practices and relations between teachers, school systems and pupils, rendering them common-sense, irrevocable and change-resistant – but not to judge these against some ideal version of practice. In theorising these forms of governmentality in and through assessment, two related ideas are in play: power/knowledge and truth. Power, according to Foucault, is enacted, not held by individuals, and

> is not exercised simply as an obligation or a prohibition on those who 'do not have it'; it invests them, is transmitted by them and through them; it exerts pressure upon them, just as they themselves, in their struggle against it, resist the grip it has on them. (Foucault, 1977, p. 27)

We emphasise that this can be a good or bad thing; power can liberate and is not oppressive per se, but either way, 'power produces knowledge' (ibid). 'Experts' in a field (teachers in their classroom settings, but also senior managers in the school as a whole, policy makers and children as 'expert pupils') produce knowledge through their language and activity which positions and exerts pressure in terms of the way it influences what can and cannot be said and done. In this sense, it forms a 'game of truth'. For Foucault, truth is not something to be found outside of relations. Rather it is something produced through such relations so that "each society has its regime of truth, its 'general politics' of truth: that is, the types of discourse which it accepts and makes function as true" (Foucault, 1980, p. 131). Thus, the question is not what the truth 'is', but how things come to be taken as true; how this is used in order to make manifest and exert power relations. This is

> the truth which does not belong to the order of what is, but to the order of what happens … a truth which is not found but aroused and hunted down: production rather than apophantic. This kind of truth does not call for method, but for strategy. (Foucault et al, 2008, p. 237)

It is through this theoretical lens that we return to mathematics assessment, and the following questions: how do teachers respond to the changes that a new curriculum and assessment system impose; and in doing so, how do they re-organise the economy, and politics, of truth in assessment practices in order to (re)empower themselves as experts?

Methodology

The project involved extended semi-structured interviews with primary teachers in 9 different schools (12 teachers in total) in the first year after the removal of levels. Teachers and schools were chosen purposively to reflect a range of ages, experience, school types and locations, but in this paper we draw on just three of the participating teachers – Ann, Jill and Mike, all working in state schools – in order to keep the analysis manageable. Mike and Jill are in their late 20s and both are coordinators of mathematics in their schools and are both on a programme of training to develop leadership in 'mastery' of mathematics. They work in a village and an urban school respectively; Jill has been teaching for 8 years and Mike for 7. Ann is in her late 30s, has been teaching for 19 years and in her current, town, school for 5 of these. She is a class teacher, but not a specialist in mathematics. Data from all the interviews were analysed thematically in relation to the substantive and theoretical framework – teachers' assessment practices, as we understood them in relation to power/knowledge and truths. Whilst we can only present a small set of data we have selected this carefully, ensuring that teachers' views, though sometimes individual, are never contradictory of the data set as a whole. Our aim is not to claim that the specifics are generalizable to every teacher beyond, or even within, the data set. Rather, the analysis is of the system of governmentality and the

dominant discourses that constitute it. We think it offers a trustworthy and useful analysis in this sense, meaning that it is likely to be generalizable to other teachers in terms of the way in which their work becomes problematized, even if not in terms of how individuals are able to respond. All our work conformed to the ethical procedures of the British Educational Research Association and were approved by our employing institutions.

Analysis – Reproducing the truth

The DfE's Commission on Assessment without Levels is very clear over the point of their removal.

> Removing the 'label' of levels can help to improve pupils' mind-sets about their own ability. Differentiating teaching according to pupils' levels meant some pupils did not have access to more challenging aspects of the curriculum. (McIntosh, 2015, p. 15)

Interestingly, this critique itself illustrates Foucault's central point about governmentality, namely that it is through labelling that subjects are categorised, normalized and objectified. They 'become' their label – and act accordingly in the common-sense, normal(ized), way that this affords. Whilst removing the language of levels is well-intentioned in order to remove such labels, we noted above that teachers' work takes place in a culture of performativity with dominant discourses of control and responsibility. Central to governmentality, they require teachers to 'know' what their pupils can and cannot do so that they can take responsibility for 'filling the gaps' in their knowledge by "identifying specific 'corrective' activities to help them do this" (ibid. p.17). These, then, become questions of truth, of what pupils 'actually' and 'really' know. However, as Foucault notes, a truth statement is "contingent on the instruments required to discover it, the categories necessary to think it, and an adequate language for formulating it in proposition" (Foucault et al., 2008, p. 236). The language of levels may have gone, but the imperatives for control remain and so a new language is needed for teachers with which to think and speak it. Our interviews suggest that the language of 'mastery', codified through other national continuing professional development programmes, has offered teachers such an alternative, so that:

> For every child you can click on an objective and say whether you are working towards it, achieved, secure, or greater depth. (Ann)

> Basically we have developed a system throughout the year. So, we haven't bought a system in. We've simply developed our own system as a school where we've given the children a grade of either 1, 2, 3 or 4. (Mike)

> When we were talking, as a school, what we were going to put for our levels, we said "what shall we call them?" We've got to have things and labelling them "emerging, developing, secure, exceeding". (Jill)

Ironically then, the notion of mastery which was meant to take teachers away from codifying and levelling has provided alternative "types of discourse which it [the system] accepts and makes function as true" (Foucault, 1980, p. 131). Classification continues, but with new levels. What is significant in terms of governmentality is that, despite the best intentions, this replacement is inevitable since it is founded in the performative discourse which underpins pedagogic activity. In English primary schools this performance is measured by 'progress'; in the past meaning the movement up levels and sub-levels of attainment. Although the removal of levels has meant that there might be a new official understanding of it – that "progress can involve developing deeper or

wider understanding, not just moving on to work of greater difficulty" (McIntosh, 2015, p. 12) – it has not removed the imperative of being able to make it demonstrable as the way in which schools are judged. In other words, knowing 'where pupils are' is still central to "the status of those who are charged with saying what counts as true" (Foucault, 1980, p. 131) and is not therefore optional.

Hunting for truth with technology

Foucault (1980, p. 131) has pointed out that the political economy of truth is characterised by, amongst other things, the form of scientific discourse, economic and political demands and the ways in which it is diffused and consumed amongst different organisations. Each school in our study has made use of some form of tracking system, either commercial software or a spreadsheet of some sort, as a technology for capturing data and in different ways teachers are looking for these technologies to help them seek the truth about the progress of their pupils. In each case, there are two technologies at work. Firstly, a tracking system recreates labels:

> you've got all the statements and you can say whether the children are working towards it, expected for it, or exceeding for it, or something. Then it breaks it down into them being, for each year group, they are beginning to access or beginning plus, working towards or working at plus, secure, secure plus. There are six basic, what would have been sub-levels. (Mike)

But, he notes, "it can't generate something that tells you your child probably is secure or probably is working at" and "it's not comparing your children to anyone else. It's not saying anything." Whilst teacher judgement is "fine and good" it does not seem to represent a sufficient truth for the accountability purposes to which it is to be put. Mike's school has therefore turned to commercially produced online tests. These give him "beautiful data" and whilst it also serves a formative purpose in identifying "gaps" it "provides a comfort blanket" because "it gives you a standardised score and it's based against however many thousand children from around the country".

Whilst Mike has turned to comparative statistics to produce knowledge of progress, Jill agrees that numbers and labels mean that "it somehow feels like it's clearer, but if it's not well-defined that's quite dangerous, really". Rather than seeking a truth in statistics though, Jill is committed to the idea of illuminating pupils' mathematical understanding and somehow mapping this onto the new labels so that they can say, "these children are where they should be and these children aren't … so that the gaps that they have got [can be] filled". Rather than comparisons to other pupils nationally, Jill's plan is to exemplify for colleagues a truth about what each label (developing, secure etc.) looks like in terms of the objectives from the curriculum that pupils can achieve. In this way she hopes that "it would be very clear where the children were and where their next steps were more clearly" and that "within the following year's teaching you can see that clear progression, and then that becomes a way for teachers to show progress". Jill's belief seems to be that professional judgement, evaluating pupils' understanding against exemplar materials, will, in time, allow teachers to learn what the new levels "feel like".

The rationale for the removal of levels and a focus on mastery was, in part, based on the assertion that "too often … teaching became focused on getting pupils across the next threshold instead of ensuring they were secure in the knowledge and understanding defined in the programmes of study" (McIntosh, 2015, p. 5). We have illustrated how levels have been recreated by teachers to serve the function of performativity, yet this is not to say that the idea of refocusing on pupils' understanding of the curriculum was not welcomed and encouraged by this move. Mike notes that alongside the

security of knowing how their pupils rank against others "we are thinking about 'OK they are working at expected levels or just below but what are their gaps and how am I going to fill their gaps?" Jill claims that "I think the move away from levels has been absolutely fantastic" because it allowed them to "take the time to sit back and actually think about the underlying maths". Ann also welcomes the focus on ensuring that "gaps are filled" and considers this as central to pupils' success. However, in her experience

> it was just a lot that had to be covered and part of it was because there were gaps that I needed to go [over]. So for example my class didn't have a very good understanding of decimals, so rather than teaching thousandths and all of what was in the year 5 curriculum, I've had to go right back to the start and doing tenths. And that is your year 3 and year 4 objectives. (Ann)

This has led to her being reluctant to say that any child is secure and to her "feeling that almost, as a teacher, you've failed", with her confidence being affected as a result. The school uses a system called School Pupil Tracker Online (SPTO) which, unlike the other systems, is meant to calculate whether pupils are emerging, developing or secure, but Ann does not trust its output.

> I just experimented with 'what if I made that [objective] mostly achieved?'. And by doing that I could see that it was literally one little click turns that level up. … I didn't like the fact that just one click sent that judgement over, particularly when it didn't look like it was right.

She notes that even if the company that runs the software alters this in the coming year "it sounds like the standard is going to slightly change every single year, which just makes it completely confusing. How can you work towards something that you don't know what it is?" This lack of clarity over the truth of her pupils' learning is leading to some tension for Ann.

> So within what I do with the children I see progress but I don't always see it in what I've got on paper, on SPTO. The progress isn't always reflected there … I thought I was a good maths teacher, maybe I'm not, because of what's coming out … In some ways I'm almost fighting against it and saying 'you will not do this to my confidence' [laughs], yeah.

A new normalizing gaze

We noted above that one intention of removing levels was to avoid labelling pupils in ways that prevented access to the curriculum. As the quote that begins this paper makes clear, however, from Foucault's position any examination "combines the techniques of an observing hierarchy and those of a normalizing judgement" which "establishes over individuals a visibility through which one differentiates them and judges them" (Foucault, 1977, p. 184). Foucault's use of normalizing here is two-fold. On the one hand it points to the standardisation and categorisation of pupils; their allocation into categories, in this case 'emerging', 'secure' etc. which are then used to define normal, and hence abnormal, and to take remedial action. Mike refers to "the ones who haven't quite got there" and Jill to those who are "where they should be" and those that are not. On the other hand, it refers to the notion of making this categorisation 'normal' practice; common-sense, inarguable, defining what can and cannot be thought and said. Thus, although removing levels is meant to avoid differentiating pupils and restricting their access to the curriculum, the need to track progress makes such differentiation necessary. To speak of progress is to speak of changes in category as the only "type of discourse which [society] accepts and makes function as true" (Foucault, 1980, p. 131); "a truth provoked by rituals, captured by ruses, seized according to

occasions" (Foucault et al., 2008, p. 237). Such rituals create a practical tension in the idea of normalisation. As Mike notes,

> It's that challenge we're set of trying to keep together and moving forward together but having children still working at a greater depth but closing the gap for the ones that are lower.

For those already 'succeeding' as secure, mathematics involves a range of activities. Mike describes "10 children who we saw as working at greater depth and they worked in groups with teaching assistants and had some really different kind of problem solving". However, Jill points out that the governmentality around floor targets means that for "the children who are almost secure but not quite, there is a real push to get them [over the threshold]".

> Yes, but I think the secure one [is key] at the moment. I think at the moment with the new system it's different, it's difficult, it's unknown. I think it's that 'where are we for secure?' (Jill).

Hence, whilst the change in the curriculum structure is meant to ensure that children move together through the content, the manner in which assessment inevitably "establishes over individuals a visibility through which one differentiates them and judges them" (Foucault, 1977, p. 184) means that the way in which they experience the subject is far from equal.

Discussion

Our analysis suggests that although superficially things might look different and teachers may feel that their practice has changed, this appears to be largely a reconstruction of the same dominant discourses in new language. Whilst the specific practices of governing might have been altered, the fundamental forms of governmentality have not and teachers are in the process of reconstituting much of what they had before. We recognise that the recent changes have opened up opportunities for discussion, collaboration and reflection within and between schools and made teachers pause and take stock of assessment in ways that feel positive to them. However, they have also reproduced pressures and tensions which can work to deflect attention away from questioning the responsibilities of policy makers and the implications for the teaching and learning of mathematics in the new system. Whilst there is a significant impact on teachers' day-to-day teaching and assessment practices, and how these are evaluated, the performative role of the teacher remains largely the same. There does seem to be more consideration of pupils' development in mathematics; though this is produced in particular ways: an atomised curriculum and filling in gaps. There are signs too that far from alleviating the problem of access to the curriculum for all children, there is a new normalizing gaze; one that focuses teachers' efforts on an even slimmer tranche of pupils who might just be normalized – literally, to the middle of the normal distribution. Similarly, only those who are 'secure' in their 'knowledge' of the subject get access to a rich version of mathematical problem solving. These points raise questions about the way in which such tightly managed forms of assessment affect pupils' relationships with the subject and about the equity of pupils' access to the curriculum. The nature of these authoritative discourses of progress, control and responsibility that make up performativity, and the version of mathematics and assessment produced within them, appear difficult for the teachers in our study to identify. All schooling operates within policy and its incumbent discourses and can never be free of it, however the value of a Foucauldian analysis is in making such discourses visible to those responsible for making changes to the assessment system.

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Towards an empirical validation of mathematics teachers' intuitive assessment practice exemplified by modelling tasks

Xenia-Rosemarie Reit

Goethe-University Frankfurt, Institute for mathematics education, Germany; reit@math.uni-frankfurt.de

The identification of intermediate steps in student solutions as a basis for assessment is a common procedure in mathematics teaching. Modelling tasks, providing more than one solution approach, are considered hard to assess. This is not least a reason for the unsatisfactory proportion of modelling in school. Assuming that task difficulty is strongly connected with assessment, the implications of a study about the difficulty of modelling tasks are discussed. Starting point for the discussion is the question whether intuitive assessment practices like the identification and scoring of intermediate steps, can be supported by empirical findings. The focus is on the influence of cognitive aspects regarding structural characteristics of solution approaches. Results indicate that a pure sequential consideration of thought structures in a solution approach lead to reasonable results and might justify its application in school due to its straightforward implementation.

Keywords: Assessment, cognitive structure, modelling tasks, mathematics teaching.

Introduction

Within the mathematics community it is mostly agreed upon the positive impact of modelling tasks on the learning of students. Mathematical modelling is promoted and there are many votes for its broader implementation in school mathematics. However, several studies provide evidence that modelling is far away from playing an integral role in everyday school teaching, in Germany and also elsewhere (Blum, 2007, p. 5). Jordan et al. (2006) confirm that the proportion of modelling in daily school routine is low. Research focusing on the teachers' point of view reveals several difficulties that teachers are confronted with. In a study of Schmidt (2010) it has been found out that 67% of the interviewed teachers indicate assessment as being the major challenge in the implementation of modelling tasks. Blum (1996) also speaks of an increased difficulty in the context of modelling tasks. These findings are comprehensible in view of the multiple solution approaches of modelling tasks.

A common opinion is that modelling tasks cannot be assessed as objectively as traditional task formats (Spandaw & Zwaneveld, 2010). However, if we want modelling to be part of mathematics teaching, it must part of the grading (Hall, 1984). With the striking quotation "What you assess is what you get" Niss (1993) also argues in favor of a provision for modelling activities in the grading. The aim of the present paper is not to discuss and contrast formative and summative assessment since the advantages of formative assessment as assessment *for* learning movement could be confirmed in different settings (e.g. Black & William, 1998) and are not denied. However, in view of the fact that summative assessment aspects hinder the implementation of modelling tasks in everyday school live, it is necessary to provide tools or possibilities in that direction.

Besides a number of assessment methods which aim at assessing modelling competence (e.g. Berry & Le Masurier, 1984; Haines, Crouch, & Davis, 2000), there is hardly any assessment instrument which can be used for assessing modelling tasks in everyday school live. In this context so far only Maaß suggests an assessment scheme which can be adapted to different modelling tasks by a variable

weighting of several categories (Maaß, 2007, p. 40). However, an empirical validation is lacking such that the assessment scheme might serve as orientation but it cannot give detailed instruction.

On the way to an assessment scheme for a mathematics task, a common procedure of mathematics teachers (in the following referred to as "intuitive assessment practice") is to identify reasonable intermediate steps in a solution which are worthwhile to be scored. On that basis an assessment scheme is set up which determines the conditions to be fulfilled for a differentiated scoring of those intermediate steps. Hence, there is a procedural difference between the phase of identifying scoreable aspects in a solution and an assessment scheme. The former is the requirement for the latter. At this point the present paper ties on by discussing the use of so called thought structures to identify reasonable intermediate steps in solution approaches of modelling tasks in connection with its difficulty. The question of identifying intermediate steps and the influence of their structure within a solution approach to its difficulty has been analysed by Reit (2016). In this study different models are developed and evaluated to determine the difficulty of solution approaches. Assuming that assessment of a mathematics task is strongly determined by its difficulty, interesting conclusions can be drawn concerning common assessment practice in school. Results of the study of Reit (2016) indicate that there is quantifiable influence of structural characteristics of a solution approach on its difficulty. However, it is also stated that a sequential model which is based upon a sequential arrangement of thought operations, similar to the intuitive assessment practice of mathematics teachers, can also be confirmed.

Theoretical framework

The core of the study of Reit (2016) is a structural analysis of students' solution approaches of modelling tasks. These thought structures of solution approaches indicate the chronology of thought operations to be done to arrive at a solution. Assuming that parallel thought operations complicate a solution approach, a non-weighting difficulty model (addition model) is contrasted with four models varying in their weighting of parallel thought operations.

Thought structure analysis

Recalling structures is a wide-spread procedure in mathematics (Bourbaki, 1961, pp. 163). In this context Breidenbach (1963) looks at the structural-substantial complexity of a word problem to decide amongst others about its difficulty. He formulates that tasks with one operation deal in the simplest case, with one issue in which three factors play a role and every factor is uniquely determined by the two others (Breidenbach, 1963, p. 200). Breidenbach named such tasks *Simple*x. A linking of several *Simplex* is called *Komplex*. Further developments of Winter and Ziegler (1969) lead to the arithmetic tree which is still used in mathematics textbooks (Figure 1). An obvious but so far empirically not validated conclusion is that a larger number of *Simplex* and a more complicated nesting of them, has an effect on the difficulty of the tasks' solution (Graumann, 2002).

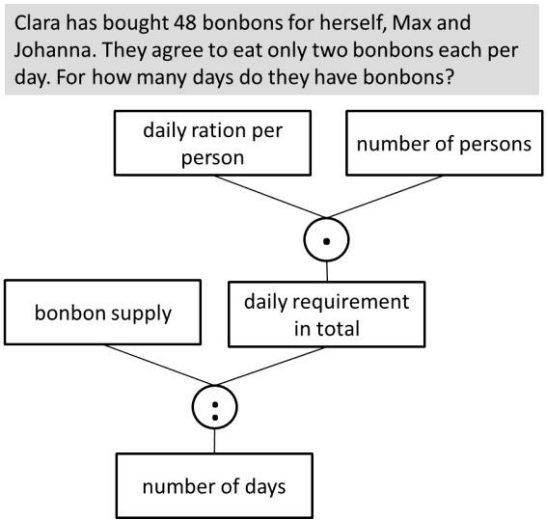


Figure 1: Arithmetic tree of an exemplary task following Winter and Ziegler (1969)

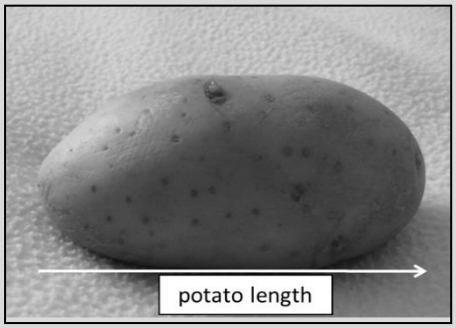
The study of Reit (2016) investigates the cognitive complexity of a solution approach on the basis of its structural complexity represented by its arithmetic tree-like structure. At that point the coherence of structural considerations and cognitive psychological theories play an important role. In a study of Fletcher and Bloom (1988) it is assumed that text comprehension is a kind of problem solving process, where the reader must find a causal chain which links start and end of a text. Furthermore they assume that information must be kept simultaneously in the working memory to be able to form such a causal chain. Results of their study show that readers must keep that information available that is the direct predecessor in the causal chain. It can be concluded that the task of the working memory is to keep information available which is necessary to link old and new information (Baumann 2000).

By relating these findings to structural considerations of a solution approach represented as an arithmetic tree, statements can be made about its cognitive complexity. On the one hand the arithmetic tree-like structure (Figure 1) can be interpreted as causal chain since the start (given information in the task text) and end (solution of the task) is linked by chain links (intermediate steps in the solution process). On the other hand direct predecessors can be identified as relevant intermediate steps. Thus, it can be concluded that the previous intermediate step must be kept active in the working memory to master the following. The assumption that the mental processing capacity is limited (Sweller, 1988) leads to the statement that several information which has to be kept active at the same time, complicate the solution process. Thus, it can be deduced that the load of the working memory is dependent on the number of intermediate steps necessary to master the current intermediate step. That means that the load of the working memory increases with increasing number of information needed at the respective point in the solution process.

Based on these considerations a study has been performed where theoretical difficulty of a solution approach is characterized as its cognitive complexity (Reit, 2016). Starting point is the so called thought structure of a solution approach which can be interpreted as kind of arithmetic tree (Figure 1). To formulate a thought structure all student solutions of a modelling task have been clustered into several solution approaches according to the mathematical model or solution process used. An aim of the study was to investigate whether the number of sequential and parallel thought operations has an effect on the cognitive complexity and thus, the theoretical difficulty of a solution approach.

Study design

Approximately 1800 grade 9 students (15 years of age) from German grammar schools took part and completed a booklet consisting of three out of five modelling tasks (see modelling task "potato" in Figure 2) under seatwork conditions. The total processing time for a booklet lay within one teaching unit.



Industrial manufactured French fries are supposed to be equal in size and the single sticks are cut out lengthwise. Therefore not the whole potato can be used. The potato tubers look similar to the picture above, are regularly formed and approximately 10 cm in length.

How many of these potato sticks can be obtained from one potato? Reason mathematically.

Figure 2: Modelling task "potato" (Reit, 2016)

Method

In the following the method in the study of Reit (2016) will be briefly explained. For a detailed description of the methodical implementation it is referred to Reit (2016).

All student solutions of a modelling task were analyzed and different solution approaches could be identified (two to four solution approaches per modelling task). These solution approaches within one modelling task differed in their underlying mathematical model used or, if similar to this, in their solution process. Every student solution was finally assigned to a solution approach (Figure 3). A structural analysis of these solution approaches then lead to individual thought structures indicating the chronology of thought operations. Based on thought structures of solution approaches different difficulty models have been developed to translate the respective structure into a scalar difficulty value. In a first step a thought structure was mapped onto a so called *thought structure vector* (Figure 3). These thought structure vectors represent the tabular-compact form of a thought structure. Each vector component indicates the number of parallel thought operations on the respective level of the thought structure.

Due to the fact that it was not clear yet if parallel thought operations lead to a higher difficulty than sequential thought operations, different operationalization of a thought structure vector into a scalar value were imaginable. Therefore different difficulty models have been set up (four accounting for parallelism of thought operations by weighting them and one non-weighting model (addition model)) which lead to solution approach specific difficulties.

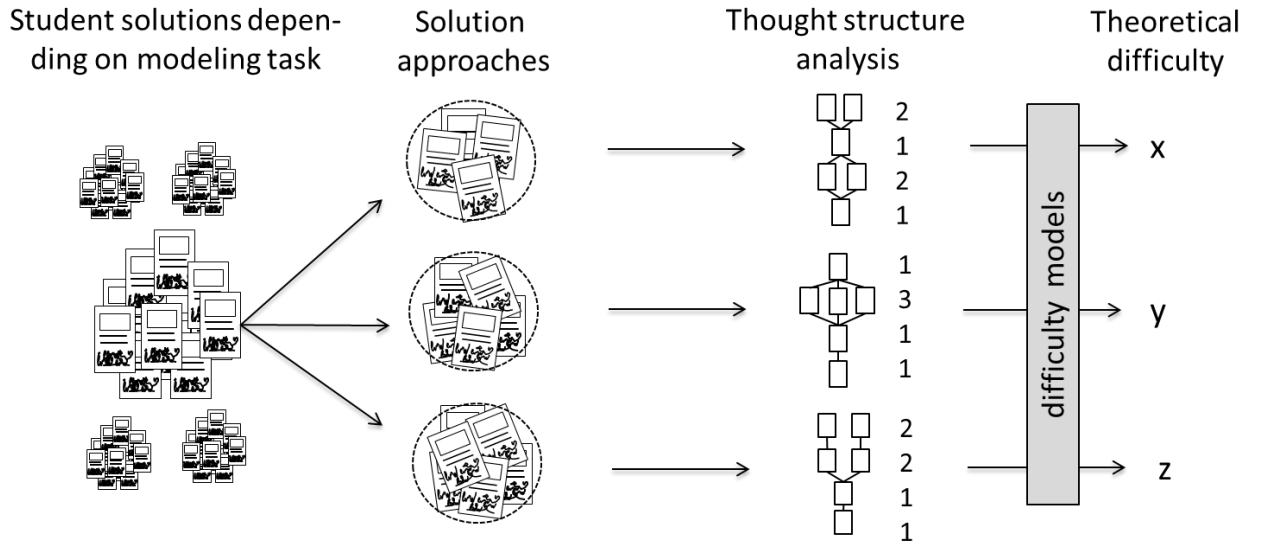


Figure 3: Identification of solution approaches, setting up thought structures (together with its thought structure vector) and applying difficulty models which lead to theoretical difficulties

Results

Whether and to what extent parallel and sequential thought operations have an influence on the difficulty was evaluated by comparison with the corresponding empirical difficulty, as a measure of the average score of a solution approach. To determine the empirical difficulty all student solutions have been assessed by two independent raters on the basis of a predefined assessment scheme set up by experts. The question was whether the theoretical difficulty reflected the associated empirical difficulty of a solution approach. In this case structural characteristics of a solution approach can be taken as a basis for assessment of modelling tasks. Of special interest are the results of the non-weighting difficulty model as analogy to mathematics teachers' intuitive assessment practice. The non-weighting difficulty model (addition model) adds up all thought operations to arrive at a theoretical difficulty as it is done, more or less intuitively, by mathematics teachers when identifying scoreable intermediate steps.

The results (Figure 4) indicate that addition and factorial model (pseudo-R^2=0.83) map the coherence of theoretical and empirical difficulty best. The factorial model weights parallel thought operations. In regard of the focus of the paper the results in Figure 4 clearly show that the addition model lead to significantly better results than the most weighting models.

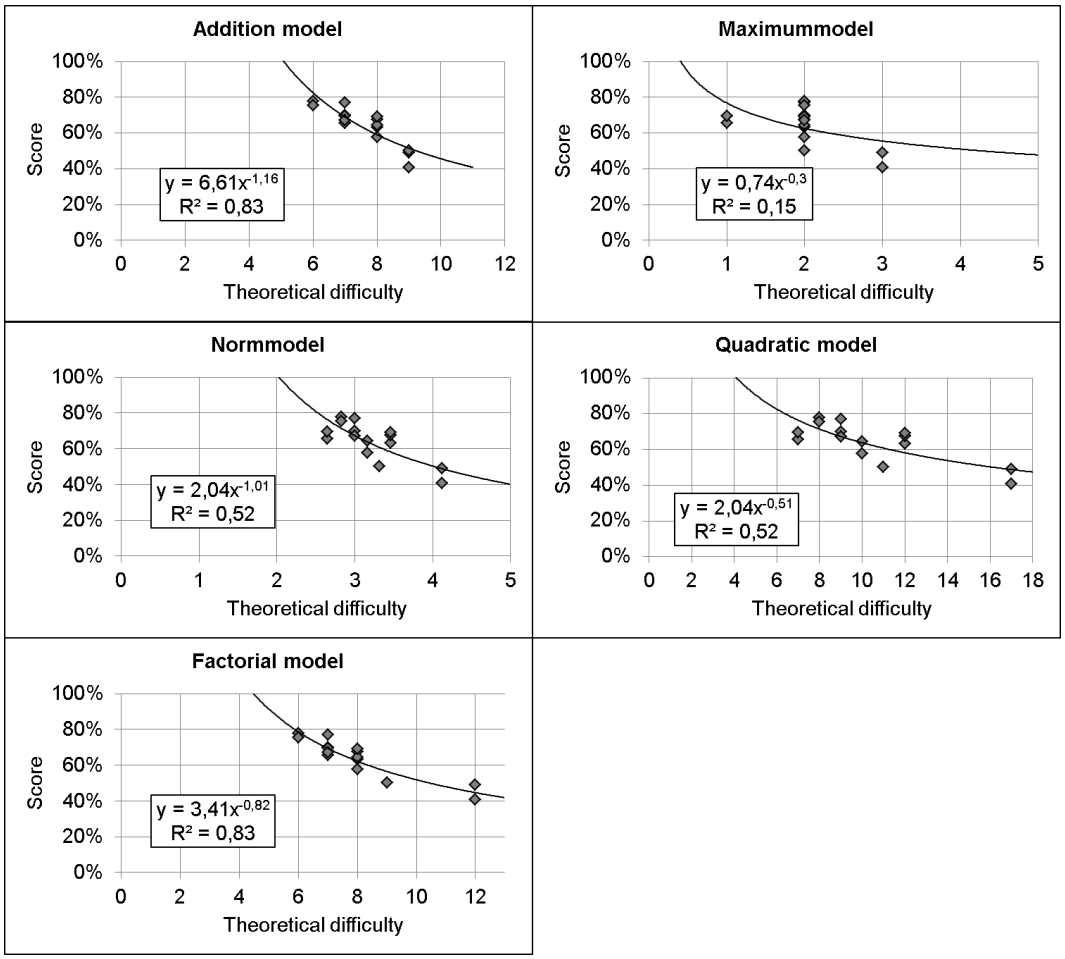


Figure 4: Comparison of theoretical and empirical difficulty of solution approaches taking account for different difficulty models

Discussion of results

Particular reference is made to established but so far not researched assessment practices in mathematics teaching. The focus is on whether intuitive assessment practices of mathematics teachers can be empirically confirmed and transferred to modelling tasks. Intuitive assessment practice means the common procedure of mathematics teachers of scoring intermediate steps in student solutions without accounting for structural-cognitive particularities. These intermediate steps usually then serve as a basis for an assessment scheme. The portrayed procedure is commonly used when assessing performance tasks in mathematics (summative assessment). Modelling tasks with its multiple solution approaches are considered to be hard to assess. This problem not least leads to the fact that modelling tasks are sparsely used in mathematics class. Results of a study investigating the difficulty of modelling tasks support the intuitive assessment practice of mathematics teachers and thus, legitimate transferring this assessment practice to modelling tasks.

In detail the results show that the addition model which treats sequential and parallel thought operations equally lead to reasonable results. This indicates that difficulty of a solution approach can be described well by the number of thought operations needed to arrive at a solution. By applying the addition model it is assumed that parallelism of thought operations has no influence on the complexity of a solution approach. This assumption is also made, more or less intuitively, in common assessment practice in mathematics teaching. Intermediate steps are identified and scored. Thus, on the one hand

the results support the everyday procedure in mathematics teaching where the difficulty of a mathematics task is often interpreted as the number of intermediate steps to complete a solution. On the other hand the results might justify a similar assessment procedure when assessing modelling tasks. In view of the widespread problems concerning assessing modelling tasks in everyday school live as part of the grading, the results clearly highlight a possible and furthermore practicable way.

In summary it can be concluded that the so far intuitive assessment practice in school of identifying intermediate steps in a solution, can be supported by empirical findings. It can be a worthwhile procedure to identify thought structures as a basis for assessment especially when assessing modelling tasks. By assuming that assessment is connected with difficulty of the respective solution approach, parallelism of solution approaches has an influence (see the results of the factorial model) but might be neglected in favour of a straightforward applicability in everyday school practice. Thus the results of Reit (2016) can serve as a basis for the development of a manageable assessment scheme for modelling tasks and might promote their implementation in mathematics teaching.

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Mathematics teachers' assessment of accounts of problem solving

Anna Teledahl

Dalarna University, Falun, Sweden; ate@du.se

In mathematics education it has been argued that traditional assessment provides insufficient evidence of students' overall achievements. Assessment of problem solving has been put forward as a more comprehensive form of assessment. This however entails a subjectivity which raises concerns regarding the reliability. This study aims to investigate mathematics teachers' assessment of mathematical problem solving. Nineteen teachers have been interviewed in five groups and asked to discuss a sample of 16 accounts of problem solving by 10-year-old students. The analysis focused on examining how five mathematical abilities, described in the Swedish mathematics syllabus, were addressed and discussed by the teachers. Preliminary findings indicate that the accounts provide teachers with very little evidence of students' mathematical abilities. One of the reasons for this appears to be that the accounts do not offer clear descriptions of the problem-solving process.

Keywords: Assessment, mathematical problem solving, abilities, teachers.

Introduction

Assessment forms a large part of teachers' practice yet studies indicate that teachers feel inadequately prepared for the task of judging students' performances, skills and understandings. (Cumming & Wyatt-Smith, 2009; Mertler, 2004). Research that has investigated teachers' assessment practices has also criticized such practices for failing to meet standards of reliability, objectivity and validity (Allal, 2012). Assessment is inherently a process of professional judgment in which the element of interpretation is salient. In mathematics education Morgan (1998) has shown that teachers can interpret the meaning of the same passages of texts, produced by students in mathematics, very differently. When teachers interpret observed test results or other types of information to come to a conclusion about a student's level of knowledge or skill such a conclusion may be referred to as inference, and although some inferences can be made with more confidence than others, no conclusion about a particular student's knowledge or skill can ever be made with certainty (Cizek, 2009). Assessment in school mathematics has always relied heavily on students' written work (Morgan, 2001b). Written responses to mathematical tasks, such as problem solving, require that students explain both their thinking and the proposed solution. For such written material to act as valid evidence, from which judgements regarding students' problem-solving processes and mathematical abilities may be inferred, it has to be clear and comprehensive. There is reason to believe that not all students possess the ability to produce clear and comprehensive accounts of their mathematical problem solving (Monaghan, Pool, Roper, & Threlfall, 2009).

This study represents a microstudy on groups of teachers' assessment of a specific set of accounts of solving mathematical problems. The accounts were collected from two classes of 10-year old students. The aim of the study is to investigate the aspects of mathematical problem solving which are addressed and discussed by the teachers and to relate these to the five mathematical abilities set by the Swedish mathematics syllabus. These abilities are related to: *problem solving, mathematical concepts, mathematical methods, mathematical reasoning* and *communication*.

Students' writing in mathematics

It can be argued that the written mathematical work of students in school mathematics typically serves two very different functions. It can be seen as a part of a learning process in which writing is used to record and perhaps reflect on various mathematical ideas; hence, the text is written by and for the student herself. It can also however, be seen as a product for the purpose of assessment; hence, written for a teacher or examiner. Unlike the work of professional mathematicians, which is often thought to be the model for school mathematics, the work in school mathematics often serves these two functions at the same time (Morgan, 2001a). When problem solving is viewed as an individual cognitive activity, students use their writing to understand, explore, record, and monitor their own problem solving (Stylianou, 2011). Several studies indicate that writing poses problems for students. Evidence suggests, for example, that it is far more common for children to experience problems with semantic structure, vocabulary and mathematical symbolism than they do with, for example, standard algorithms (Ellerton & Clarkson, 1996).

Assessing mathematical problem solving

Assessment of students' mathematical problem solving is complex. There are different definitions of what mathematical problem solving is and what constitutes a problem. A generally accepted definition suggests that problem solving can be seen as a response to a question for which one does not already know or have access to a method (Monaghan et al., 2009). This understanding is also used by OECD in the PISA 2012 Assessment and Analytical Framework (OECD, 2013). Problem solving can be seen as a goal, a process, and a skill and problem-solving activities are thought to engage students in a number of different processes such as reasoning, communication and connections (Rosli, Goldsby, & Capraro, 2013). In a situation where traditional assessment in mathematics is increasingly seen as providing insufficient evidence of mathematical knowledge and abilities beyond routine skills and algorithms there are high hopes for alternative forms of assessment of which problem solving is one (Jones & Inglis, 2015; Rosli et al., 2013). Despite its power to engage students however, problem solving has been problematic to use as a source from which to make inferences about students' mathematical achievement. Reliance on the traditional mathematics test has often been justified on the grounds of reliability and comparability, but this has often been at the expense of validity (Watt, 2005). The challenges to assessment of problem solving are several. The first is that it requires access to evidence of the process. Most test situations do not include the option of observation to provide such evidence but rather require students to produce an extended written account which includes an explanation of both their problem-solving process and their proposed solution(s). This is problematic because considerable skill is required to produce clear and comprehensive accounts of problem-solving processes, a skill that students may or may not have (Monaghan et al., 2009). The second challenge is the element of interpretation and, thus, subjectivity. As teachers read and assess students' texts, their professional judgment is formed by a set of resources which varies with their personal, social and cultural history as well as their relation to the particular discourse. These resources are individual, as well as collective, and they include: personal knowledge of mathematics and the curriculum, beliefs about the nature of mathematics and how these relate to assessment, expectations about how mathematical knowledge can be communicated, experience and expectations of students and classrooms in general, and experience, impressions, and expectations of individual students (Morgan & Watson, 2002). Individual teachers

may also have particular preferences for particular modes of communication as indicators of understanding. A study from Australia has also indicated that teachers themselves object to the use of alternative assessment methods such as problem solving on the grounds that it is perceived as too subjective (Watt, 2005). In Sweden there have been calls for national tests to be assessed and graded externally instead of by the teachers who already know the students. External grading is seen as a way to secure objectivity and fairness.

Mathematical abilities

Assessment in mathematics has many concerns, of which perhaps the most important one is: what is it that is being assessed? This issue has been dealt with and given many names throughout the history of mathematics education including numeracy, mathematical proficiency, mathemacy, matheracy and quantitative literacy, to name a few (Wedege, 2010). Competency frameworks in mathematics are constructs that build on the assumption that mathematics is a domain in which it is possible to provide a generic set of mathematical practices (Säfström, 2013). Given that mathematical activities have to be about something, arriving at a common and generic set of such skills and abilities proves a challenging task, as has been pointed out by many (see for example Jablonka, 2003; Kanes, 2002; Kilpatrick, 2001; Wedege, 1999). Some frameworks have focused on this 'something' whereas others have focused on the mental processes that are associated with mathematical activities in general. Influential examples of the latter include the five strands of mathematical proficiency introduced by the Mathematical Learning Study of the NCTM in the US (Kilpatrick, Swafford, & Findell, 2001) and the KOM project in Denmark (Niss, 2003; Niss & Højgaard Jensen, 2002).

One of the motives behind the above referenced frameworks is the clear intention to break with a traditional teaching of mathematics associated with rote learning and procedures and instead promote a more dynamic view of what it means to do mathematics (Boesen et al., 2013). In Sweden the Swedish national curricula has been influenced by the ideas from these frameworks and in the Swedish syllabus in mathematics, introduced in 2011, five different abilities which the teaching in mathematics should provide the students the opportunity to develop, are described. These include the ability to:

- formulate and solve problems using mathematics and also assess selected strategies and methods,
- use and analyse mathematical concepts and their interrelationships,
- choose and use appropriate mathematical methods to perform calculations and solve routine tasks,
- apply and follow mathematical reasoning, and
- use mathematical forms of expression to discuss, reason and give an account of questions, calculations and conclusions. (SNAE, 2011, pp. 59-60)

In the syllabus the abilities, described above, are actualized in a set of knowledge requirements which define what constitutes an acceptable level of knowledge for the grades E, C, and A, where A represents the most advanced. In the results section the five knowledge requirements are shortened to: *problem solving, mathematical concepts, mathematical methods, mathematical reasoning and communication*.

Data collection

The study sets out to investigate teachers' assessment of a specific set of accounts of mathematical problem solving and aims to identify the aspects of mathematical problem solving which are addressed and discussed by the teachers. Nineteen elementary school teachers from four schools in a middle-sized town in mid-Sweden were interviewed in groups. There were five groups of 3, 4 or 5 teachers respectively. At the time of the interview all nineteen teachers were teaching mathematics. They were initially chosen by their principals and asked to participate based on their own interest. The interviews were all recorded on video and an additional audio recorder. The teachers were presented with 10-16 accounts of problem solving produced by students, aged 10. The problem-solving was centered on two specific problems. They were both Diophantine equations involving the identification of a number of ways to distribute: a) 30 legs on 12 animals or b) 36 wheels on 11 vehicles (see figure 1). This type of problem can be formulated in this way where there is only one possible combination or as an open problem to which there are many solutions. A small number of legs or wheels also results in a small number of combinations; the problem can therefore be adapted to fit different students or age groups. The students can also be asked to demonstrate that they have found all possible combinations and explain how they know this. The problem offers opportunities to adopt a more or less systematic trial-and-error strategy, but there are also other ways to solve the problem. Given that the problem involves concrete objects it also offers students opportunities to draw. All these properties contributed to the choice of the problem type.

The teachers in the interviews were given information on the problems but very little information on the situation in which the texts were created. Being faced with an account of mathematical problem whose origin you know very little about forces a teacher to focus on the account itself and the interpretations derive to a larger extent from the account than it would had the teacher been asked to comment on their own students' written material. The teachers were asked to discuss the different accounts from an assessment perspective and to provide arguments for their reasoning. The group interview was chosen so as to create room for discussions but also for eliciting the teachers' idea of possible 'common grounds' in evaluating students' accounts. The interviews, which amounted to a total of 4 hours 26 minutes, were transcribed.

Analysis

The analysis was performed in two steps. In the first step the transcribed interviews were analyzed with the intention of identifying instances in which the teachers discussed what the students seemed to be **doing**. This focus was inspired by the understanding that knowing mathematics is doing mathematics, as described above. This analysis included identifying verbs connected to instances of action such as *understand, know, think, draw, calculate, see* and *show*.

The second step in the analysis was focused on relating the identified instances to the different abilities described in the syllabus. The five abilities *problem solving, methods, concepts, reasoning,* and *communication,* did not have to be mentioned specifically. A discussion regarding a method such as trial-and-error was considered as relating to method even if the term *method* was not used. Discussions about failed attempts or deficiencies were also considered as belonging to the category of the ability in question. Examples of quotes from the teachers are shown below together with the abilities they were thought to relate to. One quote can be related to several of the listed abilities.

Teacher: Here they have really tried…drawn all the tires… (problem solving, method, communication)

Teacher: He has counted the number of fours he has taken away and those are plus signs… it is plus 7… (problem solving, method)

Teacher: There is no reasoning to show that this is correct… (reasoning, communication)

Teacher: They cannot reason without explaining a little bit more… she has not used any concepts for example… (concepts, reasoning, communication)

Teacher: It is not enough to just write an answer…you have to be able to show in writing how you arrived at this… (communication)

Teacher: Yes but she…she does know how to solve the problem… (problem solving, communication)

Teacher: And then you try different numbers… that is how they have done it… you can see that they have erased… (problem solving, method, communication)

Preliminary results

The preliminary results are presented under headlines which are consistent with the five abilities described in the syllabus. In some cases the teachers' discussions are covering two abilities at the same time and in these cases they are either presented under both headlines or presented as a compound ability which is treated under one headline.

Problem solving

Many of the teachers' discussions are focused on the students' choice of method or strategy for solving the problem and the teachers spend considerable time trying to identify the specific method of each student. Once this has been identified however, the discussions tend to turn to other issues. A problem solving strategy is seldom judged based on its appropriateness or sophistication. Other aspects of problem solving that are addressed by the teachers include the ability to describe a problem-solving approach. The ability to describe a method, strategy or problem-solving approach can be seen as part of a problem-solving ability and this aspect is also mentioned in the knowledge requirements. This aspect however is very difficult to distinguish from the ability to account for and communicate a method, strategy or problem-solving approach. The teachers' discussions on students' ability to communicate are treated under this headline below. The ability to reason about the plausibility of results of the problem solving, or to propose alternative approaches, which is mentioned in the knowledge requirements, is not discussed.

Mathematical concepts

Very few discussions deal with mathematical concepts. The four operations are mentioned but they are referred to as calculations which illustrate the process rather than as concepts. One student is identified by several teachers as having used the equals sign in a non-standard way which can be interpreted as relating to the concept of equality but this can also be connected to the way students choose to present their calculations.

Mathematical methods

As was presented above this is the ability which many of the teachers' discussions are focused on. The method that most teachers identify is the trial-and-error method. Several teachers claim that this is the method that all students have used. There are several accounts which show different ways in which the students have carried out and represented this method but these differences are most often referred to as relating to the ability to communicate. There are examples of accounts where the trial-and-error method is not used systematically and other examples where the representation indicates a calculation that precedes the trial-and-error since the account either contains no errors at all, or displays errors that have been erased but which are still traceable. This difference stirs many discussions among all the teacher groups. They are discussing whether they can tell if a student has tried different combinations and ruled some out or if the student came by the right combination by chance or by doing mental calculations that are not represented in writing. Sometimes they agree that they cannot tell and that this is due to students' lack of ability to communicate their problem-solving processes, other times they have different opinions regarding what can be inferred.

Mathematical reasoning

There is only one teacher who addresses the students' mathematical reasoning. This teacher argues that any account of problem solving which describes a method or strategy constitutes evidence of some form of mathematical reasoning. The rest of the teachers in this group are not questioning her but they are not offering her support and the issue of mathematical reasoning does not come up again.

Communication

There are very few discussions that do not involve students' ability to account for and describe their approaches to solving the problem. Practically every instance involves a question from the teachers regarding what the students have done or what they mean. Even in cases where the teachers have identified a successful strategy along with a correct answer to the problem they still raise questions regarding the clarity and coherence of the account. The discussions on presentation are focused on the students' [lack of] logic, neatness, clearness, abstraction, accuracy, appropriateness, and comprehensiveness. When discussing students' choice and employment of different method a typical comment from the teachers is "if she had only shown…". This fictional comment summarizes the teachers' frustration with what they perceive as lack of evidence on which they can base their judgements regarding other abilities.

Discussion

The preliminary results presented above can be used in response to the calls for external grading of national tests in mathematics in Sweden, and elsewhere, as a way to ensure objectivity and fairness. The results indicate that students' lack of communicative skill makes it difficult for the teachers to use these written accounts to assess other mathematical abilities. The study thus confirms Monaghan et al's (2009) claim that students' ability to communicate, to describe and to account for, their processes or their thinking, is crucial for teachers as well as for students. In order for teachers to evaluate students' abilities, they need to understand what the students have done and why. In order for students to write in a way that reflects their mathematical knowledge they need to know

how to represent their problem-solving process along with explanations and arguments for their various choices. The fact that there are different ways to interpret what the students have written, further strengthens the conclusion that using this writing, to assess other mathematical abilities, may be problematic. The results should not be interpreted as suggesting that problem-solving should not be used to assess students' mathematical abilities but rather that both teachers and students need to know more about different ways to clearly and comprehensively account for problem-solving processes.

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Using classroom assessment techniques in Chinese primary schools: Effects on student achievement

Xiaoyan Zhao[1], Marja van den Heuvel-Panhuizen[1,2] and Michiel Veldhuis[2]

[1]Freudenthal Institute, FBW, Utrecht University, the Netherlands; x.zhao@uu.nl

[2]Freudenthal Group, FSW, Utrecht University, the Netherlands

In an experimental study with a pretest/posttest/delayed posttest and control-group design, we investigated the effects on students' mathematics achievement of using classroom assessment techniques in Chines classrooms. Participants were 47 third-grade teachers and their 608 students in Nanjing, China. The teachers were assigned to either the experimental condition, participating in two two-hour workshops on classroom assessment, or the control condition, in which the teachers followed their regular teaching plans. The workshops focused on the use of classroom assessment techniques to reveal students' understanding of multiplication and to enable teachers to adapt teaching to their students' needs. Students from the teachers in the experimental condition slightly improved their mathematics achievement scores. However, no statistically significant difference was found between the two conditions.

Keywords: Classroom assessment, student achievement, China, multiplication, teachers.

Introduction

The guidance teachers provide in their mathematics classes to their students can be more or less effective for stimulating students' learning processes, depending on whether their instruction is attuned to students' needs and possibilities for further development. Therefore, at practically every moment teachers need to know where the students are in their learning process (Wiliam, 2011). This was also recently emphasised by Schoenfeld (2014) when he wrote that "[p]owerful instruction 'meets students where they are' and gives them opportunities to move forward" (p. 407). Classroom assessment, i.e. assessment in the hands of the teachers that is interwoven with instruction and integrated in daily teaching practice, can inform teachers of 'where their students are' and as such enable them to adapt their further instruction to their students' needs.

Since the importance of classroom assessment on raising students' achievement was revealed by Black and Wiliam (1998), much attention has been paid to professional development to enhance teachers' classroom assessment practice. The rationale for this is that providing professional development to teachers on the use of classroom assessment can lead to teachers gaining more information on their students' understanding and skills. Through this information teachers can adapt their teaching to their students' needs, which in turn is expected to lead to improved student achievement. Whether professional development indeed has impact on student achievement was investigated in several studies (Phelan et al, 2012; Randel, Apthorp, Beesley, Clark, & Wang, 2016; Thompson, Paek, Goe, & Ponte, 2004; Veldhuis & Van den Heuvel-Panhuizen, 2014, 2016). The results of these studies are mixed. Facilitating teachers to use classroom assessment has been shown to lead to considerable improvement of students' achievement (Phelan et al, 2012, Veldhuis & Van den Heuvel-Panhuizen, 2014, 2016). It also happened that professional development on classroom assessment had only a small but consistent positive effect on student learning (Thompson et al, 2004)

or failed to yield any statistically significant impact (Randel et al, 2016). These mixed, but generally positive, results on the effects of professional development on classroom assessment were all found in the western educational context. As there are important differences between mathematics education in Western and East Asian countries (Leung, Graf, & Lopez-Real, 2006), we aimed to find out whether giving support to Chinese teachers on the use of classroom assessment would have an effect on their students' mathematics achievement.

In China, recently, classroom assessment has received increasing attention from primary school mathematics teachers, as evidenced by an increasing number of teacher-written papers addressing classroom assessment (Zhao, Van den Heuvel-Panhuizen, & Veldhuis, 2017). Moreover, in Chinese primary mathematics education, teachers generally agree that assessment is useful for the improvement of teaching and learning, and they assess their students at least weekly by employing various methods, for example observing, questioning and assigning textbook tests (Zhao, Van den Heuvel-Panhuizen, & Veldhuis, 2016a). It seems that classroom assessment has been widely embraced and implemented in teaching practice. However, professional development focused on classroom assessment seems sparse (Zhao et al, 2016a), let alone investigations into its effect on students achievement.

In our study, classroom assessment is conceived as the use of what we call 'classroom assessment techniques' (CATs): short teacher-initiated assessment activities that teachers can use in their daily practice to reveal their students' understanding of a particular mathematical concept or skill. These CATs have been used in earlier research in the Netherlands (Veldhuis & Van den Heuvel-Panhuizen, 2014, 2016). Our main research question was: *What are the effects of supporting Chinese primary school mathematics teachers' use of classroom assessment techniques (CATs) on students' mathematics achievement?*

Method

An experiment with pretest/posttest/delayed posttest and control-group design (see Table 1) with 47 third-grade mathematics teachers from 18 primary schools was carried out in Nanjing, China. All teachers used the same textbook, namely 苏教版 textbook published by Jiangsu Phoenix Education Publishing House (2014). Based on the participating schools' reputation, educational quality, and location, pairs of matched schools were allocated either to the control or to the experimental condition. Teachers in the experimental group participated in two two-hour workshops on the use of classroom assessment techniques, whereas the teachers in the control group followed their regular teaching plans.

Condition	January	March			May
		Week 1	Week 2	Week 3	
Control	Pretest			Posttest	Delayed posttest
Experimental	Pretest	Workshop	Workshop	Posttest	Delayed posttest

Table 1: Time schedule of the experiment in 2015

In the workshops, the teachers were introduced to eight CATs. These CATs are low-tech and low-cost, and can easily be implemented by teachers. Every technique consists of a short activity (less

than 10 minutes) and helps teachers to quickly find out something about their students' understanding of mathematics, provides indications for further teaching. Also, the teachers could adapt the techniques to their own practice; they could choose when and how to use the CATs. The focus of the assessment techniques was on the first chapter of the second semester of Grade 3, in which students learn how to solve multiplication problems of two-digit numbers mainly by written digit-based algorithm. In the following we illustrate three examples of these CATs. During the workshops, the teachers in the experimental condition were provided with a detailed teacher guide describing the eight CATs that all fitted to the content of their textbook. Detailed information about the purpose of the CATs and suggestions for how to use them was provided and discussed during these workshops. It was also explained that the teachers were free to decide how they would use the CATs in practice in the following two weeks of multiplication teaching.

CAT 1: Family problems

This CAT (see Figure 1) is aimed at assessing whether students recognize similarities among analogous problems and can use the given answer to one of these problems to solve the others.

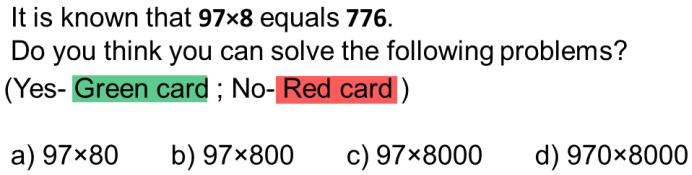

Figure 1: CAT 1: Family problems

One strategy to solve a multiplication problem with either the multiplicand or the multiplier being a multiple of 10 is making use of an analogous problem of which the answer is known or which is easy to calculate. A requirement for students to choose and use this strategy is that they understand the analogous relationship, even when the numbers involved in the multiplication are bigger than two digits. CAT 1 is meant to elicit information of whether and to what extent students have this understanding. The students are provided with the answer of 97×8 and are then asked whether they *think* they are able to solve mentally a number of other, related multiplication problems that, at first sight, are not easy to solve by mental calculation. CAT 1 differs from the regular assessment tasks in the textbook in which the students have to carry out the calculation and the focus is on detecting whether students can do this correctly. In CAT 1 it is assessed whether the students recognize the analogue structure of the problems and are aware that they can use this for solving these problems. In CAT 1, the teacher asks for every problem whether students think they are able to solve it. All students have a green card (for the answer: "Yes") and a red card (for the answer: "No") with which they can show their answers (see Figure 2). By inspecting the waving green and red cards the teacher gets an immediate overview of the students' responses and whether they see the analogy between the problems, and whether their understanding is affected by the number of zeroes in the family problems.

a) 97×80



d) 970×8000



Figure 2: Students showing their cards in CAT 1 for the problems 97×80 and 970×8000

CAT 2: Breaking down a multiplication

This CAT (see Figure 3) is aimed at assessing whether students can identify the components of a multiplication by filling in the blanks on a work sheet.

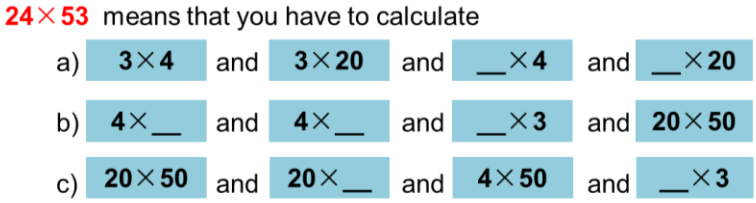

Figure 3: CAT 2: Breaking down a multiplication

Students may be able to find the correct answer of a problem like 24×53 by performing the standard multiplication algorithm perfectly; however, this does not necessarily mean that students understand what they are doing and that they understand the structure of multiplications with multi-digit numbers, which is the focus of CAT 2. This approach of requiring students to unravel multiplication problems differs from the regular approach to assessing students in which finding the correct answer of a multiplication problem receives most attention of mathematics teachers. In the case of CAT 2, the multiplication of 24 and 53 can be unpacked into four sub-multiplications, namely 3×4, 3×20, 50×4, and 50×20. The sum of the results of these sub-multiplications gives the answer of 24×53. By asking students to identify the components of a multiplication problem of multi-digit numbers it can be revealed whether they understand what is 'behind' the multiplication algorithm. For example, the student work in Figure 4 shows that Student 1 has difficulties in being fully aware of the values of the digits (having 5×4 and 5×20 instead of 50×4 and 50×20 in Task a, and having 2×3 instead of 20×3 in Task b), while Student 2 could not clearly distinguish the different components of the multiplication 24×53 (having 4×3 instead of 20×3 in Task b, and having no answer filled in Task c).

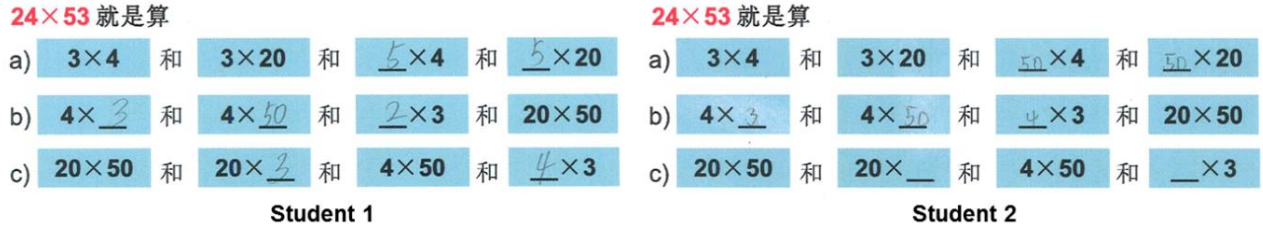

Figure 4: Work of two students in CAT 2

CAT 3: Fruit language

This CAT (see Figure 5) is aimed at assessing whether students can use the associative and distributive property of multiplication to restructure a multiplication problem.

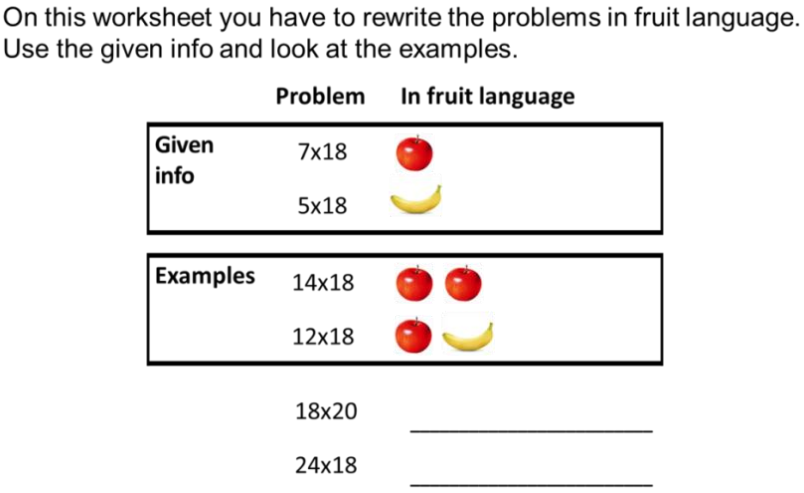

Figure 5: CAT 3: Fruit language

Making use of the associative and distributive property of multiplication is the basis of solving multiplication problems. By using these properties students can convert a difficult multiplication problem into a number of easier multiplication problems. For example, 25×36 can be solved by calculating 20×36 and 5×36 (distributive property) or by calculating 25×4×9 (associative property). For solving multiplication problems in this way it is very important that students understand the associative and distributive property of multiplication and that they can identify the possibilities of restructuring a multiplication problem. CAT 3 provides an opportunity for students to show this understanding. In order to avoid the difficulty of formal notations, fruit is used as a substitute.

The student work shown in Figure 6 reveals that Student 3 has arrived at a high level of the understanding of the associative and distributive property of multiplication and is able to notate this in a proper mathematical way, although not using a formal notation with number or letter symbols. Student 4 only 'rewrote' one of the multiplication problems (18×20) by drawing four bananas. Moreover, the worksheet of this student shows that he/she did not use the properties of multiplication but instead was calculating the multiplications and then tried to express the answer by using the fruit.

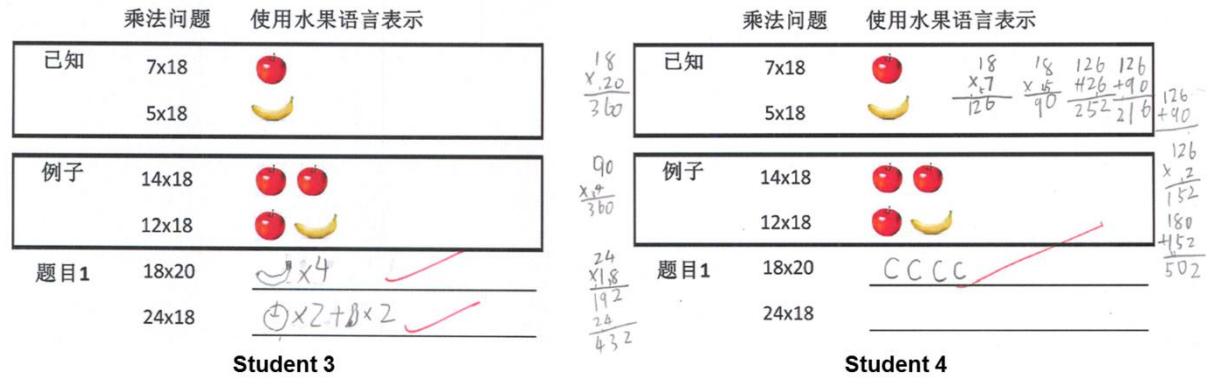

Figure 6: Work of two students in CAT 3

In order to measure students' mathematics achievement, three tests were used, which were designed and arranged by the local teaching research office. These tests have the same structure in terms of the type of questions and total score (100 points). However, the mathematical domains that are tested are different. The immediate posttest was an end-of-chapter test and focused on the multiplication of two-digit numbers. The pretest and the delayed posttest were end-term and mid-term tests, which also

included problems related to measurement, fractions, and geometry. Nevertheless, multiplication is the main focus of all the three tests (30% of the points in the pretest and delayed posttest and 90% in the immediate posttest were related to multiplication tasks).

Originally, 3040 students took the tests. Since it was found that mistakes were made when grading students' examination papers, we decided to choose 608 (20%) students systematically, based on their student number in every class, for data checking to be included in the final analysis.

Results

Unexpectedly, on average, students in both conditions had decreasing mathematics achievement scores from pretest (M_{exp} = 89.2, SD_{exp} = 8.7; M_{con} = 90.8, SD_{con} = 7.7) to immediate posttest (M_{exp} = 88.5, SD_{exp} = 9.3; M_{con} = 89.5, SD_{con} = 9.0) and to delayed posttest (M_{exp} = 86.4, SD_{exp} = 12.2; M_{con} = 87.7, SD_{con} = 11.2). When looking at the standardized scores this image becomes a bit less clouded by the different tests measuring different domains at the different time points, therefore we report the *z*-scores in Table 2. The pattern remains almost the same, with relatively higher scores in the control condition than in the experimental condition, but, in the experimental condition, a slight improvement of the scores appears after the intervention.

	Pretest score		Posttest score		Delayed posttest score		*n*
Condition	*M*	*SD*	*M*	*SD*	*M*	*SD*	
Control	0.104	0.930	0.059	0.986	0.058	0.952	278
Experimental	-0.088	1.049	-0.050	1.010	-0.049	1.038	330

Table 2: Descriptive statistics of students' standardized mathematics (z) scores per condition for the pretest, posttest, and the delayed posttest

We performed an analysis of covariance (ANCOVA) on the immediate posttest scores to see if this small improvement was statistically significant. In this ANCOVA the pretest score was entered as covariate and condition as fixed factor. It turned out that no significant effect for condition was found ($F(1, 605) = 0.08$, $p = .776$, $\eta_p^2 = 0.000$).

Discussion

The students of the teachers that participated in the workshops on the CATs only very slightly improved their standardized mathematics achievement scores after the intervention. This improvement was not significant, neither in size, nor in the statistical sense. Contrary to these findings in the experimental group, the students in the control group did not improve their standardized scores from one test to the other. However, on average the students in the control condition outperformed the students in the experimental condition on all three tests. A possible reason for the minor changes in students' mathematics achievement could be that there appeared to be a strong ceiling effect on the tests (average success scores of around 90%). Maybe students' extant high achievement level could also have caused that the use of the CATs did not further optimize the teachers' instruction. Another explanation for the small improvement in the experimental condition could be the short period of time of the intervention. In less than three weeks, the teachers in the experimental condition needed to understand how to use the CATs, to incorporate them into their teaching plans, and to reconcile the new insights into their students with their original understanding of students and teaching. For

teachers to really get used to and to make the most of the implementation of the CATs, probably more time needs to be reserved and more guidance needs to be offered in the professional development workshops.

Also the context of the experimental study may have influenced the effect of the CATs on the students' mathematics achievement. First of all, as we found in an earlier study (Zhao, Van den Heuvel-Panhuizen, & Veldhuis, 2016b) Chinese primary school mathematics teachers have detailed lesson plans and tend to include CATs in their pre-arranged lessons as extra exercises rather than implementing them as formative assessment activities. As such, the teachers may not have used the information gathered with the CATs for adapting their instruction. Another issue is that the planned lessons have for every addressed topic a fixed time schedule for instruction and practice. By including the CATs less time could be spent on teaching these topics and students may have had less practice in solving the problems as used in the regular tests. A promising finding is that despite this smaller investment in the regular program the students in the experimental condition did not perform really worse in the regular tests than their counterparts in the control condition. In this way, our study provides some evidence which may encourage teachers to go beyond the straightforward testing of the standard operations and pay also attention to examining students' deeper understanding of these operations, and use the assessment information adaptively for improving instruction and student learning.

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The complexity of mathematical thought and the quality of learning: Portfolio assessment

Fernando Luís Santos[1] and António Domingos[2]

Instituto Piaget, Almada, Portugal; fernando.santos@almada.ipiaget.pt

UIED, DCSA, FCT da Universidade Nova de Lisboa, Portugal; amdd@fct.unl.pt

This poster presents an instance of portfolio assessment by two students on a geometry problem in a pre-service teacher training course analysed through a model designed to access the mathematical thought and the quality of student leaning outcomes. This analytical model, supported by the SOLO taxonomy, uses Activity Theory as a contextual framework that integrates the different relations, namely advanced mathematical thinking concepts like procept and proceptual divide. Results allowed us to see the different pathways taken by the students to solve the same problem.

Keywords: Assessment, geometry, portfolio, quality of learning, mathematical thought.

Introduction

Portfolio assessment brings an open evaluation method into the mathematical classroom and allows the mathematical abilities of the students to grow. In this study, students chose three of 15 problems and were given one month to solve them and to explain in detail their solution process. This solution process involves brainstorming sessions centred on the best solution, and the detailed explanation necessarily involved in self-regulated learning processes. This teaching method aims to extend the mathematical knowledge of future teachers, involving them in activities more open and less structured than the traditional ones.

The data presented here were chosen because the students show a similar path (12th grade mathematics) and took different approaches to the same problem. Data was studied using the analytical model that highlights these differences and integrates the SOLO taxonomy (Biggs & Collis, 1982) with the advanced mathematical thinking theories and concepts of Tall (1991) alongside the conceptualization of the proceptual divide (Gray & Tall, 1994), and activity theory (Engeström, 2001) as a contextual structure. The SOLO taxonomy allows us to identify five progressive levels of understanding from the *prestructural* (lowest level), through the *unistructural*, the *multistructural*, the *relational* to the *extended abstract* (highest level).

The problem statement asks to find the length of $\overline{BC}$ (to the second decimal place) knowing that $\overline{AC} = 10cm$ and $\measuredangle BAC = 30^0$ (do not use any trigonometry) with the aid of figure 1.

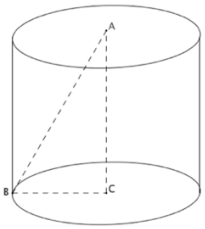

Figure 1: Visual representation of the problem

Raquel, one of the participants, struggled with this problem due to the limitation stated in the problem (do not use trigonometry), but she sketched another triangle making an isometry using $[AC]$ as a

symmetry axis, creating with B' an equilateral triangle ABB' because if $\measuredangle BAC = 30^0$ then the resulting isometry makes $\measuredangle CAB' = 30^0$ therefore $\measuredangle BAB' = 60^0$ and a equiangular triangle is also an equilateral triangle. By using the Pythagorean theorem, she calculated:

$$If\ \overline{AB} = x, \overline{BC} = \frac{x}{2}, \overline{AC} = 10\ then\ x^2 = \left(\frac{x}{2}\right)^2 + 10^2\ simplifying\ x^2 = \frac{x^2}{2^2} + 100 \Leftrightarrow 4x^2 = x^2 + 400 \Leftrightarrow$$

$$\Leftrightarrow 4x^2 = x^2 + 400 \Leftrightarrow 3x^2 = 400 \Leftrightarrow x \cong 11{,}55 \Leftrightarrow \overline{AC} = \frac{x}{2} \cong 5{,}77\ cm$$

Mariana on the other hand started calculating $\measuredangle ABC$ by subtracting the other two angles known so:

$$If\ \measuredangle BAC = 30^0, \measuredangle BCA = 90^0\ then\ 180^0 = 90^0 + 30^0 + \measuredangle ABC\ so\ \measuredangle ABC = 60^0$$

And then she made a relation between the sides of the triangle by using the ration of special triangles (a trigonometry concept) what gives the following outcome:

$$If\ \overline{AB} = 2a, \overline{BC} = a, \overline{AC} = a\sqrt{3}\ \wedge\ \overline{AC} = 10\ therefore\ a\sqrt{3} = 10\ \wedge a = \frac{10}{\sqrt{3}}\ so\ \overline{BC} = a\ so\ \overline{BC} \cong 5{,}77\ cm$$

Final remarks

These students are very familiar with being tested by closed book exams (and they were expecting that also), but portfolio assessment was a different approach and they were in an *unknown territory*.

Raquel clearly surpassed the proceptual divide and her outcome was classified as an *extended abstract.* She made connections to other concepts, explained her pathway and was able to justify the outcome, supplying evidence to support her solution. Activity theory was used to identify the contradiction arising by the use of different mediating artifacts namely the use of isometries to produce the equilateral triangle.

Mariana knew something about what she was doing, but by breaking the mathematical rules she did not surpass the proceptual divide. Her outcome was classified as *multistructural*. After the presentation of the results it is clear that her answer could be *unistructural* because she made simple connections without identifying their role. She could not describe what's the process she had followed but could not justify what she did and how she obtained the outcome. The use of activity theory revealed two major contradictions related to the mediating artifacts and the limitations.

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TWG22: Curricular resources and task design in mathematics education

Introduction to the papers of TWG22: Curricular resources and task design in mathematics education

Birgit Pepin[1], Sean Delaney[2], Sebastian Rezat[3], and Andreas J. Stylianides[4]

[1]Eindhoven University of Technology, Eindhoven School of Education, NL; b.e.u.pepin@tue.nl

[2]Marino Institute of Education, Dublin, Ireland; sean.delaney@mie.ie

[3]University of Paderborn, Germany; srezat@math.upb.de

[4]University of Cambridge, Faculty of Education, UK; as899@cam.ac.uk

In this report the themes and points for discussion of the Thematic Working Group 22 on Curricular Resources and Task design are briefly explained and summarized. These related to design principles and task characteristics, also of digital curriculum resources, and perspectives on the development of digital resources were identified. At the same time the role of teachers in task design, whether as "designers" or "partners in task design" or as mediators of tasks designed by others, and of course the role of students working with the tasks/resources, were acknowledged as crucial issues. Concerning teachers, pre-service or in-service teacher education, and generally teachers' work in collectives, were perceived as stimulating contexts for the design of and work with curriculum resources: how can teachers develop design capacity and knowledge when working with curriculum resources, and which affordances (and constraints) are provided by digital resources? Concerning students, discussions evolved around issues related to how digital resources can provide feedback for enhanced student learning.

Keywords: Curriculum resources; task/curriculum design; design principles; teacher collective work; design capacity.

Background

The TWG 22 received 25 paper and poster submissions: 17 submissions were accepted as papers, 6 as posters, and 2 were withdrawn (for various reasons: lack of funding; lack of time). The contributing authors came from 17 different countries.

The sessions were organised into five paper sessions organised under four themes, one poster session, and one report & discussion session. For each session we now briefly present the authors and titles of the individual presentations, followed by a summary of themes discussed in the sessions.

Session 1 & 2

Theme: Issues linked to the design of tasks and resources

- Gijsbers & Pepin: Context based tasks on differential equations to improve students' beliefs about the relevance of mathematics
- Nagari Haddif: Principles of redesigning an e-task based on a paper-and-pencil task: The case of parametric functions
- Borys & Choppin: Tensions between resource perspectives and trends in the design and dissemination of digital resources

- Brocardo, et al.: Tasks to develop flexible multiplicative reasoning
- Seidouvy & Eckert: Designing for Responsibility and Authority in experiment based instruction in mathematics
- Cohen-Eliyahu: The role of design in conceptual change: The case of proportional reasoning

The issues identified in these two sessions related to four main themes. First, participants discussed the relationships between design principles and task characteristics. This is particularly important, if the design is expected, for example, to afford conceptual change (see Cohen-Eliyahu), or to support student reasoning competency involving aspects of authority and responsibility (see Seidouvy & Eckert) or to enhance students' beliefs about the relevance of mathematics (see Gijsberg & Pepin), or to facilitate noticing and visualization when developing flexible multiplicative reasoning skills (see Brocardo, et al.). It is also pertinent for tasks that were re-designed from paper and pencil to e-tasks (see Nagari Haddif). Second, perspectives on the development of digital resources were identified, and tensions between conceptions of teachers' interactions with digital resources and the ways other actors (e.g. policy makers, curriculum developers) frame the purpose for and development of digital curriculum resources (see Borys & Choppin). A third point of discussion was the role of teachers in task design, whether as "designers" or "partners in task design" or as mediators of tasks designed by others. In fact, it was concluded that often teachers were left out, or insufficiently considered, in relation to their role of mediating tasks. A fourth point of discussion was related to the role of students working with the tasks/resources, which was addressed in the majority of the papers presented in these two sessions.

Session 3 (poster session)

In this session all posters were presented in order to provide additional opportunities for contributors to participate in this TWG.

- Noll, et al.: How to design educational material for inclusive classes
- Jukic Matic: Teachers' pedagogical design capacity and mobilisation of textbook
- Wynne & Harbison: Task design within the Universal Design for Learning Framework to support inclusion in the mathematics classroom
- Llanos & Otero: Changes in the images and arguing from mathematics textbooks for the secondary school in Argentina along 67 years
- Dooley & Aysel: Using variation theory to explore the re-teaching phase of lesson study
- Cizmesija et al.: Asymptotes and asymptotic behaviour in graphing functions and curves: an analysis of the Croation upper secondary education within the anthropological theory of didactics

Session 4

Theme: Issues linked to prospective teachers' work with tasks and resources

- Dempsey & O'Shea: Critical Evaluation and Design of Mathematics Tasks: Pre-Service Teachers
- Kilic, et al.: Pre-service teachers' reflections on task design and implementation

- Stylianides & Stylianides: Promoting prospective elementary teachers' knowledge about the role of assumptions in mathematical activity

The issues identified in this session related to four main themes. First, although not directly addressed in the three papers, the issue of design capacity building was raised: what teacher design might mean (see also sessions 1 & 2), who is designing what (see also sessions 1 & 2), and how design capacity building relates to teacher education. Linked to latter, a second point was discussed: the relationship between teacher knowledge (see Stylianides & Stylianides) and design capacity. As a third point for discussion, participants reflected on the implementation process in terms of pre-service teacher learning/knowledge development (see Kilic, et al.). Under a fourth issue it was discussed how teacher knowledge (in teacher education) could be enhanced through task design (Dempsey & O'Shea).

Session 5

Theme: Issues linked to design and use of resources in professional development and collective work

- Eckert: Agency as a tool in design research collaborations
- Essonnier, et al.: Factors impacting on the collaborative design of digital resources
- Gueudet & Parra: Teachers' collective documentation work: a case study on tolerance intervals
- De Moraes Rocha & Trouche: Documentational trajectory: a tool for analysing the genesis of a teacher's resource system across her collective work

The issues identified in this session related to three main themes. First, it was emphasised that teachers' learning trajectories of professional learning are typically not short-term, and hence that there is a need to research them longitudinally (see De Moraes Rocha & Trouche). Second, the dynamics in collaborative task design of teachers and teachers working/designing collaboratively with curriculum resources, were highlighted (see Eckert; Gueudet & Parra): in terms of lesson preparation, design of learning trajectories, and/or teacher re-design in class. Linked to that a third point was raised: the cultural aspects of teachers working with curriculum resources (see Essonier et al.).

Session 6

Theme: Issues linked to teacher and student use of resources/curriculum materials at primary level

- Daina: From textbook to classroom: a research on teachers' use of pedagogical resources in the context of primary school in the French
- Delaney: Children's performance on a mathematics task they were not taught to solve: A case study
- Gaio: Programming for 3rd graders, Scratch-based or Unplugged?
- Rezat: Students' utilization of feedback by an interactive mathematics e-textbook for primary level

The issues identified in this session related to four main themes. First, although not directly addressed in all papers of this session, the choice, design and use of digital as compared to non-digital curriculum resources were discussed. This was seen as particularly relevant, as often digital curriculum resources are combined with traditional materials (see Gaio), such as textbooks. Second, participants discussed the use of mathematical tasks/curriculum resources to promote particular learning goals (see Diana), in particular as textbooks often do not provide information on how to organize the teaching of particular activities. This links to the third point, the enactment/ implementation of tasks depending on teacher goals (see Delaney). A fourth point related to task design and feedback, that is ways in which teachers or curriculum developers can provide effective feedback, i.e. feedback that actually influences the development of mathematical concepts (see Rezat).

Session 7

In session 7 the following themes were identified as overarching issues:

- Task design: what does task mean; what does design mean?
- Teacher design capacity & the role of the teacher in "design" (incl. implementation)
- Digital resources/tasks/curriculum materials
- Plurality of theoretical frameworks & clarity
- Operationalization of theoretical frames
- High inference claims & evidence

Whilst the first three have been addressed under previous sections, the last three deserve a special mention. It was noted that a plurality of theoretical frames was used. This is not an issue in itself, but it becomes a problem, when the diversity of theoretical frameworks diverts from, and sometimes covers, the problem addressed, that is obscures the clarity of the research. Linked to that selected participants would have wanted a better, or clearer, description of how the theoretical frames were actually operationalized. Moreover, it was mentioned that too often the researchers made high inference claims based on insufficient evidence. This was seen as a shortcoming of such research. Overall, participants emphasized the positive and constructive atmosphere in the group, where criticism was welcomed as a vehicle for developing deeper insights and sharpening up of ideas.

Tensions between resource perspectives and trends in the design and dissemination of digital resources

Zenon Borys[1] and Jeffrey Choppin[1]

[1]University of Rochester; zenon.borys@gmail.com

[1]University of Rochester; jchoppin@warner.rochester.edu

Teachers are increasingly using digital resources to design lessons. We describe three perspectives for describing teachers' interactions with digital resources, perspectives that denote different assumptions with respect to teacher agency and the connections between capacity development and resource use. This paper examines the tensions between these conceptions of teachers' interactions with digital resources and the ways other actors – including policy makers, curriculum developers, and purveyors of online content – frame the purpose for and development of digital resources. Our analysis suggests that the assumed role of the teacher differs across different sets of actors and different visions related to the design and development of digital curriculum resources. The implications relate to the opportunities for teachers to transform digital resources to suit their purposes and to develop and grow professionally as a result.

Keywords: Digital curriculum, teacher practices, curricular resources.

Teaching is design work: Teachers actively interpret and mobilize resources to attain pedagogic and curricular goals. Moreover, teachers increasingly use, and are expected to use, digital resources to design lessons. However, there are tensions between education technology discourses, curriculum design trends, education policies, and the work teachers do with digital resources. As a case in point, in Sweden a public/private partnership endeavor created a repository of resources and lessons for teachers that was little used because the design of the repository did not take into account how teachers actually take up digital resources, illustrating the lack of alignment between teachers' professional practices, education policy, and the design of instructional resources (CERI, 2009). As Remillard (2012) notes, teachers do not simply interpret authors' intentions as they engage with curriculum resources, they engage with the artifact itself (p. 114). Given that teachers' work with resources can promote teachers' professional development and meaningful experiences for students, it is important to understand how tensions between teachers' practices and the assumptions about teaching and teachers embedded in digital resources potentially constrain teachers' opportunities to develop design capacity.

To illustrate how digital resource design and dissemination limit teachers' agency – and thus their roles as designers – we examined tensions between conceptions of teachers' interactions with digital resources and the ways other actors – including policy makers, curriculum developers, and purveyors of online content (commercial publishers, large philanthropies) – frame the purpose for and development of digital resources. These actors constitute the largest source of resources and information regarding digital resources and thus their influence merits attention from the research community. The ways digital resources are framed and promoted by policy makers and other actors may be at odds with the actual work teachers do and the needs of their students.

Conceptualizations of teachers' work with digital curriculum resources

The meaning of the term resources depends on the perspective of the researcher and what is being researched (Ruthven, 2013). Researchers frame the relationship between teachers and their use of resources in three ways: resources function as tools that mediate the act of teaching; resources function as artifacts (instruments) that signify outcomes of processes; and resources function as objects whose creation is a key component of teachers' professional work (Remillard, 2013). These are elaborated in more detail below.

The most common framing in teachers' design work is that resources are tools that mediate teachers' work, in line with sociocultural notions of tool mediation (e.g., Wertsch, 1998). The resource's function is primarily to help/aid/assist teachers in the work of teaching. Teachers' use of resources allows them to improve practices they already engaged in or, sometimes, to engage in new practices altogether. Often, the affordance or benefit of the resource is located within the resource and teachers perceive the affordance or benefit and apply it to their local context. Kasten and Sinclair (2009), for example, showed that teachers selected digital applets to help present topics in new ways. Also, Duncan (2010) demonstrated that teachers made their classroom practices more student-centered because the resources they were using automatically provided students feedback and gave students more agency over their interactions with content. These teachers crafted lessons that capitalized on this shift in the student-content relationship. In these two examples, the benefits or affordances were thought of as residing in the resource.

Considering resources as an artifact – rather than as a tool –connotes place and time: teachers' use of resources is inseparable from when they are used and teachers' purposes for using resources. The theory of instrumental genesis (Rabardel, 2002) considers three factors: the impact of the design of tools on how they are used (instrumentation); the users and their previous experiences with using the tool (instrumentalization); and the purposes and goals users assign to tool use (schemes of utilization). Acknowledging the different factors that impact a how teachers take up tools makes the processes associated with using the tool rather than the tool itself the object of study. Rabardel defined an *instrument* as an artifact coupled with the schemes of utilization users assign to it. It is important to note that the theory of instrumental genesis is focused on the resource *in use*, not the resource itself. The use of the term instrument emphasizes a process of transformation where the focus is on how teachers use resources to attain goals, not some innate quality of the resources. This is especially important when considering teachers' work with resources because teachers increasingly draw from a variety of sources when crafting instruction and use resources differently over time according to their context (e.g. Gueudet, Bueno-Ravel, & Poisard, 2014).

The third conceptualization is framing resources as objects, where a key component of teaching is to create resources. Document genesis (Gueudet & Trouche, 2009) is a theory focused on teachers' creation of resources for teaching. Stemming from the instrumental approach, document genesis explains how, in the creation of resources, teachers develop schemes of utilization that are comprised of teachers' knowledge and processes for using resources. Document genesis offers a window into teachers' evolution of practices with their resources. Since teachers often reuse resources from previous years, the documents from one year become resources for the next year and are embedded with teachers' experiences and modifications (Gueudet & Trouche, 2009). Often, when resources are considered tools, teachers are not producers: their work is of application and applies to one instance

of enactment without connection to future use. However, when treated as objects of design, the focus is on how teachers create resources to meet their local contextual challenges and on how the teachers' understanding and use of resources evolves over time.

In a complex landscape where teachers weave together various resources in the design of lessons, treating resources as an object of teachers' design work provides researchers with a theoretical grounding to make sense of and explain interactions in online spaces, such as resource repositories. The resources being downloaded and shared by teachers are not solely resources (tools) to mediate teaching: their creation and transmission constitute important facets of teaching. For example, Trglaova and Jahn (2013) examined how teachers improved resources posted to a repository based on the feedback they received from other teachers. Also, this framing allows researchers to situate teachers in broader collectives and to determine how the impact of belonging to and participating in collectives has on teachers' practices (Guedet & Trouche, 2012).

These perspectives on resources suggest different roles for the teacher with respect to how digital resources get taken up. In the first, resource as tool, the resource is considered to have innate qualities that teachers can employ to varying degrees in their lessons but not substantially revise. The latter two perspectives outline a more active role for the teacher, suggesting not simply a mediating role between resource and student, but a transformative role for the teacher with respect to the design and use of the resource. More generally, thinking about teachers' design processes positions teacher capacity as an important goal in the design and use of digital resources; consequently, perspectives on digital resources that minimize teachers' role have potential impact in terms of agency and ultimately the development of teacher capacity. Below, we explore how trends in the design and dissemination of digital resources may be in tension with conceptualizations of active roles for teachers with respect to the design and use of digital curriculum resources. We consider how different actors influence the design and dissemination of digital curriculum resources, and that these efforts may neglect the role of the teacher. An aim is to show that there are potentially conflicting values between perspectives focused on teaching as design and perspectives that promote characteristics of digital resources that constrain teachers' roles and their ability to be responsive to their local context.

Trends in the design and dissemination of digital resources

In this section, we articulate broad trends in the design and dissemination of digital curriculum materials and then connect those trends to the perspectives that emphasize the design role of teachers. We characterize these trends by focusing on the actors who emphasize particular perspectives – and exercise considerable influence – related to the design and dissemination of digital resources. We focus on the following three broad sets of actors because of the considerable influence they exercise over the design and dissemination of digital curriculum materials: designers, policy makers, and purveyors of online content (e.g., commercial publishers, for-profit educational websites, large philanthropic or corporate organizations). Included in the group of corporate entities are philanthropies and corporations not previously engaged in educational publishing (e.g., the Gates Foundation, Amazon, Mark Zuckerberg's funding efforts), who strive to influence both content as well as delivery mechanisms for that content, especially in the U.S.

These groups make a number of claims about the potential transformative features of digital curriculum materials. We focus on three features to highlight the roles of the sets of actors identified above:

- Content can be more relevant and interactive;
- High quality content can be inexpensive and widely accessible; and
- Content can be customized to meet the needs of individual students.

We selected these features because they are the primary focus of the design, dissemination, and publicity efforts of the actors described above. Below, we describe how these features are emphasized by the various actors and how they are in tension with the perspectives on resource use, with implications for how teachers get positioned as active designers who can develop increasing capacity to design and enact curriculum materials.

Content can be more relevant and interactive

Advocates claim that content in digital materials has the potential to be more interactive and relevant, as it can be updated and revised to fit the local context. In terms of interactivity, digital texts can be flexible with respect to navigation (e.g., hyperlinks) and with respect to creating documents with resources and materials from a range of sources, including the web (Zhao et al., 2010). Other kinds of interactivity include the use of sliders or buttons to manipulate parameters in a model to investigate problems or phenomena (Dede, 2000). More powerful forms of interactivity involve the use of tools with flexible purposes in open working environments, such as curriculum programs developed in Israel and Korea (cf. Lew, 2016; Yerushalmy, 2016). In general, interactivity can be conceived in terms of the choices users can make to influence their engagement with the content.

We focus on the set of actors we call *designers* to highlight how interactive features are incorporated into digital resources. We refer to designers as those who conceptualize features of digital materials based on research on learning and learning systems. Of the sets of actors described above, the curriculum resources designed and disseminated by designers are the most aligned with teaching as design perspective. These resources offer the greatest flexibility in terms of adaptation by teachers and in terms of generating interactions with students that provide opportunities to understand how student thinking develops. Designers emphasize learning experiences that augment or enhance what is possible in paper curricula. Moreover, designers emphasize the development of tool-rich workspaces that enhance interaction, communication, and exploration. Designers incorporate ubiquitous access to tools that allow for dynamically linked representations and the ability to record and curate work (Confrey, 2016; Lew, 2016; Yerushalmy, 2016). Confrey, Lew, and Yerushalmy emphasize that workspaces should provide access to a suite of tools that learners strategically select as they engage in complex problems. These workspaces facilitate the use and manipulation of multiple representations, including symbols, in ways that are intuitive and that communicate increasingly formal inscriptions of the mathematics. Furthermore, these workspaces should allow students to store and curate their work, for future reference for themselves and external audiences.

In terms of developing teachers' design capacity, and thus align with the artifact and documentation perspectives described above, designers emphasize more complicated and idiosyncratic learning paths for students in terms of deviating from a rigid demarcation and flow of mathematical activity (Confrey, 2016). Curriculum materials differ from open tools, such as Sketchpad, Cabri, or

Mathematica, in that they are intended to provide structure by bounding and sequencing mathematical activity. Integrating rich problems and work spaces provides opportunities for the kind of complex activity that involves non-linear processes (unproductive approaches may precede more productive approaches), complex interactions of tools and representations, and the collective negotiation of the viability and validity of solution paths. Such complex activity can disrupt well-defined lesson structures and allocations of time (both duration and synchronicity)(c.f. Ritella & Hakkarainen's [2012] discussion of chronotype), interrupting the potential flow of a lesson, with implications for following a prescribed scope and sequence of mathematics. Managing these non-linear activity flows calls for more prominent roles for teachers and entails developing new forms of capacity in terms of understanding curriculum progressions and coordinating (orchestrating) multiple tools and artifacts in the workspace.

The work of designers focuses on the interactive and flexible features of digital resources, while other actors – policy makers and purveyors of online content (mostly commercial interests and philanthropies operating from a neoliberal perspective) – emphasize different features, explored below. The different features emphasized by these other actors have implications in terms of the roles and capacities envisioned for teachers.

Free and open digital content

Policy makers have pushed digital content that is freely available and open source. They argue that this would make high-quality content accessible to low-resource high needs districts. Internationally, there has been a push for Open Education Resources (OER) for nearly a decade now: "The open educational resource (OER) movement aims to break down such barriers [from proprietary systems] and to encourage and enable freely sharing content" (OECD, 2007). Recently, the US Department of Education launched an initiative designed to encourage districts to adopt open resources and to share their efforts and experiences with others, in part to make access to high-quality instructional resources more equitable (USDoE, 2016). The use of open resource content, however, can be time-consuming and the resources themselves are of uneven quality. There is little quality control with respect to content, and much of it requires little interactivity or minimal educative features for explaining the design rationale to teachers. Furthermore, these efforts assume that teachers have considerable capacity to evaluate, select, and sequence content chosen from a variety of sources. Given that much of these efforts are aimed at low-resourced districts, especially in the U.S., there is an assumption that teachers can use the materials without modifications, which aligns with the resource as tool perspective. Recently, there have been efforts to curate content in ways that provide quality control and articulate curriculum progressions (Confrey, 2016); however, these efforts have yet to be coordinated with the larger policies for open resources and their impact on teacher design capacity is not yet determined.

Customizing content for individual learners

Policy makers, commercial publishers, and large philanthropies have emphasized the promise of digital content to be customized to meet the needs of individual learners. Customization has been discussed in a variety of ways. This can be achieved through systems that emphasize mastery learning, in which software dictates the sequencing of content for a learner based on the learner's performance on skills-based assessments (e.g., Means, Peters, & Zheng, 2014). Or, it could involve personalizing

the software settings so that the user has control over video and audio as well as the presentation of the text. A third way is for the teacher to make content selections within a program so that different students would see different content. Mostly, however, the personalized systems entail diagnostic assessments administered through online programs that dictate the pacing and sequencing of content (Choppin, Carson, Borys, Cerosaletti, & Gillis, 2014). These efforts, largely funded and publicized by large philanthropic or corporate entities, push to embed digital content in comprehensive learning management systems that include data reporting and classroom management systems. They also emphasize adaptive programs based largely from the mastery learning perspective (e.g., courseware funded by the Gates Foundation). Some educational websites, either for-profit or philanthropy-funded, create collections of lessons, sometimes developed by a small group or by larger author groups (e.g., Khan Academy, sofatutor.com). While these programs may eventually involve sophisticated adaptive systems and customized learning tools that allow learners to explore content in complex workspaces, the initial versions of these programs typically entail low-level content and reporting of percentage of correctly answered multiple-choice questions (Choppin, Carson, Borys, Cerosaletti, & Gillis, 2014), characteristics that constrain opportunities for learning. Furthermore, the programs minimize the role of the teacher, impacting development of teacher capacity.

Implications of tensions between perspectives

Teachers' interactions with digital resources involves agency: in the act of designing, teachers take up digital curriculum resources, interpret their purposes, and transform them to respond to their pedagogical purposes and their evolving understanding of how those resources can be used to engage students. This assumes that teachers are designers, actively interacting with and transforming curriculum resources to engage students. The three perspectives describing teachers' interactions with digital resources denote different assumptions with respect to teacher agency and the connections between capacity development and resource use. The assumed role of the teacher differs across the perspectives of the different sets of actors that relate to the design and development of digital curriculum resources.

The analysis illustrates that different perspectives on the design and dissemination of digital resources have implications for the opportunities teachers have to understand, adapt, and transform digital resources and thus develop capacity as a result. The designer perspective offers the greatest possibility of alignment between views of teachers as designers and the affordances of digital curriculum materials, though much of their focus and expectations is on creating a medium for rich student interaction rather than informing teacher use and capacity development. Other actors – such as policy makers, commercial publishers, large philanthropies – have different perspectives, and motives, for designing and disseminating digital curriculum. The programs associated with these perspectives show little sensitivity to the complexities involved with how teachers take up digital curriculum resources, and minimize the role of teachers to transform digital resources to suit their purposes. Many of these programs emphasize personalized learning, which entails little interaction with teachers, and little flexibility on the teacher's part to construct learning experiences for students. These entities exercise great influence on which programs are available, and their impact on teacher autonomy and teacher capacity needs to occupy the attention of researchers.

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Tasks to develop flexible multiplicative reasoning

Joana Brocardo[1], Jean Marie Kraemer[2], Fátima Mendes[3], Catarina Delgado[4]

[1]Escola Superior de Educação, IPS de Setúbal and
UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; joana.brocardo@ese.ips.pt

[2]CITO, The Netherlands, jeanmariekraemer@gmail.com

[3]Escola Superior de Educação, IPS de Setúbal, Portugal; fatima.mendes@ese.ips.pt

[4]Escola Superior de Educação, IPS de Setúbal, Portugal; catarina.delgado@ese.ips.pt

The project 'Numerical thinking and flexible calculation: critical issues' aims to study students' conceptual knowledge associated with the different levels of understanding numbers and operations. This paper presents the conjectural theory used to design and articulate the tasks of an explorative instructional sequence that promotes flexibility in reasoning and calculating in grades 2 and 3 in the field of multiplicative thinking. We focus our analysis of task design in the development of the first task of the sequence, illustrating how data analysis of students' solutions is used to reformulate the task.

Keywords: Task design, multiplicative thinking, flexible reasoning.

Conjectural theory

Our approach aims to foster flexibility in multiplicative reasoning and calculation by systematically developing both factual knowledge on numbers and the ability to operate with them as mental objects. The schemes and situations associated with multiplicative thinking constitute the referential field of modeling the relationship between quantities/magnitudes with multiplication and division (e.g. Greer, 1992, 2012; Thompson & Saldanha, 2003; Tzur et al., 2013; Vergnaud, 1983, 1988). Against this background, we develop a conjectural theory assuming that flexibility in the transition phase of addition to multiplication requires the ability of operating with products and quotients as mathematical objects, using different symbolisms for the same object (Gray & Tall, 1994; Tall, 2013) (e.g. 60 seen as 6 × 10; 12 x 5; 4 × 15; ½ × 120). We envision the development of this ability along the experimental sequence as follows.

*Starting point: product as representing some numbers of some amount***.** In Portuguese textbooks, as in many other countries, "times" and the notation '__ × __' is introduced as symbolization of the sequence of repeated addition executed to quantify a relative big collection of identical objects in situations where these objects are combined in equal groups (e.g. Greer, 1992). The first cycle of activities introduces the expression "*some numbers of some amount*" (with or without remainder) as common way of describing the result of combining objects into equal parts, putting an amount into parts of a given size (partitive or ratio/measurement division) and distributing an amount of something among a given number of persons (e.g. Thompson & Saldanha, 2003). It is conjectured that this should foster the later understanding of the numerical equivalence of this three processes. The symbolic notation '__ × __ + __' represents the underlying multiplicative structure. It ties conceptually and numerically the initial meaning of multiplication with the two meanings of division. It explains, at the same time, why both repeated subtraction and repeated addition allow to find how many times does an amount of things go into a given quantity - inverse relationship between

multiplication and division (e.g. Freudenthal, 1983; Greer, 2013). Last but not the least, reasoning in "some numbers of some amount*"* allows to apprehend the inverted/reciprocal relationship between the involved quantities (e.g. if sharing 52 pictures between 4 children results in 13 pictures each, each child receives ¼ of the collection, which implies that the initial amount (52) is 4 times as large as each part (13) (e.g. Thompson & Saldanha, 2003).

"Times" as multiplicative comparison. From the above understanding of the symbolic '__ × __' we conjecture that students should extend the use of familiar products of the tables to compare quantities or magnitudes reasoning in terms of "so many times as much as __". The number that in the situations above operates as `*multiplier*` gets now the sense of `factor` (e.g. Freudenthal, 1983; Greer, 1993). The above inverted relationship comes back: saying that Bernardo has 4 times as many volumes of Asterix as Fernanda, means that her number of volumes is 1/4 as large as Bernardo´s collection (and vice versa).

"Times" as operation that emerges from a rectangular arrangement. Until now, multiplicative reasoning and calculations are limited to situations where combining, partitioning and distributing are modelled with sequences of multiples (times tables). The following cycle of measurement activities extends stepwise multiplicative reasoning by mean of the rectangular arrangement of objects. First the multiplication of factors emerges from pacing concrete units in the length and width and using familiar products to find the number of units. This way of reasoning is then adapted to (1) calculate the number of tales of a given arrangement and to transform it by substituting the initial units in bigger or smaller ones; (2) to recognize the two rectangular arrangements of the starts of the American flag. The operative notion of commutativity ($12 \times 6 = 6 \times 12$) and distributivity emerges from these activities. This increases the power of multiplicative reasoning and extends the application of the available factual knowledge.

Endpoint: multiplicity goes together with divisibility and proportional relationship. The last inquiry is conceived to connect the conceptual, procedural and factual knowledge constructed along the reflection about the ways of reasoning and calculating in the above multiplicative setting. The challenging question is why hours do have 60 minutes. Students are engaged to "unroll" the time line of a clock segmented in units of 5 and 15 minutes and to symbolize the accumulation of time with the underlying numbers' pattern and the corresponding product to identify how one hour is divided in equal parts (5, 10, 15, ... → 12×5; 10, 20, 30 → 6×10; etc.). The inverted number relation met before comes back by the connection of 4×15 and 2×30 with respectively 15 minutes as "a quarter of an hour" and 30 minutes as "half an hour". Finally, by identifying all the possible ways of grouping 60 chairs in a rectangle, children discover the hide equivalents of the 'products of the clock'. 60 then appears as 'an object' that can be composed and decomposed in "many ways" (e.g. $4 \times 15 = 15 \times 4$ related to $60 \div 4$ and $60 \div 15$).

Key principles of designing task with focus on flexibility

The chosen approach prompts the systematic extension of the numerical relationships, arithmetic operations and factual knowledge using multiplication and division to model situations. We foster that students come to act and reason in a "mathematical reality" (e.g. Freudenthal, 2003; Tall, 2013; Gravemeijer, Bruin-Muurling, Kraemer & van Stiphout, 2013), manipulating flexibly mathematical

objects and relationships at hand through symbolic representation as professional mathematicians do (e.g. Gray & Tall, 1994).

We use the framework above to develop three kinds of tasks with a specific function and to articulate them transversally and vertically in cycles of inquiries (Bell, 1993). *Open task*s as "Prawn skewers" (Figure 1) prompt the exploration of a key idea (e.g. envisioning multiplicities of identical things thinking in some numbers of some amount**)** and the development of a symbolism to express it (e.g. __ × __) to express it (e.g. Bell, 1993; Back, 2011). *Numerical* tasks (without context) promote the organization of numbers as ties in a web of number relations (Van Hiele, 1985) which extend the personal factual knowledge (Threlfall, 2002) and the development of specific skills such as analyzing numbers multiplicatively and using the process-object symbolic to explore unfamiliar forms of operating with numbers as mental objects (Gray and Tall, 1994). *Conventional* tasks focus on understanding how a particular relationship (theorem-in-action) of a scheme works and can be adapted in a limited class of situations.

The constructed webs of number relations connected to the schemes of reasoning and the classes of situations form the horizontal junctions between the tasks. The same tasks are articulated vertically, considering the transitions from a lower (informal) to a higher (formal) level of reasoning, symbolizing and computing.

The task 'Prawn skewers' (Figure 1) exemplifies this approach. The task provides an opportunity to explore "partitioning" as a process of structuring quantities on the own level of understanding the relationships involved in equal group situations (see draft of conjectural theory above) and using the number patterns in the available tables of multiplication. We conjectured that the majority of the children would notice that 61 is an odd number (ending by 1), near 60 (60+1 and/or 62-1). Focusing by turn on number 60, they would in first instance notice it can be reached counting by two and/or ten which suggests partitioning the pile of prawns by way of repeated addition (2+2+2+2…; 10+10+10+10+10+10). Students that operate on a higher level of understanding two-digit numbers would associate 60 with 6x10 seen as "six tens". Then, one can explore other grouping possibilities, varying additively or multiplicatively the number of prawns of each stick or transforming a founded partition in a new one, substituting smaller composite units by bigger ones and vice versa.

Considering the current phase of development of grade 2 students, we expect difficulties with the symbolization of founded structures with products (e.g. difference between "6 times 10" written as 6 × 10 and "10 times 6" written as 10×6) and related misunderstanding in the communication about the transformation of one possible structure into another one (e.g. 6 times 10 into 12 times 5; 6×10=12×5). Finally, we expect that some students could first approach the task by describing the process of exhausting the pile of 61 prawns with an arithmetical sequence of repeated subtraction and then (quickly) invert their modeling to avoid the (arising) computing difficulties.

Methodology

The project plan is based on a qualitative and interpretative methodology (Denzin & Lincoln, 2005) with a design research approach (Gravemeijer & Cobb, 2006). The preparation of teaching experiences is a crucial aspect of the project plan.

To prepare teaching experiences we design and reformulate mathematical tasks using a three-step cyclic process: (1) design tasks, (2) analyze what children noticed in the numbers and how they use

their knowledge about numbers and operations to solve the task presented along clinical interviews and (3) reformulate the previous task.

This text refers to one teaching experiment that involved 24 grade 2 students (age 7-8) and focuses on the students reasoning to solve the task 'Prawn Skewer' (Figure 1). Students knew how to add and subtract numbers until 1000 and had some experience in solving word problems. They hadn't yet learned the multiplication tables. Data was collected through video and audio recordings of the classroom work, researchers' notes, audio recordings of the preparation and reflection meetings with the school teacher involved.

The proposed task was designed and reformulated according to the data analysis of four clinical interviews, where students (4 students, 8-year-old) solved a first version of the task (Figure 2) analyzed in Brocardo, Kraemer, Mendes and Delgado (2015).

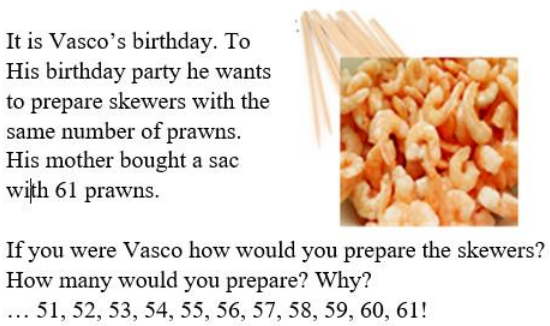

Figure 1: The tasks 'Prawn skewer' (reformulated)

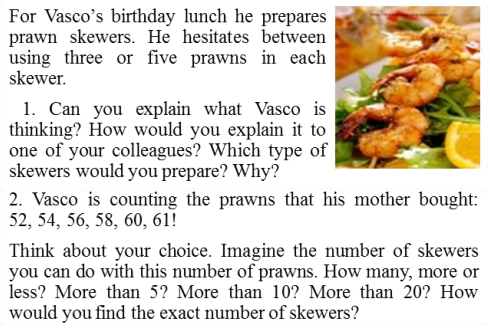

Figure 2: The tasks 'Prawn skewer' (first version)

Since the given alternatives in the first version (groups of 3 or groups of 5) seemed to hinder the *envisioning* of other ways of grouping, we decided to give students the freedom to experiment and evaluate different ways of grouping, taking into account two conditions: the given quantity of prawns and the freedom to invite more or less friends. We also `opened´ the illustration of the task, to stimulate the students' own constructions. Finally, by giving only the number of prawns, and by continuing to choose the `ugly´ number 61 (in the sense of Thompson & Saldanha, 2003), we created a problem that these students surely never encountered before.

Under these conditions, we expected that students would envision different possibilities of sticking a pile of 61 prawns, modeling from two ways of understanding "division". This is to say, reasoning in terms of exhausting the pile by a sequence of repetitive subtraction (ratio/measurement division), and/or reasoning from the converse idea of accumulating 61 by counting on n by n (division as converse of multiplication) (Freudenthal, 1982). In second instance, we expected that the choice condition of the task should stimulate students to compare envisioned ways of grouping, considering the relation between the multiplier (number of sticks) and the multiplicand (number of prawns in each stick). Exhausting, taking away a smaller set of prawns go together with making more sticks and subtracting a bigger set with obtaining less sticks. And, increasing the number of sticks by accumulating goes together with decreasing the number of prawns in each stick, as putting a bigger set of prawns goes together with obtaining less sticks, while sticking less prawns provides more sticks. Finally, we expect a great variety of modeling, verbal explanation, and calculations, according to

levels of memorizing the products/multiples of the tables, and understanding `multiplicity´ and `proportionality´ in the experienced contexts of multiplicative thinking.

Results

The analyses of the working sheets of the students and of some dialogues occurred in the classroom gave us a global idea of the patterns of reasoning that students used to solve this task.

Globally, we identified the following patterns of reasoning: trying with 18 sticks (they counted the sticks represented in the illustration); drawing the prawns' skewers one by one (with 18 sticks); putting 2 by 2 we will arrive to 60; putting 10 by 10 gives 60; adaptation of grouping by 10 making one skewer with 11 prawn (Figure 3); adaptation by "grouping by one"; putting 5 by 5 and/or 15 by 15 gives 60 (Figure 4); intuitive notion/feeling that putting by n would give 61.

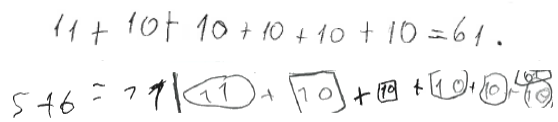

Figure 3: Adaptation of grouping by tens, making one skewer with 11 prawns

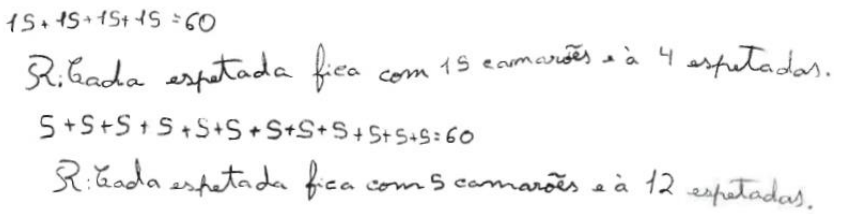

Figure 4: Putting by 15 and by 5 with verbal description of the result

The most frequent way of modeling was trying to reach 60 arithmetically, by repeated addition. Some students shortcut their long addition by mean of doubling consecutively the terms (Figure 5).

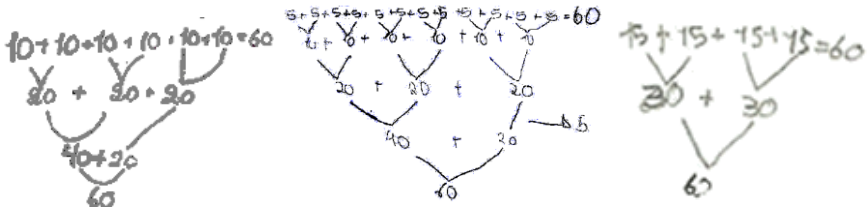

Figure 5: Shortcutting by successively doubling as way of controlling / justifying

Some dialogues suggest that some pairs of students are jumping to 60 in a kind of mental sequence of multiples, without keeping track. Having arrived at 60, they must then count the numbers of "tens" to derive the multiplier from their long addition.

Some pairs of students initially understood that they could choose the number of guests but failed to connect the given 61 prawns with the 18 sticks they counted in the picture. In a short dialog with the teacher, they envisioned ways of grouping with units, leaded by a kind of intuition (feeling) that the repeated addition they had in mind would give 61. Vera was thinking about 11 + 11 + 11 …, and moved to 5 + 5 + 5 … as response on the teacher´s question "Why 11?". Her colleague Martin was expecting that putting prawns by 7 would give 61, "since 7 + 7 = 14 and 14 + 7 = 21".

José drew the 18 sticks and then ´sticks' 4 prawns in each (Figure 6). He then symbolized the represented accumulation with the long addition 4+4+4 … and tried to determine the total number of prawns, by doubling 4 and 8, transforming 4+4+4+4 … (18 times four) into 8+8+8... (9 times 8) and 16+16+16+16+8.

Only one group used repeated subtraction (viewpoint of exhausting 61; Figure 7). They began to subtract 1 to obtain 60 but it seemed they changed their idea and begun to work with 61, subtracting 5 until they have a remainder smaller than 5.

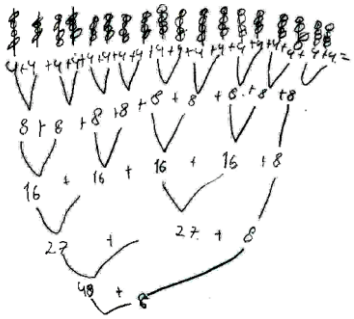

Figure 6: Wrong modeling by drawing 18 skewers with 4 prawns in each

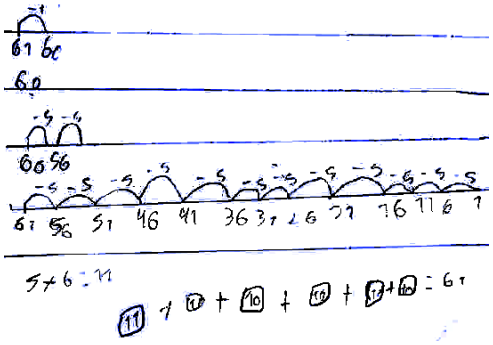

Figure 7: Modeling by repeated subtraction

Finally, two pairs of students associated the information of the story and the illustration of the task with the decomposition of 61 into two parts. One pair is thinking about a combination of 18 children (the number of sticks of the illustration) and 40 children (a multiple of 10). The other pair proposes "as 41 + 20 = 61, one has 41 and the other 20" and does not adds any other justification.

Implications for task Design

From data analysis, we may conjecture that the formulated instruction does not scaffold the expected mathematization including: (1) finding and representing adequately different forms of sticking the prawns, (2) choosing the preferred way, and (3) justifying this choice. It seems that "Why" is interpreted as an instruction to demonstrate and/or control that the modeling with the sequence of repeated addition indeed gives 60 prawns. This interpretation explains the spontaneous short-cutting of long additions (Figure 5). A solution to avoid this is to structure the task in an exploring phase asking explicitly to look for possible ways of sticking and a reflective one including the choice and the justification of the referred form of sticking.

On relating the choice of 61 as cardinal of the set, we can argument that the complexity of taking away, could explain the high frequency of the modeling by repeated addition and the single use of repeated subtraction. Since more students associate 61 with the near even number 60, and since the table of two and 4 are memorized, it would be meaningful to replace 61 by 62 to increase the chance that more students balanced between modeling by jumping back to exhaust 62 and jumping forwards to reach 62. We might expect that, being aware that this way of grouping would give a lot of `small´ sticks, more students would try to subtract a bigger quantity and finally move to jumping forwards to avoid annoying calculations.

Finally, since the representation of some stick may suggest some students to fix the multiplier and model the process directly by drawing all the set of prawns, stick by stick, we have to change the illustration to avoid the observed try-and-error approaches.

Implications for task design in the field of multiplication

In this text, we explicit how we analyze the influence of contexts, numbers and pictures/images aiming to potentially facilitate *noticing* (Threlfall, 2002) relevant numerical relations and *envisioning* different approaches that emerge from noticing.

Another aspect of our analysis is related with the importance to propose non-conventional equal group problems to develop flexible multiplicative reasoning. In all conventional equal group problems, two values are given. Consequently, children learn to identify the 'kind' of problem from the story and the given numbers. Then, they solve it applying the standard scheme of reasoning. Children´s approaches and solutions of the task 'Prawn Skewers' show the advantage of the missing multiplayer and multiplicand as stressed by Back (2011). They have to adapt their common way of thinking to the unusual conditions of the task (Vergnaud, 2009) noticing numerical relations and envisioning different approaches. Some fixed the multiplier counting the sticks of the pictures. Others inferred that they had to fix the number of prawns on each stick and to look how many skewers can be made. The vast majority envisioned the all process of sticking, using the knowledge that counting by ten leads to 60. A crucial advantage of these non-conventional problems is that the teacher can focus the reflection on the relationship between "continually putting prawns by *n*" and "taking again and again *n* prawns" and the use of asymmetric role of the multiplier and the multiplicand to symbolize both process.

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The role of design in conceptual change: The case of proportional reasoning

Noa Cohen- Eliyahu

The Hebrew University of Jerusalem, School of Education, Teacher Education Program, Jerusalem, Israel ; noa.cohen-eliyahu@mail.huji.ac.il

Conceptual change is extremely difficult to trigger in mathematics and science education. The present study shows the special role of design – in the fact that design can afford behaviors that lead to conceptual change. We chose the domain of proportional reasoning. The design was based on collaborative and argumentative scripts, provision of a tool for checking hypotheses, and arrangement of students with different strategies. Above all we created a task, the Blocks Task, that invites students with different strategies different solutions, whereby creating (socio-)cognitive conflicts with the possibility of enhancing conceptual change. The participants were 496 ninth graders. A central result is that performing the task resulted in conceptual change. It was also shown that the provision of a tool for checking hypotheses yielded substantially larger gains. We conclude by providing design principles for fostering conceptual change in mathematics.

Keywords: Conceptual change, cognitive conflict, task design, proportional reasoning, collaborative work.

The present paper

The focus of the present paper

The present paper aims to expose whether the *block task* is effective in triggering conceptual change in proportional reasoning. While triggering conceptual change in mathematics and in science education is a very challenging task. We have defined conditions for task design and we showed that they can dramatically boost conceptual change and learning in the field of proportion reasoning: Collaborative scripts, the arrangement of students in small groups and the provision of a hypothesis testing device for feedback were central design decisions. Above all, we created a special task, the Blocks Task, which invites students with different strategies in proportional reasoning (that reflect different levels in proportional reasoning) to exhibit different solutions, by such creating socio-cognitive conflicts

Theoretical framework

The process of conceptual change had been described in literature in various ways. The pioneering model of Posner, Strike, Hewson, and Gertzog (1982) had described the conceptual change of replacing the earlier premature, and somewhat naïve, conceptions (which, at times, contradicts the scientific explanations) with more up-to-date scientific conceptions. The trigger for this change had been the dissatisfactions with the earlier conceptions. Other scholars (e.g. Vosniadou, & Verschaffel, 2004) replaced Posner et al.'s model with a model they called a synthetic model which is based on the combination of a scientific concept with the student naïve preliminary concept. Cognitive conflict had been for a long time considered as a major, necessary ingredient in the development of psychological theories that explain conceptual changes. The role of cognitive conflict is important in many mathematical fields Stylianides and Stylianides (2009) related to the potential concealed in

cognitive conflict to support the development of the student knowledge. In the domain of proof, counterexamples play a great deal as a pivotal means in creating and resolving a cognitive conflict (Zazkis, & Chernoff, 2008, Stylianides, & Stylianides, 2009). However, it was also found that introducing conflicting data was generally not sufficient for triggering conceptual change (Limón, 2001). Collaborative and argumentative settings as well as the provision of feedback are among the means described in the literature which may prompt the development of conceptual change among students (Schwarz, 2009). The *block task* which was designed for the purpose of the present study has been a sophistication of the *blocks task* developed by Harel, Behr, Lesh and Post (1992). Harel and colleagues designed the original Blocks Task as a diagnostic tool to assess the level of proportional reasoning of adolescents. I modified the task as a learning task with the potential for triggering conceptual change (Cohen-Eliyahu, 2001). The modifications consisted of posing the task in a dyadic setting, providing a collaborative and argumentative script in their interactions, and providing a hypothesis testing device (a balance) for checking hypotheses. This mathematical task has the potential to create socio-cognitive conflicts based on the three aforementioned conditions. Schwarz and Lichevski (2007) showed that the design yielded conceptual change. In an additional study, Asterhan, Schwarz and Cohen-Eliyahu (2014) examined the mechanisms that govern conceptual change with the Blocks task. In the present paper, I examine the role of design in triggering conceptual change. Let us first describe succinctly the study. More information can be found in Asterhan, Schwarz and Cohen-Eliyahu (2014).

Method

Participants

The participants were 496 Israeli ninth graders from large metropolitan areas (After studying the subject proportion ratio and percentages). 16 groups were formed with different conditions for each group: individual or pair work, with or without weighing condition (the hypothesis testing device). Students were paired according to their initial solution strategies, in order to create varied social settings based on differing initial cognitive levels. Three strategies were focused on: Students with additive strategies (N=196), students with proto- proportional strategies (N=194) and students with proportional strategies (N = 106).

Tools

The Blocks tasks were designed according to three aforementioned conditions in the field of mathematics, and its aim was to facilitate learning (i.e. conceptual change) of ratio and proportion, as follows: (a) The task enables collaborative or individual work. (b) There is a tool (scales) with which one may check (or not) whether hypotheses are correct. In other words, students receive feedback on their solutions by using scales (c) The task allows the activation of a variety of strategies by which students may be differentiated according to their levels of thinking, which may facilitate teaming them up according to their levels of proportional thinking. In each task students are shown 4 three dimensional blocks constructed from bricks (A,B,C and D). They are told the bricks in A and C is identical (the same weight = the same color) and the same is true for bricks in B and D. In each trial, students are given information about the relation between the two base block constructions A and B (A is heavier than B, B is heavier than A, or they are of equal weight). They are asked to determine the relation between the weights of the two target blocks, C and D, choosing one of the following

four options: “C is heavier than D”, “D is heavier than C”, “They are of the same weight”, or “Impossible to determine”. They are required to provide a verbal explanation for their choice.

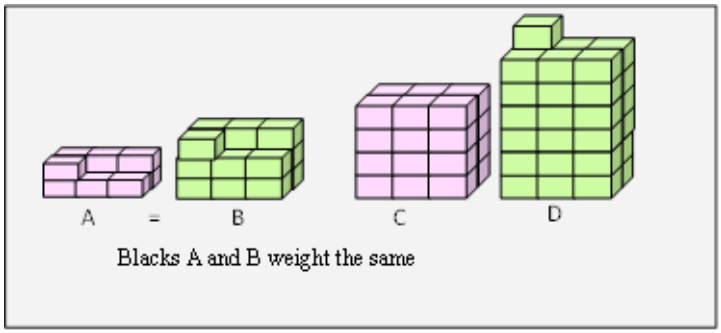

Figure 1: Blocks Task 1

In the current study seven configuration of the Blocks tasks had been designed. The design was aimed at leading wrong reasoning strategies to wrong answers. For example, a student using Visual explanations typically would claim that “it has less bricks in it". A student adopting an additive reasoning strategy would use an explanation such as "when B has 6 more bricks than A, A and B weigh the same. D has 13 bricks more than C so D weighs more than C." A student adopting a proto-proportional reasoning strategy would typically draw a wrong conclusion: For example, he would say "B has more bricks than A but they have the same weight so each brick in A is heavier than single brick in B. C is 13 bricks less than D but each bricks weigh more so it's impossible to decide." Students adopting a proportional reasoning strategy would typically predict the right conclusion that C weighs more than D. They would say something like, "10 bricks in A(10a) have the same weight as 16 in B(16b) i.e. one single brick in A weighs 1.6 times one single brick in B a=1.6b. so 24 bricks in C weigh 24*a=24*1.6b = 38.4b that is more than 37b bricks in D." We see then that the Blocks task was designed to lead students with inferior strategies to give wrong answers.

Procedure

The current study consisted in three stages: pre- test, intervention and post- test.

Stage 1: Assessment and selection– pre-test. Five Blocks Task Test was administered in pen-and paper format to all students in the ninth grade classes to assess their initial level of proportional reasoning. In each tasks the blacks' constructions (A, B, C and D) were presented to the class and trained research assistants called aloud the instruction. According to their initial reasoning strategy students were arranged in order to create socio-cognitive conflicts.

Stage 2: Learning stage. Students were asked to solve collaboratively two different Blocks Tasks according to the aforementioned condition. They were provided a collaborative and argumentative script: The dyads asked to solve the tasks together. they were invited to collaborate and to argue with each other. Additionally, students that work with hypothesis testing condition get a scale after they finished each task to check their answer. An experimenter helped them to put the blocks constructions (C and D) on the scale and told them the correct answer, and asked them to explain it. All students, dyads or individuals with or without hypothesis testing device, were interviewed while this stage by trained research assistants who, repeated the instruction again and asked the student to explain their answers.

Stage 3: post – test the student answered the five Blocks Task Test again individually.

All the participants completed the three stages in less than one month.

Results

Over the entire sample, it appears that in all conditions, conceptual learning was attained. T–test showed significant results in compering differential from pre-test to post-test to zero.

Analyses were conducted with a mixed model (SAS PROC MIXED) with random effects of dyad within condition and of individual within dyad and condition. Differentials means and standard deviations between pretest and posttest performance per condition are reported in Table 1.

	Pairing condition		
	Single	Peer	Sum
With Feedback	0.22 (.06)	0.28 (.03)	0.25 (.04)
No Feedback	0.17 (.06)	0.10 (.03)	0.13 (.04)
Sum	0.19 (.04)	0.19 (.02)	

Table 1: Differentials mean (and SD) between pretest and posttest scores on the BlocksTask (N=488)

There is significant different between Students who worked with feedback (single or peer) (M=.25, SD=.04) and students who worked without feedback (single or peer) (M=.13, SD=.04). The results confirmed that hypothesis-testing improved learning (F (1,422)= 5.10 p= 0.024). But in general there is no advantage to work with peer over individual work.

Further analyses were made in purpose to reveal what accord in different pairing condition. Focusing on students who lacked full proportional reasoning, non-Proportional students (Proto – proportional students (ProtoS) and additive –students (AddS)). Differentials means between pretest and posttest and standard deviations performance for non-Proportional students are reported in Table 2.

		Pairing condition				
		single	The same strategy-peer	Different strategy-peer	Proportional strategy-peer	Sum
hypothesis-testing condition	With Feedback	0.20 (.08)	0.12 (.06)	0.16 (.08)	0.58 (.06)	0.26 (.03)
	No Feedback	0.22 (.07)	-0.02 (.06)	0.22 (.08)	0.24 (.07)	0.16 (.03)

	Sum	0.22 (.05)	0.05 (.04)	0.19 (.05)	0.41 (.05)	

Table 2. Differentials mean (and SD) between pretest and posttest scores for non-Proportional students (N=382)

There is significant different between non-Proportional Students who worked with feedback (single or peer) (M=.26, SD=.03) and students who worked without feedback (single or peer) (M=.16, SD=.03). The results confirmed that hypothesis-testing improved learning ($F(1,239)= 4.13$ $p= 0.043$). In addition, Focusing on the non-Proportional students revealed that apart from the hypothesis-testing condition, there is significant affect to the pairing condition (($F(3, 225)=10.98$, $p<.0001$). and the Tukey-Kramer post hoc test showed that most efficient pairing was with a partner with full proportional reasoning. The interaction between the hypothesis-testing condition and the pairing condition is significant ($F(3,225)= 3.22$, $p=0.024$). The Tukey-Kramer post hoc test showed that with feedback condition students who worked with student who reached proportional reasoning gained more than student who worked with any other pairing. But, without feedback condition the result wasn't significant.

We present here the example of two students, Ido and Shira, and show how the design of the task helped them advancing their strategies in proportional reasoning. Ido and Shira both started, according to their pre-test, in a proto-proportional stage. Neither of them had a fully proportional answer to any of the tasks, but they were well aware of the importance of looking at each of the cubes separately. After the interaction, they both progressed: Shira gave one full-fledged proportional answer and Ido, two. Right from the start, elements of the proportional discourse were apparent in both Shira and Ido's interaction. Interestingly, the additive discourse co-existed with elements of the proportional discourse all through the interaction about the first task (see figure 2). Thus, for instance, Shira first suggested an additive argument:

> 24 Shira: I think, I think this (C) is bigger than this (D) because here they tell us that this (A), no, sorry, because here they tell us that this (A) is smaller than this (B) although here (A) there are more, but only by one cube more, so if here (C) there is more in three cubes, so that says that this (C) is big.

Ido, on the other hand, starts referring to multiplicative routines:

> 25 Ido: Look, if you take- this (A) is 11, this (A) is bigger than this (B) by 1 and here (C) it's bigger times 3 than this (A)
>
> 26 Shira: right
>
> 27 Ido: yeah, try multiplying both of them by 3 and you'll see what happens because it's thirty something. 10 times 3 is 30 so 10 times 3 is 30 and 11 times 3 is 33. If you multiply it by 3 it comes out 33.
>
> 28 Shira: oh, so, like I got it, I got it, so here it's one
>
> 29-30 Ido: and thirty one, two… that means if you put this (A & B) times 3 you'd get here 33 and 30 and if you add another one to each one of them so it will be 34 and 31.

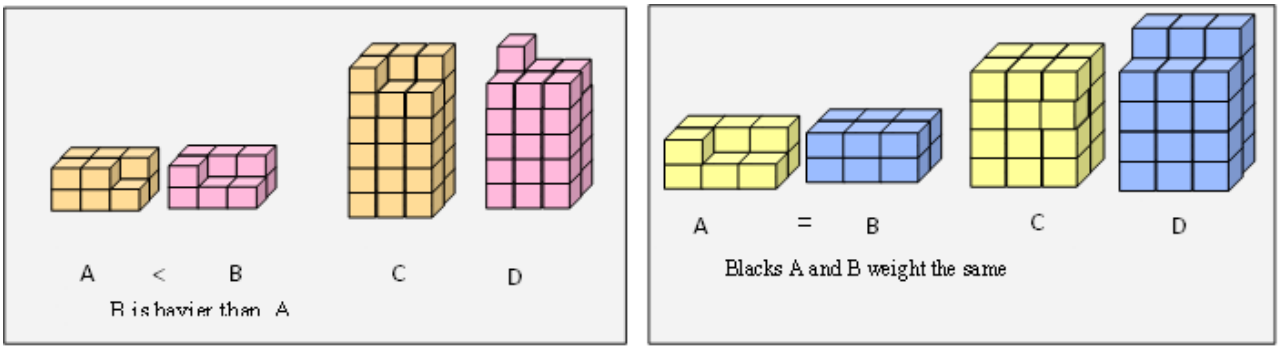

Figure 2 :Learning stage Blocks Task 1

Interestingly, Ido and Shira's solution, though not accurately proportional (Ido found a linear relationship between the block structures, not a proportional one), still leads the two students to a correct solution. In other words, it leads them to predict that C < D (by preserving the linear relationship), and the scales do not contradict it. Thus, there is no resistance of material agency. During the second phase (where 10Acubes = 12Bcubes), it is Ido's disciplinary agency (that is, the previous mathematical knowledge obtained by Ido) that resists the additive discourse. The pair starts out by trying to implement the successful routine that they had employed in the first task:

> 106 Shira: Now, OK here there are 10, can you show me the calculation again? This (A) times 3.

After some calculations, Ido realizes this solution brings them to a dead end:

> 115 Ido: OK, look, we multiplied both of them times two, it (A) turned out 20 and this (B) turned out 24. Now, if we add to both of them 3, you get this (B*2) plus 3 gives 27 like this (D) and this (A*2) plus 3 gives 23, it didn't get to 24.

According to this linear relationship, Ido and Shira agree that C and D are not equal. However, they do not have a way of determining which block (C or D) is heavier. This, because the only way they know to use the linear relationship is by applying it to affirm the relationship between the C and D blocks stays the same as between A and B. Realizing that the former routine does not work (resistance from disciplinary agency), Shira goes back to the additive discourse:

> 120 Shira: so look here (C & D) this is by 3 more (meaning D has 3 more than C) and here (A & B) it's only by 2 more (meaning B has 2 more than A), get it?

After quite a long discussion (24 turns), Shira convinces Ido of her additive solution. But as she writes up her argument on the paper, Ido continues calculating numbers. The experimenter turns Shira's attention to that and Ido explains to Shira:

> 147 Ido: look, this (A) is 10 and here (B) it's 12. Just saying, I'm not saying it's true but let's suppose each mass here (A) is 3 and here (B) each cube is 2.5 cause that's less and they are equal cause 3 times 10 is 30 and also 2.5 times 12 is 30 and the weight between them is equal, but each cube is different like they said here. If you multiply to get to 24 you'll get 72. If you multiply this (a cube in D) times twenty s… twenty..
>
> 148 Shira: Oh, I get it like 2.5 times 27.
>
> 149 Ido: yes if you multiply you'll get 67½ that means that C is bigger, (the) weight is higher than D, see?

We can see that Ido changed his mind before the balance scales proved Shira's additive solution to be wrong. We can only hypothesize why this happened. It seems the resistance that was most important here was that of disciplinary agency. Both Shira and Ido could not initially find a multiplicative routine that would satisfy them. But once such a routine was found by Ido, they happily switched to it, being aware all along of the appropriateness of the multiplicative discourse for the problem at hand. Ido and Shira were actively engaged in convincing each other and questioning each other's solutions. Especially impressive was Shira's attempt, once she was convinced by Ido's new multiplicative solution, to critically examine her previous solution:

> 181 Shira: so how could it be (true) if like this (*covers one cube from the D structure with her hand so that there is a difference of only 2 between D and C*) it's equal and like this (*raises her hand to expose the whole structure*) it (C) is bigger?
>
> 182 Ido: because the weight of the cubes is not the same.

Shira is not convinced. She repeats her argument that D minus 1 cube equals C, but now Ido challenges this assumption: "who said it's equal?" He then goes on to explain

> 188 Ido: you told me that there are two points (probably means cubes) less so you add two here and another one but you don't know, the two point here (A & B) and the two points here (C & D) are not the same.
>
> 189 Shira: The mass is different, I get it.

Thus, even though Ido is the main one to pursue the proportional solution and substantiate it, Shira is very active in her attempts to follow his logic, and to compare it with her own previous solution. Only then she is convince that the additive solution is invalid.

Conclusion

The Blocks task resulted in substantial learning which may be termed conceptual change, since under all conditions the students improved their solution strategies very significantly from the pre to the post – test. These are impressive findings that suggest the crucial role of design in triggering conceptual change. The additional results consistent with the assumptions based on previous researches (Schwarz and Lichevski, 2007) about the conditions that promote conceptual change. The very rich dialogue between Ido and Shira revealed the crucial role of task design in triggering conceptual change that lead to learning. In the pre-test both student initial strategy were proto-proportional reasoning. During the Learning stage (intervention) they both made a huge advance and promote their strategies reasoning to proportional as been founded in the post-test. The Blocks task design afforded and invited, as we saw in Shira and Ido very rich dialogue, a variety of reasoning strategies (Additive, Proto-proportional & Proportional). Shira and Ido considered those antithetic strategies by justify their claims and explain to each other their opinions this socio and cognitive conflict lead them to collaborate and argument in dyad and that lead to the desired conceptual change. Also in Ido and Shira case the scale's checking hypotheses "just" confirmed their solution. After the first task confirmation done with the scales, for Ido it was a trigger in the second task to continue searching for other option even if the AddS sound acceptable (120 Shira). The deep dialogue showed the power of the collaborative work, investigating each other ideas (an additive and a proto-proportional solutions) cause emerging of a new insight (proportional –solution). The case of Shira

and Ido exemplifies the importance of collaborative and argumentative scripts, of a tool for receiving feedback (the scale), of the arrangement in dyads, but foremost of the design of the task that led students with inferior strategies to arrive to wrong conclusions, to check them with the scale and to revise them accordingly. In conclusion, as the statistical result and Ido and Shira's dialogue showed, the task design allows the use of variety of confronting strategies that may led to socio- cognitive conflict between the dyads. In the statistical result we find that gap between students play a major role. Non-proportional students with hypothesis-testing condition is significantly advanced proportional reasoning when they work with proportional students. In Ido and Shira case also it looked they started with the same strategies we reveled in the rich dialogue gap between their strategies. when Shira used automatically additive strategy Ido looked for another option that explain his proto-proportional strategy. The hypothesis-testing condition that confirmed his first answer helped him to persist in finding satisfactory solution.

It seems that the next step is to expand the study to examine the effectiveness of the conditions defined in designing mathematical tasks in other areas.

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From textbook to classroom: A research on teachers' use of pedagogical resources in the context of primary school in the French speaking part of Switzerland

Audrey Daina

Geneva University and HEP Vaud, DiMaGe, Switzerland; audrey.daina@hepl.ch.

In French speaking Switzerland, primary school teachers use uniform textbooks edited by the government. These official textbooks are specific, they provide a source of activities but do not give information about how to organize the teaching. In this context, it is interesting to observe how teachers choose and organize in-class activities, the different ways they use textbooks, and the consequences in the classroom. Our research is a case study, which is based on interviews with teachers and classroom observation. The analysis we will present in this contribution were conducted using the model of Robert and Rogalski (2005), which allows us to describe teachers' choices as a coherent system that does not depend only on learning objectives but also on characteristics of the profession and on certain constraints.

Keywords: teachers' practice, textbooks, primary school, cross-analysis.

Introduction

School textbook is a field of research which has been widely increasing since the last three decades as described by Fan, Zhu and Zhenzhen (2013) survey study for the special issue of ZDM "Textbook Research in Mathematics Education". The authors conclude this survey by suggesting future directions in this field of research:

> it is necessary for researchers to establish a more solid fundamental conceptualization and theoretical underpinning of the role of textbooks and the relationship between textbooks and other variables not only in curriculum, teaching and learning but also in a wider educational and social context" (Ibid, p. 643)

The large-scale cross national research study TIMSS allowed us to have a wider vision on the role of textbook in connection with social or political issues as presented by Valverde and al. (2002) who analyze textbooks in terms of cross-national differences in *educational opportunities* "the configuration of social, political and pedagogical conditions to provide pupils chances to acquire knowledge, to develop skills and to form attitudes concerning school subjects" (Ibid, p. 6).

Our contribution goes in this direction by presenting the results based on our PhD theses (Daina 2013), which aimed to describe and analyze:

- the social and political context in which the uniform textbooks are designed in French speaking part of Switzerland, known as "Moyens d'Enseignement Romand pour les Mathématiques"(*MERM*);
- how five teachers in Geneva use this resource for mathematics, considering their "ordinary" practices.

Our research brings a different focus on the question of the connections between social and political issues and teachers practices by considering a qualitative methodology and a theoretical framework which allowed us to understand the practice we observe as a complex but coherent system that is dynamic and does not depend only on learning objectives but also on characteristics of the profession and on certain constraints we can infer from the study of the context (Robert & Rogalski, 2005).

Research design

Our research is a case study, which is based on interviews with teachers and classroom observation. The data were collected from April to June 2009 in five classrooms in Geneva in two different schools. In every class, we collected the following data: an interview before the teaching sequence; the observations in class with video recordings of the various activities of the teaching sequence on the theme of area measure; an interview at the end of the teaching sequence.

To understand the political and social issues connected to our context of study, we collected selected documents, which showed different aspects of the official textbook design process (institutional requirements, project design, review report, etc.)

Theoretical frameworks and research questions

Our project aimed to study how teachers use the textbook and prepare their lessons which is an "invisible" part of the teachers' practices. In addition, we wanted to observe ordinary practices, with all the complexity implied. Therefore, to build our theoretical framework and our methodology we lean on two theoretical approaches: Margolinas' model (2005) and the model of Robert and Rogalski (cross-analysis of the teacher's activity). We did not combine this two framework, instead we asked different independent questions relating to these two theoretical frameworks which we study using tools and methodology from the relevant framework. In this presentation, we will focus on the second model and the results related to it.

The model of Robert and Rogalski: Cross-analysis of the teacher's activity

This model combines two different approaches. Firstly, a didactic approach: teacher's practices are linked with the learning objectives. In this sense, the knowledge content of the teaching and the way the teacher organizes his teaching are analyzed using a didactic theoretical framework. Secondly, a psychological approach from cognitive ergonomics: the teacher is considered as a professional whose practices are subject to a professional contract, with particular goals, repertories of action, representations of mathematical objects and their learning, and, more generally, personal competencies which determine his activity.

Robert and Rogalski (2002) determined five dimensions to analyze teacher's activity.

- The cognitive and mediation dimensions which concern the set of teacher's choices about the content and the organization of the knowledge before (cognitive dimension) and during the class-time (mediation dimension).

The combination of this two dimensions allows us to trace what "kind" of mathematics is proposed in the class « la fréquentation des mathématiques qui est installée, ce qui est valorisé par les scénarios et leur accompagnement et ce qui pourrait manquer » (Robert & Rogalski, 2002, p. 514).

This first part relates to the description of the teacher activity, then we would like to interpret and highlight what determines these practices. We refer to three dimensions:

- The social, the personal and the institutional dimensions which permit to define the constraints and the personal aspects of each teaching project.

These three dimensions are studied based on interviews, deduced from the observed teaching sequence and also from the study of the institutional and local context.

This bring us the following research questions:

- What type of mathematics is promoted by various teachers, according to different scenarios and their execution in different contexts? Which kind of logic of action can we observe? Are these practices compatible with the didactic and pedagogical choices of the MERM resource designers?

- What hypotheses can be formulated concerning the dimensions (social, personal, institutional) that determine the teachers' practices of our study? How do the MERM influence the observed practices?

Method of analysis

The analysis was realised according to the following stages:

- A transcription of the lessons we observed was made using Transana, a software which allowed us to have permanently the video and the transcription on the same screen and give the possibility to introduce time codes and keywords. Referring to methodology used in various research using the double approach framework, we divide each lesson in temporal phases, which we call an episode and corresponds to a content unity.

- The scenario of the whole teaching sequence was then reconstituted and analysed to clarify on the one hand what kind of mathematical content is presented during the teaching (cognitive dimension) and, on the other hand, the dynamics in which the content appear during the classes (mediation dimension). The episodes we defined in the first stage of the analysis were coded according to the mathematical content and the teaching strategies (for more details, see Daina 2013).

- we wrote a report based on the interviews with teachers which allowed us to define the "profile" of each teacher, a synthesis of all the information we collected, which give information, among other aspects, on social, personal, institutional dimensions.

Selected results

French speaking Switzerland context and pedagogical resources

Switzerland has a highly decentralized system with no federal or national Ministry of Education. Each of the 26 cantons which composes the country has its own education legislation. However, four Regional Conferences, including French-speaking Switzerland which is represented by the CIIP (http://www.ciip.ch), have led to some effective coordination since the last five decades, drawing up common curricula, publishing material, jointly managing institutions and recognizing qualifications and admissions.

In the 1970s in all French speaking Switzerland a common official set of pedagogical resources, the *MERM,* was designed by the CIIP appointed group of experts and teachers from the different cantons. This first edition originates from a double necessity: a will of « inter-cantonal » coordination of

education (mathematics but more widely all the disciplines) and the introduction of a new curricula (CIRCE I), linked with the reform of the "modern mathematics ". In the 1990s, modern mathematics were abandoned and the pedagogical resources have been renewed in the 1990s according to a new educational paradigm based on problem solving, strongly influenced by the socio-constructivist approach.

Switzerland has therefore a long tradition of diverse cantonal educational policies but also the willingness to coordinate the educational system in order to facilitate in particular communication and student mobility. The *MERM* are the “symbol” of this process, especially in mathematics because they were the first to be done. In fact, the *MERM* are the result of a long process of discussions and compromises because they have to be approved and accepted by all the cantons of the French speaking part of Switzerland. It is necessary to allow three or four years to realize the *MERM* for one degree. We have to take into account this complex context in our observation and analysis of this resource (institutional dimension). The *MERM* must for example be compatible with all the plans of studies. They cannot thus be too prescriptive and require an opening.

Besides, the pedagogical resources are central to the reforms and innovations regarding mathematics education and more than just simple resources, they have the role of promoting innovation, in particular thanks to the teacher's textbook which describes the didactic and educational choices. They have to introduce the changes and harmonize the practices. This is also a critical element we have to take into account in our analysis (institutional dimension).

The *MERM* have been thought as a set of resources, therefore they mostly consist of a succession of activities for class, regrouped in 6 to 8 main themes, without hierarchy. Contrarily to textbooks in other countries, they do not give a day-to-day organised plan for teaching, which remains the teachers’ responsibility. To do this, the teacher must provide a considerable amount of preparation prior to the lesson and our PhD theses aimed to study the ordinary practices in connection with this resource in order to make visible this essential part of the work of the teacher.

Case study: Mathilde and Sophie

To exemplify our methodology and present some results of our research, we will now present two examples of our case study. We will first provide the chosen information about their “profile”, which gives information about the personal and social dimensions. Then we will present and compare their teaching sequences.

Mathilde and Sophie were both young teachers (4 and 5 years of experience) and they worked in the same school. The year of our experiment, they taught the same degree (6P) and they met twice a month and collaborated to develop mathematics lesson plans. Looking at the exchanges between the teachers during this meeting, we saw that they spoke little about the teaching objectives which seemed to be implicitly known and shared.

Mathilde said that the textbook was the reference and did not feel the need to talk more about the objectives. She specified during the interview that her colleagues “trust” her concerning the choice of the activity because it’s impossible, according to her, to choose the activities in the textbook without assistance. Sophie also speaks about "trust" during the interview what shows the importance of the collaboration in the preparation, by filling what is felt as a "lack" of the resource.

However, the study of Sophie's and Mathilde's profiles allows us to highlight that the objectives of both teachers are very different. For Sophie the calculating procedures in particular the introduction of the techniques of calculation of areas for triangles and parallelograms represent an important aim in the sequence. For her part, Mathilde considers that the main objective concerns the understanding of the notion of area and the formula to calculate the area of a square. Even though their objectives are different, this point remains implicit throughout their collaboration.

The scenarios analysis and their carrying out in class will allow to see how will evolve the educational projects of the two teachers whose starting point, by the way, is the same list of activities.

Figures 1 and 2 allow to have a global vision of Mathilde's and Sophie's scenarios. Mathilde provides a teaching sequence we analysed in three parts: an introductory session, three sessions dedicated to the area of square and rectangle and a session of introduction of the area of other polygons. This corresponds to what is proposed in the textbook.

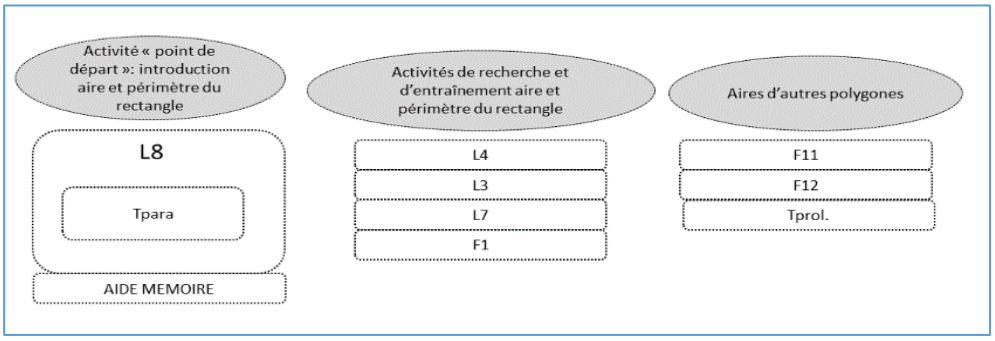


Figure 1 Mathilde's scenarios

The analysis of the series of activities proposed in Sophie's classroom show a split teaching sequence where activity from the official textbook are often mixed with improvised tasks on the blackboard.

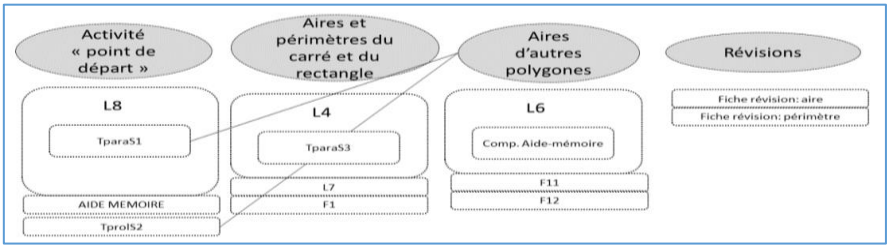
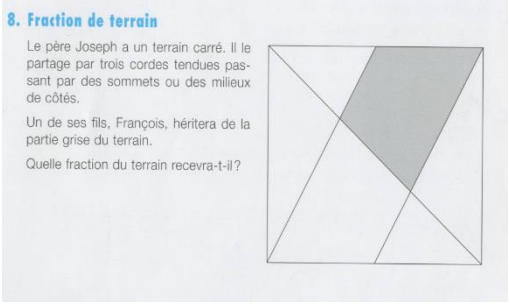


Figure 2 Sophie's scenarios

We represented a structure in four parts: an introductory activity, a series of activities on the area of square and rectangle, an introduction in the measure of areas of the other regular polygons and a session of revision.

Although Sophie and Mathilde based there teaching sequence on the same common project, we see well how both scenarios evolved. While the starting point is almost identical, the gap is widening in the course of the sessions, leading to differents mathematics promoted in each class (considering the cognitive dimension, Daina 2013). It is nevertheless interesting to note that in the interviews, Mathilde and Sophie did not seem conscious of these differences.

The analysis of the first session, while Sophie and Mathilde proposed the same activity, "Fraction of a field", allowed to see more in details this difference.

8. Fraction of a field Father Joseph has a square field. He splits it using three ropes passing through the vertices or the midpoints of the sides. One of his sons, Francis, will inherit the grey part of the field. What fraction of the field will he receive?	

The table below synthesizes the succession of the phases during the carrying out of the activity in Sophie's and Mathilde's class.

Sophie		**Mathilde**	
Phases	*Duration*	*Phases*	*Duration*
Instruction	7 minutes	Instruction	3 minutes
Pupils work on the activity	6 minutes	Pupils work on the activity	10 minutes
Pooling	3 minutes	Pooling	7 minutes
Pupils work on the activity	3 minutes	Aide mémoire	19 minutes
Pooling	8 minutes	Pupils work on the activity	39 minutes
Parallel task on white board (interruption)	5 minutes		
pooling (reprise)	8 minutes		
Aide mémoire	4 minutes		
Parallel task on the white board (interruption)	8 minutes		

Here we see very clearly that, although the same activity was proposed to the pupils, its management in class is totally different between the two teachers (mediation dimension).

In the class of Mathilde the number of phases is limited. The instruction is very short. During the pooling, the interactions testify of a discussion between the teacher and the pupils who have an active participation in advancing the discussion. Corrections are made individually.

In the class of Sophie we observe at first a longer instruction time. Sophie testifies of a will to make sure that the pupils understand well « what they have to do ». The progression of the project is managed collectively. Sophie makes regular pooling in the course of which, she directs the pupils on a strategy of resolution which is going to become common. However, the task is also diverted on secondary tasks which are in connection with knowledge bound to calculate the area of parallelograms or triangles which are central objectives for her. Sophie's more personal project enters thus in tension with the progress of the main activity which becomes a material medium to introduce new knowledge in a lecture style of teaching on whiteboard.

We can thus observe a big variability in the practices of these two teachers. What is really questioning is that these differences do not seem to worry them and in spite of their various ways of functioning, the teachers find an interest to prepare together the teaching sequences.

The space provided in this paper only allows to give a limited insight into our research result but some analysis made permit to identify tensions bound to the use of MER. The first cause of tension results from the "shape" of this resource. Indeed, the quantity of the activity proposed is too important and a novice teacher can't know them all and make a real choice. As said in the introduction, there is no supplied sequence and not much piece of information given about each activity. This leads the teachers to follow what make other colleagues, even if sometimes they did not share the same objective and teaching strategy. Furthermore, as we show in the first part, the specific status of this resource will act as a strong constraint on the teacher who will use it to be in "conformity", even if

they do not follow the socio-constructivist approach and will finally distract from the goal of the activities.

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Children's performance on a mathematics task they were not taught to solve: A case study

Seán Delaney

Marino Institute of Education, Dublin, Ireland; sean.delaney@mie.ie

A teacher documents how a task was used to elicit children's knowledge of multiplication as a precursor to learning the long multiplication algorithm. Data include samples of children's written work, and transcripts of two 30-minute lessons. Children discussed solution strategies. The teacher used time between the two lessons to select and sequence representative strategies to share. Strategies used by the children were diverse showing a range of multiplication-related knowledge among the children ranging from knowledge of repeated addition to knowledge of place value and the additive decomposition of numbers to various applications of the distributive property.

Keywords: Long multiplication, problem solving, third grade, mathematics laboratory.

Introduction

A common pattern of mathematics teaching is for teachers to present a problem to children, demonstrate how to solve it, and then set similar problems for the children to solve applying the demonstrated strategy (Lyons, Lynch, Close, Sheerin, & Boland, 2003; Stigler & Hiebert, 1999). Calls have long been made for mathematics instruction to focus on teaching thinking strategies and to promote goals like creativity, forming and changing hypotheses, and reflecting on one's own thinking and the thinking of others (e.g. Streefland, 1992). This case study describes one teacher using heuristics to introduce long multiplication to children who have completed third class.

Children in Ireland learn short multiplication in third class and long multiplication in fourth class. The study describes and analyses work done over two days in a mathematics laboratory school which took place in July 2016 in Dublin. Two research questions are addressed in this study. First, what knowledge did children draw on to solve a long multiplication problem they had not been taught to solve? Second, what knowledge does the class possess that prepares the children for future work on long multiplication? The data consist of children's written work and transcripts of two 30-minute lessons on consecutive days.

Theoretical framework

Three research areas frame this study. The first is progressive schematisation or progressive mathematisation as inspired by the work of Freudenthal and Realistic Mathematics Education. It describes how children solve problems using their own ideas and informal strategies (Streefland, 1992). These ideas and methods become more sophisticated as students begin to understand and use formal, more efficient algorithms (Treffers, 1987). Although others have written about invented algorithms (e.g. Kamii, Lewis, & Livingston, 1993), Treffers (1987) formalises the process and refers to vertical mathematisation (where algorithms are reorganised and refined) and to horizontal mathematisation (connecting the mathematics to real-life) (Selter, 1998). The second area of research relates to the conceptual field of multiplication: the grounding of mathematical analysis of situations and problems, and the development of children's ideas over time. It includes paying attention to children's past and future learning to inform present learning (Vergnaud, 1988).

The third area of research framing this study is the teaching of multiplication (Lampert, 1986). Lampert identifies four categories of knowledge used when learning mathematics: intuitive, computational, concrete and principled. Intuitive (or naïve) knowledge refers to how people in particular contexts invent ways to calculate in order to do their work; it may not transfer well to other contexts. Computational knowledge is procedural knowledge, like standard algorithms, that children typically use in school. Concrete knowledge is used when objects are manipulated to find answers. This may include rectangular grids which are sometimes used to compute answers to multiplication problems. Finally, principled knowledge is knowledge children can use, without necessarily understanding the meaning of what they are doing. Such knowledge might involve children drawing on principles such as place value, the commutativity of addition or multiplication, or the distributive property of multiplication over addition.

This framework will help to identify stages in the heterogeneous work of a class (Selter, 1998) of children as they attempt to solve problems for which they do not have a solution. It will also help to categorise the knowledge that children used and shared in solving the problems.

Method

Participants

Twenty-four children – 17 girls and 7 boys – were in the class which lasted for two hours per day over five days. The children had just completed third class in ten different schools and therefore could be expected to be familiar with short multiplication but not to have worked on long multiplication. The mathematics laboratory class was taught by the author and was observed by twenty-five teachers who were completing a summer course in mathematics. Although several topics were taught in the summer school, the focus of this study is on the introduction of long multiplication. A word problem was chosen from Van de Walle: “The parade had 23 clowns. Each clown carried 18 balloons. How many balloons were there altogether?” (Van de Walle, 2001, p.182). Children worked on this problem collaboratively in pairs; they were encouraged to solve the problem and to be prepared to justify their solution.

Data analysis

Two data sources were used: samples of children’s written work and transcripts of dialogue from the lessons. Children did their written work in squared exercise books using black pen to ensure that they would not erase work they were unhappy with or that contained errors. This rationale was shared with the children. All lessons were video recorded by two cameras – one focused on the children and one focused on the teacher. The videos were used to prepare lesson transcripts.

The research questions relate to the knowledge used by the children and available to the class as a resource for future learning. All children’s written work completed in response to the problem was studied and compared to ensure that samples of every approach used were represented in the four samples of work selected for more detailed analysis. The chosen work samples were subsequently analysed to identify categories of knowledge that were evident in the work. The four categories outlined by Lampert (1986) guided this analysis. The transcripts were analysed for evidence of student mathematical knowledge. Although the categories identified by Lampert informed this

analysis, the analysis was open (Corbin & Strauss, 2008) so references to knowledge not covered by the four categories could be identified.

Results

Lesson plans

Children were asked to work on the multiplication problem in pairs. The following day the plan was to continue working on the same problem. Prior to lesson 2, the author (as teacher) looked at the children's work to select and sequence four approaches to share with the entire class. The strategies selected involved repeated addition, repeated addition with some multiplication, multiplying using partial products and an attempt at the standard algorithm for long multiplication. The lesson plan refers to a pictorial representation of the problem that would help the children get an understanding of the dimensions of the problem. This was introduced in response to a similar approach used for various scenarios by Lampert (1986). The lesson plan concluded with the intention to ask the children to independently solve a second long multiplication problem.

Day 1

The calculation was embedded in a word problem referring to a setting familiar to most children in the culture – many clowns each holding several balloons. Two two-digit numbers needed to be multiplied. The combination of relatively low two-digit numbers and the concrete image of clowns with balloons made it relatively easy for children to draw the scenario if they decided to do so. After a class discussion of the problem conditions, children worked on it in pairs for seventeen minutes. The teacher circulated among the children monitoring the work of pairs. Teachers participating in the summer course walked around the class observing the children working. Although they were asked not to interact with the children, on one or two occasions some did.

After the children had worked in pairs, the teacher asked one pair of children – Sandra and Lisa – to tell the class how they went about solving the problem. They had added eighteen and eighteen to make thirty-six. Then they added another eighteen. The teacher asked the class if this strategy were implemented properly, would it yield the correct answer, and based on their responses reminded them of the need to be systematic in recording their work.

Two more children, Chuck and Róisín, shared a different approach. They wrote down twenty-three eighteens and multiplied twenty-three by eight and twenty-three by one. This represented an understandable mistake, forgetting that the one digit in the eighteens represents ten rather than one.

At this stage the teacher adjourned the discussion and moved to another mathematical topic. Having concluded that part of the work for the day, the teacher could examine and reflect on the children's work in order to select and sequence material for discussion in the following day's lesson.

Day 2

Overnight the teacher looked at each child's work. No one had successfully used the long multiplication algorithm suggesting that, as expected, it had not been taught to the children prior to the summer school. Four examples were selected and sequenced in a way that was anticipated to tap into the children's current understanding, to show increasing efficiency or sophistication of solutions – progressive mathematisation – and to prepare the children for subsequent work on long

multiplication. Although all children had worked in pairs, work to be shared was selected according to the clarity of the work recorded in individual children's copybooks.

First was Christine who had used a straightforward repeated addition approach. She had a pictorial representation of the problem with twenty-three faces and eighteen balloons over seven of them (see Figure 1). Next was Donal who had no pictorial representation but who also used a repeated addition approach. He had grouped ten eights where possible to multiply them and had multiplied twenty-three by ten (see Figure 2). Third was Fintan who solved the problem by calculating ten eighteens, another ten eighteens and three eighteens and then added the three calculations (see Figure 3). All three students had the correct answer of 414. The fourth student, Eileen, got the wrong answer but the strategy used seemed closer to the standard long multiplication algorithm. She wrote an account of what she did and of how she was thinking rather than just recording the calculation. She multiplied the three from twenty-three by the eight in eighteen and got twenty-four. She then multiplied twenty by ten to get two hundred. She refers to multiplying two by one and it is unclear if that is a precursor to multiplying twenty by ten (see Figure 4).

Despite the fact that Christine had set up the calculation to be solved using repeated addition (see Figure 1), she stated that to solve the problem she and her partner "drew a picture and we did loads of dots and we counted them all up." When challenged by the teacher about the repeated addition work in her copy, Christine responded that "I had a long sum but that didn't really work because I kept on losing count." This provided an opportunity to discuss a problem that arises with repeated addition, and to prepare the class for seeking more efficient ways to calculate using long multiplication. The teacher did not ask Christine why it was easier to keep track of counting the balloons individually than adding 18 twenty-three times and that may have yielded information about a system she had developed to keep track of the balloons already counted.

Donal had a more sophisticated way of working with repeated addition (Figure 2). Although his layout of the problem looks similar to Christine's, he approached it as follows

(8x10) + (8x10) + (8x3) + (10x23)

In solving it this way Donal and his partner showed understanding of the distributive property of multiplication. They noted that the ones in the eighteens represented tens and not units. However, Donal's written recording of the work and his oral explanation of it suggests that he was not yet familiar with multiplying numbers by ten. In contrast, his classmate David stated that "When I'm multiplying by tens, I just add on another zero at the end of the number." Although the wording of "adding" another zero may not be helpful, he is referring to the fact that multiplying a number by ten shifts each digit one place to the left requiring zero as a placeholder in the units place.

A more condensed understanding of the distributive property was apparent in Fintan's work (Figure 3). Unlike Donal or Christine he did not write out the calculation using repeated addition. Nor did he separate the tens and units in order to complete his calculation, which took the following form:

(18x10) + (18x10) + 18 + 18 + 18

Fintan sees his approach as being similar to Donal's and he states that "we basically did the same as Donal; we used hundreds, tens, and units." However, whereas Donal's approach was limited by

apparently not being able to multiply two-digit numbers by ten, Fintan was able to make the calculation more efficient by multiplying eighteen by ten.

When the children were asked where the twenty-three (clowns) could be seen in Fintan's strategy, two children (Katherine, Ethna) found it difficult to identify. One, Doireann, successfully constructed an explanation with the teacher in the following exchange.

Doireann: So the eighteen times ten is done twice. So that would be like twenty there. And then…

Teacher: So, you're saying that this is ten clowns with eighteen balloons, and this is another ten clowns with eighteen balloons.

Doireann: Yeah.

Teacher: Is that what you're saying?

Doireann: Yeah.

Teacher: Okay. And what then?

Doireann: And then if you add that together that's twenty…

Teacher: Twenty clowns with eighteen balloons.

Doireann: Yeah, and down the bottom there, it's three eighteens. Add them onto the twenty and it's twenty-three.

When asked to choose a preferred strategy from those presented by Christine, Donal and Fintan. Four children preferred Fintan's approach on the basis that it is quicker and it requires less writing. Two claimed to prefer the repeated addition approach because it looked less complicated.

When children solved the first long multiplication problem, twelve of them used a variation of repeated addition, three used a form of the distributed property, three a variation of the conventional algorithm, the work of three children was unclear and one used counting. Following the discussion, seven children used repeated addition, four used a form of the distributive property, five used a form of the conventional algorithm, the work of six students was unclear and one used counting.

Finally, Eileen was asked to share her approach. Initially she stated that she and her partner got the wrong answer. After reassurance from the teacher that the class could learn from the wrong answer, she shared her approach. She writes that "in her head" she laid out the problem as it would be laid out in the conventional algorithm for long multiplication. She multiplied the three units by the eight units and got twenty-four. Then she multiplied the two tens (of twenty-three) by the one ten (of eighteen). She added them together and got 224. Although the teacher sequenced Eileen's strategy after Fintan, it is conceivable that her understanding is more naïve than his because she may have been attempting to apply the algorithm for short multiplication to long multiplication without really understanding the distributive property. Nevertheless, it provided an opportunity for the teacher to introduce an illustration of the distributive property of multiplication to all children.

Eileen failed to multiply the eight by twenty and the ten by three. In order to help her and her classmates visualize this, the teacher proposed a drawing of the scenario based on Lampert (1986). In this drawing (see Figure 5) twenty clowns were on a bus travelling to the parade and three clowns

had to walk because there were only twenty seats on the bus. Twenty strings with balloons on them could be seen emerging from the bus, each string with one group of ten and one group of eight and the three clowns outside the bus held similar strings of balloons.

The class was asked how they could calculate the number of balloons held by the clowns altogether. The idea of the picture was to make it clear that the numbers that need to be multiplied are (20x10), (20x8), (3x10) and (3x8). The first two calculations correspond to the clowns sitting in the bus with the strings containing ten balloons and eight balloons and the second two calculations correspond to the clowns standing outside the bus holding strings with ten and eight balloons on them. The diagram helped children see that Eileen had neglected to calculate the (20x8) and the (3x10).

Discussion and conclusion

That multiplication was the operation needed to complete this task was uncontested by the children. In solving the problem, they drew on different categories of mathematical knowledge. Some relied on intuitive, context-specific knowledge and drew versions of the clowns in order to solve the problem. Computational knowledge of multiplication was widely held, as would be expected by children who had completed third class. Although occasional errors were made, children had access to multiplication table cards if they wanted them so that even a lack of computational knowledge would not pose a barrier to solving the problem.

Little evidence of the children using concrete knowledge emerged in the lessons. This may have been because no manipulable objects were made available to them to help them find an answer. Some children may have used the diagrams they drew as a form of concrete knowledge to support their solution but that is unclear from the data sources used.

Much evidence of principled knowledge emerged from the children. Donal and David showed understanding of place value, by separating the tens and the eights in eighteen (Donal) and in stating how numbers can be easily multiplied by ten (David). Several children had a tacit knowledge that numbers can be decomposed additively and of the distributive property of multiplication (e.g. Donal, David, Caitlin, Fintan and others). The teacher attempted to make this aspect more explicit by introducing the diagram with the clowns in and beside the bus. Students like Eileen have some understanding of the distributive property but it needs to become more explicit if she and others are to be ready to apply it in taking the next step to understand and use the long multiplication algorithm automatically. They need to grasp the principle that (a+b)*(c+d) requires multiplying a by both c and d and multiplying b by both c and d and not just multiplying a by c and b by d.

At the end of the week around fourteen children were still either using repeated addition for similar multiplication tasks, using counting or not showing evidence of how they found their answer. Nine showed willingness either to apply the distributive property of multiplication or to at least attempt the conventional algorithm. Of those who were still applying the repeated addition algorithm, four had become more sophisticated in using it (moving closer to Donal’s approach than to Christine’s) following the class discussion documented here. This highlights the importance of children learning from each other through working on and explicitly discussing their completion of a task.

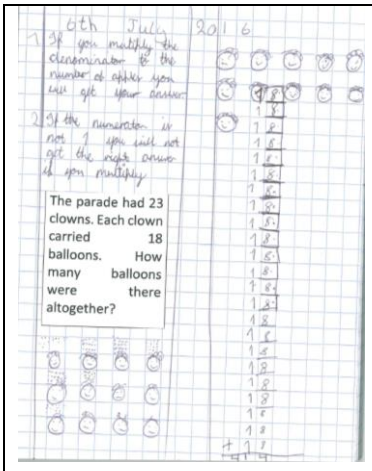

Figure 1. Christine's work.

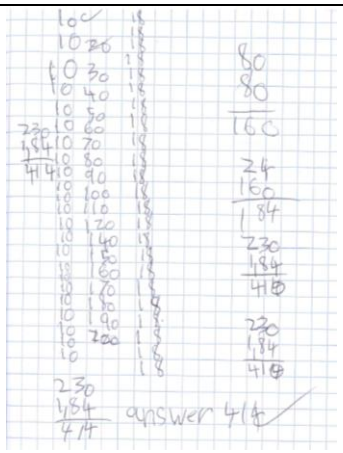

Figure 2. Donal's work.

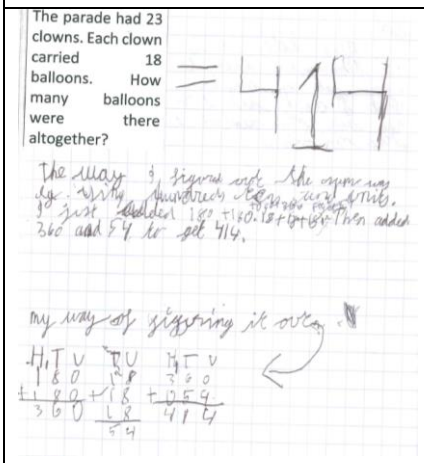

Figure 3. Fintan's work.

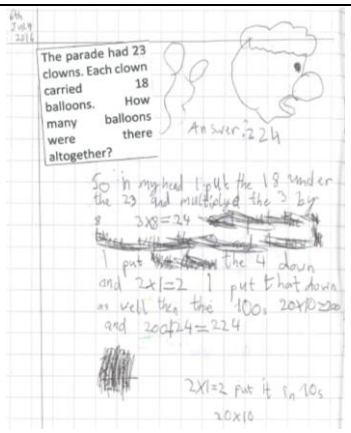

Figure 4. Eileen's work.

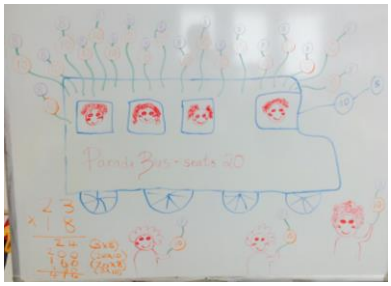

Figure 5. Visual representation of long multiplication problem.

Half the children used a form of repeated addition to solve the problem. However, additive reasoning differs from multiplicative reasoning (Nunes & Bryant, 1996). It remains unclear from this study if children reverted to additive reasoning because long multiplication was unfamiliar to them or if they have made the leap from additive reasoning to multiplicative reasoning at all. Future study could involve assessing children's grasp of multiplicative reasoning in short multiplication scenarios (Sherin & Fuson, 2005), exploring the use of arrays (Fosnot & Dolk, 2001) and introducing long multiplication tasks that would more likely elicit responses that exhibited multiplicative reasoning.

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Critical evaluation and design of mathematics tasks: Pre-service teachers

Majella Dempsey[1] and Ann O'Shea[2]

[1]Maynooth University, Department of Education, Maynooth, Ireland; majella.dempsey@nuim.ie

[2]Maynooth University, Department of Mathematics and Statistics, Ireland; ann.oshea@nuim.ie

This paper reports on a research project undertaken with a group (n=19) of Irish pre-service student teachers (PSTs) during the third year of a four-year undergraduate education course. A series of workshops were carried out on the critical evaluation and design of mathematics tasks. The research is presented as a case study using mixed methods to gather data. Through critically evaluating and designing mathematics tasks PSTs developed knowledge of cognitive demand, pedagogical design capacity and showed evidence of developing curriculum-making competences. The research highlights the need for PSTs to work together on evaluating and designing tasks.

Keywords: Mathematics, tasks, design, curriculum, pedagogy.

Introduction

This paper reports on a research project undertaken with a group (n=19) of Irish pre-service student teachers (PSTs) during the third year of a four-year undergraduate education course. A series of workshops were carried out on the critical evaluation and design of mathematics tasks. For the purpose of this research a 'mathematical task' is a problem or set of problems that address a specific mathematical idea, they are situated between teaching, learning and assessment (Smith & Stein, 1998). The types of tasks that students engage with have been shown to influence their development (Jonson et al., 2014), and studies have shown that students spend the majority of their time in mathematics classes working on tasks (Boston & Smith, 2009; Haggarty & Pepin, 2002). Furthermore, Smith and Stein (1998) asserted that the highest learning gains in mathematics were related to the how mathematics tasks were set up and implemented in teaching and highlighted the importance of students being engaged in high levels of cognitive thinking and reasoning (see also Swan, 2011; Boston, 2013). Many challenging questions arise from this assertion for pre-service mathematics teachers, such as, what is a good learning task? How is a good learning task set up? How is it implemented in a mathematics classroom? These questions are especially relevant in Ireland, given that a report on mathematics education found that traditional approaches to teaching and learning were widespread and recommended that students engage with more tasks which require higher order thinking skills such as problem-solving and justification (Jeffes et al., 2013).

Research questions

In what ways did pre-service teachers' knowledge of the cognitive demands of mathematical tasks change following their participation in a module on critical evaluation and design of mathematical tasks?

How did this knowledge impact on their competences in curriculum making?

In this paper we present a review of some key research on mathematics task design. We provide an overview of the module implemented as part of the research project and the methodology used to

collect data. We present the key findings from the research and discuss the salient themes emerging as they pertain to pre-service teacher education. Finally, we summarise our recommendations and conclusions.

Literature review

The dependence of mathematics teachers on textbooks in their teaching appears to be a phenomenon in many countries (Haggarty & Pepin, 2002; Jeffes et al., 2013.). Haggarty and Pepin (2002) write about the dominance of the textbook in the mathematics classroom and conclude that without time to prepare for teaching, and, we would add, the skills to enrich the curriculum materials available, textbooks take on a prominence in "relation to teacher thinking and planning*"* (p.588). This is of concern as a recent review of mathematics books in Ireland found that all available books fell short of the standard needed to support mathematics teaching at that time and furthermore they especially fell short on the integration of technology, approaches to teaching for understanding and problem solving (O'Keeffe & O'Donoghue, 2011).

Synder, Bolin and Zumwalt (1992) describe three teacher curriculum approaches: the *fidelity approach*, where teachers are transmitters of the written curriculum without changing it; the *adaptation approach* where the teacher adapts the curriculum to suit their context; and the *enactment approach* where the teacher develops the curriculum in action depending on the student experience. Shawer (2010) builds on this work and identifies *curriculum-transmission strategies* where the textbook and teacher's guide are the source of pedagogical instructions. He describes *curriculum-making strategies* as where the teacher develops their own materials in addition to those available in response to a needs assessment. *Curriculum-development strategies* on the other hand include experimentation, material writing and evaluation and involve both macro and micro level curriculum development. Within the Irish education system opportunities for curriculum-development strategies are limited since the curriculum is centrally devised with little space for school-based curriculum development, this coupled with a teaching culture that has a dependence on textbooks would point to a need for PSTs to engage with research on task development.

Studies have previously looked at this issue as part of professional development courses for in-service teachers. For example, Boston and Smith (2011) describe a *task-centric approach* to such courses where the focus is on teachers' ability to select and implement cognitively demanding tasks. They found that after a series of workshops, where teachers analysed both the cognitive demand of tasks and the implementation of tasks, the participants increased their ability to select high-level tasks and this improvement was sustained over time. The workshops also influenced teachers to consider the impact of the tasks they selected on their students' learning (Boston, 2013). Arbaugh and Brown (2005) used a similar approach and found that introducing teachers to criteria for high-level tasks influenced their task selection, and ultimately their pedagogical content knowledge.

A number of different frameworks have been developed to classify mathematical tasks and have proved useful in research, professional development and pre-service teacher education (Boston & Smith, 2011). In this research we used three different but complementary frameworks with the participants. The first framework is that of Smith and Stein (1998) which looks at the level of cognitive demand (LCD) of tasks. They identify two levels of LCD: Lower-level demands (with task types of *Memorization* and *Procedures without connections to meaning*), and higher-level

demands (*Procedures with connections to meaning* and *Doing Mathematics*). The distinction between tasks is relevant, as the level of cognitive demand in a task provides different learning opportunities for the learner and demands a different learning environment for the development of competences required by the task. Our second framework is the mathematical reasoning framework developed by Lithner (2008). This framework can be used to classify the opportunities for different types of mathematical reasoning afforded by tasks. Lithner (2008) describes two types of reasoning: Imitative Reasoning which consists of *Memorised reasoning* and *Algorithmic reasoning*; and Creative Reasoning which involves local and global *Creative mathematically founded reasoning*. Creative reasoning tasks fulfil the criteria of novelty, plausibility and mathematical foundation. Lithner (2008) is concerned with how tasks can be used to promote creative reasoning as opposed to imitative reasoning. He contends that the teacher's task is to "arrange a suitable didactic situation in the form of a problem" (p.271) so that the learner can take responsibility for the problem solving process, and use creative reasoning.

These two frameworks can be used to classify tasks using either the degree of cognitive effort required or the type of reasoning needed. They both divide tasks into two broad categories - either high or low levels of cognitive demand in the case of Smith and Stein (1998) or imitative or creative reasoning in Lithner (2008). They have been used in professional development to alert teachers to the effects of different types of tasks (e.g. Arbaugh & Brown, 2005). In order to help the PSTs to move from classifying tasks to designing them, we introduced a third framework. Swan's framework (2008) describes five task types that encourage concept development and provides very clear design principles to inform task development and implementation. There are many examples of Swan's mathematics tasks available on-line (see, for example, Mathematics Assessment Project, n.d.). The five task types that he posits will encourage concept development are: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and, analysing reasoning and solutions.

Methodology

Nineteen pre-service teachers in the second semester of year three of a four-year post-primary teacher education course took part in the research project. At the time of the research the PSTs were midway through their second school placement experience and were teaching a minimum of two hours per week. All participants were taking mathematics in their degree and one other science subject (either biology, chemistry or physics). The research is presented as a case study using mixed data collection methods looking at the group of PSTs as a whole, over a sustained period of time as they developed competences in task design (Yin, 2009). This allowed us to build on earlier research (Boston, 2013) and incorporate PSTs reflections on the design process. Jones and Pepin (2016) contend that when teachers interact with mathematical tasks, they develop knowledge; this is done individually in preparing and planning for teaching and collectively when they are afforded opportunities to develop and discuss tasks with peers. In designing curriculum materials PSTs need both *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK) (Ball, Thames & Phelps, 2008). With this in mind we designed a module for the group of PSTs based on task evaluation and ultimately task design.

In order to investigate any gain in knowledge for the group over the course of the intervention, we administered a pre- and a post-test designed by Boston (2013). This test asked students to classify 16 tasks as either High Level or Low Level tasks, and to give a rationale for their choice. At the end of the module, the pre-service teachers were asked to complete an evaluation questionnaire which asked them: to report on a key learning moment during the module; whether their teaching had changed as a result of the module and if so, in what way; what they would change about the module; and to indicate their level of agreement with some statements about the reading from the Mathematics Education literature. 13 of the 19 students submitted the evaluation questionnaire.

The assessment for the module consisted of the assignment outlined in Figure 1. All 19 pre-service teachers submitted this assignment and gave their consent to use it for research purposes. The tasks designed by the PSTs were analysed using the LCD and Lithner Frameworks. The classification was conducted by two researchers who were familiar with the curriculum, assessment, and textbooks relevant to the classes taught by the PSTs. The researchers used their knowledge to decide if (in the context of the PSTs' classes) the tasks should be classified as either high or low level tasks. We also looked for evidence that PSTs employed aspects of Swan's (2008) framework in their design. A general inductive approach as advocated by Thomas (2006) was taken to analyse the students' reflections on the differences between types of tasks. Analysis was guided by the research questions and a number of a priori themes (such as MKT), allowing flexibility for other themes to emerge.

Task Development

For a topic of your choice design (or significantly adapt) a series of tasks. One task/s should be suitable to be used in class while teaching, and, one for use as homework. Design an examination task/s for the topic. Present your rationale for each task based on your readings. Reflect on the differences between classroom task, homework task and examination tasks.

Figure 1: End of module assessment

Findings

PSTs' specific learning about cognitive demands

Thirteen participants completed the Boston (2013) pre- and post-tests on levels of cognitive demand; a paired t-test was used to investigate whether the mean of the group had increased significantly over the course of the module, and found that it did (p=0.037). There was also evidence for PSTs developing knowledge about cognitive demand in their response to the question on the end of module evaluation asking what was their key learning moment.

> Realising the different reasoning and thinking about the type of question. In the textbook, where homework is usually given from, questions are repeated, low demand. In the maths exam students are faced with high level conceptual questions so there is a big gap there that needs to be addressed. (S14)

Here we see that S14 is *noticing* the level of demand and reasoning in the artefacts available to them in their teaching, the textbook. This text-guided Algorithmic Reasoning (AR) is supported and encouraged by the rote use of the textbook for homework (Lithner, 2008). The recognition of this by the PSTs was notable in many comments such as:

> Having completed this module, I seriously consider what I give my students as homework. Beforehand I generally gave a list of questions at the end of the chapter but now, having seen the different levels, I generally spend more time selecting and developing questions …. (S16)

The PSTs seem to be linking the levels of cognitive demand with the level of reasoning required, bringing the two theories together in their own thinking about the mathematics curriculum and assessment. The knowledge of the different frameworks is enabling the PSTs to move from a role of curriculum transmitters dependent on the text book to being curriculum makers as described by Shawer (2010).

Pedagogical design capacity and moving from curriculum transmission

The analysis of the PSTs' end of module assignments gave further evidence of them making this transition. All students showed that they were able to design or modify tasks to get high level questions. The PSTs classification of their tasks using the LCD and reasoning frameworks demonstrated that they were competent in using the frameworks for classification.

We noted the types of tasks designed by the PSTs for the three different situations of classroom tasks, homework tasks, and examination tasks. The types of tasks designed seemed to fall into two broad camps: open-ended exploratory tasks (which were mainly found in the classroom setting) and more traditional formats (which were mainly found in homework and examination tasks). The latter types of tasks mostly consisted of word problems with a real-life context; the PSTs designed a small number of other types of tasks for use as homework or examination questions, including tasks which required students to make a conjecture, provide an example, or evaluate a mathematical statement. In addition, one PST designed a homework task which involved a pre-class investigation. The majority (13 of 19) of the PSTs used card-matching designs for their classroom tasks. These tasks were based on Swan's "Interpreting Multiple Representations" (Swan, 2008, p. 3) task type. The PSTs were introduced to this idea through the Swan (2008) article and also participated in a card-matching task (on the topic of fractions) during one of the module sessions. Three of the PSTs used games (such as 'Battleships' and dice games) to devise tasks for use in the classroom, two PSTs used investigations as the basis of their task, and one designed a series of worksheets with problems of increasing difficulty.

The PSTs showed creativity and an appreciation for tasks with high levels of cognitive demand. However, an analysis of their designed tasks showed that the design process was not without difficulty for the group. Some of the questions were not always clear due to missing or confusing instructions, and sometimes the context made the question ambiguous (this has also been a problem in state examinations in Ireland). Occasionally it seemed as if the PSTs did not have a clear understanding of the underlying mathematics themselves, possibly owing to their level of SMK, and sometimes their use of mathematical language caused difficulty (such as using the term 'equation' instead of 'expression' for something like $2x+1$). The learning trajectories for the tasks or sets of tasks were not always clear - sometimes it was not clear what understanding and what concept the PSTs were trying to develop.

Pre-service teachers' pedagogical content knowledge

The PSTs' knowledge of levels of cognitive demands challenged their deeply held view of how best to teach mathematics. Previous research with a similar co-hort of PSTs found that they focused on

content when planning for teaching and placed little emphasis on the learner or prior learning (Nolan, Dempsey, Lovatt & O'Shea, 2015). Most of the respondents said the module impacted on how they taught mathematics with the majority citing a change in how they asked questions, placing more emphasis on higher level of cognitive demand in questions. Prior to this module, these PSTs would have completed a module which included a significant input on questioning skills for teaching; they seem to have needed the knowledge of cognitive demand in order to have changed their questioning practices. It must be noted that this was reported but may not have been the reality when one takes into account the examples provided by the PSTs in their end of module assessment. However, an increased emphasis on discussing mathematics problems appears to be evident with comments such as

> I try to think more about pushing my students to reason more when completing tasks. I try to ask questions, give tasks to my students with much less information, and I want my students to rely less on me giving them the answer. (S11)

This PST also spoke about the effect of the intervention on her teaching:

> I never really thought much into the differences between the tasks that I give during class, homework or exams or the impact it could have on my students' development in a subject. Having studied and researched the classification of math's tasks and implementing my own selection/adaption of tasks into my class, I now feel that I have gained a deeper understanding into the effect my choice of tasks can have on the progression and learning... (S11)

The PSTs who implemented their tasks in their teaching placements, realised the effect that the teacher or set-up can have on the cognitive level of the task and this led to them thinking about different types of tasks or redesigning their original tasks:

> The students struggled very much with it at the beginning and due to my own fault I went through an algorithm with them and then the task immediately became a lower demand one, just requiring the students to reproduce an algorithm each time. If I were to redesign the tasks, I would change tasks E and F [card-matching tasks] to tasks where the students have to spot a mistake in a question/statement and justify their reasoning and how they would alter the question/statement … in order to encourage them to develop critical thinking skills. (S12)

This reflection would suggest that the PST is developing her thinking on organising pedagogical content, adapting materials to suit students and adopting curriculum planning and making strategies (Shawer, 2010).

Discussion

Increasing the PSTs awareness of different levels of tasks and giving them an opportunity to design and modify tasks would appear to have allowed them to develop skills such as the ability to classify tasks and design tasks at different levels. They also seem to have developed knowledge especially PCK which linked to their knowledge of cognitive demand has enabled them to adapt their practice especially around questioning. We note though that the evidence we have presented in this regard is based on self-reported data. The importance of applying frameworks in order to increase awareness of concepts such as levels of cognitive demand is significant for PST education; awareness may be a crucial first step in knowledge acquisition. Similar to findings from Boston (2013) and Swan

(2007) who worked with practicing teachers, our research appears to demonstrate the need for an awareness of cognitive demand in order for mathematics educators to be able to select and develop rich and engaging tasks. This increase in knowledge and skills seems to be crucial in order to make the transition from curriculum transmitters to curriculum makers (Shawer, 2010). The space in our intervention for discussing textbook questions and State Examinations materials was cited as being the most impactful for the PSTs. They suggested that the module could be enhanced with more time devoted to this kind of peer interaction in task evaluation and design. This need for space for curriculum making and professional learning, and, the challenges therein has not been fully explored within PST education.

This research has highlighted a gap in the PSTs' education on task design in the case study institution, and, as such will be used to make changes to the module design and implementation. PSTs' practices and beliefs around tasks for homework merits further exploration. We intend to carry out more analysis on our data such as on the tasks assigned by PSTs (pre- and post-intervention) and of their reflections on tasks linking back to Lithner's (2008) concept of sociocultural milieu.

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Agency as a tool in design research collaborations

Andreas Eckert

Linnaeus University, Sweden; andreas.eckert@lnu.se

The aim in this paper is to shed light on interactional aspects of researcher and practitioner collaboration in design research in mathematics education. Symbolic interactionism is used to gain understanding of interactional aspects as it has potential to take both individual and social aspects of the interaction into account. Aims and agencies are in focus of the retrospective analysis of the collaboration between two researchers and two practitioners as they collaborate to develop instructional design. The analysis show how referring to authoritative disciplines as the mathematics community influence agency and therefore has great potential to influence how the negotiation of meaning progress and participants acts. I argue that agency could be viewed as an indirect tool that has the potential to direct the collaboration when designing tasks based on what aim different actors put in the foreground.

Keywords: Collaboration, agency, interaction, design research.

Introduction

Collaboration is at the core of design research in mathematics education. A key characteristic is that it is research conducted with researchers and practitioners in real-world settings (Plomp, 2013). This collaboration is often between one or more researchers and practicing teachers. It is essential that this team collectively has the competence to develop the design, conduct the lessons, and perform the retrospective analysis (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). This means that we have different actors in the activity, each with its interpretation of the aim and purpose, as well as of the actual activity and the mathematics involved. Since each actor has a specific set of competences and a vital role to play in the collaboration to develop the design, the question is how their differences influence the discussions during meetings and by extension the design? McClain (2011) explores this interplay of differences in the classroom interaction between students who performed the resulting tasks and the teacher who orchestrated the discussions. She views it as interplay between the students' contribution, the task and what she calls the proactive role of the teacher. One key aspect of the interplay is where the authority over the topic, the agency, lies. Agency is thought of as a capacity to act in social interaction. McClain (2011) emphasizes that it is important for students' learning that agency shifts between different actors in the activity.

The discussion in this paper focuses on the developing phases of design research. How does this shifting of agency that McClain (2011) identifies in the design research classroom appear in the development of tasks in between lessons? The aim of this paper is to further understand how the aim and agency fluctuate between the participants and its influence on how the negotiation develops as researchers and teachers collaborate to develop instructional design. Results could be viewed as a contribution to the mathematics design research methodology discourse of how collaboration between researchers and practitioners can support the design process.

Previous research

Design research in mathematics education is described as a research design that is interventionist; iterative; process, utility and theory oriented; and has involvement of practitioners (Plomp, 2013). The idea is to develop tasks and activities, test them in real classrooms, evaluate the outcomes and then revise the design in an iterative process involving practitioners from the field. All in all, the research design enables the team to pursue multiple goals in the same project. The aim is to understand the processes involved in utilizing the developed tasks from the point of view of a chosen theory, design usable material for users in real life context and lastly to contribute to further development of theory. Plomp (2013) calls it the twofold yield of design research, producing both research-based intervention and knowledge about interventions in the form of theory.

Beside the global aims of design research, an intention to develop theories and instructional designs, there are local aims as well. The instructional design has an aim, an intention to stimulate learning, in the form of a hypothesised learning process and goal for the subjects (Cobb et al., 2003). The so-called Hypothetical learning trajectory (HLT) is defined by Simon (1995) as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (p. 136)

Cobb (1999) argues that the learning goal of a HLT should be from a group perspective rather than an individual one. Simon (1995) amongst others talk about a prediction of individual learning processes and thinking whereas Cobb (1999) call this line of thinking highly idealized at best. Instead, he proposes a focus on collective mathematical development in the classroom community. A HLT then consists of an "envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices" (p. 9).

Global and local aims are both individual interpretations as well as objects for the research team to negotiate. In this paper, it is assumed that this process is a social interaction within the team. Interaction is described with the theoretical background of symbolic interactionism (Blumer, 1986). It proposes that humans act according to the meaning that objects have to them and that the meaning of objects arise out of social interaction. This is an interpretative process where humans constantly interpret others' actions and the meanings they indicate before acting themselves (Blumer, 1986). It means that participants of the research team interact according to how they interpret the local aim of the design as well as their own global aims and interpretations of the mathematics in question. Voigt (1994) calls this the negotiation of meaning in the mathematics education context. It is a negotiation because the actors contribute to a discussion based on their interpretation of what is being discussed and at the same time re-evaluate their own understanding, thus creating a negotiation of what is viewed as the community's meaning of the objects.

As global and local aims and the design itself are negotiated, participants position themselves through their contribution to the negotiation. Burr (2003) talks of the capacity to take up positions for one's own purposes and that agency lies in responsive actions in interaction. In the mathematics classroom context, McClain, Zhao, Visnovska, and Bowen (2011) defines agency as "authority over both the mathematics being taught and the sequencing and presentation of that content" (p. 63).

Combined, they frame the agency concept for this paper. For the purpose of this paper agency is viewed as involving one's intentions, sense of responsibility, as well as one's expectations of recognition and reward in taking a particular action. As the participants of a research team interact in the development of the design, they act according to their interpretations and their positioning in a community, for example as a representative for the mathematical community, and act with the authority of that discipline. Participant act according to different agencies as fits their purposes and evaluate its impact on the negotiation, also known as the dance of agency (Pickering, 1995). As the negotiation progress, agency shift between the participants and within them. Shifting agency enables the participants to contribute in different ways and from multiple perspectives, for example as mathematicians, practitioners or researchers. The result is an effect on the design of the HLT in line with different actors' fluctuating aims and their agency to contribute according to those aims.

Method

The data used here is generated from video recordings of a small-scale teaching experiment involving probability with students from year 5 and 6 in a Swedish elementary school. The aim of the task was for the students to become able to discuss matters of relative frequency data and the law of large numbers in a probability context. Relative frequency is a way of analysing data from repeated random events, such coin flips, where the number of observations of each outcome is divided by the total number of events. The law of large numbers then states that as the sample size increases, the likelihood of a difference between the relative frequency and the actual probability of the event decrease. Thus, can the relative frequency be used as a measure of probability. The research team consisted of the author of this paper, a senior researcher (here called Paul) and two teachers (here called Karen and Tilly). The whole process was initiated by the two teachers who felt that they needed inspiration and experience in teaching probability, which they had never done before. The balance of numbers provided a sense of balance between researchers and teachers, which later has been recognized by Stephan (2015) to be an important factor to highlight teachers' unique knowledge in design research collaborations. The work was organized as such that after an introductory meeting, the two researchers drafted a design proposal in line with the requested topic. That design was further developed by a discussion within the team, which was video recorded, and then initiated by one of the teachers in the classroom. Minor adjustments were carried out between the two teachers' lessons and major changes of the lesson sequence were carried out after both teachers had used each lesson plan. A total of 5 lessons were designed, although one of the teachers divided the last lesson into two because of time management issues.

The activity and hypothetical learning trajectory

The task design originated from a teaching experiment by Brousseau, Brousseau, and Warfield (2001), where the students were asked to investigate a chance event with an unknown sample space. The aim was to introduce basic principles of the Law of large numbers from probability as well as a frequency perspective on probability theory. We used an opaque soda bottle containing an unknown amount of small coloured balls (neither the students nor the teachers knew the content of the bottles) during the first lesson. When the bottle was turned over, the colour of one ball was revealed while remaining inside the bottle. Thus, creating a constant but unknown sample space. The activity was presented as a race in the first lesson with three contestants on a six-step track. As one of the three

colours was observed on a bottle turn, that colour advanced one step down the track. The students were asked to guess which colour would first get six observations during each race. Based on the topics discussed by the students in the first lessons, the following three lessons made use of a transparent bottle with a visible sample space. Here the students were asked to discuss chance, random variation, sample space, sampling and the law of large numbers. The importance of the sample space was highlighted in the second lesson because of ideas discussed in the first lesson. The students got to return to the opaque bottle in the last lesson(s) and again, in an organized manner, investigate the unknown sample space from the first lesson with the use of the law of large numbers. By producing a large enough sample, they could reason about the sample space in the opaque bottle by translating the relative frequency of each outcome into the probability of that outcome. Overall, one class needed a total of five lessons and the other class six lessons, to reach an agreement about the unknown sample space in the opaque bottle.

Method of analysis

The analysis of the transcript in the forthcoming section is inspired by retrospective analysis from the design research methodology. It is based on open inquiry and constant comparison (Glaser & Strauss, 1967) where you retrospectively analyse and compare small instances of data from the whole set with one another to gain insights into the processes (Gravemeijer & Cobb, 2013). Trustworthy accounts of possible meanings can be developed by immersing oneself in the social setting, using participant observation, alongside systematic coding of data in retrospective analysis (Cobb, Stephan, McClain, & Gravemeijer, 2001). Instances of active contribution to the negotiation, utterances by the teachers or researchers, are coded with open codes. These instances are then compared to each other to find differences and similarities in their actions as they indicate the participant's interpretations, aims and agency. When looking at longer sequences, patterns are sought after, especially how participants' aims and agency influence the development of the negotiation. Short excerpts used for constant comparisons are presented in the text and expanded upon. Tied to those excerpts are expansions on the continuing interactions not shown in the transcripts due to space limitations.

Results

The excerpts from the transcripts presented here are all from a meeting with the two researchers and the two teachers between Karen's first lesson and Tilly's first. The purpose was to engage in a mini-cycle to evaluate the initial design and revise it before Tilly used it in her classroom. One of the researchers, the author of this paper, was present during Karen's lesson and the discussion utilized their experiences as a main source of data to analyse. In the first episode, the two teachers discuss Karen's experience during the first lesson. She had asked the students to reflect on the notions statistics, chance and probability at the beginning of the lesson and then proceeded to carry out the design outlined here in an earlier section. Notice how the focus shifts from being about implicit aspects of the activity to being about students connecting knowledge.

Karen: What fascinated me was that their engagement induced the use of the concepts that we highlighted and reconnected to what we did at the beginning. Statistics, chance and probability, well, that Kim said "This is what I think! Statistics is what

we're doing, and the balls drop by chance but you may still calculate the probability". He started…

Tilly: He added that, you didn't guide him?

Karen: No, he was like "This is what I think, I figured out this with statistics". So, it kind of extracted their knowledge.

Tilly: They latched on on the correct incident somehow.

Karen: Exactly, and they could use the concepts to describe it, what we had done.

Karen acts with an interest in the activity as an eliciting factor for a student's development of concepts. Tilly indicates that she is interested in how Karen carried out the activity in the classroom and both act as if their main aim is to further develop the task. Tilly then subtly indicates that she has shifted her focus towards the students' learning process in the second utterance. She acts as if her interpretation of the aim has shifted towards the global aim of contributing to theory on students learning processes. The negotiation takes off in another direction, initially being about gaining understanding of the design aspect towards being about understanding aspects of learning. The global aim of understanding the students' learning in respect of making connections as well as developing language is pursued long after this extract ends. The following episode picks up this chain of events further into the meeting. Karen admits that the development of the lesson had made her unsure of how she interpreted the three concepts statistics, chance and probability. We start off with her reading her own notes from what she found out from a dictionary after the lesson.

Karen: "Not be able to calculate in advance. Statistics, summarization of information, nah, Probability, the chances of getting" for example blue.

Paul: So what it becomes, Heads or Tails, aren't known in advance. Is that what you mean?

Tilly: Why did you pose that question like that? What were you thinking? Since you do this professionally… Why did you ask that question?

Paul: Well, because I thought the sentence was incomplete. "we calculate in advance", I just wanted to emphasise … what is it we can't calculate in advance?

Tilly: Aaa, okay

The initial statement was copied from a dictionary and Karen acts as if she trusts and places the authority within that community. At least in the case of chance and probability, she relies on the dictionary and achieves agency with the use of it. The group's prior negotiation of meaning of statistics makes her less confident in the case of statistics; she indicates that she gives primacy to the group's interpretation. Paul questions this authority altogether. Tilly seems to pick up on Paul's questioning and probes the nature of Paul's agency; is he acting from a mathematics education researcher perspective, or a mathematical community perspective or something else? Paul continues by acting as if there are better interpretations of these concepts by questioning the wordings. He later on continues to negotiate the meaning of these concepts by means of examples and more mathematically precise definitions and thereby achieves agency by referring to the mathematical community. This exchange impacts the negotiation towards being even more focused on language.

It also becomes apparent that Paul's agency influences how Karen and Tilly use technical terms in the remainder of the session. Paul's aims and agencies remain in focus as they are given more space and remain unquestioned as the work progresses. The following episode is from the later parts of the meeting. It shows the impact of Karen's and Tilly's interaction in the first episode regarding the topic but also how Paul is left as an authority. Notice how Paul remains unopposed even though his claims and sentences are incomplete just as Karen's were in the previous episode.

Paul: My question about statistics springs from that fact that statistics is a rather large subject… A large topic so to speak

Karen: Mm

Paul: And I think that the curriculum sort of… Even in our … our curriculum contains what this student is saying about the ratings of a TV-show. That you compare… often just think about observations, or we usually say frequencies, frequency tables and so on, so you limit the whole field of statistics to what one might call data collection, frequencies and such.

Tilly: Mm

Paul: What do you think, we could think about beginning to establish this type of concepts like… What is the frequency of blue? What is the frequency… How many observations of blue? How many observations of red? So, you insert this type of technical terms to become more precise. Specify a little bit more. That is, I imagine, a part of learning, that you learn to… You use a language and start to become a little bit more precise.

Paul acts as if he wants to shift focus to the local aim of the activity. He offers an alternative, or additional, learning goal in students developing their language through the activity. He pushes his agenda by referring to the authoritative mathematical community and therefore achieving agency. Both Karen and Tilly accept Paul's agency and leave his claims unopposed and instead adjust their use of language after the episode to fit Paul's. Paul's agency also results in a shift of focus in the following interaction. The continuing negotiation still involves interpretations of language but also aspects of the design and how language development can be anticipated and sequenced in a HLT.

To sum up the analysis, I exemplify how rearranging the aims of the activity, placing the aim of understanding students' learning in the foreground instead of the aim to develop the design, can have huge impacts on the course of the developmental process. Karen's and Tilly's interaction in the first extract refocused much of the remaining discussion towards negotiating the meaning of language use and development. An example of the dance of agency in designing educational activities has also been offered. Especially how some agency has higher authority and thus also more impact on the negotiation process.

Discussion

Shifts of agency emerged in the presented episodes. Karen and Tilly both seemed to mostly rely on personal agencies in their actions, trusting in their professional experience in teaching for learning and language development. Paul on the other hand was perceived to rely on agency achieved from highly regarded disciplines as the mathematics and mathematics education research communities

when he contributes with examples and language. It corresponds with results from when Pickering (1995) studied mathematicians in their work. He saw that mathematicians often relied on their personal agency as they created initial ideas but "surrendered" to that of the discipline, as they needed to resort to following standard procedures of (for example) proofs. A similar phenomenon was observed in the episodes. Karen and Tilly relied on their personal agency until Paul referred to the mathematics community; Karen and Tilly then surrendered to the formalized language of the discipline to a greater extent. Further on it resulted in worksheets handed to the students using this formalized language and thus re-evaluating the HLT in light of formal use of concepts. I suggest that it emphasizes how the dance of agency (Pickering, 1995) is a principle to regard in collaboration between researchers and practitioners as it has the potential to be an indirect tool to guide the negotiation process in different directions.

Shifts in aims emerged in the data. Karen initially acts in line with the global aim of developing aspects of the design but soon shifts to align with Tilly to negotiate the meaning of a student's language development. Paul later uses the mathematics community to achieve agency to shift focus towards developing the design once again and then sits back to evaluate its impact. Blumer (1986) argues that not only individual aims should be regarded but also that of the group. When multiple actors interact over a period of time a joint action is formed. It is a social construct that extends from being merely the sum of all actions, for example it also has its own aim that is negotiated by its participants. One might look at Karen's, Tilly's and Paul's shifts in aims as attempts to negotiate what aim should be in focus or in the foreground while the others remain in the background. It becomes apparent in the topics discussed that the negotiation of aims for the joint action has impact. In the extended data, patterns emerge of how shifts in the joint aim redirect the following interaction until it was renegotiated. The result was an added activity in the HLT that was meant to challenge students' ideas of chance. Reasons for why the participants chose to contribute in this way did not emerge from the existing data but one could speculate whether it has something to do with the respective role of the actors, being from a tradition of research or education. It presents a way forward to further advance our insight in the collaboration between researchers and practitioners in design research in mathematics education.

Implications

As Stephan (2015) highlights the importance of working with small groups of teachers instead of just one in design research collaborations, the dance of agency is yet another tool to create purposeful design research collaborations. In the case of Paul, agency is used to steer the negotiation of meaning towards more mathematically aligned use of vocabulary. In another setting, it is possible to rely on shifting agencies to empower teachers in the collaboration by placing it in the domain of mathematics teachers. There is also the possibility to put focus on the research agenda when discussions tend to steer away. One could also consider the interpretation that it has the potential to shift the power relations, creating an (even bigger?) imbalance between participants, making participants less likely to make substantial contributions to the development of the task. The conclusion is that conscious achievement of agency can be used as a tool for researchers in design research collaborations to manoeuvre the discussion and shape the HLT to fulfil different aims.

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Factors impacting on the collaborative design of digital resources

Nataly Essonnier[1], Berta Barquero[2], Christian Mercat[1], Mohamed El-Demerdash[1, 3], Jana Trgalova[1], Mario Barajas[2] and Pedro Lealdino Filho[1]

[1] Claude Bernard University Lyon 1, S2HEP EA4148, Lyon, France; nataly.essonnier@etu.univ-lyon1.fr, christian.mercat@univ-lyon1.fr, jana.trgalova@univ-lyon1.fr, m_eldemerdash70@yahoo.com, pedrolealdino@gmail.com

[2] University of Barcelona, Barcelona, Spain; bbarquero@ub.edu, ma.barajas@gmail.com

[3]Menoufia University, Egypt

In order to identify mechanisms that can support mediation, this paper analyses the decision making process in a collaborative design of a digital learning resource by two different Communities of Interest (CoI). It focuses especially on the influence of both the CoI contexts and the socio-technical environment. This research was carried out within the framework of the "M C Squared" European project aiming at studying social creativity in the resource design. Specific conceptual and technical tools were used in this project to ease and document social interactions in the design of innovative learning resources promoting Creative Mathematical Thinking in the users. We focus on two main forces: tools and culture, which supported the collaborative design work between two CoIs.

Keywords: Community of Interest, context, socio-technical environment, collaborative task design, Creative Mathematical Thinking.

Introduction

This paper focuses on the analysis of a collaborative design of an innovative kind of digital educational resources for teaching and learning mathematics by different teams of designers. This research took place in the frame of the European Research and Development project called "M C Squared (MC2)" (http://mc2-project.eu/) where innovative digital resources have been produced to promote creative mathematical thinking (CMT). These resources have been designed by four Communities of Interest (CoI) (Fisher, 2001) constituted within the project: the English, French, Greek and Spanish CoIs. One of the objectives of the project was studying the processes of social creativity occurring during the design of resources and uncovering factors fostering it. Moreover, as the design was carried out in four different countries, the question of the influence of the cultural and institutional context on the design choices, as well as on the processes of social creativity, was raised naturally.

In this paper, we focus on the design process that involved a collaboration between two CoIs, the inter-CoI interaction being considered as a window on contextual issues impacting the design. We report the case of a resource called "Limits" that was initially designed by the French CoI members, redesigned by the Spanish CoI, and finally redesigned again in the cross-CoI collaboration between the two CoIs. In this framework of a collaborative design of a resource, we explore the influence of the context and of the conceptual and technical tools on the design process. In other words, we are particularly interested in *how the CoI context influences the design process in a given socio-technical environment* and *which tools and mechanisms support the collaboration between different teams of designers in the process of task design.*

The paper starts by presenting the context within which this research was carried out and its theoretical and methodological background. The design of the "Limits" resource is then described and analysed and the findings are discussed bringing to the fore elements of answers to the research questions.

Context and socio-technical environment of the CoI

Communities of Interest (CoIs) and their context

According to Fischer (2001), Communities of Interest "*bring together stakeholders from different CoPs* [Communities of Practice] (Wenger, 1998) *to solve a particular (design) problem of common concern*". Four CoIs were constituted in the MC2 project gathering together, around a digital resource design, mathematics teachers, teacher educators, researches in mathematics education, educational software designers, artists, etc.

The French and the Spanish CoI, whose experience is reported in this paper, present different compositions and characteristics; we consider these as contextual aspects. The French CoI consists of 13 members with varied professional background, including researchers, school teachers, teacher educators, and educational technology developers. They share a socio-constructivist approach to mathematics learning rooted in the French didactical tradition of teaching and learning mathematics (CFEM, 2016). This approach has shaped the CoI representation of creative mathematical thinking (CMT) that manifests itself through (implicit) task design principles, such as designing tasks aiming at revealing specific students' misconceptions, using multiple representations to enhance conceptualisation of mathematical notions, fostering social aspects through collaboration between students and affective aspects through challenging problems and games, or focusing on tasks calling for generalisation. The Spanish CoI, composed of about 20 members, involves people from different communities of practice, including researchers in and out of mathematics education, secondary school and university teachers and publishers. Most of the resources designed by the Spanish CoI present many design principles that are especially important for mathematical modelling, such as proposing real questions to students in order to face linking mathematics with other disciplines (social sciences, history, etc.), articulating questions posed and mathematical tools to engage students in modelling processes, enhancing the exploration or the contrast and validation of mathematical tools and models.

The socio-technical environment and collaborative design

The design of resources took place within a specific socio-technical environment developed in the MC2 project, called C-Book technology (http://mc2dme.appspot.com/mcs/). It integrates two main tools: i) an authoring environment enabling to create digital resources, called c-books ("c" for creative), which consist from pages including texts, pictures, hyperlinks, dynamic interactive widgets, and allowing to record successive versions of the c-book units; ii) a tool, named CoICode that provides a workspace to organize and enhance interactions among designers. CoICode enables each designer to post various kinds of ideas ("contributory", "alternative", "objection", "off task" and "task organization"), each of them having a specific icon. When a designer posts an idea, the system captures several details: author's name, date, title of the idea, comments, attached resources, hyperlinks, etc. The CoICode system provides designers from a CoI or a CoI-pair (two collaborating CoIs) with a space for collaborative design. In CoICode, the discussions can be visualised in form of threaded forum or in a mind-map view (Fig. 1), where nodes are ideas, and branches of the tree model

the evolutions of an idea. The reports in the form of a graph provided by the system are the main data gathered for the study of social creativity. A voting system has been implemented in the CoICode allowing designers to evaluate in terms of creativity any idea posted by someone else. Such evaluation follows a "*middle c*" perspective of creativity (Moran, 2010), that views creativity as a competency developed through interactions between members of a community and through their participation in situations where they display their intentions and negotiate new alternatives for the interpretation of actions in situated activity systems.

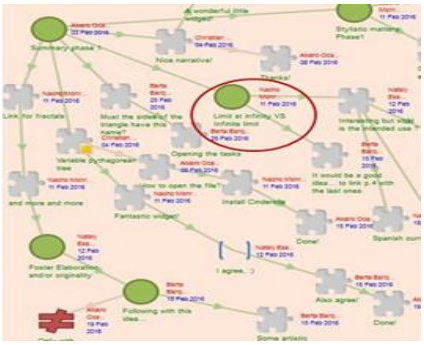

Figure 1. Excerpt of a CoICode workspace in the mind-map view.

The cross-CoI collaboration on the re-design of the "Limits" c-book was organized in the following five phases: (1) a part of the French CoI, acting as the primary designers, designed a first version of the c-book; (2) four members of the Spanish CoI (two secondary school teachers, one researcher in mathematics education and one researcher in Calculus) evaluated the CMT potential of the c-book; (3) these members of the Spanish CoI redesigned the c-book according to their own approach, which constituted the first phase of the redesign; (4) a second redesign phase was carried out by the CoI-pair comprising this Spanish sub-CoI and two members of the French CoI (one researcher in mathematics education and one secondary school teacher); and (5) four new members (two from each CoI not involved in the redesign) evaluated the CMT potential of the redesigned c-book.

Theoretical and methodological background

Documentational and boundary crossing approaches

Our focus on the genesis of the c-book resource leads us to adopt the *Documentational Approach to Didactics (DA)* (Gueudet&Trouche, 2009) and thus consider the design of this resource as a documentational genesis. The analysis of resources coming into play in this genesis and of their successive versions unveils designers' mathematics knowledge, CMT representations and culture. In addition, considering the collaboration between two CoIs, which can be viewed as two different activity systems, allows inferring the influence of the contexts on the design choices.

The *Boundary Crossing approach* (Akkerman& Bakker, 2011) enables enlightening the interactions between these contexts. It allows to highlight discontinuities, i.e. boundaries. Boundary objects (Star & Griesemer, 1989) and brokers support the communication and the understanding between and within the CoIs, allow to build new norms and a common frame of reference. Moreover, highlighting the mechanisms of *identification* (i.e., consciousness of discontinuities, awareness of multifold cultural background, which allows pointing out differences), *coordination* (i.e., creation of continuities between domains and bridges between cultures, which enables the construction of a common frame of reference), *reflection* (i.e., perspective making, perspective taking on the problem

at stake, which supports divergent thinking), and *transformation* (i.e., confrontation, recognition of a shared problem space, hybridization or combining ideas, and crystallizationor keeping a perspective, an idea) helps us to better understand the design process.

Grid for the evaluation of c-book features fostering CMT

The evaluation of the potential or affordances of a c-book to foster CMT was a central task in the MC2 project. Facing the necessity of CMT cross-evaluation, a need emerged for agreeing on and sharing common criteria, tools and methodologies, which had been developed independently in the first cycles of c-book production. A common CMT evaluation grid, which combines design criteria or principles proposed by the four CoIs involved in the project, has been elaborated by the researchers. This grid could be adapted by each CoI or CoI-pair to better fit its context, by adding specific criteria, and played a crucial role in the construction of a common frame of reference for all four CoIs.

The CMT evaluation grid is a questionnaire composed of three sections. The first and the widest section focuses on the evaluation up to what degree different dimensions of mathematical activity considered crucial for fostering CMT, such as conjecturing, questioning, evaluating, and establishing connections, are taken into account in the c-book design. With a total of 14 items expressing the indicators of different dimensions, evaluators of a c-book grade (from 1-4) their agreement on the items and explain their response according to the design being evaluated. For instance, the dimension of establishing connections is evaluated through the item: “The c-book provides users with opportunities to establish connections between various representations of the mathematical concepts at stake”, or the validation dimension through the item: “The c-book stimulates to think about, reflect, summarize and evaluate the mathematical work already developed”. The second section addresses social aspects through items like: “The c-book stimulates user's collaboration / cooperation / interaction with other users”. Finally, the third section focuses on affective aspects via items like: “The c-book actively promotes engagement by generating a perception of usefulness of mathematics, either in everyday life, or inside the mathematical context”. This grid, filled in for each c-book, provides the basis of the CMT study and development.

CoICode analytics features

In the MC2 project, a creative idea is defined as: (1) *novel* (original, unusual or new for the CoI members), (2) *appropriate*, that is it conforms to the characteristics and functions of the c-books, including their CMT affordances, bind to the CoI context, and (3) *usable*, that is available and ready to be used in the design of the c-book according to the designers’ (the CoI members’) estimation (Daskolia, 2015). CoICode voting mechanism allows any CoI member to express his/her opinion about the three attributes of any idea posted by any other CoI member. The expressed opinions are aggregated into the creative score of an idea defined as follows: “creative score of the idea i (CRi) = 0.5 x number of ‘novel’ votes + 0.25 x number of ‘appropriate’ votes + 0.25 x number of ‘usable’ votes, if the number of ‘novel’ votes is at least a half of the number of CoI members involved in the c-book design, otherwise CRi = 0”. This definition reflects the fact that novelty is the sine qua non condition for an idea to be deemed creative; this is why the corresponding weight is the highest (0.5). On the other hand, the “middle c” perspective of creativity leads to considering an idea creative if the majority of the CoI members share this opinion. Thus, the interactions recorded in CoICode allow

tracking communication among the designers during the design process and getting automatically the ranking of the ideas expressed according to their creativity score (Table 1).

USER	DATE	ID	TITLE	NOVEL	APPROP	USABLE	SC SCORE
CM	04/02/2016 11:03:51	45675	Variable pythagorean tree	4	4	4	4
NE	12/02/2016 11:01:13	45901	EpsilonChat to foster social aspects	3	3	3	3

Table 1. Quantitative measurement to identify creative ideas.

Data collection and observables for each phase

The ideas and their organisation in CoICode workspaces, the creativity score of ideas obtained automatically from CoICode, the CMT grids filled in by the evaluators of the c-book and the successive versions of the c-book constitute the main data we analyse in order to highlight the impact of context, and cultural evolutions on the design decisions taken, as well as the role of the tools in the design process.

c-book design process in the cross-CoI collaboration and its analysis

Our analysis focuses on two out of the five phases of the redesign of the c-book "Limits" (see above), namely phase (3), when the Spanish CoI redesigned the c-book and the phase (4) when the CoI-pair worked collaboratively on agreeing upon and conceptualizing the last changes of the redesigned c-book. We have chosen these two phases of the redesign process as they appear especially important with respect to our research questions.

Adopting the c-book and de- and re-contextualizing its design: mechanisms of coordination and reflection

The initial version of the c-book "Limits", designed by the French CoI, covered the notion of infinity through its meaning in solving equations, constructing the Pythagorean tree, analysing geometric sequences, comparing growth of functions, and calculating limits of real functions. The CMT representation of the French CoI members shaped the design of the c-book. In particular, it led the designers to embed tasks that enable intra-mathematical connections, generalisation, competition and challenge as levers for the CMT development. Following these principles, they proposed tasks offering various representations of the mathematical notions at stake (limits and infinity), by using algebraic, calculus, and geometrical settings, with the aim to provide students with alternative ways to make sense of these difficult notions in calculus and to generalise some properties. Moreover, the educational technology developers, involved in the CoI, enabled the development of specific widgets with features deemed as important to foster CMT, such as relevant feedback, written collaboration and discussions (a chat tool), and a framework for designing playful activities affording students' self-assessment. Hence, the involvement of software developers in the designers' team impacted the c-book design by creating new widgets in line with the French CoI culture. They also worked in close collaboration with the C-book technology developers, thus playing the role of technical brokers within the CoI.

As soon as the phase (3) started, the Spanish CoI began with the redesign of the c-book.The designers structured the workspace dedicated to the intra-CoI redesign work according to the results of their CMT evaluation with the grid (see section 3). For instance, they found the c-book improvable

regarding the connections that could be established with other disciplines or with other mathematical topics. They appreciated some characteristics of the c-book such as *connections* between several representations (numerical, geometric, and algebraic) of limits or the potential of the new widgets to simulate functions, sequences, limits, etc. and their practical use in activities focusing on evaluating students' work and progress. The decision to maintain these features can be interpreted as the agreement on the underpinning design criteria by both CoIs. They also detected several traits to further improve the c-book redesign, some of them being central for their own CMT representation; for example, they missed situations and questions that give sense and utility to the mathematical notions at stake (infinity, limits, etc.) – *questioning* or *problematisation*. Likewise, they missed a global *articulation* of some of the activities dealing with a more general narrative and questions to focus on. This led the designers to organize the CoICode workspace around the eight design criteria or indicators they considered as crucial to be prompted (validation, connections, articulation, problematisation ...) according to their CMT approach (Fig. 2, first column on the left) to orient further discussion. Therefore, the redesigned c-book urged students to investigate questions like the ones about fractal constructions and properties (guiding part 1 of the redesigned c-book), or the one about a cell phone password as the problem of the 9 points (guiding part 2), and to engage students in recognizing patterns in the process of mathematization of a problem, and in using the corresponding mathematical relations to check the validity of a conjecture.

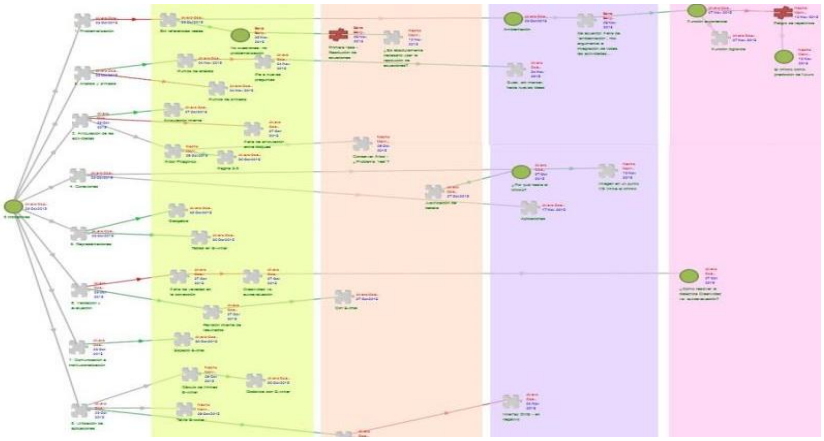

Figure 2. Excerpt of the workspace created for the intra-CoI redesign.

In this episode, we can identify a mechanism of coordination initiated by the CMT evaluation which supported the subsequent mechanism of reflexion sustained by the structure given to the workspace. We note that the Spanish CoI instrumentalized CoICode, with a strong purpose of enhancing the c-book potential to foster CMT in students in line with their culture. Hence the mechanism of reflexion enabled to open new perspectives, related to the Spanish context, by adding tasks on fractals, the problem of 9 points and the mathematization of another problem.

The CoI-pair collaboration in the c-book redesign: mechanism of transformation

The cross-CoI collaboration (phase 4), started with the translation of the redesigned c-book into English and the creation of a new workspace common to both CoIs. In order to organise and facilitate the communication between the two CoIs, the workspace was structured according to the four main sections of the c-book, and a summary of the main aims and changes introduced by the Spanish CoI in each section was added; the French team could thus compare the new version of the c-book with

its original design. During the CoI collaborative work, some design principles stemming from both CoIs were recognized and discussed to progressively become shared by both CoIs (confrontation and crystallisation of principles), such as the importance of tasks calling for conjectures, simulation, communication of results, and validation. Other design choices issued from the Spanish CoI were accepted by the primary designers of the c-book, such as the extra-mathematical connections included in the c-book or the new way of structuring and articulating activities in terms of chains of interrelated questions with increasing complexity. Furthermore, the quantitative information provided by the CoICode data analytics, in particular in terms of creative scores of ideas (see Table 1) appeared as a powerful tool to identify ideas worth to be further developed in the CoI-pair collaboration. The two ideas that obtained the highest creativity scores came from two comments made by two members of the French CoI while analysing the c-book redesigned by the Spanish CoI. The first idea was related to the first part of the c-book devoted to the study of fractal properties and the appearance of the notion of limit at infinity. A French CoI member provided a link to a widget he designed with Cinderella dynamic geometry system to simulate fractals and predict their tendency in the infinity (Table 1, idea n°45675). The widget was subsequently integrated at the end of this first section with new questions that CoI-pair members suggested. The other idea was suggested by another French CoI member concerning the possibilities embedded in chat tools, developed within the French CoI, to foster social aspects (Table 1, idea n°45901).Chat tools were subsequently integrated in the c-book to enable students to communicate their results or to pose new questions. These episodes can be interpreted as hybridisation and elaboration of ideas.

Besides the importance that this phase had on creating a new and common CoI-pair design context, our analysis shows that the CMT grid, the CoICode workspace the creative scores of ideas and the c-book versions used as boundary objects between the two CoIs constituted key meditation supports to enable the designers to agree on which ideas to accept (or not) and on the ways of further elaboration of some of these ideas, thus sustaining the mechanisms of coordination, reflection and transformation.

Discussion and conclusion

The analysis of the c-book “Limits” collaborative design shows that different CMT representations that both CoIs held, influenced by each CoI own culture and traditions, enriched the cross-CoI collaboration, acted as a boundary object and participated in the key mechanism of coordination for decisions making in the intra-CoI and cross-CoI design work (Barajas, 2016).

In the two phases of intra-CoI and cross-CoI work (phases 3 and 4), it appeared that redesigning does not mean a total transformation and complete re-contextualisation either of the initial unit, of the empirical setting envisioned or of the academic approach (Barquero, Papadopoulos, Barajas & Kynigos, 2016), but rather an improvement of some aspects, and it helped to establish confidence and trust atmosphere. On the contrary, some design principles shared by the two CoIs were reinforced, crystallised through the mechanism of transformation, such as connections between multiple representations, or making and investigating conjectures. Others, coming from only one CoI, were negotiated and became shared by both CoIs, such as extra-mathematical connections or interrelations between the c-book activities, yet others were abandoned. The CoI-pair created a new, wealthier design context thanks to two different cultures close enough to create some overlaps yielding a common frame of reference, which enabled to build understanding and fostered the mechanism of

coordination. The understanding and respect allowed to share design decisions with the help of mediation tools used as boundary objects (different versions of the c-book, CMT grid, creative scores of ideas) and to cross some boundaries (different CMT representations, school cultures, research approaches, distance collaboration). The mediation tools favoured the dialogue between the CoIs and facilitated decisions, in conjunction with common good practices and CoI moderation strategies. The workspaces in CoICode used as meditational artefact were instrumentalized to support reflection, enabling to make points of view explicit (perspective making) and to enrich ideas (hybridisation). The boundary objects, the structure of the workspaces and the moderation strategy played a major role in the mechanism of coordination, reflection and transformation.

This study brings to the fore two main forces that shaped decision making in the design process: *tools* and *cultures* (Fig. 3).

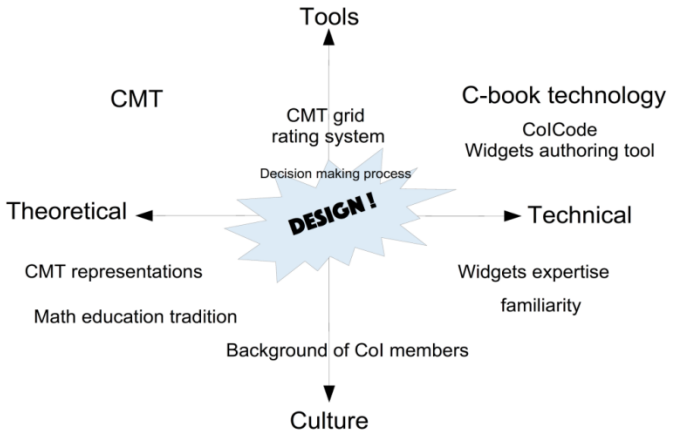

Figure 3. Main forces shaping the decision making in a c-book design process.

Both had either *theoretical* or *conceptual dimensions*, for example the CMT evaluation grid built on theoretical considerations about creativity, but they have *socio-technical aspects* as well because the C-Book technology, comprising authoring tools, widget factories and CoICode, is the MC2 project social management main tool. The cultural context of the CoI includes mathematics education theoretical tradition, composition of the CoI, familiarity and expertise with the variety of widgets. The background of the designers impacts their attitude towards these tools, their CMT representations, their position in the collaborative decision making and the widgets they use. Eased by the proximity of collaborating cultures, the interplay of culture and tools in cross-CoI collaboration had enriched the scope of the designed tasks.

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Programming for 3rd graders, Scratch-based or unplugged?

Aaron Gaio[1]

[1]University of Palermo, Department of Mathematics and Computer Science, Palermo, Italy;
aaron.gaio@dmi.unict.it

In this paper we describe a comparison between two different approaches to teach some algorithmic and computational thinking to children, mainly in 3rd grade. Children's learning is taken into main consideration and we want to analyze the difficulties students encounter using the different approaches. Before that, an introduction is done, describing the research framework and methodology, offering the background for this research and outlining the larger research project from which the paper is derived. We then describe the tasks used and look at some examples of the difficulties children face, on one side dealing with the problem of abstract thinking while programming, and on the other having troubles relating more practical activity with what the calculator does.

Keywords: Primary education, curriculum, computer science, programming, algorithms.

Introduction

Computer Science and algorithms in education are gaining more and more importance as the use of digital technologies is nowadays part of everyone's life. New educational trends are therefore emerging both from the computer science and mathematics education research community and from elementary and secondary school teachers (Franklin et al., 2015; Richtel, 2014). The question about how young children learn computer science is still a new area of research; and providing effective learning opportunities to K-5 students is a big challenge (Hills et al., 2015; Gelderblom & Kotze, 2009). Some good examples have been tried in the secondary school, while we feel that not much is present, at least in our country, in lower school grades. Topics in computer science and discrete mathematics are not clearly delimited in our curriculum and teachers are usually not aware that they actually could. We are thinking of our work as able to enhance the study of teaching and learning skills of mathematical practice through discrete mathematics problems, both general skills, such as reasoning and modeling, and skills particular to discrete mathematics, such as algorithmic and recursive thinking.

Background and context

Preliminary survey among teachers – relation with cryptography

We had a first survey, with results collected from about 150 teachers, mostly in service and quite evenly divided between primary, middle and secondary school. The survey was done with an online platform. The result analysis is mainly following a quantitative approach, the qualitative analysis was referred to the codification of some particular key words used by teachers.

Analysing the results, teachers, especially at lower levels, admit not to have the necessary knowledge to teach this in school. Question was about their previous experience in learning cryptography and graph theory, as well as the connection of these to mathematics and computer science. Some of these teachers see the connection between algorithms, cryptography and mathematics in general and computer science as quite necessary, while some don't have this idea clear in their mind. Also,

teachers were asked if they had any previous experience in teaching the topic or if "they would be interested in teaching some algorithm, cryptography and other discrete mathematics topic to students", and feeling from their answer is that this results can be taken as a first starting and promising point to make something of this into the national curriculum. A detailed analysis of these results is available in another article (Gaio & Di Paola, 2016, in press).

National Guidelines and teaching situation

The Italian Ministry for Education, University and Research published the current National Guidelines for the first cycle (kindergarten to middle school) of education (Ministero della Pubblica Istruzione, 2012). These guidelines are not any longer a detailed description of school curriculum to follow, but just want to provide concepts from which the single schools and institutes, and teachers, can take the basic goals and competences to reach. Some general standards are set with objectives for the educational achievements and learning goals. In the section talking about mathematics, there is a great importance given to reading and understanding texts with logical content, build lines of reasoning, having own ideas and defending and comparing them with others; a positive attitude towards mathematics, realizing how mathematical topics are useful in the real world. Algorithms and logical thinking as also referred to as important in the technology chapter of the guidelines, for all school grades. Following these guidelines, and our idea as well, "the first education cycle has a prominent role in the school curriculum considering the importance of this time in every student's life. Within this, the school attributes great relevance to the education and teaching methods that can fully activate energies and potentialities of every kid".

Research question

Our main general research problem lies therefore in a proposal to alleviate the substantial lack of activities in the national school curriculum about discrete mathematics and computer algorithms, especially for primary and middle school. Both in the school programs and in textbooks, activities of this kind are missing almost entirely, despite many agree that they can be really useful to improve the skills mentioned above.

The purpose of this specific paper is to deal with the introduction of programming reasoning to children as young as 8 or 9 years old. The question is whether *it is better to approach the subject with an unplugged approach and only later go on with computer-based coding or if it is ok to proceed using Scratch-based software and tools to serve the same purpose*. We do this by describing two different approaches which have been used in the teaching of these computer science and discrete mathematics topics. We will in particular analyze and focus on certain difficulties students encounter while using both.

Theory and methodology

This is an overview, referring to our whole project's methodology and background theory.

Background theory

Teaching methods follow the model of Realistic Mathematics Education (Gravemeijer, 1994) and Guided Reinvention of mathematics (Brousseau, 1997).

Guided Reinvention of mathematics is based on Hans Freudenthal concept of mathematics as human activity. Education should give students the "guided" opportunity to "re-invent" mathematics by

doing it. This means that in mathematics education, the focal point should not be on mathematics as a closed system but on the activity, on the process of mathematization (Freudenthal, 1973).

Realistic Mathematics Education (RME) is an instructional design theory which centers around the view of mathematics as a human activity (Freudenthal, 1991); "The idea is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible."(Gravemeijer, 1994). The main goal is to develop a local (i.e. domain-specific) instructional theory (LIT) that will allow students to "[invent] the mathematics themselves" (Larsen, 2008). This need two steps: Step 1, in which "students are engaged in activities designed to invoke powerful informal understandings" (Weber & Larsen, 2008); Step 2, in which "students are engaged in activities designed to support reflection on these informal notions in order to promote the development of formal concepts" (Weber & Larsen, 2008).

Research methodology

The methodology we are going to use is that of design research or design experiments (Cobb et al., 2003; Barab & Squire, 2004; Brown, 1992). For the purpose of this thesis, the developmental approach is taken into consideration (Plomp & Nieveen, 2007); development studies function is to design and develop a, research based, intervention (Steffe, 1983) and constructing design principles in the process of developing it. The goal is to explore new learning and teaching environments, to verify their effectiveness; to develop somehow new methods, instruments and teaching actions to further improve in the field of problem solving and logical thinking, using somehow unusual topics as algorithms and cryptography are for primary school students. Doing this the goal is to contribute to the development of new teaching and learning theories, taking into consideration learning processes in the specific situation, with contents and goals clearly defined. Design research is quite appropriate in this situation, as we are facing a brand new experience in an environment that we need to analyze carefully, i.e. on a local scale, considering all the different elements in the learning environment. The intended design experiment will be a classroom experiment in which the researcher (or researchers) will cooperate with the teachers in assuming teaching responsibilities. On one hand, the teacher is a part of the design team and will be a key role in the development and reviewing of the activities, on the other, they have no previous knowledge and need a guide to experiment with this new experience and new content to present.

Design, tasks, analysis and results

As a first design step, based on theoretical framework and literature, two hypothetical learning trajectories were designed for the two different approaches. One approach is Scratch-based, and has been taken from the most popular book on curricular resources about Scratch programming in our country (Coding, DeAgostini publisher, Ferraresso, Colombini, Bonanome, 2014). The second approach was developed by our research team, taking idea and inspiration from the Computer Science Unplugged project (Bell, Witten, Fellows, 1998, 2015 review) and other related sources (Casey et al., 1992), with a development, after a preliminary teaching experiment, to better adapt the activities to the school level and local situation and norms. The two HLTs are taking into account the theoretical framework presented above, both in the choice of tasks (e.g. some tasks are chosen for their RME approach,others for the group and cooperative work students have to do, and so on) and in the way of presenting them to the classroom or students.

The tasks we are going to describe are just some of the many sequences of tasks that were proposed to various schools and age groups during the 2015/2016 school year in the bigger research project. In a design research paradigm (Plomp & Nievenn, 2007), the activities were tried out many times, always with an a priori analysis together with the teachers and with a retrospective look after each lesson.

Schools	Grades	n. of Classes	n. of Students	Approach
1,2,4	3	4	78	Unplugged
1,2,4	3,4	5	80	Scratch-based
3,5	3,4	3	54	Unplugged
3,5	3,4	2	39	Scratch-based

Table 1: Classes involved in different grades, with students numbers and curriculum used

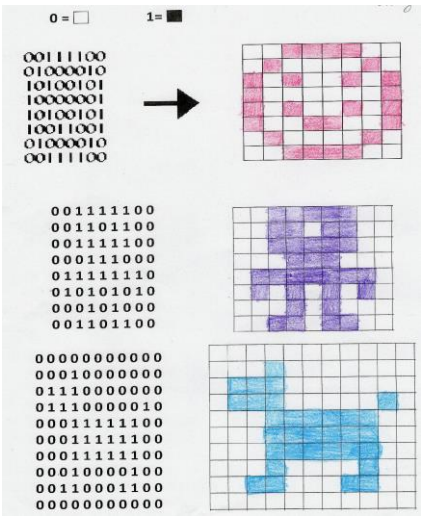
Scratch-based teaching and learning

As mentioned above, the sequence of tasks we called "Scratch-based" is taken from this Coding book, which is getting popular in our country's schools. Also we did use the M.I.T. official Scratch guide (Creative Computing, Brennan, Balch & Chung, 2014). It is following a similar approach to many school text books and even M.I.T.'s own guidelines on Scratch use and we feel it is good material for teachers. We did choose the tasks that are most popular among teachers already doing this kind of activity in their classroom, at least investigating the most popular in our area.

We did in particular choose tasks related to sequencing, selection and iteration. The goal is to have students learn basic ideas behind an algorithm (seen as a sequence of instructions), but also more complex concepts like selection instructions (i.e. do this only if something else happens) or iteration procedures. A sequence of tasks on those three topics were selected together with the classroom teacher and then tried out with the students during mathematics and technology lessons.

Unplugged approach, teaching and learning

Our "unplugged" sequence of tasks occupies 3 or 4 lesson slots of about one and a half to two hours and follows a brief introduction given on how computer works and binary numbers, in form of games (this was given also the groups using the other approach). Briefly describing the tasks, task 1 was an activity on paper, about binary image representation. Students had to color a grid which was provided with 0s and 1s and produce a drawing following the numbers. This task goal was about following instructions and beginning to understand how a computer transmits information.



Task 2 was about giving and receiving instructions. Students were divided in pairs and given a series of shapes and objects they could move on their table. One student (1) for each pair was to create a composition on his table; without looking at each other (physical barrier between the two), student 1 had to explain to the other how to reproduce the same composition with the objects and shapes. Children were required to be as precise as possible while the game went on, and to try to find out compositions that were harder to form. Only oral communication were left them, not to make them "correct" the other mistakes or looking at the other composition. Slightly different versions of the game were tried out, e.g. with just one student giving instructions to all others, or with different kinds of objects, even with just drawing something instead of moving objects, and so on.



The following tasks had the goal to make programming even more tangible for children. We wanted them to learn to give instruction as a calculator, through a path to walk on. One student (blinded) was the "robot" walking along this path on the ground and the others were the "programmers" having to give him instructions how to move to get to the end. We did this both speaking and then written. The written exercise does not give the possibility to correct the robot while you are actually giving the instructions, kind of how a real computer program works. Finally, with some of the classes we went on to construct some more complex sequence of instructions, posing different games to strengthen the concepts, but always with similar goals.



Methods of video selection and analysis

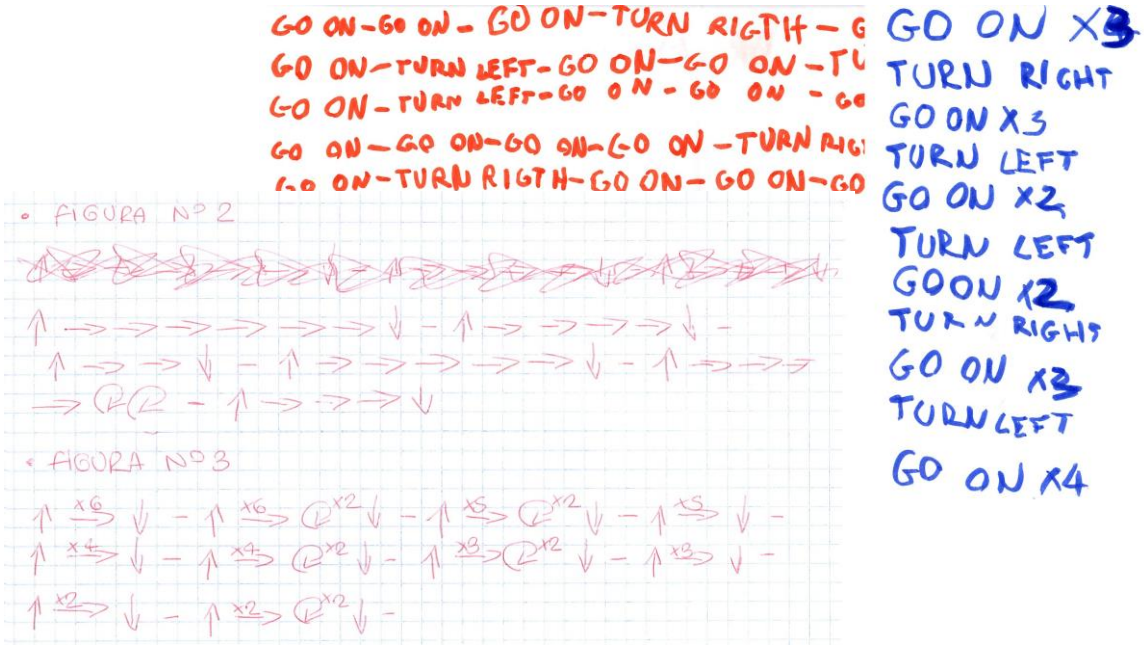
Analysis of the results is video-based, qualitative and fine-grained; both group activities and classroom discussion are recorded and we also have many of the transcripts, together with field notes, student's sheets, and interviews as other sources of evidence. As already said, focus is put on students' learning and thinking, in reaction to the different tasks proposed.

Following Zacks & Tversky theory, data is represented by events selected from the video recordings available. We used an inductive approach in video selecting, beginning with viewing the corpus in its entirety and focus on details later on. Indexing and summaries of videos, plus a content log, help in this process. Going on with the analysis some events which were particularly relevant were isolated. Although there are some recognizable recurring situation and choice of words we coded, our focus is more on a "play-by-play" description of these chosen events. We are, with this approach, analyzing selected episodes focusing on the same happening and constantly revising our finding and new hypothesis, as in Cobb and Whitenack (1996) with the involvement of "constantly reconciling provisional analytic categories with subsequent data and newly formulated categories".

Different difficulties emerging

The "events" we are focusing on does not pretend to show that one approach is better than the other, but which kind of, different, difficulties each of them can create in the children learning and thinking using the different approaches.

In the unplugged approach, children easily figure out what they are really and practically doing, drawing conclusions that they usually do not get in front of the computer. See for example the following figures where the transition from longer instruction in the first part to shorter instruction (switching to an iteration notation) in the second comes automatically. Almost every student quite naturally finds out that it takes a long time to write again and again the same instruction and is quickly asking himself if he might "make it somehow shorter". This is probably due to the fact that they were left free to develop their own language with arrows, and they feel they can adapt it to what is more appropriate and efficient for the situation. Videos showing these moments when they realize this fact has been isolated from the data.



Figures 5 and 6: Showing the transition from sequential instruction to iteration

Or, on the other hand, as an example, see the following short transcript from a video (in front of a Scratch set of instruction on the computer), where the students do not realize the usefulness of shortening a computer program to make it simple and more efficient:

Teacher: Why aren't you writing it in a shorter way (more compact?). You don't need to write an instruction 6 times, you could write "do this … times".

Student: Well, but what's the need for it? The computer is doing it anyways.

In these examples, students doing it unplugged quickly find out they are more efficient if they switch to an iterative mode of giving their instructions, while students doing it on the computer do not really realize this. On the contrary, they should learn one more command (or Scratch block) they do not already know to do it, so in the beginning it does not seem so appealing. From many events observed, quite surprisingly, this shortening is not immediate on the computer. Following our qualitative video analysis, in general, it seems that both selection and iteration instructions do not come naturally in

the Scratch environment as they do in an unplugged approach. Setting the activities and game in a real world scenario, especially at this young age, seems to give children a better idea of the advantages of iteration and the working principles of selection programming. Maybe creating some highly inefficient situation could force this process to happen in this case, too.

A second aspect to consider is errors done while writing a program. Analyzing error situation we can observe that it is actually easier for the students to spot the errors in a Computer-based environment. In the unplugged approach, sometimes, error fixing does not work at all, i.e. they cannot even spot the error if told that there is one. Our conclusion is that, as they are controlling their own game, they, more or less, unconsciously, get to a right solution even with a wrong set of instructions. On the computer-based environment, trying the program and making it running, given that the computer executes exactly what it has been told, students easily spots where the mistake is.

Another aspect we are facing at this young age is abstraction capability. As many references states (see Kramer, 2007), abstraction capability is the key to be a good programmer and to have future programming abilities. Abstraction is a very difficult process and Scratch helps a lot in this direction; on the other side, the unplugged approach makes activities somehow too distant from the abstraction of programming and makes it more difficult to children to relate what they are doing with what they will later do on the calculator, as some video excerpts from these moments show. Aspects of a real world mathematics surely can help the transition to the abstract world of programming (Futschek & Moschitz, 2011), but we have to be careful in the subtle connection between the two areas.

Conclusions

As conclusions, we could see pros and cons of both approaches, and we feel that obsessing over using just one is not the correct decision. Video and other data analysis show that there are aspects that ought to be dealt with in an unplugged way before writing them on the computer (algorithms, iteration processes and many others) and come really more natural to children if they use their own language, as can be seen in many "Aha! Moments" in the recordings. On the other side, it is really difficult for young children to relate the more real-world-oriented tasks to computer science, as we can see example when they get stuck in finding the connections. Future work will try to go into combining both approaches in a more comprehensive curriculum plan, creating new learning sequences that take both into account, which we will share with teachers and educators, in a relatively long developing process. Teachers willing to teach these topics are growing in number and they ought to be prepared for the challenge they will be facing.

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Principles of redesigning an e-task based on a paper-and-pencil task: The case of parametric functions

Galit Nagari Haddif

University of Haifa, Faculty of Education, Haifa, Israel; gnagarih@campus.haifa.ac.il

Paper tasks are often redesigned to function as digital tasks. The research and design literature (Pead, 2010; Burkhardt & Pead, 2003) has reported on the challenges of such a transformation. We report on a study exploring the design principles of an e-task, originally designed as a paper-and-pencil task and converted into an interactive diagram. We describe a paper task in the content area of parametric functions, and report on results from an experiment conducted with 39 high school students, who dealt with an e-task based on a paper task. Analyzing the results, we demonstrate that in a redesigned e-task based on a paper-and-pencil task, technology should allow self-reflection, promote learning, and guide the students to focus on the important details without unnecessary distractions.

Keywords: Design, e-task, parameter.

Goals and theoretical framework

In this paper[1], we explore design principles of an e-task that encourages exploration, based on a paper-and-pencil (P&P) task, in the area of parametric functions, which is central in algebra and is adequate for enhancing the abstraction of concrete situations (Drijvers, 2001). Solving parametric equations is different and more challenging than solving numerical algebraic equations, which are solved for an unknown that is a number. Naturally, when designing an e-task we should not translate from the paper but rather use successful principles of learning within the interactive environment to design the tasks. Research shows that many complex issues arise when transferring paper-and-pencil tasks to computers. For example, if students are not familiar with the tools, the online environment may be a potential source of an additional "cognitive load" (Pead, 2010). Interactivity can spoil some tasks: for example, by allowing students to check all their answers, or by encouraging them to persist in trial-and-error experimentation, rather than engaging in analysis (Burkhardt & Pead, 2003; Nagari Haddif & Yerushalmy, 2015). Although the transition from a paper-and-pencil task to an e-task is not trivial, there may be an added value in the use of technology. For example, multiple linked representations (MLRs) both support and require tasks that involve decision making and other problem-solving skills, such as estimation, selecting a representation, and mapping the changes across representations (e.g., Yerushalmy, 2006).

With the Cabri software, Healy (2000) introduced soft and robust construction and found that despite the intention to encourage students to build robust constructions, in practice, some students preferred to investigate a second type of Cabri-object, soft constructions, in which one of the chosen properties is deliberately constructed by eye in an empirical manner, under the control of the student. Laborde (2005) referred to soft constructions as the "private" side of the student's work, which is part of the solving process and serves as a scaffold to a definite robust construction. We suggest using soft constructions as a way of exploring and identifying dependences between

properties, and as a gateway to a definite robust construction from a purely visual solution. Below we describe a task (Figure 1) taken from Taylor (1992, p. 204), and the reasons for which we decided to redesign it and convert it to an e-task. In general, Taylor's rationale for this kind of task is to have a marked difference between being able to see ("sense") the solution geometrically and the ability to solve it algebraically. Interactive MLR technology offers dynamic interactions that can support the generalization of a graph into a family, offering sensuous support for finding an abstract parametric solution. This gap between interaction and abstraction is one of the challenges of using interactive MLR technology. The following description of the e-task design and of the experiment conducted with the students who worked with this e-task, demonstrates some basic considerations and principles of designing an e-task based on a P&P task.

The original P&P task and its possible correct solutions

Research on mathematicians' conjecturing and proving activity suggests that use of examples plays a critical role both in the development of conjectures and in their exploration, as well as in the subsequent construction of proofs of these conjectures (Lockwood, Ellis, & Lynch, 2016). Therefore, when dealing with the task (Figure 1), we can expect work that would look like Figure 2 (a): the students would sketch for themselves some exemplary lines through the origin.

At the right is the graph of the cubic equation $y = x(x-1)(x-3)$. Consider the family of non-vertical lines through the origin. How many intersections does each line have with the curve? (I) Begin by making a conjecture based on the picture. (II) Describe the family of lines algebraically, and verify your conjecture.

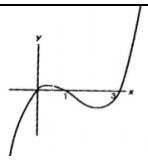

Figure 1: The original task as it appears in Taylor's book (Taylor, p. 204)

In this case, it may be difficult to make a generalization and diagnose the three different numbers of common points: one, two, and three common points between the family of lines $y = mx$ and the given function $y = x(x-1)(x-3)$. Moreover, one could start to investigate algebraically the mutual relationship between the two functions required in part (II) (Figure 1), and skip part (I).

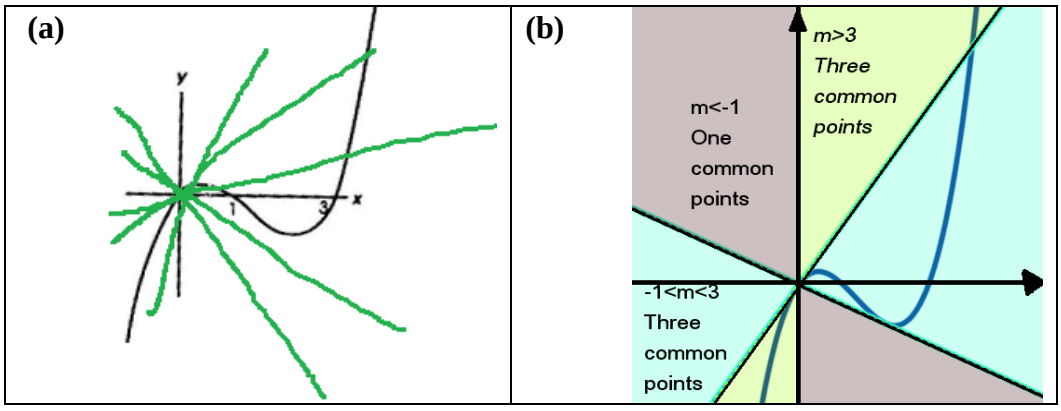

Figure 2: (a) Typical free-hand sketching used to conjecture about the intersections; (b) The domains and values of parameter m for all cases of number of common points

The abstraction and generalization needed to find and define algebraically the domains of the parameter m for each case number of common points (Figure 2 (b)) is a challenge. Generally, solving the case of two common points requires using two approaches, as shown in Figure 3.

The algebraic approach	$x^3-4x^2+3x=mx \Rightarrow x^3-4x^2+3x-mx=0 \Rightarrow x\left[x^2-4x+(3-m)\right]=0$ $x=0$ is one intersection point. Therefore, when the equation $x^2-4x+(3-m)=0$ has single solution: $\Delta=b^2-4ac=16-4(3-m)=0 \Rightarrow m=-1$ The other case is when the equation $x^2-4x+(3-m)=0$ has two solutions, one of which is $x=0 \Rightarrow 0^2-4\cdot 0+(3-m)=0 \Rightarrow m=3$
The calculus approach	$m=f'(0) \Rightarrow f(x)=x^3-4x^2+3x \Rightarrow f'(x)=3x^2-12x+3 \Rightarrow f'(0)=3 \Rightarrow m=3$ In the other case, let $(t, t(t-1)(t-3))$ be the point of tangency $m=\frac{\Delta f}{\Delta t}=\frac{t(t-1)(t-3)-0}{t-0}=(t-1)(t-3)$ $f(t)=t(t^2-4t+3)=t^3-4t^2+3t \Rightarrow m=f'(t)=3t^2-8t+3$ $m=(t-1)(t-3)=3t^2-8t+3 \Rightarrow t^2-4t+3=3t^2-8t+3 \Rightarrow t=2,0 \Rightarrow m_{1,2}=3,-1$

Figure 3: The two main ways of solving the case of two common points

In both the calculus and the algebraic approaches, there is an "easy" value of m and one that is less obvious. When grappling with this task, students should "see" (and be able to calculate) that there are two possible values for parameter *m*, for which both functions have two common points: $m=3$ or $m=-1$. Some students (see also Ron's thinking-aloud process in Figure 6) may skip one approach that reveals one of the values of m and move on to the other approach to obtain another value. A mathematical pedagogical discussion may address the manner in which the two approaches meet. This rich task concerns various mathematical concepts besides parametric functions: intersection points, tangency to a function, and mutual relationships between functions. It encourages making conjectures and aims to assess skills such as exploration, algebraic manipulation, and generalization of particular cases and examples.

Study

We redesigned the task (Figure 1) and converted it to an e-task. We conducted an experiment with 39 10th- and 11th -grade students who worked with the e-task. The students studied the standard curriculum with different teachers in the same school, without special emphasis on technology. Ron, one of the students, was thinking aloud during the solving process (Figure 6). The video recording of his thinking aloud enabled us to follow the process of task completion as it was taking place, rather than consider only its final product, and to listen to the problem-solving process.

Design considerations and possible correct solutions: Parts A and B

MLR experimentation first. In part A (Figure 4), the students used a dynamic applet that displays the function $f(x)$ and the parametric family $y = mx$ on the same coordinate system.

The following interactive diagram describes the functions $f(x) = x(x-1)(x-3)$ and $y = mx$. By dragging the red point, you can create different examples of mutual relationship between these functions.

Part A: How many common points are there for both functions? Submit three different screenshots, each one representing a different number of common points.

Part B: For which values of m does the functions have one common point? Two common points? Three common points? Indicate all the possible values.

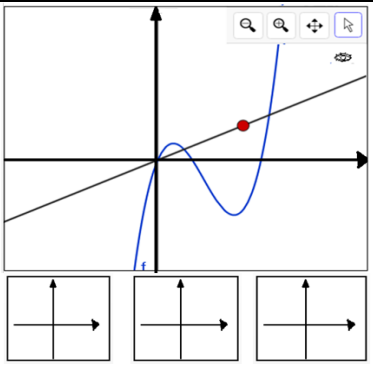

Figure 4: The designed e-task: parts A and B

Students became familiar with the activity and the givens, and were asked to submit three screenshots of three different cases of numbers of common points, in other words, three different "soft constructions" (e.g., Healy, 2000; Laborde, 2005), designed to support their generalization process and symbolic work required in part B. In the case of the parametric family in the MLR environment, any change in the value of m changes simultaneously the graphic representation of the relevant line. In designing this part, we wanted to make sure that students experimented with the applet, understood all the details and givens, and were exposed to many examples of the parametric function, so that in part B they could concentrate on the exploration activity with as little cognitive load as possible. An example of a correct solution is shown in Figure 5. The student submitted three different mutual relationships of the two functions, each with a different number of common points.

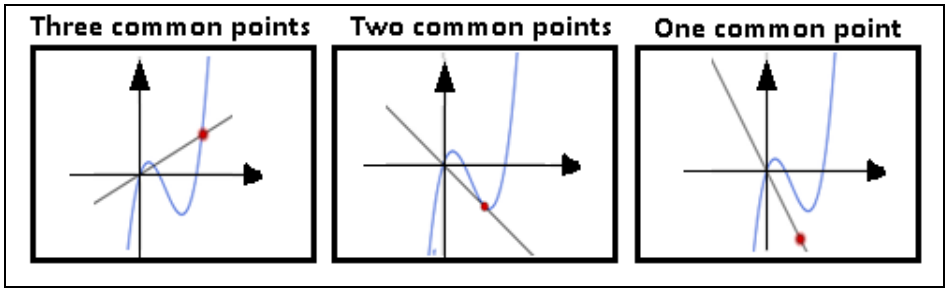

Figure 5: Example of a correct solution for part A

Minimal necessary tool set. There are deliberately very few tools available to the student: zoom in, zoom out, and move the coordinate system. The tools are designed to enable users to sense the task qualitatively, allow them to focus on the relevant parts of the graphing picture, and not to provide numeric information. This minimal design coveys the message that other parts of the tasks require numeric and symbolic calculation not provided by the interactive diagram. To summarize, the goals of part A are: (a) encourage student experimentation with the dynamic applet: feel/sense the givens and avoid cognitive load; (b) expose students to a variety of examples demonstrating the mutual relationship of the two functions; (c) engineer an experience-based conceptualization for solving the general case required in part B; and (d) assess the student's understanding of the givens and of what is expected of them (as well as some technical issues). Solutions for each number of common points appear in Figure 2 (b): one $m < -1$; two $m = 3$ or $m = -1$; three: $-1 < m < 3$ or $m > 3$.

Findings and data analysis: Parts A and B

(1) Part A: Most students submitted correct answers for part A (**Table 1**). This is not surprising, because the purpose of this part was to encourage students to "sense" the problem and its givens. But 15% of students missed one case (of one or two common points). Possible reasons for this are that students were not experienced enough with the applet, that they did not use the tools to view all cases, etc. As mentioned above (Figure 3), it is easy to reach one of the two possible values of m, but while experimenting with the interactive diagram one may notice that there is another possible value of m. In Figure 6 we describe Ron's thinking aloud about a solution for part B.

Correct answers	Correct answer	29 (74.3%)
Incorrect answers	One missing case (of one or two common points)	6 (15.8%)
	Technical problem	3 (7.7%)
Not submitted		1 (2.6%)
Total		39 (100%)

Table 1: Submission characteristics of correct and incorrect answers to part A

In the beginning, Ron solved this part algebraically and found that m=-1 is the case in which the functions have two common points. By zooming in and out, he found that there is another value for m. The interactive diagram allowed Ron to connect between the algebraic and graphic approaches (Figure 6, line 9). Through experimentation, he tried to find the other value of m (Figure 6, lines 4-9). This demonstrates the power of technology as a tool that provides students means to reflect their solution, allowing learning to take place during a test. In practice, during the experiment some students noticed the missing value of m and tried to find it (not always successfully), either by expanding the solution using the same approach, or by changing the approach from algebraic to calculus or *vice versa*, as described in Figure 3.

(2) Part B (Error! Reference source not found.): 18 of the solutions included one or both correct values for m (m=-1, m=3).

	N=39 (100%)		
Number of common points	**One**	**Two**	**Three**
Correct answer	15 (38.5%)	2 (5.1%)	2 (5.1%)
Partial answer	0 (0%)	16 (41%)	7 (17.9%)
Incorrect answer	8 (20.5%)	6 (15.4%)	13 (33.3%)
Not submitted	16 (41%)	15 (38.5%)	17 (43.6%)

Table 2: Submission characteristics of correct and incorrect answers to part B

Only two students submitted a completely correct answer; 15 students did not submit an answer. Others submitted other values, probably as a result of calculation errors or because they were guessing. Only two students submitted a correct answer for the case of three common points (the same students who submitted correct solutions for the case of two common points). This may imply that in addition to the difficulty of finding both "critical" values of the parameter m (m=-1, 3), it is also difficult to generalize and formulate symbolically the possible domains of parameter m for the cases of three common points.

	Ron's thinking aloud	Ron's actions on and with the screen
1.	This is equivalent to solving this equation. Right? Right. Zero is always one common point... then... Then I can divide simply by x. Then I investigate the quadratic equation:	$mx = x(x-1)(x-3)$ $m = (x-1)(x-3)$ $x^2 - 4x + 3 - m = 0$
2.	I check when it has two solutions, one solution, or zero solutions. These are two common points in my opinion.	He looks at the case that is close to m=3.
3.	No, these are three common points... There is a certain m... It has to be minus 1	This is the value that he got through the algebraic calculations.
4.	Then where did I go wrong?	Graphically, he sees that there is another positive m, but he got only m=-1.
5.	I need to check this equation.	Refers to $mx = x(x-1)(x-3)$.
6.	I always have one common point. Then I can simply divide by x. I have a neat equation. I have to see when it is equal to zero. We need to check when the discriminant is positive, negative, or zero. m must be different from zero to have two common points. Then m equals to -1. m=-1 for two solutions.	He checks again his calculations: $mx = x(x-1)(x-3)$ $m = (x-1)(x-3)$ $x^2 - 4x + 3 - m = 0$
7.	I need to zoom in.	Ron uses the zoom in and out buttons to explore and distinguish between different cases.
8.	I don't know what is my analytic mistake...	
9.	This is the tangent, the tangent.	He finds the graphic meaning of the case of two common points. Using calculus, Ron finds the other value of m.

Figure 6: Ron's thinking aloud while working on part B

Discussion

Our first conjecture, that a dynamic and interactive MLR environment supports the generalization of a graph into a family was only partially confirmed. The results of part B reveal the complexity of the concepts involved in the tasks. Results may suggest the presence of a permanent tension when designing mathematical e-tasks. On one hand, we want to design an e-task for exploration that can be automatically checked. Therefore, we tend to give students the opportunity to explore without any hints and without leading them to the solution. On the other hand, the task may be too difficult, and we may have difficulty assessing the students' knowledge and mistakes. In retrospect, this exploration e-task was too difficult; we should have divided the e-task into more than two stages and ranked the sub-tasks: first concentrate on the case of two common points, and only later on other cases. We are currently considering a refined design of this e- task.

Below we describe some basic design principles we gleaned from the experiment described above. Other design principles are described in Yerushalmy, Haddif, and Olsher (under review), Haddif and Yerushalmy (under review)**. When re-designing an e-task based on a P&P task, technology should provide the following: (1) Allow self-reflection:** When solving a P&P task, we have few means to reflect on our solution, especially not instantaneously. Use of an interactive diagram in an

MLR environment together with manual calculations helps students control their actions and reflect on them during assessment, and check whether they are right or wrong, without telling them directly what the correct solution is. **(2) Promote learning:** Using an interactive diagram with parametric functions allows seeing many instances of the same family. This is an opportunity to see the parameter serve as a "generator" of functions that belong to the same parametric family. Experiencing with the dynamic diagram also encourages students to make conjectures and conduct interactive exploration. As demonstrated in Ron's case, technology has the potential to create a cognitive conflict and thereby provide a learning opportunity (Figure 6, line 4). Ron tries and succeeds in solving the conflict between the manual solution and the graphic representation on the screen. Had he solved the original task, he may not have noticed the "conflict" between the graphic representation and the symbolic calculation. **(3) Guide students to focus on the important details, without unnecessary distractions:** Although it is tempting to use the varied capabilities of technology, these might distract the students and produce negative effects. Therefore, it is necessary to focus on the real needs of the students and redesign the e-task to make students concentrate on the important details, without unnecessary distractions. This also implies that students understand how to approach the question. The design must reflect in some way the purpose for which the tool was created (Yerushalmy, 1999), and the cognitive load must be reduced as much as possible. We demonstrated several ways of doing this: (a) designing the task in a way that students move in stages away from using sensory knowledge in soft constructions toward experimenting with the interactive diagram to produce robust constructions, abstraction, and generalization. (b) designing the environment and the required solution in a way that defers engagement in numeric and symbolic activity: for example, eliminating grids and the option to enter expressions conveys the message that conjecturing comes before computations. (c) determining the minimal necessary tool set that is familiar to the students and common to other e-tasks. In Yerushalmy et al. (ibid.) we described in detail the tool set needed for calculus e-tasks, which enables automatic checking of the students' submissions.

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Context based tasks on differential equations to improve students' beliefs about the relevance of mathematics

Dié Gijsbers and Birgit Pepin

Eindhoven University of Technology, ESoE, The Netherlands; d.gijsbers@gymnasiumbeekvliet.nl

Eindhoven University of Technology, ESoE, The Netherlands; b.e.u.pepin@tue.nl

Motivated and talented mathematics students are not always convinced about the relevance of mathematics. More insight into applications of mathematics can be beneficial for students in terms of preparing them for their future study and career. Using design research a particular intervention had been developed which differentiated by student interest, in order to improve students' beliefs about the relevance of mathematics. The students selected were those studying advanced mathematics at upper secondary school in the Netherlands. The intervention had been designed to teach differential equations through tasks with science-, medicine-, or economics- related contexts. The results show that the students appreciated the context-rich tasks, which provided them with insights into how mathematics can be applied in other sciences and contributed to the improvement of their beliefs about the relevance of mathematics.

Keywords: Mathematics education, task design, relevance of mathematics, differential equations, differentiation by interest.

Introduction

Students' beliefs about the relevance of mathematics has been a topic of much research over the last decades. However, in most studies the main reason for studying students' views on the relevance of mathematics has been the assumption that a positive notion of the importance of mathematics contributes to a positive attitude towards learning mathematics (Schoenfeld, 1989). Hence, most research studies conducted in this area have aimed at understanding and improving students' attitudes towards mathematics, and ultimately improving their performance in mathematics at school (Farooq & Shah, 2008; Mohamed & Waheed, 2011).

However, we start from the premise that already motivated and talented students can also benefit from a clear view on the usefulness of mathematics for their future education and career. Dutch secondary school students in an advanced mathematics course (aimed at improving students' algebraic skills and elaborating the connection between mathematics and other sciences) praised the course as being challenging and fun, but they also mentioned that it was not clear to them how the mathematics would be useful for their future study (Van Elst, 2013).

In our research project a design research approach has been used to develop and intervene with tasks aimed at simultaneously teaching a new (for students) mathematical concept (differential equations) and improving students' beliefs about the relevance of mathematics in general, and in particular with respect to their future study and career ambitions. The students sampled were those taking the advanced mathematics course (in their final year before university entry) in Dutch upper secondary schooling.

The mathematical topic of the designed intervention was the theory of analyzing, solving and interpreting first order differential equations. Differential equations are an important subject in any

undergraduate university curriculum in a broad range of domains, such as engineering, physics, biology and economics, which makes it a suitable topic for an intervention with a focus on the relevance of mathematics.

To improve the odds that the real-life problems posed in the designed tasks would appeal to the students, the designed intervention differentiated by student interest. Students were offered the opportunity to choose between different real-life problems, whilst ensuring that regardless of their choice they learned the same mathematical concepts of differential equations.

The research question of the study was the following: How does a learning strategy based on differentiation by interest for teaching ordinary differential equations (to upper secondary students in a "strong mathematics" course) improve the view of students on the relevance of mathematics for their future study and career?

In the subsequent section, we provide an overview of the relevant literature. Next, the research design including the context of the study and the data collection strategies used are described. Finally, we provide a discussion of the results and our conclusions.

Literature

Students' beliefs about the relevance of mathematics is considered as one of the factors that can play an important role in their attitude and motivation. Several questionnaires measuring the attitude of students towards (learning) mathematics use a scale for measuring students' beliefs about the usefulness of mathematics: for example, the Attitude scale towards Math (Martinot, et al., 1988) contains a scale named Relevance of Mathematics.

Most studies using these surveys do not emphasize the improvement of students' beliefs about the relevance of mathematics, as they are aimed at students' attitudes towards mathematics. However, a recent study of first year university students in engineering focused on improving perceptions of the relevance of mathematics in engineering. In this study Flegg (2012) has described the use of context-based learning by applying mathematics to real-life problems as a promising approach.

The teaching of differential equations has undergone some major changes over the past decades in favor of more contextualized, problem-based education, and a less traditional, analytical approach (Boyce, 1994). This was in line with our planned design to incorporate real-life problems in the assignments. At university level several initiatives have reported good results using this new teaching approach (e.g. Huber, 2010) and the development of new course material using Realistic Mathematics Education (e.g. Rasmussen & King, 2000). In a comparison of a traditional textbook with a textbook that incorporated discipline-specific perspectives to teach the mathematical knowledge to engineering students, Czocher and Baker (2010) conclude that a contextual approach is also more in line with recommendations from the research literature.

Task design is widely recognized as an important, albeit complex activity that is at the core of mathematics education. The term 'task' is used to describe a wide variety of student activities aimed at learning mathematics (Watson & Ohtani, 2012). In our research tasks are guided group assignments about a real-life problem.

The study

The context

The Dutch education system consists of eight years of primary education, and 4-6 years of secondary education (depending on the level of education). The highest level of secondary education is the pre-university education called VWO (voorbereidend wetenschappelijk onderwijs) with a duration of six years, which provides students access to university. For every student at VWO level mathematics is a mandatory course. However, in the last three years students can choose between two different mathematics courses: "wiskunde A" (mathematics A) and "wiskunde B" (mathematics B), the latter being the more mathematically demanding course, which is obligatory for technical and engineering studies at university.

In 2007 an advanced mathematics course called "wiskunde D" (mathematics D) was introduced to offer challenging and engaging mathematics, and where the relevance of mathematics and its connection to other sciences should become clearly visible (cTWO 2007). This course is aimed at students with an interest in sciences and engineering, and it includes mathematical topics, which are part of every first year university curriculum: e.g. complex numbers; analytic geometry; and differential equations. Students with 'wiskunde B' are offered the opportunity to take this advanced course in mathematics in addition to their regular course.

It might be expected that students who take this advanced course in mathematics are convinced about the relevance of mathematics, which is likely to fuel their obvious motivation to learn mathematics by taking this advanced course. However, studies on the implementation of 'wiskunde D' tell a different story. In a study by Van Elst (2013) students praised the course as being challenging and fun, but they also mentioned that it was not clear to them why the mathematics in the course was useful for their future study and career. According to a study by Cheung (2012) teachers of 'wiskunde D' stated that the course was well suited as a preparation for a future study in a technical or engineering environment, but they also stated that the curriculum did not emphasize enough the applications of mathematics and the connections to other sciences (Cheung, 2012).

To be able to understand the theory of ordinary differential equations, secondary school students require almost all basic mathematical knowledge taught in secondary education as pre-knowledge. Hence, this topic is scheduled near the end of the final (6th) year, to make sure all major mathematics/'wiskunde B' and mathematics/'wiskunde D' topics have been covered. It can also be expected that at that time the students have a good idea of their intended future study. This might motivate the students to choose tasks with real-life problems associated with their future study.

The real-life problems in the study by Flegg (2012) all had an engineering context, which was in line with students' study (of engineering at university). However, from our experience secondary school students taking the 'wiskunde D' course are not all interested in engineering. Students typically also enroll for medical or economics studies after graduating with 'wiskunde D'. Hence,

only focusing on real-life problems in engineering would have been a too narrow approach. To give students a good view of applications of mathematics to solve real-life problems, the learning strategy should offer differentiated instruction, based on students' interest.

Beside this "practical" reason to give students a choice of which problems they wanted to investigate, research shows that differentiated instruction based on students' interest supports their autonomy and improves students' motivation for the task at hand (Katz & Assor, 2006). Differentiation is commonly used to accommodate the different learning styles of the students; however, it can also serve to accommodate other differences between the students, such as their interests and plans for their future study and career (Tomlinson et al., 2003).

Research design and data collection strategies

For the development of the intervention, a module consisting of tasks on real-life problems, a design research approach was chosen. The design process comprised of three phases: a preliminary phase; a iterative development phase; and a final evaluation phase. The preliminary phase included a context analysis and a literature review. The learning goals for teaching differential equations were defined, based on the five strands of mathematical proficiency (Kilpatrick et al., 2001).

In the second phase a partial prototype of the module was developed. In March 2015, a small pilot test was conducted, with a prototype consisting of only three tasks. In a second design cycle a module was developed, which comprised of 14 tasks covering five different types of differential equations. Validation of separate tasks was done (1) by expert appraisals from university experts in mathematics education, focusing mainly on relevance and consistency of the design; and (2) by secondary school teachers focusing on the consistency and expected practicality of the design.

The designed intervention was carried out from January to March 2016. Three classes of 'wiskunde D' students of two different schools in the Netherlands, in total 49 students, participated. The three classes all had a different teacher, one being a researcher in this study.

The data collection strategies for the intervention included

- two student surveys (before and after the intervention) asking the students about their future study plans and their views on the relevance of mathematics, using the 8 question scale Relevance of Mathematics (Relevance scale) from The Attitude scale towards Math (Martinot et al., 1988);
- a student survey after the intervention to evaluate the module and the tasks;
- student interviews after the intervention about the module and the relevance of mathematics in general, and in particular for their own future study and career;
- the video recording of a teacher meeting about their experiences during the intervention;
- the collection of exam results after the intervention at one school. These results were compared to the results of the 'wiskunde D' students of the previous exam years, who were taught the same theory of differential equations but in the traditional classroom setting.

The intervention

The intervention module comprised of five tasks, each covering a different type of differential equation. Of each task up to three different versions were developed, which applied the same mathematical concept to entirely different contexts. Prior to the first task students were given time to read short descriptions of each problem and were given the opportunity to make, for each of the five subsequent tasks, a choice which context (real-life problem) appealed to them most.

The different contexts for each of the tasks consisted of: a science/engineering related problem; a biological/medical problem; and an economical/social problem. Regardless of students' choices of contexts for the five tasks, the students worked with and learned the same mathematical concepts during their work on these parallel tasks. The students were guided through the process of modelling the problem and exploring the mathematical model analytically, graphically and numerically. After solving the mathematical problem the students were asked to interpret the results within the context of the chosen assignment. Table 1 gives an overview of the 14 tasks.

Task	Differential equation	Scientific/engineering problems	Biological/medical problems	Economic/social problems
1	$y' = \alpha y$	Nuclear disaster	Bacterial food poisoning	Forged paintings
2	$y' = \alpha y + \beta$	Mixing water problem	Intravenous infusion	Advertising effect
3	$y' = \alpha y + f(t)$	CO poisoning	Estimating time of death	Price indexing
4	$y' = \alpha y^2 + \beta y$	Oil production	The Ebola epidemic	Population growth
5	$y' = f(y,t)$	Skydive	Blood alcohol content	

Table 1: Overview of the real-life problems used in the module

Results

40 out of the 49 participating students filled in the survey about the module. Responses to most questions were measured on a five-point scale, ranging from "I strongly disagree" to "I strongly agree". The students were generally quite positive about the whole module. Asked to grade the whole intervention on a scale of 1 to 10, they rated the module 6.5 on average. Interestingly, this mean score differed greatly between the three groups, with one group scoring surprisingly lower than the other two (see Table 2). Additional comments in the survey from this group suggest that they lacked proper guidance of their teacher. Hence, the practicality and effectivity of the intervention can substantially benefit from a good description of the role of the teacher as coach, helping the students to advance in their assignments. In the teacher meeting the same recommendation was made.

Overall, 28 out of 40 students agreed (of which 6 strongly) with the statement "The tasks gave me a good view on how mathematics is applied in other sciences", and only 3 students disagreed with this statement. The negative statement "I think that these tasks were not very realistic" was disagreed by 29 students (of which 12 strongly) and only two students agreed with this statement.

Schools	Group	Group size	#Respondents survey	Rating (mean)	#Respondents Relevance scale
School A	1	19	18	7.1	18
	2	17	14	5.5	7
School B	3	13	8	6.9	8
Total		49	40	6.5	33

Table 2: Respondents and rating of the module by group

The survey also contained open questions asking students what they liked about the module and what in their opinion should be improved. Positive points were working in groups (11 times) and the application of mathematics in other sciences (15 times) with statements like: "it was fun to experience that math is useful", "clear applications of math" and "you get a better view on how mathematics can be applied".

Some negative comments were about ICT problems encountered during the intervention (e.g. "slow laptops"). However, the majority of the feedback was about the way the mathematical concepts of differential equations were introduced to them. For example, they complained that having to read the theory by themselves made it harder to grasp the concepts. Some also missed the "traditional" teacher-led instruction, where the "basic" theory is explained by the teacher before the students start working on the assignments.

The student interviews were conducted by the teacher/researcher (one of the authors). In the interviews the students voiced the same concerns about the lack of teacher instruction and the ICT problems. However, they were positive about the tasks and the application of mathematics to real-life problems, as the extract shows.

Interviewer: Did the tasks have some added value?

Student 1: Yes, I think so. I think it adds quite a lot. It helps…

Interviewer: It helps? In what way?

Student 1: To give a lot of people the idea what can be done with mathematics. That there is mathematics behind everything. Not a lot of students would have realized that before, I think.

Comparison of the 'wiskunde D' exam results of the 36 students from school A with the results on a similar exam by the 27 'wiskunde D' students of the previous year did not show a significant change in the grades. As can be seen in Table 3, the students of 2016 scored slightly better than their peers in 2015 but that was to be expected as their average grade before the exam was also better.

The intervention was not intended to improve the exam results, and the findings showed that the dual focus on both the mathematical concepts of differential equations and the relevance of mathematics to other sciences did not affect the grades of the students in a negative way.

Exam results school A	Group size	Average grade before exam	Average exam score
Students 2016 (intervention)	36	75.2 %	72.6 %
Students 2015 (traditional course)	27	74.6 %	71.4 %

Table 3: Exam grades of the students in 2015 and 2016

From the Relevance of Mathematics scale, conducted before and after the intervention, we obtained some promising results. Only 33 out of the 49 students completed both the pre and post survey, as shown in the last column of table 2.

A one-sided paired-samples t-test was conducted to compare the answers of the 33 students on 8 items of the Relevance of Mathematics scale before and after the intervention. The result of this test for the whole group of 33 respondents did not show a significant improvement with a p-value of 0.19. But the same test on the 18 results from the high response group gave a p-value of less than 0.02 indicating a significant positive change in their response to the questions about the relevance of mathematics. Due to the low response rate of the two other groups we were not able to get any significant results from these separate groups.

We also conducted the paired t-test on the answers of the 17 students who had the lowest scores on the pretest. They had scored relatively low on their beliefs about the relevance of mathematics, all with an average score between 2.38 and 3.63 for the 8 items on the five point Likert scale. On the same test after the intervention their average scores ranged between 2.63 and 4.63. 12 of these 17 students scored higher on the test after the intervention resulting in a p-value of 0.002, which indicates a strong significant positive change in these students' beliefs about the relevance of mathematics. This result indicates that students who do not already have strong beliefs about the relevance of mathematics benefit most from the context-based tasks.

Conclusion

The intervention had positive effects on selected student views about the relevance of mathematics without affecting the examination results. Providing students with purposefully designed tasks where mathematics is applied to real-life problems cannot only challenge their assumptions about the relevance of mathematics, but also improve their awareness of its usefulness.

Teacher professional development is a topic for further study that will be the focus of a new design cycle of the module. Applications like a teacher meeting, where the goals of the module are explained and an extension of the teacher manual with guidelines how to introduce the module and how to coach the students during the group assignments, are expected to contribute to a better understanding of the role of the teacher during future interventions.

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Teachers' collective documentation work: A case study on tolerance intervals

Ghislaine Gueudet[1] and Veronica Parra[2]

[1]CREAD, ESPE Bretagne, University of Brest, France, ghislaine.gueudet@espe-bretagne.fr

[2] CREAD, ESPE Bretagne, University of Brest, France, and NIECyT. UNCPBA. CONICET; Tandil, Argentina, vparra@exa.unicen.edu.ar

In this paper we use the documentational approach to investigate teachers' collective work. We follow two teachers, preparing together a lesson on tolerance intervals for grade 11. We identify Mathematical Knowledge for Teaching (MKT) that influences the use of resources by the teachers. We evidence that their collective work fosters important documentation work; but we observe significant differences between the documents developed by the two teachers.

Keywords: Documentation work, resources, teachers' knowledge, tolerance intervals, variability.

Introduction

Teachers interact with curriculum resources in and out-of-class (TWG22 call for papers). In previous works we have identified the importance of these interactions in terms of teachers' professional development (Gueudet, Pepin & Trouche 2012), and we have evidenced that teachers often work collectively with resources. In this paper we study further this collective work of teachers with resources and its consequences.

The work presented here takes place within the French national project REVEA[1] (Living Resources for Teaching and Learning). We consider the case of two mathematics teachers at upper secondary school in France teaching sampling variability in statistics in grade 11. We firstly expose the theoretical perspective we use and our methods. Then we present the data we collected, and the context of teaching sampling variability in France. Finally we expose our analyses of the teachers' work and of its links with teachers' knowledge in particular.

Investigating teachers' documentational work: Theory and method

We use for our research the theoretical and methodological perspective of the documentational approach (Gueudet et al. 2012). Mathematics teachers interact in their work with a large range of resources (Adler 2000). Resources designed for teaching purposes like textbooks or software, resources coming from the students, e-mails exchanged with colleagues etc. Teachers choose resources, transform them, use them in class; we call this work "teachers' documentational work". In previous research, we have evidenced that this work is closely linked with teachers' professional knowledge. The choice of resources by teachers, the way teachers modify and use the resources is driven by their professional knowledge (and this is called an instrumentalisation process, drawing on Rabardel's instrumentation theory, Rabardel 1995). In a reverse way, the features of the resources used modify teachers' knowledge (in an instrumentation process). In the documentational approach, we consider that from a set of resources teachers develop a document: transformed

[1] https://www.anr-revea.fr/

resources associated with a scheme of use (Vergnaud 1998). A scheme of use comprises the aim of the activity, rules of action and professional knowledge. The development of a document is called a documentational genesis.

Teachers' Communities of Practice (CoP, Wenger 1998) have a shared repertoire that the documentational approach interprets as shared resources. In previous works (Gueudet, Pepin & Trouche 2016) we have investigated the documentation work of a CoP (Sésamath, an association of teachers in France) designing online resources and identified the development of shared documents. Here we study a more "ordinary" CoP, composed of two teachers working together for the preparation of their courses. We are interested in particular in the commonalities and differences in professional knowledge within the documents developed by these teachers. The research question we investigate can be formulated as:

How do professional knowledge and resources interact in the collective design and implementation of a lesson?

Concerning teachers' professional knowledge involved in the schemes, we are especially interested in the identification of Mathematical Knowledge for Teaching (MKT, Ball, Thames & Phelps 2008): professional knowledge linked with the mathematical content to be taught.

The documentational approach is associated with a specific method, called "the reflexive investigation method". Documentational geneses are long term processes; moreover documentational work can take place everywhere and at any time. Thus we follow teachers over long periods of time; and involve them actively in the collection of data. These data are interviews of the teacher; videos of the teacher's work in class and out-of-class (videos of collective work, if teachers work together); resources chosen and transformed by the teacher.

For analyzing the data, we start with the transcribed interviews of the teachers. We identify in them the aims of the teacher's activity. For each aim, we search in the data for the resources used, and the other components of the document (rules of action and professional knowledge) . We submit these elements to the teacher who corrects and complements them if needed. We present in the next section the data collected for the case we study here.

Data collected and context

We follow since 2014 two mathematics teachers in an upper secondary school of a middle-sized town in France: Valeria and Gwen. They are both very experienced: Valeria teaches for 34 years, Gwen for 36 years, they both regularly follow teacher education sessions and are trainers for new teachers in their school. They also regularly work together, we consider them as a CoP (Wenger 1998). In 2015-2016, they decided to take two grade 11 classes called "economics and science", a specialty they taught for the first time. We followed their work for these classes, in particular for a chapter entitled: "sampling variability" (that they both used to teach grade 11 "science", with a similar content). For this chapter, we video recorded their common preparation (one hour), their individual courses (four hours each), and for each of them an individual post-teaching interview. We collected all the resources they used and produced, and the students' productions for the final assessment of the chapter. For both teachers we identify the professional knowledge/beliefs, the

possible origin of these beliefs, the consequences in terms of the activities/resources produced and the resources used.

Sampling variability is taught in France since 2010. This teaching starts in grade 10 where the idea of sampling variability is introduced using material like coins and dice and simulations on the calculator and on the spreadsheet. In the curriculum in France, the concept of tolerance interval is central in the teaching of sampling variability. In grade 10 the students have to learn how to identify the population; the sample and its size n; the probability p of a given feature in the population, and the frequency f of this feature in the sample. A first tolerance interval is introduced, without justification: $[p-1/\sqrt{n},\ p+1/\sqrt{n}]$. If f does not belong to this interval, the students learn to reject the hypothesis "the sample follows the population's law" with a 5% risk level. At grade 11 (scientific or economics and scientific) the binomial distribution is introduced; it provides another tolerance interval, which can be found using the table of the binomial distribution produced for example with a calculator or a spreadsheet. The chapter we followed concerns the introduction and use of this interval. At grade 12, the normal distribution is presented; this leads to a new interval (the asymptotical tolerance interval).

Many research works have investigated teaching variability; they emphasized the specific nature of reasoning in probability (Steinbring 1991) and the need for particular knowledge to teach this subject (González 2013). Eckert and Nilsson (2013) used the notion of Mathematical Knowledge for Teaching Probability (MKTP) for characterizing the specific knowledge needed by the teachers in probability and statistics, and for sampling variability in particular. Finding situations in relevant contexts; emphasizing the idea of variability, using the different kinds of possible representations all require specific knowledge from the teacher. Our aim here is to investigate how the interactions with resources are shaped by, and contribute to MKTP.

Results

In this results section, we firstly consider the two teachers' documentation work, during the common preparation session, the lessons taught by each teacher and finally during the design of a common assessment. Then we present our analyses of the most important aspects of this documentation work (in terms of MKT involved), focusing on the collective-individual articulation.

A collective documentation work

During the common preparation, Valeria and Gwen used several textbooks (9 different textbooks). They did not actually chose resources together, but drew on exercises and problems in the textbook to illustrate their declarations. They also talked about resources they intended to use: exercises, software (GeoGebra, spreadsheet) and the calculator.

Their discussion during the common preparation started by stating a difference: Valeria intended to use the spreadsheet from the beginning and during the whole lesson. The students have learned, in the "binomial law" chapter to produce with the spreadsheet and read tables displaying the value of $P(X=k)$ and $P(X\leq k)$, when X is a random variable following a binomial law of parameters N and p. Valeria wanted to recall this, then to introduce the binomial law tolerance interval and the method for finding it, using the $P(X\leq k)$ table produced with the spreadsheet. Gwen, in contrast, intended to use only the calculator, and no other software. After introducing the binomial law tolerance interval,

she wanted to ask students to write and implement on their calculator a program producing the tolerance interval.

In the other aspects of the common preparation, Valeria and Gwen agreed on all the points they discussed. They mentioned in particular the need to recall the grade 10 tolerance interval and to compare it with the new interval introduced.

All these aspects discussed during the common preparation are present in the lessons actually taught by Valeria and Gwen. We analysed these lessons drawing on the observations and videos in class, the resources collected and the post-lesson interview.

Valeria started indeed by recalling how to produce and read the binomial law table with the spreadsheet. She also recalled the grade 10 interval with exercises chosen in a textbook's "revision section". Then she introduced the new interval through a problem concerning overweight in USA. This problem came from another textbook, and she modified it in particular by suppressing the table giving $P(X \leq k)$, because she wanted the students to produce it themselves with their calculator. She presented how to find the interval from the table $P(X \leq k)$. Valeria insisted on the need to formulate very precisely the decision rule. At the end of the chapter, she worked with her students on the algorithm: the students implemented it on their calculator, but this program was actually not used as a tool to find the interval in exercises.

Gwen started with a problem that she built herself, about red-haired people in Scotland (inspired by a textbook problem with a different context). The first part of the problem recalled the grade 10 interval, and more generally the idea of sampling variability. The second part of the problem introduced the new interval. Just after this session, Gwen worked with the students on the production of an algorithm and its implementation on the calculator to find the binomial law interval. Afterwards this program was always used to find the interval. Gwen said that she found the binomial law interval too technical, she did not want her students to learn how to find it. She preferred to use it as an opportunity to work on algorithms. She distributed a sheet to the students presenting the interval and a diagram. Then she proposed different exercises about decisions; in particular one exercise with samples of different sizes.

The final assessment of this chapter was also the final assessment of the year for the two Grade 11 ES classes. Valeria and Gwen wrote it together; we analyse their documentation work drawing on the resources they used and produced; the e-mail they exchanged and their interviews. The control text comprised one exercise on tolerance intervals. This exercise, with an introductory text (figure 1) and three questions, concerned the rate of twins in India; it came from a textbook. Valeria and Gwen modified the initial text which was, in their opinion, too long and complex.

" India: Kodinji, the mysterious twins village

In the state of Kerala (south-west India), there is an amazing village. The rate of twins is much higher than the national average. 440 twins live indeed in this town for 14600 inhabitants. This average is outstanding, since the national average is 16 twins for 1000 births". *Extract of an article* (Courrier International, 2009)

Let X be the random variable counting the number of twins in a 14600 Indians sample.

Figure 1: Introductory text of the control exercise (our translation)

They modified the first question: in the textbook the parameters of the binomial law followed by X were given; they wanted the students themselves to find the parameters. They also changed the second question, where the students are asked to produce the tolerance interval, to display the two different methods expected for each class: use of a table (given in the text) for Valeria's class, use of the calculator's program in Gwen's class. Question 3 was left unchanged.

42 students were present at the final assessment. In the first question, 35 students justified that X followed a binomial law of parameters n and p; 29 students determined correctly the value of n, but only 15 the value of p. In the second question, 19 students determined correctly the endpoints of the tolerance interval. In the third question, 19 students justified correctly the rejection of the hypothesis ("the Kodinji village follows the national figures").

Valeria and Gwen documentation work and use of resources: General statements

We can notice that like other research works using the documentational approach (Gueudet et al. 2012) this description evidences that Valeria and Gwen are designers of their own resources. They use various curriculum resources, but transform most of them. Only some exercises texts are left unchanged, and one textbook extract presenting the binomial law interval (for Gwen). Moreover, textbooks (on paper, they do not use digital textbooks) are central resources, used as a reference to discuss the lesson plan, and to choose exercises for practicing the new methods, for the assessment/test, or to find an introductory problem (for Valeria). More surprisingly, they did not search the Internet for resources – this can be a consequence of their common preparation: Valeria and Gwen sometimes search the Internet for preparing their lessons, but always at home. For the common preparation they were at school with no Internet access. Another result already well known in the documentational approach is that the observation of students (their written texts, or oral discussions in class) constitute a very important resource for the teachers, leading to a constant modification of the resources produced along the documentation work. Both Valeria and Gwen intend to modify this lesson on tolerance intervals next year, because they consider that the students made too many mistakes in the final assessment/test.

Documents developed by Valeria and Gwen

In this section we analyse our data in terms of documents developed by Valeria and Gwen. Since our focus is on MKT, we do not give a complete description of each scheme of use but only mention the aim of the activity and the MKT involved. We have chosen three examples of documents, corresponding to different situations in terms of similarities or differences.

Valeria and Gwen had a shared aim that can be described as "Recalling previous knowledge needed for the binomial law tolerance interval". They both considered that this new chapter must start by recalling the grade 10 interval, because they knew from their observation of students during the year that "many students do not remember this interval" (Gwen even added that some students perhaps never saw it, since some colleagues keep this content for the end of the year and run out of time). This shared MKT lead however to two different documents for this aim, because of the teachers' different resources: Valeria used a lot the classroom textbook, and thus proposed revision exercises coming from this textbook, while Gwen wrote her own problem text.

Valeria and Gwen also shared a general aim that can be presented as: "Teaching how to find a tolerance interval with a binomial law". During the previous years, Valeria has developed for this

aim a document including various resources: the spreadsheet (as software or in the calculator), exercise texts, the illustrating diagram, and MKTP: "The students must learn to find the endpoints of the interval by reading the table". This knowledge can have different sources; we claim that it comes in particular from institutional texts (the official curriculum) and from textbooks. For the same aim, Gwen has developed a different document, including: the calculator, an algorithm, the illustrating diagram, exercise texts as resources; and MKTP like: "The binomial law interval is too technical"; "there are no questions about the binomial law interval at the baccalaureate[2]"; and the MKT "it is important to work with students on algorithms". This knowledge comes from reading the texts of the baccalaureate, and from a personal mathematical difficulty: Gwen declared that she "cannot remember [herself] how to find the endpoint of the interval". Moreover she considered that this grade 11 curriculum is only a transition between the grade 10 interval and the grade 12 interval (with the normal distribution) while algorithms are always present in the baccalaureate texts. Valeria and Gwen discussed this difference during the common preparation. Valeria integrated in her lesson the programming of the algorithm on the calculator, but she did not want her students to use it, because she feared that the students do not really understand the algorithm and use their calculator as a "black box".

For the aim: "Assessing the students' ability to determine and interpret a tolerance interval", Valeria and Gwen used shared resources: they wrote the assessment text together drawing on the same textbook exercise. The choice of this exercise was guided by MKT firstly expressed by Gwen, and adopted by Valeria: "the students must learn to find information in a text". Nevertheless, the text produced was also transformed to incorporate the use of two possible methods, because of their different MKT concerning how to find the binomial law tolerance interval.

In Table 1 below, we synthesise these three documents, evidencing the common and different elements.

[2] In France, the final secondary school assessment, at the end of Grade 12.

Aim	**Resources used**	**MKT/ MKTP**
Recalling previous knowledge	Valeria: Revision exercises in the classroom textbook	*"Many students do not remember the grade 10 tolerance interval"*
	Gwen: Her own problem text	
Teaching how to find a tolerance interval with the binomial law	Valeria: Problem and exercises texts from different textbooks, the spreadsheet, *algorithm on the calculator (coming from the collective work)*	Valeria: "The students must learn to find the endpoints of the interval by reading the table"; "they must not use the calculator as a black box"
	Gwen: Problem composed herself, exercises from different textbooks, *algorithm on the calculator*	Gwen: "it is important to work with students on algorithms" "the binomial law interval is too technical"
Assessing the students' ability to find and interpret a tolerance interval	*Shared assessment text written together from a textbook exercise, but integrating two possible methods*.	*"The students must be able to identify information in a text" (shared)* + MKT/MKTP described in the above line

Table 1: Synthetic presentation of documents developed by Valeria and Gwen. Shared elements are in italics.

Conclusion

In this paper we investigated the research question: "How do professional knowledge and resources interact in the collective design and implementation of a lesson?" in the case of a lesson on tolerance intervals for Grade 11 students in France. In the frame of the documentational approach, investigating how professional knowledge and resources interact means investigating the documents developed by teachers. For each of the two teachers we followed, we observed that they developed an important design work, choosing resources, associating and modifying them. This work was guided by their professional knowledge, in particular MKT and MKTP. We observe that this MKT is mostly of the type: "Knowledge of Content and Students"; deepening the analysis in terms of types of MKT is an interesting perspective for further work. In a reverse way, resources influenced the development of MKT, and this MKT can be different for each individual teacher. For example the official curriculum influenced Valeria and Gwen in different ways: while Valeria aligned with the curriculum about tolerance intervals, Gwen attached more importance to the algorithms. She developed a personal interpretation of the official curriculum, not focusing on the chapter she taught but taking into account the whole year.

In previous works (Gueudet et al. 2012) we evidenced that collective work is present in many aspects of teachers' activity and that it is a stimulator of documentation work, especially when it takes place within CoPs. In (Gueudet et al. 2016) we analysed the common documentation work in a CoP: an association of teacher designing an e-textbook. We evidenced that they developed

common documents, drawing on their individual documents. In the present study we investigated a CoP composed by two teachers preparing their courses together. We evidenced that, in spite of the collective work the documents developed by the two teachers for the same aim are never completely identical. The consequence of the collective work is that these documents sometimes share common elements. The main reason for the differences seems to be the long experience of both teachers: they already developed in previous years documents for the same aims, and thus have MKT or MKTP associated with these aims and also specific resources. The collective work can bring new resources (the algorithm on the calculator for Valeria) or new knowledge (the students must be able to find information in a text, for Valeria again), but the previous knowledge developed during interactions with resources over many years is still present and produces differences in the documents.

These teachers will go on working together; with a longer common work, evolutions may take place, and we will try to analyse these evolutions. We also hypothesize that evolutions of practice are more likely to take place in teachers' CoPs when the members of the CoP are involved in a common design activity (for example in the Sésamath case, Gueudet et al. 2016, or in the context of professional development, Pepin & Miyakawa 20016). We intend to investigate this hypothesis in further research.

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Pre-service teachers' reflections on task design and implementation

Hulya Kilic[1], Oguzhan Dogan[2], Zuhal Yilmaz[3] and Pelin Donmez[4]

[1, 2, 3]Yeditepe University, Faculty of Education, Turkey; kilichul@yahoo.com

[4]Ulus Private Hebrew High School, Turkey; pelindonmez99@gmail.com

In this paper, we will present a part of findings of a larger study in which we investigated both pre-service teachers' pedagogical content knowledge and middle school students' mathematical skills. A total of 17 pre-service teachers worked with 7th grade students on the mathematical tasks for a semester. The data was collected through inventories, videos of task implementations and written reflections. The analysis of inventories and written reflections revealed that the pre-service teachers had positive beliefs about using mathematical tasks in the lessons. They were able to evaluate the success or failure of a task by analyzing the implementation process. They recognized the importance of preparing appropriate tasks for students and implemented them as intended. They also noted that mathematical tasks provided them an opportunity to learn about students' mathematical understanding as well as to develop their scaffolding practices.

Keywords: Pre-service, middle school, mathematical tasks, reflection, teacher knowledge.

Introduction

The studies on mathematical tasks provide an opportunity to discuss not only students' mathematical thinking and understanding but also teachers' knowledge and skills (Zaslavsky, 2007). Tasks are accepted to be one of the curricular tools that help teachers to scaffold, foster and assess students' understanding when used appropriately (Stein, Grover, & Henningsen, 1996; Watson & Mason, 2007). Research on task design and implementation shows that "good" tasks or *cognitively demanding tasks* (Stein et al., 1996) have positive impacts on students' thinking and understanding (Henningsen & Stein, 2002). Such tasks encourage students to think about what mathematical concepts are conveyed in the task, what they know about them, how they are related to other concepts and which strategies, representations or materials are helpful in terms of arriving at a meaningful solution or answer. On the other hand, designing, selecting and implementing cognitively demanding tasks are related to teachers' pedagogical content knowledge (PCK) as well as to their content knowledge (Charalambous, 2010; Liljedahl, Chernoff, & Zazkis, 2007; Stylianides & Stylianides, 2008). Teachers' decisions about whether a task is appropriate for their students in terms of its complexity or if their students possess the necessary prior knowledge or the task's goal aligns with the lesson's objectives emerge from their PCK (Hill, Ball, & Schilling, 2008). The studies on task design and implementation showed that when pre-service teachers (PSTs) were given opportunities to develop and implement tasks, they were able to prepare "good" tasks in which they used real life context and multiple representations and they asked for explanation, interpretation or justification (Ozgen & Alkan, 2014).

In this study, we aimed to investigate how PSTs' involvement in task design and implementation process influenced their views about tasks and their PCK. We used tasks as a tool to build up an environment for PST-student and student-student interactions where the PSTs observed how students worked on the tasks both individually and as a group and then joined their discussions to elicit students' thinking and support their understanding. Thus, the PSTs had opportunity to learn

about students' mathematical skills and also to gain experience about how to implement a task without loss of fidelity, to scaffold students' understanding, and to manage the group work. In this paper, however, we will present some findings about PSTs' reflections on this task design and implementation process in terms of effectiveness of the tasks and their professional gains from this intervention study.

Methodology

Research setting

This study was conducted under university-school collaboration between the mathematics education department and a local middle school in Istanbul in Turkey. A total of 17 PSTs (10 of them in Fall 2015 and 7 of them in Spring 2016) and one of the 7th grade classes (aged between 12-13) from the partner school volunteered to participate in this study[1]. We arranged the students as mixed-ability groups of 4 students based on the results of an achievement test. We randomly assigned the PSTs to these groups. The PSTs worked with students in a 40-minute lesson for each week, a total of 11 weeks during the Fall and 12 weeks during the Spring.

All PSTs were senior students who volunteered to take an elective course that was carried out specifically for this study. During the first couple of weeks of the course, we talked about the design and implementation of tasks, effective ways of communicating with students and assessment of students' mathematical understanding through samples. Then, each week, prior to implementation, we discussed the tasks that they would use with the students and made revisions on the tasks when necessary. After each implementation in the school, we met the PSTs again and discussed how the implementation went. We wanted PSTs to write reflection on each implementation by analyzing the implementation videos of their own group and students' worksheets. We also asked PSTs to prepare and implement a set of tasks for their own group at the end of the semester.

The tasks

We prepared the tasks in alignment with the 7th grade mathematics curriculum. In the Fall semester we prepared 10 tasks which were about integers, fractions, rational numbers and algebra. In the Spring term we prepared 11 sets of tasks which were about geometry, algebra and data and statistics. Each set of tasks consisted of 3 to 4 sub-tasks with some additional problems. The tasks were set up around a common theme which related to the use of mathematics in daily life. For instance, in one of the data analysis tasks, we conducted a short survey on students' preferences in the class such that we gave them a list of foods where each student would choose 5 of the foods from the list. Then we wrote down the frequencies for each food on the board and then asked them to answer questions about their preferences such as finding out the most popular food, making a bar graph of the drinks, etc. The majority of the tasks that we used were selected from the ones that we implemented in the previous years. However, we revised those tasks before the implementation based on the needs and performances of the students in our new sample.

[1] The school was located in a crowded neighborhood in terms of school-age children. There were 40 students in the class, average size according to national education statistics, but in the Spring semester 4 students dropped out. Since we had only 7 volunteers in the Spring semester, two researchers from the research team worked with two of the groups.

Data collection

The data was collected through pre and post questionnaires, videos of PSTs' task implementations and group discussions, and PSTs' written reflections and assignments. One of the questionnaires (Kayan, Haser, & Isiksal-Bostan, 2013) consisted of 26 5-point Likert-type items asking for PSTs' beliefs about mathematics and mathematics teaching. The other questionnaire was developed by the research team and it consisted of open-ended questions where in the pre questionnaire the PSTs were asked to write about their prior teaching experiences and their expectations from the study and in the post questionnaire they wrote about whether their expectations were met or not and what they had learned from this intervention, etc. In their written reflections, we wanted PSTs to discuss about how the implementation went. Among the other questions, we asked them to comment on whether the students were able to complete the tasks, whether the tasks achieved their goals or not and why, and what they would do as follow up. At the end of the semester, we wanted them to discuss in which tasks the students experienced the most difficulty and what their suggestions would be to revise those tasks. Furthermore, we asked them to prepare tasks for their own groups and provide the rationale behind those tasks.

Data analysis

We have not analyzed all data yet. However, because the knowledge of students' thinking and understanding, the ability of selecting or developing appropriate tasks, and using appropriate teaching strategies for particular groups of students are counted in teachers' PCK (Hill et al., 2008), we are basically looking for any instances that could be counted as an influence of task design and implementation process on the PSTs' PCK and their views about tasks. For instance, whether they paid attention to the students' earlier performances while preparing their own tasks, how accurate assumptions they were able to make about students' performances on the new tasks, etc. Yet, we did document analysis such that we analyzed the items related to task design and implementation from the questionnaires, written reflections and assignments. We attempted to figure out the frequencies of common issues that emerged from those instruments. We found out the mean scores of the items in the Likert-type questionnaire but we did not compare pre and post results because of low number of participants. We examined pre and post open-ended questionnaires, the assignments asking for making overall evaluation of the study and the reflection reports including PSTs' reflections on the implementation of their own tasks. We developed a coding scheme for the common issues that emerged from the reflection reports. For instance, when discussing the reasoning behind the success of the tasks, if the PSTs wrote that "they liked it" or "they enjoyed it" or "they had fun with it" then we coded that reasoning as "enjoyable" (see Table 1). However, out of possible 194 reflection reports, 6 of them were missing. Therefore, we coded a total of 188 reports. We achieved 90% agreement in initial coding. We discussed the discrepancies by re-reading the PSTs' reports and then we came with an agreement. Moreover, we all agreed on the common issues that emerged from the open-ended questionnaire and the assignments.

Findings

PSTs' thoughts about use of tasks in mathematics

The analysis of items in the Likert-type questionnaire showed that the PSTs agreed that teachers should encourage students to be active learners (The average of 3 related items; pre $\bar{x} = 4.38$; post

$\bar{x} = 4.45$), the tasks are important for students' learning and understanding (The average of 2 related items; pre $\bar{x} = 4.36$; post $\bar{x} = 4.61$) and manipulatives and materials facilitate students' understanding (The average of 3 related items; pre $\bar{x} = 4.09$; post $\bar{x} = 4.60$). Their answers in the open-ended questionnaires were compatible with these results. The PSTs noted that doing mathematics through tasks enables students to participate in the lesson ($f = 5$), love mathematics ($f = 4$), discover or review mathematical concepts or facts ($f = 4$), and use materials or manipulatives ($f = 4$).

PSTs' reflections on the implementation of the tasks

In the reflection reports, we asked the PSTs to comment on the following questions: 1a) Were the students able to complete the task? 1b) Did the task attain its goal or not? 1c) Why did the task attain its goal or not? 2) What more would you like to do about this implementation? 3) What is your suggestion for the follow up of the task? The PSTs gave various answers to these questions. For instance, for question 1a they noted that some of the students completed the task or just one student could not finish all of them, etc. Although we coded them separately, we re-coded them in terms of whether their answers were most likely "Yes" or "No". Similarly, we defined 12 different answers for question 1c. However, we preferred to present only the most frequent ones. In Table 1, the frequencies of PSTs' answers to these questions are given.

Q1a: Were the students able to complete the task?

	No	**Yes**
Fall	30 (29%)	75 (71%)
Spring	15 (18%)	68 (82%)
Total	**45 (24%)**	**143 (76%)**

Q1b: Did the task attain its goal or not?

	No	**Yes**	**No Comment**
Fall	38 (36%)	67 (64%)	
Spring	15 (18%)	63 (76%)	5 (6%)
Total	**53 (28%)**	**130 (69%)**	**5 (3%)**

Q1c: Why did the task attain its goal or not?

	No Comment	**Enjoyable**	**Lack of Knowledge**	**Use of Materials**	**Recognize own Mistakes**
Fall	28 (27%)	10 (10%)	32 (30%)	11 (10%)	14 (13%)
Spring	19 (23%)	5 (6%)	19 (23%)	1 (1%)	14 (17%)
Total	**47 (25%)**	**15 (8%)**	**51 (27%)**	**12 (6%)**	**27 (14%)**

Q2: What more would you like to do about this implementation?

	No Comment	**Nothing more**	**Discuss more**	**Teach for Understanding**
Fall	9 (9%)	17 (16%)	37 (35%)	29 (28%)
Spring	1 (1%)	58 (70%)	15 (18%)	6 (7%)
Total	**10 (5%)**	**75 (40%)**	**52 (28%)**	**35 (19%)**

Q3: What is your suggestion for the follow up of the task?

	Nothing more	Review for Exercise	Review for Learning	Easier Tasks
Fall	5 (5%)	46 (44%)	31 (30%)	6 (6%)
Spring	16 (19%)	42 (51%)	12 (14%)	3 (4%)
Total	**21 (11%)**	**88 (47%)**	**43 (23%)**	**9 (5%)**

Table 1: The frequencies of pre-service teachers' answers to the questions about task implementation

As seen in the table, the PSTs noted that in a 40-minute lesson the students were able to complete the given tasks and discuss their answers (76%). However, some of the PSTs wrote that the tasks were difficult for their students and they only answered one of the sub-tasks. For some of the tasks, the PSTs noted that although the students completed the tasks, there was no time for themselves to discuss students' answers as a group discussion.

The PSTs wrote that some of the tasks did not attain their goals (28%) mainly because of lack of students' prior knowledge (27%). In a few cases, they noted that the students did not understand the task because the text was unclear (3%). Some of the PSTs attributed the success of the tasks to use of materials (6%) and context of the tasks that attracted students' attention (8%). Furthermore, some of the PSTs wrote that the tasks attained their goal because during the group discussion the students recognized their own mistakes and learned from each other (14%). However, in 25% of the reports, the PSTs did not write anything about the reasoning behind the success or failure of the task. Furthermore, we analyzed the pattern in PSTs' perceived cause-and-effect relationship between Q1b and Q1c. Among the total of 141 responses to Q1c, the PSTs wrote that the tasks achieved their goals because students learned from each other while engaging in the task (f: 25), they used materials (f: 12) and they enjoyed it (f: 15). They noted that the tasks were not successful mostly because of students' lack of prior knowledge (f: 38).

As an answer to Q2, the PSTs noted that they were able to do whatever they wanted to do during the implementation (40%). However, some of them noted that there was not enough time to explore how students thought about the given tasks or they were unsure whether the students understood the reasoning behind the answers to the tasks or not, therefore they would like to discuss more about those issues (28%). In some of the cases, especially in the Fall term, because of students' lack of knowledge about fractions and rational numbers, the students could not finish the given tasks. In such cases, the PSTs wrote that they would like to teach about those concepts before or after the implementation if they had enough time (19%). On a total of 6% of the reports, the PSTs did not write relevant answers but summarized how the implementation went.

For the third question, 12% of the reports (not shown in the table) included irrelevant answers such as the PSTs suggested encouraging students to read more books to improve their reading skills or they criticized themselves in terms of not managing time better. In a few reports, they wrote that the tasks were difficult for the students; there should be easier tasks on the same content domain (5%). However, they mostly suggested making review exercises for students to reinforce their understanding of the tasks as well as the content domain (47%). When they realized that the students did not know much about the content, they suggested remedial lessons for them (23%).

PSTs' reflections on the tasks and the intervention study

At the end of the semester, we asked the PSTs to evaluate this intervention study including the effectiveness of the tasks and the contributions to their own professional development. We specifically asked them to determine five mathematical issues in which students had experienced the most difficulty and two tasks that they would like to revise. We also asked them to give the reasoning behind the selection of those tasks and their revisions.

According to the PSTs, among the others, the students had difficulty in the tasks which were about fractions and rational numbers ($f = 16$), algebraic expressions ($f = 9$), solving equations ($f = 7$), area problems ($f = 7$), and discovering patterns ($f = 5$). They wrote that the students could not do these tasks completely because of their lack of prior knowledge ($f = 17$), lack of attention ($f = 7$), lack of review exercises done at home ($f = 5$), lack of understanding of the tasks ($f = 3$), and personal insecurity ($f = 2$). Specifically, they noted that students had difficulty in fractions, rational numbers and area problems because they did not know the algorithm for four operations with rational numbers as well as the area formulas of quadrilaterals and circle. They wrote that students failed in doing operations and discovering patterns because they did not practice enough at home or did not pay attention to the operations and procedures. However, the tasks that they would like to revise were mostly about integers ($f = 7$), algebraic expressions ($f = 4$), fractions and rational numbers ($f = 3$), area problems ($f = 3$), and transformations ($f = 3$). The PSTs wanted to make revisions to the tasks about integers not because of students' lack of knowledge but because the context of the tasks was confusing for the students. Therefore, they noted that they would rephrase the text and change the order of the sub-tasks in those tasks. They decided to make the tasks about fractions and area problems easier because students did not possess the required knowledge to complete the tasks. Furthermore, in some cases the PSTs preferred to change the tasks that were done by the students but they were uncertain whether they understood them thoroughly or not.

The PSTs prepared a set of tasks for their own group of students at the end of the semester. Their rationale for their tasks was either to focus on the issues where the students experienced the most difficulty ($f = 13$) or to make an overall review of what was done during that semester ($f = 4$). In parallel with their comments about the students' mathematical difficulties, they prepared tasks about integers, fractions and rational numbers, area problems, algebraic expressions and solving equations. Some of the PSTs kept their tasks as simple as possible because of their students' poor performance on earlier tasks. Furthermore, 7 of the PSTs prepared their tasks around a common theme as we did in this study but 10 of them prepared separate problems related to the content domain that they focused on.

In the post open-ended questionnaire, the PSTs wrote that this intervention study contributed to their professional knowledge in several ways. Among the others, they noted that they practiced how to scaffold students' understanding without directly telling them the solution or answer ($f = 7$), they got better in anticipating students' possible difficulties in mathematics ($f = 5$), they learned to be patient ($f = 4$), and they learned what kind of tasks attract students' attention more ($f = 3$).

Discussion

The findings of this study have potential to contribute to the relevant literature that teachers learn from their own task design and implementation practices in terms of better understanding of students' mathematical thinking and how to use or tailor tasks to scaffold students' understanding (Zaslavsky, 2007). Although we have not yet analyzed all data, we recognized that the PSTs were able to evaluate the task implementation process in terms of the facts related to the task itself, the organization, the students and their own professional development.

When we asked the PSTs to write their thoughts about the success or failure of the tasks, they were able to make reasonable inferences from the implementation. They recognized that the tasks were successful because the students were actively involved in the solution and discussion process (14%, see Table 1, Q1c), the materials were appropriate for the tasks (6%, see Table 1, Q1c), the tasks were aligned with the 7th grade curriculum and they were in the role of facilitator of group discussion (28%, see Table 1, Q1c and results of post questionnaire). In only a few of the cases, they noted that the students failed to complete the task because the task was unclear for the students (3%). For the other cases, they did not blame the tasks but the students because they did not possess necessary prior knowledge that they learned in previous grade levels or in their regular mathematics lessons (27%, see Table 1, Q1c). The PSTs' interpretations revealed that they began to recognize students' mathematical understanding as well as the importance of preparing tasks both aligned with the curriculum and appropriate for a particular group of students (Hill et al., 2008). For instance, while preparing their tasks, they preferred not to use higher cognitive demand tasks but the easier ones because some of their students did not know about the procedures required to solve given tasks, such as the rules of four operations with rational numbers. The PSTs also commented that students should revise the issues discussed in their mathematics lessons in order to understand the mathematics conveyed in the tasks (47%, see Table 1, Q3). Moreover, as we inferred from their answers in the questionnaire they appreciated the facilitator role of the teacher during the task implementation. They stated that they suppressed their feelings of telling and teaching when students could not figure out the solutions. Briefly, their reflections on the tasks and the implementations indicated that they were aware of the task implementation process which begins with selection of appropriate tasks, continues with implementation of the tasks and ends with evaluation of students' learning (Stein et al., 1996). Because orchestration of task implementation process is involved in teachers' PCK, such awareness of the PSTs can be counted as a sign of their PCK (Charalambous, 2010). However, because we have not yet analyzed the implementation videos, we are not able to validate PSTs' reflections on the tasks implementation process, especially whether they were able to implement the tasks without loss of fidelity and manage the group discussions appropriately. Indeed, it is hard for teachers, even more so for PSTs, to keep the cognitive demand of the tasks such that they might have provided hints or helped students when the students did not possess the required knowledge (Stylianides & Stylianides, 2008). Therefore, we are not able to comment on their "PCK in practice" even though we could make inferences about their PCK from their written reflections.

Finally, the analysis of pre and post questionnaires revealed that the PSTs had positive beliefs about use of student-centered teaching strategies, mathematical tasks and manipulatives while teaching mathematics and such beliefs sustained and even increased throughout the study. They recognized

that tasks provided an opportunity for them to elicit students' mathematical understanding and they could be used as a tool to foster students' mathematical understanding.

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Students' utilizations of feedback provided by an interactive mathematics e-textbook for primary level

Sebastian Rezat

University of Paderborn, Germany; srezat@math.upb.de

Feedback is acknowledged as an important influential factor on learning and achievement. An affordance of digital learning tools is that they provide different kinds of feedback to students. Research on the effectiveness of feedback has mainly focused on different forms of feedback and its timing assuming that different students react homogeneously to feedback. This paper provides a qualitative in-depth analysis of two third grade students' responses to feedback in an interactive e-textbook environment. Students responses to feedback are conceptualized in terms of utilization schemes within an instrumental approach. Results indicate that students utilize feedback differently, which has consequences for the effectiveness of the feedback.

Keywords: Feedback, e-textbook, instrumental genesis.

Introduction

Feedback is widely acknowledged as an important influential factor on learning and achievement (Hattie & Timperley, 2007). The fact that interactive digital learning tools constantly provide feedback to the learners' actions with the contents is indeed one of the most emphasized advantages of learning with digital tools (e.g. Mason & Bruning, 2001). In fact, an outstanding defining aspect of interactivity is that users get immediate feedback to their actions with the tool.

According to Hattie and Timperley (2007, p. 81) feedback is understood as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding". The goal of feedback is to support understanding and/or performance. In line with this, Shute (2008, p. 154) defines formative feedback as "information communicated to the learner that is intended to modify his or her thinking or behavior for the purpose of improving learning". According to Hattie and Timperley (2007, p. 87) effective feedback has to address three questions: "Where am I going? How am I going? Where to next?".

Research related to feedback aims at identifying features of feedback that increase its efficiency. Two aspects seem to be important for effective feedback: 1) the information provided by feedback and 2) the timing of feedback. Mory (2004, p. 753) distinguishes five categories of feedback regarding information complexity: "1. No feedback means the learner is presented a question and is required to respond, but no indication is provided as to the correctness of the learner's response. 2. Simple verification feedback or knowledge of results (KR) informs the learner of a correct or incorrect response. 3. Correct response feedback or knowledge or correct response (KCR) informs the learner what the correct response should be. 4. Elaborated feedback provides an explanation for why the learner's response is correct or incorrect or allows the learner to review part of the instruction. 5. Try-again feedback informs the learner when an incorrect response and allows the learner to one or more additional attempts to try again."

Research has shown that both, the wrong form of feedback and the wrong timing might even have negative effects on learning and achievement (Fyfe & Rittle-Johnson, 2016a; Hattie & Timperley,

2007). The majority of studies in this context quantitatively measures effect sizes of different forms or timings of feedback in order to draw conclusions regarding the effectiveness of feedback. The underlying assumption in these settings is that students will react consistently to the respective form or timing of feedback. While different conditions of providing feedback that might influence its effectiveness have been studied, e.g. prior knowledge (Fyfe & Rittle-Johnson, 2016b) or feedback specificity (c.f. Shute, 2008) students' individual ways of responding to and making use of feedback have scarcely been studied in mathematics education. Bokhove (2010) presents an exception. He reports from a study where student inquiry about desired feedback was used in order to develop a feedback design of a digital tool to learn algebra. He concludes that "asking students when to use what feedback can improve a digital tool" (Bokhove, 2010, p. 125). Most of the research on feedback is based on experimental testing (Shute, 2008, p. 156). Shute (2008, p. 156) summarizes that „the specific mechanisms relating feedback to learning are still mostly murky, with very few (if any) general conclusions". The aim of this paper is to contribute to the understanding of the mechanisms relating feedback to learning. In particular, the focus is on two research questions: 1) How do students individually utilize feedback in order to improve their understanding and performance?; 2) What are the consequences of students' individual utilizations of feedback with regard to the efficiency of feedback?

Theoretical framework and methodology

In this paper, feedback is regarded as an artifact, which is developed in order to improve students' learning and achievement. According to Rabardel (2002) an artifact is transformed into an instrument in use. An instrument is a psychological entity that consists of an artifact component and a scheme component. In using the artifact with particular intentions the subject develops and adjusts utilization schemes, which are shaped by both, the artifact and the subject. This process is referred to as "instrumental genesis" (Rabardel, 2002). According to Vergnaud (1998, p. 167) a scheme is "the invariant organization of behavior for a certain class of situations". Vergnaud (1998) suggests that schemes are in particular characterized by two operational invariants, which refer to the knowledge included in schemes: theorems-in-action and concepts-in-action. The difference between both operational invariants is that of relevance and truth. While "concepts-in-action are relevant, or not relevant, or more or less relevant, to identifying and selecting information", "theorems-in-action can be true or false" (Vergnaud, 1998, p. 173).

With regard to research question 1, students' utilization of feedback is conceptualized in terms of their concepts-in-action and theorems-in-action, which guide their utilization schemes of the feedback. The situation that the scheme refers to is defined by the type of task with respective feedback.

Methods

The study presented in this paper is part of a larger study, which aims at understanding students' learning with interactive e-textbooks at primary level. Therefore, we used an interactive e-textbook that is available on the German market. The e-textbook "Denken und Rechnen interaktiv 3"[1] was the only interactive textbook for primary level that was available on the German market when the study

[1] http://www.denken-und-rechnen-interaktiv.de

was carried out. The e-textbook was not developed for the sake of this investigation, but by one of the leading German publishing companies for textbooks. Consequently, the design principles for the feedback are not known and do not necessarily take into account the current state of research in this field.

In this paper, a case study with altogether 12 cases is presented. All students were at the end of third grade with an age between 8 and 9 years. Each case works on a tablet in experimental conditions on tasks from one unit of a beta-version of the web based interactive mathematics e-textbook. During data collection the students encountered the interactive e-textbook for the first time. The students were asked to verbalize their thoughts (thinking aloud). Additionally, the interviewer asks questions in order to understand the students' actions and thoughts. The interviewer also gives hints in order to assist students' instrumental genesis. The work of the students was video recorded. Data was transcribed and analyzed in terms of concepts-in-action and theorems-in-action. As the name of these concepts indicates, these are mainly inferred from students' actions. Only sometimes students explicate the concepts and theorems guiding their actions verbally. Concepts-in-action were inferred from the data by constantly asking 'what are the concepts / relevant categories / notions guiding the student's action?'. Accordingly, theorems-in-action were inferred from the data by asking 'what assertions / beliefs assumed as true by the student guides the student's action?'

Since utilization schemes are defined as "invariant organization of behavior" (Vergnaud, 1998) it might seem questionable to investigate them with children that encounter an e-textbook for the first time. However, utilization schemes do not develop from scratch, but can be understood as adjustments (accommodation) of existing schemes. In fact, this experimental setting allows for analyzing the instrumental genesis in terms of accommodation of existing schemes.

Analysis

Analysis of the artifact

An analysis of the task and related feedback is a prerequisite in order to understand students' utilization schemes of the feedback, because they are influenced by the affordances and constraints of the artifact in the instrumentation process (Rabardel, 2002). Task no. 2 on page no. 73 and related feedback is analyzed for the scope of this paper. It is depicted in figure 1. The task is to find solids in the picture ("Which solid shapes do you find?"[2]) and to enter their names into the empty fields. After pressing the OK-button on lower right corner of the screen the software provides knowledge of results (KR) feedback. Correct inputs are shown in green color with a green frame, wrong inputs stay as they were entered for a few seconds. Afterwards they disappear and the empty fields are shown again (figure 2). Correct answers stay on the screen and the student can enter new inputs into the empty fields. Students get the opportunity to correct their wrong inputs twice. After their third try knowledge of correct response (KCR) feedback is provided by showing all the correct answers in green color. The ones that were entered correctly by the student are framed. An answer is evaluated as correct if

[2] This was the task in the beta-version of the interactive textbook. In the latest online-version of the textbook the task was changed to "Enter befittingly sphere, cube, cylinder or pyramid".

the student entered the correct name of the solid in the correct spelling. The feedback does not differentiate between incorrect spelling mistakes ore incorrect solid names.



Figure 1: Task

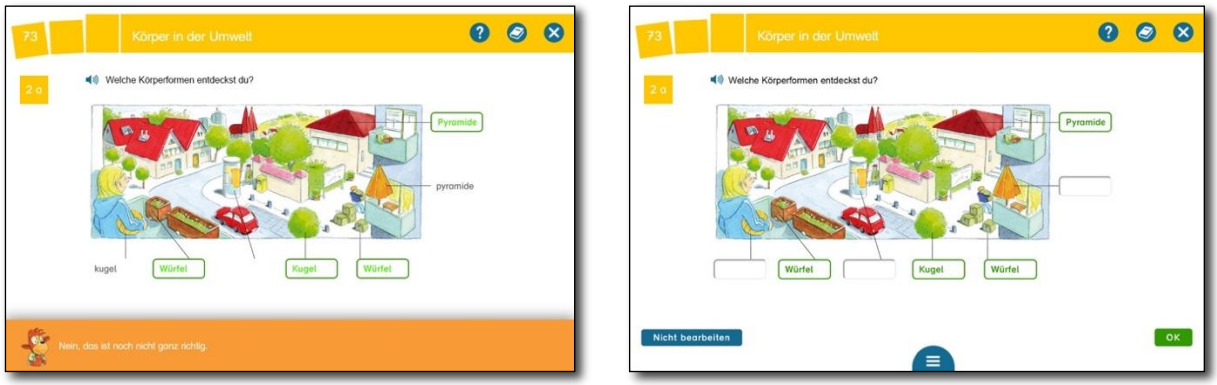

Figure 2: Task-level-feedback (KR) and screen for second try

Analysis of students' utilization schemes of the feedback

In this section data of two cases will be analyzed. The analysis of both cases starts at the moment, when the students have filled in most of or all empty fields and press the OK-button at the lower right corner of the screen. After pressing the OK-button the KR-feedback appears on the screen.

Case 1: Farrell

On Farrell's feedback screen four out of seven answers are depicted in green color and with a frame and stay on the screen while three of his answers remain as they were entered and disappear after a few seconds. He gets the feedback "Sorry, wrong".

13'03"	Interviewer	What happened?
13'08"	Farrell	Correct answers are green and what is gone now was wrong
13'18"	Interviewer	Ok. And do you have an idea why it was wrong?
13'22"	Farrell	Wrong spelling? And wrong entry.
13'30"	Interviewer	Could be. … Think about it. What else could you write or how could you write it differently if you think it also might be because of the spelling
13'35"	Farrell	*Tips into an empty field and starts typing 'Cube' in correct spelling.*
13'54"	Interviewer	Uhum, now you say it's a cube. What did you enter before?
13'56"	Farrell	Square.

13’57”	Interviewer	And why do you think it is the cube now?
13’58”	Farrell	Because, before, the square is not an object, no symmetrical figure
14’12”	Interviewer	Ah, ok, and now you believe it’s the cube.
14’15”	Farrell	Yes, because the square can only be seen and not touched
14’19”	Interviewer	Ah, ok.
14’35”	Farrell	*Types ‘cuboid’ in one empty field in the same wrong spelling as the first time.*
14’08”	Interviewer	Which form could that be?
15’12”	Interviewer	Can the book help you somehow?
15’21”	Farrell	*Presses the ‘?’-button and choses “help” (the help screen appears). He reads the help screen.*
15’45”	Interviewer	Does that help you somehow? What do you see?
15’52”	Farrell	There is everything that could give me a hint.
15’56”	Interviewer	Hmm, ok. … And does that help you for the task?
16’02”	Farrell	No.
16’04”	Interviewer	No. Is there maybe another function that could help you?
16’09”	Farrell	I will check. *Presses the ‘?’-button and points on the option ‘Lexikon’.* Lexicon.
16’13”	Interviewer	Yeah, click it.
		Farrell explores the lexicon. The interviewer asks questions, which are related to the use of the lexicon. He looks for ‘form’ and “solid”.
18’43”	Farrell	I believe it starts with a ‘Q’. *At the same time, he opens the letter ‘Q’ in the lexicon and looks at the entries.*
18’58”	Farrell	No, but I found something different.
19’02”	Interviewer	Ok. What did you find?
19’04”	Farrell	The cuboid (*in German: Quader*) *Returns to the screen/tab with the task and tips into the field with the entry “Qader”.*
19’09”	Interviewer	You already wrote cuboid there. So what did you find?
19’12”	Farrell	That in-between the Q and the A there is a U.
19’18”	Interviewer	Ah, that means the lexicon helped you a little.
19’28”	Farrell	*After completing to type the word ‘cuboid’ he presses the OK-button. All his entries are shown in green color with a frame. He gets the feedback “No, not correct yet.”*
19’30”	Interviewer	And, what now?
19’31”	Farrell	I looked if what I wrote now is already correct.
19’38”	Interviewer	And, how does it look?
19’38”	Farrell	Correct.

The relevant concepts that guide Farrell’s revision of his answers are verbalized in the beginning of the episode: Farrell verbalizes his interpretation of the feedback at 13’08” and also explicates two concepts of possible mistakes at 13’22”. Accordingly, Farrell’s utilization scheme of the feedback is guided by two concepts-in-action: 1) Correct answers are shown in green; wrong answers are shown in red (not explicit, but likely) and disappear (13’08”); 2) His concept-in-action of mistakes indicates that two kinds of mistakes are possible: Wrong names of the solids or wrong spelling of the names

(13’22”). The latter concept-in-action is supported by his way of proceeding with the task. On the one hand he thinks about different entries (13’35”) and on the other hand he is sensitive about the spelling (18’58”-19’12”).

Case 2: Edda

Edda gets the feedback “No, that it not quite correct”. On the screen, the fields around her entries disappear and all her entries stay on the screen. An analysis of her answers reveals that five of her answers (from left to right and bottom to top: 1, 3, 4, 5, 7) named the solid correctly but were spelled wrong with no capital first letter, while two of her answers (2, 6) also contained a wrong name of the solid.

23’51” Edda	It says that it is all not quite correct. *Looks at the screen*
23’55” Edda	*All (wrong) entries disappear. All fields are displayed empty again.* Ah, boy!
24’02” Edda	*Looks at the screen.* Hm.
24’07” Edda	But this must be a sphere.
24’12” Interviewer	Do you have an idea what may could have been wrong? …
24’16” Edda	Yes, this and this (*points at the two empty fields at the right side of the picture)*
24’20” Interviewer	… because everything disappeared.
24’27” Edda	Well, this can’t be a cone. … Then I simply do again … It’s a sphere, definitely. *Types ‘sphere’ in the same (wrong) spelling as before in the empty field.*
25’00” Edda	Hm, it’s bad, now I don’t know what was wrong and now one cannot …
25’10” Interviewer	What is it that you can’t?
25’12” Edda	… ehm, know what is wrong, I know … ehm … I have to do everything from scratch … (not understandable) this and this was wrong (*points at the two empty fields at the right side of the picture*)

Edda’s feedback screen shows all her entries in the way that she typed them. She seems frustrated that all her entries disappear from the screen (23’55”). Her answer to the interviewer’s question at 24’12” reveals that she does not infer from the feedback that all her answers were wrong, but has her own beliefs about her wrong answers (24’16”). Her belief that her two answers on the right side of the picture were wrong seems to be stable throughout the episode (25’12”). However, these beliefs are not congruent with her actual performance on the task. At 25’00” it becomes apparent that she does not have the concepts available to make sense of the feedback given by the tool.

Although she does not say it explicitly, her actions seem to be guided by the concept-in-action that she has to enter the correct names of the solids. There is no indication of her being aware that it is not only the name of the solid that is relevant, but that correct spelling is also a relevant aspect of the name related to this task.

Discussion

The analysis of students’ utilization schemes of feedback shows that both students do not activate the same concepts-in-action when they utilize the software’s feedback (research question 1). While Farrell’s utilization scheme is guided by the two relevant concepts-in-action “name of the solid” and “spelling of the name”, Edda’s interpretation only refers to the “name of the solid”. Edda seems surprised and disappointed that all her entries disappear on the screen and her belief that only two of

her entries were wrong seems to be stable. This indicates that she does not seem to have relevant concepts available in order to utilize the feedback.

However, it is important to note that the different utilization schemes of Farrell and Edda appear under different conditions. The intended interpretation of the feedback occurs in a situation when the student has got correct and wrong answers. In the case of Edda, it becomes apparent that she has difficulties to utilize the feedback. This is supported by findings from other cases in the study that have got all answers wrong in their first attempt. Two hypotheses can be inferred from this observation (research question 2): 1) In order to make effective use of the feedback it is important that students have both, correct and incorrect answers. If all answers are wrong it is more difficult to utilize the feedback, because it is more difficult to make sense of the feedback. 2) On the other hand, the findings might hint at an overall connection between the mathematical ability of the students and their ability to utilize the feedback effectively. Students who need the feedback most in order to improve their mathematical performance, because they have got many answers wrong in their first attempt have the most difficulties to utilize the feedback for improvement.

Conclusions

The analysis of two cases' utilization-schemes of feedback in an interactive mathematics e-textbook, which was used for the first time, is too limited in order to draw far reaching conclusions. However, the results show that the feedback of this task can be optimized. For some students it seems to be important to get more detailed feedback about the kind of mistake they made, especially if mistakes from different domains such as mathematics and language are relevant for the evaluation of the solutions. Altogether, the results support the call for adaptive feedback systems in digital learning systems (Vasilyeva, Puuronen, Pechenizkiy, & Räsänen, 2007).

The fact, that the feedback of the software is not sensitive to different kinds of mistakes can in fact be appraised differently depending on the pedagogic perspective. From the perspective of concept development, it is a constraint of the feedback that it is not sensitive to the kind of mistake, because it does not provide detailed information related to the question "Where to next?", i.e. detailed information about "what is and what is not understood" (Hattie & Timperley, 2007, p. 90). Research has shown that "that feedback is significantly more effective when it provides details of how to improve the answer rather than just indicating whether the student's work is correct or not" (Shute, 2008, p. 157). From the perspective of integrated mathematics and language learning it might even be an affordance that the name and the spelling of the name of the solid have to be correct in order to be evaluated as a correct answer.

On the other hand, the reconstructed concepts-in-action and theorems-in-action that guide students' utilization schemes of feedback indicate that students interpret feedback differently. Therefore, the efficiency of feedback is not only a question of the features of the feedback, but also a question of students' utilization-schemes of feedback. Like with any other artifact, students have to instrumentalize and develop utilization-schemes of the feedback so that it becomes an instrument for improving understanding and performance.

Acknowledgement

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Documentational trajectory: A tool for analyzing the genesis of a teacher's resource system across her collective work

Katiane de Moraes Rocha and Luc Trouche

École Normale Supérieure, French Institute of Education, Lyon, France

katiane.de-moraes-rocha@ens-lyon.fr; luc.trouche@ens-lyon.fr

Our contribution to TWG 22 is dedicated to discussing teachers' interactions with resources for planning their classroom instruction, particularly in the context of collective work. Each teacher during his or her professional life uses and creates many resources. To analyse the history of teachers' work with resources, we propose the concept of the "documentational trajectory". This idea is based on and aims to contribute to the development of the documentational approach to didactics. We will present a case study of one middle school teacher. The data related to this teacher's work allowed us to consider her documentational trajectory. We then used the teacher's documentational trajectory to analyze her professional development. The teacher's documentational trajectory demonstrates a strong participation in collective work, in particular a collective named SESAMES, has an essential role. This participation contributes to the emergence of a particular resource, called a metaresource, to structure her documentation work.

Keywords: Documentational approach to didactics, documentational trajectory, metaresource, reflective investigation, professional development.

Introduction

The new possibilities arising from communication and information technologies have had a significant impact on discussions on mathematics education, due to their impact on resources available to teachers and the way of designing them. This has led to a new conceptualization of teaching resources (Adler, 2000) and the consideration of teachers' new relationships with these resources (Remillard, 2005). Drawing on these studies, Gueudet and Trouche (2009) have analyzed teachers' professional development through the lens of resources: they introduced the *documentational approach to didactics* to analyze how teachers select, use, and produce their resources, along with a process called *documentation work* (ibid, 2009). More recently, Bozkurt and Ruthven (2015) have shown how digital resources structure classroom practices, evidencing five main features: *working environment, resource system, activity format, curriculum script*, and *time economy*. In this article, we will rely on and develop these approaches for analyzing teachers' history of their resource usage. We will propose a new concept for modeling this history: the teacher's *documentational trajectory*. We will mainly discuss here *the effects of a teacher's collective work on his or her documentational trajectory*. In order to do this, we will organize our contribution in four sections. In the first, we will introduce the keys concepts that structure our analysis. In the second, we will present our methodological choices. Next, we will develop a case study based on a middle school teacher, Anna. And in the last section, we will propose some final considerations and openings for future research.

Theoretical framework and research issues

We will present and develop in this section the theoretical approaches that form the basis of our analysis: the documentational approach to didactics, the structuring features of teachers' practices, and the notion of *thought collective*. Then we will present our own propositions.

As stated by Gueudet and Trouche (2012, p. 24) "teachers interact with resources, select them and work on them (adapting, revising, reorganizing, *etc.) within processes where design and enacting are intertwined*". This process is the central focus of the Documentational Approach to Didactics (Gueudet & Trouche, 2012) that grounds our work. In this approach, the term *resource* is used in a broad sense, as everything that nourishes teachers' work. To prepare their teaching, teachers work on resources, and the result of this process is called a *document*. A document is made of *resources* that have been modified and re-organized, and by the *knowledge*, both guiding, and produced by, teacher's work. The *document* is therefore subjective, because it is created through a process of knowledge development on the part of the subject. And the new resources generated or used in this process take place in a very structured set of teachers' resources, called a teacher's *resource system*. To develop this system, some resources introduced by Prieur (2016) as *metaresources* play a critical role.They design resources that support and guide the creation of other ressources and, beside that, favor teacher's reflection about their documentation work.

Teachers' documentation work grounds teachers' classroom practices, structured, according to Bozkurt and Ruthven (2012) by five *features:* the *working environment,* where class takes place (infrastructure, social organization, etc.); the *activity format,* which comprises the body of work in the classroom, such as routines and models of interactions between teacher and student along teaching and learning; the *resource system,* which gathers tools and materials for class. The *curriculum script* is to be understood in the cognitive sense of structured organization of activity guiding a teacher's work in the classroom: goals, actions, activities, potential difficulties of students, among others. And *time economy*, based on the comparison between the teaching time and the learning time of students. Bozkurt and Ruthven use this framework for analyzing teachers' integration of new digital resources into classroom practices. We will extend this usage for analyzing teachers' design and usages of resources. Thus, we will retain the notion of a *resource system* as proposed by Gueudet and Trouche (2012):

> "we consider here as resource system does not fully coincide with Ruthven's definition, because of the broader meaning of resources we retained. The resource system comprises material elements, but also other elements that are more difficult to collect, like conversations between teachers" (p. 27).

We consider that teachers' interactions with colleagues are likely to foster their documentation work and professional development (Gueudet & Trouche, 2012). This is the reason why we give a primary importance to *collectives*. We have retained the broad definition of *thought collective* (Fleck, 1934, p. 44) that exist when "two or more people are exchanging thoughts" and generating a *thought style* "characterized by standard features in the problems of interest to a thought collective, by the judgment which the thought collective considers evident, and by the methods which it applies as a means of cognition" (ibidem, p. 99). We will differentiate the natures of collectives, according to their duration

(stable vs. unstable), their organization (formal vs. informal) and type of participation (voluntary vs. obligatory).

Finally, our proposition for modeling a teacher's history with resources based on the notions of resource, collective, and *event*. An event is something that happens in a teacher's professional life, and that he or she has remembered as important regarding her documentation work. We define the teacher's *documentational trajectory* (Rocha, 2016) as the interplay, over time, between events and resources.This interplay is socially situated, because it happens in schools or collectives, or because the events or the resources themselves are social products. The design of a teacher's documentational trajectory is then a way to analyze *when*, *where*, *why*, *how* and *which* resources are created. We will focus in this article on the documentational trajectory as a tool for analyzing the development of a teacher's resource system across their collective work. We describe, in the following section, our methodological choices for such a design.

Methodological design

Our methodology is inspired by the four principles of *reflective investigation* proposed by Gueudet and Trouche (2012): "*long-term follow-up"*, "*in- and out-of-class follow-up"*, "*reflective follow-up*" and "*broad collection of the material resources".* This methodology gives a major importance to specific drawings made by a teacher (for example the 'schematic representation of their resource system'). To retain this way of reflective investigation by the teacher of her documentation work, we propose some evolutions: instead of the word "representation", we propose (Rocha, 2016) the word "mapping", integrating the metaphor of a *progressive exploration* of a new territory. And we propose two kinds of mapping: *reflective* (made by the teacher herself) vs. *inferred* (made by the researcher) mapping. We differentiate then Reflective, vs. Inferred Mapping of a teacher's Resource System; Reflective vs. Inferred Mapping of a teacher's Documentational Trajectory. For the design and analysis of a documentational trajectory, we also use: interviews, follow-up of lesson preparation, and classroom observation, and a logbook filled by teachers.

Our current research is mainly based on case studies. To choose teachers, we searched teachers that had *Sésamath* textbooks as official textbooks in their class. Sésamath (http://www.sesamath.net/) is an association of mathematics teachers in France that collaboratively designs online resources (software, textbooks, etc.) at a very large scale, opening a window for us on advanced teachers' design, use and sharing of resources. We present here the case of Anna, a middle school mathematics teacher, whose school had chosen a Sésamath textbook. She has a strong partnership with a colleague from her school, Cindy, and both of them participate in various collectives and use a lot of digital resources beyond the Sésamath textbook.

Our work with Anna started in March 2015, when we followed her 6th grade class for three months. During this period, Anna created and shared with us a Dropbox folder, where she uploads resources that she used in her lesson or to prepare it. In addition, we also recorded four moments of interaction involving her documentation work. In the first one, she made a reflexive mapping of her documentational trajectory. In the second one, she reviewed her reflexive mapping (focusing on a particular resource and a specific year). In the third one, she prepared a lesson with Cindy about a new curricular subject. In the last one, she spoke about her usages of a particular digital tool, a padlet

(https://padlet.com/) used to save and organize resources found online. Also, we had her use a logbook to complement our data collection when we could not follow her documentation work.

We will explore in this paper different mappings of Anna's documentational trajectory: inferred and reflexive mapping. To obtain the initial reflexive mapping, we asked her to write down above a timeline the main events that have influenced her use of teaching resources teaching, or the way of conceiving them, and to write down, below the same timeline, the resources associated with the event at stake. To help Anna, we gave her a sample of possible events: the arrival of a new person in her school; the participation in a new collective, an unexpected interaction with a student or colleague; a change of program; a change of teaching level or of textbook; a training course or the discovery of a new resource related to mathematics teaching (book, movie, website, etc.).

Our method of analysis is in development. In this article, we transcribed our first interview with Anna and we did a digital transposition of her reflexive mapping. Afterwards, we identified on the map, among the events she exposed, those related with collective work, following our hypothesis that knowledge is socially situated. After that, we identified the effects of collective work on her documentation work through associated resources at the event. Afterwards, we inferred from Anna's speech her role in collectives and the nature of each of these collectives, following the hypothesis that some features could inform us on the collective effect on a teacher's documentation work. For example, a collective where teachers are voluntary and have a long or permanent engagement nourishes teachers' work differently than an obligatory collective where they have a short time engagement. Then, we searched collectives that have an important status in Anna's documentational work. For this purpose, we looked for the collectives that appeared more frequently and related to other events in the map. Afterwards, we analyzed how these collectives nourished her documentational work exploring associated resources, and identifying in her words features relating resources and collective work. We will present the main results of this analysis in the following section. This work is still in progress.

Analysis

This analysis is divided into two parts: (1) Anna's participation in collectives along her documentational trajectory and the structuring role of a particular collective, SESAMES in it; and, (2) the structuring role of a particular resource, "Mise en train", on Anna's individual and collective documentation work.

Anna's documentation work in collectives along her documentational trajectory and relationships with SESAMES."

We start analyzing the first reflexive mapping drawn by Anna (Figure 1). The analysis of the events evidences Anna's strong involvement in collective work (she says: "I cannot work alone"). Eight (E6, E7, E8, E9, E10, E12, E13 and E14) over 14 events are related to collectives. For deepening the analysis, we study the *properties* of collectives, the *roles* of Anna in these collectives, and the *functions* of the resources that are designed.

The collectives have different natures:

- some of them are transient, as a short episode of coworking with Sésamath (E4), or Assist Me (E12, linked to a European project), or M@gistère (E13, linked to the design of a

teacher training path); some of them are 'permanent' (meaning that, once Anna enters this collective and stays in it), as APMEP (E10/E11, the French national mathematics teacher association), or LéA (E9, collective linking Anna' middle school and the French Institute of Education), or SESAMES[1] (E6, a team associating researchers and teachers for renewing Algebra teaching);

- some of them are obligatory (as E14, meeting with parents), some of them are voluntary (as E7 the close partnership with Cindy).

In these collectives, Anna can have six different roles: *member*, reading and using their resources (E8 and E10); *author*, conceiving articles and resources for readers external to the group (E6, E8, E10, among others); *teacher trainer*, training middle school teachers (E6, E8, and E10); *teacher researcher*, reflecting about mathematics teaching (E6, E12 and E8); *partner*, exchanging and co-producing resources with colleagues (E6 and E10).

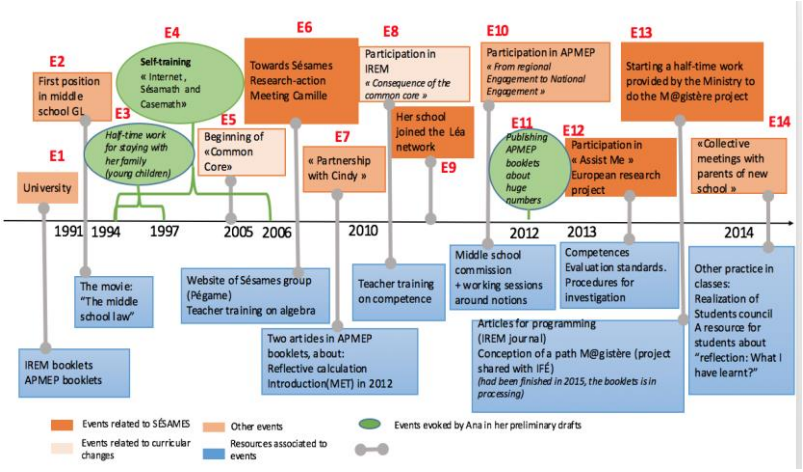

Figure 1. **Anna's reflective mapping (October, 22nd 2016) of her documentational trajectory**

Each collective contributes in different ways to her documentation work. However, they are entangled in a way that it is difficult to attribute a single function to each of them. The interviews with Anna help us to distinguish some structuring features of Anna's documentation work: elaborating her resources for teaching algebra; elaborating activities for teaching mathematical concepts and interpreting curriculum materials; elaborating resources for developing and evaluating students' competencies; creating lessons and curricular scripts for her class; reflecting about using digital resources; creating new resources according to pedagogical changes in the school, supporting her participation in other collectives outside of school, writing papers, and teacher training.

These functions are not supported by all collectives, but all of them are exploited in Sésames, where Anna and Cindy used to work together. In addition, SESAMES gives Anna new possibilities for participating in new collectives and establishing new partnerships. Figure 2 shows how SESAMES resources nourish the resources of other collectives. Develop a critical thinking on their practice. When she was invited to join SESAMES, Anna hesitated "*that was a change, anyway. I accepted, finally, to join ... to join SESAMES. […] It was a real challenge…*".

[1] In spite of the likeness of the acronyms, Sésamath, a mathematics teacher association designing resources at a large scale, is totally different of SESAMES, a small team gathering researchers and teachers for re-thinking algebra education.

SESAMES [2] had a big impact on Anna's collective work. We can see (Figure 1) that she joined Assist me, M@gistère and Léa as a consequence of her engagement in SESAMES. It gave the opportunity for a new partnership with Camille. It helped her understand the competencies emphasized in the new French curriculum, leading tothe design of a teacher training path at IREM, resulting in a chance to join this institute.

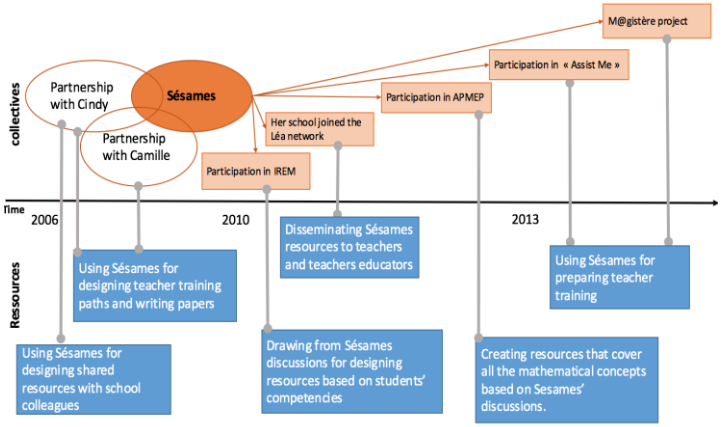

Figure 2. Inferred mapping about the collectives and resource system impact

SESAMES has two sets of *principles* guiding Anna's documentation work (cf. the SESAMES website, Pégame: http://pegame.ens-lyon.fr/), mirroring the thought style of SESAMES. The first set is composed of three principles for teaching algebra: justifying computation throughout algebraic rules; proposing proof activities and exploiting formulas to introduce the concept of a function. The second set is composed of four principles for teaching mathematics: providing students with sufficiently rich and open problems; giving them a chance to explore; giving them a chance to speculate; giving concrete meaning to concepts taught.

These principles gave birth to resources emblematic of SESAMES thought style, guiding then the whole process of collaborative resource design in this group. It's exactly the characteristic of the metaresources we have already introduced in this paper. One of them is *Mise en train*, and we will analyze in the following section its impact on Anna's documentation work.

The structuring role of a metaresource on Anna's individual and collective documentation

The *Mise en train* (MET) corresponds to a specific activity format: it aims to organize teacher's work at the beginning (around 15 minutes) of each class. The expression *Mise en train* has three meanings: the direct one is *warming up* (like for an athlete at the beginning of his training); the second one derives from a literal translation, "*put on a train*", meaning 'cutting a mathematics subject in short successive parts (allowing to store them in the successive wagons of a train); the third meaning derives from an acronym (created by Anna): Travail de Recherche ou d'Approfondissement avec prise d'INitiative (Research and Deepening Work with Initiative Taken). The global meaning of Mise en Train has to be understood as the compilation of these three interpretations. In the following, we have chosen to keep this acronym MET, incorporating this global meaning. This global meaning evidences some features of MET resources' design. MET is exactly a metaresource, as it gives a way to produce new resources and stimulates teacher's reflection on their documentation work and its effect on students' activity.

[2] Science Education: Modeling Activities, Assessment, Simulation (SESAMES, Situations d'Enseignement Scientifique : Activités de Modélisation, d'Evaluation, de Simulation).

MET appears as emerging, in SESAMES, from the documentation work of Anna and her colleagues. Anna explains factors leading them to create this metaresource: the loss of time at the beginning of each class (teacher being mobilized by administrative tasks); the good experience with the short sections of reflecting calculations; and her exchange with English teachers dividing students' activity in short articulated moments for a more dynamic activity format.

Once created, MET deeply changed Anna's documentation work (cf. Figure 4). It affects all five structuring features of classroom practice. The working environment changed, e.g. students entering class late did not disturb class activity. The activity format is also altered, because the class is divided into two moments: MET vs. main class.

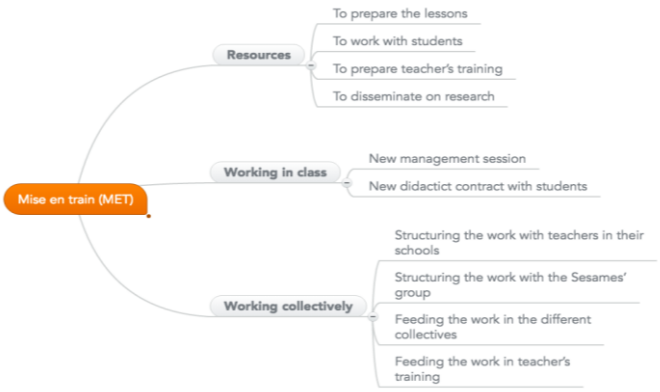

Figure 3. The impact of MET on Anna's documentation work

The *curriculum script* is modified, including new goals and activities. Anna has then three possibilities for developing a lesson: MET then the regular course; the regular course, then MET; or beginning with MET… and going on with MET, for giving more responsibility to students for the advancement of the knowledge in the classroom. Regarding Anna's *resource system*, new resources are created (new curriculum script, new notebooks for students, new lesson plans with MET activities, slides that contains MET activities linked to a given notion, new articles (APMEP, IREM, Pégame website) for disseminating SESAMES resources. Last, but not least, the *time economy* changed, for example, Anna removed the initial "call to students" at the beginning of each lesson.

MET also affects Anna's work in other collectives: in her school, the new curricular script is shared by all teachers, as Cindy and Anna explain the principles of MET, and present their interest form their practice; outside of her school, Anna disseminates this metaresource in SESAMES training, IREM group, and training, APMEP group and training. Finally, this metaresource, initially constructed in SESAMES group to teach algebra, was extended to other mathematical topics. For us, the metaresource *MET* is a point of convergence between Anna's need and SESAMES' interest.

Final considerations and perspectives

Our original question was: *What are the effects of a teacher's collective work on their documentational trajectory?* Our initial analysis of Anna's documentation work gives us some clues. Our exploration of Anna's documentation work in collectives allows us to understand her resource system better. We saw a diversity of collectives that she participates or participated in, and how her different roles contributed to her work. Among them, SESAMES appears as an important collective, having a strong impact on Anna's documentational trajectory. It contributes to developing new collective work, resources and thought styles. In this collective, she contributes to create a metaresource that structures her documentation work afterwards: this metaresource is exploited in various collectives and structures her way to create resources.

We proposed the concept of documentational trajectory for modelling a teacher's history with resources. In this modeling, the reflective and inferred mapping of the documentational trajectory allows us to evidence some critical aspects of this history. It should be noted that these maps constitute a picture at a given moment, and in a given context. This temporal aspect is linked to the fact that Anna's documentation work is still ongoing. The context aspect is also linked to the relationships the researcher can build with the teacher.

The combination of the *Structuring Features of Classroom Practice* and the *Documentary Approach to Didactics* helps us to analyze teachers' documentational trajectory, demonstrating the structuring role of SESAMES and a metaresource associated to a thought collective. Finally, we retain from this analysis that the development of the concept of documentational trajectory was relevant for analyzing interactions between resources, collective work and teacher's practice.

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Designing for responsibility and authority in experiment based instruction in mathematics: The case of reasoning with uncertainty

Abdel Seidouvy[1] and Andreas Eckert[2]

[1]Örebro University, Sweden; abdel.seidou@oru.se

[2]Linnaeus University, Sweden; andreas.eckert@lnu.se

Abstract: This study examines principles of task design concerning the concept of uncertainty in the area of statistics. A purpose is to promote and support students reasoning competency involving the aspects authority and responsibility. By using inferential role semantics as a background theory, we examine students' reasoning by means of how they show authority and responsibility for statements in the reasoning process. Statistical tasks where students generate and analyze their own data formed the basis for this pilot study conducted with seventh grade students in Sweden. The students were able to reflect on how their actions and consequences of their actions influence their reasoning with uncertainty. The study describes the findings, and presents principles to inform the design of innovative learning environments that promote authority and responsibility in reasoning in the domain of uncertainty.

Keywords: Design principles, uncertainty, responsibility, authority.

Introduction

Designing lessons and tasks has been an important part of developing theories of instruction in mathematics for over a century (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) and emerging from this practice is the design-research methodology. Plomp (2013) speaks of a twofold yield in design research-projects, both producing an intervention that offers a solution to a problem in the practice and what he calls re-usable design principles. In the present study, we are aiming for the latter. Design principles are to be seen as heuristics as they do not guarantee success. The principles rather suggest design elements that could support the development of a prototype design of a task (Plomp, 2013). Design principles are guides that answer questions like "What should the lesson look like?" or "How should the lesson be developed?" (Van den Akker, 1999). They should connect task design with both a practical and theoretical understanding of the topic and inform the teaching practice as well as the research practice.

This paper begins in the idea of developing tasks for experiment based instruction in mathematics and the special case of stochastic. Groth (2013), amongst others, argues that stochastic education should be researched in particular since the nature of mathematical reasoning is largely deterministic whereas that of stochastic is with uncertainty. Our task design principles are therefore not focused on the creation of new knowledge but on eliciting stochastic reasoning and developing students' understanding. In the present study, we examine design principles in relation to how meaning making is described in the semantic theory of inferentialism (Brandom, 1994). Our guiding research question is:

- What are the design principles that trigger students' authority and responsibility in reasoning with stochastic uncertainty?

Plomp (2013) stresses the importance of reviewing similar examples when articulating design principles for the first time. Hence, the following section will elaborate on previous examples of experimentation-oriented tasks for stochastic education in the research literature to create a deeper understanding of design principles in stochastic education. Furthermore, the theoretical background of inferentialism gives the tools to see authority and responsibility in student's reasoning connected to the design of the task.

Related works

In the Dutch tradition of RME, three principles inform task design: guided reinvention, didactical phenomenology, and emerging modelling (Freudenthal, 1973; Gravemeijer, 1994). According to the principle of guided reinvention, mathematical tasks should offer students opportunities to experience a process similar to that linked to the invention of a specific topic. On the didactical phenomenological account, task designers should consider how mathematical "thought objects" will be used by the students to structure and organize phenomena in reality. Emerging modelling emphasizes the process on progressive abstraction, from a model of a situation (experimentally real for the student) to a more general mathematical model (Gravemeijer, 1994). Overall, in RME, mathematical tasks are often rooted in Freudenthal's vision of "mathematics as a human activity", and are designed to resemble realistic problems in context. Another school of thought close to RME is authentic teaching and learning in mathematics. The design principles of an authentic practice are used to perform certain actions and procedures, and knowledge as a tool to perform and achieve particular goals (Dierdorp, Bakker, Eijkelhof & van Maanen, 2011). The underlying conjecture in authentic mathematics is that students will be motivated and engaged in rich discussions if authentic practices are used as a source of inspiration in designing mathematical tasks. Ainley, Pratt and Hansen (2006) used *purpose and utility* as design principles in mathematical tasks. According to Ainley, Pratt and Hansen (2006), purpose means that a meaningful outcome of a task is crucial for student learning. Utility refers to acknowledging the power of mathematics ideas. Chance-maker microworld (Pratt, 2000) is an example of a learning environment in which purpose and utility are implemented. The chance-maker is a microworld program with a series of gadgets, simulations of everyday random generators such as spinner, coins and dice. The students are challenged to find the gadgets that according to them do not work (The students were told that the gadgets were programmed) and mend them. The purposeful activity, for the students, of mending the gadgets led to the understanding of the utility of representations (connection between probability and data distribution), and the importance of the law of large number. The students are supposed to discover the relevance of mathematical ideas through realistic situations created in classrooms, real life (Ainley, Pratt and Hansen, 2006) or in the microworld (Pratt, 2000).

Our hypothesis is that a mathematical task that can trigger students' authority and responsibility increases student awareness on the data generation process and, has the potential to develop students' reasoning with uncertainty.

Theoretical framework: The Game of Giving and Asking for Reasons – GoGAR

The framework of our research is the game of giving and asking for reasons (GoGAR); it is a metaphor used by Brandom (1994, 2000) to describe the linguistic practices in inferentialism, which

is the background theory of this study. An inferentialist view on knowledge entails giving priority to inference in reasoning in account of what it is to grasp a concept:

> To grasp or understand […] a concept is to have practical mastery over the inferences it is involved in – to know, in the practical sense of being able to distinguish, what follows from the applicability of a concept, and what it follows from. (Brandom, 2000, p. 48)

It corresponds to the practical mastery of concept and, to the increasing awareness that reasoning is central to statistics and statistics education (Bakker & Derry, 2011). "The game of giving and asking for reasons is an essentially social practice" (Brandom, 2000, p.163), and the purpose of GoGAR is to make explicit reasons that are implicit in our linguistic practice (Brandom, 2000; Bransen, 2002). According to Brandom (1994, 2000), one way of understanding how reasons are made explicit in talk is in terms of interaction of inferentially articulated authority and responsibility. Authority of a claim is a process, capturing the influence of a claim in the GoGAR. However, "Authority is not found in nature" (Brandom, 1994, p.51) but is gained in taking responsibility by providing evidence for one's claims. Students are expected to make claims that are related in a certain way, justify and explain their claims. Responsibility can be defined as a quest for authority, and it also expresses the quality of requirement for performing and maintaining authority (Hansson, 2010). By using GoGAR we aim to show how, independently of the quality of the reasoning, elements of task design eliciting authority and responsibility influences how students reason in the domain of uncertainty.

Method

The present paper focuses on reaching an understanding of the data generation process connected to a mathematical task (Cobb et al., 2003). The data generation process involves clarifying the significance of the phenomenon under investigation, delineating key aspects of the phenomenon that should be measured, and considering how they might be measured (Cobb and McClain, 2004, p.386).

The analytical process of this paper is that of an abductive approach (Alvesson & Sköldberg, 2009). The aim of this approach is to create an initial analytical lens to view the data through, and then allow for the emerging design principles to influence the initial theorization. The specific analytical tool is retrospective analysis by the use of constant comparisons (Gravemeijer & Cobb, 2013). Instances of data are compared to find similarities and differences related to authority and responsibility in the data.

The data set used in this analysis are transcripts of video recorded lessons in a pilot study. The purpose of the study was to document our starting point (Cobb & Gravemeijer, 2003) prior to an initial cycle of a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The design experiment utilizes the idea of engaging students in actual data generation and asking them to reason about the results. Our aim is that the task would trigger students to develop authority and responsibility due to

their active involvement in the data generation. The empirical study was performed with a class of 20 students in grade 7 (aged 12/13) in Sweden using an experiment based task focusing on statistics. The "helicopter task" (figure 1) is a modified version of a task that originates from Ainley, Pratt and Nardi (2001). It involves constructing an auto-rotating helicopter out of paper and measuring five flight times per rotor length. The class dealt with the helicopter problem in two lessons. In the *first lesson*, the students were involved in the data generation by e.g. testing paper-helicopters of varying rotor lengths (3cm to 14cm) and measuring flight times. In the *second lesson*, the class interpreted and evaluated the data in groups, guided by questions provided by the teacher. This was followed by a whole class discussion. The transcripts used here are meant to illustrate the ideas developed in the paper and are chosen because they portray typical student reasoning elicited by the task while using a relativly small amount of space in the paper. The reader is then invited to evaluate the plausibility of our interpretations and thereby assess the trustworthiness of our claims as an alternative to reliability and validity more suitable for this type of research (Lincoln & Guba, 1985)

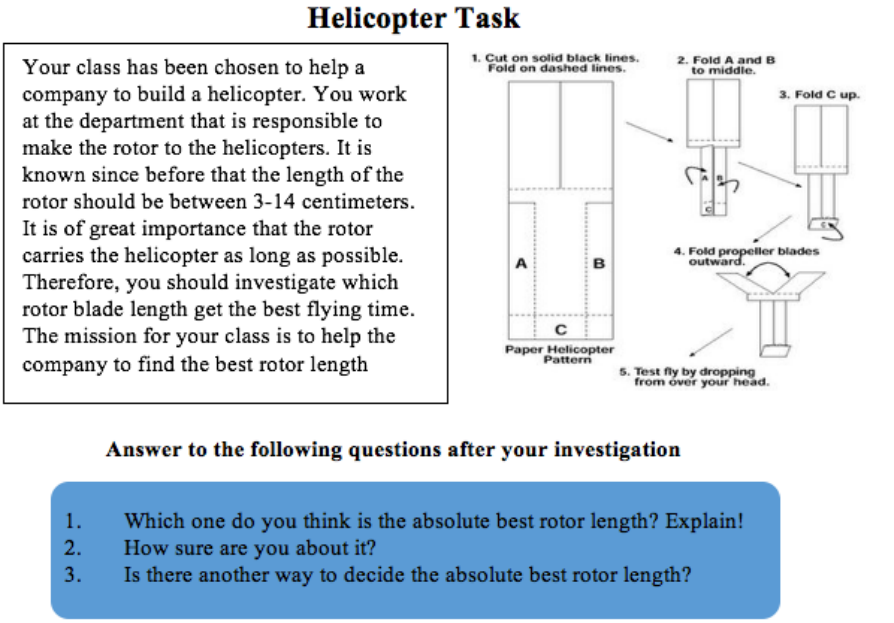

Figure 1 Instruction for the helicopter task translated from Swedish to English

Results

The transcript elaborated on below is used to shed light on aspects of how responsibility and authority, embedded in the task, elicit reasoning with uncertainty. It is part of a whole class discussion in the second lesson where one of the groups presents their findings. The questions from the helicopter task in focus here are: *Which is the best rotor length? How sure are you?* (…) means that portions are inaudible and […] means that we omitted a segment because of space limitation that we feel does not add to the reader's understanding of our analysis. Table 1 contains the students' results.

Table 1 Measured flight time for the helicopters in seconds

	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
6 cm	2.5	2.3	2.5	2.3	2.4	1.9	2.6	2.9	2.6	1.8
7 cm	1.8	1.9	2.7	2.2	2.1	2.4	3.1	2.3	2.3	2.8
8 cm	2.6	1.9	2.6	2.5	2	2	2.4	2.4	3	2.9
9 cm	3.1	3	2.9	2.9	3	1.6	1.3	1.9	1.8	2.1
10 cm	2.2	2.6	2.4	2.8	2.4	1.6	2.5	2.3	2.2	2.2
11 cm	2.2	2.8	2.9	2.1	2.7	1.3	1.9	1.9	1.3	2.8
12 cm	2.8	2.2	2.6	2.4	3.3	1.3	2.3	1.8	1.7	2
13 cm	2.6	2.2	2.5	2.3	1.7	1.7	1.6	1.8	2.4	2
14 cm	4	3	3	3.1	2.1	2.3	1.9	2.2	1.7	2.3

[1] Teacher: Before we move on, may I ask how you noticed that it was a failure with the rotors

[2] Gabriel: Because sometimes were like three, and then some were only 1 second

[…]

[3] James: We aren't so sure because the experiment is done in a small scale therefore we don't have enough data on the functioning of the helicopter, which makes it hard to know which length is the best.

[4] Eric: it also depends on the angle of the rotors (…), it depends on the height differences.

[5] Eric: (…) some might be 4 seconds and others might be 1.48.

[6] Jennie: It also depends on how you drop the rotors, wait, this? (…) You could've calculated it with the median which we should have done.

[7] Maria: But we remembered that afterwards.

[8] Jennie: But we did it with the mean which made it take longer time.

[9] Teacher: why should you have measured the mean then?

[10] Jennie: Because the differences were so big, and the times (…)

[11] Teacher: would the results have been better if you used the median?

[12] James: Yes

[13] Teacher: Why do you think so?

[14] Jennie: Because the differences were so big and the times

[15] Teacher: okay, why did it become such big differences? Was it the failure or?

[16] Maria: Because maybe you drop it from the exact same spot, maybe you drop it further down one time and another time further up.

Responsibility in justifying uncertainty

If we compare instances [2], [4], [6] and [16] there are similarities and differences. All four instances are parts of GoGAR about the level of uncertainty. There are differences as well, for instance how their claims are accepted in the GoGAR and how they situate the responsibility for their claims. In [2], Gabriel reasons that the failure of the rotors becomes apparent in his data, but the reasoning is underdeveloped and no one else acknowledges his authority in the following discussion. In the following lines, not included because of space limitation, instead focus is put on the definition of margin used earlier. Eric's reasoning in [4] on the other hand situates responsibility explicitly in the activity of data generation. He draws on the experience of having been a part of the data generation and claims that both the angle of the wings and the drop height could be sources of uncertainty. Jennie acknowledges Eric's authority on line [6] by referring to the action of releasing the helicopter. We call this *Responsibility in justifying uncertainty*. Both students and teacher recognize that there is a measure of uncertainty in the results and use various informal concepts to indicate it, for example "hard to know" on line [3] and "differences are so big" on line [10]. Both Eric's and Jennie's giving and asking for reasons for the level of uncertainty situate responsibility in the act of data generation. Maria acknowledges that Eric's and Jennie's claims still have authority in the GoGAR on line [16] by relating the mathematical discussion of mean, median and uncertainty to the act of data generation.

Exercising authority by making claims

From our analysis, it appears that students in this study exercise their authority through two types of claims: terminology related claims and context related claims. While the terminology claims are related to students' previous formal knowledge, context related claims are from observations through perception. *Terminology related claim:* We noted that all the students in the group were given the opportunity to express themselves. As whole, the group took the chance and made a considerable number of claims in a limited time period. As mentioned earlier, making claims is one way of

exercising authority. In [2], Gabriel answers to the teacher's question in [1]. A close look at Gabriel's utterance, indicates that Gabriel's authority is grounded in comparing one second to three seconds. In other words, formal logic is what lies behind his authority. In [5], Eric acknowledges Gabriel's claim [2] and undertakes it. We, therefore conclude that "Gabriel lends his authority to Eric". The same line of reasoning is observed in [10] and confirmed in [14]. Other students in their reasoning about uncertainty in GoGAR can use these claims. In [7] and [10], concepts such as mean and median are brought to the discussion. The students' involvement in data generation creates conditions in which students can make use of their previous formal knowledge. In this case the consequence of their actions in generating data, and the uncertainty that follows activates the use of mean and median while talking about uncertainty. *Context related claims:* it is evident that Eric's claim in [4], "it depends on high difference", is also accepted by the class. Hence [4] is based on observations during the data generation. One argument that supports our interpretation is that (here and otherwise in the data) the whole class has reported that it was almost impossible to drop the helicopter from the same height. Eric's claim is licensed by a context. The same observation is made by Jennie in [6]. Further, Eric points out that the angle of the rotor is reason for variation and uncertainty. This observation was specific to Eric's group. However, we believe the rest of the group will use it as a premise in their reasoning with uncertainty if needed. The context created by the data generation enables students to make claims that can be used in reasoning with uncertainty.

Discussion

We organize the discussion around three questions connected to the design that seem to trigger authority and responsibility.

Which different opportunities in the task are given by the students to accept responsibility for their claims? The students reason naturally with uncertainty since they have the practical experience of manually generating data in the task to situate their responsibility. In contrast, Pratt (1998) found that many students in his study based on the chance-maker microworld software had to convince themselves that the gadgets in the game were indeed random and that the mode of reasoning should be with uncertainty rather than a deterministic reasoning. As long as the students had the opportunity to influence, for example, the strength of a simulated dice throw, they were more prone to accept that it was random. One explanation could be that the students working with the chance-maker had to situate responsibility inside a black box, being the software, which led them to believe that the results were predetermined. Our analysis in [2] and [4] show how the task provides opportunity for the students to situate the responsibility of the failure in the measurement. The cause-effect opportunities created by the task initiate fruitful GoGARs in the domain of uncertainty, which also seems to fit with Pratt's (1998) findings.

Which different opportunities in the task give the students the ability to create authority for their claims? From Roth's (1996) study, we know that students show difficulties in analyzing data when they have not been actively involved in the data generation. This view is supported by Noss, Pozzi & Hoyles (1999). Our analysis showed in [2] [3] and [4] that the task enabled the students to act as experts. The students were able to connect data analysis and inferences to data generation while reasoning with uncertainty. The experiment based instruction, which entails manual data generation, opens up for students to take control over the process. Leaving the data generation process to students, triggers their authority and responsibility and, students are more motivated than if they would just be

limited to data collection and get "the right data" (Cobb & McClain, 2004). Getting students to reflect on how their own actions influence the results is the first step in acting with authority and responsibility.

Which different opportunities in the task are given to the students to shift responsibility and authority between different domains? Designing tasks with manual data generation based on principles of authority and responsibility is not the general solution to improve statistics education. In fact, creating large data sets, suggested by Pratt (1998) as being a corner stone in statistics instruction, becomes almost impossible with this focus on manual data generation during lessons. We merely suggest that there are merits to these design principles in statistics education as it naturally elicits GoGARs in the domain of uncertainty and empowers students to take responsibility and use mathematics to back up their claims. In [6] and [8] our analysis in [6] and [8] show how the task provides opportunity for the students' quest of authority as they shift responsibility between the empirical and theoretical domain. We show that designing tasks with manual data generation elicit fruitful situations for shifting authority and responsibility between context and concept levels. Thus the students used their previous concepts to act upon the context, and also looked for evidence in the context to strengthen their authority and responsibility. In sum, creating a learning environment where students can exercise authority by using their previous formal knowledge and most importantly "our imperfect perception" can promote a fruitful GoGAR in the area of uncertainty, and in mathematics in general.

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Promoting prospective elementary teachers' knowledge about the role of assumptions in mathematical activity

Andreas J. Stylianides[1] and Gabriel J. Stylianides[2]

[1]University of Cambridge, Faculty of Education, UK; as899@cam.ac.uk

[2]University of Oxford, Department of Education, UK; gabriel.stylianides@education.ox.ac.uk

Although the notion of assumptions is important in the discipline of mathematics and permeates (often tacitly) mathematical activity in school classrooms, instruction in the elementary school pays little attention to it. This situation is unlikely to change unless elementary teachers have a good understanding of the role of assumptions in mathematical activity and an appreciation of the pedagogical implications of that role. In this paper, we investigate how teacher education can promote this dual goal by illustrating a promising, task-based approach to supporting prospective elementary teachers to develop pedagogically functional mathematical knowledge about the role of assumptions in mathematical activity. We developed the approach in a 4-year design experiment we conducted in a mathematics course for prospective elementary teachers in the United States.

Keywords: Assumptions, elementary grades, prospective teachers, task design, teacher knowledge.

Introduction

Assumptions denote the statements doers of mathematics use or accept (often implicitly) and on which their claims are based (Fawcett, 1938) and, thus, assumptions are fundamental to any mathematical activity at all levels. In school mathematics, however, assumptions often receive little explicit attention, especially at the elementary school level. The notion of assumptions has received also relatively little explicit attention in the mathematics education literature.

A small number of studies addressed the role of assumptions in the particular area of proving at the secondary school (Fawcett, 1938; Jahnke & Wambach, 2013) and elementary school (Stylianides, 2016) levels. However, there is a scarcity of relevant research at the teacher education level. This is problematic: unless teachers are supported to develop a good understanding of the role of assumptions in mathematical activity and an appreciation of its pedagogical implications, it is unlikely that teachers will offer to their students productive learning opportunities in the area of assumptions.

In this paper, we investigate how teacher education can promote prospective elementary teachers' knowledge about the role of assumptions in mathematical activity. We focus on the design and implementation of tasks that can support *Mathematical Knowledge for Teaching* (*MKfT*).

Research background

The notion of MKfT (Ball et al., 2008) denotes the kind of "pedagogically functional mathematical knowledge" (Ball & Bass, 2000, p. 95) that teachers need to have to be able to manage the mathematical demands of their practice. It has been noted that "there is a specificity to the mathematics that teachers need to know and know how to use" (Adler & Davis, 2006, p. 271) and that teacher education should aim to create learning opportunities for prospective teachers (PTs) that would enable them "not only to know, but to learn to use what they know in the varied contexts of

[their] practice (Ball & Bass, 2000, p. 95). Thus mathematics courses for PTs cannot lose sight of the domain of application of the targeted knowledge (i.e., the domain of mathematics teaching).

We consider next what might be essential elements of MKfT about the role of assumptions in mathematical activity. We begin by describing two elements we identified based on consideration of the role of assumptions in the discipline of mathematics (e.g., Fawcett, 1938; Kitcher, 1984) and mathematical analysis of classroom practice where the notion of assumptions received explicit instructional treatment (Fawcett, 1938; Jahnke & Wambach, 2013; Stylianides, 2016).

- *Element 1:* Understanding that a conclusion is dependent on the assumptions on which the argument that led to it was based.
- *Element 2:* Understanding that different legitimate assumptions can lead to different conclusions that, although on the surface may appear to be contradictory to each other, may nevertheless both be valid within the set of their underpinning assumptions.

These interrelated elements are central to mathematical work. The different sets of axioms in Euclidean and non-Euclidean geometries, including associated results, offer an illustration of both of these elements in the discipline of mathematics or in the upper secondary school (Fawcett, 1938). Similar ideas are central also to the elementary school (Stylianides, 2016) and are consistent with recommendations in curriculum frameworks (e.g., NGA & CCSSO, 2010, p. 6). Furthermore, the two elements are essential for mathematical knowledge that is also pedagogically functional. For example, they can allow a teacher to recognize that, when two students offer inconsistent answers to a task, this does not necessarily mean that one of them is wrong; both students may have applied sound reasoning based on different assumptions about the conditions of the task. We capture this kind of pedagogical functionality of elements 1 and 2 under a third element of MKfT:

- *Element 3:* Ability to recognize possible ways in which elements 1 and 2 might apply in mathematics teaching.

Regarding how to promote PTs' MKfT, in Stylianides and Stylianides (2014) we discussed and illustrated a special kind of *mathematics* tasks that we called *pedagogy-related mathematics tasks*. These tasks have two major features: (1) A mathematical focus, which relates to mathematical ideas that are important for teachers to know; and (2) A substantial pedagogical context, which is an integral part of the task and essential for its solution. Notwithstanding the importance of pedagogy-related mathematics tasks, there is also a need for the use of other kinds of tasks in mathematics courses for teachers. In this paper, we focus on a task sequence that illustrates another approach we followed to promote MKfT. Unlike pedagogy-related mathematics tasks where mathematics and pedagogy are intertwined in the same task, this task sequence illustrates an approach to promoting MKfT in which (1) initial work on a non-contextualized mathematics task can create a productive space for pedagogical reflection and (2) follow-up instruction can foster the intertwinement between mathematics and pedagogy. We elaborate on the task sequence in the Method section.

To conclude, in this paper we address the following research question: What task sequence can offer a productive learning environment for prospective elementary teachers to develop the three elements of MKfT that we described earlier about the role of assumptions in mathematical activity?

Method

Research context

This research derived from the last cycle of a 4-year *design experiment* (e.g., Cobb et al., 2003) in a semester-long mathematics course (3hrs per week) for prospective elementary teachers in the United States. The design experiment comprised five research cycles of implementation, analysis, and refinement of task sequences and associated implementation plans that aimed to promote PTs' MKfT. The students were undergraduates who majored in different fields and were taking the course as a prerequisite for admission to the master's level elementary teacher education program. The task sequence we focus on was implemented toward the end of the semester and was the only one that explicitly targeted PTs' MKfT about the role of assumptions in mathematical activity. We introduced the task sequence in cycle 4 because we felt an explicit intervention was needed to adequately promote PTs' MKfT of elements 1–3. Analysis of how the task sequence played out in cycle 4, alongside our developing understanding of how things "worked," led to modifications of the task sequence, culminating in the form described below which took place in cycle 5.

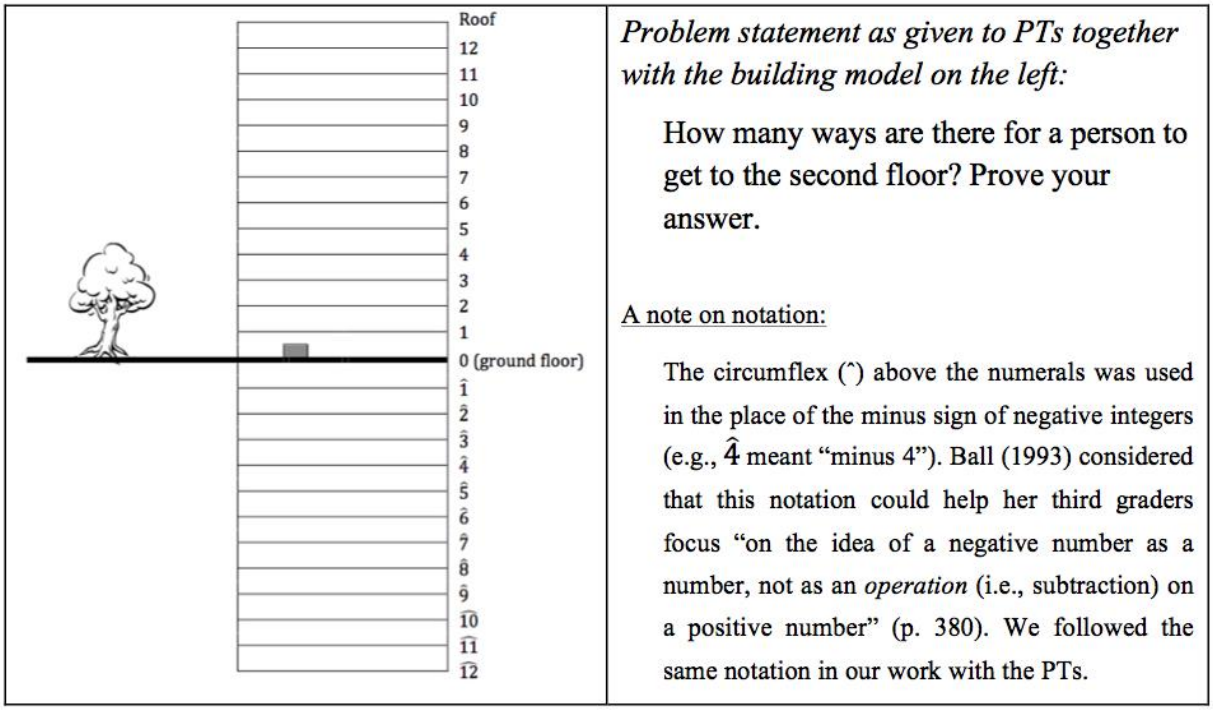

Figure 1: The "Floors Problem" (derived from Ball, 1993) in part A of the task sequence

The task sequence

The task sequence comprised three parts. In part A we used the "Floors Problem" (Ball, 1993) in Figure 1. Two task features made it suitable for our goals to promote elements 1 and 2 of MKfT. First, the task conditions were ambiguous and thus subject to different legitimate assumptions; a major ambiguity concerned whether or not the person in the task had to travel directly to the second floor. Second, different assumptions about the task conditions could support different arguments for answers to the task that might appear to be contradictory but might actually be correct given their underpinning assumptions. In addition to promoting elements 1 and 2 of MKfT, we hypothesized

that PTs' mathematical experience with the task in part A could offer a productive context for pedagogical reflection (cf. element 3). We capitalized on this potential of the task in parts B and C.

In part B we used two *conceptual awareness pillars* (or simply *pillars*; Stylianides & Stylianides, 2009) to which the PTs responded individually and in writing. (There was a third pillar whose discussion we omit due to space limitations and a lack of direct relevance to our focus.) The pillars are presented in the first column of Table 1 (see next section). With them we aimed to amplify PTs' mathematical learning (see pillar 1 in relation to elements 1 and 2 of MKfT) and its intertwinement with pedagogical reflection (see pillar 2 in relation to element 3). As per the definition of pillars (Stylianides & Stylianides, 2009), the non-directive prompts in them could increase PTs' awareness of key realizations they might have developed (possibly in tacit form) during their work on the task.

Finally, in part C we engaged PTs in small-group and whole-class discussion around the three prompts in Figure 2 that aimed to further support PTs' reflective insights from part B and help them consider the applicability of those insights beyond the particular task. The prompts raised issues relating to elements 1–3 of MKfT. For example, in prompt 1 the first question related to elements 1 and 2, while the second question related to element 3. Each small group was asked to collectively produce a written response to each prompt prior to a whole-class discussion. Due to space constraints we report only findings from our analysis of the small groups' written responses.

1. "Conclusions are 'true' only within the limits of the assumptions on which they are based." How do you understand this statement? Do you think it is important for elementary school students to develop a sense of the role of assumptions in mathematics? Explain.

2. "Teachers should always make sure that the mathematical tasks they give to their students have unambiguous conditions." What do you think about this statement? Explain.

3. There may be situations where teachers do not realize that a mathematical task they give to their students has ambiguous conditions. What might happen in these situations and how might teachers handle the situations?

Figure 2: Discussion prompts used in part C of the task sequence

Data and analysis

The task sequence in research cycle 5 was implemented in two parallel (independent) classes of the course that were both taught by the second author. At this stage of our analysis we are using data from only one of the classes: videos and transcripts of the implementation of the task sequence in this class with 16 PTs attending on that day; these PTs' written work including their individual responses to part B and their group responses to part C; and field notes from a research assistant documenting the work of one small group. Our data analysis was guided by elements 1–3 of MKfT that we aimed to promote. Specifically, we conducted qualitative content analysis of the transcripts and field notes in part A to examine whether and how the PTs developed understandings relating to elements 1 and 2, and also of PTs' responses to parts B and C to identify themes in their responses and examine whether and how these themes corresponded to elements 1–3 of MKfT.

Implementation of the task sequence and discussion

Part A: implementation of the Floors Problem

Stylianides showed the problem statement and the building model, and explained the notation for negative numbers. He also established a common notation with the class about representing trips. After that, he asked the PTs to work on the problem first individually and then in groups of four. He also said he would not answer any clarifying questions about the problem, an intentional aspect of our instructional design. The discussion in the small group where the research assistant was taking field notes is indicative of the way the PTs engaged with the problem (all names are pseudonyms):

Amanda: So… can you pass the second floor and then go back to it? Or do you have to stop, because you've technically gotten there? So… you just have to look at how to get *directly* there? If not, it's going to be infinity!

Beth: That's what I did…the direct. I just counted how many up and how many down…

Monica: But that doesn't say that you have to come in the entrance… So you can start anywhere?

At this point Stylianides passed by the small group and Monica asked him whether the trips had to start from the ground floor. Stylianides reminded her that he would rather not respond to clarifying questions and he moved on to a different group. The small group discussion continued as follows:

Victor: So… the solution of this problem depends on our assumption.

Amanda: Well, fine then. I say 15… because I'm assuming that you start on the ground floor and can't pass [floor] 2 and can't change direction more than once.

Victor: I say 25 because… [He was interrupted by the start of the whole-class discussion.]

As illustrated by this exchange, the decision not to respond to clarifying questions allowed for the notion of assumptions to emerge naturally in the discussion: had the instructor specified an interpretation, the PTs would not have considered alternative interpretations. The whole class discussion started with the small groups explaining their answers to the problem: 25, infinity, 15, and 51. Sherrill explained the answer of 25 by noting how each floor could constitute a separate starting point for a direct trip to the second floor. Sophie then tried to explain infinity as an answer:

Sophie: Well, we picked infinity because […] if you're saying you can go up and down and up and down as many times as you wanted before you reach the second floor, I think that the question isn't specific enough. Like they [Sherrill's group] assumed you can only go up or down one time. […]

Stylianides: So you say [referring to Sophie] if we don't assume that [what Sherrill's group assumed] and [we assume] you're allowed to travel up and down as many times as you want, then the answer would be infinity? [To the class:] What do you think about that? Is one of them wrong? Are both of them right? […]

Lindsey: I think […] they just made different rules, and like, thought processes. […] You could have a direct route and that's it, no going back and forth… like, you *could*, when you were talking about going up and down and up and down and that's not

really a direct route… So really, you could do anything that you wanted to, but it just depends on what the person… what the problem is looking for, I guess.

Sherrill: It's up to the interpretation of the reader... Like I read it and I just assumed that it had to be a direct route, but it clearly doesn't state that… […]

Infinity as an answer was discussed again later, after the class had considered also 15 and 51 as possible answers. Overall, the whole-class discussion mirrored the previous small-group discussion. Sophie, Lindsey, and Sherrill's comments show an increased understanding of elements 1 and 2: the PTs acknowledged that the task allowed for different legitimate interpretations of its conditions and that their conclusions depended on their assumptions (element 1), while realizing that the different answers they came up with were all correct based on their underpinning assumptions (element 2).

Part B: PTs' responses to the pillars

Analysis of PTs' responses to the two pillars gave rise to the themes summarized in Table 1. Each response could receive multiple codes. The frequencies are offered to show how prominent each theme was among PTs' responses, not for generalizability. Regarding pillar 1, PTs' responses under themes 1–4 related to both elements 1 and 2 of MKfT. The responses below illustrate all themes:

Sherrill: I was surprised at the multiple assumptions made. I had not even considered the fact that answers other than 25 existed because of the way people interpreted the problem. I now realize how greatly making assumptions can alter mathematics. (themes 2, 4)

Lorri: [A]ll of our groups had different interpretations of how many ways there were […] Some of the other groups' answers […] I could understand their rationales as to why their answer was correct. Our answers were not wrong under this ambiguous problem. (themes 1, 3)

Pillars	Themes and frequencies (in parentheses)
Pillar 1: Is there anything that particularly stood out to you from our work on the "Floors Problem"?	1. Multiple legitimate interpretations (8) 2. Multiple assumptions (5) 3. Multiple correct answers (4) 4. Importance of assumptions (4) 5. Other (2)
Pillar 2: The "Floors Problem" was purposefully designed to be ambiguous. Why might a teacher use a problem like that in his/her classroom?	a. To encourage sensitivity to language (7) b. To increase awareness of the role of assumptions (6) c. To enhance appreciation of different possible interpretations of the same (ambiguous) text (6) d. To enhance appreciation of the interdependency between different assumptions/interpretations and different answers and solution paths (5) e. To encourage discussion, explanation, or proof (5) f. Other (3)

Table 1: Summary of results from PTs' responses to the pillars in part B of the task sequence (*n*=16)

Regarding pillar 2, PTs' responses under themes a–e showed ability to recognize how their mathematical insights from the problem could apply in teaching. Thus the responses offered evidence related to element 3 of MKfT. We offer two responses that illustrate some of the themes:

> Lorri: A teacher could use this ambiguous problem to get their students to understand that sometimes their answers are not wrong and that there may be other answers to a problem. Students will be able to explore their different ways of interpretation and must be open to others' interpretations […] They will understand that there will be different […] answers that will depend on the information given in the problem, which may be ambiguous. (themes a, c, d)
>
> Helen: To learn about assumptions and understand not only the importance of specifics […] but also the freedom for one to think along the terms of their own assumptions. (themes a, b, c)

Part C: responses of small groups to the discussion prompts

Overall the PTs' responses to part C were pedagogically sound and were informed by the powerful impact the Floors Problem had on their mathematical learning. Also, one can see parallels between PTs' responses in part C and the part B pillars; these were supported by our instructional design.

Prompt 1: An illustrative explanation of the first question in prompt 1 was: "You have to start with a base and then make assumptions from that. So your conclusion can only be based on those assumptions. Therefore your conclusion can be only true or false for those assumptions." Regarding the second question in prompt 1, all groups noted the importance of helping students develop a sense of the role of assumptions in mathematics. The groups justified their response with reference to the following: the importance of assumptions in the discipline; knowledge of assumptions can help students understand that different legitimate interpretations of a problem can lead to different valid conclusions; and knowledge of assumptions can highlight the need for students to explain their thinking. These justifications relate to element 3 and are underpinned by the understanding of elements 1 and 2 that was evidenced in PTs' responses to the first question in prompt 1.

Prompt 2: All groups disagreed with the statement and highlighted, in a similar manner, that the phrasing of a task should align with the teacher's goals. Here is an illustrative response: "If a teacher is giving a math test, or graded evaluation where they are looking for a specific answer then yes, the tasks should be unambiguous. However, if the teacher is trying to prove a point about assumptions or different approaches to the same problem, an ambiguous task may be beneficial."

Prompt 3: Two major points emerged from the responses to prompt 3. The first point related to teachers being able to recognize that some students' approaches to a task that appear as "faulty" may be in fact mathematically sound based on unforeseen (to the teacher) legitimate assumptions. The second point related to teachers asking students to explain their thinking to see whether the students indeed operated on different assumptions rather than just focus on the final answer. Again, these justifications relate to element 3 but are also based on understanding of elements 1 and 2.

Conclusion

In this paper we investigated how teacher education can promote prospective elementary teachers' knowledge about the role of assumptions in mathematical activity, with attention to the design and implementation of a task sequence that aimed to support three elements of MKfT. We illustrated a task-based approach to promoting MKfT whereby a mathematics task, although not embedded in a

pedagogical context, can nevertheless create a productive space for pedagogical reflection through the creation of a powerful mathematical experience for PTs. Analysis of data from the fifth research cycle of a design experiment in a mathematics course for PTs provided evidence for the promise of this approach and highlighted the important role of the teacher educator in implementing deliberate (pre-planned) instructional moves to help amplify PTs' mathematical learning and enhance the pedagogical functionality of their acquired knowledge. Future research can investigate whether and how the rich insights (both mathematical and pedagogical) that PTs may develop through the task sequence about the role of assumptions in mathematical activity can inform their practice.

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Asymptotes and asymptotic behaviour in graphing functions and curves: An analysis of the Croatian upper secondary education within the anthropological theory of the didactics

Aleksandra Čižmešija[1], Ana Katalenić[2] and Željka Milin Šipuš[3]

[1]University of Zagreb, Faculty of Science, Zagreb, Croatia; cizmesij@math.hr

[2]University of Osijek, Faculty of Education, Osijek; akatalenic@foozos.hr

[3]University of Zagreb, Faculty of Science, Zagreb, Croatia; milin@math.hr

Keywords: Asymptote, asymptotic behaviour, anthropological theory of didactics (ATD).

Introduction and methodology

Asymptotes and asymptotic behaviour play an important role in many traditional and modern mathematical disciplines and are supported by a rich and well-established abstract theory. Some basic aspects of these concepts can be utilized already in elementary mathematics as powerful tools in graphing and analyzing behaviour of elementary functions at infinity and near singularities and in graphing simple plane curves such as hyperbola. Therefore, this body of knowledge is a common part of upper secondary mathematics curricula worldwide. Here, we investigate its didactical transposition to the Croatian general upper secondary education. The survey is conducted within the theoretical framework of the Anthropological theory of the didactics (ATD), developed by Y. Chevallard (Chevallard, 1981, 2007). The main idea of the ATD is to determine a relation $R_I(p,O)$ between a body of knowledge O and a person that occupies a position p in an institution I. Accordingly, mathematical knowledge and activities are described in terms of a *praxeology* $[T, \tau, \theta, \Theta]$, where its practical component, *praxis*, is represented with a *task* T and a *technique* τ and its discursive or theoretical component, *logos*, with a *theory* Θ and a *technology* θ. Research within the ATD should include relevant data about the relation $R_I(p,O)$ for all institutions I involved in the educational process i.e. the process of didactic transposition (Bosch, Chevallard & Gascón, 2005). Questioning and understanding *conditions* and *constraints* on the relation $R_I(p,O)$ is necessary for setting up attainable, justified and significant educational interventions (Barbé, Bosch, Espinoza, & Gascón, 2005). Hence, as relevant for our setting, we analyzed and compared the relations $R_B(p,O)$ and $R_S(p,O)$, where O is a graphical representation of an elementary function and a hyperbola in a square coordinate system, together with corresponding techniques regarding asymptotes, while institutions considered are two actual Croatian mathematical gymnasium textbooks B and the cohort of 40 the final, fifth year mathematics education students S at the largest mathematical department in Croatia.

This is a part of a more comprehensive study regarding asymptotes and asymptotic behaviour in the Croatian pre-university education which also included the institution of academic mathematicians. The methodology included a praxeological analysis of the textbooks as representatives of the *knowledge to be taught* and three questionnaires with open-ended questions for the prospective mathematics teachers to provide an insight in related *knowledge available* to students as a potential *taught knowledge.* Based on this, a reference epistemological model (REM) is proposed and then verified and improved with scholarly knowledge gained from semi-structured interviews with two

academics. Here, we focus on textbook topics, tasks, techniques and discourses related to *O* and on three particular tasks from the questionnaires administered: graphing and describing a simple rational function and a shifted exponential function from a real world problem, given by formulas, and graphing a hyperbola and its asymptotes.

Results and conclusions

The results of the textbook analysis show that graphing a function *f* pointwise, that is, by plotting some corresponding points (*x*, *f*(*x*)) and connecting them by a smooth curve dominates all other available techniques for graphing functions given by formulae. A technique of graphing a function or a curve regarding its properties recognized from a corresponding algebraic expression (formula or equation) occurs only in relation to the tangent and cotangent function and to a hyperbola, while the tasks on graphing polynomials and rational functions appear only as common practical activities of utilizing derivatives (calculus). Finally, a technique of transforming a prototype graph of an elementary function to get a graph of its composition with a linear function (by translations and dilations) is rarely implemented. Although textbook praxeologies of graphing functions often elaborate function properties and flow, discursive accents are set on establishing a function's domain, monotonicity and symmetry of its graph, neglecting its asymptotic behaviour. Asymptotes are seen relevant only for tangent and cotangent functions. Results of the questionnaire analysis completely reflect those of the textbooks. The students applied techniques and provided discourse to the same extent as it is given in the praxeological organization of the textbooks. Namely, their dominant techniques are drawing curves through their points and drawing graphs regarding function properties determined by using calculus, their chosen techniques are not the most efficient for the task in question, and asymptotic behaviour is available to them but not fully utilized in praxeologies relevant to graphing functions or curves. Considering this, it is our suggestion for the teaching practice and for an ongoing curricular reform in Croatia that: all relevant function properties should be emphasized when describing its behaviour; common properties of a function should be more related to its algebraic representation and utilized for its graphing; functions should be graphed by graph transformations, whenever fitted; and asymptotic behaviour should be more emphasized, adequately graphically represented and described by formal and informal mathematical discourse.

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Using variation theory to explore the reteaching phase of Lesson Study

Thérèse Dooley and Tugba Aysel

Institute of Education, Dublin City University; therese.dooley@dcu.ie

Keywords: Variation theory, Lesson Study, task design, fractions.

Japanese Lesson Study is now internationally accepted as a powerful form of professional development, particularly in the field of mathematics. It involves a cycle in which teachers collaborate to plan, observe, analyze and refine actual classroom lessons (Lewis, Perry and Murata, 2006). Frequently the lesson is retaught, although rarely by the same teacher to the same students (Fernandez and Yoshida, 2004). Reteaching of a lesson allows teachers to see how small variations in, say, the presentation of a task can deepen learners' understanding of mathematical concepts. This suggests that variation theory might be a useful lens for exploration of this phase of the Lesson Study cycle. From this theoretical perspective, learning is defined as "a change in the way something is seen, experienced or understood" (Runesson, 2005, p.70). Thus small changes in the design of a task can result in changes in what it is that students discern or notice. Watson and Mason (2006) refer to *dimensions of possible variation* (what is possible to vary) and a *range of permissible change* (perceived constraints on the extent and nature of change in any of the dimensions of variation): "Teachers can … aim to constrain the number and nature of the differences they present to learners and thus increase the likelihood that attention will be focused on mathematically crucial variables." (p.102)

In this paper we report on a research project in which one primary and three post-primary teachers participated in Lesson Study in order to examine transition issues in mathematics. In particular, there is reference to the mathematical written work of two different groups of students who engaged with a similar task. The teachers had identified the topic of fractions as one that poses both teaching and learning challenges at primary and post-primary levels. They developed a research lesson and taught it first to a junior post-primary class (students aged 12-13 years) and later to senior primary class (students aged 11-12 years). The goals for both lessons were the same and centred around students'

1. development of confidence in comparison of fractions;
2. utilization of their own approaches to solve the problem;
3. discussion of their own ideas and opinions with each other; and
4. motivation to engage in further such mathematical tasks.

The particular task they chose (sourced from nrich.maths.org) concerns the identification of the greatest amount of chocolate in a room where there are three bars on one table, two on another and one on the last table. It is assumed that an unknown number of people will enter a room in turn and that each will decide which might be the best table at which to sit 'at that moment'. Thus the first person should choose the table at which there are three bars. The comparison faced by the seventh person is mathematically more complex than that encountered by the first six people as, up to this point, comparison is between whole numbers and fractional amounts less than 1. For example, a choice that might have to be made by Person 6 is ½ or ⅔ or 1 bar of chocolate. However assuming

that Person 6 opts for the table at which there is one bar, the choice for Person 7 is ½ or ⅔ or ¾. While much planning time was spent anticipating such complexities and how pupils might deal with them, the teachers did not foresee that poor organisation of work by students would impinge on their solution processes. A change introduced in the second iteration of this lesson was that students would be encouraged to present their solutions in tabular form, and, in fact, on inspection of their written artifacts, most did so. However, while the use of this table facilitated fraction comparison (Goal 1), it did not encourage the exploration of various approaches (Goal 2) to the same extent. It appeared that this small variation in the task had considerable impact on students' focus of attention.

Conclusions

A broader *range of permissable change* appears to have been conveyed in the post-primary than in the primary class due to the introduction of the tabular format in the second lesson. However, students perceived a narrower dimension of variation than was suggested by either teacher, evident in the frequency of use of decimals for recording in the post-primary class and use of fractions/tabular format for recording in the primary class This may have been influenced by the teacher's representation of the task on the whiteboard in each lesson and might be explained by its endurance (and thus propensity to be noticed) over the course of each lesson. Such representations seem to have a considerable impact on the (enacted) object of learning and deserve focused attention in the design and analysis of mathematical tasks. More generally, while other theories have explanatory power in the consideration of the different outcomes of two similar lessons, variation theory has a significant role to play.

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Changes in the images and arguing from the mathematics textbooks for the secondary school in Argentina along 67 years

Viviana Carolina Llanos and María Rita Otero

Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN). Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina; vcllanos@exa.unicen.edu.ar
rotero@exa.unicen.edu.ar

This work analyses the changes in the relationship between arguing and images from the mathematics textbooks for the secondary school in Argentina along 67 years. The textbooks have been published in the period 1940 thru 2007. The analysis is done by (N=137) textbooks based on three meta-categories in an inductive way. A factorial analysis of multiple correspondences was performed to find the main similarities and differences between the textbooks and to make a cluster analysis and one possible classification.

Keywords: Textbooks, mathematics education, secondary school.

Introduction and conception of arguing

In 1994 an educational reform was performed in Argentina. The syllabus was changed and the scholar textbooks were adapted to the new educational system. The main modifications were realized in the properties of the images more than in the content. The characteristics of the text books, the ideas about argumentation and the characteristics of the images in the books are analysed. This research adopts an idea of arguing that emphasizes the relevance of the divergences between different points of view and the epistemological function of arguing, proposed by Leitão (2007). Different from other theories as Schwarz, Hershkowitz & Prusak (2010), Driver, Newton & Osborne (2000); Leitão proposes that arguing has to be analysed based on three elements: *"argument, argument against and response"* in order to generate confrontation between *argument* and *argument against*, to achieve construction of knowledge and transformation of perspectives in the subject (*response*). These processes occur into face to face situations, or in negotiations of the different points of view with ourselves, in this case, when we are reading a textbook.

Methodology, categories of analysis and some results

A set of (N=137) mathematics textbooks is selected by means of purposive sampling techniques. The analysis was performed starting from a previous qualitative inductive categorization based on three meta-categories:

A- Characteristics of arguing. A1- *Commencement of arguing*: *Questions* or situation, which will be answered later; *Definition*, using to introduce knowledge; and *Examples* to formulate knowledge. A2- *Type of arguing*: *Deductive formal*, used deductive mathematics argument (definition, theorem, hypothesis, theory, demonstration, etc.); *Deductive informal*, they do not reach the formalism of the demonstrations; and *Inductive* that generalize knowledge from a single case. **A3**- *Degree of arguing*, cognitive conflict promoted by the text is analysed in three levels: *High*, books that generate explicitly confrontations, without solution in the text; *Low*, textbooks generate explicitly a cognitive conflict, solved later; and *Absent*, textbooks that inform without questioning.

B- Relationship between the images and arguing (Otero, Moreira and Greca, 2002): **B1-** *Use of the image: Ornamental*, images used with a decorative aim, not related to the content; and *Argument*, used as source of information, knowledge can be derivative. **B2-** *Type of image: Mathematical representations*, use mathematical systems of representation; and *Non-mathematical representations*, images not related with mathematical content. **B3-** *Grammatical style of the images*: *Conceptual*, represent relations and fixed characteristics between the represented elements; and *Narrative*, identify actions between objects that can represent a relation between them in the image. **B4-** Relationship with the "real world": *Naturalist*, images referring to the empirical world, detailed and complex; and *Abstract*, not referring to the world that we experience.

C- Characteristics of the textbooks: **C1-***Date of publishing*: *Period 1*, 1940 thru 1973, *Period 2* 1974 thru 1994 and *Period 3*, after the reform, until the year 2007. **C2-** *Educational level*: refers to the educational level the textbooks. *Level 1*, students between 12 and 14 years old; *Level 2*, students between 15 to 17 years old; and *Level 3*, older than 18 years old. **C3-** *Mathematical traditions* (Klimovsky & Boido, 2005): Computational, emphasis in the resolution of problems and calculation with numbers; *Axiomatic*, present the mathematics demonstrations steps; and *Structuralist*: books that search of regularities that meet the same conditions.

Using this categorization, a qualitative description is made, which originated a first analysis. Then, the categorization is transformed in a group of nominal variables and modalities using Exploratory Data Analysis (Lebart, Morineau, 2000). A Factorial Analysis of Multiple Correspondences allowed the selection of one possible classification in three classes. In addition, a test of randomness to analyse the reliability of the sample was performed using the statistical software SPAD.

The analysis explains changes in the images and arguing, given by: books that propose questions, or only definitions and examples, by the way to conceive and validate to mathematical knowledge, and mainly by the changes in the images and the relation between images and knowledge. The goal of most books seems to be informative. This explains the absence of questioning and discussing about several points of view, and the low level of arguing and conflict found within them.

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Teacher's pedagogical design capacity and mobilization of textbook

Ljerka Jukić Matić

Department of Mathematics, University of Osijek, ljukic@mathos.hr

Keywords: Textbook mobilization, pedagogical design capacity, mathematics teacher.

To investigate, understand and explain the relationship between teachers and curriculum resources, in our context a textbook, Remillard (2009) derived a model of the teacher-curriculum relationship, from earlier Brown's (2009) Design for Capacity Enactment Framework. The center of the model represents the participatory teacher–curriculum relationship designating what the teacher and the curriculum bring to the teacher-curriculum interactions. In this model, teacher resources include concepts of pedagogical design capacity, human and social capital, and agency and status. Curriculum resources include several components: mathematical topics and tasks that are structured in deliberate ways, embedded supports for the teacher to guide pedagogical decision-making and presence of a pedagogical orientation or emphasis embedded in instructional strategies and lesson structures. In domain of teacher resources, an interesting and important characteristic is the pedagogical design capacity (PDC). PDC describes the capacity of teachers to perceive and mobilize existing resources to create productive instructional episodes (Brown 2009), which to some extent, depends on the used resource and on the ways of working with that resource (Gueudet, Pepin & Trouche, 2013).

In this study, we wanted to examine the relationship between the mathematics teacher and the textbook in the classroom. Therefore, we formed the following research questions: What characterizes the nature of the teacher–textbook relationship and why? What is the level of the teacher's pedagogical design capacity?

Remillard (2009) suggested that individual PDC can be measured by examining dimensions of human and social capital, therefore in this study, two measures are used to examine the teacher's level of PDC; the one proposed by Remillard (2009) and the one the one described by Leshota (2015). Leshota (2015) proposed that one possible way for measuring teacher's PDC is examining whether teachers make injections of mathematical content into the lesson, omissions of mathematical content from the textbook and mathematical errors. Using those criteria one could determine whether teacher has low or high PDC.

We observed four lessons in Mrs. D's classroom and conducted an interview with the teacher. In terms of textbook content, Mrs. D offloaded, adapted and improvised in the lessons, but not to the same extent in every lesson. Those types of textbook mobilization were dynamically interchanging within a lesson. Mrs D. made several injections of content that are not in the curricular outlines or in the textbook for this grade level. She had no critical omissions in the lessons.

It seems that the teacher regards the textbook as a vital resource for the students' learning, but not crucial for her teaching. The established relationship between the teacher and the textbook could be regarded as a two way process in which both participants communicate. The outcome of that process is a product that fits the students' needs and the teacher's goals. The teacher positioned herself as having instructional authority in the classroom, regardless of what her colleagues in the school do or think. Recognizing the textbook's affordances and constraints allowed her to place

herself as an authority over the textbook. Mrs D. showed a high level of PDC in her teaching. She omitted content from the textbook, like activities or worked examples which were not crucial for learning mathematics. She injected content that is not usually introduced until the following grade because it was applicable to the topic being taught.

From our perspective, the term PDC seems to be more efficient when examining teaching expertise than Shulman's pedagogical content knowledge (PCK). In a way, PDC has more dynamic nature than PCK. Therefore, from the perspective of the textbook utilization, teachers' development of PDC is an important and critical part of their interactions with the textbook. Our study showed that interplay between curricular knowledge, professional development, mathematical knowledge, knowledge of the textbook's characteristics are important aspects in ability to craft pedagogical beneficial lessons. More studies on teachers' PDC would be beneficial for new and inexperienced teachers.

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How to design educational material for inclusive classes

Anna Noll, Jürgen Roth and Markus Scholz

University of Koblenz-Landau, Germany; noll@uni-landau.de

Keywords: Inclusion, symbols, easy-to-read, fractions.

Inclusion is based on international legitimate developments like the Salamanca statement, which emphasizes the right of education for all (UNESCO, 1994). Furthermore, several studies showed that coeducation can have a positive effect on the development of performance of pupils with and without special educational needs (Markussen, 2004). But to include all students regardless of their physical, intellectual, social or other abilities the educational framework conditions need to be adopted at first. One important step into this direction consists in providing all pupils access to the assignments by enhancing their readability.

Theoretical background

Readability can be enhanced by linguistic simplifications like the application of the *easy-to-read guidelines* (Netzwerk Leichte Sprache, 2006). Easy-to-read language has been established to facilitate understanding in everyday life for people with disabilities. It is, for example, used for the simplification of manifestos in order to support participation in society. So far, these guidelines are not verified scientifically, but show several similarities to empirically based linguistic simplification rules, like the *Hamburger Modell* (Langer, Schulz von Thun, & Tausch, 2011). The authors state, among others, that only one statement should be made per sentence. Another possibility to facilitate the comprehension of assignments is the use of symbols. A symbol can be defined as a graphical image conveying a single idea or concept (Detheridge & Detheridge, 2002). Little empirical data about the use of symbols to foster the readability of texts exists (Jones, Long & Finley, 2007; Poncelas & Murphy, 2007). Nevertheless, a positive influence can be assumed, e.g. because of the multimedia principle (Mayer, 2009). It indicates that people learn better from words and pictures than from words alone. An explanation is given by the cognitive theory of multimedia learning (ibid.) which assumes that pictorial and verbal information are processed in two different channels in our brain. When words and pictures are presented, both channels are used and the cognitive load on the limited capacity of the working memory is reduced.

Methodology

Does the use of easy-to-read language and/or enriching text with symbols facilitate students' performance in mathematical tasks? This research question shall be answered with the following methodology. The tasks of this study deal with introducing fractions. These are taught in activity-oriented manner with hands-on material. The tasks are divided into two complexes. The first complex aims at the conduction of more basic actions like counting. Then, the pupils receive an input about fractions by watching a video. The pupils use the information of the video for more complex mathematical considerations which are necessary in the second task complex. Thus, the pupils' conceptualization of fractions is fostered, e.g. by the naming and comparison of fractions. In November and December 2016, a pre-study was conducted with 30 students in grade 5, 6 and 7. The sample consisted of pupils with learning difficulties and students without special educational needs. The students worked on the tasks individually and participated in a subsequent interview. Data was

also collected by use of eye tracking and thinking aloud. A first result of the pre-study is that the symbols are used by the students without explanation. This result can be exemplified by the following excerpt of the interview and the corresponding eye-tracking data. The different colors of the heatmap represent different durations of the fixations:

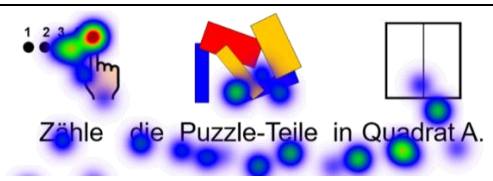
Student: The little pictures helped me, because I could see how it works. Because sometimes I didn't understand the text and then I watched the pictures and they helped me.	

Figure 1: Primary data insights

Deeper qualitative analyses, which focus on how the symbols are used by the students as well as on the linguistic comprehensibility of the tasks, will follow. For the main study a posttest-only design, which includes two experimental as well as one control group, is planned. While experimental group 1 receives a linguistically and pictorially simplified version of the tasks, experimental group 2 works with a variation which is linguistically simplified only. The control group receives a not simplified version. After working with the exercises, the students' knowledge about fractions will be measured. The participants' reading ability and their IQ will be elevated beforehand. These control variables shall help to build comparable groups.

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The application of the Universal Design for Learning framework to task design in order to support lower attaining children in the primary school mathematics classroom

Caitríona Wynne[1] and Lorraine Harbison[2]

[1]Church of Ireland College of Education (a college of Dublin City University), Ireland; caitrionawy@gmail.com

[2]CASTeL, Institute of Education, Dublin City University, Ireland; lorraine.harbison@dcu.ie

Keywords: Universal task design, mathematics education, inclusion of lower attaining children.

Background to the study

It is a requirement that all schools in Ireland administer standardized attainment tests in mathematics when children are in 2nd Class (aged 7 – 8). On the basis of these results, children who are deemed to have underperformed nationally (Standard Score <90), are usually offered additional support by means of withdrawal from their mainstream classroom for mathematics lessons to work on a complementary program with the learning support teacher. Although it is argued that such provision is offered to less able children as they tend to simply 'give up' in classrooms in which they find the mathematical tasks too challenging, the concern is that that these children will fall further and further behind. It is the contention of the authors, that it is not the tasks per se that are too challenging for the children but rather the nature of the tasks, which all too often tend to emphasize traditional practices of rote procedure and drill.

In this study, the researcher, who was working as the learning support teacher, and the class teacher planned together to integrate children back into their classroom by co-designing mathematically rich tasks. Over a four week period, three children who had formerly been withdrawn from the classroom were put to the fore in the planning of the tasks. Particular consideration was given to their learning styles. The mathematical tasks were inspired by the three key principles of the educational framework, Universal Design for Learning (UDL), which constitute Multiple Means of Representation; Multiple Means of Action and Expression; and Multiple Means of Engagement (Rose & Meyer, 2000). The term 'universal' is particularly pertinent in the design of the tasks as they were developed in line with specific mathematical learning goals for 'all' learners from the beginning rather than implementing a standard 'one size fits all' set of tasks and differentiating the tasks to cater for the marginalized, less able children, at a later stage.

Therefore, in order to create learning tasks that will engage all children by design, this research sought to ascertain:

1. How can the principles of UDL be used to design mathematically rich tasks?
2. Do UDL informed tasks engage and support children of low-ability in mathematics?

Methodology

This case study took place over four weeks with one class of 32 children for 45 minutes per day.

The children were aged between eight and nine years old.

It was a detailed body of work comprising collaborative universal lesson design on the topic of 'Measurement', implementation of lessons, critical analysis of tasks and peer review. Each lesson was assessed using an adapted scoring rubric developed by Spooner, Baker, Harris, Ahlgrim-Delzell and Browder (2007). Frequencies of different events were tabulated. Formal observational instruments were developed to recognize and discern certain types of behaviors such as children's degrees of engagement. Observations were supplemented by photographs. Teachers' daily reflections were analyzed using the analytic technique of pattern matching. A matrix of categories was developed and evidence placed within each classification. Information was put in chronological order. A follow up interview was held with the host teacher at the end of the intervention.

Results

The 14 lessons scored 82 points out of a maximum of 84 points on the adapted scoring rubric (Spooner et al., 2007) implying that a very high level of the UDL approach was used in the task design and implementation. Diversity was the starting point in planning the tasks, with lower ability children being accommodated within and enriching the regular class. The development of positive learning profiles for the three target children, such as, 'needs assigned role during group tasks', or, 'needs to have basic equipment available such as a pencil and a ruler prior to task allocation', helped to remove barriers to and enable participation in learning. UDL tasks offered the children various ways of acquiring information and knowledge; provided alternatives for demonstrating what they knew; tapped into children's interests, gave appropriate challenges, and increased motivation. Multiple means of representation, action and expression, and engagement were used in task design. Video clips were used on five occasions, concrete materials on eight occasions, ICT (PowerPoints, images, interactive stylus and interactive tools) were used during eight lessons, a parallel ICT mathematics program was set as homework for the children on each of the 14 days and the local environment was used on seven occasions. ICT was also found to be a key component that engaged children who were previously observed to be challenged by mathematics.

Discussion and conclusion

This research revealed that tasks which take into consideration UDL instructional goals, assessments, methods and materials are usable and accessible from the outset rather than having to retrofit the tasks to children's needs as an afterthought. Crucially, results from this study found that by intentionally creating flexible learning opportunities, less able children were engaged and understood difficult mathematical ideas when they were provided with UDL informed tasks.

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TWG23: Implementation of research findings in mathematics education

Introduction to the papers of TWG23: Implementation of research findings in mathematics education

Uffe Thomas Jankvist[1], Mario Sánchez Aguilar[2], Jonas Bergman Ärlebäck[3] and Kjersti Wæge[4]

[1]Aarhus University, Danish School of Education, Denmark; utj@edu.au.dk

[2]Instituto Politécnico Nacional, CICATA Legaria, Mexico; mosanchez@ipn.mx

[3]Linköping University, Department of Mathematics, Sweden; jonas.bergman.arleback@liu.se

[4]Norwegian Centre for Mathematics Education, Norwegian University of Science and Technology (NTNU), Norway; kjersti.wage@matematikksenteret.no

In this introduction, we briefly present selected theoretical constructs relevant for the Thematic Working Group 23 (TWG23). We first address the topic of "implementation research" by looking into other research fields and domains where this topic is well-developed. Drawing on a taxonomy of so-called "implementation science" in health-care, we attempt to categorize the papers and posters of TWG23 according to their "implementation research aim" (Nilsen, 2015). Using this taxonomy, we elaborate on future perspectives for the TWG by relating to ongoing discussions in mathematics education research.

Keywords: Research findings, research results, theory to practice, implementation research.

A need for creating a new CERME Thematic Working Group

With almost five decades of accumulated knowledge, research findings, theoretical frameworks and experiences, the field of mathematics education research now has quite a bit to offer to the ongoing teaching and learning of mathematics in primary, secondary and tertiary education. Regardless of the well-known long and winding "journey" that research results must travel before finding an actual foothold within practice, results from mathematics education research nowadays seem to be present to a much larger extent in practice than ever before – not to say that there is anything strange in this, but rather it is probably only natural given that the field has matured and become successively more and more established over the years.

Indeed, as researchers in mathematics education, we are frequently involved in putting previous empirical results and findings as well as theoretical constructs based on these to good use in mathematics classrooms, mathematics programs, mathematics teacher education, in-service teacher training, etc. In several countries there are ongoing developmental projects that rely heavily on previously documented research results. However, as researchers we often find ourselves in a peculiar situation when wanting to report on these activities, since such accounts do not necessarily fall under the usual paradigm of "research in mathematics education" and do not in a clear-cut manner qualify as either "empirical" or "theoretical research". To assist in closing this "gap" the purpose of creating a new thematic working group focusing on aspects and issues related to the implementation of research results and findings, is to provide a venue for discussing, collecting and advance the "implementation research" aspects of our activities.

Implementation research and its aims

In relation to research on actual implementations, mathematics education may profit from other disciplines or areas, where research on implementations is further ahead. Healthcare is one such area, where since 2006 an entire journal has been devoted to "implementation science" (the name of the journal as well). Although the journal of course publishes several empirical studies related to various aspects of implementations, it occasionally also offers theoretical studies focusing on "implementation research" itself. One such theoretical contribution is the study by Nilsen (2015), who proposes a taxonomy consisting of five categories of theoretical approaches in order to make "sense of implementation theories, models and frameworks" (p. 1). Nilsen describes implementation science as "the scientific study of methods to promote the systematic uptake of research findings and other EBPs [evidence-based practices] into routine practice to improve the quality and effectiveness" (p. 2). Although Nilsen focuses on the case of health care, the definition seems adaptable and applicable also to the field of teaching and learning of mathematics. As part of the background for the taxonomy, Nilsen states that "Implementation science has progressed towards increased use of theoretical approaches to provide better understanding and explanation of how and why implementation succeeds or fails" (p. 1). According to Nilsen, the theoretical approaches used in implementation science have three overarching aims:

1. describing and/or guiding the process of translating research into practice (*process models*);
2. understanding and/or explaining what influences implementation outcomes (*determinant frameworks*, *classic theories*, *implementation theories*); and
3. evaluating implementation (*evaluation frameworks*).

The five categories of theoretical approaches consist of those provided in parentheses following the aims above. For aim 1, *process models* serve the purpose of breaking down the translation process into smaller steps, stages or phases. For aim 3, *evaluation frameworks* serve the purpose of assisting in evaluating the success – or lack thereof – of a given implementation. More interestingly, perhaps, are the theoretical approaches associated with aim 2. *Determinant frameworks* specify types of determinants that act as barriers and enablers that influence implementation outcomes, or even specify relationships between types of determinants. *Classic theories* are defined as those originating from fields external to implementation science, e.g. psychology, sociology or organizational theory that can be applied to understand or explain aspects of implementation. Finally, *implementation theories* are defined as those which have been developed from scratch within the field of implementation science.

Implementation research aims of the papers and posters in TWG23

TWG23 received 16 papers prior to the congress. At the congress 14 papers and 1 poster were presented. In the light of Nilsen's three aims of implementation research (or science), most of the studies presented in TWG23 at CERME 10 in Dublin concerned aim 1, addressing aspects of how to adapt research results and findings to practices in schools or other learning situations. A few of the presented studies touch upon aim 3, i.e. evaluation frameworks. Although it is seldom the main focus of the studies presented, aspects related to aim 2 occasionally surface. We shall return to

potential reasons for this distribution, but first we will use Nilsen's framework to categorize and briefly discuss the papers and posters in TWG23 presented at CERME 10.

In line with Nilsen's aim 1 to guide the process of translating research into practice, **Ärlebäck** describes and discusses the framing of, and experiences from, a project that combines research, practice, and teachers' professional development based on the tenets of the "Models and Modeling Perspective" on teaching and learning. Besides providing a general description of the methodological considerations in the project design, the paper describes how the accumulated results and experiences in the research literature on so-called model eliciting activities are used to inform the design, implementation and evaluation of activities aiming at introducing functions to grade 8 students. The focus of the paper is on the implementation, and aims to show how the teacher in question realized the offered perspective and tools in practice. The work presented by **Aguilar, Castañeda and González-Polo** aligns with aim 1 as it illustrates how research results generated in the field of mathematics education can be implemented in the design of mathematics textbook tasks. In particular, it is shown how research findings related to representation registers and to the conceptualization of the concept of function as a process are used in the design of textbook tasks for upper secondary level. The poster by **Chandia and Montes** matches with aim 1, since they report a professional development strategy for teachers focused on improving students' and teachers' problem solving skills. The professional development strategy by Chandia and Montes incorporates research results related to the creation of professional development systems in mathematics. **Bulien** presents theoretical and methodological arguments for the design and implementation of a research based course for pre-service teachers aimed at clarifying and strengthen the connection between didactical and mathematical theories to in-school teaching activities. Drawing on a design experiment methodology and the theory of communities of practice, Bulien elaborates on a "Mathematics Didactics Planning Tool" for teaching in different classroom situations. Thus, this study also relates more closely to Nilsen's aim 1. **Jankvist and Niss** deal with the research-based design and implementation aspects of a so-called "detection test" in relation to upper secondary school students' difficulties with mathematical conventions, concepts and concept formation, in particular those related to equation solving. In a similar manner, **Ahl** addresses the design of a detection test related to students' difficulties with proportional reasoning. Hence, both these studies deal with Nilsen's first aim, that is, translating carefully selected research results from mathematics education into suitable test items. Based on the answers of 405 Year 1 upper secondary school students, Jankvist and Niss also address aspects of evaluation (aim 3). Another research report that corresponds to aim 1 is the one by **Kjeldsen and Blomhøj**. In their work, research findings on students' concept formation and the digital tutorial genre are brought to use in the teaching of a first year calculus course. They present and discuss a theory-based design and its implementation for students' productions of video tutorials aimed at supporting their understanding of the limit concept. It could be said that this work is also related to the aim 3 delineated by Nilsen, since the study examines whether the designed learning environment supports the students' formation of key concepts in calculus or not.

The paper by **Valenta and Wæge** touches on both aim 1 and aim 2 of Nilsen's taxonomy. The paper describes a course aimed at supporting in-service teachers' learning of ambitious mathematics teaching. The design of the course is based on a project called "Learning in, from, and for Teaching Practice Teacher Education Project" (aim 1). In addition, the particular question addressed in the

paper is coupled to aim 2, since it focuses on the learning potential in the interactions between in-service teachers and course instructors during the public rehearsals that are the key innovative feature of the designed course and manifested through cycles of enactment and investigation. The theoretical paper by **Nilsson, Ryve and Larsson** align with Nilsen's second aim (understanding and/or explaining what influences implementation outcomes). They draw upon a systematic literature review on productive classroom practice to construct a framework for categorizing theories aiming at supporting teachers' actions in mathematical classroom practices. They do so by relating to theories and literature on educational policy research, professional development research and implementation research. Related to a larger scale early intervention program, **Lindenskov and Kirsted** touch upon aspects of Nilsen's aim 2. More precisely, they discuss teachers' perception of "theory" and barriers these may lead to, when implementing research results in practice. In addition, they also address aspects of the translation of theoretical constructs to the teachers as well as the suitability of these constructs provided a given context of practice. The study reported by **Tamborg, Allsopp, Fougt and Misfeldt** clearly falls with Nilsen's category of studies related to developing determinant frameworks (aim 2), since it investigates the of role the local supervisor (enabler) in the implementation of a mathematics teacher training program.

Amit and Portnov-Neeman's work can be related to the aims 1 and 3 proposed by Nilsen. They report on the implementation of a methodology used to teach reading and mathematics called "Explicit Teaching Method" focused on teaching students the "working backwards strategy" for solving non-routine mathematical problems; on the other hand, the effect of using the explicit teaching method as a means to learn the working backwards strategy is evaluated. **Koichu and Keller** position their paper as so-called design-based implementation research (DBIR) (see later). They present an evaluation framework (aim 3) to analyze and theorize their attempts in creating and sustaining online exploratory problem-solving discussion forums using the conceptual tools provided by Rogers' "Theory of Diffusion of Innovation". **Ejersbo and Misfeldt** report on a design-based research (DBR) project related to developing numeracy in grades K-3. This study too focuses specifically on evaluation aspects (aim 3), not least in terms of improving the design being implemented as well as the future of the project at the local school. **Kuzle**'s work somehow touches all three aims outlined by Nilsen. She reports on a collaborative project between educational researchers and practitioners with the goal of developing a problem-solving curriculum for grade 6 students using DBR. The curriculum was developed and implemented based on problem solving research and theory, and through the evaluation of its implementation objective and subjective factors that inhibited the full-implementation of the curriculum were identified.

Implementation of research findings in mathematics education

As seen above, a few aspects of aim 2 were touched upon in the papers and posters of TWG23, and some papers also considered aim 3. Still, aim 1 appears to be the dominant one among the reported studies. This, however, is not so strange since actual "implementation research" within the field of mathematics education must be regarded as a relatively new trend. This is of course due to the field of mathematics education itself not being much older than fifty years, but at the same time it is mature enough to have produced a sound basis of research results to actually be implemented into the practice of teaching and learning mathematics. Engaged in such implementation-oriented endeavors, researchers in mathematics education work systematically at different levels to establish

evidence-based solutions to the problems and challenges faced by practitioners and learners. Whether the research carried out is empirical or theoretical in nature, *implementation* of research findings and results is at the core of the research activities, either in the form of evaluating and furthering actual practices or materials etc., or to deepen our theoretical understanding to facilitate, guide and support various future implementations. Hence, and as already illustrated by the papers of TWG23, implementation of research findings and results in mathematics education can take many forms and expressions. Further examples from the literature are: in the design of experiments (e.g. Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) and mathematical tasks (e.g. Margolinas, 2013); as tools for professional development of in-service and pre-service mathematics teachers (e.g. Tsamir, 2008; Sánchez, 2011). The research findings and results that are implemented as part of systematic research are typically empirical results, theoretical results in terms of frameworks of different kinds, or some mixture of the two. Still, such findings and results usually fall within Nilsen's first and third aims, whereas results directly concerning aim 2 are scarcely touched upon.

As seen from the presented research studies of TWG23, implementation of research findings may have connections with research areas already existing in the field of mathematics education. One such example, although not reported in TWG23 at CERME 10, is that of *lesson study*. In the lesson study approach, lessons are designed and analyzed as a means to improve mathematics teaching in the classroom, but also as a means for professional development of mathematics teachers. Another existing research area, represented in TWG23, is that of task design. Hence, from the presented papers, it is clear that "implementation research" encompasses different kinds and formats (textbooks, apps, software, etc.) of didactical designs and products, stretching from task design, over teaching modules, courses, to entire programs – on all educational levels. Yet an example is that of design-based research (DBR), where results might take the form of a teaching module that successively and iteratively have been envisioned, designed, applied, analyzed and redesigned. The result is the final design as well as measures of how successful the design has proven to be. However, to focus on the implementation aspect of DBR means to not only focus on the end product and its success in achieving what was set out to do, but also to seriously take into account the "design phase" of the design research cycle. That is, the phase where the researcher identifies a learning problem and then uses available research results to design a (preliminary) product or tool that can help students in overcoming this learning problem. A primary concern then becomes to focus precisely on the way research knowledge is applied to generate some type of educational product. Elements of these concerns are addressed by Fishman and colleagues (2013), who forefront the implementation aspects of DBR in a research approach they call design-based implementation research (DBIR) – a framework also used in a few of the papers presented in TWG23. In short, DBIR has:

> "(1) a focus on persistent problems of practice from multiple stakeholders' perspectives; (2) a commitment to iterative, collaborative design; (3) a concern with developing theory and knowledge related to both classroom learning and implementation through systematic inquiry; and (4) a concern with developing capacity for sustaining change in systems." (Fishman and colleagues, 2013, pp. 136-137)

More generally, an important aspect when implementing research findings and results into practice is to focus on what Burton (2005) has called the *methodology* of the research conducted. Burton

argues that researchers in mathematics education in general pay little or no attention to explaining and motivating the rationale for the actual research design they apply to be able to draw the conclusions they report when writing up their research. This "craft knowledge" of the researcher is in a way silent. In Burton's opinion, accounts of research is full of descriptions of how results were obtained (i.e. what the explicit methods applied were), whereas elaborations on *why* choices were made and decisions taken in order to arrive at conclusions are rarely found. The *how*-question concerns the *methods* used by the researcher to undertake his or her research, while the *why*-question focuses on the rationale for the research design, i.e. the *methodology*. That more emphasis should be put explicitly on the methodology has also been put forward by for example Wellington (2000), who describes methodology as "the activity or business of choosing, reflecting upon, evaluating and justifying the methods you use" (p. 22). He further argues that it is necessary to know the methodology of a piece of research to be able to impartially judge and assess it. TWG23 provides a venue and forum for researchers to discuss *how* to best put research results to use in practice alongside the accompanying rationale for *why*. In this sense, TWG23 has as one of its primary foci *methodologies* for initiating and institutionalizing *research-based implementation designs*. Over time, the activities of such a group could also make us wiser on the actual usefulness of our various research results, constructs, and frameworks.

Perspectives for the TWG at future CERMEs

Although the main focus of TWG23 seems currently to be on Nilsen's first aim, and to some extent the third aim, in time the TWG may potentially contribute much more to the second aim: in identifying determinants across various countries; in identifying relevant classical theories external to mathematics education, which may help to understand or explain implementations; and last but not least in developing homegrown *implementation theories of mathematics education*. This was also reflected in the evaluation of the TWG, where the question was asked: What shall be TWG23's contribution of knowledge to the field of mathematics education? The participants of the TWG collectively phrased the following "vision" for the group:

> "We want to explore a wide variety of 'good examples' of implementing research findings and results (back) into practice in order to improve the teaching and learning of mathematics at all educational levels on a research-based foundation. Over time we may begin to look into the aspects of research on implementations, potential requirements for these to function, etc."

Hence, for the future of TWG23, it may be envisioned that the TWG could come to consist of a core of researchers interested in these aspects (Nilsen's second aim). But at the same time, a group like TWG23 is also a place for mathematics education researchers to go when wanting to report and discuss on any "intermediate" activities of either designing new research projects or developmental work, before the activities may result in more traditional research to be reported in other TWGs. In this sense, TWG23 also provides a forum for mathematics educators at CERME to "come and go" from one congress to another.

To put it a bit boldly, it is our hope that this TWG can assist in filling the "gap" of where to report on implementation activities in our research community, while at the same time act as a "bridge" between research and practice.

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Research findings associated with the concept of function and their implementation in the design of mathematics textbooks tasks

Mario Sánchez Aguilar[1], Apolo Castañeda[2] and Rosa Isela González-Polo[3]

[1]Instituto Politécnico Nacional, CICATA Legaria, Mexico; mosanchez@ipn.mx

[2]Instituto Politécnico Nacional, CICATA Legaria, Mexico; acastane@ipn.mx

[3]Instituto Cumbres Toluca, Mexico; rosaiselag@gmail.com

The aim of this paper is to illustrate how the research results generated in the field of mathematics education could be implemented in the design of mathematics textbooks tasks. First we present research findings related to the concept of function, particularly findings related to representation registers and to the conceptualization of function as a process. Next we illustrate with examples obtained from a high school textbook, how these research findings can be implemented in the design of mathematical tasks. We close the manuscript with a reflection on the implications for research that this kind of implementation may have.

Keywords: Textbook development, functions, task design, implementation of research findings.

Introduction

As noted in the call for papers for the Thematic Working Group 23, *implementation research* can encompass a wide range of different kinds of didactical designs on a broad range of formats. In this work we focus on the use of research findings in the design of mathematics textbooks. More particularly, we will address research findings related to the learning of the mathematical concept of function and its implementation in the design of tasks included in a mathematics textbook for upper secondary level.

Textbooks play an important role in the teaching and learning of mathematics. For example, textbooks can affect teaching strategies by conveying pedagogical messages to mathematics teachers (Fan & Kaeley, 2000); also, mathematics textbooks can contribute to the creation and strengthening of students' misconceptions (Kajander & Lovric, 2009), and even the content selection and presentation of materials in a textbook appear to influence learners' participation and success in mathematics (Macintyre & Hamilton, 2010). Due to the huge influence that textbooks can exert in the dynamics of the mathematics classroom, over time there has been a growing interest in developing new and high-quality textbooks, however as pointed out by Li, Zhang & Ma (2014), "there are a very limited number of studies available that examine and discuss textbook design and the process of textbook development" (p. 306). Even more rare are the studies considering the implementation of research results in the design and development of mathematics textbooks, which it is the main focus of this manuscript.

The aim of this paper is to illustrate how research results produced in the field of mathematics education can be implemented in the design and development of tasks for mathematics textbooks. To achieve this we identify some research findings related to the concept of function and illustrate how they were used in the development of tasks for a high school mathematics textbook that is currently in use in the Mexican education system.

On the concept of function and research findings associated with it

We have focused our attention on the concept of function because it is a fundamental concept in the mathematical knowledge and as such plays a major role in mathematics textbooks (Mesa, 2000), particularly in those of middle and upper level. Because of its status and importance in the corpus of mathematical knowledge, educational research around this concept has been developed for several years and its results could be useful in the process of designing textbooks. In the next section we mention some research findings associated with this concept that we ourselves have implemented in the design of mathematical tasks included in upper secondary level Mexican textbooks.

What do we mean by «research findings»?

The products or results generated by research in mathematics education can be varied and fluctuate in a range that goes from tangible products (such as textbooks, educational activities, software) to more abstract products (such as constructs and theories). When we use the term «research findings» in this work we refer to information that has been obtained or discovered through empirical research and that can be expressed through observations, identification of obstacles and students' modes of thinking, didactical suggestions, etc.

We argue that this type of information may be applicable in designing mathematical tasks included in textbooks. Next we present some examples of research findings connected to the learning of the concept of function, whose implementation will be illustrated below.

What research says about the learning of the concept of function?

Different representations of the concept should be encouraged. Some students' difficulties connected with the concept of function can be attributed to the procedural emphasis with witch this concept is taught, and also to the lack of variety in the representation contexts in which this concept illustrated and manipulated.

Research has shown that many students possess prototypical visions of the concept of function. For instance, they tend to assume that functions are linear or quadratic in cases where this assumption is unwarranted, so for example they tend to think that "u-shaped" graphs are parabolas (Schwarz & Hershkowitz, 1999); Carlson & Oehrtman (2005) suggest that this difficulty may be related to the fact that many teachers introduce the concept of function through prototypical examples, which often are linear or quadratic. Thus, they suggest that the concept of function instruction should include more opportunities to experience different types of functions emphasizing different contexts of representation.

This is in line with the observations of Duval (2000). He claims that the conceptual understanding of a mathematical object becomes more robust when there is coordination between representation registers. Each register highlights certain characteristics and properties of the mathematical object, and the interaction between these registers allows for a broader conceptual understanding, so it is important to promote tasks that favor the transit between such representation registers, particularly the transit in directions that are not usually addressed in school, such as transit from the graphical register towards the algebraic one.

A dynamic vision of function as a process should be promoted. A common practice in the teaching of mathematics is to represent functions as static objects, however it has been suggested that

students must possess dynamic interpretations of this concept in order to favor a conceptualization of function as a process (Carlson, Oehrtman & Engelke, 2010). For instance, Figure 1 represents the area under a curve defined by a function *f* (*x*), and it does not promote a dynamic conceptualization of the area function; to achieve such dynamic conceptualization the student should imagine that point *b* moves and in doing so the shaded area S increase or decrease its size.

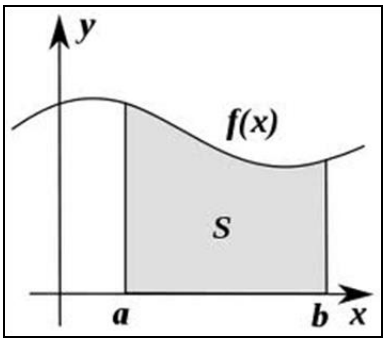


Figure 1: «Static» representation of an area under a curve

This type of static conceptualizations is closely related to an *action view of functions* (Dubinsky & Harel, 1992):

> An *action* conception of function would involve the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). (p. 85)

However, an action view of functions may result in an impoverished conceptualization of the concept; for instance, students with an action view often think of a function graph as being only a curve, a fixed object in the plane, they do not think the graph as defining a general rule where a set of input values are mapped to a set of output values (Carlson & Oehrtman, 2005). It is desirable to move students from an *action view of functions* to a *process view of functions*:

> A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. (Dubinsky & Harel, 1992, p. 85)

Is difficult to achieve this transition from static to a dynamic view of functions, however it has been suggested that technological tools can help in this transition. For example Borba & Confrey (1996) have suggested an approach to the study of functions based on visualization and the use of software; the approach focuses on the relationship between graphs and tabular values, and on the relationship between graphs and algebraic representations. For instance students could be asked to use the software to graph and explore how the coefficients of a quadratic function relate to translations, stretches and reflections of its graph.

Examples of implementation of research findings in the design of mathematics textbooks tasks

As we have claimed before, we believe that research findings as those previously presented can be implemented in the design of tasks for mathematics textbooks. We are aware that there may be different types of «implementation of research findings»; although it is not our intention to discuss such distinction here, we do want to clarify that in this work the «implementation of research findings» is interpreted as taking results or suggestions produced through research, and to use them as a source of inspiration for the design of mathematical tasks. To illustrate this point, next we present examples of tasks that were designed taking into consideration the research findings previously discussed. These tasks are included in the text González-Polo & Castañeda (2014), which was developed by the second and the third authors of this article. This is a textbook for high school level that is currently in use in the Mexican educational system; high school in Mexico traditionally consists of three years of education divided into six semesters, and this book is used in the fourth semester. Its print run for 2015 was 10,000 copies. The tasks proposed in the textbook are unpublished, but some of them have been inspired by tasks used as tools in the development of research in mathematics education.

Tasks to study functions in different representation registers

As mentioned before, research suggests that the concept of function should be studied and manipulated in different representation registers, but also should be promoted the transit between such representation registers, especially in directions that are not habitually addressed in school.

To implement this research-based suggestion, we have designed activities that require the student to transit from a graphical representation register to an algebraic register, when the usual is to ask students to start from an algebraic expression to generate a table of values, and from this table trace the graph of the function. Figure 2 shows an example of this kind of task.

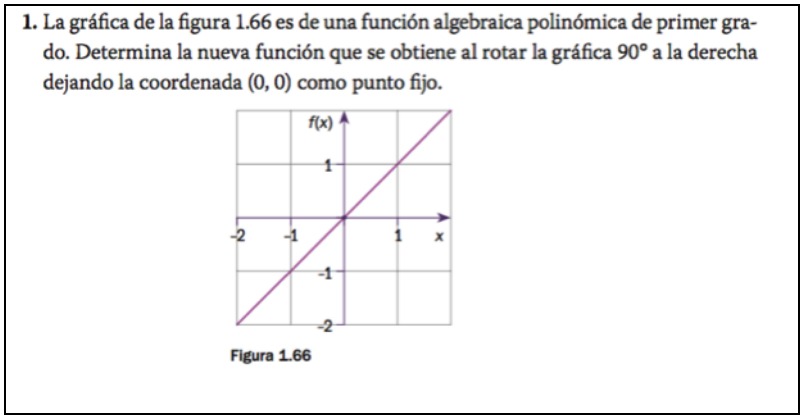
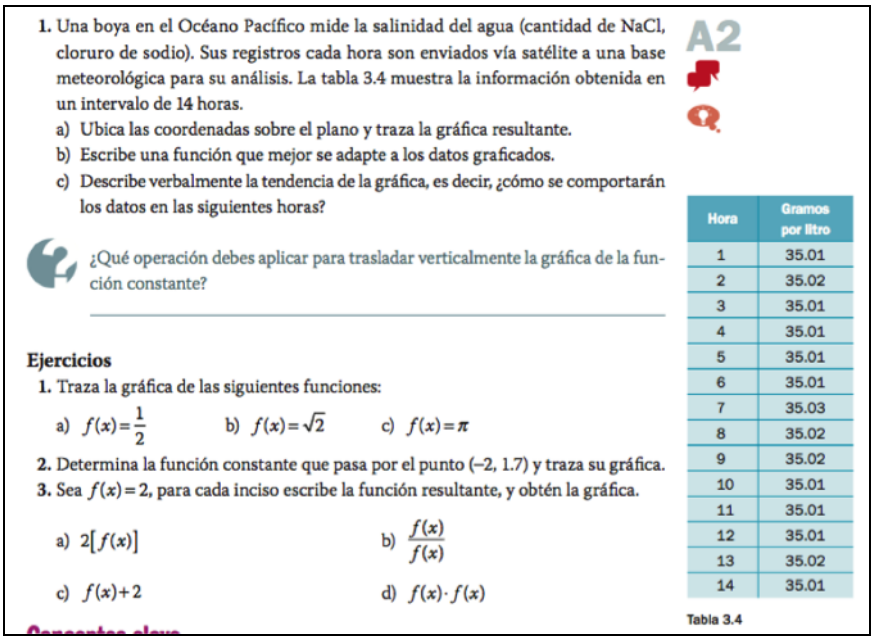


Figure 2: Task that requires the student to transit from a graphical register to an algebraic register

The English translation of the task instruction is as follows: «The graph in figure 1.66 belongs to a first degree polynomial function. Determine the new function obtained by rotating the graph 90º to the right leaving the coordinate (0, 0) as a fixed point». In this task the student must start working on a graphical register—rotating the graph 90º clockwise—and then determine the algebraic expression that defines the new function, which in this case would be $f(x) = -x$.

Another example is shown in Figure 3. This is an activity in which the study of the concept of constant function in different representation registers is promoted, although contrary to usual, the student is required to transit from a numerical register—a table of values—to an algebraic register.



1. Una boya en el Océano Pacífico mide la salinidad del agua (cantidad de NaCl, cloruro de sodio). Sus registros cada hora son enviados vía satélite a una base meteorológica para su análisis. La tabla 3.4 muestra la información obtenida en un intervalo de 14 horas.
   a) Ubica las coordenadas sobre el plano y traza la gráfica resultante.
   b) Escribe una función que mejor se adapte a los datos graficados.
   c) Describe verbalmente la tendencia de la gráfica, es decir, ¿cómo se comportarán los datos en las siguientes horas?

A2

¿Qué operación debes aplicar para trasladar verticalmente la gráfica de la función constante?

Hora	Gramos por litro
1	35.01
2	35.02
3	35.01
4	35.01
5	35.01
6	35.01
7	35.03
8	35.02
9	35.02
10	35.01
11	35.01
12	35.01
13	35.02
14	35.01

Tabla 3.4

Ejercicios

1. Traza la gráfica de las siguientes funciones:
   a) $f(x)=\frac{1}{2}$ b) $f(x)=\sqrt{2}$ c) $f(x)=\pi$
2. Determina la función constante que pasa por el punto (–2, 1.7) y traza su gráfica.
3. Sea $f(x)=2$, para cada inciso escribe la función resultante, y obtén la gráfica.
   a) $2[f(x)]$ b) $\frac{f(x)}{f(x)}$
   c) $f(x)+2$ d) $f(x)\cdot f(x)$

Figure 3: Task that requires the student to transit from a numerical register to an algebraic register

The English translation of the task instruction is the following:

«A buoy in the Pacific Ocean measures salinity (the amount of NaCl, sodium chloride). The measures are sent every hour via satellite to a meteorological database for analysis. Table 3.4 shows the information obtained during an interval of 14 hours.

a) Locate the coordinates on the plane and trace the resulting graph

b) Write the function that best fits the data graphed

c) Describe verbally the trend of the graph, i.e., how the data will behave in the next few hours?»

The interpolation requested in paragraphs b) is somewhat facilitated because the function that best fits the data provided is a constant function, which in this case could be the function $f(x) = 35$. It is important to note that the activity is complemented by the question: «What operation should you apply to vertically translate the graph of the constant function?». This question attempts to engage the student in a dynamic conceptualization of the constant function, that is, to be able to understand that the graph of a constant function $f(x) = k$ will move vertically by adding another constant c obtaining thus the graph of the function $f(x) = k + c$.

Tasks to promote a process view of functions

Inspired by the approach proposed by Borba & Confrey (1996) in which functions are studied with a strong emphasis on the visualization of their graphs, we have included tasks where students are asked to explore the graphical behavior of functions using software. It is assumed that these kinds of activities promote a dynamic conceptualization of functions where the graph is not interpreted as a fixed or static entity. An example of this type of task is shown in Figure 4.

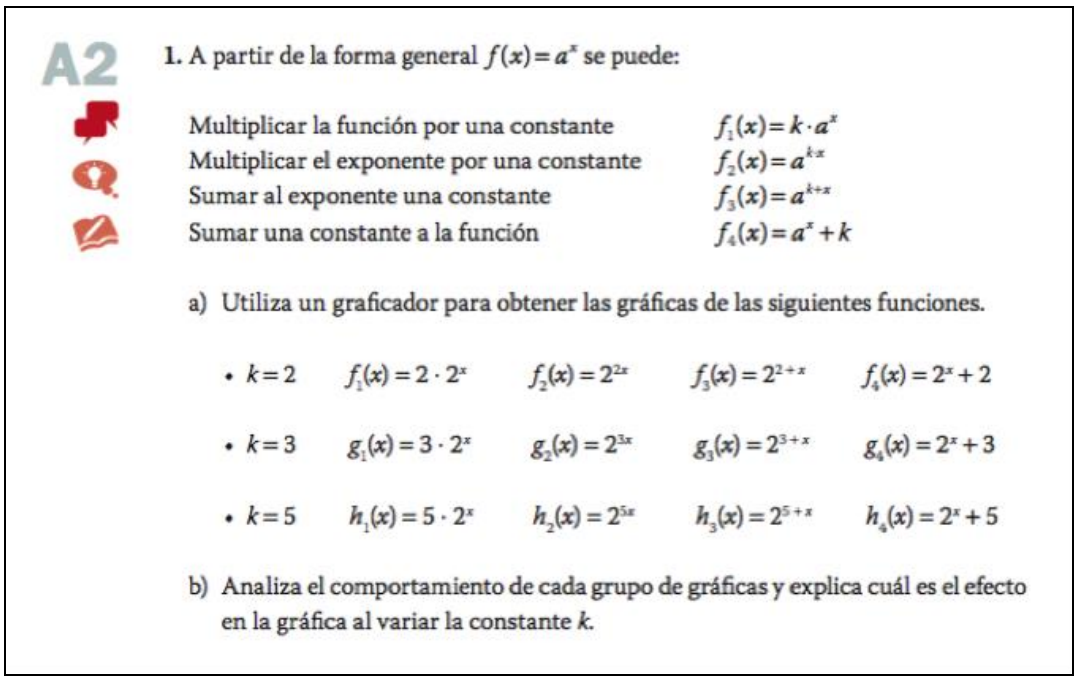


Figure 4: Task requesting the student to use graphing software to explore the effect of parameters on the graph of a function

The task takes as its starting point the function $f(x) = a^x$ and the constant k. Then the student is asked to use software to explore the effects that different integer values of the parameter k produces in the graph of the following functions:

$$f_1(x) = k \cdot a^x$$

$$f_2(x) = a^{k \cdot x}$$

$$f_3(x) = a^{k + x}$$

$$f_4(x) = a^x + k$$

Discussion

In this article we have tried to illustrate how some research findings related to the learning of the concept of function can be implemented in the design of tasks for mathematics textbooks. If one of our aims as mathematics educators is to bring products that are generated in our discipline closer to the school society (teachers, students, administrators, parents, etc.), then the textbooks are a privileged outlet for this purpose since it allows to bring research findings into the heart of formal mathematics instruction: the mathematics classroom.

Our enthusiasm as authors of textbooks and as researchers in mathematics education incline us to think that these tasks with a research-based design can be productive and beneficial for students' mathematical learning, but what evidence is there to support these enthusiast assumptions? It would be necessary to develop studies from different perspectives that could allow us to understand how the textbook mathematical tasks are enacted in the classroom, and the type of conceptions and perspectives that they produce on students.

Regarding the actual design of the tasks, in her reflections on textbook design, Yerushalmy (2015) has suggested that the tasks appearing in textbooks—or more precisely the mathematical concepts involved in such tasks—can be organized around objects and operations that can mathematically and pedagogically support a variety of progressions and sequences. For example, in the case of functions, it can be considered an organizational map that clarifies the type of mathematical object involved in the task (like linear or quadratic functions), but also the type of operations required in the task such as represent, modify, transform, analyze, operate or compare), where each operation can take place in symbolic, graphic, or numeric representations (see figure 5).



Figure 5: Example of an organizational map for the tasks included in a textbook. Taken from Yerushalmy (2015, p. 243)

This type of organizational maps can make the design of tasks more transparent, that is, to render explicit the mathematical objects involved in the tasks as well as the operations that are performed on them. These maps can work as a framework that helps both textbook designers and users to identify gaps in the presentation of concepts, and produce a sequencing of tasks that addresses the largest possible number of operations and contexts of representation with the intention to provide students with a richer picture of the mathematical objects studied.

Finally, it is important to note that in addition to the textbook González-Polo & Castañeda (2014), there is a teacher's guide explaining in more detail the theoretical background on which the design of the tasks is based, as well as their purpose. This kind of guide represents a fundamental support to achieve a classroom implementation of mathematical tasks that is faithful to the intentions of the task designers.

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Designing a research-based test for eliciting students' prior understanding on proportional reasoning

Linda Marie Ahl

Kriminalvården, Section for adult education, Sweden; linda.ahl2@kriminalvarden.se

Mathematics education in the Swedish prison education program is struggling with a high rate of students that fail to pass the basic mathematics courses. One of the main issues seems to be the challenge for the teachers to elicit students' widespread prior mathematical knowledge. The consequence of this is that the teachers cannot meet the students' educational needs with meaningful teaching activities. Focusing on the most pervasive mathematical idea in these courses, proportional reasoning, a test was designed that aimed to elicit students' mathematical reasoning. This paper illustrates that by making use of accumulated and selected research results and findings, we can gain valuable information on students' proportional reasoning competency. This information may be used as an access point for individualized instruction.

Keywords: Adults, individualized instruction, proportional reasoning, prison education.

Introduction

In the Swedish prison education program, only two out of ten students finish and pass their mathematics courses[1]. This is disturbing in itself, but particularly so given the resources available. The teachers are university trained upper secondary school mathematics teachers, and the students sign up voluntarily and typically are highly motivated. Moreover, all courses are individually designed for each student, which should ensure good teaching and learning conditions. However, a challenge is that the student group shows significant variation in age, ethnicity, socioeconomic background, school background and life experience in general. For the basic mathematics, the mathematics in compulsory school and the first course in upper secondary school, there exists no such thing as one course design that suits all students' different backgrounds. In 2015 the Swedish prison education program in mathematics had 728 students enrolled, spread across 47 prisons, and 80% of these were found in the basic mathematics courses

A plausible reason for the low pass rate in the basic courses is that the teachers fail to make proper use of the individualization possibilities. A prerequisite for actual individualization is that teachers have the opportunity to find out students' prior mathematical understanding and adapt the teaching accordingly. Realizing that this opportunity hinges on the teachers' competencies, e.g., they need to put their didactic and pedagogical *teaching competency* to play (Niss & Højgaard, 2011). But teachers' possibilities to individualize instruction might also depend on various forms of support. Inspired by Jankvist and Niss (2015), I report on a research-based effort to develop such support: a test for identifying beginner students' prior mathematical understanding. The test needs to provide information on students prior understanding in two ways: vertically, in relation to progression throughout school years, and horizontally, throughout taught topics in compulsory school. Hence, a major design decision was to focus the content on proportional reasoning. As will be argued below, proportional reasoning permeates the basic mathematics courses in a systematic way, which means that probing students' competencies in this area gives a good access point for individualized teaching. The foundation for the test is the accumulated and selected research results and findings related to proportional reasoning, since proportionality may be the most important, pervasive and powerful idea in elementary school mathematics (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007).

Constructing such a test involves several design decisions involving the content and form of the test, as well as constructing, collecting and adapting test items that realize these design decisions. The

[1] Data from administrator Gunilla Jonsson, personal communication, July 12, 2016.

main question elaborated on in this paper is: *How can research findings inform the development of a test that elicits students' prior understanding on proportional reasoning so as to provide teachers with an access point for designing individualized teaching?*

Theoretical underpinnings for the development of the test

Mathematical reasoning is one of eight competencies for identifying and analyzing students' mathematical understanding, described in the Danish KOM-project (Niss & Højgaard, 2011). "The *mathematical reasoning competency* consists, first, of the ability to follow and assess mathematical reasoning, i.e., a chain of arguments put forward – orally or in writing – in support of a claim." (Jankvist & Niss, 2015,. p. 264). The kind of mathematical reasoning called proportional reasoning is a prerequisite for successful further studies in mathematics and science, since multiplicative relations underpin almost all number-related concepts studied in elementary school (Behr et al., 1992; Lamon, 2007). A proportion is defined as a statement of equity of two ratios a/b = c/d. Proportion can also be defined as a function with the isomorphic properties f(x+y) = f(x) + f(y) and f(ax) = af(x) (Vergnaud, 2009). A function, A(x,y), can also be linear with respect to several variables, (n-linear) functions. For example the area functions for a rectangle with sides x and y is bilinear (2-linear) since A(x,y) = xy and it is easy to check that this function is linear with respect to each of its variables when the other is considered constant.

Proportionality is a key concept in mathematics and science education from elementary school to university (Lamon, 2007). Despite the pervasive nature of proportional reasoning throughout the school years it is well known that children around the world have considerable difficulty in developing the mathematical competency to reason about fractions, percentages, ratio, proportion, scaling, rates, similarity, trigonometry, and rates of change (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007). Typically, proportional reasoning problems come in the shape of a missing value problems or comparison problems (Lamon, 2007). In the former, a multiplicative relation is present where three elements are provided and the fourth is to be found. The latter asks the student to compare which ratio is the bigger or smaller.

From accumulated research, some *key points* for the developing of proportional reasoning and the building of multiplicative structures can be identified (c.f. Behr, Harel, Post, & Lesh, 1992; Fernández et al., 2012; Lamon, 2007; Shield & Dole, 2013; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010; Vergnaud, 1983). Students need to:

1. Be able to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present;
2. Be able to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions, a:b = c:d;
3. Recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures;
4. Acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity.

Key point 1. Research studies and findings show that the ability to distinguish additive from multiplicative comparisons constitute a major stumbling block for students (Van Dooren et al., 2005). Students need to be able to recognize that a proportional situation exists when the comparison is multiplicative (Shield & Dole, 2013). In Sweden, students get acquainted with additive strategies for reasoning about quantities in grades 4 to 6. For example, an increase in price by 10% can be calculated in two steps. First, calculate how much 10% is and then add this to the original price. A transition from an additive to multiplicative thinking approach is introduced in grades 7 to 9. The new price can now be approached in one multiplicative step: the original price multiplied by the factor 1.1, to find

the new price. Far from all students embrace this new idea of approaching percentage change. The additive approach works well for calculating a single increase or decrease, while they may lack motivation to change strategy.
Fernández et al. (2012) found that the error of using additive strategies on proportional situations increased during primary school and decreased during secondary school. A desirable development in students' reasoning would be that they, after being introduced to multiplicative reasoning, still hold on to their ability to use additive strategies when appropriate. However, research findings show that once students have been introduced to multiplicative strategies they tend to overuse this approach on everything that resembles a proportional situation (Van Dooren et al., 2005). Further, non-integer ratios cause more errors than integer ratios (Fernandez et al., 2012; Gläser & Riegler, 2015), while the non-integer situations can be considered to require a more developed understanding of rational numbers.

Key point 2. Many situations require that students can relate to part/part ratios and part/whole fractions (Vergnaud, 1983). For example, if a company employs 11 women and 31 men, the part/whole fractions 11/42 and 31/42 represent the relation of women and men related to the whole. If asked to determine the company's gender distribution, it is instead the part/part ratio 11:31 between women and men that is relevant. When a ratio connects two parts of the same whole, students may not adequately recognize the difference between part/part and part/whole relationships (Clark, Berenson, & Cavey, 2003). It is not easy for students to approach situations that require shifting from part/part to part/whole situations. Moreover, students need to connect mathematical ideas. Since ratios can be written in fraction form, they obey the same mathematical laws as fractions (Shield & Dole, 2002).

Key point 3. Another stumbling block for students is that they tend to apply linear proportional reasoning on scaling, without considering the nature of the item. Van Dooren et al. (2010) found that students tend to use linear proportional reasoning even when it is inappropriate e.g., in word problems where a real word context is required to solve the problem. For example: *Farmer Gus needs 8 hours to fertilize a square pasture with sides of 200 meters. Approximately how much time will he need to fertilize a square pasture with sides of 600 meters?* Recognizing this as a missing value problem i.e., three values given and one unknown, this problem will trigger a cross-multiplication type solution which gives the wrong answer of 24 hours. Since scale is one of the major themes that span mathematics, chemistry, physics, earth/space science and biology it is crucial for students to gain understanding of the concept of scale. Scale in one, two, and three dimensions is a central unifying concept that crosses the science domains, crucial for understanding science phenomena (Taylor & Jones, 2009).

Key point 4. Proportionalities can be represented in different ways, e.g., with words, pictures, algebraically, with graphs or tables. Shield and Dole (2013) enhance the use of a range of representations to promote students' learning. If students are given the opportunity to work with graphs, tables and other diagrams that illustrate the proportional situation present in the mathematical task, their conceptual understanding is promoted (Vergnaud, 2009). Further, their ability to see connections between problems that are based on the same mathematical idea is enhanced, e.g. to see that missing value problems on similarity, proportional functions and speed problems can be illustrated with different representations but approached with the same mathematical idea.

Several concepts are in play when students reason with proportional quantities. The intertwined concepts required for the development of proportional reasoning makes up a *conceptual field* (Vergnaud, 2009). A conceptual field is a set of situations and concepts tied together. As the theory of conceptual fields show, together with other well-known theoretical frameworks for conceptual understanding, the meaning of a single concept does not come from one situation only (Sfard, 1991; Tall & Vinner, 1981; Vergnaud, 2009) but from a variety of situations demanding mathematical reasoning related to the concept in question. The conceptual field of intertwined concepts in play in

proportional reasoning cover at the least "linear and n-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division" (Vergnaud, 1983, p. 141). It is the complexity of the concepts in play together with the pervasive nature of proportional reasoning from elementary school to university that makes proportional reasoning suitable for the design of the test.

Design of the test on proportional reasoning

An important design choice for the test was to use a multiple-choice design. Even though open response tests are a powerful method to elicit students' understanding, the advantages of multiple-choice tests were in this case considered to be the best option. An open response test can be a negative experience for students with low prior understanding, since they may be unable to supply any answers. Since the students often have bad experiences from school mathematics, we want to avoid negative experiences in the beginning of a mathematics course. A multiple-choice test, on the other hand, is easy to take for the students. Even when they do not have the mathematical competencies to reason and solve an item, they can still provide an answer by intuition or chance. The test is designed to be followed up with student interviews. This is an important step since many students do not have Swedish as their mother tongue, which of course may cloud their interpretation of the items. Many of the students also have concentration difficulties, so a written test may not give a satisfactory picture of students' prior understanding.

A downside of multiple-choice is the possibility to choose the right answer by chance. For this reason, a two-tier design was chosen (see examples below) yielding only 0.125 probability to pick both the right *true* or *false* value and the right claim. A pilot version of the test, consisting of 22 items, was tried out in April 2016. Feedback from the participants informed me that the test was too long and that some of the items were difficult to interpret. After revision and further testing, the resulting test consists of 16 proportional reasoning items. The final version of the test takes about 20 to 40 minutes to complete, without any time pressure.

The items in the test were chosen from published research papers, with the intention to draw on knowledge from the research field on proportional reasoning. The rationale for my choices is as follows: a) the items have already been proved to work well for giving information on students' understanding, and b) extensive background information of the nature of the mathematical reasoning in play are provided as well as analyzes of students results. Referring to the *key points* presented in the theory section, the potential reasoning related to each item involves several concepts and abilities, yet the items can still be categorized as referring mainly to one or two of the four presented key points:

Key point 1. Students' ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present, and is always required for carrying out proportional reasoning, however mainly tested by items 1, 5, 6, 7, 11 and 16.

Key point 2. Students' ability to draw connections to the algebraic rules for fractions when working on part/part ratios, part/whole fractions and proportions, a:b = c:d, is mainly tested by items 2, 4, 12, and 13.

Key point 3. Students' ability to recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures is mainly tested by items 3, 8, 9 and 15.

Key point 4. Students' ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity is mainly tested by items 8, 10, 13 and 14.

Several errors on items referring to the same *key point* indicate a lack of understanding that should be investigated further in the following student interview. The test items are also adapted to mirror the progression throughout the basic course. Items 1 and 4 refer to content taught in part two of the basic

course. Items 2, 3, 6 and 10 deal with content from part three and part four is reflected in items 7, 8, 9 and 11-16.

Key points Basic course	Distinguish additive from multiplicative reasoning	Draw connections to the algebraic rules for fractions	Recognize and use a range of concrete representations	Acknowledge the properties of geometrical objects
Part 2	Item 1	Item 4		
Part 3	Item 5 Item 6	Item 2	Item 3	Item 10
Part4	Item 7 Item 11 Item 15 Item 16	Item 12 Item 13 Item 16	Item 8 Item 9 Item 15	Item 8 Item 13 Item 14

Table 1. Schema over items in relation to key points and progression in the basic courses

The sources for the test items are: Hilton, Hilton, Dole, and Goos (2013); Fernadéz et al. (2012); Niss and Jankvist (2013a; 2013b); and Gläser and Riegler (2015). The items from Hilton et al. were already designed as two tier multiple test items. The other items were adapted from their original design to a multiple-choice design, using erroneous answer alternatives either reported in the original studies or answer alternatives recalled from my experience from teaching.

Examples of test items

In what follows, I will exemplify how research results on common difficulties on proportional reasoning are guiding the choice of the test items. To illustrate, items included to elicit students' difficulties to discriminate *additive* from *multiplicative* situations and difficulties with *scaling* are displayed below.

Consider this item, adapted from Fernadez et al. (2012):

> Loading boxes: Petra and Tina are loading boxes in a truck. They started together but Tina loads faster. When Petra has loaded 40 boxes, Tina has loaded 160 boxes. ***When Petra has loaded 80 boxes, Tina has loaded 200 boxes.***
>
> **True** or **False** because (choose the best reason)
>
> a) Tina will always be 120 boxes ahead of Petra.
> b) Petra loads faster than Tina.
> c) Tina loads 4 times faster than Petra.
> d) Tina loads with double speed.

This is a proportional situation where Tina is loading 4 times faster than Petra, so the claim "***When Petra has loaded 80 boxes, Tina has loaded 200 boxes."*** is false. Students should consider whether it is appropriate to use additive reasoning, that is, if Tina has still loaded 120 boxes more than Petra. If the students answer *a) Tina is always 120 boxes ahead of Petra*; further investigation of their reasoning strategies is required, though the answer indicates that there may be a lack of transition from additive to multiplicative thinking. This suspicion is further strengthened if the student is successful in items requiring additive reasoning, like in the item below, from Hilton, et al. (2013):

Running laps: Sara and Johan runs equally fast around a track. Johan starts first. When Johan has run 4 laps, Sara has run 2 laps. ***When Sara has completed 6 laps, Johan has run 12 laps.***

True or **False** because (choose the best reason)

a) The further they run; the further Johan will get ahead Sara.
b) Johan is always 2 laps ahead of Sara.
c) Johan completes double the laps of Sara.
d) Sara has run 3 lots of 2 laps to make a total of 6 laps, so Johan must have run 3 lots of 4 laps to make a total of 12 laps.

This is an additive situation where Sara and Johan run at the same speed. Students should consider whether it is appropriate to use multiplicative reasoning, that is, if Johan runs 3 times faster than Sara. If the students answer d) *Sara has run 3 lots of 2 laps to make a total of 6 laps, so Johan must have run 3 lots of 4 laps to make a total of 12 laps*, further investigation of their reasoning strategies is required though the answer indicates that a difficulty to discriminate multiplicative from additive situations exists.

The two examples above illustrate how research findings on proportional reasoning have been used in the design of the test. By including items requiring multiplicative reasoning as well as items requiring additive reasoning you may elicit the students' ability to discern when a multiplicative situation is present.

The Dice- and the Circle item below are adapted from Niss and Jankvist (2013b), The Dice item is originally phrased: *A cube of wood with all edges 2 cm weighs 4.8 grams. What weighs a cube of wood, where all edges are 4 cm? Justify your answer.* [En terning af træ med alle kanter lik 2 cm vejer 4.8 gram. Hvad vejer en terning af træ, hvor alle kanterne er 4 cm? Begrund dit svar.] I added the claim: **"*A wooden dice where all edges are 4 cm weight 19.2 g.*"**, and the response alternatives.

Dice: A wooden dice where all edges are 2 cm weighs 4.8 g. ***A wooden dice where all edges are 4 cm weight 19.2 g.***

True or **False** because (choose the best reason)

a) The weight increases 4 times if the edge doubles.
b) The weight increases 6 times if the edge doubles.
c) The weight increases 8 times if the edge doubles.
d) The weight doubles if the edge doubles.

Circle: ***Simon says that if you draw a new circle with half the diameter of another circle, the new circle will have half the perimeter and half the area of the other circle.***

True or **False** because (choose the best reason)

e) If the diameter is halved, the perimeter and area is halved.
f) The area will be ¼ and the perimeter ½ of the original.
g) You cannot know without knowing the length of the diameter in the new circle.
h) You cannot know without knowing the length of the diameter in the original circle.

Students may fail to interpret the effects on volume from a doubling of the edges, while further investigation on the students' conceptualization of geometrical objects needs to be undertaken. To reason about the circle item, the students need to consider the conjunction that both the perimeter and the area are halved. Since (area scale) = (length scale)^2; a halving of diameter will result in a ¼ size of area while the perimeter halves. An error on these items may indicate difficulties to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity.

Reflection

There are many reasons why educational research tends to be isolated from practice. Research results and findings need to undergo a number of transformations from theory to practice, before they can be adapted to teaching practice, as illustrated in the design of the discussed in this paper. The test was designed with considerations to a special prison context and early results from using the test shows that it provides valuable support for the teacher when eliciting students' prior understanding of mathematics. Although, the test focuses on the *mathematical reasoning competency* it also informs us of students' *mathematical thinking competency*, *problem-handling competency* and *modeling competency* since these competencies are intertwined and overlapping. Together these four competencies create one out of two overall competences associated with mathematics: *The ability to ask and answer questions in and with mathematics* (Niss & Højgaard, 2011). The other overall competence: *The ability to deal with mathematical language and tools*, covers the intertwined competencies *representing competency, symbol and formalism competency, communication competency* and *aids and tools competency*. The scope of the test does not cover *the ability to deal with mathematical language and tools*. These competencies are left to be tackled within the course design, as well as the further development of the students' *ability to ask and answer questions in and with mathematics*.

A fundamental idea of educational research is that research findings should be put in play in teaching practice to help students to succeed with their studies in mathematics. I have discussed the design of a test for supporting teachers when pursuing the goal of finding an access point for individualized instruction. Through making use of accumulated and selected research results in the area of proportional reasoning in the design of the test, we gain a more thoughtful idea of the students' prior understanding.

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‘Explicit teaching’ as an effective method of acquiring problem solving strategies - the case of ‘working backwards’

Miriam Amit and Yelena Portnov-Neeman

Ben- Gurion University, Department of Science and Technology Education, Israel;

amit@bgu.ac.il ; yelenapo@post.bgu.ac.il

It has been shown that children who control strategies are better able to direct their own learning and knowledge. Seeking for an effective teaching method to achieve this goal, we experimented with the Explicit Teaching method vs. a traditional school one, using both to teach the Working Backwards Strategy. The study was conducted amongst 57 mathematically talented students participating in a unique mathematics program called Kidumatica. A mixed method analysis showed that Explicit Teaching produced better results regarding students' ability to use the strategy, though it did not affect the students' ability to recognize the strategy. This indicates that young students can understand when to use this powerful tool and, with further guidance, can also master their ability to use it.

Keywords: Explicit teaching, strategy, working backwards strategy

Introduction

Since Polya (1957), who claimed that students who control many strategies will become more effective and intelligent problem solvers, other researchers have advocated integrating problem solving strategies into school mathematics (English, 1993; Steiner, 2007), especially for talented students (Lee, 2014). The development and use of strategies is definitely not intuitive, and students need proper instruction, guidance and encouragement in order to systematically implement strategies in different domains - especially in problem solving (Tishmen, Perkins & Jay, 1996). Unfortunately, teachers face immense difficulties when it comes to teaching strategies in an effective way (Zbiek & Larson, 2015). This fact led us to look for an effective method to teach (talented) students some basic problem solving strategies. Based on interviews and talks with “professional mathematicians," we decided on four such strategies: Trial and Error, Proof by Contradiction, Working Backwards, and Recursion, which were taught to students using the “Explicit Teaching method.” In this paper we focus on just one example, showing how Explicit Teaching can be used to teach students the "Working Backwards Strategy” for solving non-routine mathematical problems.

Explicit teaching method

Explicit Teaching is a systematic methodology that is currently used primarily to teach reading and mathematics. This method is described as “highly organized and structured, teacher-directed, and task-oriented” (Archer & Hughes, 2011). All stages of the learning process include mediation between the teacher and the learner, in which the teacher transmits an external understanding of certain information to the learner, who then processes that pre-determined understanding (Olson, 2003). Nevertheless, using Explicit Teaching does not predetermine or confine learners’ thinking; on the contrary, it can help them become more active solvers and foster independent thinking (Portnov-Neeman & Amit, 2015). The methodology consists of a five step model (Figure 1). The steps described below are performed sequentially by the instructor in order to efficiently pass on specific information to the learner with as little ambiguity and room for error as possible (Rosenshine, 1986).

Orientation: Each lesson begins with a clear instruction about the purpose of the lesson. Learners need to understand what they going to learn and how it connects to previous lessons.

Presentation: The lesson material is divided into small units that fit the learners' cognitive abilities. The teacher uses a model or schema to guide them through their problem solving process.

Structured Practice: The instructor gives a direct and detailed explanation of the problem solving using the model or schema that was presented in the previous step. During this phase, it is critical that the instructor asks learners questions and encourages class discussion in order to check and assess their understanding of the material and clarify any confusion.

Guided Practice: In this practice the instructor addresses individuals' questions and misconceptions one-on-one, and tailors responses to meet the individual needs of each learner. Students can work in small groups in order to develop their ideas together and help each other with the new material.

Independent Practice: In this step, learners are asked to complete an assignment on their own and without assistance. They are not expected to have a flawless understanding of the lesson, but they must understand the steps involved in the process. This step should continue until learners gain full independent proficiency with the materials.

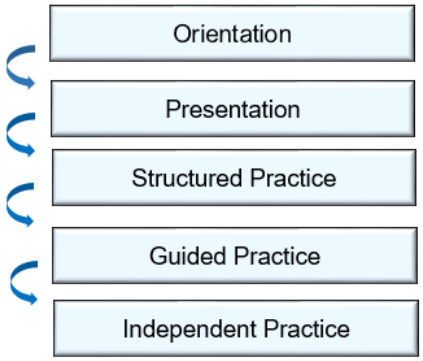

Figure 1: Model of Explicit Teaching

'Working Backwards'

'Working Backwards' is a useful and efficient strategy for solving problems in many aspects of our lives, in which an achievable outcome is known, but we have not yet determined the path towards achieving it (Newell & Simons, 1972; Portnov-Neeman & Amit, 2015). When dealing with word problems, for example, the information given in the problem can appear like a complex list of facts, so it is sometimes helpful to begin with the last detail given (Wright, 2010). The Working Backward strategy is illustrated in Figure 2 and explained step by step in detail below:

1) Read the problem from beginning to end and identify all its components and steps.

2) Check the final outcome of the problem.

3) From the final outcome, start reversing each mathematical operation in each step until you reach the beginning of the problem. For example, reverse the adding operation and replace with a subtraction operation.

4) After reversing every step, resolve the initial state of the problem.

5) Check the answer by starting from the initial state and working through the steps to see if the final outcome is achieved (Amit, Heifets & Samovol, 2007).

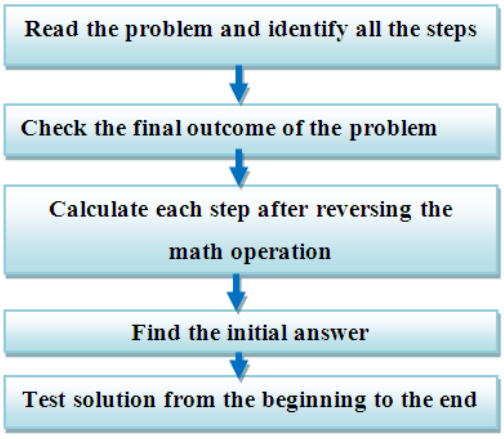

Figure 2: Model of the Working Backwards strategy (Amit, Heifets & Samovol, 2007)

Methodology

The study presented here examined the effect of using the Explicit Teaching method to learn a new strategy, specifically the Working Backwards Strategy for mathematical problem solving. The research questions are: To what extent does Explicit Teaching affect:

a) The ability to recognize when the Working Backwards Strategy is needed for problem solving?

b) The ability to use and implement the Working Backwards Strategy?

Context

The study was conducted in the framework of "Kidumatica". Kidumatica - the math club for excellence and creativity - is an after school program for talented students in the 5th to 11th grades who are interested in mathematics, but require further tools to reach their full potential (Amit, 2009). Fifty-seven (N= 57) 6th grade students were divided in two groups: an Experimental Group (EG = 30 students) and a control group (CG = 27 students). Over a period of six months, these students learned different mathematical strategies, including the Working Backwards Strategy. The EG learned through the Explicit Teaching method, while the CG was taught using the traditional school one. None of the students in this study had been research subjects in previous studies involving the Working Backward Strategy, and none had learned the strategy before.

The 'Explicit Teaching' Group (Experimental Group)

Students in this group studied all the strategies by means of the Explicit Teaching method. Each strategy, including the Working Backwards Strategy, was taught for four weeks by one of the researchers, according to the model illustrated in Figure 1. The teacher had an integral part in the lessons. She clearly and explicitly outlined what the learning goals are for the student, and offered clear, unambiguous explanations of the skills, information and the problem solving process. As the lessons progressed, the teacher's role reduced, until students were able to solve problems independently. It was like riding a bicycle, were the instructor gradually releases his hold of the bike and the child rides off by herself. The first lesson started with an explanation of the strategy, including its importance as well as where and how it should be implemented. The teacher showed the students the model of the strategy and explained the role of each step in the solution process. The following lessons were dedicated to structural, guided and independent practices. During the structured practice the teacher gave a direct and detailed explanation of the problem solution using the Working Backwards model (Figure 2). The teacher encouraged discourse between the students and asked questions to assess their understanding and clarify any confusion. In the guided practice, students worked in smaller groups or by themselves on different working backwards problems. The teacher walked around the students and addressed individuals' questions. When the teacher felt confident enough of a student's abilities, that student was allowed to start working individually and begin the independent practice step. At the independent practice stage, students were asked to complete several assignments using the working Backwards Strategy, and to solve complex problems on their own.

Traditional teaching control group

The control group studied the Working Backwards Strategy for the same period of time as the EG, but they studied the strategy in the traditional school method. This group differed from the EG in the following ways:

1. Lessons were mainly dedicated to students' work. The teacher's part was smaller than in the EG. Her role was to give short explanations about the lesson activities. She did not use the word "strategy" in her explanations, or explain that a special approach is needed for solving working backwards problems. Most of the lesson was dedicated to independent time, so that students would develop their own strategy toward those problems. It was important that students draw their own conclusions, create their own conceptual structures, and assimilate the information in the way that makes the most sense to them.

2. The teacher did not show the model of the strategy and did not name the strategy explicitly. Instead, students could develop their own model and meaningful name based on the teacher's examples and their own experience.

3. The practice process in the CG was mainly independent, in contrast to the three levels of practice used in the EG. That led to less room for discussion and collaborative work between the students, unlike the EG, where time was allotted for these during the structured and guided practice.

Data collection and analysis

Data was collected via pre- and post-tests, students' products, short interviews during and after the lessons, and teacher's notes. The pre/post-tests were administrated to both groups before and after the learning program. Both tests included working backwards problems. This paper will discuss two problems from the pretest (problems 1 and 2 below), and two from the posttest (problems 3 and 4).

***1. Card Problem**: "Yael, Danny and Michael played cards. At the beginning of the game each one had a different amount of cards. Yael gave Danny 12 cards. Danny gave Michael 10 cards and Michael passed Yael 4 cards. At the end each one of them had 20 cards. How many cards did Yael, Danny and Michael have in the beginning?"*

***2. Mangoes Problem**: "One night the King couldn't sleep, so he went down into the royal kitchen, where he found a bowl full of mangoes. Being hungry, he took 1/6 of the mangoes. Later that night, the Queen was hungry and couldn't sleep. She too found the mangoes and took 1/5 of what the King had left. Still later, the first Prince awoke, went to the kitchen, and ate 1/4 of the remaining mangoes. Even later, his brother, the second Prince, ate 1/3 of what was then left. Finally, the third Prince ate 1/2 of what was left, leaving only three mangoes for the servants. How many mangoes were originally in the bowl?"*

***3. Weight Problem**: "Four students in the class weighed themselves. Cobi was 15 kilograms lighter than Adi. Gaby was twice as heavy as Cobi and Jenya was seven kilograms heavier than Gaby. If Jenya weighed 71 kilograms what was Adi's weight?"*

***4. Baseball Problem**: "The Wolverines baseball team opened a new box of baseballs for today's game. They sent 1/3 of their baseballs to be rubbed with special mud to take the gloss off. They gave 15 baseballs to their star outfielder to autograph. The batboy took 20 baseballs for batting practice. They had only 15 baseballs left. How many baseballs were in the box at the start?"*

At the end of each test, students were asked to write what method they had used to solve the problems. The purpose of the pre-test was to determine the homogeneity of the two groups. The post-test examined the effect of the teaching methods at the end of the learning process. A five point scale was used to rank students' answers (5 points = full and correct answer, 0 points = no answer). For example, if students identified all the steps, calculated each one by doing the opposite mathematical calculation and wrote the final answer correctly, they received 5 points. Figure 3, for example, shows a five point solution for the "Weight Problem." The problem has three steps: (1) Jenya was seven kilograms heavier than Gaby; (2) Gaby was twice as heavy as Cobi; (3) Cobi was 15 kilograms lighter than Adi. The student calculated the weight of each person by working backward through every step of the problem. Figure 4 shows an example of a 2 point solution. The student started with the last detail given and calculated Gaby's weight correctly. However in the next two steps he did not reverse the mathematical operations and got an incorrect answer.

	Final phase	Operation	
Adi	47	Cobi was 15 kilograms lighter than Adi	32+15 = 47
Cobi	32	Gaby was twice as heavy as Cobi	64:2 = 32
Gabi	64	Jenya was seven kilograms heavier than Gaby	71-7 = 64
Jenya	71	Jenya weighed 71	

Answer: Adi weights 47 kilograms.

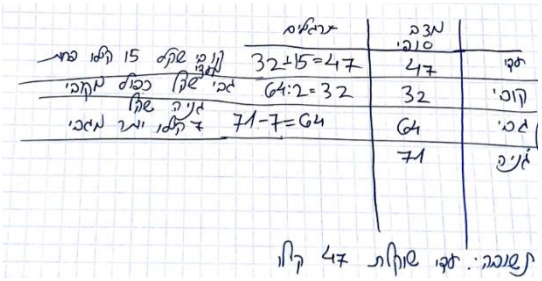

Figure 3: Example of a five point answer to the weight problem

Jenya	Cobi	Adi	Gabi	Categories
1 71	2 64x2=128	4 64+15=79	3 71-7=64	Weight
71	128	79	64	Answer

Adi weight 79 kg

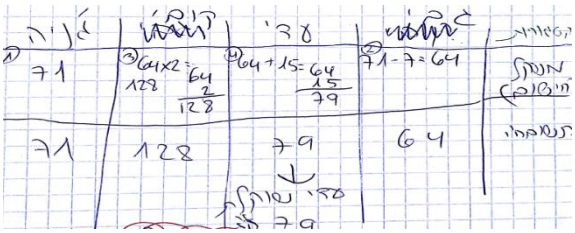

Figure 4: Example of a two point answer to the weight problem

Findings

Findings from the pre-test showed that in both problems, there was no significant difference between the groups, which indicates that both groups had the same level of homogeneity. After six months of learning strategies, the average scores in the post-test for both problems were higher among the EG than the CG. In Table 1 we can see a significant difference in the post-test between the two groups in both problems. Figure 5 indicates that students' ability to recognize the strategy improved after the learning process, but that both groups had similar results in the pre and post-test.

Table 1: Results from pre- and post-test in the EG and the CG

Test	Problem	Group	N	Mean	SD	Sig
Pre test	Card Problem	EG	30	3.50	1.676	*P = 0.68*
		CG	27	3.30	2.035	
	Mango Problem	EG	30	1.77	1.977	*P = 0.96*
		CG	27	1.74	2.141	
Post-test	Weight Problem	EG	30	4.97	.183	*P = 0.00**
		CG	27	3.56	1.987	
	Baseball Problem	EG	30	3.77	1.547	*P = 0.03**
		CG	27	2.89	1.553	

**P<0.05*

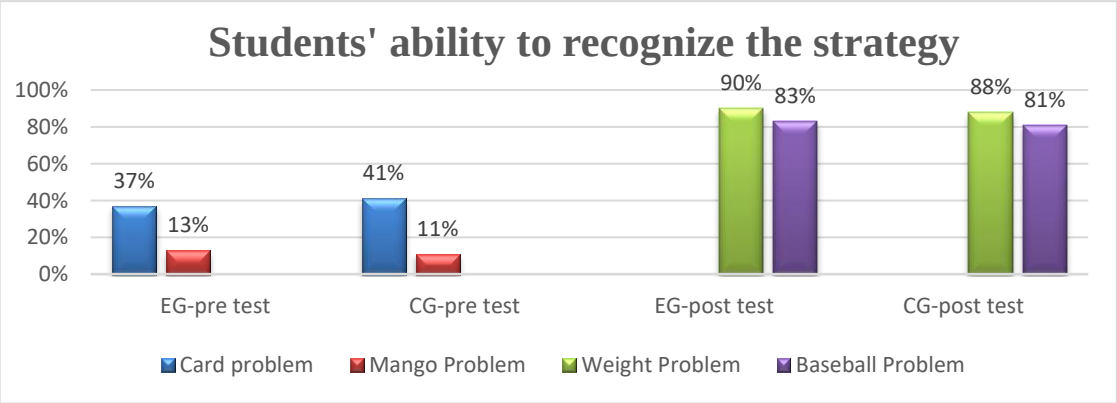

Figure 5: Amount of students from the EG and CG that recognized the Working Backwards Strategy in pre- and post-tests

Our qualitative analysis of students' solutions revealed that the EG students reversed the mathematical operations much better and more easily than those in the CG, and were thus able to solve the problem correctly. Moreover, while the EG explicitly stated the name of the strategy they had used when asked, the CG students were very creative in naming the strategy, coining names such as, “going in through the back door”, “reverse manual” etc. Finally, the EG students used the model of Working Backwards Strategy in a very efficient way, sometimes adjusting the model to make it easier to use.

Discussion

Strategies are undoubtedly an important tool for goal-directed procedures in problem solving. Introducing them at a younger age can improve learners’ mathematical ability (Polya, 1957) as well as their understanding and thinking skills (English, 1993). To achieve this goal, it is important to use a specific teaching method (Tishmen, Perkins & Jay, 1996). In this study, that method is the Explicit Teaching method, through which we introduced the Working Backwards Strategy. The study examined the effect of this method on students' ability to recognize and solve working backwards problems. Fifty-seven sixth graders were divided into two groups, an experiment group (EG) that studied with the Explicit Teaching method and a control group (CG) that studied with a traditional school one. The strategy was unfamiliar to both groups and the findings from the pre-test showed that both groups had a similar starting point. At the end of the learning process, the group that studied explicitly showed higher results than the control group. The structured steps of the Explicit Teaching helped the students to have a better, clearer understanding of the strategy (Anhalt & Cortez, 2015). Qualitative analysis revealed that students who studied explicitly were more much resourceful in their solutions. They understood how the strategy works, adopted it and changed it to make it easier to solve. We believe that this ability developed due to the discourse and the collaborative work in the structured and guided practice. We saw how students’ understanding of the strategy and its use improved over time. They asked more questions, listened to other students’ answers and learned how to avoid misconceptions. In addition, the integral role of the teacher in this method helped students gradually to build their confidence. Thus, these students were more prepared to work on working backwards problems by themselves. Our previous study showed that teaching explicitly can help students become active learners and foster their independent thinking (Portnov-Neeman & Amit, 2015). The current study supports this conclusion, showing that Explicit Teaching did not limit students' thinking by fixing it on a particular process. On the contrary, students understood the core principle of the Working Backwards Strategy and then applied it creatively in whatever way seemed

best to them. Both groups showed improvement over time in their ability to recognize when and why the Working Backward Strategy is needed. The percentage of students that recognized the strategy before and after the learning process was similar. This is very encouraging, since it may indicate that the teaching method does not affect students' ability to recognize strategies. We can assume that with additional practice, all of the students could potentially master strategies and develop their understanding and their strategic approach to problem solving, but this has to be tested and researched. In this study, we experimented in 'laboratory conditions' with talented students, and found that the method works. Further research is needed to confirm its effectiveness outside of the Kidumatica Mathematics Club, in the 'real world' of education.

Conclusion

Mathematical strategies are complex concepts to learn and understand, and we as educators must search for the most effective way teach them. In this study, we used a systematic and structured methodology called Explicit Teaching, and found that students who studied with this method had higher scores than students who studied with a traditional school method. Introducing strategies like these to students is important, since they can help students evolve into better thinkers and develop their ability to solve problems. We believe that strategies can and should be introduced from a younger age so they can be developed over time. We have found that younger children are capable of acquiring the basic tools. Given time, they will be able to develop their tool kit of strategies further, and eventually master them all. It is our obligation as educators to teach our students how to use strategies correctly, and the sooner we do so the better.

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Using a Models and Modeling Perspective (MMP) to frame and combine research, practice- and teachers' professional development

Jonas Bergman Ärlebäck[1]

[1]Linköping University, Department of Mathematics, Sweden; jonas.bergman.arleback@liu.se

This paper describes and discusses the framing of, and experiences from, a project that combine research, practice- and teachers' professional development based on the tenets of the Models and Modeling Perspective on teaching and learning (MMP). Besides providing a general description of the methodological considerations in the project design, the paper describes how the accumulated results and experiences in the research literature on so-called model eliciting activities are used to inform the design, implementation and evaluation of activities aiming at introducing functions to grade 8 students. The focus of the paper is on the implantation and aim to showcase how the teacher in question realized the offered perspective and tools in practice.

Keywords: Practice development, teachers' professional development, models and modeling perspective, model eliciting activities.

Introduction

Clarke, Keitel and Shimizu (2006) have shown that much of the teaching and learning of mathematics in many counties are centered around, and dominated by, a traditional use of textbooks. This practice seems to strengthen as the students progress though the educational system, as does the fact that many students lose their interest in, and motivation for, learning mathematics with increasing age. TIMSS 2007 for example shows that the attitudes towards mathematics in Sweden expressed by grade 4 students generally are much more positive compared to the attitudes among grade 8 students (Skolverket, 2008). This situation in combination with the declining results of Swedish students on the international assessments PISA and TIMSS, are reflected and frequent in the public debate as well as in many of the ongoing drives, projects and programmes trying to realize changes in schools. Learning mathematics is complex (Niss, 1999), and now more than ever is the role teachers play stressed for what mathematical understanding and knowledge the students develop in schools (Hattie, 2009; 2012).

However, the challenges that teachers meet in their everyday mathematics teaching are numerous and of various kind and nature. Students' lack of interest in combination with too monotonous (and "traditional") forms of teaching seems to be part of the reasons for the Swedish students' declining performances as well as interest in mathematics. Often, a proposed strategy to reverse theses trends is to try to change the prevailing norms in the classroom (Yackel & Cobb, 1996) by increasing student interaction and the overall student activity. Teachers are encouraged to try to vary their teaching, increase students' activity levels and strive to make students 'talk more mathematics'. But how is this going to work in practice in everyday teaching? How do we get students to 'talk more math' and to be more engaged and (inter-)active in the mathematics classrooms?

The concerns mentioned above are part of the motivation for the installment for a joint collaborative initiative between two municipalities and a university, from which this paper will discuss some aspects. The overarching question for the initiative is: *How can we organize the mathematics*

teaching so that students are given the opportunity to develop their conceptual, procedural/ methodology and reasoning abilities in, for the students, interesting and engaging ways?

A project combining research, practice- and teachers' professional development

As a response to the situation and challenges briefly outlined above, a collaboration between two municipalities and a university was initiated with the aim to establishing a long term and sustainable collaboration as well as to seek for ways to counterbalance the current trends. The initiative rest one three strands, namely to simultaneously (1) combine and produce research with (2) the development of teaching practices in schools and (3) to serve as professional development for the teachers in the municipalities. The project involves two researchers focusing on different grade levels: grades F to 6 (6- to 12-year-olds) and grades 7 to 12 (13- to 18-year-olds). The two researchers have autonomy in how they define, plan and conduct the work within the given boundaries defined by the university and the municipalities.

The project focusing on grades 7 to 12 runs a series of semi-parallel 1-year projects were the researcher in each project works together with 4-6 teachers from different schools and grades as *partners* (c.f. Jaworski, 1999) in, what ideally could be described as, a *co-learning agreement* (Wagner, 1997). Each project departs from the practices of the participating teachers and the possibilities and challenges they see in their everyday teaching. Based on the teachers' experiences a discussion leads to the formulation, planning and implementation of a 1-year long project with specified aims and goals. The research carried out in the projects is centered around the participating teachers own everyday teaching, and their engagement in research and developing their own practices constitute the professional development for the teachers. Within the context of the initiative two key questions then become: *How to coordinate the experiences and result from the individual projects?* and *How to communicate then?* Note that these questions also are at the heart of mathematics education more generally (aka the accumulation of research and the dissemination of knowledge; a main topic for CERME 10's TWG23). For the 1-year project discussed in this paper, the studied question was: *How can we create and work with joint classroom activities that challenge all students regardless of their levels of mathematical understanding and capabilities?*

The models and modeling perspective on teaching and learning

The *models and modeling perspective on teaching and learning* (Lesh & Doerr, 2003), MMP for short, sometimes given as an example of a so-called *contextual perspective* in the discussion on modeling (Kaiser & Sriraman, 2006), draws on and traces it's lineages back to Vygotsky, Piaget and Dienes as well as influences from the American pragmatists' tradition represented by Mead, Peirce and Dewey (Kaiser & Sriraman, 2006; Mousoulides, Sriraman, & Christou, 2007). The central notion in this perspective is that of *models*, which are conceptual systems used to make sense of situations and phenomena. Models are considered to be human constructs which are fundamentally social in nature and can be described as systems consisting of elements, relationships, operations, and rules that can be used to predict, explain or describe the behavior of some other system. In the MMP learning is equated with model development, in which the role of modeling activities is to support this development by engaging the students in purposefully developing, understanding, modifying, and using their models to make sense of different situations and contexts (Lesh & Doerr, 2003).

The adaptation of the MMP at the macro level for all grade 7-12 projects establishes a common perspective and vocabulary that facilitate communication within as well as between different projects and levels of stakeholders in the initiative. The inherent recursive complexity of the MMP (researchers developing models of teachers' models for teaching and supporting students developing their models) connects the work and results from the different projects and levels. The inclusive and accessible notions models and model development (understood in a more mundane way) facilitates communication with teachers, high municipality officials and policymakers.

Model eliciting activities

Model eliciting activities (MEAs) are purposefully designed activities where students need to develop a model that can be used to describe, explain or predict the behavior of, for the students, meaningful contexts, phenomena and situations. Traditionally, much work within the MMP have been centered around so-called *model eliciting activities* (MEAs) developed by Lesh and colleagues (Lesh, Hoover, Hole, Kelly, & Post, 2000). Although originating in mathematics, MEAs have in the last 15 year been used to support and investigate the development of students' models (conceptual systems) in a range of disciplines and contexts (Diefes-Dux, Hjalmarson, Zawojewski, & Bowman, 2006; Iversen & Larson, 2006; Yildirim, Shuman, & Besterfield-Sacre, 2010; Yoon, Dreyfus, & Thomas, 2010).

The research involving MEAs have resulted in six design principles for MEAs, which also to some extent capture the essence of the MMP: (a) *the reality principle* – the MEA connects to students' previous experiences and is meaningful; (b) *the model construction principle* – the MEA induces a need for the students to develop a meaningful model; (c) *the self-evaluation principle* – the MEA permits the students to assess their work and models; (d) *the model documentation principle* – the situation and context in the MEA requires the students to externally express their thinking (models); (e) *the model generalization and sharable principle* – the elicited model in the MEA is sharable, generalizable and applicable to similar situations; and (f) *the simplicity principle* – the situation in, and formulation of, the MEA is as simple as possible (Lesh et al., 2000; Lesh & Doerr, 2003).

Teacher working with MEAs have proven to provide rice opportunities for professional change and development. Schorr and Lesh (2003) found that teachers working with MEAs in their classrooms

> (a) changed their perception regarding the most important behaviors to observe when students engaged in problem activities; (b) changed their views on what they considered to be strengths and weaknesses of student responses; (c) changed their views on how to help students reflect on, and assess their own work; and (d) reconsidered their notions regarding the user of the assessment information gathers from these activities. (Schorr & Lesh, 2003, p. 157)

These experiences and results suggest that MEAs might provide a productive tool to address the question about how to create mathematics teaching that is challenging for all students. The teachers in the project found this a promising approach and especially expressed the following aspects of MEAs appealing: MEAs build on and respect what the students bring to the classroom in terms of prior knowledge in a fundamental way; MEAs focus on the students' sense making of meaningful situations, representations and connections between representations; working with MEAs naturally includes a range of classroom organizations (working one-by-one, in pair, group or different whole class interactions); MEAs have the students work on explicitly formulating and expressing their

thinking using mathematics. In addition, the six design principles for MEAs come to play a few different roles: tools for design; tools for analyzing tasks; evaluative tools for students work in class; and tools for thinking about one's own view of mathematics, teaching and learning.

The MEA and its' implementation

We now end the paper by showcasing the result of one teacher's implementation of an MEA given as an introduction to linear function in grade 8.

The context and the design of the MEA *Candy time!*

The teacher wanted to use an MEA to introduce linear functions in grade 8. Functions, and different representations of functions, are something that the students have been exposed to more or less consciously in different forms during the majority of their mathematical schooling. In other word, students already have ideas and models for what graphs and tables are and how and when to use them. So, rather than systematically treat these concepts in a traditional manner, the teacher wanted to challenge the students to use their previous experiences and to see and explore the connections between tables, graphs and diagrams by engaging in a more exploratory activity (aka an MEA).

The context of the problem was chosen by the teacher to be about buying candy. In Sweden, there is a often practiced tradition, that the children on Saturday do their weekly candy shopping, called *lördagsgodis* – "Saturday's Candy." Not seldom, the candy is bought in candy stores where you pick 'n' mix candy after your own preferences and liking, and pay by the hectogram (100 grams) or the based on the actual number of pieces of candy you picked.

In the design of the MEA the teacher stressed four of the guiding principles as especially important for this particular purpose: (a) *the reality principle:* the choice of context and situations (the candy store) was made in order to be familiar to the students in that it should facilitate the students in making connections and interpretations between different representations; (b) *the model construction principle:* the intention with the activity *Candy time!* is for the students to build on their previous experiences and knowledge in order to connect and coordinate them further; (d) *the model documentation principle:* to facilitate for the students to document their work, a in that they make their models visible and objects for discussion, a worksheet was developed with easy-to-read instructions, questions and diagram as well as generously with space for writing answers and comments; (e) *the model generalization and sharable principle:* to promote that students share ideas as a means for furthering their models, the MEA was designed to have the students working along as well as in pairs or small groups, and engaged in whole class discussions. Note that the four principles not are independent, and that they contribute to make the students' previous experience and knowledge the basis for the activity, to make students' ideas and thoughts (models) visibility, and to facilitate that the students' models are confronted with other students' models so that they through discussion can refine and develop their ways of thinking.

Implementation

After the teacher started the lesson and introduced the first part of the activity, the students began to work individually on the first part of the task about the three stores A, B and C; see Figure 1 below. However, it only took seconds until the students spontaneously started discussing with each other about which store would given them the most candy for their money, and they spontaneous formed

pairs and small groups. The teacher observed the students' work and listen to the students' discussions while walking around in the classroom and making sure all students understood the task, but otherwise intentionally kept a low profile.

Candy time!

It's Saturday and you're thinking about which of the three stores A, B and C you'll go to and spend 2,50 € so that you'll get as much candy as possible. Compare all three stores and motivate your choice.

Store A

You'll pay 1 € for a bag of 32 pieces of candy.

Store B

Pieces candy (#)	Price (€)
5	1,5
10	
15	

Store C

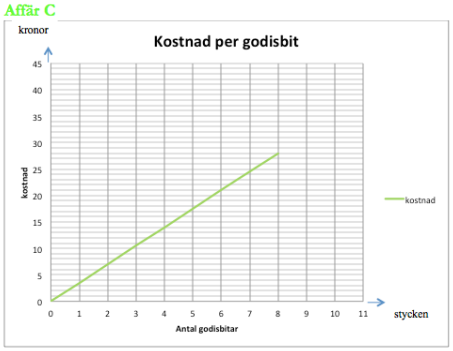

Figure 1: Part one of the activity *Candy time*!

The idea was that the students' should use an experimental approach and try different ways and strategies to approach the problem. If the teacher noticed that some of the student got stuck she approached the student with encouragements like *"Try to fill out the table for Store B!*", *"What would it look like if one plotted the table-values for Store B in the same diagram displaying Store C's pricing?"*, or "*What would Store A 'look like' in the Store C diagram?"* When the majority of the students had decided in which Store to do their shopping of Saturday's candy, the teacher focused and pulled the class together by asking *"What would the graphs for Store A and Store B look like if you plotted them in the same diagram as the graph for Store C?"*.

When all the students had decided on which store gave them the most value for their money, the teacher, based on her observations in the classroom, chose a few of the students to orally present their solution for the whole class. The selected students showed, motivated and explained what method they used to approach and solve the problem. In the whole class discussion that followed the students' presentations, the teacher, based on continuous inputs of the students, showed what the graphs for the different stores would look like if they were plotted in the same diagram.

The discussion continued in smaller groups were the students were engaged in thinking about and explaining: *What use does one have of graphs and tables? What are the differences and similarities between the three stores? What factors other than the price can affect where one choose to buy one's candy?* Looking at the students' answers, there is a tendency to consider graphs as suitable tools for comparing things (*"when you want to compare something"*, *"you can see the differences in prices"*) or to illustrate how something develops over time (*"when you wanna show something along a timeline"*). Tables on the other hand the students put forward as good tools for presenting different kinds of compiled data or results (*"as for example results from sports"*, *"to present one's findings"*, *"sport results, lengths, weight, sizes, ages, sexes, opinions"*).

Regarding the differences and similarities between the stores the students mostly commented on directly observable features like *"all are selling candy"*, *"the price goes up with the number [of pieces of candy you buy]"*, *"all have different pricing"*. The selection of available candy in the different stores, both with respect to and quality and quantity, as well as to the geographical location of the store, were factors the students identified as things influencing where one buy one's candy.

After the students had discussed and compared their answers for a couple of minutes, the teacher introduced part two of the *Candy time!* activity; see Figure 2 below:

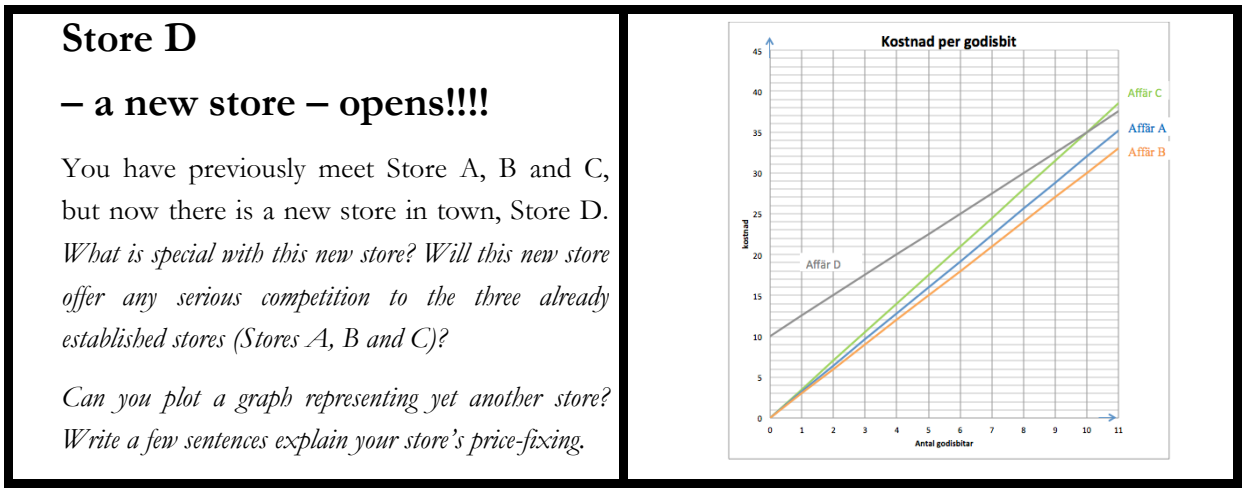

Figure 2: Part two of the activity *Candy time*!

While working on the second question in the second part of the activity, *Will this new store [the Store D] offer any serious competition to the three already established stores (Stores A, B and C)?*, the students concluded *"well, it depends on how much you buy!"*. Many of the students argued that Store D not would be any competition to the other stores if you as in the first part of the activity, only spent 2,50€. However, if you were spending a greater amount of money, then Store D should be the preferable choice. (*"No, this [Store D] is more expensive that the others [Stores A-C]. But this [Store D] becomes more affordable if you buy a larger and larger amount"*). The fact that the graph for Store D intersect the *y*-axis at *y=10* some of the students interpreted as *"you have to pay 1€ to enter the store, like an entrance fee"* or that you pay for the box or bag you put the picked candy in: *"Surely it's some kind of fancy candy store where you have to pay for the boxing. That's is probably one of the reasoning people will come [and shop in the store] – that it's a fancy shop that is"*.

The last task in the activity set lose the students' creativity, drawing graphs describing other imaginary store's pricing (see Figure 3). Most of the students draw in multiple stores and the most commonly pricing was a model giving the price proportional to the number of pieces of candy bought, as exemplified by Store H: *"Every single piece of candy costs 0,10 € each"*. One of the students wrote *"In this store the only sell giant pieces of candy"* (Store E) to explain the steepness of her graph. Store G was described by another student as *"I've made a cheaper store - one where you'll get one piece of candy for free!"*, explaining the meaning of the graph intersecting the *x*-axis at *x=1*. Although the diagram only display the price for between zero and 11 pieces of candy, some of the students physically prolonged the lines representing the cost in Stores A – D and concluded that if you buy a large enough amount of candy, then Store D is the most price-worthy store. The

students also constructed stores that had price-fixing represented by a line with negative slope (*"The price decreases, and after 11 pieces the candy becomes free"*, Store I), and stores with a flat rate price-fixing (*"Take as much candy you want for 2,80 € ☺"*, Store F). After the lesson the teacher noticed and expressed her surprised over how much the students own examples of stores' pricing showed and reveled about the students' creativity and proficiency to interpret linear functions $y = kx + m$ with positive ($k > 0$), negative ($k < 0$) slope as well as zero slope ($k = 0$) in the given context of the activity.

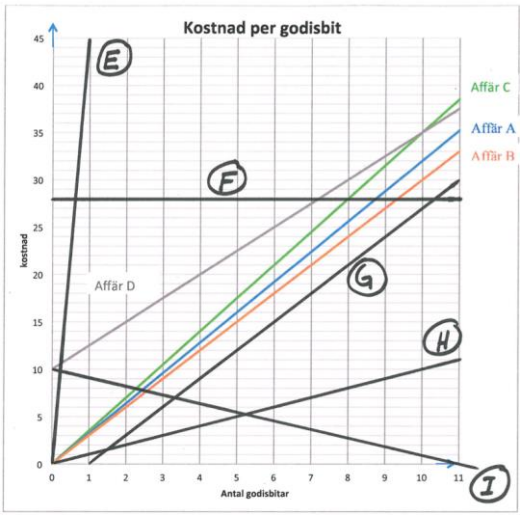

Figure 3: The students' own stores (Stores E – I)

The students' worked on the second part of the activity till the lesson ended. The teacher then collected all the students' written work, and followed up the activity the following lesson, after having read and summarized the students' explanations, with a whole class discussion about the students' conclusions, interpretations and price-fixing of their own stores. The teacher was surprised over the interest and engagement the students showed when working on the activity as well as over the wide range of solutions and explanations the students offered. The fact that the activity allowed for a variety of solutions resulted in almost all students wanting to share their solution and thinking at the whiteboard in the whole class discussion. In a few instances the students asked for how to name certain concepts such as *origin* and *intersection point* to be able explain their thinking more precise and clear to their peers. In other word, the students wanted to express themselves mathematically correct.

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Providing a tool for lesson planning in pre-service teacher's education

Tone Bulien

University of Stavanger, Faculty of Arts and Education, Norway; tone.bulien@uis.no

This paper discusses the theoretical and methodical arguments for the design and implementation of a research based course for pre-service teachers. The course program was carried out five times matching the methodology of design experiment and the theory of communities of practice. Students' assignments in the course were designed to focus on planning teaching and to give the pre-service teachers experiences of how to use and connect didactical and mathematical theories to practical situations, and hence provide them with a tool for teaching in different classroom situations.

Keywords: Pre-service teacher education, implementation, design experiment, lesson planning, mathematics teaching and learning.

Introduction

The implantation of a new teacher-training program in Norway (Kunnskapsdepartementet, 2010) presented an opportunity to think about new ways of teaching a pre-service teacher course in mathematics within the program at a Norwegian university. Previous to this reform the corresponding course was organized more traditionally entailing plenary lectures, workshops and various student assignments in which the pre-service teacher students (PST) actively had to participate. However, feedback from the PSTs showed that they found it difficult to see the connection between the course work at the university and the in-school teaching practicing activities, especially with respect to the implementation of didactical theories in the everyday mathematics classroom practice (Bulien, 2008; Solstad, 2010). Therefore, the design and implementation of a revised course focused on and included activities that aimed to enhance and make the connection between theories of teaching and learning and the actual practice of teaching clear and explicit for the students. This paper discusses the central idea used to realize this connection in the redesign of the course, namely, to focus on lesson planning by adapting and introducing the so-called *Mathematics Didactic Planning tool* (MDP).

Besides presenting the MDP, the aim of this paper is to describe the research design and the theoretical background for the educational design experiment in which the MDP was developed, implemented and evaluated.

Introducing the MDP, the setting and background

The course that was subjected for redesign is a 30 credit compulsory course in the teaching and learning of mathematics given within a full time teacher education program for grades 1–7 (6–12 years old). The students take multiple courses simultaneously and the course in teaching and learning mathematics spans over a period of 2 years (7,5 credits each semester). Even though the program is a full time program, most of the students are not at campus all the time. Rather, the students travel and are physically present at joint seminars at the university for three separate weeks each semester. In each week, the students studied mathematics for 7 hours spread out over two days. The rest of the semester, the PSTs and teachers communicate via different media over the Internet.

In the first two semesters (credits 1-15) of the course in teaching and learning mathematics, the students focus on theories and frameworks related to the role of a mathematics teacher, teaching and learning mathematics, numeracy and early training of algebra (pre-algebra). This experience of different theoretical approaches to teaching and learning mathematics provided an advantage when they was introduced to the MDP in the third and fourth semester of their teacher-training program. During the second year of the course (credits 16-30) the PSTs worked through three MDPs, each organized in three different parts or phases named A1, A2 and A3, see Figure 1 (left).

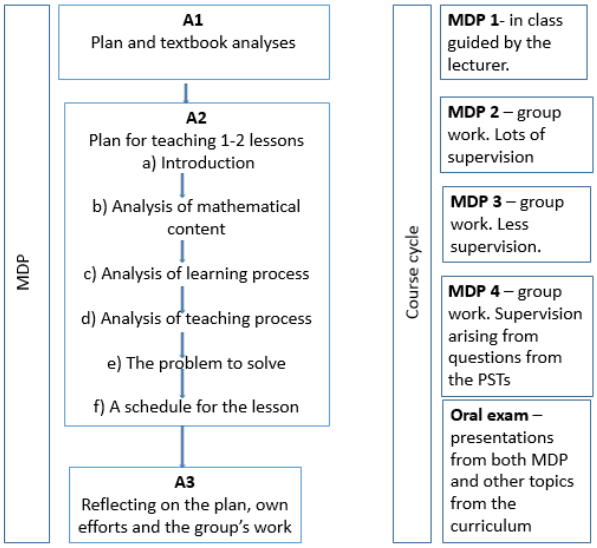

Figure 1: MDP model (left) and the course cycle of PST implementing MDP (right)

Phase A1 is about analysing textbooks and making a brief plan for teaching mathematics covering a whole school year. In phase A2, a much more detailed plan for teaching a lesson given a particular content is worked out. The last phase, A3, time is spend on reflecting back on the previous phases. The assignment students are obliged in phases A1 and A2 are written assignments carried out as group work, whereas the assignment in phase A3 is individual reflecting text.

In this second part of the compulsory course (semester three and four), the structure of the MDP was used as a frame to organize of both lectures and students' assignments, and during the year the PSTs worked through three MDPs focusing on different mathematical topics as well as on different grades (1–7). The mathematical topics were geometry, measurement, statistics, probability and functions.

The introduction to the MDP model was given in the first seminar in the third semester at campus where the teacher and the PSTs together planned a fictitious lesson for teaching using the MDP framework. The following three MDPs were mainly supervised using Google Disk (GD) and through Internet seminars, but also in face-to-face seminars with the teacher when the students met at campus. Additionally didactical and mathematical theories were provided to the students through lectures, both at campus and in online video lessons made and hosted by the teacher. Having the PSTs iteratively working through three MDPs made it possible for the PSTs to familiarize themselves with the framework, and hence their need for support and guidance successively declined, aiming at the PSTs to be more and more independent. In other words, writing the assignments (A1,A2 and A3), the supervision in GD was more actively during the first MDP, less in the second and only by questions from the PSTs in the third (Bulien, 2013).

Methodology

The methodology of the project to implement the revised course on teaching and learning mathematics based on this new tool for teaching (the MDP) was founded on experiences from previous teaching similar courses; research about PSTs' education experiences (e.g. Bulien, 2008; Solstad, 2010); and, students' feedback on the content in similar and previous courses. Based on these, special emphasis in the revised course became the application of theory in practical teaching situations. In short, the aim of the project was to present a more visible connection between the theoretical work at campus and the practice at schools.

During the four-year teacher education program, the PSTs experience rather short periods of actually practicing teaching in schools, and each time they must focus on three or four different subjects which makes it difficult to go into depth on each subject during their practice time in the classroom (Solstad, 2010). Solstad (2010) found that PSTs' ability to connect theory with practice increase the longer they attended the education program, but the students still wanted more supervision in implementing theory into their teaching practice. This increasing theoretical understanding might be a result of engaging in multiple cycles of supervised practice at schools and evaluated written assignments at the university, which suggests that a similar iterative learning model would be the preferred teaching and learning environment for the new course. However, closer collaboration and involvement with the actual teaching practices done in schools by the students was not feasible. Instead, a more active collaboration between the PST and the teacher in the spirit of communities of practice (Wenger, 1998) was integrated in the design of the course.

To guide and structure the overall re-design, implementation and evaluation of the new course, something that can be considered to be a classroom design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), a *design experiment (Cobb et al., 2003)* methodology was adapted.

> Most classroom design experiments are conceptualized as cases of the process of supporting groups of students' learning in a particular content domain. The theoretical intent, therefore, is to identify and account for successive patterns in student thinking by relating these patterns to the means by which their development was supported an organized. However, different classroom design experiments may set their focus on different constellations of issues (Cobb et al., 2003, p.11)

The theoretical intent of implementing the MDP was to focus on the PSTs' development in lesson planning using didactical theories and mathematical knowledge, which could be evaluated in the assignments of each MDP and the PSTs' reflection notes. The teaching and learning process involved in the implementation of the course was continually evaluated through out the five years based on the PSTs' work and reflection as well as the teachers reflections notes and written logs. The course was modified and adjustments were made both during the semester and when before starting a new course for a new group of PSTs.

In the design and development of the MDP and the new course, Cobb et al. (2003) five crosscutting features of design experiments were used to structure and guide the work during the five-year period of this project.

First, design experiment involves both the *goal* and the *process* (Cobb et al., 2003). The goal was to educate prospective mathematics teachers so they could give their students a grounded mathematical

education based on mathematical knowledge and research about how to teach mathematics. The process, which should facilitate the goal, had two different parts. Initially the process was about developing a class of theories that supported the ideas of using the MDP as an artefact for the PSTs' studies, and to get an overview of the implementation of the new ideas in the course. Next it was the process of defining the roles of the PSTs and of the teacher, in which the theory of communities of practice (Wenger, 1998) was introduced.

Since the collaboration between the PSTs and the teacher was supposed to be less like the usual teaching situation and more like an ongoing working together process, communities of practice was a suitable framing since "… the property of a kind of community created over time by the sustained pursuit of a shared enterprise" (Wenger, 1998, p.45). Further, Wenger claims that in educational design "Learners must be able to invest themselves in communities of practice in the process of approaching a subject matter" (Wenger, 1998, p.270). In this project the subject matter was the MDP that could be seen as a reification of a teaching and learning situation in the sense of "… giving form to our experience by producing objects that congeal this experience into "thingness"" (Wenger, 1998, p.58). For the MDP, the process was to present a learning trajectory to the PSTs that realized the possibility to adapt theoretical perspectives of teaching to classroom situations. A corollary to this involved a change in the teacher's role from less lecturing to more supervising, which supports Wenger's idea of self-investment.

Second, design experiment is about being *innovative*, and there should be a *discontinuity* between tradition and new ideas to test (Cobb et al., 2003). Expecting a more active contribution from the students and more supervision from the teacher changed the traditional lectures to seminars. In between seminars at campus, the supervision was implemented online using Google Disk (GD) and short video-films made by the university teacher. The focus for the supervision was to encourage the PSTs to evaluate didactical theories and mathematical knowledge and decide how to use them in teaching and learning situations.

Third, design experiment should involve both *prospective* and *reflective* situations (Cobb et al., 2003). In this project, the prospective aspect was manifested in the hypothesis or conjectures developed in terms of lesson plans in the written assignments in phases A1 and A2 of the MDP. In Phase A3 provided the possibility for reflection based on reflective knowledge and building on experience and research. Here, the MDP and the change of teaching and learning situations, build on previous teaching courses in mathematics and new ideas about learning, especially focused on social constructivism and the community of practice.

Fourth, design experiments include an *iterative* design. In this project, there are two levels of iteration (Cobb et al., 2003). A first is that the PSTs repeat the MDP three or four times during the course period, whereas the other is that the course itself was repeated five times with different students. The PSTs repeating the task hopefully provides better learning and the teacher had the possibility to adjust teaching according to the students' needs for supervision. The iteration of the course focused on the design of both the MDP and the teaching on a meta-level.

Fifth, design experiment involves *analysing* and *evaluating* the course to search for potential new theories (Cobb et al., 2003). Each year the PSTs were asked to evaluate the course and their responses together with the exam results and logs written by the teachers during the year, were used to adjust

the plan for the course for the following year. Although analyses took place during the five years, a retrospective analysis is yet to be fulfilled. Since then the designed course with the MDP has been repeated five times, and it has additionally been used in other courses, might give broader information about the ability of the use of MDPs. This adoption into other courses and continued use lead one to assume that the analyses made after each year have given positive results, but there is still need for more thorough analyses to quantify the effects of the implementation of the MDPs and the changes in teaching. A retrospective analysis of all available written material collected during the five years is necessary and is currently in progress.

Theoretical discussion of the MDP

The Danish textbook for teacher education, Delta (Skott, Jess, & Hansen, 2008), was part of the syllabus for the course and in this book there is a presentation of Gomez cycle of didactic analysis (Gomez, 2002). Based on this work of Gomez an artefact for lesson planning named *The Mathematics Didactic planning tool* (MDP) was constructed by changing the original model to fit a Norwegian teaching and learning context informed by the work of Ball, Thames & Phelps (2008) and Niss and Højgaard (2011). Hence, the content of the MDP considered both mathematical, didactical and methodological theories that was based on the curriculum of the course, but focusing on different mathematical subjects for each assignment aiming at a (fictitious) classroom situation.

Gomez' (2002) model for didactical analysis was inspired by the *teaching trajectory* of Simon (1995), which is a cyclic planning tool for mathematics teaching, and Shulman's (1986) work on pedagogical content knowledge which later informed the development of *Mathematical content Knowledge for Teaching (MKT)* (Ball et al., 2008). MKT is divided into different theoretical issues like areas of *knowledge of content and curriculum, content and students, content and teaching, common content knowledge, specialized content knowledge,* and *horizon content knowledge* (Ball et al., 2008, p.403). In the various parts of the MDP (Figure 1) these areas was conceptualized in terms of *the mathematical tasks for teaching* from Ball et al. (2008, p.400), such as for instance *using mathematical notation and language*, *asking productive mathematical questions*, and *finding an example to make a specific mathematical point.* Another theoretical framework used were the didactical and pedagogical competencies with specific regards to mathematics from the KOM-project (Niss & Højgaard, 2011). These are the eight mathematical competencies concerning mathematics as a discipline (chapter 4 and 5), and the six forms of specific competencies which a mathematics teacher should possess (chapter 6 and 7). The areas of knowledge concerns the students' *competences of representing, symbol and formalism, communicating, aids and tools, mathematical thinking, problem-tackling, modelling,* and *reasoning* , and the teachers' *competences of curriculum, teaching, revealing learning,* and *assessment* (Niss & Højgaard, 2011). In the following text these theoretical notions will briefly be discussed how to be used in the design of the different phases of the MDP; see Figure 1 for the presentation of the model.

A1 introduces and presents topic, grade and focus points from the National curriculum's perspective. This provides the background for the PST choosing two or three textbooks, which the PST should analyse according to the mathematics topic and the focus points in the national curriculum. After which, they should make a short plan for teaching mathematics over a school year focusing on how to structure and organize given topics and argue for their choices. For instance, it would be wise to have worked with fractions and multiplication before an introduction to probability. A1 focus mainly

on theoretical aspects from *knowledge of content and curriculum* (Ball et al., 2008) and *the teacher competency of curriculum* (Niss & Højgaard, 2011).

A2 is the main part of the MDP where the PSTs works with their understanding of both knowledge about mathematics and didactical theories. It is important for the student to notice that the different parts of the plan (again see Figure 1) are meant to be understood and worked through in a hierarchical way. The theoretical aspects mainly focus on the MKT (Ball et al., 2008) and the competencies of teaching and learning (Niss & Højgaard, 2011), but of course additional theory is added when relevant.

The aim for this part is to formulate a plan for teaching the subject given, e.g. geometry, over a period of two to five lessons. In the first part (A2a), they present focus points and quality frameworks from the national curriculum to illustrate the frames and goals for the teaching and learning. Since the PSTs had no real class to teach, they had to make assumptions based on the information given in the national plans and other relevant sources such as e.g. textbooks. Analysis of mathematical content (A2b) illustrates "all" the mathematics that the PSTs knew about the subject given, including symbols, algorithms, different representations, modelling, etc. The goal of this part is to provoke the students to go deep into their own mathematical knowledge and analyse different aspects of mathematics without thinking about teaching or learning. In these analyses of mathematical content, the PSTs theoretical framework is knowledge about *representing competency*, *symbol and formalism competency*, *communicating competency* and *aids and tools competency* (Niss & Højgaard, 2011), which also illustrates *general and special subject matter knowledge* and *horizon content knowledge* (Ball et al., 2008).

Analysis of learning process (A2c) is about the students learning process *in the specific subject given*, e.g. geometry, not about general pedagogical learning theories. This part focuses on *knowledge of content and students* (Ball et al., 2008) for instance illustrated by knowledge about misconceptions, what could be difficult to understand and why, or how to support the learners that needed an extra challenge building on the mathematical issues discussed in A2b. Some of the same issues were the basis for the analysis of teaching (A2d), but this time the focus is teaching and to predicate how they can meet different situations that probably occur in teaching the mathematics subject given were, such as *asking productive mathematical questions* (Ball et al., 2008). This part is illustrating *knowledge of content and teaching* (Ball et al., 2008), and the teacher competencies about *teaching*, *revealing learning*, and *assessment* (Niss & Højgaard, 2011). Since the A2 is hierarchical built, the three analyses should be written in the way they are presented, the next building on the information given in the previous. A final and important task in A2 is to construct an open mathematics problem (A2e) with different levels of cognitive demands (Stein & Hiebert, 2009) which is based on knowledge gained from the three analyses. All the work is summed up in a schedule for the lesson(s) (A2f).

A3 is an individual written reflection assignment where the PSTs reflects on the decisions and analysis made in A1 and A2. Since A1 and A2 are carried out as group work, the final reflection gives each student the possibility to add his or her own perspectives on the process and the result of phases A1 and A2. The last phase, A3, has been a valuable resource for the on-going and continuing development and implementation of the MDP in the new course.

Final comments

Using different theoretical perspectives from previous research about teaching (Ball et al., 2008; Niss & Højgaard, 2011; Stein & Hiebert, 2009), the research design method for planning new teaching models (Cobb et al., 2003), and research-based models of lesson planning (Gomez, 2002), have been useful in in designing and developing the new course in which the MDP serves as a backbone.

Although the PSTs' practice in schools could not be integrated within the course, it could be argued that focusing on lesson planning by working with the MDP had a close connection to the practice of teaching mathematics, which was one intention of using lesson planning as a focal idea. Another was to elucidate the importance of theoretical argumentation when planning (and teaching) mathematics lessons and the intention of repeating the MDP several times became an important choice of performance since especially the complexity of A2 needed to be exercised several times. A profit of all the work was that all the MDPs together with the teachers' comments were shared on a learning management system to be available for all the PSTs attending the course. When the PSTs finished the course, the students thus had a number of lesson plans ready to use (with adjustments) in their future career.

The empirical material available for analysing different aspects of the project is teachers' logs, planning documents, students' reflective papers from the course, students' papers on didactical and mathematical analysis, and electronic records showing supervising dialogue during the writing of papers. The preliminary analysis of all the empirical material available, five years of teaching experience, and reading some of the PSTs' final reflection notes indicates that the majority found the MDP challenging but educational, and that they experienced writing MDPs' assignments to be valuable. Many argued that it would be too much work for planning lessons when working as a teacher, but that the experience of having done so during their education will influence their work as teachers.

There were a couple of practical changes made during the five years of study, such as to change the number of group members from groups up to 6, to working in pairs, and lowering the number of MDPs the students should work through from four in the first year to three the years following. Since the students only were on campus for a short time, much of the contact had to be via the Internet and became the common form of communication. Supervising the process of writing papers in this way had clear implications on the process. This is documented in several reflection notes where the PSTs explicitly wrote that they would not have gone so deep into the material if they had not been pushed by the supervisor.

In summing up, it seems like the MDP can be a good mediator for MKT and competencies. Further and more rigorous analysis of the empirical material will be valuable in documenting the proof of this indicative claim, and it should additionally provide more information and suggestions on how to enhance the MDP and the role of the supervisor.

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From theory to praxis

Lisser Rye Ejersbo[1] and Morten Misfeldt[2]

[1] Aarhus University, DPU, Copenhagen, Denmark; lre@edu.au.dk

[2] Aalborg University, Copenhagen, Denmark; Misfeldt@learning.aau.dk

This article is a description and discussion of a design research project in which we introduced a research idea about the influence of language on number concepts development into praxis on a school in grades K-3. Danish children have difficulties remembering the Danish number names because the Danish language resembles a primitive number concept in mathematical thinking. In the project, we renamed the numbers between 11- 99 after the base-10 system. Our hypothesis was that this system would help Danish students to get a more secure concept about the base-10 system. The project lasted for three years ending in spring 2016. Our results were so convincing that the school decided to continue using the mathematical number names, and other schools that heard about the project seem to be interested in using the system as well. In other words, the project goes from being a top-down project to a bottom-up project.

Keywords: Design research, base-10 system, grades K-2, top-down and bottom-up project.

Introduction

In this article, we describe a design research project, which took place in a public school in the primary grades K-3. We started with a hypothesis, and then it moved on to a design and then contact with a public school. We ran the project for three years and then worked with the school on its continuation. The main concern of our paper is that of existing and anchoring projects that have proven successful and engaged (part of) the organization with which we have collaborated.

Our research idea was provoked by the fact that the Danish number names (Ejersbo & Misfeldt 2011; 2015) have an etymology and wording that are both peculiar and impractical. For example, in Danish, the number 73 is *treoghalvfjerds* (three and half-four), the number 32 is *toogtredive* (two and thirty), and the number 16 is *seksten* (sixteen). These old roots are unknown to most students. Furthermore, the number names are abbreviated. In Danish, the number 70 or *halvfjerds* (half-four) was once named half-four-times-twenty (in Danish halvfjerdsindstyve), but the *times-twenty* (in Danish sindstyve) has been lost in the counting numbers but retained in the ordinal numbers (for further explanation, see Ejersbo and Misfeldt, 2011). Another concern is the irregularity of the number names between 10 and 20, where 11 and 12 have unique names while 13 to 19 each end with a ten. Also, the two-digit numbers from 13 to 99 have an inversion property (the ones are said before the tens), and the tens have names inspired by a 20-base system.

Even though Danish number names are particularly irregular, most European languages break away from the clear regularity of the base-10 place value system. So understanding the effects of such irregularities on mathematics teaching and learning is interesting. With that motivation, we developed a hypothesis of how speaking about numbers with specific regular words that resemble the base-10 system would be beneficial for learning about the numbers in an easier way. This hypothesis has been tested with a three-year long intervention at a school in the Copenhagen area.

The project has ended, and the intervention was to a large extent successful. The data about student learning confirms that the classes where the new words for numbers system was successfully implemented have very strong performances in the areas related to number sense and arithmetic. Therefore, some of the teachers and the school organization would like to continue the work. We consequently discuss the question of how to exit intervention projects while anchoring relevant practices from the intervention into the organization.

In the paper we explore how this initial hypothesis has been activated as a way of using language as a didactical tool in a design research project. We will describe the state of the art that allowed us to develop the project idea, and how the project proceeded. Then we will briefly state our results in terms of a more well developed and tested set of hypotheses. Finally, we will describe our *exit strategy* for leaving a better practice at the school and reaching out to other schools and teachers.

Number concepts, base-10 and number names

We know from the research literature that there are major differences in the kind of system and regularities a language uses to describe numbers. Most European countries have an irregular naming system for numbers between 11 and 20; both the German and Dutch systems feature an inversion property of the numbers between 13 and 99, similar to the Danish one. These inversion effects were studied by Moeller, Pixner, Zuber, Kaufman, and Nuerk (2011) for two-digit numbers, showing how inversion-related difficulties predict later arithmetic performance (for an overview, see Ejersbo and Misfeldt, 2011, 2015).

Different studies (Miura & Okamoto, 1989; Miura, Okamoto, Chungsoon, Steere, & Fayol, 1993; Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999) compared Japanese, Chinese, Korean, and English-speaking American first graders' (6–7 years old on average) cognitive representations and understanding of place value. The findings confirmed that the Asian-language speakers showed a preference for using base-10 representations to construct numbers, whereas English speakers favored using a collection of units. Note that a significant difference between American and Asian number names appears between the numbers 11 and 19, exactly when the base-10 system starts to use two digits. In Miura and Okamoto's (1989) study, children were asked to construct the numbers 11, 13, 28, 30, and 42 from sets of ten and unit wooden blocks. The results showed that 91% of the American first graders used unit blocks to represent the numbers on their first attempt. In contrast, about 80% of the Asian children used sets of ten blocks when representing the numbers on their initial attempt. These differences in cognitive representation were mainly ascribed to language (Miura et al., 1993).

Learning to count and understand the base-10 system are cognitive challenges involving many small steps. We have chosen to focus on oral counting, the cardinal principle of combining a name with a cardinal value, and the combination of words for a number, its cardinal value, and the digit sign.

Children typically learn the names of numbers as a long list of words and demonstrate knowledge of the stable order principle by almost always saying number words in a constant order while emphasizing the last number (Goswami, 2008). The names are developed as sounds connected to the number of objects in the sets.

The developmental shift to understanding the number name as a cardinal value requires a qualitative shift in children's representation of numbers. The cardinal principle requires comprehension of the logic behind counting (Goswami, 2008) and the ability to judge the size of a set. It relies on a representation of quantitative information in which the coding of smaller quantities is different from that of larger quantities (Goswami, 2008). Children's conceptual understanding of numeration depends on their ability to make a connection between a number name and its cardinal value, which they learn to do by grouping and quantifying sets of objects (Thomas, Mulligan, & Goldin, 2002).

Learning how to connect the number name, its cardinal value, and the digit sign is another challenge. As discussed, two different systems must be combined with different representations. Becoming an expert at combining these two systems means developing rapid access to an automatic use of written numbers and simultaneously being able to multitask to solve other problems in parallel. If the two systems are iconic and support each other, the child encounters less difficulty in learning this skill, as is the case for Japanese-speaking children. If the two systems are irregular and therefore conflict with each other, it is more problematic for the child to understand and remember the connection among the name, the cardinal value, and the sign. Duval (2006) described this situation as a conversion between registers and observed that the congruent conversions seem the easiest for students, meaning that the representation in the starting register is transparent to the target register. One obvious solution is, therefore, to use a fully regular approach to saying the names of the numbers, which means saying "one-ten and four" instead of fourteen and so on. It is possible and easy to *create* such a logical system for naming the numbers in Danish, and thus this became our main project idea.

This reasoning helped to form the project idea of using such logical number names as a didactical tool.

The design of a research idea

Occupied by these issues of why and how different languages influence number concepts and perhaps even the ability to learn simple arithmetic, we designed a three-year project to take place at a Danish public school in the suburbs of Copenhagen.

Using design research (Cobb & Gravemeijer, 2008), we formulated our hypotheses for empirical investigation. The hypotheses were grounded in our initial understanding of the difficulties that Danish children experience with the Danish number names. The research builds on the following two hypotheses:

1. Number names function as cognitive artifacts; hence, a concordance between spoken and written language is sensible.
2. Language constitutes concepts, which is why clear terminology seems effective in developing lucid concepts.

Project intervention

To address the question of the influence of number names on number concepts, we contacted a Danish public school that could be interested to run the project together with us. We already knew the school, which made the access easier. We were invited to a meeting with the leading team of the

school, including the headmaster, together with a small group of teachers from the school. We presented our project, and the participants accepted it for one year as a start. An evaluation would decide if it should continue additionally for two more years. We decided to involve all 10 classes—three grade 2, three grade 1, four kindergarten classes—and 9 teachers in the primary section of the school. The project combined the renaming of numbers with supporting the teachers in instructing the students. In each class, 20–35% was children who had migrated from other countries, but all the children spoke Danish, and all the teaching was in Danish. The entire research project was planned to last for three years. The data consists of students' performance in classroom observations, a number understanding test, teachers' portfolios, interviews with teachers and students, and notes from collaboration with teachers. At the end of the projects, we used the national test for evaluating the students' competencies in Algebra and Numbers.

The first year

The cooperation with the teachers and the classes were only possible because of the positive attitude from the headmaster of the school. She left it to us, the researchers and the teachers, to run the project. But she and the leading team was helpful and showed interest the whole time.

An in-service course for the involved teachers was the first thing to arrange and run. At this course many questions came up and were discussed. Should the teachers always rename both the names of the numbers? How should the fractions be named? Would the student get to know the normal Danish number names? We made a lot of decisions that day and agreed that the teachers should write a log with further questions that we as researchers should answer, either by discussion or by recording answers in the log.

The participants were now ready to start the next school year with the mathematical numbers.

Kindergarten: The Kindergarten (K) teachers were used to cooperating with each other and continued this work with the mathematical numbers. We observed the classrooms regularly and had follow-up meetings. The K teachers used both the mathematical number names and the normal number names when they named a number, or the students read a number. They also arranged joint counting for all using mathematical numbers, and they made materials for student use that helped the students to be aware of the base-10 position system, and the students became very familiar with the mathematical numbers. The parents were informed at a meeting with the kindergarten teachers only; all in all, they implemented the mathematical numbers very easily.

Grade 1: The project proceeded differently in the first-grade classrooms. All these classes had new teachers, which is normal for students in the first grade in Denmark. We were in a real-world situation with all the mess that exists there. The three first-grade classes had three different teachers who did not work together very often, and none of them continued the work done by the K class teachers. So the routines disappeared. The big difference in practice between first grade and the K class caused some chaos during the first two months. Furthermore, some of the first-grade teachers left the school or their classes during that period. But new teachers came, and during November and December the classes also worked regularly with the mathematical numbers. We were lucky that one of the newcomers believed in the project idea and was very involved with it. He became a teacher in two of the three first-grade classes, and his presence was a great benefit for the project.

Grade 2: We never observed any of these classes, but met with the teachers and discussed how they could implement the mathematical numbers in the best way. These students had already been in school for two years, and we decided that while we could not expect that they would naturally use the mathematical numbers, they should know them.

Evaluation: In the first year we were very busy collecting data, observing the seven classes, and trying to find the best ways to implement the mathematical numbers. The teachers' log idea never caught on, so we solved any problems during our meetings with the teachers. At the end of the year, we tested all the students. The outcome of the test showed us that the student used both names for the numbers quite naturally. There was a slight tendency that the students were more secure from the spoken mathematical numbers to written numbers than from the spoken normal numbers to written numbers. In the K classes, we noticed that the students were much more secure in correctly recognizing and naming numbers between 10 and 20 than was normal for these classes.

During the year, we used the design research method as a way to evaluate the actual lesson related to how the whole project was running. We exchanged good ideas and noticed the progress and difficulties. We solved the difficulties in different ways and changed some of our means; but not our goal. We made a report of the first year with our results and data. It was positively received by the leading team, who decided to let the project continue for the next two years.

In our plan for the second year we decided not to observe in the K classes, but only meet with the K teachers. We would do a brief orientation for the new mathematics teachers in the four new first grades, and we would follow the second grades more intensively.

The second year

For each year, we expanded our research with new K classes and with that also thirds grates. The K teachers could more or less develop and repeat with their new students what they had done first year. The newcomers in the K teacher group were taught by especially one of the K teachers taking the major responsibility for informing the new teachers. As we discussed in our meetings with the teachers, the work in the K classes went very smoothly.

The four new first grades had new teachers, luckily only two teachers with two classes each. They did not know anything about the project before they chose to teach first-grade mathematics, but cooperated from the start. We met with them and introduced them to the project, and visited their classrooms several times during the year.

The three second grades were the most interesting, because the students were in their second year of the project. Their teachers were very engaged and consistent in the use of the mathematical numbers. Each time we observed the classroom we talked with the students and asked how they felt using two different names for one number. The answers were surprising:

Student 1: It is fun, and we like to use the mathematical number names, because then we always are able to remember the names.

Student 2: We also know the cardinal at once.

Student 3: It is a help to remember the names of the normal number as well.

We were a little overwhelmed but agreed with the students that they should explain to their parents why we used the mathematical number names. We decided that no one could do it better than them.

At the end of the year, we used the same test we had used in the first year but only with the first and second grade. We noticed again that the students' understanding of numbers was very good, both for the use of normal number names and for the mathematical number names

Due to the time we had to do the project, and how it developed in the school, we decided to minimize the observation to only the three third-grade classes, but we still met with the teachers from second grade. We did not observe the first grades or meet with their teachers. We continued with the K teachers' meetings.

The third year

We mostly concentrated our research on grade three and the kindergarten classes. The use of the mathematical number names seemed very natural for the students together with the normal number names, and they were bilingual in the numbers from 10–99. In third grade, the students were so familiar with the base-10 value system that they could transfer the knowledge to the decimal numbers, which meant that they easily answered questions correctly when asked to compare numbers like 0.4 and 0.25. In May 2016, the three classes had the national test in mathematics for third grade. Compared to the average of all the third graders in Denmark, one of the third-grade classes—the one that was observed most frequently—had an average score in Numbers and Algebra that was far above average. The other third-grade classes also showed a better result in Numbers and Algebra than the average third-graders.

The conclusion we draw was that the students using the new system showed a better understanding of the base-10 system. We saw these competencies, and met our goals, in all the classes which used the mathematical numbers. And because the national test investigates additional competencies in Numbers and Algebra, we dared to conclude that the students gained from using two names for the numbers.

During the following year, the school had a new headmaster and a new leading board, but because of the results, she decided to continue with the mathematical number names even though the research project stopped.

From Top-down to Bottom-up

With the decision that the school wanted to continue using the mathematical number names after our exit, we needed to design a plan for how it could be possible. Inspired by the research (Jarvis, 1999; Nielsen, 2001) we suggested the following plan:

1. All the teachers at the school should know that the project stopped as a research project, but that the project would continue as an intervention project with the teachers as the drivers.

2. One K teacher should be responsible for introducing the methods in the K classes for incoming teacher and for ideas to be exchanged among the K teachers.

3. A mathematics teacher should be responsible for orientation of the mathematics teachers in first grade each year and arrange a course at the beginning of the new school year, which everybody could join in.

4. There will still be access to the researchers for questions and other things; we are interested in the continuing process.

This plan was first discussed with the involved teachers who agreed to the work they should do, and then it was presented to the headmaster. She also agreed and was willing to find time for the teacher support.

We started the process with a course for mathematics teachers in the K-3 classes in August 2016. There were about 15 people at the course, which was organized and run by a K teacher, a mathematics teacher, and one of the researchers. At this course, we made a quick run through the ideas behind the project and how it had run in the previous three years. The teachers who had previously taught classes and been involved in the project exchanged ideas and views of the learning processes with the mathematical numbers. The K teacher told how she was at an in-service course for K teachers in the Copenhagen region and told about the project and how she and the math supervisor at the school videotaped how she used the mathematical number names in the K classes. The other participants at the in-service course showed a big interest in the project. A similar course will be held again, and the math supervisor has told us that there is already a big interest in this in-service course.

Discussion

This article is less concerned with the actual results of the investigations that we conducted in the school and more concerned with the transition from an intervention driven by research curiosity to an ongoing project driven by the school itself.

The project was large with many classes involved, and we must admit that it was a little too big for only two researchers on a very low budget. We shared the work, so at times only one of us made the observations, conducted the interviews, and participated in the meetings with the teachers. Everything was documented with taping, pictures, and materials that we analyzed together; we still have data waiting for deeper analysis.

In spite of the low budget and the few researchers, or maybe because of these limitations, we saw some teachers taking over the project in an especially engaged way. One teacher, in particular, took a lot of responsibility and during a period when other teachers were out sick, he taught all the three classes in third grade. Without him, we are not sure the project would have had the success it had. It would neither be possible to run such a project without the support and interest we had from the leading team including the head master, who played an important role.

Perspectives

As it looks now, we hope that the use of mathematics number names will spread to other schools and continue to develop. Even though we officially stopped the project at a meeting with all the

teachers at the school, we will continue with some kind of support if necessary. We will also stay in touch for our own sake.

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The notion and role of "detection tests" in the Danish upper secondary "maths counsellor" programme

Uffe Thomas Jankvist[1] and Mogens Niss[2]

[1]Aarhus University, Danish School of Education, Denmark; utj@edu.au.dk

[2]Roskilde University, IMFUFA, Denmark; mn@ruc.dk

This paper presents and discusses a specific aspect of the Danish "maths counsellor" programme for upper secondary school, namely that of detection tests. *More precisely, the purpose and design of a detection test is presented, as is the prospective counsellors' use of the test. In the description, emphasis is placed on the ways in which detection tests assist in informing the maths counsellors in their work with students experiencing learning difficulties in mathematics.*

Keywords: Detection test, "maths counsellor", learning difficulties.

Introduction

The words "test" and "testing" are omnipresent in educational research and practice in general and in mathematics education in particular. However, the actual notions covered by these terms are very diverse, as are their roles and uses. Basically, the terms mean "critical examination of a person's or a thing's qualities". There is an abundance of different purposes, goals and objects of testing, as well as a multitude of different approaches to and instruments for testing. Without attempting to outline a comprehensive theory of test and testing, one distinction is worth introducing in the context of mathematics education, a distinction between direct and indirect testing. In *direct* testing, the test object directly epitomises the very purpose and goal of the testing. If your purpose is to find out whether a given person can actually drive a car, a direct test consists of taking the person to trial in real car driving. In *indirect* testing, the test object is devised as an indirect indicator of – a probe into – something that is not identical to the test object itself, because this "something" is either inaccessible in direct terms or too large or too complex to be fully represented by the test object. So the test object becomes a proxy for – a representative of – the underlying, but necessarily indirect, object of testing. Mathematical learning, understanding, reasoning, modelling, and problem solving are just a few examples of such underlying objects of testing, for which a wide variety of test objects are only more or less well-chosen proxies. Mathematics education research and practice make extensive use of indirect testing, alongside direct testing.

The key issue concerning indirect testing is what relationships can be established between respondents' responses to the test object and the real underlying object of testing. The generic question is: What do responses to the test object tell us about the respondent's qualities in relation to the real underlying object of testing. Since, in indirect testing, these objects are not identical it requires a non-trivial amount of clarification of concepts, of interpretation and analysis, and oftentimes of independent empirical research to account for the inferences that can be justifiedly drawn from responses to the test object onto the underlying object of testing. The "detection tests" in focus in this paper are instruments for indirect testing. So, the above-mentioned generic question has to be specified for our context. Our *research question* then is: *In what respects and to what*

extent does the detection test presented below allow for the detection of students with mathematics specific learning difficulties regarding mathematical concepts and concept formation?

Due to space limitations we are not able to fully answer this question here and to fully corroborate the answer. Instead, we confine ourselves to providing some key points in an answer, that is by describing the context in which the detection tests are used; by providing an overall description of what a detection test is; and finally by means of an illustrative and authentic example. It should be noted that although the actual content of the detection tests is based on research findings and issues considered in the mathematics education literature, the notion and role of "detection test" in the sense presented here have been introduced by us and thus have not been described previously.

The "maths counsellor" programme

This section is based on (Jankvist & Niss, 2016). The maths counsellor in-service teacher programme at Roskilde University (Jankvist & Niss, 2015) runs part time over three semesters (in total 30 ECTS – European Credit Transfer and Accumulation System), during which the upper secondary teachers – ideally – have a reduced teaching load at their schools. Each semester has an overarching theme: (1) concepts and concept formation in mathematics; (2) reasoning, proofs and proving; (3) models and modelling. These themes were chosen both because they are significant to upper secondary mathematics education in Denmark, as is spelled out in the national curriculum documents, and because they epitomise key aspects of the eight mathematical competencies in the Danish KOM-project (Niss & Højgaard, 2011), which constitutes the theoretical foundation of the maths counsellor programme. The teachers' work in each semester is structured in terms of three different phases: (1) to *identify* (i.e., detect and select) students with genuine learning difficulties in mathematics; (2) to *diagnose* the learning difficulties of the student(s) identified; and finally (3) undertaking *intervention* according to the diagnosis arrived at with respect to the individual student.

At the very beginning of each semester, the teachers are equipped with a theme-specific *detection test*, consisting of questions and tasks for the students in relevant classes at their schools. As will be exemplified below, these tests are developed by us and are informed by research literature regarding the specific theme. The purpose of the test is to assist the teachers in detecting students with potential learning difficulties in mathematics. Usually, each teacher detects several such students, some of whom are selected for being offered maths counselling with the aim of rectifying or reducing the observed difficulties during the semester. This typically leads to the *identification* of 1-4 students per class in need of, and also interested in, receiving counselling. When speaking of mathematics specific learning difficulties, we rely on our previous definition given in (Jankvist & Niss, 2015, p. 260), i.e. "those seemingly unsurmountable obstacles and impediments – stumbling blocks – which some students encounter in their attempt to learn the subject. These stumbling blocks include, but are not limited to, a wide range of misconceptions, misinterpretations, misguided procedures, inadequate beliefs etc. with regard to established notions of mathematics. We do not include general learning disabilities, cognitive or affective disorders and the like." The purpose of the counselling is not to motivate unmotivated students, but to assist those who work hard in mathematics on a daily basis but do not succeed.

In the diagnosing phase, the participating teachers – strongly assisted by the research literature they read as part of the programme (see Jankvist & Niss, 2015) – employ self-constructed tasks,

interviews, etc., to come to grips with the nature and origin of the students' mathematics specific learning difficulties. Taking the diagnosis as the point of departure and with support from the research literature and supervision by us, the teachers design and implement an intervention scheme for the students selected. The intervention scheme also includes steps which enable the counsellors to "measure" in what respects and to what extent the intervention has worked as anticipated for the selected students. For each semester, groups of 2-3 teachers write up a report. After the completion of the third semester, all three reports are combined into one, along with an introductory chapter. This final report forms the basis for a final oral exam at the university. The teachers who pass receive a diploma as certified maths counsellor.

What is a detection test – and what is it not?

A *detection test*, as designed for the Danish "maths counsellor" programme, is a set of maths questions to be answered by upper secondary student classes (grades 10-12) within a time frame of 60-90 minutes without time pressure. The questions are short, both in their formulation and in the sense that they neither require lengthy procedures or computations nor longwinded explanations. Moreover, the questions do not involve conceptually complex or technically involved mathematics beyond standard upper secondary school mathematics. However, the questions are usually not routine questions either. On the contrary, many of them are deliberately posed in such a way that they break the "didactical contract" of upper secondary mathematics and require students to think and act independently. Danish upper secondary school takes three years and students usually enter at the age of 16 after having completed ten years of mandatory comprehensive primary and lower secondary schooling. Upper secondary students can choose to have mathematics for one, two or three years; three years being the advanced level. Danish upper secondary school covers three streams: general, technical, and business.

The primary purpose of a detection test is to be one among several instruments for *detecting students* possessing genuine learning difficulties in mathematics, within the relevant theme of the programme. So, the focus is not primarily on detecting the difficulties themselves – even though the tests do have something to offer to that end as well, because the questions in a detection test are composed such that wrong answers, individually or in combination with others, may suggest the potential presence of particular kinds of learning difficulties with a student giving these responses. As mentioned, a detection test is not meant to stand alone. When it comes to detecting students with learning difficulties, other sources of information, e.g. the teacher's prior knowledge of the students have to be taken into account as well. More precisely, a detection test may be seen as having three different roles. Firstly, in cases when the test, within a certain area or theme, points out students who by the teacher/counsellor were already suspected to have difficulties within that area, the role of the detection test is to *strengthen* the teacher's observations. Secondly, in cases when the test singles out students who were not already detected by the teacher, the test serves to amplify and sharpen the teacher's attention and to supplement his or her own observations of the students. Thirdly, it is also a purpose of the detection test to provide an initial support in pointing out the *specific sub-domains* within the test's theme, in which a detected student displays difficulties. Of course, students' test responses may not only indicate difficulties within particular mathematical topics; students' response patterns may also suggest overarching difficulties of a more principal or

general nature. Thus, this third role of a detection test then typically is to provide inspiration for the following "diagnosis" (cf. later sections).

It is important to keep in mind that a detection test is *not* meant to be a fair test of the students' attainment levels in the subject of mathematics, neither when it comes to content knowledge, skills, and proficiency, nor when it comes to mathematical competence at large or to inventiveness or special mathematical talent. Due to the fact that detection tests are designed with a different purpose in mind, several important aspects of the usual handling of mathematics – e.g. familiarity with concepts and facts, computational skills, or proficiency in solving standard routine tasks – are not in focus of the tests. Similarly, the test cannot be used as a *screening test* in the usual sense, attempting to chart students' possession of various mathematical competencies. However, employed on a larger population of students, e.g. a year group in a given school, the test may of course be used as a screening test for the potential presence of mathematics specific *learning difficulties pertaining the theme of the test*, within this population, but the test is still much more focused than a general screening test for attainment level or competencies.

Even though the test contributes to singling out students with potential learning difficulties, it *cannot determine*, in itself, whether a given student actually possesses such difficulties. It is certainly possible to encounter poorly performing students whose erroneous answers are not due to mathematics specific difficulties, but to ill-will and shoddy job, lack of accept of the didactical contract with or in the test (e.g. because the test is not supposed to influence teachers' marks, or because the questions are of a different nature than usually encountered by the students), a bad day on the time of testing, or maybe to much more general learning difficulties (or disabilities) that manifest themselves in several subjects, not only in mathematics. To determine whether a student detected by the test actually possesses mathematics specific difficulties, supplementary means must be applied as well, not least the teacher's knowledge of the student.

Beside the fact that the test, for a student who has been "detected" by it, may provide important indications for a subsequent diagnosis of mathematics specific difficulties, the test is *not a diagnostic test*. It requires an independent diagnostic process to uncover the specific nature of observed learning difficulties as well as the sources actually responsible for them. Oftentimes, preliminary hypotheses concerning the nature of the difficulties, and what may have caused them, must be supplemented with – or even replaced by – other hypotheses as the diagnosis proceeds. This may be due to much more deeply rooted difficulties than the ones observed at first, e.g. regarding more fundamental mathematical conceptions and beliefs than those in focus of the detection test.

An illustrative example of algebraic equations and equation solving

As mentioned above, in each semester of the programme the maths counsellors are equipped with a detection test related to the theme of the semester. Hence, *detection test 1* concerns mathematical concepts and concept formation (we intent to discuss detection tests 2 and 3 in subsequent publications). This test consists of some 57 questions (and sub-questions) on selected topics relevant for Danish upper secondary school. These include: concepts of number (including fractions, decimals, negative numbers, irrational numbers); percent; algebraic expressions and transformations; equations (first and second degree, with different types of numbers as coefficients and solutions, and with the unknown on both sides of the equal sign); simple functions and aspects

of the coordinate system; and finally a selection of mathematical conventions such as: different symbolic notations for fractions; the equal sign; the inequality sign; minus and negative numbers. Out of the 57 questions (with sub-questions) around ten questions concern equations and equation solving. In the following we shall focus on examples of this.

As suggested by various researchers (see e.g. Kieran, 2007), students' difficulties in solving algebraic equations are of two rather distinct kinds. The first kind is related to transformation of equations – and algebraic expressions – by means of permissible operations, eventually leading to solutions. This not only involves knowing and understanding the scope and legality of the operations at issue, it is also to do with the nature and structure of the number domains implicated, the meaning of the equal sign, and the arithmetic operations involved, etc. The second kind of difficulty is to do with what an equation actually is, and what it means for an object to be a solution to an equation. Detection test 1 includes the following questions, among others: [17] Are there any values of a such that $a^2 = 2a$? [18] Are there any values of b such that $4b = 4 + b$? [20] What is the solution(s) to the equation: $3x – x = 2x$? [25] Is $x = 0$ a solution to the equation: $3x – x = 2x$? [35a] Solve the equation: $3x + 20 = x + 64$. [36a] Solve the equation: $–6x = 24$. [37] For what x do we have $38x + 72 = 38x$? Our purpose here is to illustrate two things: what the maths counsellor may learn from using the test on a larger population of students; and what the maths counsellor may learn about a single student from his or her answers to the test questions.

When a maths counsellor gives the detection test to a group of students, perhaps a larger cohort of students – say a class or a year group – certain patterns are likely to reveal themselves. For example questions 17 and 18 may tell us something about the students' algebraic understanding, e.g. the students' perception of how variables may and may not be denoted (anything other than x is often rejected as a variable). Questions 35a and 36a address the first kind of difficulty of solving algebraic equations, namely the operational aspect in relation to the number domains involved. Question 35a is an example of what Filloy and Rojano (1989) call a "non-arithmetical equation", referring to the fact that the unknown appears on both sides of the equal sign. Question 36a may give rise to difficulties due to the appearance of the negative coefficient and division by a negative number, but also the situation of having to accept a negative number as a solution. On the one hand, questions 17, 20, and 25 may tell us about the second kind of difficulty mentioned above, i.e. knowing what a solution to an equation means, as well as about the consequences of the fact that an equation may have infinitely many solutions. On the other hand, they may also tell us something about the students' conception of equality in relation to equations and equation solving. From extensive experience, we know that Danish students have difficulties with equations that have either no solutions or any number as a solution. Question 37 addresses another aspect of the second kind of difficulty. Despite the fact that the vast majority of students are not able to correctly answer question 20, a large number of students will say that 0 is indeed a solution to the same equation in question 25. More interesting, perhaps, are those students who are able to answer that all numbers satisfy the equation $3x – x = 2x$, but still answer "no" to 0 being a solution. This may have to do with a belief that solutions are positive integers or be an aspect of more fundamental difficulties with 0.

To illustrate what an overview of a large student population may reveal, we provide table 1, which displays a binary ("correct-incorrect") coding of 676 Danish upper secondary students' responses from 2012 and 2013 (from all three levels and streams). For the 405 1st year students participating in

the study we may confirm that questions 20 and 37 are indeed difficult ones, since 92.8% and 85.4%, respectively, cannot answer them correctly.

As an illustration of two maths counsellors' use of the test in regard to equations and equation solving, we present the story of student Å (Christensen, 2016). Student Å followed the mathematics programme at intermediate level at a general upper secondary school. The two maths counsellors spotted student Å at the beginning of Year 1, and then worked with her for three consecutive semesters, while they themselves were enrolled in the maths counsellor programme. In relation to the above questions on equations, student Å answered incorrectly on both questions 17 and 18 ("no"), she left question 20 unanswered but answered question 25 incorrectly ("no"), and left questions 35, 36, and 37 unanswered. The two maths counsellors initially interpreted this as if she had difficulties with the transformation of algebraic equations and with algebraic expressions in general, since she also gave incorrect answers to: [6] What is $(a / b) \cdot (b / a)$? (Where neither a nor b is 0.) (Å: "a^2 / b^2".) and [50] If $a = b$ is then $b = a$? (Å: "no.").

	[17]	**[18]**	**[20]**	**[25]**	**[35]**	**[36]**	**[37]**
1st year (405)	148	285	376	193	168	149	346
Error rate (%)	36.5	70.4	92.8	47.7	41.5	36.8	85.4
2nd year (196)	45	166	172	87	54	44	139
Error rate (%)	23.0	84.7	87.8	44.3	27.6	22.5	70.9
3rd year (75)	12	64	49	25	25	18	43
Error rate (%)	16.0	85.3	65.3	33.3	33.3	24.0	57.3

Table 1: Binary coding of 676 student answers to selected questions from detection test 1 (coding by Morten Elkjær Hansen as part of his master's thesis at Aarhus University, 2016)

Based on interviews which confirmed that student Å most certainly had difficulties in solving algebraic equations, and handling algebraic expressions in general, the two maths counsellors designed a series of small interventions focusing on solving various equations, arithmetic as well as algebraic ones, etc. (Filloy & Rojano, 1989). Soon, however, the maths counsellors began to suspect that Å's difficulties had indeed deeper roots. Student Å found that negative numbers as well as fractions were "ugly", and on one occasion she uttered "0 that's not a number!" Sometimes student Å had difficulty at distinguishing the operations of addition and multiplication. When trying to find the difference between two numbers, she counted on her fingers. When having to find how many times 8 divides 24, she answered "four" by counting "8, 12, 16, 24", and even double checked the result by repeating the same count. Having to perform the division 24/6, she eventually gave up and replied "I don't know when the numbers are so big." It turned out that student Å had fundamental difficulties with the concept of number, including understanding of numbers, number domains and handling of numbers. She appeared only to be on safe ground when operating with small natural numbers, where calculations can be performed on her fingers. Student Å's difficulties extended to her (in)ability to correctly apply basic mathematical terms. Thus, on a later occasion she used the term "diameter" (of a pizza) as just another unit along with centimeter and millimeter.

Upon revealing the depth of student Å's difficulties and acknowledging these to be the cause of her symptomatic difficulties with equation solving, the obvious question becomes whether the detection test might have provided us with some indications of this. In hindsight, even if there were several questions that Å left unanswered, the ones she answered erroneously do seem to corroborate her subsequently revealed learning difficulties: [7] Is the number $– a$ positive or negative or is it not possible to decide this? (Å: "negative"). [13] Which number is larger: 13/3 or 13/4? (Å: "13/4"). [15a] What is $x + 0$? (Å: "0x"). [26] Round off 148.72 + 51.351 to an integer. (Å: "149 + 51 = 191"). [27] Which of the following fractions are equal: 1/4, 4/16, 4/12, 2/8? (Å: "4/12 and 2/8"). [52] If $a < b$ is $b > a$? (Å: "no"). [53] Is $(a – 1) / (b + 1) = (a / b) – 1$? (Å: "yes"). 55. Is $(a + 3) / (a + 4) = 3/4$? (Å: "yes"). 56. Is $(a – 1) / (b – 1) = a / b$? (Å: "yes"). In total, the occurrence of the above erroneous answers, together with the responses to the previous questions on equations, indicates the presence of a manifest learning difficulty syndrome with student Å.

The revelation of mathematics specific learning difficulty "syndromes"

We now return to our research question, i.e. in what respect and to what extent do detection tests allow for the detection of students with mathematics specific learning difficulties – here exemplified by concepts and concept formation regarding equations and equation solving? As we saw in the case of student Å, she most certainly was *detected* to possess the potential learning difficulties as suggested by the test. Clearly, the counsellors' first hypothesis concerning Å's difficulties regarding concepts and concept formation was insufficient. However, we should keep in mind that this was the first time ever that these counsellors used the instrument of an indirect detection test. Once the maths counsellors become accustomed to the instrument and skilled in using it, they tell us that they are able to make much more accurate initial hypotheses – or even preliminary "diagnoses". Indeed, having experienced a case like student Å, our two maths counsellors are able to make much more qualified initial hypotheses concerning students' difficulties. Seeing the answers to the questions above on numbers, conventions, etc., these maths counsellors will no longer suspect a student "merely" to have difficulties with solving first degree equations; they will see this as a likely symptom of more deeply rooted and fundamental difficulties.

Indirect tests, such as the detection test outlined above, may mislead the interpreter of test outcomes in several ways. In the case of student Å we saw that the maths counsellors at first mistook the student's apparent difficulties for her real, more fundamental difficulties. Another example, which we have also seen time and again, is where students are perfectly able to solve algebraic equations in an instrumental manner, but do not understand the relational aspects of the operations they perform or the very meaning of the solutions they arrive at (for references, see Jankvist & Niss, 2015). This is to say that if the aim is to "train monkeys" to find solutions to equations, then this is certainly possible. Our aim with the indirect detection test is to go deeper, since "our" object of learning is more complex than to mechanically obtain a solution. Our aim is to pave the way for drawing conclusions that are much broader than what the test questions ask, taken at face value, e.g. we insert "spot probes" into aspects of students' mastery of numbers and algebraic expressions and attempt to come up with hypotheses concerning their concept of number in general: if a student comes up with this and that erroneous answer, (s)he most likely possesses such and such learning difficulties; or if, on the contrary, the student can give correct answers on this particular set of

questions, then it is fair to assume that (s)he has actually grasped, in a relational manner, something significant about the entities involved.

As illustrated above, an indirect test such as a detection test may function both on an individual student level and on larger populations. For example, in the case of student Å we noticed that she answered incorrectly to question 15 and question 50 (cf. above). This we interpret as an indication of student Å not believing 0 to be a number and possibly possessing misconceptions of equality. But how special are these misconceptions for a 1st year student like Å? The coding among the 405 1st year students displayed in table 1 revealed an error rate of 18.3 for question 15 and 13.6 for question 50. In addition, question 14, asking what $0 \cdot x$ is, which student Å answered correctly, has an error rate of 18.8 among the 405 1st year students. This is to say that if questions 14 and 15 are taken as markers of difficulties with the number 0, and if the population of the 405 students is representative, then it might be expected that more than one sixth of the students in a class at the beginning of Year 1 will have the number 0 as a "stumbling block" in some sense. The indirect detection test may suggest the presence of syndromes, on an individual level as well as on the level of populations.

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Students' video tutorials as a means for supporting and analysing their reflections on the limit concept

Tinne Hoff Kjeldsen[1] and Morten Blomhøj[2]

[1]Department of Mathematical Sciences, Copenhagen University, Denmark; thk@math.ku.dk

[2]Department of Science and Environment, Roskilde University, Denmark; blomhoej@ruc.dk

We discuss how a theory based design for students' production of video-tutorials explaining problem solving activities in a first year calculus course can support the students' conceptual learning. We focus on the limit concept and show how the production of tutorials can facilitate the students' interactions and reflections and at the same time provide a rich source for analyzing their learning difficulties in relation to key concepts. The theory based design of students' productions of tutorials and the following analyses can inform and support the development and implementation of the learning environment in order to facilitate better the students' conceptual learning.

Keywords: Limit concept, calculus, student produced video-tutorials, student-student interactions, theories on the learning of mathematical concepts, learning environment.

Introduction and research questions

The many papers in the literature of mathematics education devoted to calculus signify the difficulties with the teaching and learning of calculus with the limit concept as one of the fundamental challenges (Rasmussen & Borba, 2014). In this paper we present and discuss a theory based design and its implementation for students' productions of tutorials aimed at supporting the students' learning of the limit concept. The research was related to a teaching experiment in a first year calculus course at Roskilde University, where the students produced tutorials with the app *Explain Everything* or similar apps[1]. Building on theories explaining general learning difficulties with mathematical concepts such as the notion of students' concepts images (Vinner & Dreyfus, 1989) and Sfard's (1991) model for formation of mathematical concepts and the importance of communication in the learning of mathematics (Sfard, 2008) a didactic tutorial activity was designed. The purpose was to get the students to express their images of key mathematical concepts in dialogues with their fellow students and to support their concept formation process in particularly with respect to their use of symbolic representations and their reification of key concepts such as the derivative, integral and limit concepts as mathematical objects. The activity was designed and implemented as an assignment; namely to produce in small groups video tutorials explaining the group's solution to selected exercises and the mathematical basis hereof, see figure 1. In the assignment specific requirements for the workflow of the students' production were given in order to support the didactic purpose. The tutorials were used as a resource by the students at the course and in preparing for the final.

In the design and implementation of the experiment the basic idea was to use the digital tutorial genre to create a theory directed learning environment that encouraged the students to communicate with each other about mathematical theory, concepts and techniques. The tutorial genre mediated

[1] The pedagogical set up was developed in collaboration with Maja Bødtcher-Hansen, University of Copenhagen.

student-student interactions and hereby supported students' reflections on key mathematical concepts. The didactical assumption was that by focusing the students' attention on the production of tutorials, aimed at supporting their fellow students' learning, it would be possible to create situations where small groups of collaborating students would activate and express their own understandings and images of the mathematical concepts in focus. In addition, the digital tutorials could be (and was) used by students for their own retention of the problem solving techniques and their mathematical foundation – particular in preparing for the final. Furthermore, re-viewing and discussing the tutorials could support the students' reflections on their conceptual learning or be used as a basis for common reflections.

With respect to the scope and focus of the working group *Implementation of Research Findings in Mathematics Education* (see the introduction to TWG23), the implementation aspect of this paper and further related work is the design and organization in the classroom of the digital tutorial in creating a learning environment in which research findings on students' concept formation and the importance of communication are brought to use in the teaching of a first year calculus course. We are developing our practice of mathematics teaching through interplay with research – and in this process we are also engaged with research, so in the present paper, implementation of research results and research are intertwined.

We audio recorded the students' reflections during their work with the tutorials. The transcripts of these records, the tutorials, the students' performance at the course and observations during the course form the empirical basis for addressing our three didactical research questions (RQ):

1. In what sense and degree can a learning environment focusing on the students' productions of tutorials and other digital products with subject matter content support the students' formation of key concepts in calculus?
2. Which learning difficulties related to the key concepts in calculus can be revealed and theoretical explained through analyses of such student activities and products?
3. How can the findings from such analyses be used to further develop the learning environment and the way in which the students' products are used in the calculus course?

In this paper we address the three RQs focusing on the limit concept.

The context of the experiment

The experiment was performed at the calculus course in the natural science bachelor program at Roskilde University. There were 32 first year students in the course, which was taught in English using a typical American calculus textbook (Adams & Essex, 2013). During the previous few years, it had become evident that the students' had problems explaining and applying the concepts and methods of calculus in their subsequent courses. One possible explanation for the increasing difficulties for the students in developing their conceptual learning in the calculus course could be changes in their prerequisites concerning key mathematical methods and concepts from high school, which are fundamental for calculus such as algebra, variable, function and the limit concept. CAS and graphing software for calculators and computers are used intensively in mathematics teaching at high school in Denmark and students learn instrumented techniques to solve standard problems by means of such tools. Similar experiences are reported in other educational systems. Barbé et al.

(2005) have analyzed the situation in Spanish high schools, in particular the incoherence in the mathematical organizations of the theory of limits at high school level. Similar incoherencies have been found with other key elements of upper secondary level calculus in Denmark and American textbooks for first year calculus courses at the university level, due to the elimination of the topological parts of mathematical analysis (Winsløw 2015, pp. 200-203). In general, the extended use of it-instrumented techniques, in particular CAS-based techniques, poses challenges for the teaching of calculus in high school and at introductory courses at university entrance level. Gücler (2014, p.4) addresses the role of the teacher in students' learning of the limit concept in such a teaching practice and pinpoints the importance of the teacher in challenging the students to consider limits both from a process and an object perspective in different mathematical contexts. In particular, the students need to communicate their conceptual understanding in order to really experience both of these perspectives and typically they will need specific support in order to grasp the idea of the limit of a function as a mathematical object.

Our approach to this challenge is to experiment with the learning environment so as to make the students produce and publish their own mathematical products in the form of tutorials in the digital genre through the use of various app's and video-recordings with tablets. The students' products combined oral and written explanations and presentations of their mathematical task and its content matter with visual and dynamical elements.

Implementation of the experiment: mediatized presentations and assessment

The students' fabrications of the mediatized tutorials were connected to their usual homework of sets of traditional problems and exercises. During the course, the students handed in three sets of homework that each consisted of between 10 to 21 standard problems and exercises from the text book. The students were allowed to discuss the solutions to the exercises with each other, but they handed in an individual presentation of solutions to the exercises. In order to participate in the final, a student had to have all three homework sets approved by the teacher. After the deadline for handing in a homework set, the exercises of the set were distributed between the students who in small groups produced a video tutorial of a solution to one of the exercises using the app *Explain Everything* or by other means. The video assignments were accompanied by a set of requirements for the product and a specific work flow which the students had to follow. The instructions given to the students can be seen in figure 1.

Each student also participated in a project work in which the students worked in groups with different subjects from the text book that were not covered in class. Each group wrote a technical report of two pages, which was supplemented by various video and/or other visual and oral digital products.

Finally, the assessment criteria were changed. The written test towards the end of the course was replaced by an oral test. The oral examination consisted of two elements: (1) a 12-15 minutes group presentation of the group's project work. Every student was required to participate actively in the presentation of his/hers project work; (2) an individual examination of 8-10 minutes following the group presentation. After the presentation of the project work, the group left the examination room and hereafter each student was called back in for the individual examination. Each student drew by chance one of the three homework sets of exercises, and made a 5-6 minutes presentation of

selected parts of the subject matter covered by the exercises in the set. The presentation by the student was followed by 4-5 minutes of questioning in the remaining parts of the course and/or homework sets.

Video Assignment for portfolio

You must prepare a tutorial for the exercise you have been asked to work with.
A tutorial is a visual, instructive guide with added speak. The genre is used primarily for technical explanations on the internet and 'Walk Troughs' to computer games. Here we use it to communicate mathematics, both to understand the content and partly to train oral communication with and about mathematics (and as preparation for the final).



Your tutorial must be prepared in the app: "explain everything" (download in the app store). You can see a tutorial (!) to the program's most basic functions here: https://www.youtube.com/watch?v=9yvA4bzfrl4 See it before you come to class on Thursday the 23. of October, so you can skip most of the technical problems.

The assignment corresponds to the individual part of the final, where you draw a portfolio set and have to characterize it in relation to the syllabus and then focus on specific, representative exercises that illustrate the concepts, techniques and issues. Taken together, all your video tutorials constitute a 'bank' of small videos for each portfolio set.

Product requirements for the video:

The video may last 3-5 minutes. All group members must play an active role in the preparation. The video must contain the following:

1. Introduction
   An introduction to your tutorial (we are xxx, this tutorial is aimed at xxx)
   An introduction to the topic (e.g. xxx)
   What kind of math is involved?
   What do you calculate / use it for / describe?
2. Problem Solving
   A demonstration of how to solve the exercise
   In your demonstration you must explain and highlight every step both verbally and visually
3. The "bottom line"
   Finish the video with a short summary
   Reference to the introduction (e.g. we now have an example of ....)

Workflow:

Before you begin, you should all have downloaded "explain everything" and seen the accompanying tutorial (link above).

1. Formulate an introduction to the exercise. You may write it out completely in prose, so the text can be read aloud.
2. Write down the solution to the whole exercise
   Divide the solution process into clear 'steps' - preferably with a number.
   Elaborate on each 'step' with the speaking that needs to be added in the video.
3. Formulate "the bottom line" – remember to refer to the introduction
4. Check with Tim
5. Make a small test movie where you test the sound and get familiar with how to draw and write in the program.
6. Record the movie.
7. Upload the video to Moodle with the file name: "exercise ?? in chapter ?? "or" review exercise ?? in chapter ?? " according to which kind of exercise you have worked with.

Figure 1: Assignment, product requirement and workflow for video production. 'Tim' in step 4 of the work flow refers to the teaching assistant. The students were requested to perform a quality test through his approval before they continued to step 5

The alignment of the requirements for the students' work during the course and the final was an important aspect of the design of the new learning environment. The oral communication skills which were need for the final were trained through the student-student interactions in the design phase of the video productions as well as in the actual fabrication of the videos of the homework exercises and the mediatized parts of the project work. The mediatized products of the project report were directly aligned with the first part of the final, and the video tutorials were directly aligned with the second part of the final. The bank of video tutorials of the total amount of exercises in the portfolio sets that were produced by the students during the course helped the students prepare for the second part of the oral examination. The students carefully designed tutorials that explained

clearly and in depth how to solve the exercises, because they were to be used by their fellow students in their preparation for the final.

Analyzing the students' reflections on the limit concept

The students' tutorials were examined during the course in order to identify those which were rich enough to be analyzed for answering RQ 2 with respect to the limit concept. Among them, a few were selected for discussions with the students in the classroom. Later, the material might be used for analyses that focus on other key concepts. Here, we illustrate with one case how we have analysed the students' work with the tutorials for answering RQ 2 focusing on the limit concept. Guided by the theories, we looked for indicators of students' image concept, of students revealing a conceptual understanding, and of students using a process oriented conception when we analysed the data. The data is transcripts of the audio recordings of one group of students while they produced their first tutorial of an exercise from the first homework set. The students' video is also included in our analysis.

The students' objective was to explain the limit concept using the exercise of finding the limit:

$$\lim_{x\to 3}\frac{x^2-6x+9}{x^2-9}$$

To begin with, the students observed that the limit cannot be found simply by inserting x=3 in the expression, since the denominator is zero for x=3. They mentioned the technique of factorization in order to handle this challenge. This technique is used in several examples in the text book. During their work, the students had software for graphical and CAS analysis at their disposal. However, before implementing this technique they began to reflect on the meaning of the limit concept. The students realized that the video should be aimed at "all the students that are struggling with calculus", as student 2 phrased it. Accordingly, they took on the challenge of explaining the limit concept in their tutorial. Their dialogue went as follows:

S1: Wait a second and now I think I'm lost a little bit. What is this limit thing actually doing?

S2: We have to watch the limit, so we have to watch the value that our function is approaching, when it's, when the axis approach …[S2 draws a graph of the function on the computer.]

S1: Ah, yeah so we can see here, that's true. Okay trace, so we can see here, so we take *x* value 3, eeh.. to here, so it's pretty close to 0.!

S1: Makes a lot of sense. We can also zoom in!

S1: I think you just say that a limit is a number that is (inaudible) by. We have a value that is approaching a number ... right?

S2: A variable that is approaching a number?

S1: We have a function. We have variables in a function that is getting approached by a number. .. It's like we're getting infinitely close, we're getting infinitely close to a number in the function without reaching it. That's a limit.

Here we see that there is a close interplay between, on the one hand, the students' understanding of the particular problem in hand and their ideas and images for the limit concept in general, and on the

other hand, the challenge of producing an explanatory tutorial for their fellow students. The situation forces the students to activate and verbalize their images of the limit concept. It becomes clear for the students that their understanding is not sufficiently developed for explaining the limit concept clearly in their own words in the tutorial. In the process of writing the manuscript for their tutorial the students consulted the text book of the definition of the limit concept. The dialogue continued as follows:

S1: yeah, okay, but let's first answer the first one. ... What's an introduction to the topic. I think an introduction would just be like "A limit is an."!

S2: value that

S1: it's actually like…

S2: a value or number, maybe a number. It's a number that…

S1: yeah it is a number that is getting approached, but is never reached!

S2: it can be reached, it can be reached. I just. You don't have to say if it's reached or defined or undefined, you just say that it's a number that the function approaches as *x* approaches. As *x* gets closer and closer to the *c.* [Referring to the notation in the textbook]

S1: but I mean it doesn't reach the limit, that's the idea.

S2: it has… it could. It can because you can write, because that's only the...

S1: it's still the number it's going towards but…

S2: you can, you can say limit of *x* when *x* approaches 2, it's (over) the limit.

S1: yeah okay, but the limit is numbered, but it never reaches that number, that's not the idea. The limit is just taking it to the limit.

S2: okay!

The students are getting into a deep discussion of how to understand the limit concept. Student 1 is quite persistent in his process-based thinking of a limit, and he has difficulties with accepting that the limit of a function can actually be reached. Student 2, however, switches to a more object-based thinking in their discussion of how they should introduce the limit concept in their tutorial. In some of her utterances she is getting close to formulating the definition of a limit in her own words, and she seems to be aware of the fact that the value of the function can be forced arbitrary close to its limit in a certain point *x=c* by choosing *x* to be sufficiently close – but not equal – to *c*.

After a while student 1 begins to switch into a more object-based thinking of limits. He says:

S1: a limit is a number that is being. A limit is a value, you have (on there), the *y* axis, when you're approaching a number on the *x* axis…. But is it mathematically correct to say that? It's like if you have a function, and you go towards the number.

In their tutorial the students explain in detail doing the algebraic manipulation by hand how to get to the identity: $\frac{x^2-6x+9}{x^2-9}=\frac{(x-3)^2}{(x+3)(x-3)}$, which for x≠3 is equal to $\frac{(x-3)}{(x+3)}$.

They explain in the tutorial how the limit of the given expression can be evaluated by substituting 3 for *x* in "the redefined function" and that this yields the result 0. The students also explain that the

given function is not continues for x equal to 3: “It has a hole in its graph for x equal to 3, while the redefined function is continues in this point and has the value 0”.

However, the crucial mathematical argument is not stated or explained explicitly in their tutorial; namely that since the two expressions are equivalent for all real value of $x \neq 3$, then

$$\lim_{x\to 3} \frac{x^2-6x+9}{x^2-9} = \lim_{x\to 3} \frac{x-3}{x+3}$$

Discussion and some initial conclusions

Focusing on the limit concept, our findings in relation to the three RQs are:

Ad (1): The learning environment encourages the students to articulate and communicate their understanding of key concepts such as the limit concept in various situations and forms of representations. As illustrated in the analysis presented, in the process of producing the tutorial, the students engaged in dialogues about the limit concept using their own language. In their tutorial they used oral, written and visual forms of expressions in communicating their conceptual understanding and the techniques they used in solving their problem. The crucial didactical feature of the learning environment is the focus on the students’ productions of the tutorials aimed at their fellow students. Together with the prescribed work flow and the requirements specified in figure 1, the focus on the productions of the tutorials, encouraged and enabled the students to express and reflect on their own conceptual understanding. In general, from analyzing the students’ dialogues during their production of their tutorial it is evident that only when the students began to work on the video production they became fully aware of what their task really was about, how the technique they used could be explained, and why their results were correct.

Ad (2): From the analysis it is clear that the students found it very difficult to really understand and use the formal definition of the concept of a limit of a function from the text. Especially student 1 insisted on understanding the limit concept as a process or actually as two connected processes. He did not focus on the quality of the limit of a function, and he emphasized wrongly that the limit is a number which will not be reached by the function. This phenomenon can be understood by means of the concept of students’ concepts images (Vinner & Dreyfus, 1989). Their research explains and evidences that formal concept definition only become meaningful to students to the extent in which they are unfolded and concretized by personal experiences. The phenomenon can be further analyzed by means of the process – object duality of mathematical concepts (Sfard, 1991).

In the tutorial, the students explain how to use the technique of factorization to find the limit of the given function. They reached a new expression for the function, which is equivalent to the given expression for all real values of the independent variable x except for x=3. Only the new expression is defined for x=3 and can therefore be evaluated in x=3, which yields 0. However, in the tutorial, the students did not really explain why the limit for $x \to 3$ have to be the same for the two expressions since they are equal for all values of x except for x=3. Güçler (2013, 2014) found and analyzed similar difficulties in students’ learning of the limit concept.

Ad (3): In general, analyses of the students’ productions of the tutorials and the tutorials themselves allow pinpointing learning difficulties related to key concepts in calculus such as the limit concept, which are theoretically explainable. The analyses can provide ideas for variation of problems to be

dealt with in the tutorials in order to invoke and challenge the students' different concept images of the key concepts in calculus. Such analyses can inform the development of the learning environment in order for the theories to be used in the practice of teaching for identifying and help overcoming students' learning difficulties. Moreover, anchored to the students' experiences with the tutorial, theories can help students develop a sound meta-learning related to the formation of mathematical concepts.

Also, the tutorials can be used in whole class teaching as a point of departure for discussing the relations between the techniques used to solve the different types of problems addressed and the mathematical theory explaining them and hereby contribute to the development of the mathematical organization in the calculus course.

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Implementation enterprise through the lens of a theory of diffusion of innovations: A case of online problem-solving forums

Boris Koichu[1] and Nelly Keller[2]

[1]Weizmann Institute of Science, boris.koichu@weizmann.ac.il
[2] Technion – Israel Institute of Technology, nellyk@tx.technion.ac.il

The goal of this article is to present and theorize our more successful and less successful attempts to create and sustain problem-solving forums, in which exploratory discourse takes place. The main argument is that many implementation-related phenomena that we have encountered when working with seven high-school classes for one or two school years can be characterized and explained with the aid of conceptual tools provided by Rogers' Theory of Diffusion of Innovation. The most successful process of forming an online forum in one of the classes is presented in some detail, and the parallel processes in the rest of the classes are presented in the form of an aggregated summary. Implications for future design-based implementation research are drawn.

Keywords: Problem solving, online forums in social networks, diffusion of innovations.

Introduction

This article presents an implementation aspect of a research project entitled "Heuristic and engagement aspects of learning through long-term collaborative mathematical problem solving"[1]. The main research goal of the project (on-going until September 2017) is to produce a model of learning through mathematical problem solving, which would be attentive to cognitive, socio-affective and contextual aspects of this activity. In particular, the model is supposed to attend to interactions between variations in heuristic behaviors (Koichu, Berman & Moore, 2006; Koichu, 2010) and socio-affective engagement structures (Goldin et al., 2011) activated when high-school students collaboratively cope with challenging mathematics problems for relatively long time.

An implementation aspect of the project consists of designing and sustaining a special learning environment, in which long-term problem solving might be investigated. The intended environment comprised of a particular combination of problem-solving lessons in a classroom and out-of-classroom work supported by online asynchronous discussion forums. Its design was strongly informed by past research on affordances of online learning environments. Past research tells us that students in online environments can actively participate in solving complex problems for 2-3 weeks almost without teacher interventions (Moss & Beatty, 2006) and that some of those students who tend to be silent in a classroom can actively participate in online discussions (Schwarz & Asterhan, 2011). In addition, there is evidence that online discussions enable students to meaningfully use their mathematical knowledge, enhance self-regulation skills and support knowledge construction (Nason & Woodruff, 2003; Tarja-Ritta & Järvelä, 2005; Stahl, 2009).

Schwarz and Asterhan (2011) attribute the benefits of online discussions to their unique traits, such as: fostering divergent rather than linear interactions, enabling flexible time schedules of

[1] Selected findings of the project are reported in Lachmy & Koichu (2014), Koichu (2015ab), Keller and Koichu (in press).

participation in the discussions over relatively long periods of time, encouraging explicit and accurate expression of the ideas in writing. Koichu (2015ab) argues that many affordances of an online problem-solving forum stem from its fundamental characteristics of being a *choice-affluent environment*, that is, an environment, in which the students are empowered to make informed choices of: a challenge to be dealt with, a way of dealing with the challenge, a mode of interaction, an extent of collaboration, and an agent to learn from. In brief, past research on online collaborative problem solving presents many evidence-based cases of successfully working online forums.

However, little is known from the professional literature about how to put the forums into work and sustain them. Our experience in the aforementioned project taught us that this enterprise is truly challenging. The goal of this article is to make sense and theorize our more successful and less successful attempts to create and sustain problem-solving forums, in which exploratory rather than expository problem-solving discourse (this distinction is due to Stahl, 2009) takes place. The main argument is that many implementation-related phenomena that we have encountered when working with seven high-school classes (grades 10 and 11) can be characterized and explained with the aid of conceptual tools provided by Roger's (2003) Theory of Diffusion of Innovation.

Conceptual framework

Approach: Design-Based Implementation Research (DBIR)

Werner (2004) refers to *implementation research* as the systematic study of the implementation of innovations. Fishman, Penuel, Allen, Cheng and Sabelli (2013) point out that this type of research encompasses studies of fidelity, of variations in implementations as well as studies of conditions under which programs can be implemented effectively. They further refer to implementation research and to design-based research as antecedents of a new, emerging research model, which they name Design-Based Implementation Research (DBIR). The core principles of DBIR are as follows:

> (1) a focus on persistent problems of practice from multiple stakeholders' perspectives; (2) a commitment to iterative, collaborative design; (3) a concern with developing theory and knowledge related to both classroom learning and implementation through systematic inquiry; and (4) a concern with developing capacity for sustaining change in systems. (Fishman, Penuel, Allen, Cheng and Sabelli, 2013, pp. 136-137)

In addition, DBIR calls for breaking down barriers that isolate those who design and study innovations and those who study the diffusion of innovations. We find the DBIR concept and principles well-adjusted to the needs of our project.

Vocabulary: Selected elements of the theory of diffusion of innovations

The notion *innovation* is frequently used in the literature on implementation research as a self-explanatory one (Fishman et al., 2013). However, there exists a branch of the professional literature that explicitly focuses on innovations and the processes of their diffusion. In particular, Rodgers (2003) defines *innovation* as "an idea, practice, or object that is perceived as new by an individual or other unit of adoption. It matters little, so far as human behavior is concerned, whether or not an idea is 'objectively' new as measured by the lapse of time since its first use or discovery" (p. 11). In our case, the idea of stretching the boundaries of a classroom by means of an online problem-solving forum was an innovation because it was new to the students and the teachers.

Rodger's (2003) theory of diffusion of innovations meticulously characterizes the *innovation-decision process,* in which individuals (or other decision-making units) decide whether to accept an innovation or not. In particular, Rodgers distinguishes five stages of the process: *knowledge, persuasion, decision, implementation* and *confirmation*. The stages are briefly presented below.

At the *knowledge* stage, potential innovation adopters are exposed to the innovation's existence and obtain some information about how it functions. Sometimes individuals become aware of an innovation by accident, and sometimes they actively look for it in order to fulfill particular needs. It is also possible that the needs are formed as a result of one's exposure to an innovation.

At the *persuasion* stage, an individual forms a favorable or unfavorable attitude towards an innovation. This stage presumes affective involvement with the innovation. In particular, the individuals may mentally apply the new idea to their present or anticipated future situation. They seek to answer such questions as "what are the innovation's advantages and disadvantages in my situation?", and seek the answer mostly from their near-peers, whose opinions based on their personal engagement with an innovation, are the most convincing. There is a discrepancy between forming a favorable attitude towards an innovation and an actual decision to adopt it. Adoption of an innovation can be influenced by a *cue-to-action,* an event that crystallizes an attitude into overt behavioral change.

At the *decision* stage, an individual adopts (i.e., makes full use) or rejects an innovation. Any decision is not final however. The rejection can occur even after a prior decision to adopt; in Rodgers' terms, this phenomenon is called *discontinuance*. The theory distinguishes between *active* and *passive rejection*. The former type of rejection consists of considering adoption of the innovation and then deciding not to adopt it. The latter one consists of never "really" considering the use of the innovation. The decision stage frequently includes a small-scope trial. The actual sequencing of the three stages presented so far can alter. Namely, both *knowledge–persuasion–decision* and *knowledge–decision–persuasion* sequences are possible.

At the *implementation* stage, an individual puts an innovation into systematic use. Even though the decision has been made, the adopters may still feel a certain degree of uncertainty about the consequences of the innovation. In addition, problems of how exactly to use the innovation may emerge. Sometimes the adopters change or modify (in Rodgers' terms, *re-invent*) the innovation at this stage. The implementation stage can be lengthy, but it ends when the idea that has once been innovative becomes institutionalized and regularized in the adopters' normal functioning.

Finally, at the *confirmation* stage, an individual constantly seeks reinforcement for the decision to adopt or reject an innovation that has already been made. As a result of positive or negative messages about the innovation, the decision can be reversed. Rodgers points out that the *change agents* (i.e., those who influenced one's decision to adopt an innovation) have responsibility of providing supportive messages to the individuals who have previously adopted the innovation.

Methodological aspects of the project

Participants and the project's activity

Two experienced mathematics teachers and two of their 10th grade classes took part in the first year of the project (2013-2014); five more teachers and their corresponding five classes joined the

project during its second year (2014-2015). Each participating teacher acted in the project as a member of the research group and took part in the meetings of the group. In addition, each teacher worked in contact with an additional member of the group who was responsible for the technological support and documentation of the activity. Mathematics in all participating classes was studied for five hours a week, in accordance with the Israeli high-level curriculum (see Leikin & Berman, 2016, for details). For the concerns of this article, it is enough to mention that geometry was studied two hours a week and that its study included systematic work on proving tasks.

We planned that each participating in the project class would be engaged, at least three times during a school year, in the following activity. The students cope with a series of preparatory tasks during a 90-minute lesson and are offered an especially challenging geometry problem at the end of the lesson. They then engage, for 5-10 days, in solving the problem from home in a closed (that is, available only to the students of a participating class and the members of the research group) online forum. Different technological platforms, including Google+ and WhatsApp, were tried in different classes. When the problem is solved or, alternatively, when the students give up, a 90-minute lesson is conducted in the classroom in order to get closure. The lesson consists of whole-class and small-group discussions, during which the students share their experiences with the problem.

Documentation of the project

Forty-two meetings of the research group were audiotaped (about 100 hours) and, in addition, documented in the protocols of the meetings (more than 100 pages). The documents produced by the group and all relevant email exchange were stored. Every member of the group was required to keep a diary. The diaries were for writing anything their authors deemed important for the project, including their thoughts and feelings in relation to the project's events. The diaries were stored in shared Google Drive of the group and were available for reading and commenting by the members. In addition, 14 lessons were videotaped, the content of the forums was stored (more than 3000 posts), interviews with students and teachers were conducted (about 15 interviews), and the students' written feedback on different aspects of the project was collected.

The story of NK's class, which is presented below in some detail, is produced using *narrative inquiry methodological tradition*. As Clandinin and Caine (2008) explain, "Narrative inquiry is marked by its emphasis on relational engagement between researcher and research participants" (p. 542). This approach was chosen because we (hereafter, BK and NK) had been active members of the processes under exploration; in particular, NK was a mathematics teacher of the class. An aggregated summary of the stories in the rest of the classes is produced using a *general inductive approach* (Thomas, 2006), which enables researchers "to condense extensive and varied raw text data into a brief summary format" (p. 238).

Findings

A (success) story of NK's class

The main events at the *knowledge* phase of the project in NK's class consisted of: (1) a conversation between NK and BK following BK's observation of one of NK's lessons; (2) a conversation between NK and her students. Because of the first conversation, NK decided to take part in the project mainly because the idea to stretch the boundaries of a classroom by means of an online forum resonated well with NK's constant need to enrich her teaching repertoire in order to create

valuable learning opportunities for her students. In Rodgers' terms, NK acted as a *venturesome innovator* who is able to cope with high degree of uncertainty about an innovation, and BK acted as a *change agent*. In her conversation with the students, NK acted as a *change agent*, and the students were potential innovation-adopters to be *persuaded*. NK argued that developing problem-solving skills was a strong benefit of participating in the project, and appealed to the students' curiosity to try something new and be a part of an interesting initiative. The students' reaction to the information about the project was favorable, though not exactly for the reasons that NK had presented.

The first mathematical problem of the project is presented in Figure 1. It is representative of most of the problems of the project. In particular, it looked similar to geometry problems the students were familiar with from classwork and homework. As such, the problem "invited" the students to approach it by means of mathematical ideas that worked well in the past. For instance, the students might think of including the angles, whose equality is to be proved, in a pair of triangles and attempt proving their congruence by finding some equal elements. However, such a general plan was insufficient; something else (e.g., a clever auxiliary construction) should have been invented.

Nine-Square Problem: There is a net of nine congruent squares (see the drawing). Prove that the two angles denoted in the drawing are equal.

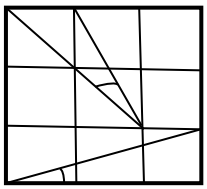

Figure 1: The first problem of the project

When the problem was uploaded to the Google+ forum, three students worked on it. Their three-hour-long brainstorming session was unsuccessful. As a result, the forum was non-active during the next two days. The following day NK met the students at school and asked: "Why did you stop solving the problem? It is not too difficult". The students showed NK their hand-made drafts as evidence that they had tried. NK asked the students to upload their drafts to the forum and continue solving the problem together. That evening eight students entered the forum, cooperated and eventually solved the problem. Two solutions to the problem by the active participants of the forum and an additional solution by a student who was a silent observer were presented at the mathematics lesson following the forum. The students' voluntarily expressed their suggestions as to how to further run and improve the forum by the end of the lesson,

In Rodger's terms, the first three students acted as *venturesome innovators*. Rodgers points out that this category of adopters is important for launching a new idea, but they have little influence on other individuals' decision to adopt or reject the idea. The conversation between NK and the students in school was crucially important as a *cue-to-action* for eight students, who acted as *early adopters*. Rodgers characterized this category of individuals as *respectable*, that is, well-integrated members of a local community whose opinion about the innovation matters for the potential adopters. The mathematics lesson, in which these eight students shared their positive experience at the forum with the rest of the class, was another crucially important *cue-to-action*.

The next two problems of the project were approached on the forum by about the same group of students. The students learned to share their half-baked ideas, and even developed some rules related to publishing the full solutions at the forum. In brief, they agreed that a student who obtained

the full solution should not publish it early, in order to not "spoil the fun" for others. The forum was indeed exploratory rather expository in nature. Three months later, collaborative problem solving at the forum became a well-established practice for six students; most of their classmates joined the forum occasionally and constituted the *early majority.* It seems that each student has *decided* how and to which extend to use the forum. The *implementation* stage (three to eight month from the beginning of the project) was characterized by gradual two-directional diffusion of social and socio-mathematical norms developed in the forum and in the lessons. In particular, there were several forum-like lessons initiated by the students (see Keller & Koichu, in press, for details).

The evidence of *confirmation* of the students' decision to adopt the innovation came from the following sequence of events. As mentioned, the Google+ forum flourished for several months, but it we have not yet mentioned that then it was deserted. NK inquired with the students about this fact and discovered that the activity moved from the Google+ to WhatsApp, a popular social network in Israel since about 2014. The students granted NK access to their WhatsApp forum, and we were happy to find there many autonomous problem-solving discussions of exploratory nature. The WhatsApp forum flourished in NK's classroom until the students' graduation in 2016.

An aggregated summary of seven stories

An aggregated summary of the conduct of the project in all participating classes, by Rodgers' phases, is presented in Table 1.

	NK class	**AP class**	**AH class**	**ES class**	**RN class**	**OG class**	**LA class**
Knowledge	+ +	+ +	+ +	+ +	+ +	+ +	+ +
Persuasion	+ +	+ +	+ +	+ +	+ +	-	+
Decision	Accept	Passive R	Accept	Passive R	Accept		Active R
Implementation	+ +	+	+ +	+	+		
Confirmation	+ +						

Table 1: The implemented stages of the project in seven classes

The sign "+ +" in the table means that the stage is fully realized; "+" means that the stage is partially realized (e.g., the forum was active only as a trial or only few students were active); "-" means that there were no conditions for realizing the stage; "Active R" and "Passive R" stand for active and passive rejection, respectively. An empty cell means that the project did not arrive at that stage.

As Table 1 tells us, only NK's class went through all five stages, up to the point when using the online forums for problem solving stopped being an innovation and became a routine. Implementation of the project's idea at the scope comparative to NK' class occurred in one other class, and partial implementation – in three classes. In two classes the project did not reach the implementation stage, despite of much effort made by the teachers and the research team.

Concluding remarks

Recalling Tolstoy's seminal assertion, *happy families are all alike; every unhappy family is unhappy in its own way*, we can argue that there is a unique story behind each cell of Table 1.

Unfortunately, we cannot tell these stories here due to the space constraints. In brief, sometimes school conditions or classroom norms were inappropriate for realizing the project's idea, sometimes a particular *cue-to-action* event did not happen at the right time or was not appropriately designed, and sometimes our decisions and actions as a research group were far from being optimal. We have also observed, more than once, the phenomena of *discontinuance* and of *passive rejection* for which we do not have convincing explanations, despite the extended data set in our possession.

We intend to continue the aforementioned project, and one of the lessons learned so far is that the DBIR concept and theories like the Rodgers' theory of diffusion of innovation should be taken seriously. Either detailed or aggregative analysis of implementation of the project idea is helpful for us as a tool for refining the roadmap of the project. In addition, we now better understand that creating conditions for implementation of an innovative pedagogical idea in a school reality should be given full attention prior to delving into a pursuit for "traditional" research questions, such as questions on cognition and affect in mathematical problem solving that have been the main *research* questions of the project. Based on the accumulated experience, we call for reporting and analysing not only those cases where an innovative idea is being fully implemented, but also those case where the implementation was partial or did not occur as planned. We conclude by suggesting that systematic attention to implementation issues, by means of DBIR, may have not only practical, but also fundamental theoretical significance in mathematics education.

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From theory through collaboration into practice: Designing a problem- solving curriculum for grade 6 students

Ana Kuzle

University of Potsdam, Germany; kuzle@uni-potsdam.de

Teachers are continuously confronted with instructional endorsements, such as the inclusion of problem solving in school mathematics. However, adoption of problem solving is still not a reality. One reason for it is the lack of practical teaching materials based on research findings to achieve the goals stated in the standards. In the context of this reform agenda, collaborative work between educational researchers and practitioners in a real setting working on issues of everyday practice is crucial in order to overcome the gap between theory and practice. In this paper, I focus on such theory based problem solving curriculum for grade 6 students that was developed using design-based research. At the end, I discuss factors inhibiting the implementation of the curriculum.

Keywords: Word problems, material development, mathematics activities, problem solving.

Introduction

The (inter-)national educational standards (e.g., KMK, 2003; NCTM, 2000) have strongly endorsed the inclusion of problem solving in school mathematics. Empirical studies, however, portray a different picture. Students are often unable to solve problem tasks (e.g., Schoenfeld, 1985). In addition, quality analyses in the German school system contend to a poor problem solving culture. As reported in Kuzle and Gebel (2016), problem solving tasks got rarely introduced. When this was the case, they were primarily done by the teachers; mostly routine tasks dominated the lessons, and problem solving strategies were explicitly applied in one third of examples only. The biggest problem reported by the school's teachers was the lack of practical teaching resources to achieve the goals stated in the standards (Kuzle & Gebel, 2016). In the context of this reform agenda, the development of materials for students and teachers is of great importance for overcoming the gap between theory and practice (Jahn, 2014). One urban school recognized this deficit and set as a goal promoting problem solving instruction centered around curriculum material, developed through collaborative work between educational researchers and practitioners.

Here I report on a small part of SymPa[1]-project (Systematical and material based development of problem solving competence) focusing on collaboration between practitioners and researchers with the goal of developing a problem solving curriculum for grade 6 students using design-based research (DBR). The guiding question is: What factors inhibit implementation of research-based problem solving curriculum in practice? In the following sections I outline relevant theoretical and methodological underpinnings used to design a problem solving curriculum, before showing how these got implemented, and report on its evaluation (initial DBR-cycle). As a result of evaluation, I discuss the curricular redesign that might allow more effective implementation in practice.

[1] SymPa stands for „**Sy**stematischer und **m**aterialgestützter **P**roblemlösekompetenz**a**ufbau". Inga Gebel (researcher) and Christian Conradi (practitioner) initiated and participated in the project likewise.

Theoretical foundation guiding the design process

Plethora of research on problem solving undergoing since the 1970s identified several pivotal areas for a problem solving curriculum. I outline here only a small portion of this research that was crucial for the project based on German standards' conception of problem solving (KMK, 2003).

Problem solving competence

Problem solving competence relates to *cognitive* (here heuristic), *motivational* and *volitional* knowledge, skills and actions of an individual required for independent and effective dealing with mathematical problems (Bruder, 2002; Kuzle & Gebel, 2016). Accordingly, students should a) learn approaches (heuristics) for solving mathematical problems and how to apply them appropriately in a given situation, b) develop reflectivity for own actions, and c) develop willingness to work hard (cf. Bruder, 2002; KMK, 2003). As problem solving competence encompasses so many different facets, problem solving curriculum should account for the following research areas: (a) teaching approaches and concepts on problem solving, (b) theories of self-regulated learning and self-regulation in problem solving, and (c) theories of motivation which are outlined below.

Teaching approaches and concepts on problem solving in combination with self-regulation

There are at least seven practices for problem solving curriculum that researchers (e.g., Kilpatrick, 1985; Pólya, 1945/1973; Schoenfeld, 1985) have claimed to be important for helping students grow in their problem solving ability: (1) osmosis (give lots of problems), (2) give "good" problems, (3) memorization (teach specific or general heuristic strategies (heurisms)), (4) imitation (model problem solving), (5) cooperation (limit teacher input by having students work in small groups), (6) reflection (promote metacognition by asking metacognitive questions or encouraging students to be reflective), and (7) highlight multiple solutions. In the recent years, Bruder (2002) developed a problem solving teaching concept focusing around Lompscher's (1975) idea of "flexibility of thought". *Flexibility of thought* is expressed by one' ability to

1. reduce a problem to its essentials or to visualize it by using visual and structuring aids, such as informative figures, tables, solution graphs or equations (*reduction*).
2. reverse trains of thought or reproduce these in reverse, such as by working backwards (*reversibility*).
3. simultaneously mind several aspects of a given problem or to easily recognize any dependences and vary them in a targeted manner (e.g., by composing and decomposing geometric figures and objects, by working systematically) (*minding of aspects*).
4. change assumptions, criteria or aspects in order to find a solution, such as by working forwards and backwards simultaneously or by analyzing different cases. Such ability prevents "getting stuck" and allows new approaches and insights (*change of aspects*).
5. transfer an acquired procedure into another context or into a very different one by using analogies for instance (*transferring*).

These typical manifestations of mental agility can be related to analyses of heuristic approaches by Pólya (1945/1973). Untrained problem solvers, however, are often unable to access the above outlined flexibility qualities consciously. Moreover, not only the knowledge of different heuristics (flexibility of thought) is needed when problem solving, but also self-regulatory abilities which evolve gradually through a 5-phase model (Zimmerman, 2002). Problem solving can be trained by

learning heuristics corresponding to these aspects of *intellectual flexibility* in combination with *self-regulation*, which according to Bruder (2002) and Bruder and Collet (2011) develops through the following five-phase concept:

1. *Intuitive familiarization*: This phase builds on Pólya's (1945/1973) model, in which a teacher serves as a role model when introducing a problem to students. Thus, the teacher moderates behaviors typical for the problem by engaging in self-questioning pertaining to different phases of the problem solving process (before, during, and after). For example questions such as, "What is the problem asking for?" "What information am I given?" "Is there anything I don't understand?" "Am I headed in the right direction?" may help guide the students (Kuzle & Bruder, 2016). At this point the heurism in focus is not specified.
2. *Explicit strategy acquisition*: During this phase the students get explicitly introduced to the heurism in focus on the basis of a reflection from the first phase. Here the particularities of the heurism get discussed and it gets a name (exemplification). Here prototypical problems get used for introducing a heurism in focus, so that the students can more easily recognize its main ideas and more easily remember their specific steps for future problem solving.
3. *Productive practice phase*: During this phase the students practice solving the problems using the heurism in focus. Here is important that the problems do not reproduce type problems, bur rather expand the possibilities from the first two phases. In addition, differentiation should be a guiding concept during this phase, so that students can choose at what cognitive level they want to work and adapt the observed learning behavior.
4. *Context expansion*: In this phase the students should practice the use of heurism in focus independent of a mathematical context. In that way, the students learn to flexibly, unconsciously and independently of a context use the heurism in focus.
5. *Awareness of own problem solving model*: The aim of the teaching concept is that the problem solving model of the students gets expanded, so that they are in a position to solve problems better using different heurisms. Awareness of own problem solving model can be induced by having students reflect on and document their problem solving process.

Lastly, students' willingness to work hard is a major factor for the successful problem solving process. Without an effort from the learners, there will be no successful learning. For that reason, the criteria such as, understandable and clear problem, age-appropriate choice of context, and visible competence growth (Bruder, 2002) are crucial when designing problem solving curriculum.

To summarize: the problem solving curriculum was developed around the operationalization of the terminology "problem solving competence". This included the teaching concept of problem solving by Bruder (2002) in combination with Zimmerman's (2002) self-regulation model taking into consideration the criteria for motivating tasks (Bruder, 2002).

Curriculum development

The problem solving curriculum was developed in collaboration between the two researchers (author and young researcher) and one practitioner (teacher from the project school). More concretely, the researcher team developed the curriculum based on the outlined theories and school's contextual factors (see below), which were discussed up-front. Curriculum materials (e.g., problems, textual parts, figures, colors) were either separately developed by the researcher team and

discussed afterwards with the teacher or the entire team met together and developed them. The final decision about the problem solving curriculum (e.g., content, problems) was met by the teacher.

Enactment

For the design of curriculum contextual factors played a great role, in which theoretical ideas had to be operationalized. Students of 6th grade were chosen to participate in the project lasting one school quarter (ca. 16 lessons, 1 lesson = 45 min). The implementation of the curriculum took place during two parallel phases (see Table 1). During the first DBR-cycle 13 students participated. Teacher A initiated the project, had previous experience in problem solving (e.g., attended professional development courses on problem solving, read literature on it, and implemented problem solving tasks occasionally in his teaching practices). The second DBR-cycle started parallel to the first DBR-cycle, and was led by another mathematics teacher. In total 12 students participated. Teacher B had practically no experience with problem solving or teaching problem solving.

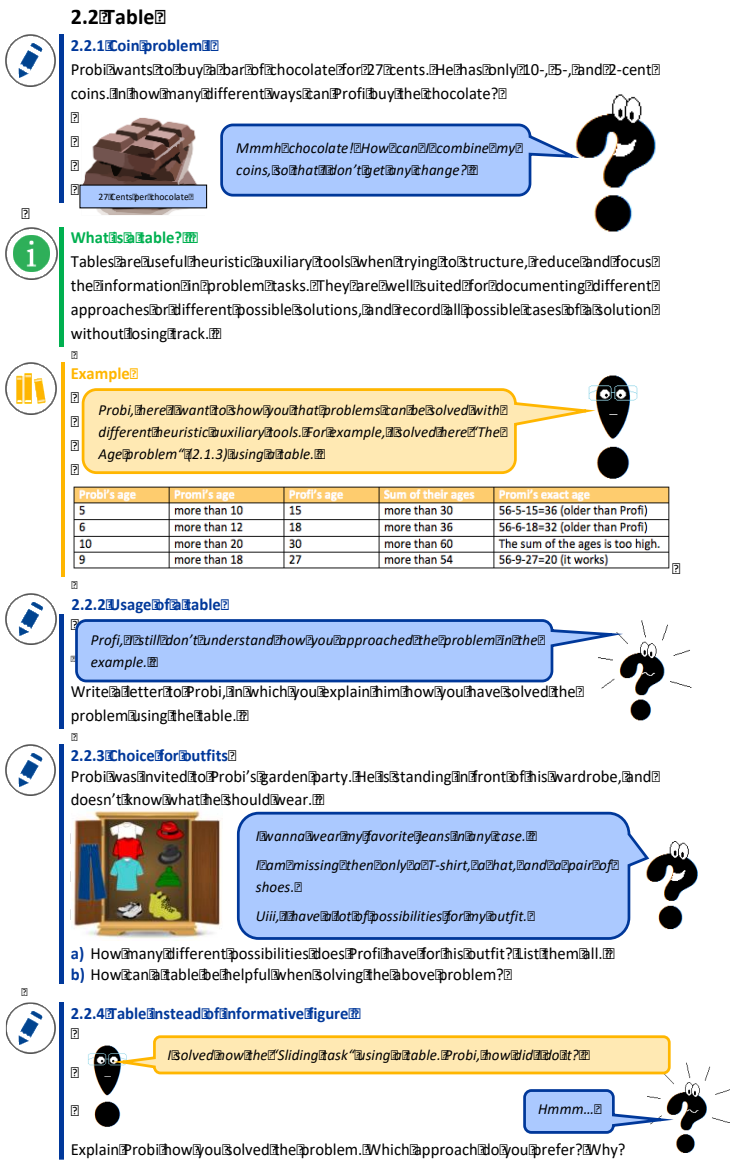
1st DBR-cycle	2nd DBR-cycle
every 14 days (8 meetings), Fridays, double period, teacher A	weekly (17 meetings), Mondays and Tuesdays, single periods, teacher B

Table 1: Parallel enactment cycles

With respect to heurisms, focus laid on those heurisms prescribed by the school's own curriculum, namely heuristic auxiliary tools (informative figure, table, solution graph), heuristic strategies (working systematically, working forwards, working backwards), and heuristic principles (composing and decomposing). Thus, all flexibility qualities were addressed. With respect to mathematical content, problems covered the content areas of 5th and 6th grade mathematics (operations with natural numbers and fractions, combinatorics, geometric and numeric patterns, measurement pertaining to 2- and 3-dimensional figures). Based on the project time frame, each heurism was covered within two lessons, but followed the above underlined problem solving concept. For one exemplarily operationalization with references to theoretical base see Figure 1.

During the implementation phase data collection took place on three different levels: student level, teacher level and classroom level. With respect to the student level, data from student textbooks (intermediate reflections, final reflection) and their workbooks (student productions) was collected. With respect to the teacher level, data from continuous communication with the teachers (e-mail, telephone calls), teacher textbook and semi-structured interview at the end of the project was collected. Concretely, continuous communication allowed the researcher team to support the teachers during the implementation phase with respect to pedagogical and/or methodological questions (e.g., discussion of different solutions, cooperative methods), by answering questions of content nature (e.g., questions about particular heurism), and through flexible and stepwise redesign of the curriculum after each lesson. Lastly, with respect to the classroom level, observations allowed for analysis of student-teacher interaction, and students' interaction with the curriculum.

Curriculum on the heuristic auxiliary tool of table	Theoretical foundation



In the phase of intuitive familiarization, students solve a representative problem for the heurism in focus together with the teacher, who serves as a moderator. Here the imitation of teachers' behavior takes place through self-questioning. The problem represents the students' first encounter with the heurism in focus.

In the phase of explicit strategy acquisition, the heurism in focus gets formally introduced through a short student-centered information text and an example.

In what follows, at least three problems of different cognitive level are presented that serve as a productive practice phase. This allows for differentiation, where each student can solve as many problems as he or she can. In addition, problems from different mathematical content areas are covered, to allow for transfer (context expansion phase), which pertains to the fourth phase of the teaching concept.

In addition, the heurisms are interrelated, so it is important that the students comprehend this. The last task allows students to make this connection by comparing the two heurisms and reflect on it.

Figure 1: A sample page from the problem solving curriculum on the heuristic auxiliary tool of table

The problems focused on contexts that are motivating and appropriate for young students. In addition two figures were introduced to support students' willingness to work hard; they could identify with Probi (shape of a question), who asks questions and gets stuck. Profi (shape of an exclamation mark) offers then support to students, who illustrates a professional problem solver.

Evaluation

Qualitative-content analysis was used to analyze the collected qualitative data as outlined in Patton (2002). This method is particularly suitable for research activities, in which the knowledge is low and a study has more of an exploratory character. Thereby, the aim was to systematically analyze the

qualitative data and produce a category system by focusing on factors inhibiting the implementation of the curriculum. The deductive analysis was performed based on the theoretical foundation, which was then refined in the inductive analysis by emerging issues and additional codes. The situations were interpreted as inhibiting when they allowed for a limited implementation of the curriculum only as reported explicitly by teachers and students and/or was observed by reviewing the collected data. As a result four categories were produced (see Table 2, for more detail see Kuzle & Gebel, 2016). The category system was then used to interpret the results of the study with respect to the research question. All data got analyzed by the two researchers independently.

Category	**Description**	**Subcategory**
Language	All language-based comments that influenced the understanding of the problem were assigned to the language category.	Problem solving terminology Figure names Problem enumeration
Level of performance	Barriers influencing the level of performance during problem solving process (e.g., content barriers, subjective barriers) were assigned to the level of performance category.	Motivation and differentiation Curricular difficulties Increased level of performance
Learning pedagogies	The evaluations of the curriculum in terms of the didactic ideas about learning how to solve problems are listed in the learning pedagogies category.	Communication ability Reflective ability
School and personal influences	Any feedback that aimed at external factors influencing the implementation was assigned to the school and personal influences category.	Teacher attitude Professionalism Organization

Table 2: Four content categories inhibiting implementation of the problem solving curriculum

Table 2 shows that also school and personal influences, which were not part of the design process, influence the extent to which the curriculum gets implemented. Thus, teachers also inhibit successful implementation of the curriculum, despite being part of the design process. I focus on this category by giving different subcategory examples since this factor was most surprising.

Figure 2 shows student's work solving one problem where the heuristic auxiliary tool of informative figure was to be used. Instead of representing a graphical illustration of the situation to be resolved from which the solution can be "read" (informative figure), it represents a sorting of information. Hence, the student work is a reflection of teacher's B lack of knowledge of problem solving, despite this problem being discussed in the teacher manual, which she glanced only once (professionalism). This was seen in all students' notes. Moreover, the analysis of students' workbooks revealed that teacher B avoided introducing problem solving terminology and allowed students to solve problems as they wished (teacher attitude). She explained that she did not want to burden them with formal terminology and constraint their problem solving process.



Figure 2: Teacher's professionalism reflected in a sample page from one student's workbook

Also teacher B criticized heavily the name of the figures in the curriculum (Profi, Probi), which was reflected in students' final reflection, where they urged for the change of figure names. Thus, teacher's negative attitude about a design element transferred negatively onto the students. Such behavior was not observed with the students led by teacher A, who questioned the chosen names, but never criticized them during his instruction. Teacher B in comparison to teacher A was not part of the design team, but was assigned to implement the curriculum due to organizational school context. The case of teacher B shows that negative attitude and lack of professional behavior inhibited successful implementation of the curriculum. Hence, teacher as an inhibiting factor should not be neglected during the design and implementation process.

Conclusion

Problem solving must gain more importance in school mathematics. Although several teaching concepts and practices are known (e.g., Bruder, 2002; Kilpatrick, 1985; Pólya, 1945/1973; Schoenfeld, 1985), these get rarely implemented. Moreover, curriculum based on existing and empirically tested problem solving pedagogies is non-existent. To overcome this gap SymPa-project was grounded. Teachers participating in the project reported improvement in students' problem solving competences with respect to deliberate and mindful use of different heurisms when problem solving. In addition, the teachers not participating in the project, reported students using these heurisms in regular mathematics classes. Hence, it was possible to develop curriculum that met the local demands with the aim of supporting a systematical development of problem solving competence. However, different objective and subjective factors inhibited full-implementation of the curriculum. With respect to the former (language, level of performance, learning pedagogies), changes done in the re-design phase of the DBR-cycle will shed light to which extent these were enough for successful implementation in the upcoming phases of enactment. With respect to the latter (school and personal influences), it became clear that the curriculum alone does not guarantee the implementation of the teaching concept. Substantial knowledge base of the content and pedagogical ideas seem necessary to teach in accordance with the theoretical foundation. Confidence and experience in teaching problem solving played a crucial role likewise. Likewise school organizational factors should not be ignored. Since the teachers were assigned to teach the problem solving lessons and received no compensation for the participating in the project, a lack of motivation may develop, which influences the willingness to teach, the lesson quality and with it the students' acceptance of the curriculum.

In this paper I demonstrated that DBR-paradigm allows creating novel teaching environments in which theory and practice were not detached from one another, but rather complemented each other. Here the efforts were made to design, use and do research on problem solving curriculum in a real

setting. This promoted adoption of the innovation – problem solving curriculum – which became an official part of that school's curriculum. Moreover, close collaboration during the design and enactment phase, and the re-design of the materials by the researchers as a result of teachers' feedback allowed them to develop a sense of ownership for the designed curriculum. Future work should use similar methodologies to ensure implementation of research into practice, adoption of research into practice, which would then allow research on implementation projects. These components may build a fundamental step to overcome the gap between theory and practice.

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Theoretical constructs for early intervention programs in mathematics: Who cares? – A Danish example

Lena Lindenskov & Katrine Kirsted

Aarhus University, Danish School of Education (DPU), Denmark

lenali@edu.au.dk, kaki@edu.au.dk

"There is nothing so practical as a good theory". The statement from Kurt Lewin is frequently cited, also in mathematics education. The statement invites for and requires close cooperation between different agents, whatever their specific relation to practice and theory is. It is not a straightforward endeavour. One reason is that the term theory as well as the term practice may very well be given different meanings by different agents. This variation is, in our view, to be considered in "implementation research" and Lewin's statement ought to be qualified by two questions: "Who cares about a good theory?" and "What makes a good theory good for whom?"

This paper explores the variation of how theory is perceived by mathematics teachers and by mathematics researchers involved in a developmental project on early intervention in mathematics education in Denmark. The paper exemplifies how agents' different work conditions and work requirements seem to constitute qualitatively different needs for theoretical constructs, despite some common interests.

Keywords: Early intervention programs, teaching principles, theoretical constructs.

Background for the early mathematics intervention project

We noticed a long tradition of integrating early mathematics intervention programs into compulsory education practice like, e.g. Mathematics Recovery (MR) (Wright, 2006; Wright et al, 2007) and Extending Mathematical Understanding Intervention Program (EMU) (Gervasoni, 2016), in Australia, Ireland, the UK and the USA. A similar tradition exists in Denmark for early reading intervention programs, as such programs are implemented at a regular basis in all schools in Denmark. Either as part of a municipal policy or a matter of choice, schools launched early (from the first grade) intervention processes in reading to support individual children, who show signs of reading difficulties.

At the ministry level in Denmark concerns were raised about pupils failing at mathematics in the National Official Guidelines in the 2003 National Mathematics Curriculum from the Ministry of Education (UVM, 2003). In 2004 the need to support failing pupils in mathematics in the first school years was emphasised (Mortimore et al.) and the national official guidelines to the revised 2009 national curriculum (UVM, 2009) described, for the first time, the issues in detail. Still, no intentions of integrating programs for early mathematics intervention into compulsory education practice were seen in Denmark until recently (Lindenskov, 2007).

In 2009 the material Early Intervention in Mathematics [Danish: Tidlig Indsats i Matematik, TIM], written to primary school mathematics teachers was published and used in some places. Just before, in 2007 local politicians and school authorities in the municipality of Frederiksberg in the capital area, decided to give priority to mathematics teaching and learning in their 9 public schools in the period 2007-2013. Priority was given to a development project on early mathematics intervention

for their 9 schools in collaboration with the researchers Lena Lindenskov and Peter Weng. It soon became clear that the existing intervention framework and written materials solely focused on numbers and arithmetic, which were insufficient to comply with the Danish Mathematics Education Philosophy and Curriculum. Also, approaches in the existing intervention framework and written materials were insufficiently inquiry and problem based to correspond with the curriculum. Finally, teachers' freedom and responsibility to adapt materials to their own students were too limited in the existing frameworks and materials.

With this background a research-based developmental project with four design cycles was prepared in order to develop a Danish program for early intervention in mathematics that would fit into the Danish Mathematics Education Philosophy and Curriculum. The private Danish fund Egmont showed interest in a Danish early mathematics intervention program (personal communication), and together Egmont, Frederiksberg Municipality and Aarhus University assured the budget for this project. The project was named Early Mathematics Intervention at Frederiksberg with the Danish abbreviation TMF [Danish: Tidlig Matematikindsats Frederiksberg].

Perceptions of what is good theory – by mathematics school teachers

Our research question is, *what is a good theory for whom - teachers as well as for researchers?* Our analyses draw on our communication with the 18 teachers involved in the design cycles of the project. In the following these teachers are called pilot teachers. All pilot teachers were chosen by their school principal as among the most qualified and motivated mathematics teachers at the school. Some also were 'Mathematics Counsellors' with a one-year diploma course.

It is the use of theoretical constructs in the four cycles, which is analysed in the following. We analyse interactions between teachers and researchers. The data come from written materials and recorded minutes and notes from seminars[1], training sessions, coaching sessions and e-mails

Generally, Danish teachers have a relatively high self-confidence and a strong wish to influence. It is shown, for instance, that Danish teachers, more than British teachers, prioritise their students' personal development and see their students' mathematical development as a means for personal development (Kelly, Pratt, Dorf & Hohmann, 2013). Because of the way the 18 pilot teachers were chosen by their principals, we anticipated that the teachers would be involved, to a high degree, in the project's four design cycles. The specific choice of mathematical and other aspects for the framework and written materials for the early intervention was actually made in dialectic processes involving the researchers and the 18 pilot teachers. Further descriptions of the cycles are found in Lindenskov & Weng (2014).

[1] Mathematics Recovery Programme (MR) is a source of inspiration for the developmental project, see Wright et al., 2007. This is why we included a teacher seminar with Ms. Noreen O'Loughlin from Mary Immaculate College, University of Limerick focusing both on some hypotheses and issues in MR and on specific concerns at Frederiksberg.

The framework and materials developed through four design cycles

(1) From January 2009 to September 2009, Weng and Lindenskov developed the first draft material, based on theory, empirical results and their knowledge about mathematics in life and in primary and lower secondary schools in Denmark. They initially doubted whether the teachers would find it relevant to study the rationales and theory behind the choice of mathematical areas, materials and evaluating and teaching principles. That is why only a few theoretical constructs and justifications were in the first draft communicated to the teachers.

But, as the structure and each part of the draft material were critically explored and discussed during the teacher training sessions from 14 - 18 September 2009, this expectation of the teachers' perceptions of their needs for theory was wrong. The teachers endorsed the underlying ideas, but actually asked for further explanation of rationales and theoretical constructs. The teachers also asked for an extensive introduction to the program as such. The time teachers were expected to use in the development processes did not include reading articles, so the researchers presented articles orally and provided printed extracts or copies of some articles as handouts. The main printed materials were newly developed diagnostic test materials and the problem solving materials to be used with their students, plus general introduction and justification for the choice of mathematical areas and instruction approach.

At the end of the week, the teachers suggested that measurement as a mathematical area and the use of measurements in other mathematical areas should be expanded in the next draft.

(2) Lindenskov and Weng developed a second draft of material based on the pilot teacher feedback and feedback from the research assistant. This meant that for more mathematics areas further justification for and explanation of rationales and theoretical constructs were included. The second draft was sent to each school October 2009 for experimenting. Each pilot teacher tried out specific parts of the material in the fall of 2009. The distribution of the parts to each school was decided through discussions among all pilot teachers. Each pilot teacher was requested to try out two or three activities with as many pupils as possible. The age of the pupils was not important. If possible, more material was to be attempted. The pilot teachers were given a specific task in order to evaluate the materials: they were asked to document in as much detail as possible - by writing in premade tables - how each mathematics task and each mathematics and attitude question led to pupil-teacher conversations which could indicate the pupil's thinking. The experiments were concluded with a seminar on 3 December 2009, where each pilot teacher presented results. Anything that had particularly surprised the teachers was also presented and some common concerns were then discussed. It was put forward that the pages with descriptions of justifications and theoretical constructs were helpful, 'or else we do not know why the chosen mathematical areas, concepts and competences are important to focus on.'

(3) Based on these results Weng and Lindenskov developed the third draft material and introduced it at a seminar on 28 January 2010. Justifications and theory were included for all mathematical areas. In the following months, each pilot teacher tried out parts of the material with a number of pupils. This time all the pupils were in the second grade. The aim was to allow the pilot teacher to experience the structure of the material and to practice pupil-teacher conversation. Peter Weng

visited and coached every teacher once and the teachers were invited to contact the researchers at any time during the pilot study.

At a midway seminar on 9 March 2010, the teachers described their general impression as positive and generally considered the material adequate. Several pilot teachers said they found it motivating to work with the material together with the pupils and that they had heard from the pupils' ordinary mathematics teachers that the intervention seemed to have a positive impact on the pupils' learning process.

The individual schools' prioritization of subjects was also discussed: how to decide between pupils' participation in a class excursion or a TMF session?

At the seminar a representative from the Egmont Foundation was present, as the Foundation had decided to fund the project. In the developmental project, the choice of pupils was left to the schools and the criteria differed between schools. The Egmont representative was particularly interested in the discussions on ethical issues: Whether pupils with very weak home support should be chosen over pupils with better support from home, who would probably benefit more? It is well known from research that socioeconomic factors are important for pupils' learning and development. Maybe particular pupils need this intervention the most, but are they really the ones chosen?

Issues regarding the scope and range of the material were discussed, for instance how to prioritise between presentations of many mathematical aspects or assuring success in fewer mathematical areas. The risk that the material put severe strain on teachers, especially when they were unfamiliar with it, was also discussed. To illustrate this discussion, we have listed three pilot teacher transcripts and one researcher transcript below:

Teacher 1: I feel pinned down by the material. I feel like, 'Now I must do this, then I must do that,' and you have to look for concrete material yourself. It is very restraining. While I look for extra material, I give the pupils small tasks on the computer to work with, OK.

Teacher 2: The material could be constraining. But the material is important as a database of ideas. The material gives me ideas. It supports my own inspiration process and it helps me to include everything in my practice.

Teacher 3: The material is useful, when I prepare the intervention sessions.

Researcher Weng:

Try to think about the material as something that provides you with opportunities and inspiration. We invite you to a flexible adaptation to specific pupils.

(According to the minutes, authors'translation)

The final seminar on 27 May 2010 discussed organisational and psychological issues in detail. The teachers wanted organisational and psychological aspects of individual pupils' learning and instruction to be emphasised as equally as the mathematical aspects.

Also the teachers again asked for more geometry and measurement in future versions, as well as a compiled list of recommended materials, but they did not mention any further need for justification and theory. (According to the minutes, authors' translation)

(4) The fourth draft was developed by 12 August 2010 and was to be used from 2010 onwards in the regular TMF for individual second grade pupils in all of Frederiksberg's public schools. The research assistant, Tina Kjær, examined the material and ensured that the teachers' suggestions were taken into account. Strongly supported by the pilot teachers' feedback, organisational and psychological aspects of individual pupils' learning and instruction were included as just as important as mathematical aspects.

Example of how researchers' theoretical understanding is communicated to teachers

As an example of how the researchers communicated their theoretical understanding underpinning the developmental project to the pilot teachers, we have chosen the mathematics area "Basic Strategies for Numbers in Addition and Subtraction". The table below shows in the left column four of the theoretical constructs chosen by the researchers to underpin the project. The right column shows how the constructs were being communicated and discussed between researchers and teachers. The right column consists of citations from the final written materials, which was published in 2013 and meant to be studied and discussed among teachers involved in intervention projects.

Researchers' choice of theoretical constructs and justifications	**Citations from the published intervention materials (Lindenskov & Weng, 2013)**
Relational understanding (RU) and instrumental understanding (IU): Although IU in its own context is often easier to understand and gives correct answers with less knowledge involved, RU is more adaptable to new tasks and easier to remember. (Skemp, 1976/2006).	When the pupil experiences a productive development in his/her basic strategies in addition and subtraction, it opens the pupil's possibilities of becoming capable in doing relevant addition and subtraction and to use it in many contexts. Also, potentially this experience will contribute to another highly relevant math competence: good estimating skills for big numbers.
Constructivist teachers' primary activity is communicating with students. In the constructivist view, teachers should continually make a conscious attempt to "see" both their own and the children's actions from the children's points of view. (Cobb & Steffe, 1983).	Some teachers might, for the last decades, have misunderstood the core of constructivism. Some teachers might have been inclined not to interfere when the pupils calculated and developed their calculation skills and strategies. Some teachers might have believed that the pupils by themselves would develop at the pace that was most optimal for them individually. But we know, it is a risky affair.

Pupils who engage in strategy development decisively perform better in the long run than pupils who do not. (Ostad, 2008).	Pupils, who from an early age, start developing his/her strategies, tend to continuously improve existing strategies and increase the number of strategies. In contrast, pupils, who stick to their strategies, tend not to start improving them until later on. It is shown that pupils, who stop developing their strategies, will toil hard and will still lag behind.
Strategies, strategy development and teaching strategies should be the core of fundamental mathematics instruction and learning. (Ostad, 2013).	Do not just present materials for the students to acquire new further learning. Let the pupil use materials and activities in order to consolidate what is almost or recently learnt as a means to improve the pupil's self-confidence and realistic self-perception of addition and subtraction skills. We recommend that the teacher talk with the pupil about his/her strategies, i.e. by regularly asking how long this strategy has been used, if the strategy leads to the right results, if the pupil uses other strategies, too, or thinks other strategies could be used. Appropriate further learning may well be about strategy development.

Conclusion

This paper has shed light on what is a good theory for whom - teachers as well as researchers, how to explain theory and justifications to the pilot teachers in a meaningful way and how to develop material in collaboration between researchers and teachers? During the four development phases the pilot teachers endorsed the underlying ideas of the intervention project and asked for the rationale behind every included aspect to be explicitly communicated. They encouraged more extensive introduction and to expand the included measurement aspects into two measurement aspects.

The teachers explicitly endorsed the theoretical construct and justifications in the material, as they said it helped them to acknowledge many opportunities to help the pupils and to identify pupils' potentials and motivation while exploring and developing their mathematical needs. The teachers appreciated that the material gave a firm frame and at the same time invited and inspired the teachers to adapt and further expand the materials to the specific learning situations with the pupils. The teachers recommended the material to be expanded with more mathematical concepts and competences, which are considered relevant in the Nordic contexts (Dalvang & Lunde, 2006; Niss & Højgaard, 2011) and by the teachers as potentially troublesome for the weaker pupils, and to expand the materials on measurements and part-whole.

The teachers asked for further ideas and materials which could be used as they were or could be adapted in order to fit their own pupils' needs and motivations. The teachers did not suggest more clarified theoretical constructs and justifications underpinning the program than were communicated to them already.

For the educational researcher the task was to find and select theoretical constructs to underpin the intervention and communicate these to the teachers, as it is further described in Lindenskov et al (2016). It could not be communicated as in scientific journals, but as justified practical advices. Both theorists and practitioners care for theory, but in very different ways.

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Characterizing theories aimed at supporting teachers' mathematical classroom practices

Per Nilsson[1], Andreas Ryve[2] and Maria Larsson[3]

[1]Örebro University, School of Science and Technology, Örebro, Sweden; per.nilsson@oru.se

[2]Mälardalen University, School of Education, Culture and Communication, Västerås, Sweden; andreas.ryve@mdh.se

[3]Mälardalen University, School of Education, Culture and Communication, Västerås, Sweden; maria.larsson@mdh.se

In this paper we draw upon examples from a recently published systematic literature review (Ryve et al., 2015) on productive classroom practice to contribute to the research on the nature of theories for action in mathematics education. By relating the results from the review to theories and literature on educational policy research, professional development research and implementation research we construct a framework for categorizing theories aiming at supporting teachers' actions in mathematical classroom practices.

Keywords: Theoretical frameworks, theories for actions, mathematical classroom practice.

Introduction

The development of frameworks and theories[1] that aim at guiding the actions of teachers have rendered much recent attention in educational research (e.g., McKenney & Reeves, 2012; Ruthven, Laborde, Leach, & Tiberghien, 2009). Within research areas such as curriculum material (Davis & Krajcik, 2005), planning and implementing whole-class discussions of cognitively demanding tasks (Smith & Stein, 2011), professional developments programs (Borko, 2004; Desimone, 2009), and improvement of mathematical instructions at scale (Cobb & Jackson, 2012) cumulative work has been conducted to establish theories for action. Further, frameworks for supporting teachers' actions and thinking such as curriculum materials, theories and models have been put forward as essential components of effective professional development (Cobb & Jackson, 2012) and for establishing productive mathematical classroom practices (Franke, Kazemi, & Battey, 2007). However, we need to know more about how theories should be designed to facilitate implementation; to be used and do real work in supporting and constructing teachers' actions (McKenney & Reeves, 2012). In this paper we relate result from a recently published systematic literature review (Ryve et al., 2015) on productive classroom practice to literature on educational policy, professional development and implementation research in order to construct a framework for understanding and facilitating the implementation of theories and research results aiming at supporting teachers' mathematical classroom practices.

Relevant research

The approach of the present study is based on the assumption that 'theory matters' for teachers, to enhance their ability to develop rich mathematical classroom practice (Charalambous & Hill, 2012).

[1] There are many ways to denote theories serving the purpose of guiding actions and this is further discussed below.

For instance, by adopting and making use of theoretical tools teachers are supposed to enhance their ability to establish productive routines in their classroom practice (Franke et al., 2007), develop and continuously adjust a learning trajectory and the means to support that trajectory (Cobb, Confrey, Lehrer, & Schauble, 2003), become more sensitive to notice instructional opportunities in the moment and be methodical without being mechanical. However, Burkhardt and Schoenfeld (2003) argue that most theories that have been applied to education are quite broad, lacking the specificity that helps teachers to guide and understand the design and analysis of learning activities. Cobb et al. (2003) adhere to this view, claiming "General philosophical orientations to educational matters – such as constructivism – are important to educational practice, but they often fail to provide detailed guidance in organizing instruction" (p. 10). So, there is this dilemma; theoretical constructs are supposed to enhance teachers' capacity to teach but, to do such work, theories need to be of a certain kind.

Perspectives on theories

diSessa and Cobb (2004) detail the nature of different theories relevant for research in mathematics education. They distinguish between grand theories, orienting frameworks, frameworks for action, domain-specific instructional theories and ontological innovations. Skinner's behaviourist theory provides an example of a grand theory. Even if grand theories have a prominent position in educational research, they appear to be too general to provide guidance for explaining and supporting the learning of mathematics. Orienting frameworks, such as constructivism (Von Glasersfeld, 1995) or communities of practice (Lave & Wenger, 1991), provide general support for specifying issues of learning, teaching and instructional design whereas frameworks for actions concern analytical constructs of a more or less general *prescriptive* character (diSessa & Cobb, 2004). Domain-specific instructional theories are also of a prescriptive nature as they are typically specific to a domain or even learning trajectory of certain content and the means by which this trajectory can be supported. An ontological innovation is descriptive in nature. It is about developing analytical categories by which aspect of a phenomenon can be discerned. The framework of Socio-mathematical norms (Yackel & Cobb, 1996) exemplifies an ontological innovation.

diSessa and Cobb's (2004) categorization not only labels the nature of different frameworks, it also points to the *descriptive*, *explanatory*, *predictive* and *prescriptive* purposes of different theories. **Firstly**, theories could be used to describe the world and many theories and frameworks within mathematics education serve such a purpose. The contribution to research in engaging in describing or characterizing objects or processes as certain phenomena could be understood in terms of new or unconventional lenses for viewing the world. **Secondly**, a further purpose of theories is to explain relations between phenomena and as mentioned above this purpose is often stressed as absolutely central for theories. A prerequisite for explaining those relations is to explicitly characterize each phenomenon. Therefore, theories used for explanatory purposes build upon or encompasses descriptive theoretical contributions. **Thirdly**, in a similar vein predictive theories necessitate explanations and clear descriptions of phenomena. Predictions include foreseeing effects of certain actions under certain conditions. **Finally**, prescriptive theories are used to identify and articulate productive ways to make decisions and performing actions. This kind of theory integrates descriptive, explanatory and predictive knowledge to guide actors in constructing and establishing

interventions. Within design research prescriptive theories are often denoted design principles but neither the term nor the nature of those design principles are settled (Ruthven et al., 2009).

Theories for action

In this paper we are particularly interested in theories for actions and what McKenney and Reeves (2012) denote the prescriptive role of theories. Both the characteristics of theories of actions and prescriptive theories and ways of denoting them are not settled in educational research as indicated above (cf. McKenney & Reeves, 2012). As becomes apparent in (Ruthven et al., 2009), the relation between the terms used to denote theories for action is not just connected to neutral ways of denoting the same phenomenon but instead accentuates particular features and characteristics of such prescriptive theories. For instance, Ruthven et al. (2009) shortly muse about the relation between the design tools they introduce and design principles. They suggest that the conceptual set up of grand theories, intermediate frameworks and design tools introduced in Ruthven et al. (2009) stresses *theoretical* underpinnings for *sensitizing* researchers to critical issues while design principles from US often *prescribe* certain course of actions and are typically more loosely anchored in theoretical perspectives. One may ask, should theories for actions prescribe and sensitize teacher? In general, the development and understanding of design principles is weakly developed and in summarizing the most urgent issues for educational design research McKenney and Reeves (2012) suggest "a worthy challenge facing educational design researchers is to further the development of predictive and prescriptive theories" (p. 212). We want to add to this research.

Method

The design was framed by a ten-step process for systematic literature reviews (Gough, Oliver, & Thomas, 2013): (1) Need, (2) Review questions, (3) Scope, (4) Search, (5) Screen, (6) Code, (7) Map, (8) Appraise, (9) Synthesize, and (10) Communicate. In the project we engaged in processes 1-7 and 10. Our *review questions* were: (a) What characterizes research on classroom teaching practices, teaching approaches and teaching methods in mathematics? (b) What characterizes research on teachers' instructional strategies used to establish classroom practices in mathematics? and (c) What does research tell about teaching for the learning of mathematical competencies?

We *searched* in title, keywords and abstract in Web of Science[2]. Search strings[3] were iteratively developed while reading some abstracts. In total, we had 622 hits that we *screened* for relevance according to our *scope* defined by our inclusion criteria[4]. The screening was made in two steps based on: (1) title, keywords, and journal name, and (2) abstracts. Uncertain cases were discussed and decided upon collectively among the three researchers. After the two screening steps, 242 articles remained potentially relevant for the scope of the review. Simultaneously as the screening of

[2] We searched Web of Science Core Collection as a way to focus on high-quality journal articles. We limited our search to the year span 2008-2014 and to the document types "article" and "review".

[3] See Appendix A in Ryve et al. (2015) for the exact search strings.

[4] For an article to be included in the review, the article must be: (1) about mathematics teaching/learning/education, (2) related to compulsory school (grade 1-9), and (3) about the teacher's role.

abstracts, the articles that remained relevant were *coded* on Object of study, Method, Number of participants, Context, Results, and Implications for practice. When needed, we also read other parts of the articles apart from the abstract. 201 articles remained relevant for the review based on the inclusion criteria. The next step of the *mapping* was to structure and characterize trends and interests within the discourse of research in mathematics education that focus on teaching methods, classroom practice and teacher's role in classroom practice. Therefore, we looked closer at the abstracts of the 201 articles, categorizing them in relation to object of study. See Ryve et al. (2015) for detailed descriptions of steps and rationales in the processes.

Results

	Student knowledge		Student attributes (SA)	Practice (P) (Teaching approaches)
	Processes	Products		
Interactional strategies (IS)	3	10	4	11
Teaching approaches (TA)	9	42	8	
Learning material (LM)	5	14	2	7
Background variables (BV)	1	16	2	34
Characterization (Char) Total: 33				

Figure 1. Categories and analytical relationships of the mapping

Of the 201 articles, we found 168 to be structured according to an analytical relationship between an outcome variable and a design variable. In the remaining 33, the object of study did not follow such a structure. The design variables we distinguished are gathered in the left column of the matrix (Figure 1), while the outcome variables we found are categorized in the top row of the matrix. In elaborating on the mapping we intend not to go through the entire mapping. We highlight and provide examples on some specific findings, which we then follow up on in the discussion. In the discussion we construct a new framework, to be used for categorizing theories to be implemented in mathematics classroom practices. On this account, and due to space limitation, we are not backing up all claims and findings by references from the mapping.

Elaborating on theories within the mapping

Student knowledge

Our review reveals a clear bias towards research that focuses students' product knowledge. Mathematical products relate to conventions, symbol systems, concepts and calculation techniques of mathematics. Looking at studies that emphasized a product view of mathematics and those emphasizing a process view we also notice a methodological difference. In the product view knowledge is expressed in the language of mathematical products and students understanding are profiled and ordered in accordance to the mathematics itself. Connected to such a conceptualization of knowledge, students' performances are often measured by standardized tests (Desimone, Smith,

& Phillips, 2013). In the process view, qualities and progressions are not explicitly elaborated on. These frameworks are descriptive in nature, specifying a set of analytical categories, which is used to sensitizing (Ruthven et al., 2009) the analyst on some certain characteristics of, for instance, students' ability to communicate and reason in and with mathematics.

Interactional strategies

Studies belonging to this category explicitly refer to teachers' moves and actions. The teacher takes an active role in these studies; how he/she acts in interaction with the students, is central to the investigation. The focus is on how teachers communicate and engage with their students, and what role the communication and engagement play in students' learning of mathematics. It could be about, for instance, how a teacher uses gestures and questions (Shein, 2012) and follow up on students' ideas in order to develop the mathematical classroom practice (Akkus, 2013).

Teaching approaches

Teaching approaches refer to studies taking a broad perspective on classroom teaching in mathematics. The teachers' actions and interactional behavior are not the main object of investigation. The teacher may be important, but it is the more general and overall structures of teaching that are the object of study. In our review we found different examples of teaching approaches, such as technology-based teaching, mathematical games, problem-based teaching, and contrasting ways of using textbooks in mathematics teaching.

Learning material (task design)

Some studies focus on how a specific artifact or design principle can support or challenge students' learning in mathematics. In these studies, the teacher takes a passive role. Focus is on the students' interaction with the learning material and the role of the teacher is basically to execute the lesson. Studies belonging to this category may investigate the role of visualization or simulation in the learning of mathematics (David & Tomaz, 2012). In this group we also include issues of task design; types and sequences of tasks (Hattikudur & Alibali, 2010) and instructions for solving mathematics tasks (Orosco, 2014).

Background variables

Several studies did not connect classroom practice and students' learning to any didactical design variable. These were studies giving accounts of personal attributes such as teachers' beliefs, attitudes and knowledge in order to explain classroom practice and students' performance.

Characterization

In 33 articles of the articles, the object of study did not follow the structure of an analytical relationship between two didactical variables. In this group of studies, to describe a certain practice or teaching approach is the focus in itself. The goal is to provide *descriptive* accounts of analytical categories of a teaching/learning phenomenon, which can be used to sensitizing researchers and teachers to critical issues of the phenomenon in question (Ruthven et al., 2009). It may concern the characterization of curriculum material (Sherin & Drake, 2009), the orchestration of math-talk with interactive whiteboards (Beauchamp, Kennewell, Tanner, & Jones, 2010), mapping the mathematics in classroom discourse (Herbel-Eisenmann & Otten, 2011), or profiling students' understanding or strategies of specific subject matter content (Wagner & Davis, 2010).

Discussion

By relating the results of our mapping to literature on educational policy, professional development and implementation research we construct a framework for understanding and facilitating the implementation of theories and research results aiming at supporting teachers' mathematical classroom practices.

In studying the papers it is apparent that teachers are ascribed different roles in different research studies. While quite a few studies within the category of learning material position teachers as administrators of tasks and computer programs other studies highlight the role of expert and orchestrator of classroom practices. Within the latter categories of articles, the role of teachers is central in asking questions, explaining content and acting formatively to support and challenge students' mathematical thinking. In understanding the implementation of theories and results aimed at improving classroom practices and students' mathematical learning it seems essential to consider how **theories construct the role of teachers in classrooms.**

Ruthven et al. (2009) notice the distinction between theories that prescribe teachers' actions and theories that aims at sensitizing teachers to essential aspects of classroom practices. In a similar vein, while some theories and studies in our review are clearly prescriptive towards teachers (e.g., theories belonging to Instructional strategies and Learning material) and what they should do in classroom practices, others aims at sensitizing and empowering teachers (theories belonging to the Characterization category). We are not normative about these different ways and suggest that both could be productive for different teachers in different context. Further, we hypothesize that these two strands are correlated with research methodologies in that many studies within mathematics education taking an educational design perspective aims at empowering teachers while studies taking a stricter experimental approach prescribe and praise clear prescriptive instructions. However, to what extent and in which ways **theories prescribe or sensitize teachers** seem relevant to consider for anybody collaborating with teachers.

Cobb and Jackson (2012) stress that tools and frameworks within educational policies play a prominent role. When it comes to designing and using tools Cobb and Jackson suggest that it is important that the tools can be used by agents immediately in that they are easy to access, but at the same time harmonize with the planned reorganization of the practices. In addition, in developing frameworks, theories and tools it is essential to consider the amount and type of learning that are required for teachers to develop in order to use them in an appropriate and reliable way. Of course, such using requires good mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). However, there is also reason to believe that certain types of frameworks are easier then other to use and apply in a mathematical teaching practice. For instance, in our survey we notice how the product perspective dominates research focusing on student knowledge. That the product perspective has a long tradition in the field is probably the major reason for this. However, taking a closer look to the mapping we can also understand this dominance as if there is a higher degree of transparency in how to use product knowledge theories compared to process-knowledge theories. From this we learn that, in implementing theories to school practice, we need to consider how user-friendly, accessible and transparent different types of theories are to teachers. In other words, we need taking into account to **what extent and which kind of teacher learning is necessary** for productively implementing theories or frameworks to mathematical classroom practices?

Desimone's (2009) put forward *coherence* as a critical feature of professional development programs. The concept of coherence refers to the relation between the PDP and teachers' knowledge and beliefs. This raises questions about the extent to which theories should be coherent with teachers' knowledge and beliefs. In other words, should theories strengthen teachers' knowledge and beliefs or should it challenge their knowledge and beliefs? Should theories aim at strengthening classroom practice or should theories aim at reorganizing classroom practices (Cobb & Jackson, 2012)? The reorganization of practice could include working with new types of mathematical problems, new roles for students and teachers, and the establishment of new classrooms. Hence, in examining and choosing theories, frameworks and models, mathematics educational researchers working with teachers should consider whether the aim is to **strengthen or reorganize ongoing practices** and, consequently, consider how frameworks are supportive for such endeavors.

To conclude, as a complement to categorize theories for actions in terms of content areas we suggest it is productive for researchers working with teachers to consider theories in terms of: the positioning of teachers in classroom practices; the positioning of the teacher as a receiver of the theory; the amount and type of teachers' learning required; and if theories primarily function to strengthen or to reorganize practices.

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Mapping the logics in practice oriented competence development

Andreas Lindenskov Tamborg[1], Benjamin Brink Allsopp[2,] Simon Skov Fougt[3] and Morten Misfeldt[4]

[1,2,4]Aalborg University, Department of Learning and Philosophy, ILD LAB, Copenhagen, Denmark
[1,3]Metropolitan University College, Department of School and Learning, Copenhagen, Denmark
[1]alt@learning.aau.dk, [2] ben@learning.aau.dk,[3]sifo@phmetropol.dk,[4]misfeldt@learning.aau.dk

In this paper we investigate the role of the local supervisor when implementing a mathematics teacher training program based on Action Learning (Misfeldt et al. 2014, Plauborg et al. 2007). Using data from interviews of teachers, local supervisors and school managers we examine the arising expectations on the local supervisor and how these expectations influence the program's ability to support teachers in their professional development. We do so by using Clarke's (2009) Situational Analysis and Arcform notation (Allsopp 2013) to map the actors' relation to the supervisor. We see that the local supervisor is caught in a tension between expectations from the Action Learning method and the school managers. This hinders schools in anchoring Action Learning as a teacher training method and thereby benefitting its full potential.

Keywords: Teacher training, Action Learning, theory and practice, Arcform.

Implementing teacher capacity building through collaboration

A crucial aspect of a number of initiatives to improve mathematics education is the ability for teachers to collaboratively question and improve their own teaching (Stigler 1998), sometimes involving resource persons such as researchers or teacher educators. For such in-service training or capacity building to be efficient and scalable it is important that they are anchored in the school organization and not solely dependent on enthusiasts. Initiatives like Lesson Studies and development of own practice (alone or in collaboration with researchers), are examples that requires systemic and organizational attention (Lewis, Perry & Murata 2006). Systematic approaches often mean that teachers take on certain roles in relation to each other's practice in order to maintain initiatives. Such structures and roles that connect in-service training to practice and build directly on the school organization and culture are important. Research has found that it is very difficult to make sustainable changes with teacher training initiatives (Shear, Gallagher & Patell, 2011; Henriksen et. al 2011), and Maurer (2010) has estimated that 70% of teacher training projects fail in changing teachers' practices within the given time frame of the project. This is a major challenge to the mainstream implementation of research findings in mathematics education. According to research literature the difficulties with changing teacher practice through in service training are associated with a lack of connection between training programs and teachers' existing practices (Fixsen et. al. 2005) as well as with the fact that teacher training programs often lack a focus on establishing active, collegial relations among teachers, which are crucial in order to build sustainable development (Hargreaves 2000, Sølberg, Bundsgaard & Højgaard 2013). When trying to address these concerns certain employees often take on certain roles towards their colleagues in the sense that they advocate for, manage and nurture certain projects. In this paper we explore this challenge by investigating a case where the distance between training activities and day-to-day teaching is very small and where the collegial relations are supported in a direct fashion. We do so by describing an Action Learning case as it has been implemented in a Danish municipality.

Context

In 2012 a municipality in Denmark launched a teachers training program aimed at developing teaching practices and enabling schools in the municipality to develop teacher competencies independently from external resources. The training program involved every school in the municipality and a total of 3500 participating teachers. The program consisted of a combination of a summer university in which teachers where given thematic lectures on a variety of subject-specific and pedagogical topics (implemented at 80% of the schools) and a method called Action Learning (implemented at approximately 20 % of the schools). Action Learning is a teacher-training program developed as an alternative to traditional course based in-service training. It draws on inspiration from Action Research in that it is based on an assumption that solutions to practical problems require practical understandings, which must be gained though iterative attempts to solve the actual problem (Plauborg et. al. 2007). The "action" in Action Learning refers to a pedagogical or didactical intervention that address *real* classroom issues as the individual teacher experiences them. While the "problem" is defined by the individual teacher the intervention is developed collaboratively by a group of teachers engaging in a so-called "team-conversation". Local supervisors from each school were designated the responsibility to facilitate professional discussions among the teachers in the team conversations. The local supervisors were teachers from the school who had a mathematics supervisor education, and who provided didactical support on a daily basis. The local supervisors also had the responsibility of anchoring the collaborations among the teachers at the schools to enable their ongoing professional development. The characteristics of Action Learning therefore seem to counter many of the challenges identified in the research literature about teacher training programs; active collegial teams are established, the team conversations are facilitated by the local supervisor in order to maintain an academic focus in the conversations and the starting point is the teachers' existing practices. In this context these initiatives however relied heavily on the local supervisors who were designated a key part in facilitating the team conversations and in anchoring team collaboration at the schools. In our research we therefore investigate the expectations arising to the local supervisors in the implementation of Action Learning and their possibilities meet these expectations (for further information about the program see the full evaluation report (Misfeldt et al. 2014).

Method

Our research draws on interviews of key actors in the training program from two schools, namely the participating teachers, local supervisors and school managers. We interviewed 6 teachers, 2 local supervisors and 2 school leaders coming from 2 different schools. Our interviews explored the actors' experiences of the training program and their understandings of the role of the local supervisor in the program. We also collected documents and literature that describes the Action Learning Method and documents from the municipality describing how Action Learning was to be carried out. All of these sources were considered with the goal of identifying how the role of the supervisor was perceived. We analyzed this data by using Clarke's Situational Analysis (Clarke 2009). Situational Analysis is rooted in Grounded Theory, but modified according to postmodern assumptions that "boundaries are open and porous; negotiations are fluid and usually ongoing" (Clarke 2009). There is no a priori assumption that human actors are of greater importance than either non-human or discursive actors (Clarke 2009), which allows us to view the Action Learning

concept as an actor in itself. In Situational Analysis, Situational Maps provide a methodological approach to organize and visualize empirical data by foregrounding situations (Clarke 2009). We initially processed our data by using a sub type of Clarke's Situational Maps: Relational maps (den Outer 2013). Like other situational maps these aim at foregrounding situations rather than individual actors or their actions by mapping all actors (human as well as non-human and discursive) that occur in any situation, but go further than this by showing relations between actors. Relational maps use a type of network notation where actors are represented by labeled nodes and relations are represented by un-labeled lines drawn between the actors/nodes.

We drew our relational maps while reading our transcribed interviews and other relevant documents. We began by listing the relevant actors that appeared in our data and their relation to each other. We produced many versions of some maps, modifying them as some actors and relations grew in prominence in our analysis. The messy nature in our data was easily overviewed with the simple structure of relational maps and thus they played an important part in opening up our data and thereby prompted our analyses. However, beyond a certain point they seemed to counteract rather than support overview. Clarke stresses that though situational maps are useful tools for beginning analyses, they are not necessarily an appropriate end-product of analysis (Clarke 2003, 563). We experienced two related problems: Firstly, it was difficult to draw some types of relations between actors and secondly they became difficult to understand/interpret, especially when returning to the analyses after several weeks. To overcome this limitation of our relational maps, we chose to visualize the situations through Arcform notation (Allsopp 2013). Like relational maps, Arcform maps do not visualize our data, but rather the results of our ongoing analysis. Arcform maps differ from most relational maps, but resemble many network notations by supporting direction and labels on relations (arcs). Thus relations like “local supervisors coach teachers” are clearly visible as an arc labeled “coach” pointing from an actor node labeled “local supervisors” to another actor node labeled “teachers”. However, Arcform also differs from most other network notations by allowing arcs to point from or to other arcs. In this way more complex relations like “teachers see local supervisors as coaches” can be drawn as shown in Figure 2.

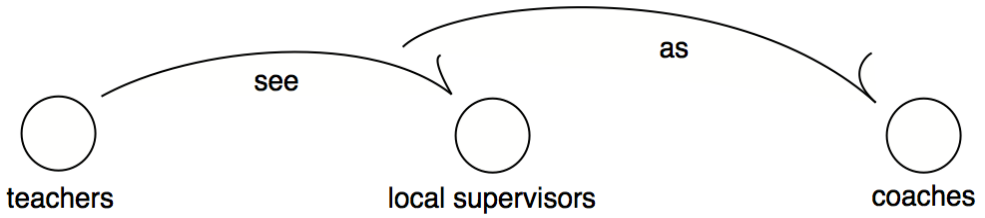

Figure 1. The sentence “Teachers see local supervisors as coaches” expressed in Arcform.

Results

As our analysis progressed it became clear that the actors in our data articulated their relation to the local supervisor quite differently, and that they had different conceptions of the main job of the supervisors in the action learning project. Though these actors were all engaged in the same project at the same school, their ways of participating and their relation to the supervisor was rather different and seemed at first glance to be related to their role in the school outside the project. Besides being a part of a project, the actors were respectively also teachers, supervisors and school managers, and this fact seemed to be of importance. Our maps also revealed that this meant that the actors had different expectations on the supervisor and that these expectations could intersect with problematic consequences. In order to refine our analysis of these preliminary results we decided to use a notion of cultural logics developed by Nielsen (2012), which we will introduce below.

In a study on teachers' learning from collaboration in teams, Nielsen develops a view of teacher collaborations as having a dynamic stability (Nielsen 2012). It is dynamic because it involves numerous ongoing activities that are oriented towards one or more objectives. It is stable because it involves a perceived regularity in actors' actions suggesting a stable understanding underlying these activities. Such logics effect peoples' objectives and can be difficult for externals to change, because they reflect the every-day phenomena which are experienced as urgent by the actors involved. For example, although teachers most likely find the learning processes of students an important objective to orient their collaboration towards, so too may they find the practicalities that make a well-settled lesson (Nielsen 2012). In situations where there are multiple cultural logics we can expect actors sometimes to be caught in a tension between these logics.

The notion of cultural logics is highly useful in our context as the Action Learning training program is a project in which several actors', who occupy diverse positions, participate. Viewed this way the role of the supervisor is at risk of being caught in a tension between multiple cultural logics. As the local supervisor is a key actor in implementing and anchoring the Action Learning method, such tensions and their implications are of particular interest in this study. We identified three dominant logics. We refer to these as the workplace logic, the curriculum logic and the project logic. The cultural logics are characterized by situations in which certain aspects of the training program are foregrounded over others which translate into a set of expectations on the local supervisor. In brief the logics translate in to the following expectations:

- In the **workplace logic** the supervisors are expected to manage the project and to avoid delays in the project.
- In the **curriculum logic** the supervisors are expected to be willing and able to guide the teachers academically in their professional development.
- In the **project logic** the supervisors are expected to initiate and support the teachers' professional development in a coaching-manner where an equal relation between supervisor and teacher is crucial.

The map below illustrates how the role of the supervisor is formed by the different cultural logics.

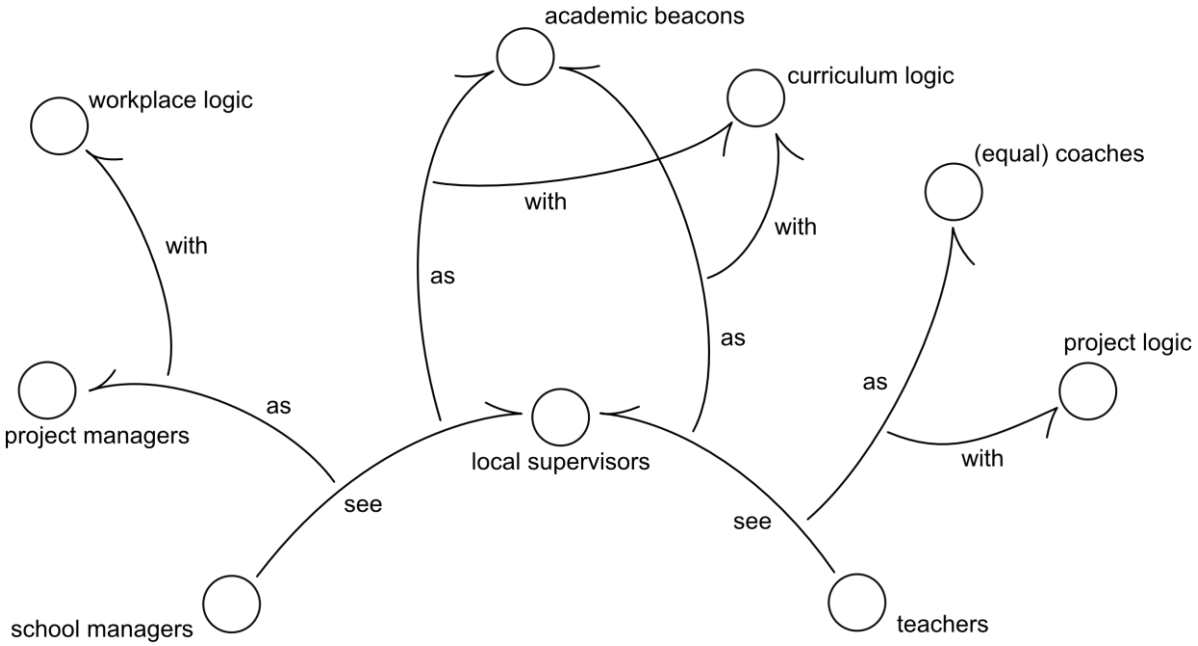

Figure 2. An Arcform map showing how two actors (school managers and teachers) see local supervisors as three different roles (project managers, academic beacons and (equal) coaches) with three different cultural logics (workplace logic, curriculum logic and project logic).

The goal in the project logic are progress and development of the school, while the curriculum logic foregrounds the quality of teaching. Though these logics can be complementary, our mapping revealed that the supervisors are faced with a tension due to a collision between the project logic and the workplace logic. In the following two sections we will therefore further describe the dynamic stabilities of the two latter cultural logics and unfold the tension emerging from here.

The project logic

The project logic concerns the cultural logic of the Action Learning training program as it occurs in documents describing the Action Learning concept and the expectations to the role of the supervisors emerging from it. In the Action Learning concept, the primary priority is the competence development of the teachers participating. In this, the supervisors are first and foremost expected to have the will to develop the school and the teachers and to do so as an equal coach rather than as a managerial authority. The supervisor is expected to initiate the Action Learning collaboration and to support the teachers in their development - not to lead/manage them. This is crucial as it is an acknowledgement that it is the teachers themselves who are experts on their own practices – the role of the local supervisor is therefore to facilitate conversations that creates the best setting for this knowledge to be shared (Plauborg et. al. 2007). The statement below from a local supervisor illustrates her view of the Action Learning project suggesting that she embraces the project logic and that she is capable of seeing the potentials in the method.

Local supervisor: (…) there were some 3rd grade teachers who said: "We have already tried this method. Why do we have to go through it again?" And my argument was that even though we have tried the method before, it is not implemented at our school. We don't use it as a method as things are now.

The statement indicates that the supervisor views competence development as ongoing and Action Learning as a way to enable such ongoing developments. She therefore argues to her colleagues that Action Learning is not a syllabus which you only have to read once and then move on – rather, Action learning is a concept that involves specific ways of collaborating which are not implemented at the school. The statement thereby demonstrates a will to develop the school that resonates with the expectations embedded in the Action Learning concept. It also tells us that the supervisor has the skill needed to spot and to articulate that the crux is to integrate Action Learning as a way of collaborating.

The workplace logic

The workplace logic concerns the main objective of the training program from the school managers' view and their expectations to the local supervisors' role in the project. From the interviews with school managers the training program appears as a project among many other projects in which the main priority is to safely navigate the school through it and to avoid any delays. Though the school managers presumably also have an interest in developing the competencies of their teacher staff, safely getting through the project appears as the dominant cultural logic. Interviews with school managers show that this logic translates into an expectation that the local supervisors will be managers of the project due to high trust of the professionalism of the supervisors. The statement below from a school manager illustrates how the supervisor is referred to through the workplace logic.

> School manager: I highly trust my supervisor's skills. Our supervisor is very professional and she is currently going through a pre-leader course. (…) and I thought that she therefore was better qualified to manage the project than I was.

In what appear as an acknowledgement of a supervisor's skills, this supervisor is given the responsibility to manage the project. At this particular school a group of teachers refused to participate in the training program due to short notice and discomfort about having to be observed as a part of the project. As the school managers had distributed the responsibility to manage the project, this became an issue for the local supervisor to handle. Consequently, the supervisor was obliged to "persuade", as she puts it, another group of teachers to participate in the project.

> Local Supervisor: I didn't lure them but… I just told them that it wasn't optional. They just had to do it, you know.

As the responsibility of managing the project was designated to the supervisor through the workplace logic, these project management issues become a task for the supervisor to handle. This implies that the supervisor is required to draw on a formal leadership mandate by reminding the teachers that participation in the project is mandatory. As the supervisor describes in the following excerpt, this incident resulted in an uncertainty among the teachers about the role of the supervisor:

> Local Supervisor: I think that this made it very unclear for the teachers what my part in this project was. Am I here to check if they are doing a bad job? Will I go to my manager and say: "That teacher does a bad job. She is really bad at teaching math". Or whatever it might be.

The supervisor's task of managing the project is not necessarily problematic in itself. But as a group of teachers refuse to participate in the training program, this is an issue that becomes a task for the supervisor to handle. In order to handle this issue, the supervisor is obliged to find another group of

teachers that are willing to participate. As no other teachers were willing to participate in the project the local supervisor was obliged to emphasize to a specific group of teachers that they were obliged to participate as this was necessary for the Action Learning project to carry on.

Implementing research findings in practice – emerging problems and prospects

Our analyses have identified three cultural logics, two of which we have unfolded above. Each of these cultural logics produces a certain set of expectation to the supervisor in terms of how he or she adequately should participate in the project. What becomes evident from our analyses is that the local supervisors are met by mutually exclusive expectations as a consequence of these logics; the project logic expects the supervisor to support the teachers as an equal peer whereas the workplace logic expects the supervisor to manage the project as a superior. This has at least two consequences. Firstly, the supervisor's delegated management role triggers an uncertainty among the teachers of the intentions of the supervisor and raises the questions of whose errands he or she is running. Is the supervisor's main task to support the teachers in their professional development or to monitor their work on behalf of the management? This uncertainty makes it difficult to draw on the supervisor as an equal facilitator. A key component in the Action Learning concept is the joint observations of each teacher's practice, which subsequently are meant to be the starting point for a conversation aiming to develop the teachers' understandings of their own practices. Such an uncertainty among the teachers in respect of the supervisor's role represents a substantial barrier in creating a safe environment in which the teachers can learn from their own practices. Secondly, the coexisting logics cause a tension on the supervisor as he or she is expected to fill many roles at the same time. Each of the cultural logics influence the actors' expectations to the role of the supervisor according to their own dynamic stability, thus tying the supervisor to different, incompatible priorities at the same time. As the potential for anchoring the Action Learning concept is closely connected to the role of the supervisor, there seem to be little chance that the supervisors are capable to do so under such difficult circumstances.

Our analysis also points to more general issues related to implementation of research findings in practice. Though Action Learning addresses what seem to be the main challenges in gaining long-term results from teacher training programs, the different expectations arising on the local supervisor complicates the implementation and the anchoring of the training program. The training program investigated in this paper exemplifies how many actors are at play in a school setting and that each participating actor may have different agendas in and around such projects. Though this perhaps is no surprise, our research suggests that difficulties in implementing research informed training programs can be the result of the differing actors' agendas outside the project. In Action Learning, as in many other training approaches, some actors are of immense importance in order to harvest the potential results of research informed approaches. The professional development of teachers involves and affects many others than the participating teachers and enters the professional lives of actors, which may have different priorities, agendas and available resources. A main problem about the issues identified in our research is that the co-existing cultural logics and the expectations arising to the supervisor thereof largely remain tacit. Though different agendas and the effects of such cannot be eliminated by simply making them explicit, an increased awareness and joint management of expectations would most likely be a step towards overcoming such hurdles.

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Rehearsals in work with in-service mathematics teachers

Anita Valenta[1] and Kjersti Wæge[2]

[1] Norwegian Centre for Mathematics Education, Norwegian University of Science and Technology (NTNU), Norway; anita.valenta@matematikksenteret.no

[2]Norwegian Centre for Mathematics Education, Norwegian University of Science and Technology (NTNU), Norway; kjersti.wage@matematikksenteret.no

An approach that has shown to give pre-service teachers rich opportunities for learning to teach mathematics is through a cycle of enactment and investigation. An important part of the cycle is rehearsal where novices rehearse their plans for enacting particular instructional activity in front of their peer pre-service teachers. The peers and the course instructor take part in the rehearsal as students, and every participant can stop the activity for discussion on different aspects in teaching. We build on the approach developed for pre-service teachers, and work on the adoption and development of the approach for work with in-service teachers in Norway. This paper reports from a pilot that was implemented with a group of in-service teachers. Our research question concerns interactions between in-service teachers and course instructors during the rehearsals and in-service teachers´ opportunities for learning in rehearsals.

Keywords: Rehearsals, in-service teachers, ambitious teaching.

Introduction

The aim of mathematics instruction is development of broad mathematical proficiency characterized by conceptual understanding, procedural fluency, adaptive reasoning, strategic competance and productive disposition to mathematics (Kilpatrick, Swafford, & Findell, 2001). This ambitious goal leads to a more demanding, and thus ambitious, conception of mathematics teaching. In this paper we aim to add to the existing knowledge base about how teacher education can support in-service mathematics teachers to learn the work of ambitious mathematics teaching.

In the Mastering Ambitious Mathematics Teaching (MAM) project we develop a course for in-service mathematics teachers in Norway. In designing our intervention, we take our lead from the Learning in, from, and for Teaching Practice (LTP) teacher education project (see Kazemi & Hubbard, 2008; Kazemi, Lampert, & Franke, 2009; Lampert et al., 2010; Lampert et al., 2013; Kazemi & Wæge, 2015). Central in the LTP-practice based approach is work with specifically designed instructional activities (IAs) in a cycle of enactment and investigation. A key innovative feature of the design is the use of public rehearsals. In a rehearsal, the pre-service teacher is responsible for teaching an IA to a group of peer pre-service teachers acting as students, with the course instructor offering guidance.

This paper reports on our work on rehearsals with *in-service* teachers in a pilot study. We ask: What characterizes the interactions between in-service teachers and course instructors during the rehearsals in the study, and in what ways might rehearsals support in-service teacher´s learning of ambitious teaching?

Rehearsals within a cycle of enactment

Ambitious teaching entails mathematical meaning making, identity building and creating equitable learning experiences for children. It requires teachers to engage deeply with children's thinking - by eliciting, observing, interpreting and responding to student reasoning, language and arguments. Attending to students' experiences and designing instruction to enable each child to do rigorous academic work in school is also a central principle of the approach (Lampert et al., 2013).

In their work on ambitious mathematics teaching, Lampert et al. (2010) build on the study of Leinhardt and Steele (2005) who identified some routines skilled teachers used in leading instructional dialogues and argued that expressing the routines explicit make them teachable for course instructors. Lampert et al. (2010) use the notion "routines" to denote well-developed practices which have shown useful in teaching, which respect the complexity in mathematics, mathematics teaching and learning. They argue that focus on learning to use these routines/practices can provide novices an opportunity to hold something constant in a process of further learning to teach. The teaching practices that are central in ambitious teaching include aiming toward a mathematical goal, eliciting and responding to students' mathematical ideas, orienting students to each other's ideas, setting and maintaining expectations for student performance, positioning students competently, assessing students' understanding, and using mathematical representations (Kazemi et al., 2009; see also Hunter & Anthony, 2012). Teachers who are novices in teaching mathematics ambitiously need to learn to enact the practices in their teaching. They also need to develop the mathematical knowledge needed to teach ambitiously at a particular grade.

Grossmann, Hammerness and McDonald (2009) argue for incorporation of "pedagogies of enactment" and use of "approximations of practice" in teacher education in order to help pre-service teachers develop knowledge, skills and professional identities as teachers. As a type of approximation to practice, Grossmann, Compton at al. (2009) suggest use of rehearsals where novices rehearse a particular instructional activity in front of a group of peers. Kazemi, Lampert and Franke (2009) develop instructional activities (IAs) that are designed to be "containers" for the practices, principles and mathematical knowledge that novice teachers need to learn and be able to use in interaction with students (see Kazemi & Wæge (2015) for descriptions of the IAs). The structure of the IAs offers the novices a scaffold in eliciting and responding to student thinking and understanding. The novice teachers learn to teach IAs – through repeated investigation, discussion, rehearsal, enactment and observation. Each cycle of enactment and investigation consists of six stages (Lampert et al., 2013), as illustrated in Figure 1:

1) Observation of a teacher educator´s enactment of a particular IA
2) Collective analysis of the observed enactment
3) Preparation to teach the IA to a group of students
4) Rehearsal of the IA by novice teachers
5) Enactment of the IA with a group of students
6) Guided collective analysis of the enactment

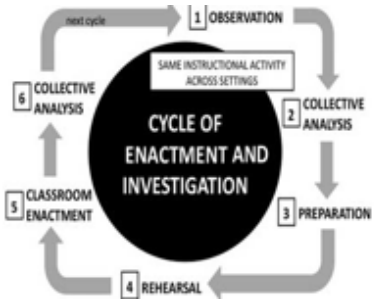

Figure 1: Cycle of enactment and investigation

In stage four of the cycle, selected novice teachers publicly rehearse their plans for enacting an IA in front of their peers and with feedback from the course instructor. During the rehearsal, the course instructor or a peer may stop action to ask questions or suggest possible alternative courses of action. The course instructor may also act as a student, by asking and answering questions or by making errors that students are likely to make (Lampert et al., 2013). Rehearsals within repeated cycles of enactments and investigation can be considered as an approximation of ambitious teaching.

Lampert et al. (2013) argue that a rehearsal is an important setting for building novices motivation and commitment to teach ambitiously (p. 239-240). They analyzed 90 rehearsals of IAs by pre-service teachers. The study revealed that rehearsals not only allow pre-service teachers to work on routine aspects of ambitious teaching, but also to attend to more complex aspects of it. The study also showed that rehearsals give the pre-service teachers an opportunity to learn the principles of ambitious teaching while the course instructor guide their progress.

Design of course

The course consisted of seven sessions (each four hours in length) during a period of four months. The sessions were held in a fifth grade classroom of an elementary school[1].

Session 1: The in-service teachers (ISTs) were introduced to the principles and practices central to ambitious teaching and the instructional activities they would work on during the course.

Session 2-6: In these sessions ISTs were divided into three teams of 4-5, and the teams worked together in planning, rehearsing, enacting, and debriefing course tasks: 1) Teams of ISTs came to class prepared to teach an IA; 2) Teams of ISTs rehearsed the IA under supervision of a course instructor (CI); 3) ISTs observed one of the CIs teach the subsequent session's focal IA to the whole group of fifth graders. This was part of the preparation for the following session; 4) One IST from each team taught a small group of fifth graders the IA that they had come to class prepared to teach. A CI also observed the enactments; 5) After a break, ISTs met in their teams to do a collective analysis of the day´s enactment with their CI; 6) Each team debriefed what they had learned; 7) The CIs prepared the class for the following session's focal IA and, as part of that, shared some reflective comments on the whole group lesson that was taught.

Session 7: The last session was devoted to concluding discussions and try outs.

Method

Participants and data

There were 14 in-service mathematics teachers from three different elementary schools participating in the pilot study. The three schools are partner schools of the Norwegian Centre for Mathematics Education. Some of the ISTs in the study had only a few years of experience as mathematics teachers, while others were experienced teachers. A group of six course instructors (including both authors)

[1] Due to practical reasons we had to make some changes to the cycles of enactment and investigation proposed by Lampert et al. (2013) and illustrated in Figure 1.

from the Centre participated in the study. The course instructors had little experience in leading rehearsals.

Rehearsals were carried out in three teams at five of the sessions. All rehearsals were videotaped, but two of the recordings were damaged. Our data is therefore consisting of 13 recordings. Each recording is about 25 minutes.

Coding and data analysis

A rehearsal consists of parts where an IST is teaching the activity, and parts where IST(s) and CI interact. We denote the interactions between IST(s) and CI during the rehearsal as *CI/IST exchanges*. To understand what characterizes the CI/IST exchanges during rehearsals, we take a CI/IST exchange as the unit of analysis and we analyze: 1) the substance of exchanges between CIs and ISTs, and 2) the structure of exchanges between teacher CIs and ISTs. In our analysis we used a priori codes adopted from Lampert et al. (2013). Table 1 shows a list of the substance codes and Table 2 shows a list of the structure codes that we built our analysis on. We used Studiocode video-analysis software which allowed for detailed coding of the rehearsals. For each rehearsal, we created a timeline for each video-recorded to capture the substance and structure of exchanges. Coding the video directly allowed for both verbal and visual cues to be considered, such as written representation, gesturing, and movements.

Results

Substance of CI/IST exchanges

In Table 1 we present an overview of frequency of the various substance codes in all CI/IST exchanges in the data[2]. The most frequent codes in our data are *representation*, *student thinking*, *content goals* and *elicit and respond.* These codes were also among the most frequent in the rehearsals by pre-service teachers analyzed by Lampert et. al (2013).

Substance codes	Description	% of all exchanges
assessing understand.	Assessing what a student knows and understands about the mathematics	16,1
attending to IA	Drawing attention to the structural aspects of the IA, particularly to help novice teachers´ understanding the entire IA	23,1
body/voice use	Attending to how one uses body and voice while teaching	0
closing the IA	Bringing the IA to an end	3,5
content goals	Attending to the specific mathematical content goals of the lesson	31,5
elicit and respond	Eliciting, interpreting, responding to student mathematical work or talk	31,5
launching the IA	Introducing and beginning student engagement with the IA	5,6
manage space	Attending to issues of classroom space while engaging students	0,7
manage timing	Moving through the lesson in a way that manages timing and pacing	3,5

[2] Due to the complexity of CI/IST exchanges, an exchange can be coded by several substance codes. As a consequence, the percentages do not sum to 100%.

mathematics	Working on and understanding the mathematical content, particularly for IST learning	22,4
orienting students	Orienting students toward each other´s mathematical ideas	7,0
process goals	Attending to the specific mathematical process goals of the lesson	16,1
representation	Representing mathematical ideas in writing and making connections between talk and representation	39,2
student engagement	Managing the intellectual and behavioral engagement of students	12,6
student error	Surfacing and responding to student errors	4,9
student thinking	Attending to the details of student mathematical thinking	32,9

Table 1: Substance codes: description and frequency as percentage of all CI/IST exchanges in the data

Many of the CI/IST exchanges involved more than one substance code, and the same combination of substance codes were frequently found together across different exchanges. For example, *student thinking*, *elicit and respond* and *representation* appear repeatedly in the same exchange. The combination of *content goal* and *representation* is also very common, in many cases together with *mathematics*. The frequency and the combination of the codes indicate that the main substance in CI/IST exchanges consist of

1) attending, representation, eliciting and responding to student thinking
2) content goals and representation of mathematical ideas in the activity

The following example is representative of the first category above:

Example 1. The IST who is teaching during the rehearsal shows the image of three groups of eight dots and asks the "students" how they see it. One of the other ISTs in the team suggest an answer.

IST2: I see eight times three. In the first group I saw four plus four, eight. I have eight three times.

IST: [Circles three groups of eight. See Figure 2.] So first you have one times eight, so one times eight, so one times eight. [Writes 1x8+1x8+1x8=3x8.] Some other suggestions?

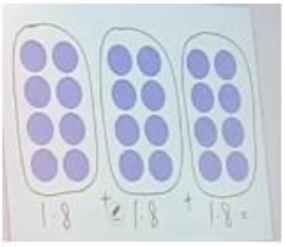

Figure 2: Drawing on the whiteboard

CI: Can we stop for a moment? Hmm, this is not so easy. The student presents her thinking rather imprecisely, and now we need to illustrate it on the image and also write it symbolically. We lose the part about seeing eight as four plus four in the way you represent her thinking. Can we try to represent her idea more accurately?

IST: I can circle four and four…

IST3: But just in the first group. She said that she saw it in the first group and then just multiplied by three. And it is not clear whether she thinks eight times three or three times eight, she says both.[3]

IST: Yes, right. I tried to make "eight times three or three times eight" clearer by leading to one times eight, and so on. Because it is three times eight.

CI: Maybe you can ask the student how she would represent it? Or, if you find her explanation too vague, you can ask her to say more?

The IST2 (who plays the role of the student) says that she sees eight as four plus four and she says "eight times three". Later she says "eight three times" which is more in accordance with the image and the convention. The IST takes no notice of the first parts of the utterance, and he grabs hold of the last part which is more in line with his goal. The CI´s first question is about *attending to* and *representation* of student thinking. The IST simplifies and changes the student´s contribution through the visual and symbolic representation. Further on, he asks for other suggestions and thus indicates that the discussion is finished. The CI´s second utterance explicitly addresses *eliciting and responding*.

The combination of *content goals* and *representation* of mathematical ideas in the activity appear also often within exchanges in our data, and the combination is illustrated in the following example:

Example 2. The IST´s goal is to use a string of problems to discuss multiplication by ten, hundred and thousand with the students. He starts by four times three, and asks the students for a story that would fit the arithmetic problem. A student (one of the ISTs) suggests four groups of three apples, and the IST draws the illustration as shown in Figure 3.



Figure 3: Illustration on the blackboard

CI: Are you planning to use money in the discussion? Your illustration reminds me of money.

IST: Yes, I have been thinking about it. Money can be useful here, when we discuss multiplication by ten, hundred. One can use tenths and talk about 12 tenths in the next step. Same with hundreds. But, another story came up.

CI: As a teacher, you have decided what the content goal is, and you have been thinking about what representation would be appropriate. You can ask about a story *with* money from the start to get the representation you want in the discussion.

These kinds of exchanges, where *representation* and the *content goal* are combined, appear frequently in the data. In Example 2, the main substance discussed is the type of representation that could be appropriate for a given content goal and how to introduce it. In some other exchanges in the data that are coded with these two codes, the discussion is on ways to represent mathematical ideas so that the representation emphasizes the the relations that are targeted in the activity.

[3] In Norway, when the multiplication is interpreted as equivalent groups, the meaning of "eight times three" is eight groups of three.

Structure of CI/IST Exchanges

The structure codes used in our analyses are the same as those used by Lampert et. al. (2013). Table 2 shows the description and frequency of the various codes in our data. Similarly as with substance codes, an exchange can be coded using several structure codes. For instance, In Example 2 above, the exchange starts by "CI facilitates discussion" and develops to "CI gives directive feedback".

Structure-codes	Description	**% of all exchanges**
CI facilitates discussion	CI lead a discussion and reflection raised by CI or ISTs	50,4
CI gives directive feedback	CI suggests new move or think aloud about possible next move	35,7
CI gives evaluative feedback	CI make evaluative comment	7,7
CI scaffolds enactment	CI takes the role of teacher or student, scaffolding the enactment by either increasing or reducing the complexity	23,1

Table 2: Structure codes: description and frequency as percentage of all CI/IST exchanges in the data

Considering the structure of the rehearsals, the analysis shows that half of all CI/IST exchanges in our data can be characterized as *discussions facilitated by CI* (at least partly, in cases where several structure codes are used in the same exchange). In rehearsals analyzed by Lampert et. al. (2013), the code "CI facilitates discussion" is the least frequently appearing code, whereas "directive feedback" is the most frequent. This indicates that the structures of the rehearsals in the two studies are quite different. One reason can be that our study concerns in-service teachers while Lampert et al. (2013) report from their work with pre-service teachers. In-service teachers have more experience with teaching than pre-service teachers, and it is reasonable to expect that their skills in teaching and their identity as mathematics teachers are more developed. A consequence can be that both in-service teachers and course instructors working with them might feel more comfortable in discussions than with CI giving directive/evaluative feedback or scaffolding enactment.

Discussion

Further work with in-service teachers and further data collection will take place in the coming year, and the results presented here are preliminary. However, the pilot study has already yielded a number of valuable insights. The study shows that the interactions between ISTs and CIs during rehearsals are mainly in form of discussions on some central principles and practices of ambitions mathematics teaching - using mathematical representations, aiming toward a mathematical goal, attending to student thinking and eliciting and responding to students' mathematical ideas. More specifically, we have found that multiple substance and structure codes are present within individual rehearsal exchanges, indicating that rehearsals offer in-service teachers the environment and opportunity to work simultaneously on a variety of aspects of practice.

The study of Lampert et. al (2013) shows that rehearsals give opportunities for pre-service teachers to learn to enact principles, practices and knowledge entailed in ambitious teaching. The findings in our study indicate that rehearsals function as an approximation of ambitious teaching in work with in-service teachers too, even though the structure of rehearsals is different.

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Incorporating mathematical problem solving in the Chilean school curriculum

Eugenio Chandia[1] and Maria Luz Montes[2]

[1]Universidad de Chile, CIAE, Santiago, Chile; eugenio.chandia@ciae.uchile.cl

2SIP, Santiago, Chile; lulu.monteslira@gmail.com

Problem solving is one of the skills that is expected to develop in children who learn mathematics. To do this, school mathematics curriculum worldwide has incorporated this skill for teachers to promote it in classrooms. Though research results show what should be made, these have been parceled and make difficult an effective incorporation of the skill, its promotion and students' learning in schools. Therefore, the following work presents a professional development strategy that incorporate diverse research results related to the promotion of problem solving in math class. This allows to achieve a professional development complex system whose main axis is the school teacher.

Keywords: Problem solving, math curriculum, professional development.

Introduction

In Chile, the General Law of Education states that "*students develop knowledge, skills and attitudes that allow them to understand and use basic mathematical concepts and procedures, related to numbers and geometric forms, in everyday problems solving, and to value the contribution of mathematics to understand and act in the world*", (article 29, section f). However, in the Curricular Study Programs of Chilean Mathematical Education there is no concrete proposal for teachers to promote such skills and attitudes in the classroom. Then, this poster shows one way to develop and include the problem solving in the Chilean Curriculum National of Math Education. To do this, is necessary to know what is needed for a successful curricular proposal implementation in math. The NCTM (2014) suggests the following points to implement a successful curricular development proposal: a) Teachers need to have a professional development of high quality to maximize the efficacy of the materials and activities they propose in the curriculum, since even the best textbooks and resources can be misinterpreted or misused. b) Collaboration among teachers throughout the school year may result in appropriate adjustments and activities adaptations for dosing topics to address the strengths and weaknesses of each student. For this reason, along with thinking about the curricular development proposal, a professional development proposal that supports teachers to incorporate in a better form the new requirements that are demanded need to be thought. On the other hand, Marrongelle, Sztajn and Smith (2013) indicate that the characteristics that make more effective a professional development program are: to be performed regularly and connected to practice; to be focused on students learning; to be directed to the teaching of a content; to be aligned with objectives of the school; and to build strong links among teachers. Considering the perspective mentioned above, in the following parts will be described the process for the implementation and the incorporation the PS in the curriculum.

Methodology

The project aims to impact the: instructional practices of participant teachers related to the delivery of opportunities to develop skills in the students, focused in problem solving; teachers' skills to solve problems; teachers' skills to modify and adjust activities to promote the development of skills in their students; students' skills to solve problems; students' skills to represent, report, discuss, argue and explain mathematically. For this purpose, a proposal for curricular development that was adapted to the reality of the Chilean school system in Mathematics Education was designed. The project includes two main stages. In the first one a proposal for curricular development based on a literature review was designed, based on the Chilean curricular framework. The proposal included classroom activities for students, instructional methodologies for teachers and an annual curriculum integration plan for each participating school. In the second one, there was implemented a pedagogical technical support model in order to teachers appropriate the activities and methodology to be performed later in the classroom. The curricular development proposal will be implemented in 4 schools, which compromised their participation of teachers and students from 1st to 4th grade and the time necessary to do so. It had a population of 24 teachers and 600 students from 1st to 4to grade in total.

Discussion and conclusions

The project is still being conducted, but there are preliminary results on its implementation. In the stage 1 of the proposal design was observed that while the curricular proposals from countries such as Singapore or Finland have among their curricular objectives to promotion of mathematical skills as the problems solving, the curricular resources available for teachers in those countries do not guarantee by their own, the skill promotion and not the acquisition of teachers' instructional practices. Again, research results corroborate the lack of professional development strategies that allow the skills acquisition by teachers (Bunyi, 2013). Given this lack and the need of the Chilean teachers in terms of mathematical knowledge, as well as in pedagogical skills to promote mathematical skills and attitudes towards it in classrooms, it is that the strategy of weekly and monthly support was crucial, both for the acquisition of pedagogical skills and mathematical knowledge.

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TWG24: Representations in mathematics teaching and learning

Introduction to the papers of TWG24: Representations in mathematics teaching and learning

Elisabetta Robotti[1], Anna Baccaglini-Frank[2], Gérard Sensevy[3], Torsten Fritzlar[4]

[1] University of Torino, Department of Philosophy and Science of Education, elisabetta.robotti@unito.it

[2] University of Pisa, Department of Mathematics

[3] University of Western Brittany, School of Education

[4] Martin Luther University Halle-Wittenberg, Faculty of Education

TWG24 made its first appearance as a new Thematic Working Group at CERME10, focusing on representations of mathematical concepts or mathematical objects because of their constituting an "integral part of the doing of mathematics" (Presmeg, 2002) and thus an important part of teaching and learning mathematics. Indeed, representation has been a crucial topic in research, for instance, in PME groups, in a special issue of ESM, in a special issue of ZDM, in ICME 13 in 2016. In the group's "Call for papers" the term representations referred to thinking tools for doing mathematics encompassing graphs, tables, diagrams, formulas, symbols, texts, concrete models, and, in a broader sense, even gestures, videos, sounds etc.

Keywords: Representation, visualization, imagine, visual-spatial abilities, visual-spatial image.

Introduction

This Thematic Working Group explicitly welcomed papers from a variety of different theoretical approaches and methodological frameworks addressing the role of representations of different types in teaching and learning processes, in particular those involving visualization (considered here as defined by Arcavi (2003)). In TWG24 there were 24 participants (authors, co-authors, and some other participants), from 13 countries (these included Chile, Denmark, Finland, France, Germany, Italy, Mexico, The Netherlands, Portugal, Sweden, Switzerland, Turkey, the UK) with 16 accepted papers and 2 accepted posters. The most part of the 16 papers, were empirical studies (related to primary and secondary school). The 2 posters reporting empirical studies conducted at the primary and secondary school levels. The poster concerning primary school described what students learn in mathematics lessons when different representations of fraction are used; and the poster concerning secondary school described how a variety of multi-sensory activities allowed 14 year old students to familiarise with some pivotal mathematical concepts such as prime and irrational numbers. The structure of the timeslots was designed in order to stimulate interaction and collaboration among participants: all participants were asked to read all papers, and prepare reaction-questions to two papers in particular that had been assigned ahead of time by the TWG leaders. After a 10-minute presentation by the presenting author, the prepared questions were posed and a general discussion was initiated and conducted for 25 minutes: first the authors of the paper would reply to the reaction-questions, then there was a discussion on issues related to the general list of questions designed for TWG24's call for papers. Posters were also allocated a few minutes of presentation time within the working group, and a short follow-up discussion took place after each of them. The last session was completely devoted to summing up the main issues that had emerged from the

group discussions. One of these was that certain key words, present in the literature on representations and visualization in mathematics education, were not being used consistently by the participants. Therefore a list was put together with the suggestion for the upcoming CERME of making explicit the definitions used in each study. Among these (in alphabetical order): figure, gesture, mental imagery, metaphor, representation (including the distinction between internal and external), sign, symbol, visualization, visual-spatial abilities, visual-spatial image.

Gestures and representations

The group agreed on the following: gestures can be considered as a way to create temporary external visualizations of internal imagery or structures, to explain or communicate thinking; movement involved in the gesture can connect physical properties and theoretical properties; different kinds of artifacts affording (or fostering) the use of gestures can be involved (such as the movement within dynamic geometry software). The importance of gestures in the context of representations in mathematics education was evident in TWG24, because many of the papers presented included a focus on gestures. Okumus and Hollebrands investigated how middle school students created 3–dimensional objects from 2–dimensional figures using an extrusion method, and they identified students' strategies for forming 3–dimensional objects with a focus on their gestural signs. The paper by Joffredo-Le Brun, Morellato, Sensevy, and Quilio focused on the relation between gestures and (other kinds of) representations (and metaphors), through the analysis of an extract from a lesson proposed in primary school during which the students work on the notion of difference, introduced with the help of several systems of representation. Ferrara and Ferrari also considered the relation between gestures and (other kinds of) representations, presenting the diagrammatic activity of secondary school students exploring motion through graphing technology, which captures a pair of space-time graphs on a single Cartesian plane. Indeed, the use of computers and technology was another transversal theme present in many papers and group discussions.

Technology and representations

TWG24 discussed the issue of how technology can change the dynamics of teaching-learning by offering specific kinds of representations. The paper by Okumus and Hollebrands presented findings from a study conducted during a summer enrichment program, in which students used manipulatives and a dynamic geometry program (Cabri 3D). Miragliotta and Baccaglini-Frank presented analyses of excerpts from a set of activities designed and proposed within the context of a 2D dynamic geometry software (Geogebra) for a group of 9th grade students. Schreiber and Klose focused on the role of artifacts and different forms and modes of representation when learning mathematics at primary school level, through an interactive approach, in which mathematical audio-podcasts were produced. A perspective on teachers'competencies in the context of multimedia-based representations was presented by Ollesch, Grünig, Dörfler and Vogel. Their study described findings from a project in which they used video-vignettes in order to assess the competencies of mathematics teachers for multimedia use in mathematics lessons. Taking a closer look into how technology can change the dynamics of teaching-learning by offering specific kinds of representations, a study by Garcia Moreno-Esteva, White, Wood, and Black showed how eye movement can be tracked and used as a window to cognitive processes involved with use of representations in mathematical activities.

Theoretical frameworks used in the papers and posters presented

Several different theoretical frameworks were referred to in the papers and posters presented: Arzarello's Semiotic Bundle theory (Bini; Robotti); Balacheff's theoretical notion of epistemological validity (Hoyos); Bartolini Bussi and Mariotti's Theory of Semiotic Mediation (Okumus and Hollebrands; Robotti; Schou; Schreiber and Klose); cognitive psychological approaches, applied in the problem solving context, such as Bayes' (Böcherer-Linder and Andreas Eichler); or Vergnaud's framework, (Serrazina and Rodrigues); Duval's registers of representation and theory of apprehension (Miragliotta and Baccaglini-Frank; Robotti; Hoyos, Bini); Enactivism (Ferrara and Ferrari; Soto-Andrade and Diaz-Rojas); Fischbein's Theory of Figural Concepts (Miragliotta and Baccaglini-Frank); Goldin's definition of representation (Sveider); the Joint Action Theory in Didactics (JATD) (Joffredo-Le Brun, Morellato, Sensevy and Quilio); Lakoff and Núñez's conceptual metaphors (Finesilver); Mishra & Koehler's Technological Pedagogical and Content Knowledge (TPACK) (Ollesch, Grünig, Dörfler and Vogel); psychological approaches such as Bruner's approach (Ott; Finesilver); or Ainsworth's approach (Böcherer-Linder and Eichler; Ollesch, Grünig, Dörfler and Vogel); Krutetskii's approach (Olgun and Ader); Tall and Vinner's Concept Image and Concept Definition (Schou);

According to these, the authors developed different kinds of empirical studies: intervention studies (short term and long term studies; with attention to the teacher's role or focused on learners); and observation studies (observing learners in different educational settings; observing teachers; observing classroom processes). In one case, a paper attempted to make some steps forward in elaborating a new theoretical framework emerging at the intersection between cognitive psychology and mathematics education (Miragliotta and Baccaglini-Frank). In another paper, Ferrara and Ferrari conceive mathematical thinking as a place of events instead of objects, and they bring forth inventive and speculative possibilities for learners to encounter and problematize spatio-temporal relationships, rather than seeing them as ways of being mistaken.

Concluding remarks

We conclude this summary with the two questions, from the general list, that seemed to arise the greatest interest of the participants, and sketch out the main comments advanced by the Working Group.

What aspects of the use of different types of representation, imagery and visualization are effective in mathematical problem solving at various levels?

Participants of TWG24 suggested that a representation does not stand alone, and it cannot be separated from how it is used. Thus, it is important to take into account interaction between the individual and the representation (both its external as well as its internal – though difficult to access – component) and between representations and context in which they are used (Joffredo-Le Brun, Hoyos, Schou). Moreover, representations are used within a social context, partly (but not only), for communication of ideas; it is important to encourage learners to express themselves using their own representational strategies, and appreciate multiple representations of information and of their ideas (Finesilver; Olgun and Ader; Robotti; Okumus). Through a careful and appropriate use of representations it is possible to increase positive affect towards mathematics and inclusion (Soto-Andrade and Diaz-Rojas; Robotti). However, there is a tension between the advantages of flexible

representation (and specific useful reps) and pushing students to use representations, which do not come naturally to them (Finesilver).

How can teachers help learners to make connections between visual and symbolic representations of the same mathematical notions (mathematical object)?

In response to this question participants of TWG24 suggested that there are certain registers of signs that are considered conventional (by teachers), and others which are less conventional. Indeed, teachers may be less familiar with the various alternative ways of representing, and either not accept alternatives as legitimate (e.g. drawing), or not be conscious of how they are being used (e.g. gestures) (e.g.: Bini; Olgun and Ader; Ollesch, Gruenig, Doerfler and Vogel; Schou). Finally, in various occasions, the group discussed the issue of low achievers and use of representations both by them and by teachers involved in their education processes. These discussions were fueled especially by the papers by Finesilver and by Robotti. In her paper Finesilver drew on qualitative data from problem-solving interviews with very low-attaining secondary school students, focusing on the visuospatial organization of elements in four types of non-standard student-created and co-created representations. She discussed these four types of representations in terms of relationships between representation type, scenario, calculation success, and the students' developing understanding of multiplication and division concepts. On the other hand, Robotti presented a didactical sequence involving the use of various artifacts, introduced by the teacher, to solve tasks on fractions. She analyzed how the representations, fostered by the artifacts, produced by the students, and then picked up by the teacher, contributed to students' development of mathematical meanings around the notion of fraction.

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Representing subset relations with tree diagrams or unit squares?

Katharina Böcherer-Linder[1], Andreas Eichler[2] and Markus Vogel[3]

[1] University of Education Freiburg, Germany; katharina.boechererlinder@ph-freiburg.de

[2] University of Kassel, Germany; eichler@mathematik.uni-kassel.de

[3] University of Education Heidelberg; vogel@ph-heidelberg.de

In this paper, we refer to the efficiency of different visualizations for mathematical problem solving. Particularly, we investigate how set relations that are potentially important in probability are made transparent by two different visualizations, i.e. the tree diagram and the unit square. In this paper, we use these two visualizations as representations of statistical information. First, we analyze theoretically the quality of visualizing set relations by tree diagrams and unit squares. Second, we briefly report a published study with students of mathematics education (n = 148) where the unit square outperformed the tree diagram when the perception of subset relations was regarded. A main focus of this paper is a replication of the aforementioned study with n = 58 undergraduate students. Finally, we discuss the significance of our results, specifically for the teaching and learning of conditional probabilities.

Keywords: Representation of statistical information, set relations, tree diagram, unit square.

Introduction

In mathematics education, it is widely accepted that representations and visualizations could have a considerable impact on students' learning. For example, Duval claims that visualizations of mathematical concepts are "at the core of understanding in mathematics" (Duval, 2002, p. 312). However, research in mathematics education and cognitive psychology gave evidence that visualization does not necessarily foster students' understanding. For this reason, a crucial question in research in mathematics education is to identify the one of potentially different visualizations that is most efficient referring to students' learning. We refer to this question concerning competing visualizations aiming to support students' learning in statistics and probability, but also in fractions. The transparency of set relations plays a crucial role in probability (Böcherer-Linder & Eichler, 2017), but is also important in many other domains of mathematics education, such as the teaching and learning of fractions.

Theoretical framework

In the context of Bayesian reasoning, research in cognitive psychology has shown that the transparency of set relations in the visualization of statistical information impacts on the performance in tasks concerning the Bayes' rule: "any manipulation that increases the transparency of the nested-sets relation should increase correct responding" (Sloman, Over, Slovak, & Stibel, 2003, p. 302).

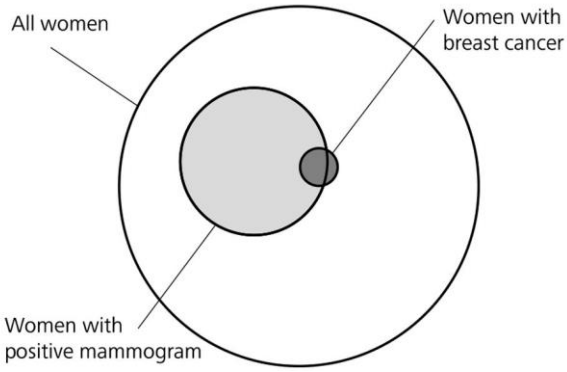


Figure 1: Nested sets of the Bayesian situation of a medical diagnosis visualized by an Euler diagram (cf. Binder, Krauss & Bruckmaier, 2015).

Proponents of this point of view, called *nested-sets account*, attribute the difficulties of Bayesian reasoning to the fact that some sets of events are nested (Lesage, Navarrete, & Neys, 2013; Sloman et al., 2003). For the example of a medical diagnosis, Figure 1 illustrates the nested-sets situation. "Transparency of set relations" means that it is easy to see, how many elements are in the sets and how the sets relate. The cognitive model into which the *nested-sets account* has been incorporated is the dual process theory (Barbey & Sloman, 2007): Representing the statistical information in a standard probability format (without visualization) obscures the nested-sets structure of the problem and, therefore, triggers the associative system which may lead to biases. Representing the statistical information with natural frequencies and / or appropriate visualizations in contrast trigger the rule-based system because nested sets relations are made transparent, enabling people to reason consciously and according to the logic of set inclusion (Barbey & Sloman, 2007). For the design of effective visualizations, proponents of the *nested-sets account* claim that visualizations are helpful to the extent that they make the nested set structure of the problem transparent (Barbey & Sloman, 2007; Sloman et al., 2003). There are different competing visualizations that claim to visualize efficiently set relations or situations that necessitates applying Bayes' rule and it is an open question which visualizations are the most efficient and which properties explain these visualizations' efficiencies.

In this paper, we investigate how set relations are made transparent by two competing different visualizations, i.e. the tree diagram and the unit square. We use these two visualizations as representations of statistical information (Venn diagrams are not considered in this paper because we focus on the visualization of statistical information and Venn diagrams are pure set representations but not representations of statistical information). First, we analyze theoretically the quality of visualizing set relations by tree diagrams and unit squares. Second, we briefly refer to a published study (Böcherer-Linder & Eichler, 2017; n = 148 undergraduate students) where we investigate whether the tree diagram or the unit square is more efficient to support the perception of subset relations. Since we used a new approach to explain the effectiveness of visualizations of the Bayes' rule, we conducted a replication study (n = 58 undergraduate students) which results are in the main focus of this paper. Finally, we discuss the significance of our results, specifically for the teaching and learning of conditional probabilities.

Visualizing set relations

"A flower girl is selling red and white roses and carnations." We use this situation as an example to illustrate how the tree diagram and the unit square visualize set relations. In this situation, we have sets (for example the set of all roses) and subsets (for example the subset of all red roses) and subset relations (for example the red flowers among the roses). If we attribute some numerical values to the number of roses and carnations, the situation can be visualized by showing absolute numbers in the tree diagram and the unit square:

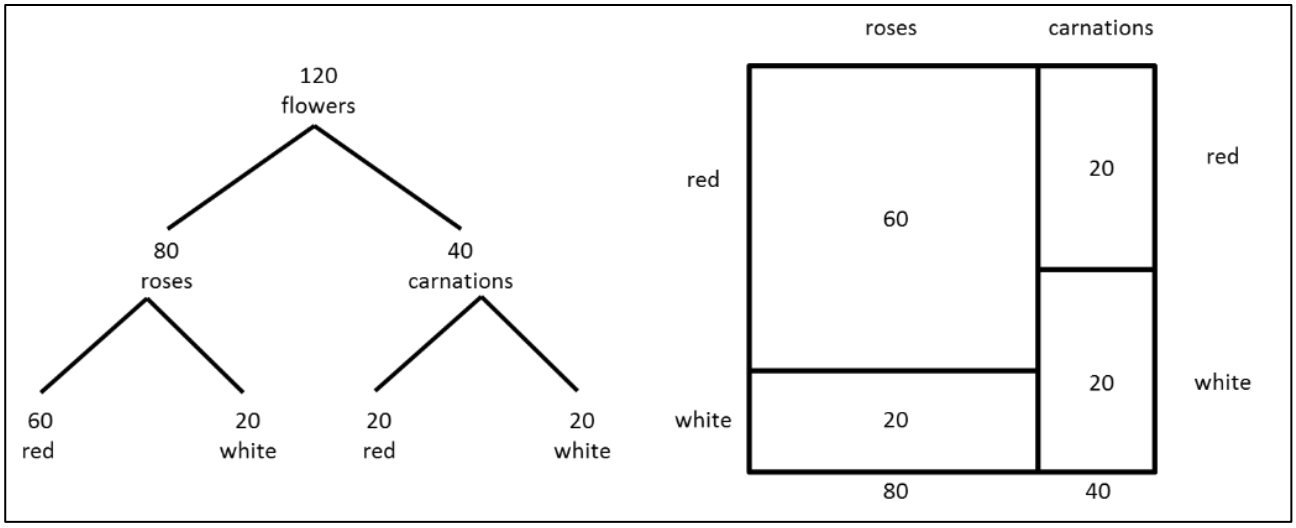


Figure 2: Representing statistical information with the tree diagram and the unit square

Both, the tree diagram and the unit square can be seen as nested-sets representations. In the tree diagram, the logical relations between sets and subsets are visualized by lines. The subsets are on a lower level than the sets in the tree and the branches connect the subsets with the sets. For example the subset "red roses" is on a lower level than the set "roses" and the branch connecting "red roses" and "roses" visualizes the relation between both sets. The tree implies a hierarchical structure which means that subsets are always on a lower level than sets. Therefore, only those subset relations that are in line with the hierarchy are salient. For example, the subset relation of "roses among the red flowers" where the "roses" are the subset and all the "red flowers" are the including set is not visualized by a branch in the tree and thus, is not transparent.

In the unit square, subset relations are visualized by areas being embedded in other areas. For example the subset of "white roses" is represented by a partial area of the rectangle that represents all roses. In contrast to the tree diagram, the unit square implies no hierarchy. That means that subset relations can be grasped vertically (e.g. "white roses among the roses") as well as horizontally (e.g. "roses among the red flowers"). Therefore, all subset relations that are possible in this situation are transparent in the unit square.

Because of these differences in the properties of the two visualizations, we expected a difference between the tree diagram and the unit square when the perception of different subset relations is regarded. Therefore we hypothesized:

If the subset-relation is not in line with the hierarchy of the tree diagram, the unit square is more efficient to make the subset-relation transparent (*hypothesis 1*). If the subset-relation is in line with

the hierarchy of the tree diagram, there is no significant difference between the unit square and the tree diagram (*hypothesis 2).*

Method

The method in our first study and the replication study was the same. In the first study, we administered a questionnaire to 148 students who were enrolled in a course of mathematics education. In the second study, the test was administered to 58 students who were also enrolled in a course of mathematics education. In both studies, we asked the students (among other questions concerning conditional probabilities) in one task that we indicate below to calculate proportions and to indicate the result in form of fractions:

$$proportion = \frac{\#\text{subset}}{\#\text{set}}$$

In this way, we could analyze if the right subsets and right sets have been grasped from the visualization. The questionnaire had two versions, one showing tree diagrams, the other showing unit squares to represent the statistical information. The rest of the test-items remained constant and the participants were randomly assigned to one of the two groups. Thus, any potential difference in the results could directly be attributed to the influence of the visualizations.

To assess the influence of representation on the perception of subset relations we designed test-items that each addressed structurally different subset relations. In Figure 3, we show the questions that were accompanied by either the tree diagram or the unit square shown above. Note that the item (d) addresses a subset relation that is not in line with the hierarchy of the tree diagram and therefore a higher performance for the unit square was expected. The items a, b, c and e address subset relations that are in line with the hierarchy of the tree diagram and therefore no significant difference between the tree diagram and the unit square was expected. We rated correct answers with 1 and incorrect answers with 0.

Flowers: A flower girl is selling red and white roses and carnations. Altogether, she has 120 flowers. Calculate the following proportions. Indicate the results in form of fractions.

The proportion of

a) red carnations among all carnations.

b) white roses among all flowers.

c) white flowers among all flowers.

d) carnations among the red flowers.

e) roses among all flowers.

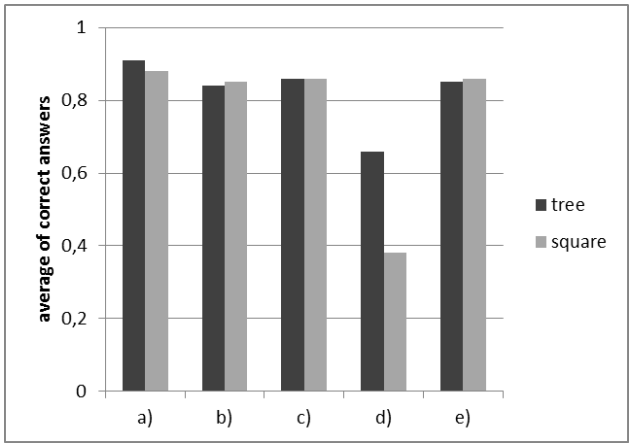
Figure 3: Items to assess the perception of subset relations

Results

Figure 4 shows on the left side the results that we reported in Böcherer-Linder and Eichler (2017). As we hypothesized in this study, the unit square (M = 0.66, SD = 0.44) was more efficient than the tree diagram (M = 0.38, SD = 0.48) for the item (d) that addressed a subset relation that is not in line with the hierarchy of the tree diagram. The difference for the item (d) was significant (t (146) = 3.579,

p <.001) with an effect size of d = .58. In the replication study there is again a significant difference referring the item (d) that addressed a subset relation that is not in line with the hierarchy of the tree diagram (unit square: M = 0.63, SD = 0.43; tree: M = 0.31, SD = 0.39; t(54) = 3.101, p<0.01, d =.82). Thus, we replicated our result referring hypothesis 1. Further, it is interesting that the ratios of correct answers are very similar in the original study and the replication study. However, there are also unexpected differences for the items (b) and (c).

Results of the first study, n = 148



Results of the replication study, n = 58

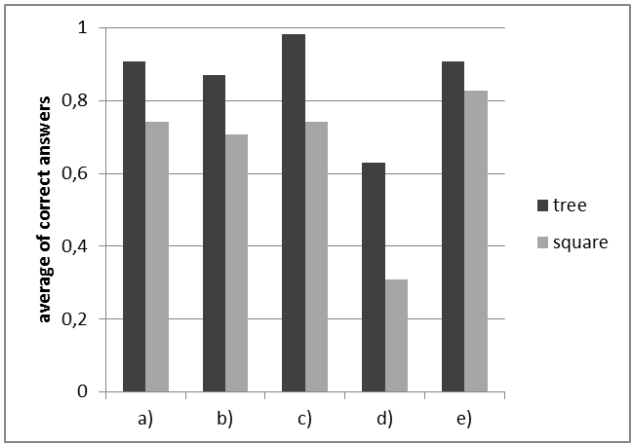


Figure 4: Participants performance in the first study (left side) and in the replication study

In our first study, we investigated also the differences in the other four items that addressed subset relations within the hierarchy of the tree diagram. For these items a t-test for the accumulated score referring to these four items (α = .739) yielded no significant difference between the tree diagram (M = 3.46, SD = 1.023) and the unit square (M = 3.46, SD = 1.036), t (146) = 0.000, p = 1.000. In the same way none of the items yielded a significant difference between the unit square and the tree diagram when investigated individually. Thus, there was no reason to reject our *hypothesis 2* in our first study. However, we could not replicate these results in the second study. When the items were regarded individually, a t-test yielded significant differences for item (b) (p<0.05) and (c) (p<0.01). Also, the accumulated score referring to all four items (α = .634) yielded a significant difference between the tree diagram (M = 3.01, SD = 1.11) and the unit square (M = 3.67, SD = 0.52), t (54) = 2.762, p = 0.008).

In the first study, the mean values of correct answers for the tree diagram were almost equal for all of the four items a, b, c and e addressing subset relations that were in line with the hierarchy of the tree diagram (88%, 85%, 86%, 86%). In contrast, in the second study for every item differences appeared. However, the performance for item (d) was lower for both visualizations. This might indicate that the subset relation (d) is more difficult to perceive than the other subset relations and that the visualization with the unit square is more helpful in this case.

Discussion of the results

For the situation of the flower girl, our results show a very clear effect in favor of the unit square. This was the case in the first study and, with very similar results, in our replication study. Nevertheless, we suggest for future research to prove this effect also for other contexts. In another study with 143 students of electrical engineering, we replicated the effect for subset relations that are not in line with the hierarchy of the tree diagram for two more different contexts. It is further desirable to investigate the effect of those subset relations used in the items a, b, c and e in more depth. This is especially the case since our results for this kind of items seems to be ambiguous. The replication study yielded significant differences referring to the efficiency of the tree diagram and the unit square in supporting students solutions in tasks where the subset relations are in line with the hierarchy of the tree diagram. Although this result was not expected and we further hypothesize that this result will not be replicated in further studies, it agrees with our overall hypothesis, i.e. the supremacy of the unit square to visualize situations in which the Bayes' rule has to be applied.

There are further aspects that that could be investigated in more detail. For example, there is the question of the order in the sequence in the tree diagram. It might be interesting to study the effect of the transposed order (roses / carnation on a lower level than red / white) and to compare it with a rotated unit square (roses / carnation arranged vertically and red / white arranged horizontally). This setting could be clarified if the hierarchy of the tree actually is the reason for the results in our study.

Moreover for the context of Bayesian reasoning, the results of Binder et al. (2015, p.6) suggest an advantage of the 2×2-table compared to the tree diagram, although no statistical difference between 2×2-tables and tree diagrams was reported. Thus, it is an open question if 2x2-tables are equally efficient than unit squares to make subset relations transparent or if there is an additional effect of the unit square due to the redundant geometrical and numerical representation.

Finally, there are further possibilities for visualizing set relations. One of these possibilities that was used in mathematics teaching is the double tree (Wassner, 2004). Thus, it could be interesting if a specific version of the tree diagram is able to decrease the weakness of the tree diagram to identify appropriately set relations.

Implications

The main result of our research seems nearly trivial: Visualizations have to visualize the main aspects of a mathematical concept if they aim to support students' understanding of this concept. Accordingly, a subset relation must be transparent when the aim of the visualization is to represent subset relations. However, it is by no means at all trivial to identify the crucial aspects of a mathematical concept. Actually, the tree diagram is very prominent in statistics education research (Veaux, Velleman, & Bock, 2012) and also psychological research (Binder, Krauss, & Bruckmaier, 2015) for visualizing Bayesian situation that necessitates applying Bayes' rule. However, our research gave evidence that – compared to the unit square - the tree diagram is not efficient to visualize the set relation that is crucial to understand the structure of a Bayesian situation since it requires a subset relation that is not in line with the hierarchy of the tree diagram.

Our results have firstly some consequences if statistics education is regarded. Sloman et al. (2003) expressed that bringing out nested set structure has been identified as being important for the improvement in Bayesian reasoning tasks. Thus, restricted to teaching and learning probability, our

results imply to reconsider the role of the tree diagram to support students' learning referring to probability and Bayesian reasoning. This would be a considerable shift in statistics education (c.f. e.g. (Gigerenzer, 2014; Wassner & Martignon, 2002). A little bit more globally, it could be considered if proportions, and in particular proportions of proportions could be appropriately visualized by a unit square to emphasize the connection between proportions of proportions and conditional probabilities. Thus, the unit square could potentially be understood as a visual connection between fractions and probabilities.

More generally, our results imply to focus the discussion of visualizations on the structure of visualization and on its relation to the structure of the represented mathematical concept. While the superiority of visualizations is a consensus in mathematics education as we outlined in the introduction, it is a crucial objective to find out which visualization best fits to a mathematical concept, especially in situations where several competing visualizations exist as it is the case for Bayesian reasoning situations. For example, Binder et al. (2015) show that the tree diagram supports Bayesian reasoning compared to pure symbolic representation, whereas our results imply that the required subset relation is not transparent in the tree diagram. Indeed, in recent research, the unit square outperformed the tree diagram in Bayesian reasoning tasks (Böcherer-Linder & Eichler, 2017; Böcherer-Linder, Eichler & Vogel, in press). Therefore an ongoing task of educational research should be to precisely identify the relation of a visualization and its structure and the mathematical concept and its structure. One main message of our paper is that this relation is not sufficiently investigated, but could considerably impact on students' learning.

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Enactive metaphors in mathematical problem solving

Daniela Diaz-Rojas[1] and Jorge Soto-Andrade[2]

[1]University of Chile, CIAE, Santiago, Chile; diazr.daniela@gmail.com

[2] University of Chile, CIAE, Faculty of Science, Mathematics Department, Santiago, Chile; sotoandrade@uchile.cl

We are interested in exploring the role of enactive metaphoring in mathematical thinking, especially in the context of problem posing and solving, not only as a means to foster and enhance the learner's ability to think mathematically but also as a mean to alleviate the cognitive abuse that the teaching of mathematics has turned out to be for most children and adolescents in the world. We present some illustrative examples to this end besides describing our theoretical framework.

Keywords: Metaphors, enaction, representation, visualisation, cognitive bullying.

Introduction

Our concern in this paper is the role of metaphor, more precisely enactive metaphor, in the teaching and learning of mathematics, particularly in mathematical thinking arising in, or triggered by, problem posing and solving. Fostering mathematical thinking in the classroom is a widespread aim in mathematics education indeed (OECD, 2014), but in our viewpoint we have a much more severe and dramatic issue to address: not only mathematical thinking is not fostered in our classrooms (Chilean and worldwide), but mathematics has turned out to be a tool of torture for millions of children, who cannot escape from it. This has been recently acknowledged as "cognitive bullying" or "cognitive abuse" in the English literature (Watson, 2008; Johnston-Wilder & Lee, 2010). a practice that is "at best marginally productive and at worst emotionally damaging" (Watson, 2008: p. 165). We thus echo Tillich's famous statement: "the fatal pedagogical error is to throw answers like stones at the heads of those who have not yet asked the questions". To tackle this complex and systemic problem, a multidisciplinary approach is most wanted, where a first diagnosis emerges: traditional (and abusive) teaching of mathematics tends to thwart cognitive brain mechanisms installed during millions of years of evolution, that we would need on the contrary to recognise, appreciate and tap into in the context of learning, to wit: metaphorising, enacting, collaborating…

Our main hypothesis is that practice of metaphorising, especially enactive metaphorising, in the classroom, might be a fundamental means to contribute to alleviating this situation of cognitive abuse towards students without forsaking their access to mathematical thinking, but on the contrary fostering it. It is our hypothesis that the way a mathematical situation is metaphorised and enacted by the learners strongly determines the ideas and insights that may emerge in them and may help to bridge the gap between the "mathematically gifted" and those apparently not so gifted or mathematically oriented. A big challenge is then trying to figure out under which conditions enaction and metaphorising, more precisely enactive metaphorising, impact on mathematical thinking processes as hypothesised above.

We intend here to pursue our recent research on metaphorising and enacting (Diaz-Rojas & Soto-Andrade, 2015, 2016; Soto-Andrade, 2015). Our earlier work on metaphor in the learning of mathematics was presented already in CERME5 (Soto-Andrade, 2007).
In this paper we focus on some examples and case studies that illustrate the role that metaphorising and enacting may play in the spectrum of mathematical thinking elicited by problem posing and solving. The contextual background of our case studies involves a variety of learners in Chile: prospective secondary math teachers, in service primary and secondary math teachers, first year university students majoring in social sciences and humanities, undergraduate and graduate students in mathematics, primary and secondary students.

Research questions

Which is the role of metaphorising and enacting in mathematical problem posing and solving at various levels? To which extent do they influence mathematical thinking elicited by the problematic situation, in particular moving amongst various mathematical registers of representation to change the problem? Do they shape our understanding of the processes involved, notably the relation between problem and learner, emotional overtones included?

How does the interplay between affect and metaphoring helps in alleviating cognitive bullying in the teaching of mathematics, and even lead to enjoyment of learning and doing mathematics?

Theoretical framework and state of the art

Metaphorising in cognitive science and mathematics education.

Widespread agreement has been reached in cognitive science that our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Gibbs, 2008; Johnson & Lakoff, 2003). In mathematics education proper it has been progressively recognized during the last decades (English, 1997; Lakoff & Núñez, 2000; Sfard, 2009; Soto-Andrade, 2007, 2014, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in grasping or even building new concepts, as well as in solving problems in an efficient and friendly way. See Soto-Andrade (2014) for a recent survey. We may visualize (conceptual) metaphors (Lakoff & Núñez, 2000) as mappings from a more down-to-earth "source domain" into a more abstract "target domain", carrying the inferential structure of the former into that of the latter. For the learning of mathematics we emphasize the "poietic" role of metaphor that brings concepts into existence, through "reification" (Sfard, 2009). In view, here lies the main difference between representation and metaphor: we *re-present* something given beforehand but we *metaphorise* to try to fathom something unknown or a concept not yet constructed. For instance, we construct the concept of probability when, studying a symmetric random walk on the integers (a frog jumping on a row of stones in a pond), we see the walker *splitting* into 2 equal halves instead of going equally likely right or left (Soto-Andrade, 2007, 2015). In what follows we will use metaphorical language as a meta-language to describe cognitive or didactic theories of interest to us, since – we claim – a theory is essentially the unfolding of a metaphor (Soto-Andrade, 2014).

Enaction in cognitive science and mathematics education

Varela metaphorized enaction as the laying down of a path in walking (Varela, 1987, p. 63), as in Machado's famous poem, when he introduced the enactive approach in cognitive science (Varela,

Thompson, & Rosch, 1991). In his own words: "The world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call cognition as enaction since enaction connotes this bringing forth by concrete handling" (loc. cit). Less radical enaction in mathematics education may be traced back to Bruner (1966), inspired by Dewey's "learning by doing" (Dewey, 1997), who characterised enactive, iconic and symbolic modes of representation. For recent significant theoretical and practical developments see Proulx (2013). In what follows we are especially interested in enactive metaphors, where the learner is whole bodily engaged, as opposed to "sitting metaphors" in the sense of Gallagher and Lindgren (2015).

(A)didactic situations and didactic contract

The theory of didactical situations (Brousseau & Warfield, 2014) might be described as the unfolding of the *emergence metaphor* in math education: mathematical concepts we intend to teach should emerge in a suitable challenging situation the learner is enmeshed in, as the only means to "save his life". No real learning is possible if mathematical concepts are airborne from Olympus. Such a situation is called a *didactic* situation, because of the didactical intent of the teacher who set it up. It becomes an *adidactic* situation when the teacher definitely steps back to let the learners interact on their own with the setting, with no hope of fathoming beforehand her didactical design or the mathematical content she is aiming at. Key metaphors are likely to emerge, as sparking voltaic arcs, in and among the learners, when enough "didactical tension" is built up in an adidactic situation for them. The notion of *didactical contract* (Brousseau, Sarrazy, & Novotna, 2014) is also of interest to us, in the context of the teacher-student relation. It is in fact a keen metaphoric description of the mainly implicit and unspoken mutual expectations, beliefs and commitments regarding the actions and reactions of the partners involved in a didactic or adidactic situation.

Affect in mathematical problem solving

The role of affect in mathematical problem solving is often neglected in spite of its significant incidence in learner's performance (Mason, Burton, & Stacey, 2003; Hannula, 2014). Here we are specially concerned by the role of negative emotions that trigger a learner's emergent metaphorising that can transform a problem that is a tool of cognitive bullying into a friendlier one. The outcome of this may be, for most learners, a positive feeling of liberation from the Procrustean bed of arithmetic and algebra, for instance (see example 2 below).

Methodology and experimental background

Our research includes an experimental facet, where our methodology mainly relies on qualitative approaches, to wit: Case Studies (Stake, 1995), Participant Observation techniques and Ethnographic methods (Spradley, 1980).

In all, 4 cohorts of learners have been involved in our teaching and learning according to our metaphoric and enactivist approach from 2014 to 2016. They include prospective secondary school physics and mathematics teachers in a one-semester course in number theory at the University of Chile; students in a one semester first year mathematics course in the social sciences and humanities option of the Baccalaureate Programme of the same University; in service primary and secondary school teachers engaged in one week professional development workshop in the South of Chile, in service primary school teachers engaged in a 15 month professional development programme (mathematics option) at the University of Chile at Santiago; graduate students working towards a

Ph. D. in Didactics of Mathematics, at the Catholic University of Valparaiso (UCV), most of them secondary school math teachers holding a Master in Didactics of Mathematics. They were chosen because they constituted a broad spectrum of learners we had access to while performing our usual teaching duties at the University of Chile, besides some invited workshops elsewhere, with whom our overarching approach could be tested. Learners, working most of the time in (random or spontaneous) groups of 2 to 4, were observed by the teacher or facilitator and an assistant, the latter assuming the role of participant observer or ethnographer (Spradley, 1980; Brewer & Firmin, 2006). Among aspects observed were: level of participation, questions and answers, horizontal (peer) interaction, emergence of metaphors, especially idiosyncratic ones, spontaneously or under prompting, gestural language of learners and teacher, expression and explicit acknowledgement of affective reactions from the learners. Snapshots of their written output in problem solving activities were taken and videos of their enacting moments were recorded.

Illustrative examples and case studies

We present and discuss here two paradigmatic examples, in geometry and arithmetics, that we have come across during our teaching at the University of Chile, to illustrating important aspects of our theoretical perspective, often neglected in usual approaches. The case of randomness has been dealt elsewhere (Diaz-Rojas & Soto-Andrade, 2015). Our geometrical example deals with the exterior angles of a polygon and their sum: a typical geometrical notion often abusively and gratuitously introduced, with no context or motivation. In arithmetic, we recall the consecutive sums of positive integers problem, thoroughly discussed in the literature (e.g. Mason et al., 2003)

Example 1. The sum of the exterior angles of a polygon

We have observed that almost every in service and prospective secondary mathematics school teacher in our country, after introducing exterior angles coming out of the blue *after* inner angles and explaining them in terms of the latter, calculates dutifully their sum from the sum of the inner angles, that depends on the number of sides of the polygon. Doing a bit of algebra they finally wind up discovering that the sum of all exterior angles is 360°, independently of the number of sides! Surprising! This traditional way to "get into" the task (Proulx, 2013), is not very appealing for most students, that experience it as "blind calculation" (a case for cognitive bullying). When trying to fathom out exterior angles of a polygon however a first thing to do – from our perspective – would be to metaphorise it, to get into the task in a more transparent way. Not just reciting its formal definition, of course. Among the metaphors emerging amongst the learners we work with, the most frequent are "a polygon is an enclosure between crossing sticks" (most popular among primary school teachers) and "a polygon is a closed path, made out of straight segments". Enacting the first metaphor triggers the idea of manipulating the sticks, as to make clearly visible the exterior angles first and then shifting them parallel to themselves to get smaller and smaller homothetic polygons. In this way teachers *see* that the sum of all exterior angles is 360°, instead of blindly calculating. Enaction of the metaphor "polygons are closed paths" by the learners themselves, literally lying down a polygonal path in walking, enables them to immediately "see" that the sum of the polygon's exterior angles corresponds to a complete revolution (Diaz-Rojas & Soto-Andrade, 2015). In this way they realise that exterior angles, not inner angles, are the convenient data for the walker to inflect or bend his path as wanted. Analogously for the sum of all acute angles of a pointed star… We noticed that metaphorising a polygon is an unusual challenge, almost a violation of the

didactical contract, for both students and teachers. But once they feel they are allowed to, even prompted to, metaphors begin to arise, shyly at first. The enactive metaphoric approach conveys here a completely different experience of mathematics than the traditional one, including the role of gestures, movements and, more broadly embodiment, in the learning of mathematics, particularly in problem solving (Libedinsky & Soto-Andrade, 2015).

Example 2. Which numbers are consecutive sums? Just an arithmetic problem?

The question is: Which numbers are sums of a string of consecutive (positive) integers. An unexpected question to many learners, however familiar with Gauss well known trick to sum 1 + 2 + …+ 100 in a wink. From our perspective it is interesting to observe how easily this question (or any question) emerges in the learners, once their attention is drawn to this sort of sums. Our hypothesis is that learners' reaction here is heavily dependent on their previous schooling and the amount of cognitive abuse they have endured. As a typical example we recall an informal short meeting to chat about "the mathematical experience" with a class of 12th graders from a Waldorf school, to whom we told about consecutive sums (just what they are), period. After a few seconds, a girl said: Which series of numbers do you obtain that way? We claim than in usual problem solving this is an often neglected aspect: enactively letting questions emerge instead of asking them… Another often neglected aspect in problem solving is the affective reaction a problem elicits in the learner. This sort of arithmetic problem quite often triggers a feeling of distaste, especially in adolescents. This negative emotion may have the immediate positive effect of stimulating the learner to metaphorise, to transform the problem into a more attractive or friendly one, i. e. a creative reaction! This is very rarely observed in our prospective teachers and Ph. D. students. Apparently didactical contract weights heavily here: learners are not supposed to transform or metaphorise the problems they receive, nor are they supposed to *like* or *dislike* mathematical objects or procedures, just to understand them or not.

We observed that every learner tackled the problem arithmetically first, doing some experimenting (some calculating small consecutive sums, others following the opposite path: checking whether 2, 3, 4, 5, etc. might be consecutive sums). Some got a closed formula for a consecutive sum but did not see what numbers are so obtained. Those who checked numbers one by one arrived quickly to the (surprising) conjecture that powers of 2 cannot be reached. In fact they re-traced Mason et al. (2003). The proof of the conjecture remained elusive until some noticed that powers of 2 do not have odd divisors and so devised an algebraic proof of the conjecture. The fact that conversely a number which is not a power of two must be a consecutive sum remained in the shadow for 45 minutes or so. At his point, we asked prospective teachers whether they liked the way we were tackling and solving the problem. Two of them said that the conjecture was interesting and that they understood the algebraic procedure but that they were not very happy about it. Fernanda said that she was not fond of this algebraic yoga, although she was able to carry it out. For Enrique this algebraic approach was easy but he was unhappy because he had noticed (giving private lessons to secondary school students) that for most students algebraic calculations are not appealing at all. So both were motivated to look for different, may be geometric, approaches.

For Ph. D. students, didactical contract played in the opposite direction: after working on metaphorising some months before, when asked now whether they were satisfied with their discovery regarding powers of 2, several students thought about metaphorising the problem, looking

at the numbers as quantities of dots arranged in clever ways. Andrea, an insightful female student, drew a trapezoidal house with a slanted roof (of slope 1) and so transformed the problem to a question about the area of this trapezoid. Some tried to remember the area formula, but others, like three clever prospective teachers, saw by rearrangement or compensation that their trapezoid could be turned into a rectangle with the same base. But then they realised that this worked only "half of the time": for an odd basis! In that case the area has an obvious odd divisor. For the even case, some conjectured that they could get a two-step horizontal roof, each step of equal length and so the idea emerged of slicing vertically the trapezoid into two "halves" of equal base and putting one on top of the other. Some went into distinguishing the cases: half base odd or even. But others had the idea of putting one "half trapezoid" topsy-turvy on top of the other, getting in this way a rectangle of odd height and half base. Then, a prospective teacher had the idea of borrowing from scratch a copy of the original trapezium and coupling both to obtain a rectangle with either odd base or odd height and whose area is twice the original one! The proof of the converse conjecture was left open. Regarding liking or disliking, graduate students at UCV were more enthusiastic about the geometric approach than prospective secondary teachers. Roughly two thirds of the latter said that they did not feel confortable with geometry so that they preferred calculating algebraically! In fact even when trying to think in geometric terms, they quickly reverted to algebraic calculation. On the other hand happy visualizers realised that the trapezium area may have any value in the continuous case but not in the discrete one, because something pops up that has no continuous analogue: Parity! A surprising fact for them, who knew, from their laptop screens, that the discrete models the continuum well. The question remained open as to whether we can see geometrically that the area of a rectangular trapezium cannot be the volume of a hypercube.

From our perspective this is an emblematic example of the possible "unfolding" of an "arithmetic" problem that that can be solved by some algebraic yoga (that many students do not appreciate at all) but can also be metaphorised as a geometric problem, more appealing to others. This metaphorising "prompts" us to jump naturally into the continuous world and get some inspiration there. We realise also the hard way that there is a tricky property of discrete shapes with no analogue in the continuous world: parity! Discrete lengths, areas or volumes may be odd or even, although asymptotically however parity vanishes… Remarkably, even insightful discussions of this problem found in the literature (e. g. Mason et al., 2003) remain confined in the arithmetic-algebraic realm, not taking advantage of the avenues and possible generalisations that metaphorisation may open up.

Discussion

We have shown several important aspects of the role of metaphorising and enacting in mathematical thinking elicited by problem posing and solving. First, we have seen that the way we metaphorise and enact determines the ideas and insights we may have when tackling a problem. Then, how metaphorisation triggered by distaste of the problem may allow the learners to move from one realm to another, instead of remaining confined in just one. By so doing, they may take advantage of different intuitions and handlings, eventually much friendlier to them, that enhance their mathematical thinking and also alleviate the cognitive abuse they have been exposed to. Indeed, an acknowledged negative affective reaction to a proposed problem may trigger creative metaphorisation to change it. In this way metaphorisation appears as a means to empower students to transform an unappealing problem given to them, something especially relevant for adolescents

who otherwise have the feeling of being abused by being forced to follow prescribed rules to solve nonsensical tasks (Watson, 2008). Also visualization appears as concatenation of metaphors: In the case of consecutive sums: “numbers are quantities”, “summing is putting together”, “factorizing is rearranging to form a rectangle” etc. Furthermore, it appears that usual problem solving, as found in the literature, tends to neglect, the important role of metaphorisation and enaction, as a learner’s first reaction when tackling a problem that looks opaque to him or her. Not only because this may allow the learner to solve an otherwise unyielding problem but also because it may allow him or her not just to solve the problem but to “see” a solution, turning a hitherto blind calculation into pellucid insight. Finally the enactive and metaphoric perspective reshapes our understanding of the relation between problem and learner in problem posing and solving, that appears as a quite more circular and entangled process than usually acknowledged, where each one codetermines the other.

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Diagrams and mathematical events: Encountering spatio-temporal relationships with graphing technology

Francesca Ferrara and Giulia Ferrari

University of Torino, Italy; francesca.ferrara@unito.it, giulia.ferrari@unito.it

This paper presents the diagrammatic activity of some secondary school students exploring motion through graphing technology, which captures a pair of space-time graphs on a single Cartesian plane. Focus is on a written task about the connections between two imaginary movements and (between) the corresponding graphs. Drawing on a vision that conceives mathematical thinking as a place of events instead of objects, we discuss three unexpected diagrams for how they bring forth inventive and speculative possibilities for learners to encounter and problematize spatio-temporal relationships, rather than seeing them as ways of being mistaken.

Keywords: Graphing technology, movement, diagram, event, problematic.

Introduction

In this paper, we deal with the issue of how students might learn about a representational system in which temporospatial relationships are the ground for the mathematical doing. Our interests are also in how visual, proprioceptive and kinaesthetic aspects of experiencing these relationships might move the learning of mathematics in unexpected and unconventional directions. We follow here de Freitas (2013) in rethinking mathematics as the place of events, instead of objects, where creativity and contingency prevail and the problematic—rather than the axiomatic—better capture the vitality of mathematical activity. The idea is that deduction "moves from the problem to the ideal accidents and events that condition the problem and form the cases that resolve it." (Smith 2006, p. 145). Thus, mathematics is concerned with the occurrence of events more than with the existence of objects, and attention is on the material encounters with the mathematical.

In this perspective, we present an activity that was carried out with grade 9 students working with graphing motion technology to study function. In particular, the technology requires that two remote controllers of the Nintendo Wii game console (Wiimotes) are moved at the same time in front of a sensor bar, and it displays two space-time graphs on the same Cartesian plane. In the graphs, space is given by the distance of each controller from the bar. Thus, the software captures the movement of the Wiimotes over time. Our focus is on a written task that asks the students, divided into groups, to draw a space-time graph related via movement to a given graph. The task is called "Rob and Bob". In it, Rob and Bob are the names of two little robots that are imagined to be moving the controllers in front of the sensor. The graph associated to Rob's movement is given on paper, together with the instructions with which Bob is supposed to be moving with respect to Rob. The students are expected to complete the task adding Bob's graph on the Cartesian plane. We will discuss how three different graphs are presented as solutions to the task from different groups, and we will develop how we think that these are significant in terms of the novel mathematical meanings that the students are articulating. In the meanwhile, we will also draw attention to aspects of the experience with the technology that might support this novelty, raising issues about the role of perception on the one side, and about the features of the technology on the other side.

Theoretical highlights

The representational system we refer to in this paper is the (space-time) Cartesian plane, which our students encounter through activity with the technology. However, we want to trouble traditional ideational assumptions that conceive such system as inhabited by mathematical figures or functions that, in their essence, are representations (particular instantiations or attributes, concrete instances) of some form, inert, transcendent, abstract and disembodied. In fact, claims de Freitas (2013), "the process of instantiation fails to capture the creative and material act of individuation that is entailed when we do mathematics." (p. 586). We instead embrace an animate vision of the mathematical drawing/creation or act of drawing/creation of a figure or a function as event-structured, full of potentiality, temporality and movement, immanent, contingent to material circumstances, and incidentally subject to transformation. This positions us in the broad discussion on the theorising about the embodied nature of mathematics thinking and learning, which attempts to look at knowledge in non-representational ways and to overcome body/mind Cartesianisms (see e.g. Nemirovsky et al., 2013; Sinclair, 2014; Ferrara, 2015; Roth, 2016). According to this view, learning is much more about encountering concepts than about recognizing concepts. Cutler & MacKenzie (2011) might argue that thus the challenge is to treat learning as an ontological rather than an epistemological problem, staying away, we would add, from opposing the mathematical and the physical. The issue of representations is crucial here. As Sinclair (2014) points out, it is not that "symbols, diagrams, programming languages and even gestures" (and any other system, we would add) "do not at times *function* to re-present mathematical concepts and relations", rather "they are inevitably bound up with bodies and discourses and thus potentially poised to open up new meanings." (p. 174, *emphasis in the original*). Our own reading of this makes sense as regards our commitment to a mobile view of mathematics and mathematical doing that tries to escape concrete versus abstract and matter versus thought divides (Ferrara & Ferrari, 2016). In Ferrara (2015), these divides are challenged through a vision of perception and creation in/of mathematics for which perceiving is conceiving, thinking is acting and creating is learning. The work of philosopher Gilles Châtelet (1987, 1993/2000) on inventive diagramming was provocative to us in considering the centrality of mobility or virtuality to bridging the physical and the mathematical. The virtual is the necessary link between the two realms. Roth (2016) also draws on Châtelet to underline how one way of thinking about dynamic systems is just in terms of the virtual. We can better understand this if we take the examples that Châtelet (1987) makes about historical contributions of new ideas by Leibniz and Abel. Leibniz theorised differential calculus thinking of points as if they were alive, as powers of explosion ("puissances d'explosion"), while Abel saw the curve not as fixed but in terms of its power of receiving intersections ("comme puissance à recevoir des intersections"). The virtual restores concepts to mobility, granting them inventive force and power. For de Freitas (2014), Châtelet shows us "how we might study a particular practice for how its lines of flight flourish and act generatively in unfolding new *intensive* dimensions." (p. 290, *emphasis in the original*). The virtual is that which nourishes encounters with mathematics, linking the concrete and the abstract and allowing recoding the indeterminate contours of the sensible and the intelligible. This has to do with the potentiality or virtuality that is always entailed in perception: "We never just register visual information from that which is in front of our eyes: we see potentiality, relationality, mobility, occurrence. Students are not seeing an object; they are seeing an event" (de Freitas, 2014, p. 298).

In this paper, we take this perspective to look at the students' mathematical encounters with spatio-temporal relationships, focussing on the material and virtual dimensions of these encounters.

Method and activity

The activity, which is the focus of this paper, is part of a classroom-based intervention (Stylianides & Stylianides, 2013) aimed at introducing the concept of function through the use of graphing technology. The wider research had the main purpose of investigating how learners might articulate meanings for functional relationships through modelling motion, and how their embodied activity with the technology might affect these meanings. A class of 30 grade 9 students and their regular mathematics teacher participated in the study, which lasted for a period of about three months with weekly sessions. During this time, the students worked on individual tasks, in groups of three people and in pairs of groups, taking part in class discussions. The authors were both present in the classroom and two cameras were used to film the mathematical activity of the students during all the sessions. Data for the research analyses are based on the films and students' written productions and diagrammatic activity. A microethnographic methodology (e.g. Streeck & Mehus, 2005) is essentially chosen for studying interactions and discourse in the classroom through strands of semiotic and representational activity over short periods of time, drawing attention to the material circumstances of the mathematical events.

The technology the students used in the case we consider here is WiiGraph, an interactive software application, which leverages two Wiimotes to display the space-time graphs of two users moving the remote controllers in front of a sensor bar. WiiGraph provides several challenges and composite operations, including shape tracing, maze traversal and ratio resolution. Choosing the plain visualisation (Line), the software captures the distances of the controllers over time and two graphs appear, in real time and with different colours, on a single Cartesian plane. Figure 1 shows a case of this type of visualisation for a 30-second default time and two students who move the controllers.

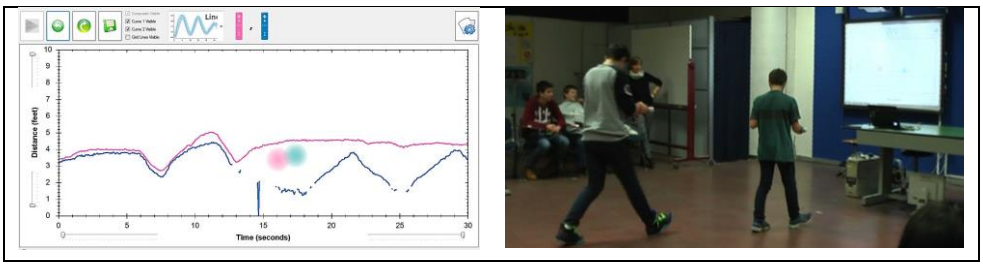

Figure 1: The graphical system in Line and two students moving

Visual and bodily (especially proprioceptive and kinaesthetic) interactions partake in the students' encounters with the graphical system in relation to experiencing spatial and temporal aspects with the technology. We will not refer to other types of graphical activity, since this is the one of interest in the case of Rob and Bob.

The task was given in a written form to the class during the second session and did not imply direct use of the technological devices. In the first session, the students explored Line and its graphical potential, became acquainted with the devices and started discussing about pairs of functions (for example, horizontal or slanted straight lines), with graphs originating in real time and projected on an interactive whiteboard. The activity of Rob and Bob was designed with the purpose of unfolding the

slope/speed relation (early insights emerged out of class discussion in the first session), and how it may reveal relationships between two space-time graphs (functions).

Rob and Bob

The task was faced by the students divided into groups of three people, and followed by a class discussion led by one of the authors. It focuses on an imaginary experience with WiiGraph in which two little robots move (the controllers) in front of the sensor bar, but only the graph associated to one robot's movement is given (Figure 2a). The text of the task is the following:

> *Rob and Bob are two little robots, which can be taught to move in front of the sensor very precisely. Suppose that, in response to Rob's movement, WiiGraph produces the line below* (Figure 2a). *Imagine that Bob also moved: it started together with Rob, at the same distance from the sensor, but moved at a double speed and in the opposite direction.*
>
> - *Which graph would WiiGraph show for Bob's movement?*
> - *Did Rob and Bob meet again after the start?*
>
> *Justify your answers.*

The task has an unconventional nature with respect to the representational system offered by the technology, because it does not ask the students to merely reason on the model to motion, or motion to model, shift. Instead, information about the missing graph is given in terms of the relationships between the two robots' movements ("double speed", "opposite direction"), so that the students are moved to think about the relationships between the two graphs (double slope with opposite sign), through their perceptual and bodily experience with the tool. In addition, the simultaneity of the two movements, which by the way recalls the usual way of using the tool, is embedded in information about the starting instant/point ("it started together with Rob", "at the same distance").

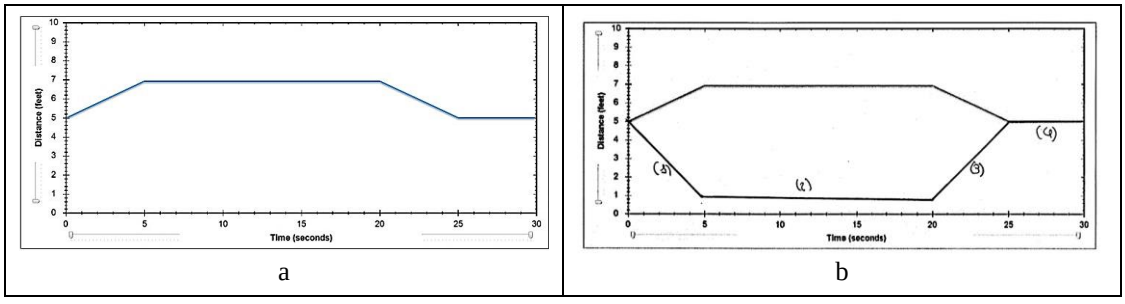

Figure 2: (a) The given graph, (b) The expected solution to the task given by one group

The given graph is that of a piecewise function made up of four pieces, which capture alternate ways of moving by Rob: stepping further from the sensor for the first five seconds, stopping for the next fifteen seconds, returning to the starting position in other five seconds, and stopping for the last five seconds (Rob keeps constant speed in each time interval). We expected the students to complete the Cartesian plane drawing a graph like the one in Figure 2b. It is the graph of a piecewise function again made up of four pieces, defined on the same sequence of time intervals as the given graph. These pieces correspond to four ways of moving by Bob: getting close to the sensor for the first five seconds, stopping for the next fifteen seconds, returning to the starting position in five more seconds, and stopping for the last five seconds (however, Bob is supposed to cover double space with respect to

Rob, according to the constraint of moving at a double speed. Of course, this is true when he moves, and trivially when he does not, since the distance covered is null).

Instead of looking at the expected graph as the correct one and speaking of difference in terms of being mistaken, we dwell on different unexpected solutions emerged from the groups about their potential to bring forth new relational possibilities for the two robots' movements as well as for the pair of graphs. In the next section, we take these solutions as the problematic actualizations of the mathematical events that the groups encounter in solving the task. It is this idea of novelty that speaks directly to inventive mathematics and makes students alive to their engaging with the task.

Graphs and discussion

The groups worked on the task for half of the time, then they took part in a collective discussion in which their graphical solutions were compared. Only one group drew the expected solution (Figure 2b), while eight out of ten created one of the three unexpected lines shown in Figure 3 (For the sake of ease, we refer these lines to three graphs labelled with numbers 1, 2 and 3).

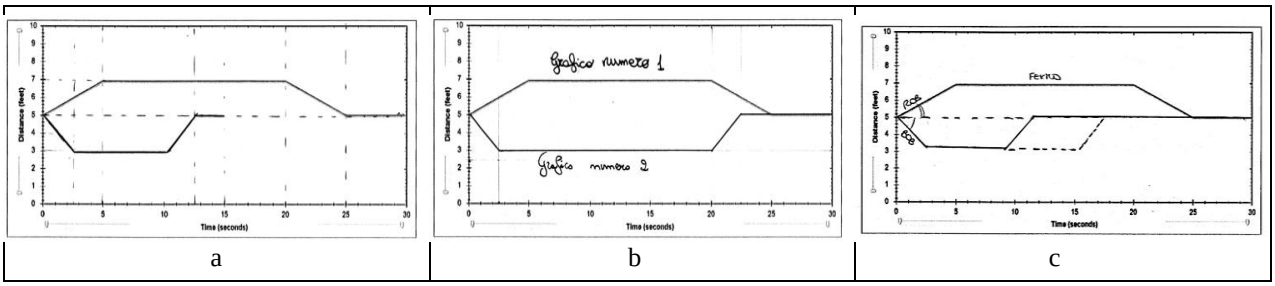

Figure 3: Unexpected solutions — (a) graph 1, (b) graph 2 and (c) graph 3

The three graphs added for Bob's movement have some similarity. They all show that taking into account information about opposite direction and capture it visually in the diagram is not an issue for the students. Each added graph is made up of four pieces, which embed the opposite way of moving with respect to Rob: first getting close, then returning to the start (first a decreasing piece, then an increasing piece). Not even slope is an issue: the double speed of movement is double slope in the three diagrams. However, the duration of Bob's movement is problematic for the students. In fact, while there is correspondence between ways of moving there is no embodiment of duration: there is no correspondence between time intervals in which both robots either move or stand still. The lengths of the horizontal pieces are different from each graph to the other, and the constraint for Bob to move at a double speed with respect to Rob is no longer preserved. Thus, the problematics of duration evolved along various accidental threads for the students, driven by their perceptual and bodily engagement with the task. These broke with causal connections and direct determination, opening up to speculative and inventive investments and to a generative movement, implicating the perturbation of spatio-temporal relationships. For example, in the case of graphs 1 and 3 (Figures 3a and 3c), some encountered the event for which Bob already stands still while Rob is still moving and, later, Bob moves towards the starting position while Rob is still standing still. Some groups introduced the new event in which the second robot stops just after fifteen seconds, in the very middle of the experience with WiiGraph, and ideally disappears from the view of the sensor, so that the second graph might accidentally stop in the middle of the diagram (Figure 3a). Almost all the students engaged with the kinaesthetic question of Rob and Bob always covering the same space, no matter the time spent, as

shown in the three diagrams. These threads are actualized through the groups' written explanations, then during class discussion. Types of explanation are the following:

> Graph 1: "*The line we represented is half of Rob's line. The lines are steeper because speed is doubled and Bob moved faster than Rob and in the opposite way. The graph ends at 15 because Bob, moving at a double speed, stopped at half of 30.*"
>
> Graph 2: "*The two configurations are different from each other, indeed slopes also change since times change: Rob is slower. So, covering the same space in different time, there will be a higher steepness.*"
>
> Graph 3: "*Bob goes at a double speed with respect to Rob, so it finishes "the lap" before Rob. The rest of the way it stood still and at the end it met Rob. Speed changes between the ways of Bob and Rob, indeed Bob has to cover the same space backwards using half of the time.*"

The logical equivalence between double speed as double distance in the same time and as the same distance in half of the time is lost, and the problematic of covering a fixed space in less time drives students' perception and visualisation in the diagramming of the missing graph. Graph 1 (Figure 3a) is the most coherent in respect to the axiomatic way of reasoning about double speed but at once the most incoherent in relation to kinaesthetic actions with the technology. Briefly speaking, it is nothing but a temporal shrinking of the given graph. Instead, graph 3 (Figure 3c) is in line with the usage of WiiGraph, because it embraces all the thirty seconds of the modelling process. The same occurs in the case of graph 2 (Figure 3b), which is particular though, since it struggles to depict the simultaneity of the two robots' movements. In the discussion, different students actualize in different ways the problematics that sustain the mathematical events that occurred in solving the task. Below, Lorenzo, Luigi, Giulio and Oliver bring forth in the discourse the issues of duration and simultaneity of movements, of moving at a double speed and of covering the same space, issues that are entangled in their diagrammatic and written activity. Lorenzo speaks about graph 1 (Figure 3a), Luigi and Giulio refer to graph 2 (Figure 3b), while Oliver argues about graph 3 (Figure 3c).

Lorenzo: 'Cause, moving at a double speed, distance remained constant, even though it was the opposite, but maybe, if Rob performed a movement in 10 seconds, Bob performed it in 5 seconds because speed was double.

Luigi: For me, hem, the graph took up the same time because, moving simultaneously, maybe, at the time they were moving, it took less time for one than for the other one to cover the same space, to move in the same space, but then one stood still until the other one also did move again, so both graphs last for 30 seconds. (...)

Giulio: For me, it [the graph] finished at 30, 'cause it's not that Rob [Bob] could know Bob's [Rob's] movement in advance, so it [the graph] cannot finish at 15, it [Bob] has to wait for it [Rob] to perform the same but opposite movement, 'cause we did see Bob's graph (*miming it in the air*) but if they move simultaneously, it means that one cannot anticipate the movements, so it cannot finish at 15 seconds. (...)

Giulio: We depicted slope at 2,5 seconds but then we stood still until 20 seconds, 'cause anyway it's true that time is halved, but Bob doesn't know what Rob will do later, so it has to wait for it.

Researcher: Did you say that time is halved?

Giulio: Yes, 'cause it [Bob] does things in half of the time, it's true, however it's not that he can know what it [Rob] will do later, hem, 'cause we do know it, but if they move simultaneously...

Oliver: Bob does our graph with the movements with which it's been set up, and then, in the end, it's not that it waits for Rob, it goes on just as it likes and wants, then at a certain point, when movements are finished, it stops and the line keeps straight for the rest of the time.

Researcher: Are you saying that time is halved?

Oliver: Yes, because of the double speed. (...)

Lorenzo: Indeed, for explaining a little the graph, I said that if there's a distance to cover, and that distance is 50 kilometres and you go at a speed of 50 kilometres per hour, it takes you 1 hour to cover that distance. Instead, if you go at 100 kilometres per hour it takes you 30 [minutes].

Giulio: Yes, for me he's right about the first piece, but then if you stand still, so speed is null, zero times two is always zero and so speed has to be equal in the positions in which it stands still, for me.

Discourse with the researcher unfolds the event-nature of unexpected threads traversed in solving the task. We see how the students inscribe themselves into the temporality of imaginary situations with the robots. The ways of perceiving this temporality are different for different (groups of) students: some imagine that one robot has to "wait for" the other to know what to do (Giulio, Luigi); for others, coordination is not needed (Oliver) or considered (Lorenzo). Time is duration and simultaneity of movements: both aspects become problematic for learners. Both are crucial in making sense with WiiGraph of time as the independent variable in space-time functions.

Conclusive remarks

In this paper we have discussed some unexpected graphical solutions to a given diagramming task based on modelling motion through the use of Wii graphing technology. We have focused not on how these solutions were incorrect with respect to the expected diagram, but on ways in which they brought forth new possibilities for the students to encounter spatio-temporal relationships. In so doing, we looked at visual, proprioceptive and kinaesthetic aspects of experiencing the technology as that which sustained the occurrence of new mathematical events in the classroom, bringing into being problematic perturbations of the given situation, like shrunk graphs as well as not coordinated movements and fixed paths, which break with the conventional visualization and activity of the representational system in use. The written, the diagrammatic, the discursive and the bodily, as the groups attempted to grapple with the task (to make sense of it), have to be seen as that which animated the task without ever exhausting it or fully determining it. The temporality of the events speaks directly to the material contingency of learning: the students are dynamically affected by the diagrammatic activity while telling stories of motion related to graphing technology.

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Emerging and developing multiplicative structure in students' visuospatial representations: Four key configuration types

Carla Finesilver

King's College London, UK; carla.finesilver@kcl.ac.uk

Visuospatial representations of quantities and their relations are widely used to support the understanding of basic arithmetic, including multiplicative relationships. These include drawn imagery and concrete manipulatives. This paper defines four particular configurations of nonstandard representation according to the spatial organization of their visual elements. These are: unit containers, unit arrays, array-container blends, and number containers, all of which have been observed to support developing multiplicative thinking, allowing low-attaining students to work with the equal-groups structures of natural number multiplication- and division-based tasks. Student-created examples are discussed, and pedagogical and diagnostic implications considered.

Keywords: Visuospatial representation, multiplicative thinking, arithmetic, low attainment.

In their early encounters with quantitative relationships, children become aware of concepts such as conservation of number, counting, etc., through interactions with collections of objects. For example, addition as the joining of collections and subtraction as removing a subset of objects from a collection – in which the ordering of individual objects is unimportant – can be considered conceptual 'grounding metaphors' (Lakoff & Núñez, 2000). Various models of children's arithmetical problem-solving development indicate a broadly similar progression from early concrete/enactive-based reasoning, to imagic/iconic, to abstract/symbolic reasoning (e.g. Bruner, 1974; Piaget, 1952). Within this broad outline, the actual external representations of learners' thinking during problem-solving include many possible sub-varieties (e.g., sets of actual objects, pictures of objects, tally marks in different configurations, dot arrays, etc.), and many possible categorizations of these for analytical purposes. The construction of appropriate analytical frameworks is necessary for the discerning of inter-individual differences and intra-individual trajectories (Meira, 1995; Voutsina, 2012). This is particularly the case when studying atypically-developing learners (Fletcher et al., 1998).

This aim of this paper is to share one aspect from the qualitative analytical framework for student- and co-created visuospatial data used in Finesilver (2014), delineating four particular types of visuospatial representation and demonstrating their use with selected examples. The project took an essentially grounded analytical approach, and so whilst this paper does not report results as such, a sample of research data is included with brief description of the process.

Theoretical background

To understand multiplication and division represents a significant qualitative change in learners' thinking compared to understanding addition and subtraction (Nunes & Bryant, 1996). These authors, amongst others, have recommended a *replications* model of multiplication, which is highly relevant both to counting-based strategies and to unitary drawn or modelled representations of multiplicative relationships. A central concept for considering this particular aspect of representation is *spatial structuring*:

> We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. The process [. . .] includes establishing units, establishing relationships between units [. . .] and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). (Battista & Clements, 1996, p.282)

There are two main forms of spatial structuring with which unitary visuospatial representations of multiplicative relationships emphasise their replicatory structure: by creating some kind of boundary to separate groups of units from each other, or by organising them in a pattern based on regular spacings. These two organisational strategies roughly correspond to Lakoff and Núñez's (2000) grounding metaphors *Arithmetic as Object Collection*/*Construction*, and to two of the common unitary configuration types I introduce below, *Unit containers* and *Unit arrays* (see Figures 1Figures 2).

Creating *container* configurations – i.e. visible boundaries within which the individual units of each group may be in any configuration – is particularly intuitive. Research that includes container representations (or equivalent) has been mainly focused on young children and their intuitive concrete models, such as sharing items (e.g. Carruthers & Worthington, 2006; Kouba, 1989). Rectangular *array* configurations, in which the groups are structured and defined by a configuration of all units in regular rows and columns – are also widely used in educational contexts. Research including array representations generally focuses on older children, grid arrays, and involves content such as rectangular area measurement; however, dot arrays have been shown as a powerful tool for supporting work in multiplication (Barmby et al., 2009; Harries & Barmby, 2007; Izsák, 2005; Matney & Daugherty, 2013), and, less frequently, division (Jacob & Mulligan, 2014). No prior studies were found that included both container and array representations, focused on the secondary age group and allowed freedom of representational strategy across multiple interviews and tasks.

Data

The data discussed below, including all examples, derive from a larger research project using microgenetic methodology to elicit and study emerging and developing multiplicative structure in low-attaining students' visuospatial representations within a flexible context (Finesilver, 2014).

There were thirteen participants, aged 11-15, attending mainstream schools in London, and identified by their teachers, educational histories, and initial sifting assessments as particularly numerically weak compared to their peers. Although having complex individual etiologies and patterns of arithmetical issues, they had in common difficulties experienced at the particular stage of moving from additive to multiplicative thinking (as highlighted by Nunes and Bryant, above).

The representations were produced during individual or paired problem-solving interviews carried out by the author (four per participant). Participants worked on tasks based within two multiplicative scenarios chosen for their ease and likelihood of visuospatial representation. These were 'Biscuits' (numbers of biscuits shared between numbers of children) and 'Passengers' (numbers of different-sized vehicles required to transport numbers of passengers). There were also some calculations presented symbolically with no scenario. The representational media available were multilink cubes, coloured pens and paper. Some representations were co-created by student and researcher at 'cognitive snapshot' points (Schoenfeld, Smith, & Arcavi, 1993), i.e. when a participant was unable to proceed further independently, and support was given in the form of a minimal 'nudge' prompt;

(e.g. ringing or counting a group aloud). Due to project methodology, support cannot be easily quantified (especially gestural interaction) and is not attempted in this paper. Documentation was via audio recording, photographs, scans of students' papers, and field notes.

Four key types of representational configuration

Over 200 visuospatial representations were collected (exact figures cannot be given as participants re-appropriated whole and parts of prior representations for subsequent tasks and expansions). The great majority were found to group into four types; inclusion criteria, as defined below, were allowed to emerge, then refined, as part of a grounded analytical process. The most common types, *(unit) containers* and *arrays*, will be familiar. A smaller substantial proportion combined both container and array elements, and a further type emerged which I call *Number Containers*. (There is only space to include a few examples here; more will be included in this paper's accompanying presentation, or see Finesilver (2014) for a complete set.)

Unit Containers (UC)

Criteria: Groups of two or more units enclosed by visible boundaries. Includes representations where units are aligned in rows and/or columns, but these do not represent divisor/quotient or multiplier/multiplicand.

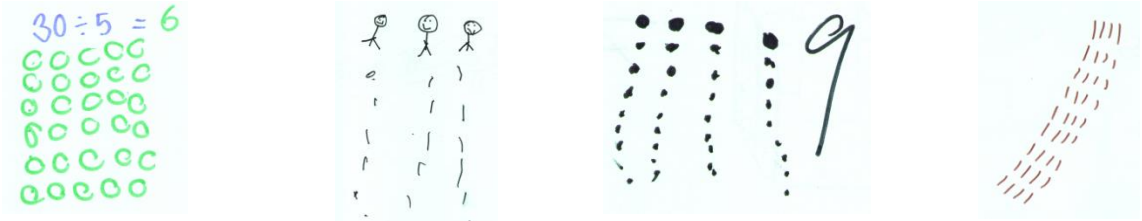


Figures 1(a-d): Examples of Unit Containers

Overall, this was the most common type (106 instances); eleven of the cohort chose to draw unit containers at some point while working on a task, although some much more frequently, and even the least able could sometimes use them independently. For the students with the severest arithmetical difficulties (e.g. dyscalculia), who could not make any start independently, visuospatial prompts were provided, e.g. drawing a set of circles ("plates") for 'Biscuits'. UCs were for the most part drawn, often with various scenario-based decorative elements, but some made use of mixed-mode, mixed-media representations with cubes or other physical units placed in drawn containers (see Figure 1d).

Unit Array (UA)

Criteria: Groups of two or more units aligned in rows and columns, where number of units in the rows/columns represents divisor/quotient or multiplier/multiplicand.



Figures 2(a-d): Examples of Unit Arrays

Plain unit arrays (of dots, tally marks, etc.) were used frequently (47 instances), the majority being produced independently by nine of the cohort, and an almost exclusive choice for three participants. All were drawn, and none constructed with cubes. (This may be surprising, as it is easy and visually effective to produce cube arrays. However, in general it was the arithmetically weakest students who made greatest use of concrete media, and that group also tended to prefer container representations.)

With a shift of perspective between vertical and horizontal structure, a learner may see that both rows and columns are formed of a set of equal groups, which underlies the commutative principle. This was independently noticed by some participants; e.g. on being asked to work out 28 biscuits shared between four people followed by 28 shared between seven, some re-used the same array, while others produced both 4×7 and 7×4, only realizing the equivalence after completion.

Array-Container Blend (ACB)

Criteria: Unit array representation with additional containing rings, where number of units in each row/column/container represents divisor/quotient or multiplier/multiplicand.



Figures 3(a-d): Examples of Array-Container Blends

While 47 instances of successful ACB use were collected, many of these were co-created and/or drawn during one particular task (see below); however, 27 were otherwise produced independently by participants. These were used mainly in ‘Passengers’ and the bare tasks, usually (although not always) with each row or column being counted out then ringed before proceeding to the next. Taking the additional time and effort to superimpose rings onto an array was thus clearly considered advantageous for certain participants on certain tasks. One student in particular began with a strong preference for plain dot arrays, but once she had seen an ACB, switched almost exclusively to that representation type for subsequent tasks.

In one particular (and uncharacteristic) task on multiplicative relationships, students were directly encouraged to produce an ACB which had both rows and columns ringed. A certain behaviour was observed with this representation type alone: some students independently looked back at it during later tasks and interviews for reference, in some cases ‘bookmarking’ it. As the numbers involved were different to those in their current task, and they only took a brief look, I suggest the images were functioning as an instant visual reminder of the commutative property of multiplicative structures.

Number Containers (NC)

Criteria: Container representation with numerals (rather than unit marks) representing the number in each group written inside, or close by, each container.



Figures 4(a-d): Examples of Number Containers

Unlike the previous three configuration types, NCs were not found in the literature or theorized prior to fieldwork, and some students introduced them spontaneously. Having observed their successful use, I included them in some later interactive support occasions, but of the 30 instances collected (from 9 participants), 22 were entirely independent. This change from unitary (iconic) to non-unitary (partially symbolic) representation is very significant cognitive step. Note, however, that some participants still chose to incorporate decorative elements from the task scenario (i.e. the vehicles were still depicted, although individual passengers were not).

Discussion

Students' use of the four types of representational configuration

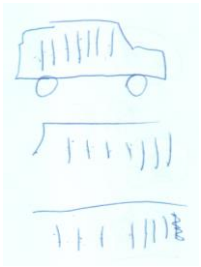


Figure 5: Transitional representation

Unit container representations allowed those students with the greatest arithmetical difficulties to create manipulable simulacra of imaginable scenarios, with as much visual resemblance as they preferred, to carry out organized sharing and grouping distributions and record their thinking. Unit array representations (with or without rings) allowed those students with a grasp of equal-groups structures, but who were not yet confident working symbolically, to perceive and make use of replicatory patterns spatially structured along two dimensions. However, the split between participants choosing to include container and/or array structuring elements also indicated personal preferences as a separate factor to arithmetical ability. (This has potential for further investigation, involving testing participants' visual pattern recognition).

While some individuals displayed firm preferences for container- or array-based forms throughout, others' representational strategy choices changed over the course of interviews, and sometimes intra-task. For example, Figure 5 shows a student's representation for calculating the number of 7-seater vehicles needed for 21 passengers, starting with a container resembling a car, then immediately discarding decorative elements and containers, in transition towards an array format.

Increasing the quantities within tasks (for those students judged likely to cope with the challenge) sometimes resulted in strategic change, in particular the introduction of number symbols. However, the general persistence of container elements surrounding those symbols (i.e. Number Containers) is striking. As seen in Figures 4b and 4d, non-mathematically-functional decorative elements (bus wheels, aeroplane wings) were included inconsistently. From a purely calculation-based viewpoint, students using NCs might as well be using plain columns of numbers – therefore the container elements clearly fulfil some other, non-enumerative, yet important, function. I suggest containers forms are a powerful visuospatial/perceptual phenomenon relating to equal-groups number structures and relationships, which persists later than might be expected. It is reasonable to expect that as

confidence is gained, the containers begin to disappear (but could be retrieved as a reassuring strategy at times of low confidence – for example, when tasks increase in difficulty).

Obviously, all types of representational configuration were used to a great extent for the enumeration of quantities, and for the visuospatial organization of these quantities so that the correct set of objects (units or groups) could be enumerated. However, it is worth noting that the representations created were not immediately rendered useless once a task solution was found. Students completed visuospatial patterns when an incomplete pattern would have been sufficient to obtain an answer; they sometimes added further organizational (or decorative) detail after giving an answer. Occasionally they even created a whole new representation to record their working retrospectively, or to help them explain an exciting discovery they had just made about numerical relationships (e.g. the commutative principle). The fact that these representational activities were important to the students for their own sake (i.e. not just for obtaining the answer in a single task) suggests that they can be an important part of these students' developing arithmetical reasoning, and their real and perceived agency in this development.

Representational configurations and developing multiplicative thinking

> Representations of mathematical objects […] can be seen as concretizations of abstract mathematical concepts and at the same time as representations of real objects. (Wittmann, 2005, p.18)

The four related types of representational configuration defined and discussed above integrate numerical and spatial concepts to form visuospatial mathematical objects that allow such a dual role: concretizing numerical relationships and representing real-life objects referred to in scenario tasks.

Whilst all four types represent equal-groups arithmetical structures, they do not fall along a single line of progression (see Figure 6, below). In the same way that concrete representations (e.g. modelled with cubes) are not necessarily less mature than iconic ones (e.g. drawn images), different types of configuration have different affordances which may be relevant at certain points. Number Containers, being non-unitary, are a clear progression from Unit Containers in terms of calculation, by requiring step-counting or repeated addition rather than unitary counting. However, Unit Arrays better instantiate the two-dimensional, reversible, nature of multiplicative relationships, whilst the ringing of rows or columns in ACBs could link procedural and static conceptions of multiplication/division.

The analysis of a set of relatively open-ended, student-generated, qualitative data based on their use of four key types of representational configuration highlighted a particular aspect of these students' late- and slow-developing multiplicative thinking: the many small adjustments that together can indicate a gradual change of focus of attention from units to groups, all happening within what is often considered to be a single stage of 'counting-based strategies'. Whether a task is multiplication- or division-based, there is a total quantity which is made up of, or can be separated into, equal groups. In terms of enumeration, the most basic strategies involve counting without any awareness of the repeating structure, while the more advanced ones make use of it. In terms of representational strategy, the most basic involve manipulating concrete or drawn units individually, to seeing and using visuospatial repeating patterns of units, to manipulating component groups as though they were units, to – eventually – focusing on these groups as new, composite units.

An individual's progress in this move from units to groups as main focus may be diagnosable via their representational strategic choices, along various possible trajectories (see Figure 6). (The bracketed items are likely or potential subsequent steps which, however, did not feature in the project from which this data derives.)

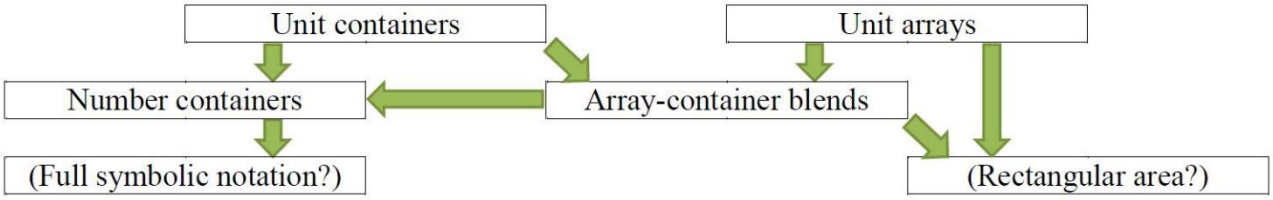


Figure 6: Potential developmental trajectories through representation types

Regarding this change of focus, there is a particular point of interest in ACBs: although they are still unitary representations (i.e. every unit is visibly present and countable), the visual and enactive emphasis on ringed subgroups serves to shift the student's level of visual focus, drawing attention away from the units and towards the groups. Thus, it encourages the possibility of seeing containers (enclosing well-aligned sets) as the new 'units' for manipulation. Meanwhile, with NCs, the replacing of (iconic) units with (symbolic) numbers is not only important for its progression toward standard notation, but as another part of this change of focus from units to groups – the change from using one mark to stand for one thing, to using one mark to stand for a collection of many things.

Even from a small sample of students it is clear that their patterns of capability, difficulty, and the representations which work best for them, are complex, interrelating, and individual. There is no single ideal path through from, for example, dealing out a pile of physical items to a set of actual present people, and carrying out a fully symbolic division calculation. However, from a teaching/learning perspective it appears important that at no stage is the leap too wide or too hasty, and that there are visual links when moving from more intuitive to more abstract representational strategies. From an analytical perspective, I suggest that tracking students' use of these four key representational configuration types in their arithmetical problem-solving (both in their initial choice of type, and in the emerging and developing spatial organization of elements within representations) may be beneficial in further study of the progression from additive to multiplicative thinking.

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Identifying key visual-cognitive processes in students' interpretation of graph representations using eye-tracking data and math/machine learning based data analysis

Enrique Garcia Moreno-Esteva [1], Sonia White [2], Joanne Wood [2] and Alexander Black [2]

[1]U. of Helsinki, Faculty of Educational Sciences, Helsinki, Finland; enrique.garciamoreno-esteva@helsinki.fi

[2]Queensland University of Technology, Brisbane, Australia; sl.white@qut.edu.au

We present a mathematical and computational analysis, partially based on machine learning techniques, of the visual scan-paths obtained during a graph interpretation task which allows us to identify when the problem solver succeeds in solving the problem with a fair degree of accuracy, and helps to understand the visual-cognitive processes at work during the problem solving task.

Keywords: Graph interpretation, eye-tracking, machine learning, mathematics education, gaze metrics

As a way of introduction: About the task and machine learning

Eye-tracking is quickly becoming an established technique for investigating cognitive processes involved in the learning of mathematics and other subjects (Lai, et al., 2013). Unfortunately, the analysis of eye-tracking data is difficult and laborious, often involving frame by frame analysis (Garcia Moreno-Esteva, Hannula & Toivanen, 2016). We partially overcome this difficulty here, with the use of machine learning and other mathematical techniques. Using a desktop eye tracking system, children completed a mathematics problem that incorporated a bar graph. The visual scan-paths and the accuracy of the response are analyzed in order to understand how a child "reads a graph". We are trying to gather from our data and its analysis, a story of what happens when several children are confronted with such a task. What do they look at? Do the gaze patterns influence the success or accuracy when responding to the task? With this information we may be able to more reliably infer the cognitive processes completed by children.

The problem-solving task

In Brisbane, Australia, a group of 113 children (mean age 8.67 years), all in the second half of year 3 in school, completed the graph problem solving task. As part of a larger project, children completed a series of eye tracking tasks (reading, mathematics) in a quiet room near their classroom. The mathematics tasks included odd-even judgement, magnitude comparison, and problem solving tasks: interpreting a bar graph and navigating a coordinate grid. The focus of this presentation is the graph problem solving task. This task was designed based on the Grade 3 Australian Curriculum Mathematics where Grade 3 children are expected to be interpreting and comparing data displays (ACARA, 2016). A similar graph interpretation task features in a Grade 3 Australian standardized achievement test. The children were shown the following: a) a bar-chart, where the height of each bar indicated a the number of hours worked by Sarah during a given week; b) a labeled coordinate system, where the x-axis had the week number labels, and the y-axis had numbers corresponding to hours; c) a sentence indicating Sarah's hourly wage; d) another sentence indicating the task to completed related to Sarah's wages in Week 3. Curcio (2010) describes a sequential framework for children's

data comprehension, this framework includes; *understanding, interpretation and prediction* with data. The current graph task required each child to *understand* and *interpret*: reading the question and basic details of the graph (*understanding*), and then reading between the different elements of information (*interpretation*) in order to complete the computation and arrive at the correct solution for Sarah's Week 3 earnings. A Tobii eye-tracker operating at 300 Hz recorded the locus of focus of their eyes throughout the activity - including the initial *understanding*, and steps involved in *interpretation*. The threshold for fixations was set at 100 ms (Tobii Technology, 2014). It was hoped that the eye tracking information (fixations and saccades) might shed light on the different cognitive steps involved in the task. Initial qualitative evaluations of the eye movements demonstrated children who did not progress past the first *understanding* stage, as they did not identify the question being asked or relevant information on the graph. Other children were able to *understand* the task and progressed to specific *interpretation* of relevant information - with a variety of behaviors demonstrated. For example, some children had high numbers of fixations and saccades around relevant areas, whereas others had fewer and longer fixations on relevant areas. These initial qualitative observations were systematically investigated using machine learning techniques.

The data included 113 visual scan-paths (for purposes of the forthcoming discussion, the *inputs*), and 113 answers (the *outputs*), considered as correct (1) or incorrect (0). The visual scan-paths consisted of sequences of pairs, each pair including the duration of a fixation in milliseconds (ms), and the location of the fixation. The visual scan-path information can be visualized as a video (or a static picture) in which the fixations appear as a sequence or red dots that have a size proportional to the duration of the fixation, and which are connected by lines to neighboring fixations.

After inspecting the visual scan-path videos, it was evident that it would be difficult to disentangle patterns of visual processing that might reveal cognitive processing of different children. It was decided that further mathematical/computational analysis of the data might provide further insight. Since the nature of the input data is sequential, classifying the visual scan-paths and test results (inputs and outputs) with a Markov model based machine learning technique was selected as an appropriate analytic method.

A word about machine learning techniques

The proprietary algorithm (*Mathematica*'s Classify function) was used to do the machine learning analyses, using a Markov model method (Wolfram Language and System Documentation Center, 2016). In this analysis you select a subset of the sample (input – visual scan-path - and output – result - data) to analyze (*classify*) with the machine learning algorithm. From that analysis a *classifier* is then used on all the inputs (visual scan-paths) to predict the outputs (0 or 1, incorrect or correct). The predicted outputs from the classifier are then compared to the real outputs, and the percentage of correctly classified outputs can be calculated (some examples are provided in subsequent sections).

Our research question

Our research question is simply, what can we learn or infer about cognitive processes related to the graph interpretation task described with mathematical/computational/machine learning based analysis techniques of the eye-tracking data, and maybe, could these techniques be of further help in analyzing the data pertaining to other well defined mathematics problem solving tasks?

Our techniques are general, in that they can easily be applied to other eye-tracking data consisting of a sequence of fixations given by the coordinates and the durations of the fixations as inputs, and a set of two or even more categories as outputs. We hope to make the programs available to other researchers wanting to undertake this kind of analyses at a later stage or our research.

The analyses and corresponding results

In this section we will describe three kinds of analysis for which we obtained encouraging results. Other possible analyses will be discussed in a later section pertaining to directions of future work.

We partitioned the visual stimulus (the graph on the screen; Figure 1) into areas of interest (AOIs), where the most critical areas of interest are labeled as A1 (wage information), A2 (week number), A3 (week 3 bar), and A4 (number region containing the number of hours corresponding to week 3), and other areas of interest which are less critical, or irrelevant, are labeled with letters B and C and a number, respectively. In addition, we labelled the whitespace around the critical areas as ZZ.

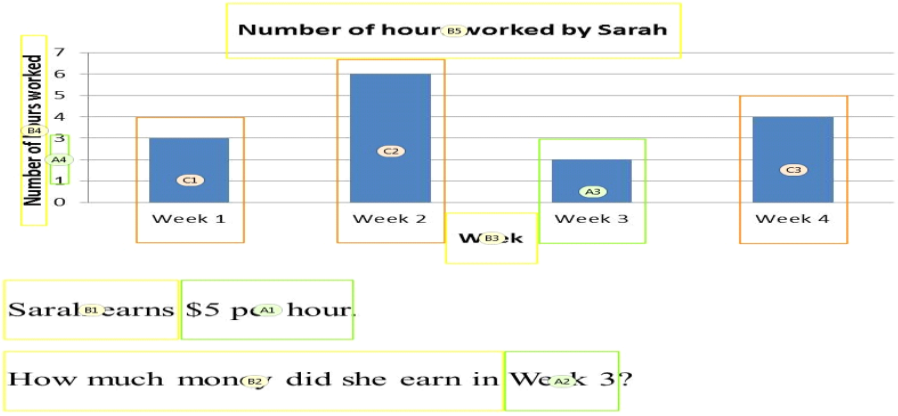

Figure 1: partition of the task sheet into areas of interest (AOI's).

As a result, data items look like the following:

{{227, A1}, {563, B2}, {267, C2}, … , {287, C2}, {517, A1}, {1443, A3}} -> 1,

Figure 2: a typical data item with pairs of elements corresponding to durations in milliseconds (numbers) and AOI's (letter and number juxtaposed) of the fixations, and the result after an arrow.

In the example above, the first fixation occurred on area of interest A1 and lasted 227 ms, the second one on AOI B2, with a duration of 563 ms, and so on. At the end the arrow with a 1 after it indicates that the child solved the problem correctly.

Finding a small and highly representative subset of data (developing a training set)

In order to find small and highly representative sets of data items corresponding to correctly and incorrectly solved instances of the task, we tried to find the smallest subsets of data items (henceforth called *training sets*) on which we could generate classifiers that predicted outcomes with a high degree of accuracy. After a building classifiers based on randomly selected subsets of data items, we could generate a classifier that correctly predicted up to 75% of the test results, and this was using only four data items in the training set (3.5% of the *sample*). It would have been impossible to test all sets of four data items out of 113 (there are 6,438,740 such combinations) so we made a number of

classifying testing runs for randomly selected subsets of size 4, and chose some of those sets which yielded classifiers with a high prediction rating. We then inspected the videos of some of these sets and tried to observe what might have been visually outstanding in these. Our prediction rate is marginally better than human experts can do after training on very large data sets. In the world of machine learning, a rating of 75% with a training set of size 3.5% is an extremely good result in what is called supervised learning (since the training set we found is so small, this is called semi-supervised learning (for machine learning principles, consult Hastie, Tibshirani & Friedman, 2009).

From this inspection, we detected parameters to investigate further with machine learning and other techniques, including sequencing, duration and number of fixations and other more elaborate metrics.

Analysis type 1: the order of fixations in the sequence – does it matter or not?

One question we had was whether the order of fixations in the sequence matters, or whether there is something else at work. Some literature in psychology indicates that the order of fixations affects certain cognitive function such as memory (e.g. Bochynska, & Laeng, 2015; Rinaldi, Brugger, Bockisch, Bertolini, Girelli, 2015). First, we tested overall order, building a classifier using the entire sample data. Its predictive rate is over 99% (using this technique we get only one mismatch between predicted and real outputs, due to a faulty item which we were able to locate through the application of the classifier itself). We then permuted the order of the fixation duration and AOI pairs at random in the visual scan-paths, and passed the permuted input data through the classifier we obtained using the entire sample. Even with the permuted data, we obtain a classification rate which is over 97%. From this we cautiously concluded that the order of the fixations in the sequence has little impact on whether the child responds to the question accurately.

As an additional check, we investigated whether the order of fixations within critical AOIs mattered. If this were occurring, it might distinguish *understanding* and *interpretation* of the graphical information (Curcio, 2010). In order to study this, we extracted just the pairs of elements corresponding to critical elements, and eliminated the rest of the data elements. With these modified data items, we built a classifier, using training sets of size 13 (approximately 11% of the sample size), and passed the rest of the modified data items through the classifier. This resulted in a prediction rate of up to 66%, which is good but not nearly as good as we had hoped. This indicates that the order in which students inspect critical areas might be of some importance, and it deserves further study. This also led us to a different form of analysis (type 3), even though much more needs to be done than we did here.

Analysis type 2: number of fixations and duration of engagement on task

The number of fixations and their duration (see figure 3) for the subjects is extremely revealing even though the analysis is less complex. These fixation duration profiles could be interpreted like a simple fingerprint of student engagement and ability. Our analysis of the number of fixations and their duration gives a clear indication that visual scan-paths can be quite revealing about what the students can or actually do. To state the results briefly, children who respond correctly take a short amount of time (under 30 000 ms) to provide and answer and have a smaller number of fixations (mean of 69) than children who respond incorrectly. Most of the children who respond incorrectly take at least 35 000 ms to respond or have more than 69 fixations. The statistically significant duration averages for children who respond correctly and those who do not are 30 000 ms and 35 000 ms respectively, and

69 fixations vs 77 fixations respectively. Interestingly, a few children (34 out of 113) who take a short amount of time and have a small number of fixations, typical of children with a correct response, provided an incorrect response. In most of these cases children had gathered the correct information from the graph but had made a calculation error. There are 17 children for which we have not yet determined an adequate explanation of their performance. Had those children read the graph incorrectly? Had they understood the task? When interpreting the graph and performing the computation, did concepts become confused? We found that these 17 children completed the task very quickly relative to the other participants, with a mean response time of approximately 25 000 ms. This information leads us to speculate that these children may not have been fully engaged in the task or in some respect confused or wandering. In summary, we can pick out, in each case, the children according to their response from a quantitative analysis by looking just at the duration of their engagement and the number of fixations during their involvement in the task. In the future, we plan to do an Artificial Intelligence based cluster analysis of the number and duration of fixation profiles only, hoping that they will separate out into four categories: those of children who respond correctly, those of children who do not read the graph correctly, those of children who read the graph correctly but miscalculate, and those of children who "do something else". There is interest and possibly a growing body of work around this topic, whether it is possible to classify gaze patterns according to the state of mind of the participant subject. It is definitively one of our goals in this and future research (e.g., Horrey, Lesch, Garabet, Simmons, Maikkala, 2017).

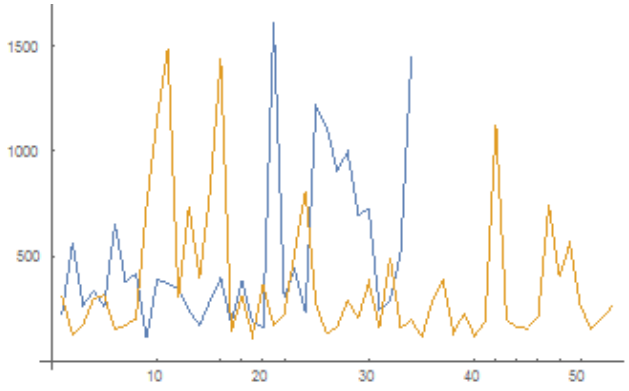

Figure 3: number of fixations and duration profiles of successful child (blue) and unsuccessful child (orange) – the x-axis is the number of fixations, the y-axis is time, the duration of fixations, in ms

Analysis type 3: duration ratios and frequency ratios

From viewing the videos it appeared that children who get the problem right seem to spend a substantial amount of time looking at critical data, and seem to look at such data more frequently. These parameters were assessed quantitatively, making a distinction between the importance of the area of interest (e.g. A, B, C), and not between the areas themselves (e.g. A1, A2, A3 etc.). Thus, we measured the total amount of time a subject spent looking at critical AOI's (with labels Ax), and non critical areas (Bx, Cx, and ZZ), and also measured the frequency with which a subject inspected an AOI labeled with A, B, C, or ZZ. The total duration of fixations on areas A, B, C, ZZ became DA, DB, DC, and DZZ, and the we considered the ratio DA/(DB+DC+DZZ). We then computed the means of this ratio for the students who successfully solved the problem and for those who did not. The means were used to compute a threshold value and make predictions as to who would

successfully solve the problem or not. The same approach was used for frequencies (call the total frequency on A-critical areas FA, FB for B-critical areas, FC for C-critical areas, and FZZ). We computed an analogous ratio where the quantities FA, FB, FC and FZZ were weighted by coefficients 1, .5, .25, and 0, respectively. The rationale for using weights in the case of frequencies is to account for the fact that looking at less critical AOI's, for example, whitespace (ZZ), can easily occur as a result of distraction while inspecting the graph or while moving from a fixation in an important area to another one, and therefore, they are overrepresented and should carry a smaller weight in the frequency count. We acknowledge there are alternative approaches that could be used.

With the two thresholds used in combination one can predict the results with an accuracy of 77%. The thresholds were combined in such a way that if a child spent both, enough time on critical areas, and looked at them frequently enough, the result would be success, and otherwise, it would result in an incorrect response. So it seems that both these parameters are indicative of a child's ability to successfully solve the graph interpretation task. A post-hoc statistical analysis was done on the means obtained for the duration ratio and the frequency ratio to show that they differ in a statistically significant way. Assuming a normal distribution of the duration ratios, the means of children who were successful and unsuccessful were 1.13 and .76, with a standard deviation of .43 and .42 respectively. These means are statistically significantly different $p << .001$. Similarly, having tested for the normal distribution of frequency ratios the means are 1.81 and 1.33, with standard deviations of .53 and .54, and $p << .001$, showing again a very significant difference.

A note about validity and reliability

The results discussed here would need to be validated with further experimentation. For example, do the results hold if the experiments are repeated with systematic variations, changing the height of the bars, the number of the week, and the salary for Sarah? Similarly, do the results remain invariant cross-culturally? We have thought of replicating the experiments, with children of the same age and/or background knowledge, in different English speaking countries and in different cultures with different languages. This work remains to be done. The reliability of these results is given in as much as the calculations are straightforward and easy to check, and the data is clean data as provided by a commercially tested device. It is hoped that in the future, a functional version of the paper can be republished in a way that the reader can verify the programs and use the programs with his/her own data.

Conclusions and direction of future work

In this report we have discussed the kind of visual processes that might be at work when a child is solving a graph interpretation task, a discussion derived from a machine learning analysis of eye-tracking data collected during the problem solving sessions. It would seem that there is strong evidence to support the claim that the order of the fixations during the problem solving session plays almost no role in the child's ability to succeed in the problem solving task. It would also seem that the amount of time and the number of times spent looking at areas where there is information which is critical for the solution of the problem relative to the amount of time and frequency of glances at other areas is definitively an important indicator of a child's ability to successfully complete the task.

As to how these results would affect teaching practices, one could conclude that it is important that

the teacher directs the student attention to what the critical information might be, where it might be located, and how to use it when teaching how to interpret graphs of this sort.

There are many other measures that can be studied (or have been studied, but are not reported here). We mention just a few, without further explanation: string edit analysis, lag analysis, cluster analysis, longest common sequence analysis. The limit in how to analyze gaze tracking data is our imagination.

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Epistemic and semiotic perspectives in the analysis of the use of digital tools for solving optimization problems in the case of distance teacher education

Veronica Hoyos

National Pedagogical University, Mexico; vhoyosa@gmail.com

The purpose of this work is to revisit from an epistemic and semiotic point of view the use of technological tools for solving problems of optimization accomplished by in-service secondary teachers participating in an online course intended for their professional development. This approach allowed to analyze teacher limitations on operational resolutions, and drew hypothesis about difficulties for experiencing processes of reflection on their own resolution without the implementation of adequate collaboration tools for working between them or by themselves and at a distance.

Keywords: Epistemic and semiotic framework, teaching and learning of mathematics at distance, resolution of optimization problems using technological tools, online teacher professional development in mathematics.

Theoretical frame

In the work that it is presented here it is reviewed new data on the resolution of in-service secondary teachers using mathematics technology (*GEOGEBRA* in this case) to solve optimization problems. These teachers were participating in an online course for their professional development, mainly in relation with the incorporation of mathematics technology into their practice, but primary they were learning to use technology and learning to do mathematics with technology[(i)]. In this context, it has been important to know teachers' strategies during the resolution of complex mathematical tasks using technology, because these allowed to identify the mathematical resources displayed by teachers, as well as their understanding of the content that were at stake.

Is in relation with the teacher (or student) understanding of the mathematics content at stake that this paper deals with the role of representations in mathematics teaching and learning, because in according with Duval (1994), "there couldn't be understanding of the content represented without coordination of the representation registers, regardless of the representation register used. Because the peculiarity of mathematics in relation to other disciplines is that the objects studied are not accessible independently of the use of language, figures, schemas, symbols[(ii)]…" (Duval, 1994, p.12)

Moreover, school optimization problems in general are designed for the modelling of real situations. However, the mathematical representations that come into play (e.g. formulas, graphs or symbols, and the treatments or operations carried out with them) obey a set of rules and operative principles within a context of mathematical theories previously established. Thus, when a statement is made in mathematical terms, the validity or not of such statement comes into play, and this within a well-defined theoretical context (Habermas 1999, quoted in Balacheff 2010, p. 5/36). Balacheff expresses this complexity of mathematical work as follows: "mathematical ideas are about

mathematical ideas; they exist in a closed 'world' difficult to accept but difficult to escape" (Idem, p. 5/36).

Finally, it is from the perspective of determining the domain of epistemological validity[iii] of the computing devices for human learning (Balacheff, 1994), that Balacheff & Sutherland (1994) have found a way to characterize a computational learning environment with reference to a given field of knowledge, but also the forms of analysis that make sense of the differences between distinct computational environments, as well as its potential contribution, specific to the enterprise of teaching and learning of mathematics.

Therefore, for the sake of the work that is being presented here it is worth to be mentioned that in order to specifically analyze what the teachers were mathematically doing when solving the optimization problems with the technological tools (see more in Hoyos 2016), it was only possible using Balacheff and colleagues' theoretical notion of epistemological validity (e.g. Balacheff 1994-2004; Balacheff and Sutherland, 1994), and Duval's work on the coordination of representation registers of mathematics, specifically of graphs (Duval, 1994). These authors have illustrated the different contributions certain software has in different virtual learning environments (Balacheff et al, 1994-2010), and here it is noteworthy not only that the teacher (or student) learns to recognize those different register of representations (Duval, 1994) that are put in play by distinct computational devices or digital tools, but also the need of the coordination of representation registers than an appropriate use of computational devices involves when the validation of a solution is in question.

Collection of data and analysis

The data that are going to be showed here were part of performances of the in-service secondary mathematics teachers participating in certain online courses implemented by the MAyTE (an acronym for Mathematics and Technology) team (Hoyos, 2009-2012). These data are presented here for identifying what these teachers accomplished specifically for their mathematics learning during the six-month of online training courses they were participating into. In the MAyTE team's courses, the mathematical activities were developed around an understanding of concepts, learning procedures or mathematical techniques that relied mainly on asking participants in the program for the resolution of some specific mathematical problems, while only providing a brief list of instructions and explanatory text on the mathematical content. In this context, these courses generally did not include tutorial indications related to the mathematical resolution of the tasks requested.

It is noteworthy to emphasize here that it was possible until now to analyze the means or the strategies that participant teachers displayed using *GEOGEBRA* to solve the problems or learning situations provided by the MAyTE team by applying the constructs of epistemological and didactical validity of computational environments. Briefly, in the online MAyTE program (see Hoyos, 2012), the activities consisted of using digital tools that were freely available on the Internet to solve math problems. The mathematical content was approached synthetically through a capsule of the content, and the digital tools for solving the mathematical problems consisted of a variety of mathematical software, particularly software of dynamic geometry (SDG). Next it is described one of the prototypical math problems to be solved by the teachers participating in the program, the

practical context in which it was proposed and how the teachers finally managed to solve the tasks involved. Two of the selected optimization problems that teachers should solve are as follows:

1) A refinery can process 12,000 barrels of oil per day and it can produce Premium [high octane] and Magna [unleaded] gasoline. To meet the demand, the refinery must produce at least 2200 barrels of Premium and 1500 of Magna. The distribution center for the Premium is 30 km from the refinery and the Magna distribution center is from 10 km. The transportation capacity of the refinery is 180,000 barrels/km per day (This means that 180,000 barrels are transported 1 kilometer per day). If the benefit is 20 pesos per barrel of Premium and 10 pesos per barrel of Magna, how many barrels of gasoline should be produced daily to maximize the benefit?

2) A certain animal fodder is a mixture of two food products, namely A and B. Each kilogram of A has 100 units of protein, 18 of fat, and 400 of carbohydrates. The kilogram of B has 200 units of protein, 2 of fats, and 300 of carbohydrates. The aim is to make bags with a mixture of A and B products, each of which should contain at least 500 units of proteins, 18 of fats, and 1500 of carbohydrates. If the kilogram of A costs 3 pesos and the one of B costs 4 pesos, determine the number of kilograms for each of these products that must contain each bag of food so that the cost is minimal. (Week 8, Task 1. *Geometry and Algebra* Course. Specialization MAyTE, Hoyos et al., 2009-2012).

Most teachers' solutions to these problems were based on the identification and formulation of several algebraic expressions that modeled the given actual situation, and they were accorded to the data provided; as well as doing a graphical representation using GEOGEBRA, based on the algebraic expressions that firstly were elicited. Such procedures were needed to determine the region of feasibility and the coordinates of the points from which it was possible to obtain the maximum or the minimum cost, depending on the initial conditions of each problem. The solution that is going to be showed here (a data table and a graph in GEOGEBRA) was taken from the documents the teachers uploaded to the platform, and were evidence of a solution strategy composed of these elements: translation from the initial conditions to algebraic expressions, and representing the data through the software GEOGEBRA. Therefore, the teachers obtained a representation of the feasibility region from which the value of maximum benefit should be deducted (in the case of the first problem). In their graph, the feasibility region was shaded, and the problem in all cases was still unsolved after the graphic was made, because a point (with coordinates (x,y)) needs to be found by means of exploration and through calculating the values of the function of two variables f(x,y), and that in the region of feasibility (for attaining or not the benefit maximum for Problem 1 or the minimal cost for Problem 2).

Next it is shown that the teacher solution starts by constructing a table to organize the information of the data included in the text of the problem, and immediately afterwards it refers to the graph included in his solution.

T1 [One of Participant Teachers]:

If the benefit is 20 pesos per barrel Premium and 10 pesos per barrel Magna. How many barrels of gasoline must be produced each day for maximum benefit?

	Minimum Production	Distance	Benefit
Premium (y)	2200	30	20
Magna (x)	1500	10	10

Solution:

According to the problem data, inequalities are the following

$x + y \leq 12{,}000$

$10x + 30y \leq 180{,}000$

$x \geq 1500$

$y \geq 2{,}200$

Where x is the number of barrels of Premium, y the number of barrels of Magna, the function that gets the maximum benefit is

$f(x)=20x+10y$

To obtain the benefit maximum, it should be drawn the graph of the following functions using 'geogebra'

$x+y=12{,}000$

$10x+30y=180{,}000$

$x=1{,}500$, $y=2{,}200$

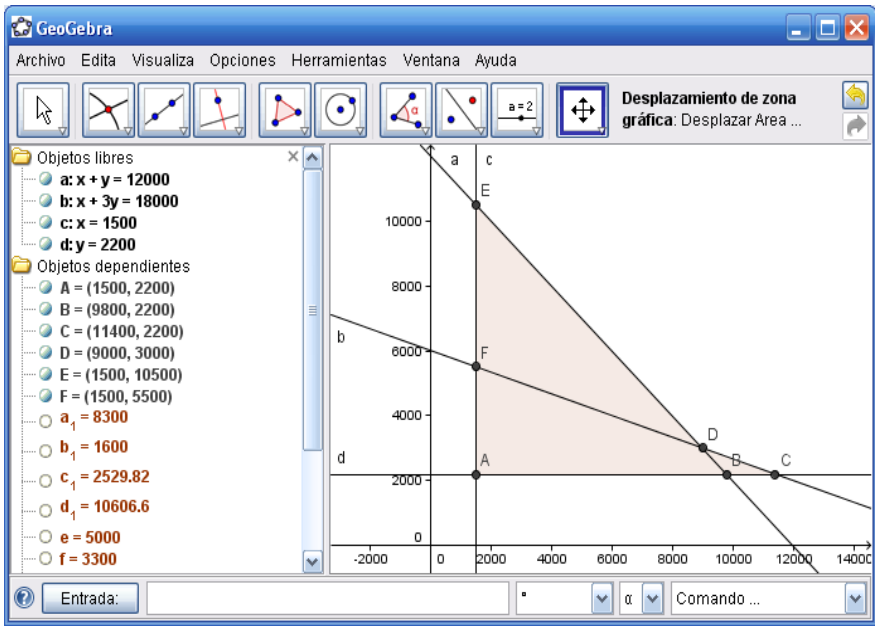
To find the value of x and y that maximizes f(x) function, we take the points that meet the initial conditions of the problem. Using the graph, you can see that the solution set is drawn within the limits of the lines and the shaded region.

In the graph, we can see that the points A, B, C, D, E, and F are some of the points that are possible solutions, but points C and D are not feasible, so substituting each of these points in the function to maximize the number of barrels of Premium gasoline, the solution obtained is x=9,800. And the number of barrels of Magna is: y= 2,200.

Giving a maximum benefit for f(x) = 218,000 bpd

The reader should note that the image of *GEOGEBRA* showing the representation of the feasibility region is part of the teacher's solution and it shall be inserted here.

Also it should be noted that seeing the image, in first place it must be considered that in T1's solution there is a mistake related to the notation that the teacher T1 chose, by which instead of denoting the function of two variables as f(x,y), the dependence of the function f was only indicated on a single variable, insofar as writing "f(x) = 20x + 10y" to refer to the function from which the maximum benefit will be established. Note that T1, after having adequately defined the region of points that satisfied the initial conditions, ended by not carrying out an exploration of the values in the region of feasibility, question that would bring T1 to obtain the requested maximum value.



Moreover, what is perhaps most interesting is to note that for the computer learning environment in question, in this case constructed mainly for exploration and use of *GEOGEBRA* and for the conversion of mathematical representations (Duval, 1994) required to solve the problem, an epistemological change in the conditions of the teacher (or student) is raised (Balacheff, 1994-2010) linked to the use of the software in the situation or problem proposed, and to the mathematical complexity of the task involved. A sign of this could be that a real graph of a function in two variables should be represented in a three-dimensional space. The difficulty then in the problem of optimization posed resides in that for solving the situation in question it is also required reflection, specifically when following the suggestions included in the capsule of content. These suggestions appeared beneath to the text of the problem: (i) to draw lines parallel to the axis and intersecting inside the feasibility region, and (ii) to explore the variability of the values of the benefit function f(x, y) within the feasibility region and/or on points of the indicated parallel lines, which would allow to calculate the maximum value requested in Problem 1.

Of course, there is another way to solve this problem, for example, by associating any point within the feasibility region to the value of the benefit function, such exploration could thus be carried out directly using *GEOGEBRA*, starting by dragging the point over the feasibility region and verifying the increase or decrease as the chosen point were varying. Moreover, this type of exploration would also help to reduce or eliminate the confusion T1 had concerning the double variability upon which the function f was dependent. For example, for point E with coordinates (2507.66, 6006.86) the value of the function f(x,y) equals 110,222 approximately. And it can be proven that the value of maximum benefit is f(x,y) = 149,924.05 when the approximate values for x and y are x = 2993.81, and y = 9004.79.

Briefly, to finally arrive to solve a problem posed in a classical school context of optimization problem solving, the teacher (or student) shall find that there is a very close relationship between the modeling of a real situation, the use and treatment of mathematical representations that come into play and the coordination of the representation registers in use. Thus, a possible trajectory for the resolution of such problems could consists of: (a) surpassing the initial difficulty of the formulation of a series of mathematical statements that model the real situation by the usage of mathematical

representations to model the actual situation; (b) advancing to get a diagram where the possible solution could be found by putting into play certain digital tools (in this case *GEOGEBRA* inherent digital tools) for the mathematical treatment or the conversion of such representations (Duval, 1994); (c) formulating a new mathematical statement, namely the possible mathematical solution to the posed problem. However, it is noteworthy that from a mathematical point of view the teacher (or student) having passed by (a), (b) and (c) has not yet concluded with the mathematical task in question, mainly because in the mathematics 'world' it is always necessary to carry out the validation of any mathematical statement last obtained or formulated (Balacheff, 2010, p. 19/36).

Conclusions

Because the difficulty to solve resides in to reflect on at least one of the three following possibilities: (i) on f as a function of two variables and that its graph would then be in R3 and not in R2, while plotting the region of feasibility is being carried out in R2; or, (ii) to reflect on the sense of the instructions and/or suggestions given at the end of the text of the problem, suggestions that concerned with the construction of parallel lines to explore the maximum value of the function f; or, (iii) on the possibility of carrying out an exploration using *GEOGEBRA*, starting by dragging a point over the feasibility region and verifying the increase or decrease of the function f as the chosen point was varying; it is worth to see that all of them are entirely relied on a necessity of feedback or teacher (or student) control of their activity within the software (see Balacheff & Sutherland, 1994, p. 15). But this control usually is relied on the coordination of the representation registers or on the comprehension of the mathematical content in question, which is usually not accessed directly by working alone within the software and at a distance.

Briefly, in a trajectory to finally arrive to solve a problem posed in a classical school context of optimization problem solving, the teacher (or student) shall find that there is a very close relationship between real situation modeling, the use and treatment of mathematical representations that come into play and the coordination of the representation registers in use. Therefore, the principal results in the analysis that has been instrumented here are as follows:

(1) The learning environment was in part defined using a computational device (in this case *GEOGEBRA*) as a procedural tool for the conversion, use and treatment of the different mathematical representations (Duval, 1994), in this case the equations and graphs that came into play in the given situation of optimization;

(2) However, to transit from a procedural context where a possible solution was found to a theoretical one to validate it, an epistemological change is required (Balacheff, 2010, p. 6/36). In this case, it consisted in instrumenting reflective tools, which are not automatically available within *GEOGEBRA* by itself.

Therefore, this work has allowed to advance a hypothesis of necessity of digital collaboration according to specific participant's (teacher or student) activity, a support to accomplish the epistemological change already mentioned. It would be included in the computational device, or otherwise it would be provided by tutorial intervention (e.g. Soury-Lavergne, 1997).

These final remarks mean it is not enough to have access to mathematics technology and/or Internet free resources to achieve expertise or comprehension of certain mathematical content addressed. Yet

perhaps what is most interesting is that the analysis from this epistemic and semiotic perspective sheds light on how to move forward by correcting the design, incorporating elements missing in the programs reviewed, instrumenting teaching guides (or constructing hypothetical learning trajectories) or working with digital materials as collaborative tools that could promote exploration and reflective thinking to be applied in the solution of certain mathematical tasks, as those that were showed now in the situations under study. In other words, in the same way that social interactions do not in principle have an impact on learning but rather depend on the content and forms of interaction chosen, the use of Internet digital tools and computational devices will have an impact on teachers and teaching (or students and learning) when instrumentalization of Internet resources had been exercised to gain knowledge, or to teachers (or students) get control of the activity within the software (Balacheff & Sutherland, 1994, p. 15), specially by themselves.

Notes

(i) Both modes are at the beginning of the incorporation of innovation at the school, according to the PURIA model. Following this model implies that teachers should experiment with the mentioned modes to advance toward successfully incorporating technology into classrooms (Zbiek and Hollebrands, 2008; Hoyos 2009-2012).

Briefly, the PURIA model consists of five stages named the Play, Use, Recommend, Incorporate, and Assess modes: "When [teachers are] first introduced to a CAS… they play around with it and try out its facilities… Then they realize they can use it meaningfully for their own work… In time, they find themselves recommending it to their students, albeit essentially as a checking tool and in a piecemeal fashion at this stage. Only when they have observed students using the software to good effect they feel confident in incorporating it more directly in their lessons… Finally, they feel they should assess their students' use of the CAS, at which point it becomes firmly established in the teaching and learning process" (Beaudin & Bowers 1997, p.7).

(ii) French is the original language of this quote: "il ne peut pas y avoir de compréhension du contenu représenté sans une coordination des registres de représentation, quel que soit le registre de représentation utilisé. Car la particularité des mathématiques par rapport aux autres disciples est que les objets étudiés ne sont pas accessibles indépendamment du recours à un langage, à des figures, à des schémas, à des symboles…" (Duval, 1994, p.12)

(iii) The domain of epistemological validity of a computational environment is characterized by at least four dimensions: (1) the set of problems that the device can propose; (2) the nature of the tools and objects that provide its formal structure; (3) the nature of the phenomenology that is displayed on the interface that is accessed directly by the user; and (4) the kind of control available for users in the computational environment with the feedback that the latter provides (Balacheff & Sutherland, 1994, p. 15).

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Continuity of the students' experience and systems of representation, an example of teaching and learning in mathematics first grade students

Sophie Joffredo-Le Brun[1], Mireille Morellato[1], Gérard Sensevy[1], & Serge Quilio[2]

University of western Brittany[1], Brest, France, s.joffredolebrun@gmail.com, mireille@syrah.fr, gerard.sensevy@espe-bretagne.fr,

University of Nice, Nice, France[2], serge.quilio@gmail.com

Our study is based on a national research project called Arithmetic and Cooperation at Elementary school (ACE). The main objective of this research is the designing of a curriculum for first grade students. This communication focuses on the analysis of an extract from a lesson proposed by this curriculum. During this lesson, the students work on the notion of difference, which is introduced with the help of several systems of representation. These systems are already known by the students. The analysis of the extract shows that the past and future situations can be related to each other. We point out that the students' continuity of experience can produce knowledge growth. This growth specifically occurs when the teacher's uttering activity directs the students' actions while they use the systems of representation. This uttering activity allows a reshaping / remodeling /modification of former knowledge through the systems of representation.

Keywords: Elementary school mathematics, continuity of education, semiotic representation .

This paper focuses on the continuity of epistemic experience in mathematics. We argue that systems of representation constitute a prominent way of achieving such continuity. We sketch the role of a specific system of representation (the number line) to build of the number sense at first grade. This research is based a larger French national research, arithmetic and cooperation at Elementary School (ACE). It offers a complete arithmetic program to 6-7 year-old-students (First grade). The conception of this curriculum relies on available scientific knowledge in different areas (Cognitive Neuroscience, Science of Education, Developmental Psychology and Didactics of Mathematics). A specific part of the conception of the curriculum is built within what we call a cooperative engineering (Sensevy, Forest, Quilio, Morales, 2011; Morales, Sensevy, & Forest, in press). This engineering consists of two spheres. The sphere 1 gathers a multi categorical team (PhD, teachers of study classes, researchers, teacher's trainers, pedagogical advisors) and the sphere 2 is constituted by 120 experimental classes. In the first year of the experiment (2011-2012), the sphere 1 designed the mathematical situations of the curriculum. These situations were implemented in the « study classes », and redesigned on line in the course of the implementation process. The first year of the experiment, the sphere 1 designed eleven modules corresponding to forty-five sessions. The second year of this experiment, this curriculum has been implemented in 60 experimental classes (*versus* 60 control classes) and in 120 experimental classes (*versus* 120 control classes*)* the third year (2013-2014). The experimental classes' involvement in this curriculum and their feedbacks allowed numerous improvements of the initial design proposed by the research team situations.

This cooperation between researchers and teachers showed a willingness to create a didactic continuity in student's experience through the use of representation of the number systems that are present in the progression throughout the year. We assume that in the joint action between students

and teacher, the systems of representation may authorize the continuity of the student's experience. But, how precisely this can be built in joint action between teacher and students? Here we can rely on Dewey's conception of continuity. "The principle of continuity of experience means that every experience both takes up something from those which have gone before and modifies in some way the quality of those which come after" (Dewey, 1998, p.27).

The Joint Action Theory in Didactics

Our analyses will build on the Joint Action Theory in Didactics (JATD) originated in comparative approach in didactics (Sensevy, 2011; Sensevy, Gruson, & Forest, 2015; Ligozat, 2009; Tiberghien & Malkoun, 2009; Venturini & Amade-Escot, 2013). Among the theoretical tools provided by the "JATD", we use mainly the contract/milieu dialectics and the reticence/ expression dialectics. When facing a new problem, students are confronted to what we call to a *milieu* (Brousseau, 1997; Sensevy & Tiberghien, 2015), as the epistemic structure. This milieu can be seen as the state of problem, what "has to be known" (Sensevy et al, 2015). Students have to face a rather enigmatic set of elements that they have to relate in order to build a system of meanings, in the knowing of what has to be known » (Sensevy, Gruson, & Forest, 2015). For example, for someone who has to do something with a representational system, the milieu is a specific symbolic organization of the system of representations itself. The milieu offered opportunities of enquiry, in which students have to connect elements of knowledge. They deal with this milieu by relying on the knowledge built in the preceding didactic joint action, *the didactic contract*, what "is already known". This relationship between contract and milieu is a dialectal one, because the understanding of a given milieu depends on the nature of the contract that guides the student's efforts.

In order to enable the student to learn, teacher enacts strategies to engage student's action. Interactions between teacher and students are determined by the didactic contract (Brousseau, 1997). In fact, the teacher knows the knowledge that students will have to learn. But he must not tell directly all what he knows. Thus, he has to make choices, in his teaching, about the equilibrium between saying/showing (expression) and remaining tacit/hiding (reticence). This is the reticence-expression dialectics. The two dialectics (contract-milieu and reticence-expression) are entangled, in that expression or reticence can be oriented to "contract " ("what is already-there"), or "milieu " ("what has to be known").

The research on the using of manipulatives and representations focus on the necessity of enabling the students to rely first on manipulative and concrete "objects", then to study iconic (analogical) representations of numbers (Bass, 2015; Schmittau, 2005, Davydov, 1975) then to write down equations in canonical form. This process seems very close to the tradition in Chinese textbooks (Bartolini Bussi et al., 2011; Sun, 2011; Ding & Li, 2014) and can be thought of as "concreteness fading" (McNeil et al., 2012; Fyfe et al., 2014). In this communication, we will try to show how in the new situation in which the notion of the difference (subtraction) is introduced, the systems of representation of number can guarantee a kind of continuity of experience. In fact, the "translational principle for representations systems" in a representational game (Morales, Sensevy & Forest, 2016) can allow students to understand the concept of the difference between two numbers.

Methodology

To discuss these questions, we focus on a specific moment of teacher's practice in a study class, with an experimented teacher, who belongs to the research team (sphere1). The data were collected in december 2013, in a first grade classroom of a French primary school. The twenty five students of this classroom were aged 6-7. This study follows a qualitative approach.

In this extract, the students collectively search the difference between two additive writings with two terms. In this communication, we focus on the introduction of a new piece of knowledge, the notion of difference between two numbers. In the preceding sessions, the students orally compared the production of two hands ads (students showed a number on the two hands, the statement) and a launch of two six-sided dice. The statement wins if it was bigger (in some cases smaller) than the two-dice throw. Then, students compared the two additions with two terms in reference to the situation of the "Statements" (fingers and dice). The result of this comparison was written in the form of a quality or an inequality with the mathematical signs « =, ≠, >, < ». These two additions were represented in two number lines to solve or prove the comparison, as we can see below (figure 1).

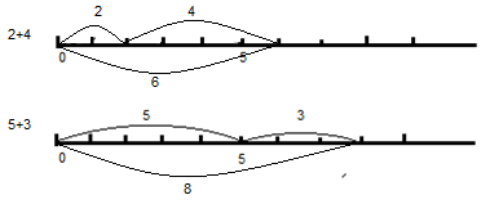


Figure 1: An example of comparison between 2 + 4 and 5 + 3 on the number line

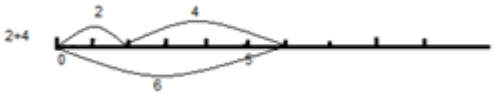
The choice of this extract is motivated by the following reason: this extract shows how the continuity of student's experience could be developed through the use the systems of representation. This extract can be considered as a mesoscopic level of the description, the pivotal level (Sensevy et al., 2015), which allows relationship between what preceded and what is going to follow. So, we can analyze the didactic transactions *hic et nunc*. We can characterize and describe the motives and the forms that directed teacher and students 'action. This description can show the teacher's strategies to make the didactic time forward.

Analyze

Presentation

This part of the curriculum "ACE" is organized around a connected series of situations. The initial situation of this curriculum is the situation of the "Statements game" (fingers and dice). One die (marked with standard dot patterns for 1-6) is about to be thrown. Beforehand, the students use their fingers to make a "statement" (for example, a student shows two fingers on her right hand, and three fingers on her left hand). The die is thrown. The students compare their statement with what is indicated by the die. If the sums are equal, the pupils have won. After this oral comparison, students compare an addition (two terms ≤ 5 with a number ≤ 6). The progressive complexification of the situation guides students to increasingly rich comparisons: the number of hands (students) is increased, so the number and the nature of dice (1 to 10 dice are played with), the rules of the game are changed (for example a pupil no longer wins because he has the same number as in the

statement, but because he has a lower or higher number). These connected situations should allow the students to build a real mathematical experience, particularly in the handling of representation and symbolic writing systems, as we will show in what follows. A number line is also introduced on which students represent the numbers. Indeed, the students manipulated a concrete object (the fingers) and they translate fingers by an iconic (analogical) representation of number (the number line) and wrote down equation in canonical form (2 + 4 = 6). For example, a student shows two fingers on his left hand and four fingers on his right hand, then she draws these numbers on a number line:



And she writes down this addition: 2 + 4 = 6

To understand various properties of numbers, students had to compare different representations of the same mathematical reality to become progressively able to recognize the differences and the similarities between these representations.

Since the beginning of the year, students acquired knowledge related to compare numbers. This comparison is performed first orally with the production of «two or three hands ads" and a launch of six-sided die. Then, students compared an addition in two or three terms with a throw of dice. They used the mathematical signs « =, ≠ <, > » (for example, to compare 3 + 1 et 5, students write 3 + 1 < 5). This situation become more complex when students have to compare two additions in two terms. Finally, students deal with the question of the subtraction on the basis of the comparison between two additions, in the continuity of the previous situations. The study is accompanied by the use of the number line.

The students have built a semiotic knowledge to represent the comparison between two numbers. The number is seen as a measurement. It refers to "the quantity of fingers" in two hands. The number line shows the number like a length measurement. This ancient knowledge is the *didactic contract*, the habits of action with which teacher and students are going to approach the new knowledge, the difference between two numbers. The extract of the session that we chose introduces the difference between two numbers from the comparison of two additions and the terms “larger than and smaller than”. The difference is a gap between two numbers, two length measurements. Four episodes will be analyzed. Here is a synoptic view of this analyzer.

Episodes	**Content**
Episode 1	Presentation of the instructions by the teacher
Episode 2	a) Comparison, looking for the difference between “1 + 3” and “1 + 1”. b) Proposal of two students: (tdp 15) “1 + 3” is larger than “1 + 1”. c) Proposal of another student: the difference between “1 + 3” and ”1 + 1” is 3.
Episode 3	a) The difference (tdp 53). b) Introduction of the two hands by the teacher to confirm the difference 3.
Episode 4	Introduction of the number line by the teacher to search the difference between “1 + 3” and “1 + 1” (tdp 68).

Table 1: Extract division

We abstract the four episodes and provide a short analysis.

Search for the difference between two numbers (episode 1, 2 3)

The teacher asks the students to look for the difference between “1 + 3” and “1 + 1”. He presents the instructions like this:

> Teacher: Today, we are going to begin a new game. It is always a game with statements. But today what we are going to make, it is to compare ours statements. We look for which is the larger statement, the smaller statement but that I would know how much more and how much less (…) we are going to find, this calls in fact the difference.

Actually, the students meet difficulties to find the difference between “1 + 3” and “1 + 1. They compare the two numbers and look for the larger number or the smaller number with the term-by-term strategy or by computing. They do not focus on "difference". Confronted to the difficulties of the students, the teacher suggests to illustrate « the two additions “1 + 3” and “1 + 1” » by a statement with both hands. The following picture (figure two) shows such a statement comparison.



Figure 2: Translation of “1 + 3” and “1 + 1” by two statements

Unfortunately, this *translation* between the mathematical symbolic writing and the hands statements in a game of representation does not bring the students to produce an adequate answer. "The semiotic habit" of the contract, which considers fingers as instruments to compare numbers, impedes a new designation of numbers, the difference.

Using the number line

Therefore, the teacher introduces in a milieu two number lines on which students have to represent both additive writings. A student writes a first bridge above the first interval of the number line (hence representing the number "1") and a second bridge above three intervals (between the second and a fourth graduation, hence representing the number "3"). He writes down above these bridges the numbers 1 and 3. Then, he draws below the number line a bridge of four intervals and writes down the sum number (4).

> Teacher: It is good the statement makes four as Neil shows us. On the second line, what are you going to draw?

On the second number line, the same student draws two consecutive bridges and a bridge of two intervals, signifying the sum number (2)

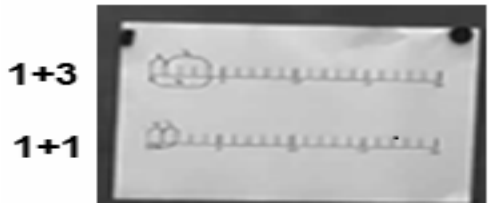
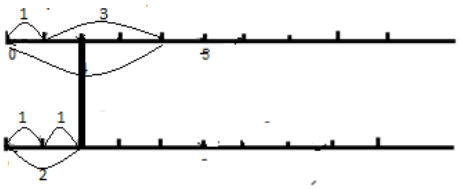


Figure 3: the representation of "1 + 3" and "1 + 1" on the number lines

Teacher: What do you see in the two number lines? Do you see if the 1+3 is largest than 1+1?

The students provide answers different answers: "three", "four" and "two". The teacher asks one student to show how he knows that his statement is largest.

Student: because here is the two [He slides his finger from the third graduation (number "2") on the first number line to the third graduation (number "2") of the second number line] It's a part of four…



The student draws a bridge between the second and the fourth graduation on the first number line.

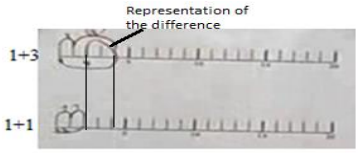


Figure 4: Representation of the difference on the number line

The teacher's expression encourages a translation between the symbolic writing and the number line. But, in the same time, she's reticent because she doesn't say how to draw these additions on the number lines. The teacher thus refers to the preceding contract. She just says: "What do you see in the two number lines? Do you see if the 1+3 is largest than 1+1?" This question can be seen as a ''milieu-oriented situation'. The action of the student is moved toward the effective representation of the difference on the two number lines. But, though this expression, the teacher is reticent because she remains silent when the student searches the difference. She indicates where the student must look but the research stays under the responsibility of the student. The number line affords to show concretely the gap between the two numbers, the difference between the two length bounded by the bridges, the sum of the two additions. The students investigate an instrument, the number line, on which they know already how to play (in the situation of comparison). By using this instrument, they achieve not yet explored potentialities of this semiotic system. In particular, the number line shows a number included in another.

Results

The students investigate different systems of representation to find the difference between two numbers: symbolic writing, concrete representation with the hands, number line. All these

representations are known by the students. First, the symbolic writing is translated by statements on the two hands. Then these statements are translated into two number lines in what we may call a translation game, which is a particular representational game (Sensevy, 2011; Morales, Sensevy & Forest, in press). The preceding semiotic knowledge is "re-experienced" by the students. In this way, we can say that the systems of representation are instrumental (Dewey, 1938/1998) in that they allow investigating new knowledge. The semiotic habits are accommodated in a new situation, a new knowledge, allowing the continuity of experience. However, the re-experience of a semiotic system for introducing a new knowledge requires a subtle enunciative work for the teacher, given that the different strategic systems in teacher's action can be a contract oriented or a milieu-oriented transactional activity. In this communication, we have shown how the expression-reticence game of this teacher enable her to introduce in the milieu a system of representation (the number line) already known by the students (contract) in order to understand new properties of this system (then considered as a milieu). This teacher's strategy enables the students to investigate the difference between two numbers while leaving them the responsibility of this enquiry (Dewey, 1938/1998).

Discussion and conclusion

However, these results ask to be worked. It is necessary to explore on a long duration this continuity of the experience of the students in mathematics through the systems of representation (Joffredo-Le Brun, 2016). In particular, it is necessary to note that such continuity can be built only through a real epistemic continuity of the knowledges within the curriculum. The design of such a curriculum has to be performed through the effective experience of the teachers, within an iterative process, as it is the case in the ACE research.

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Visuo-spatial abilities and geometry: A first proposal of a theoretical framework for interpreting processes of visualization

Elisa Miragliotta[1] and Anna Baccaglini-Frank[2]

[1]University of Ferrara, Department of Mathematics and Computer Science, Ferrara, Italy; elisa.miragliotta@student.unife.it

[2]University of Pisa, Department of Mathematics, Pisa, Italy; anna.baccaglinifrank@unipi.it

We propose a theoretical interpretation of visuo-spatial abilities, as classified in the field of Cognitive Psychology, in the domain of Euclidean Geometry. In this interpretation we make use of Fischbein's theory of figural concepts and of Duval's cognitive apprehensions. Our interpretation lays the foundations for a new theoretical framework that we propose as a tool for qualitative analysis of students' processes of visualization as they carry out geometrical activities. In particular, we present analyses of excerpts from a set of activities designed and proposed in a didactical intervention aimed at strengthening visuo-spatial abilities of a group of students identified as the weakest from a selected 9th grade class of an Italian high school.

Keywords: Geometric reasoning, spatial thinking, visualization, visuo-spatial abilities.

Introduction

Research in the domain of *visualization* and *spatial thinking* has pursued several purposes: understanding the different imaginative strategies used by students (Owens, 1999); studying the effects of teaching practices, aimed at encouraging processes of visualization (Presmeg, 2006); developing theoretical constructs, useful for the interpretation of students' perception of geometric shapes and how this perception improves in learning geometry (Duval, 1995; Fischbein, 1993, Mariotti, 2005). Some ideas in this field have been developed from the psychological studies on *mental imagery*. Since the advent of Cognitive Psychology and contemporary Neuroscience, researchers have been elaborating models to describe processes related to visualizing and using mental imagery, and they have listed sets of *visuo-spatial abilities* involved (e.g., Cornoldi & Vecchi, 2004). However, a shared definition of these abilities does not exist yet. Nor have the fields of Mathematics Education and Cognitive Psychology been able to elaborate common grounds to study visualization processes, in which they are both interested.

In this paper we propose a theoretical interpretation of visuo-spatial abilities, as classified in the field of Cognitive Psychology, in the domain of reasoning in Euclidean Geometry, that was developed as part of a study that has been recently carried out (Miragliotta, 2016; Miragliotta, Baccaglini-Frank & Tomasi, submitted). The study had two main objectives: on the one hand, we attempted to give a theoretical analysis of some visuo-spatial abilities in the context of learning Euclidean Geometry; on the other hand, we used such theoretical interpretation to study the effects of a set of activities proposed (for the most part) using a Dynamic Geometry Environment (DGE) in terms of strengthening the students' visuo-spatial abilities (as it is widely accepted that DGEs yield great potential in fostering processes such as visualization, as well as in mediating, in general, the learning of geometry: e.g., Mariotti, 2005; Baccaglini-Frank, 2010; Leung, Baccaglini-Frank & Mariotti, 2013). In this paper we will concentrate on the description of theoretical analysis of the

visuo-spatial abilities considered and on its power as a tool for qualitative analysis of students' behavior as they carry out geometrical activities. As an example of how the framework can be used, we will analyze an excerpt taken from a question (not involving the use of any DGE) of the post-intervention interview of a student from the group of students identified as the weakest of the Italian high school class of involved in the study. Since for what we present in this paper the role of the DGE is marginal, and space quite constrained, we will not discuss visualization in a DGE.

Theoretical background

According to Clements and Battista (1992), *spatial reasoning* "consists of the set of cognitive processes by which mental representation for spatial objects, relationships, and transformations are constructed and manipulated" (ibid., p.420). Referring to Kosslyn (1983), these authors observe that geometrical reasoning requires spatial reasoning, which includes four classes of image processing: generating an image; inspecting an image to answer questions about it; transforming and operating on an image; maintaining an image in the service of some other mental operation. In particular we are interested in processes involving two-dimensional geometric objects.

From the perspective of Cognitive Psychology, generating and processing mental images take place within a complex process of acquisition and use of cognitive abilities, including those denoted *visuo-spatial abilities*. A list of these appears in Cornoldi and Vecchi (2004, p. 16). We elaborated our theoretical interpretation starting from the following set of abilities: *visual organization*, the ability to organize incomplete, not perfectly visible or fragmented patterns; *planned visual scanning*, the ability to scan a visual configuration rapidly and efficiently to reach a particular goal; s*patial orientation*, the ability to perceive and recall a particular spatial orientation or be able to orient oneself generally in space; v*isual reconstructive ability*, the ability to reconstruct a pattern (by drawing or using elements provided) on the basis of a given model; *imagery generation ability*, the ability to generate vivid visuo-spatial mental images quickly; *imagery manipulation ability*, the ability to manipulate a visuospatial mental image in order to transform or evaluate it; *spatial sequential short-term memory*, the ability to remember a sequence of different locations; *visuo-spatial simultaneous short-term memory*, the ability to remember different locations presented simultaneously; *visual memory*, the ability to remember visual information; *long-term spatial memory*, the ability to maintain spatial information over long periods of time.

To interpret how these general cognitive abilities might come into play during reasoning in the specific context of Euclidean Geometry, we referred to theoretical constructs elaborated in mathematics education to this purpose.

Fischbein's theory of figural concepts

The *Theory of figural concepts* (Fischbein, 1993) describes geometrical figures as follows:

> A geometrical figure may, then, be described as having intrinsically conceptual properties. Nevertheless, a geometrical figure is not a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the mental representation of space property. […] all the geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties. (ibid., pp. 141-142).

According to Fischbein *figural concepts* "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities - like ideality, abstractness, generality, perfection" (ibid., p. 143); a geometrical figure is made up of two fundamental components: the figural component and the conceptual component. From the developmental point of view, initially the visual aspect is dominant, and gradually the role of formal constraints becomes more important, until the construction of *figural concept* is reached (Mariotti, 2005).

Duval's types of cognitive apprehension

Today the importance of visualization in mathematics is widely recognized. Since several studies have addressed visualization in different ways, we clarify that our interpretation is in line with the definition given by Arcavi (2003).

> *Visualization* is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (ibid., p. 217, emphasis added)

Peculiarities of visualization in geometry have been highlighted by Duval (1995) in describing different approaches to dealing with geometric figures: *cognitive apprehension* stresses that "there are several ways of looking at a drawing or a visual stimulus array" (ibid., p. 143). Duval speaks of four cognitive apprehensions. *Perceptual apprehension* responds to the laws of figural organization and identification of form, and helps to "recognize something (shape, representation of a thing,…) in a plane or in depth" (ibid., p. 145) at first glance. In a perceived figure we can also recognize sub-figures that do not depend on its construction. *Sequential apprehension* "is required whenever one must construct a figure or describe its construction" (ibid., p. 146). Here the sub-figures emerge in a specific order, depending on the geometrical construction, on technical constraints of the instrument used and on mathematical properties. Furthermore, Duval (1995, p. 146) claims that "mathematical properties represented in a drawing cannot be determined through perceptual apprehension", indeed, "a drawing without denomination or hypothesis is an ambiguous representation". So, indications given through speech help us to identify properties of a perceived geometrical figure, thanks to the *discursive apprehension*. Here we are in the domain of deductive reasoning. The apprehension that has a heuristic function in problem solving is the *operative apprehension*. This apprehension depends on different ways of modifying a figure that happen only within the figural register and that are independent from mathematical knowledge.

Each type of apprehension seems to be related to different cognitive processes that could be accomplished through the coordination of different visuo-spatial abilities as we hypothesise below.

Grounding for a new visuo-spatial abilities framework

While maintaining the classification proposed by Cognitive Psychology, we selected a subset of visuo-spatial abilities and provided an interpretation in the specific context of geometrical reasoning. We used the *Theory of Figural Concepts* to interpret the terms "model" and "image" as follows: *image* is the figural component of a geometric figure; *model* is a synonym of figural concept in which image and concept realize their dialectic. Since our interpretation aims at being a

stronger lens for analyzing students' processes than the visuo-spatial abilities as described in the Cognitive Psychology literature, there is not always a one-to-one correspondence with such abilities.

- *Visual organization* is the ability to recognize figural concepts from incomplete or not perfectly visible representations.

Visual organization seems to be an ability that intervenes in tasks that require the recognition of figures within another figure, or in the recognition of a simple figure within a more complex figure. This ability echoes Duval's *perceptual apprehension.*

- *Visual scanning* is the ability to recognize the properties of a figure starting from its representation.

This representation can be static or dynamic. It depends on the task and on the context in which it is proposed. For example, in the case of a dynamic figure in a DGE, *visual scanning* is involved in the recognition of properties that are invariant under dragging (see Leung et al., 2013). This ability echoes Duval's *perceptual apprehension*, but we also recognize in this ability some aspects of his *sequential* and *discursive apprehension*. For example, when observing a quadrilateral obtained through steps of a specific construction starting from two perpendicular lines, we can notice that quadrilateral seems to have a right angle. However, to recognize the property "having a right angle" only observing the figure on the screen, one needs to look at its written geometrical construction and deduce that the point one vertex is at the intersection of two perpendicular lines.

- *Visual reconstructive ability* is the ability to reconstruct, in a given representation, the figural component of a figural concept, starting from written or verbal instructions, or staring from partial representations.

For instance, the reconstruction could be realized following a sequence of construction steps given explicitly, using appropriate tools (ruler and compass, primitives in DGE, …), otherwise it could be realized planning these construction steps. It involves the ability to correctly visualize the relationships between the elementary figural units involved (such as points on lines, perpendicular lines) following the steps of a geometric construction or creating a new construction. This ability echoes Duval's *sequential apprehension* and his *discursive apprehension*. The *visual reconstructive ability* seems to intervene, for example, when carrying out the construction steps of a known geometric figure; when completing the steps of an incomplete construction; when following the steps of a given geometric construction.

- *Imagery generation ability* is the ability to instantly mentally reproduce the figural component of a figural concept recovering it from memory or generating it anew.

This ability seems to intervene when one is asked to visualize a geometric concept, for example, while imagining a a sequence of construction steps. Coupled with *long-term spatial memory*, this ability seems to be involved in the retrieval of the prototypes (that is, in Kosslyn's terms, a "stored model of a shape") of geometric shapes and of their properties. Coupled with *spatial sequential short-term memory*, it seems to intervene in the identification of particular geometric loci.

- *Imagery manipulation ability* is the ability to use the properties of a figural concept or to manipulate figural aspects of a figural concept, taking into account the theoretical relationships between elementary figural units of which it is composed.

This ability is involved in tasks that require mental manipulation of a figure in order to transform it into a new one. This ability echoes Duval's *operative apprehension*, but also differs from it. The mental manipulations on figure are tightly connected to the figure's conceptual component. Indeed, to manipulate a figure maintaining given properties, strong conceptual control over it is required, as highlighted also by Arcavi (2003), who emphasizes, as well, the high cognitive demand involved:

> Seeing the unseen may refer to the development and use of an intervening conceptual structure which enables us to see through the same visual display. (ibid, p. 234)
>
> When visualization acts upon conceptually rich images (or in Fischbein's words when there are intervening conceptual structures), the cognitive demand is certainly high. (ibid, p.235).

- *Spatial sequential short-term memory*: this ability seems to be present in various processes of geometric reasoning; here we consider it, in particular, as the ability to remember different configurations assumed by the figural component of a figural concept during an observed or imagined manipulation.
- *Long-term spatial memory* in our interpretation this refers, in particular, to the ability to maintain in long-term memory the figural components of a figural concept.

The last two abilities are involved in solving geometric tasks and are always used in combination with other visuo-spatial abilities. For example, combined with the *imagery generation ability*, *spatial sequential short-term memory* seems to be involved in tasks that require recognizing a particular geometrical locus. Combined with the *imagery manipulation ability*, *spatial sequential short-term memory* seems to be involved in tasks that require remembering the configurations assumed by a figure during an imagined manipulation.

When a solver faces a geometrical problem, s/he interacts with visual or mental images in different ways; a process that seems to occur frequently is imagining the consequence of a (mental) manipulation of the figure. Such process can be carried out through the use of the various abilities listed above that expert solvers combine in an immediate and automatic way. So consider this as an ability in its own right, that we will call *geometric prediction,* intending the identification of certain properties or configurations of a new figure, arising from a process of manipulation. This process appears to be coherent with respect to the notion of *anticipatory image* (Piaget & Inhelder, 1966), which suggests an ability to make predictions, orienting both perception and imagination, in the presence of a specific goal.

Visuo-spatial abilities framework as a tool of analysis

In this section we use the framework to analyze an excerpt taken from a question of the post-intervention interview of a student; he was part of a group of students in the 9th grade (students aged 14-15) class of an Italian scientific high school (Applied Science option), identified through a pre-test as having low performance on geometry tasks heavily involving visualization processes; the teaching intervention lasted five lessons and had been carried out using open problems, mostly proposed in the DGE *GeoGebra*. The post-intervention interview involved tasks both in the context of a DGE and with only pen and paper (if requested by the student). In the excerpt the student is solving a task proposed outside the DGE setting. The analysis has the aim of showing the power of

the framework in identifying the proposed visuo-spatial abilities and showing how they can come into play, shedding light onto visualization processes.

Activity: the student is given the following task and allowed to use paper and pencil:

> Imagine a quadrilateral. Focus on the midpoint of each side. Trace the segments that join the midpoints of consecutive sides. What can you tell me about the figure that is formed?

Below is an excerpt describing what the student says [and does].

Student: It is a quadrilateral, which… which looks like a rhombus, so to speak. [Initially he closes his eyes. Then he places four finger tips (two thumbs and two indexes) on the desk to form what looks like a square, and then, moving along two parallel lines in opposite directions, a non-square rectangle. He drags his fingers back and forth between these two positions.] If quadrilateral is a square it forms a rhombus with congruent diagonals, but if is a random figure…I mean, it depends on the figure. It changes depending on how the points are placed.

Interviewer: Draw it. What are you drawing?

Student: Four scattered points. [He draws (freehand) a quadrilateral with different sides, as shown in Figure 1]

Interviewer: Can you say more about the figure that is formed?

Student: It is a quadrilateral. Mmm…it is a parallelogram!

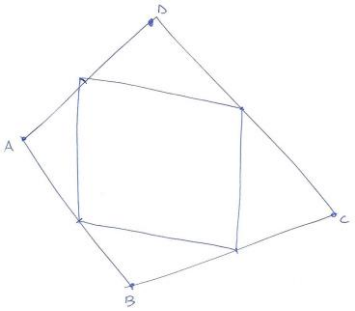

Figure 1 The student's drawing

In addition to what the student says, the excerpt is interesting also for what he does, which gives further insight into visuo-spatial abilities he may be using. After the first answer, he keeps his eyes closed and moves his fingers on the desk. This seems to suggest that a purely mental process is taking place, and the gesture on the desk seems to be a windows onto this process. In order to answer the question, first of all, we would say the student is using the *imagery generation ability* for imagining the first configuration. To this end he needs to recall a prototype of the quadrilateral that is as general as possible (this involves the *imagery generation ability* and *long-term spatial memory*); then he needs to visualize the required elementary figural units (*imagery generation ability*) and go through the steps of the construction (*imagery reconstructive ability*). Now, the student's use of his fingers on the desk is an extremely insightful window onto processes he could be enacting. Our interpretation is that he is using the *imagery manipulation ability*, helping himself with an external image (the quadrilateral with vertexes at his four finger tips) that he can act upon. What is visible of this manipulation are the positions (and their continuous change) of the vertexes. As he moves his fingers (forming what look like various rectangles) he is using *geometric prediction*, possibly aided by *visual scanning*, to visualize the quadrilateral with vertexes at the midpoints of the sides of the manipulated quadrilateral. This interpretation is supported by the fact that the student moves his fingers on his desk seamlessly, he never lifts them up from the surface, and then he selects a position which is coherent with respect to the configuration that he wants to (mentally) observe, and starts to move fingers again. The student seems to be able to manipulate the figure in a manner that goes beyond the kind of transformation described by *operative*

apprehension. Indeed, the manipulation recalls much more dragging of the vertices, as can be accomplished in a DGE. This cognitive effect could have been promoted by the kind of problems proposed within the DGE during the activity sessions. The student seems to be looking for extra external support for his *imagery manipulation* and *geometric prediction* abilities.

Moreover, this excerpt is very interesting because of what the student then decides to draw on the sheet of paper when invited to so do. Although he has only mentioned the case in which the quadrilateral is a square and realized with his fingers various cases of it being a rectangle, he draws a much more general convex quadrilateral. This behavior supports our previous hypothesis that the student seems to need external support for his *imagery manipulation* and *geometric prediction* abilities. On paper it is as if he gains confidence, possibly because the cognitive load from the conceptual control he would need to exercise over the general figure is lowered this way. Once he sees the general quadrilateral and sketches the midpoint quadrilateral he recognizes (*visual scanning* and *conceptual control*) a parallelogram.

Conclusion

The fields of Mathematics Education and Cognitive Psychology share various research interests; one of these is the identification and classification of strategies and processes involved in visualization. According to Cognitive Psychology, generating and processing mental images take place within a complex process of acquisition and use of abilities, including those denoted *visuo-spatial abilities*. Attempting to interpret visuo-spatial abilities in the context of geometrical reasoning could be beneficial to both fields. In our attempt to give a theoretical interpretation of some visuo-spatial abilities in the context of learning Euclidean Geometry, we used theoretical constructs from the field of Mathematics Education, which led to the introduction of an ability different from the basic visuo-spatial ones, *geometric prediction*, and they also led to highlighting the fundamental contribution, in solving geometric tasks, of *geometric conceptual control* over figures.

This interpretation, which can be seen as groundwork for a new theoretical framework, has allowed us to: (1) design an educational intervention aimed at strengthening visuo-spatial abilities of a group of students identified as the weakest in a selected class; (2) gain insight, through qualitative analysis, into students' geometrical reasoning. We believe that this kind of research can provide new insight into students' difficulties in learning Geometry, and be used to design educational material for strengthening students' visuo-spatial abilities.

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Middle school students' employments of gestural signs: A case study

Samet Okumus[1] and Karen Hollebrands[2]

[1]Recep Tayyip Erdogan University, College of Education, Rize, Turkey;
samet.okumus@erdogan.edu.tr

[2]North Carolina State University, College of Education, Raleigh, NC, United States;
karen_hollebrands@ncsu.edu

In this research study, we investigated how middle school students created 3–dimensional objects from 2–dimensional figures using an extrusion method. In a summer enrichment program, students used manipulatives and a dynamic geometry program (Cabri 3D). We identified students' strategies for forming 3–dimensional objects with a focus on their gestural signs. The results demonstrated that they most often employed dynamic–pointwise and dynamic–objectwise gestures to demonstrate the lateral faces or edges of 3–dimensional objects. Also, students linked their gestural signs and the Segment tool of Cabri 3D to indicate their reasoning.

Keywords: Gestural signs, 3–dimensional objects, extrusion, middle school students.

Introduction

Gestural signs are important to understand how students make sense of mathematical problems (Arzarello, 2006; Bartolini Bussi & Baccaglini–Frank, 2015; Radford, 2009). Signs (e.g., verbal or oral texts) are characterized as "something which stands to somebody for something in some respect or capacity" (Pierce 1932; 2.228). Also, students produce signs using an artifact such as manipulatives. Bartolini Bussi and Mariotti (2008) refer to signs that are produced using an artifact or any action related to the use of it as *artifact signs*. Students reason about tasks with the use of an artifact and exploit *mathematical signs* (e.g., mathematical definitions, proofs). Research studies address that students link artifact and mathematical signs employing gestures (Bartolini Bussi & Baccaglini–Frank, 2015; Maschietto & Bartolini Bussi, 2009). Gestures that are interwoven with other sets of signs facilitate mathematical communication between students (Arzarello, Paola, Robutti, & Sabena, 2009; Bartolini Bussi & Baccaglini–Frank, 2015; Radford & Sabena, 2015). In the current study, with a focus on students' gestural signs, our research question is: "In what ways do middle school students relate the features of 2–dimensional figures and 3–dimensional solids that are created using an extrusion method?"

Theoretical framework: Semiotic mediation

Semiotic mediation "sees knowledge–construction as a consequence of instrumented activity where signs emerge and evolve within social interaction" (Mariotti, 2009, p.428). In a cultural learning environment, students use an artifact or a set of artifacts during a *semiotic activity*, so a piece of mathematical knowledge is mediated. The teacher is aware of the affordances (and constraints) of the artifact and assists students in *using the artifact as a tool of semiotic mediation*. In other words, students produce artifact and mathematical signs (or hybrid signs), so personal signs transformed into the mathematics culture's signs (Bartolini Bussi & Mariotti, 2008).

A student may exploit personal signs that may be unclear for others. Having the role of *cultural mediator*, such signs become meaningful for others under the teacher's supervision. For example,

Bartolini Bussi and Baccaglini–Frank (2015) addressed that a preschooler used a non–existing Italian word "quadratizzato" referring to the movement of a programmable robot that made turns in a circular motion. The researchers translated it into English with another non–existing word as "squarized O." They identified children's and a student teacher's turning gesture and spiral arrows with reference to the word "quadratizzato." The personal sign linked artifact and mathematical signs, and became a meaningful sign in the mathematics culture. The researchers characterized it as a *pivot sign*.

Pivot signs "may refer both to the activity with the artifact; in particular they may refer to specific instrumented actions, but also to natural language, and to the mathematical domain" (Bartolini Bussi & Mariotti, 2008, p. 757). They link artifact and mathematical signs or an artifact/mathematical sign may become a pivot sign (Bartolini Bussi & Baccaglini–Frank, 2015; Bartolini Bussi & Mariotti, 2008). Suppose a student is given a rectangular card and asked to draw the container he should order to hold the stack of the identical rectangular cards until it reaches a certain height. The student may produce a mathematical sign and draw a rectangular prism denoting its edge lengths. On the other hand, he may illustrate the stack of objects and draw a deck of rectangular cards (pivot signs). The deck of cards links the rectangular cards (artifact signs) and the rectangular prism (mathematical sign).

Research studies demonstrated that students' gestural signs were intertwined with other sets of signs such as verbal signs (Radford & Sabena, 2015; Sabena, 2008). For example, Sabena (2008) characterized high school students' production of signs while they were engaged in tasks about derivative of functions. Sabena found that students emphasized the dynamic character of derivatives employing gestural signs with other sets of signs. Also, previous research indicated that students' gestural signs linked artifact and mathematical signs (Bartolini Bussi & Baccaglini–Frank, 2015; Maschietto & Bartolini Bussi, 2009). Bartolini Bussi, Boni, Ferri, and Garuti (1999) characterized the geneses of students' gestures as pointwise or global (objectwise) with a focus on artifacts (gears) students used. Maschietto and Soury-Lavergne (2013) addressed that artifacts created for the same mathematical topic resulted in exploiting different signs. Primary school students were given to a hands-on tool (gear train of five wheels) and its digital counterpart for learning the place values of numbers. Students employed different gestural signs with the use of each artifact because they had different semiotic potentials. For example, students employed gestures to start the turning mechanism of the wheels using the hands-on tool and denoted these gestures in their written texts. The decimal values were explored in the counterpart digital artifact by mouse clicks. So, using a set of artifacts gives students an opportunity for mathematical learning.

Methods

The research design was a case study in which the data were collected from a bounded system – a summer enrichment program that took place in a state in the southeastern United States (Stake, 1995). The unit of analysis was semiotic activities. The participants of this research study were selected from a group of rising seventh and eighth grade students (three boys, five girls) attending a summer enrichment program that aimed at promoting students' thinking in different STEM (Science, Technology, Engineering and Mathematics) areas. A five–day instructional unit was planned. Eight middle school students (three rising seventh grade students, five rising eighth grade students) participated in the research study. Their ages ranged from twelve to fourteen.

On the first day of the program, students were given a spatial ability test and an open–ended survey that indicated their experience with geometry, computers, games, etc. The participants were characterized by these instruments and students with different spatial abilities were paired up. On the second and third days of the program, students created 3–dimensional objects from 2–dimensional figures using an extrusion method (e.g., stacking identical circular cards on top of each other and forming a cylinder). Students were given worksheets and they answered the questions in pairs. In the semiotic activities, they were asked to identify containers (3–dimensional objects) that would hold the stack of identical objects or objects decreasing in size (e.g., coins with a different radius). They used manipulatives and a dynamic geometry program, Cabri 3D. Students were provided pre–image and image figures in pre–constructed Cabri 3D sketches and manipulatives. They were asked to drag objects in Cabri 3D sketches and make an observation. Afterwards, they identified the resultant 3–dimensional objects for holding the stack of 2–dimensional figures. Students formed right and oblique prisms, cylinders, pyramids and cones.

Prior to the summer enrichment program, a pilot study was conducted. Students were allowed to use the *Trajectory* tool of Cabri 3D that allowed tracing points and objects. However, at times, they traced the objects and Cabri 3D showed the answer. Based on the feedback from the pilot study the *Trajectory* tool was disabled from the menu bar. However, the teacher activated the *Trajectory* tool on his computer that was hooked up the projector during he generated a whole-class discussion.

Data included videorecordings of four dyads during each class session including whole class discussions, and students' written and oral responses to the tasks. Also, a program that recorded the computer screen when students used Cabri 3D was used. The researchers watched the videos of students while they worked on the semiotic activities in groups. Their computer screen recordings were watched synchronously. Some screenshots and students' gestures were inserted into the verbatim transcripts.

We focused on students' written/oral responses to semiotic activities and their use of artifacts (manipulatives and Cabri 3D). Students' artifact, pivot and mathematical signs were identified. In the current research study, we analyzed students' strategies for forming 3–dimensional objects from 2–dimensional figures with a focus on their gestures. Students' gestural signs were categorized as static and dynamic taking into account how they were employed. If students' static or dynamic gesture signified an object, it was identified as objectwise. If the gesture stood for a point or a set of points, then we categorized it as pointwise gesture. Four gestural signs were identified: objectwise –dynamic, objectwise –static, pointwise –dynamic, and pointwise –static.

Results

We identified students' strategies for forming 3–dimensional objects from 2–dimensional figures with a focus on their gestural signs. Students' gestures that linked artifact and mathematical signs were characterized as pivot signs. Also, students produced other sets of signs that were characterized as pivot signs (e.g., graphical signs, verbal signs). However, we give students' strategies in which gestural signs were characterized.

Some students suggested 3–dimensional objects (containers) that allowed no extra space and focused on the exact fit. Students employed objectwise–dynamic gestures to demonstrate lateral surfaces of

solids. For example, Charlotte emphasized that the container for identical triangular cards would be a triangular prism. She employed an objectwise–dynamic gesture as shown in Figure 1 and said:

Charlotte: …because the prism is gonna come around the edges of the triangles.

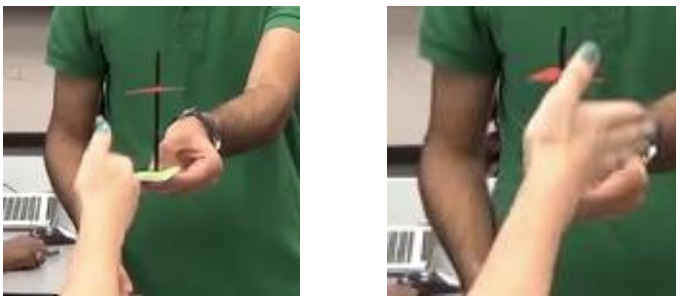

Figure 1: Charlotte makes an objectwise–dynamic gesture to denote the triangular prism

Some students focused on geometric objects in the artifact and suggested solids based on how the geometric shapes were stacked on top of each other. They interpreted artifact signs and employed static or dynamic gestures to indicate the resultant 3–dimensional objects and focused on the parts of the given artifact. For example, Vince had difficulty identifying the container (rectangular pyramid) for rectangular cards decreasing in size. He kept the height almost in the middle and positioned the rectangle. Afterwards, he employed an objectwise–static gesture to demonstrate the container as shown in Figure 3. He said:

Vince: For this one, I was trying to figure out so if it is – it is coming– it is going like that (makes an objectwise–static gesture as shown in Figure 2). As it decreases it's going like this, it's going like that. Do you see? It's going inward like that.



Figure 2: Vince makes an objectwise–static gesture to demonstrate the rectangles decreasing in size

Students connected the geometric shape on the bottom and top in the given artifact producing a gestural sign. They most often made a pointwise–dynamic gesture and connected the pre–image and image points. For example, Sloane identified the resultant object for holding the stack of identical circles employing a pointwise–dynamic gesture. She posited:

Sloane: Because there is a circle on the top and the bottom and that makes a cylinder. Because there is like a straight line right here kinda (Figure 3).



Figure 3: Sloane makes a pointwise–dynamic gesture to denote a cylinder

On the other hand, some students made hybrid or objectwise–dynamic gestures to demonstrate the 3–dimensional objects. Virginia denoted the top and bottom of the cylinder with a static–objectwise gesture. Then, she made an objectwise–dynamic gesture and demonstrated the lateral faces of the cylinder as shown in Figure 4. She said:

Virginia: Because basically the same reason as the triangular one (triangular prism). If you place a figure on the top and the bottom, and you connect them, you would create a cylinder (Figure 4), which is basically just a 3–dimensional flat version.

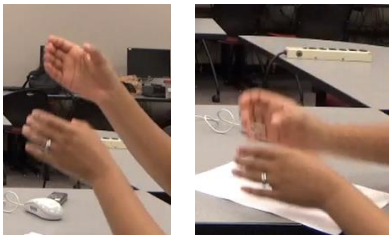

Figure 4: Virginia's objectwise–dynamic gesture for denoting the lateral faces of a cylinder

Students' gestural signs resulted in using the *Segment* tool of Cabri 3D. On the one hand, some students connected pre–image and image points using the *Segment* tool to demonstrate their thinking (e.g., Figure 5). On the other hand, with the prompt of the teacher, some students used the *Segment* tool and connected pre–image and image points to demonstrate the 3–dimensional objects.

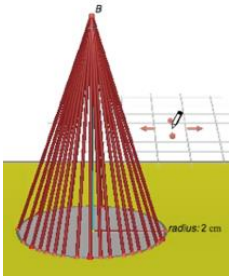

Figure 5. Vince connects pre–image and image points

During the whole-class discussion, the teacher exploited the semiotic potentials of Cabri 3D. For example, he activated the *Trajectory* tool and described the 3–dimensional as a collection of two-dimensional objects (Figure 6). He made a stacking action employing an objectwise-dynamic gesture to demonstrate the extrusion (Figures 6). He said: "continuously, I am adding more and more [triangular cards], right?" and emphasized the continuous motion of stacking the figures on top of each other.

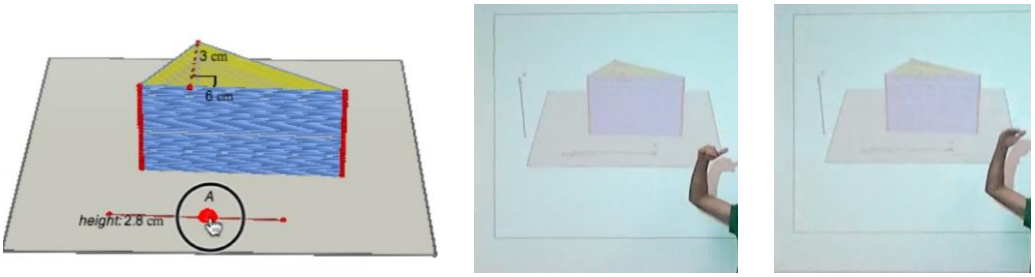

Figure 6. The teacher activates the Trajectory tool of Cabri 3D and makes a stacking action

Some students used a metaphor and referred to a real–life object or entity to describe the shape of the 3–dimensional objects for holding the stack of 2–dimensional figures. For example, Vince made a connection between the given artifact and a daily life object with which he was familiar. He used a metaphor and made a pointwise–dynamic gesture to denote a CD container. He posited:

Vince: Like a CD. Do you have like a… It's kind of a thing when you… It's like a plastic you put on CD's and you put a nob to stack all the CD's. It's like that."

We identified objectwise–static gestures during students demonstrated the top and bottom of geometric objects or a cone/rectangular pyramid. On the other hand, students most often employed objectwise/pointwise–dynamic gestures. Their gestures signified the lateral faces or edges of 3–dimensional objects.

The aforementioned strategies were not disjointed from each other. We identified that students (during group work or whole–class discussion) used multiple strategies to support their claims. For example, Sloane and Stan had difficulty naming 3–dimensional objects. The researcher having the role of cultural mediator handed the artifact to them and elicited their thinking about the resultant 3–dimensional object for holding the stack of identical triangular cards. Sloane said:

Sloane: Just imagine there are a bunch of triangles in the middle. And then it'd be like (she makes a gesture as shown in Figure 7). It'd have like a longer length on the outside probably be slanted but I don't know… I don't know. It's kind of like a roof to a house. And, I think it would be like a 3D trapezoid. That's kind of how I see it.

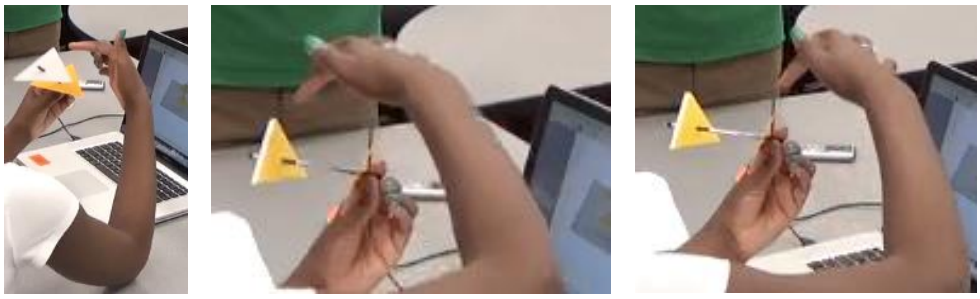

Figure 7. Sloane makes a pointwise–dynamic gesture to denote a triangular prism

Sloane thought of the 3–dimensional object as a collection of 2–dimensional figures. Then, she produced a pivot sign making a pointwise–dynamic gesture and connected the vertices of the top and bottom triangles. After her gesture, she used a "roof to a house" metaphor to describe the 3–dimensional object (triangular prism). She had difficulty naming the object and called the resultant object a 3–dimensional trapezoid producing an invented signifier. In her reasoning, Sloane used three strategies: focusing on the collection of figures, connecting pre–image and image figures, and using a metaphor. A pointwise–dynamic gesture was employed during she connected pre–image and image figures.

Discussion and implications

In the current study, we identified students' strategies for forming 3–dimensional objects from 2 –dimensional figures producing gestural signs. Students employed gestures frequently to demonstrate extrusion of objects. Similar to what Bartolini Bussi et al. (1999) found, students made objectwise/pointwise and dynamic/static gestures. Students' gestures linked artifact and mathematical

signs, and were characterized as pivot signs. Similarly, Bartolini Bussi and Baccaglini–Frank (2015) identified gestural signs that linked artifact and mathematical signs. Students' gestural signs were intertwined with other sets of signs such as metaphors. For example, students produced gestural signs and a used a metaphor when they had difficulty naming an object. Researchers found that gestures facilitated communication between students and teachers (Bartolini Bussi & Baccaglini–Frank, 2015; Arzarello et al., 2009; Radford, 2009; Radford & Sabena, 2015; Sabena, 2008). Gestures facilitate communication, in particular when students produce an invented signifier (Bartolini Bussi & Baccaglini–Frank, 2015).

Students' strategies for forming 3–dimensional objects were associated with each other. When students reasoned about the extrusion activities, they used multiple strategies and defended their conjectures about the resultant 3–dimensional objects. Students used the *Segment* tool of Cabri 3D and connected pre-image and image points to demonstrate 3–dimensional objects. On the other hand, some students used the *Segment* tool with the prompt of the teacher. Students should be given an opportunity to interact with an artifact in a longer period, so they become more comfortable using it and exploit mathematical knowledge.

In Cabri 3D, when a transformation is made, one can see initial objects/points (pre-image) and transformed points/objects (image). We enabled students to see pre-image and image figures in pre-constructed sketches and hands–on tools. New research may demonstrate how students' approaches may differ if they are not provided pre-image figures to identify 3–dimensional objects using an extrusion method.

We were unable to built hands-on tools in which rectangles/circles decrease in size to demonstrate a rectangular pyramid and cone. Students most often used Cabri 3D and changed the height of objects during semiotic activities. They interpreted artifact signs and produced mathematical signs. However, students most often exploited gestural signs during they used hands-on tools. As Maschietto and Soury-Lavergne (2013) emphasize, counterpart artifacts lead to a co-emerging of signs and using a variety of artifacts that have different semiotic potentials gives students an opportunity for mathematical learning.

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Preservice mathematics teachers' types of mathematical thinking: Use of representations, visual-spatial abilities, and problem solving performances

Beyza Olgun[1] and Engin Ader[2]

[1]Boğaziçi University, Faculty of Education, İstanbul, Turkey; beyza.olgun@boun.edu.tr

[2] Boğaziçi University, Faculty of Education, İstanbul, Turkey; ader@boun.edu.tr

The main purpose of this study was to examine preservice mathematics teachers' types of mathematical thinking and to investigate whether there are any differences between different types of thinkers based on their problem solving performances, use of representations and visual-spatial abilities. The sample of the study consisted of 113 preservice mathematics teachers in a private and four public universities. The results showed that although problem solving performances were similar for each type of mathematical thinking, preservice teachers who adopted harmonic and geometric types of mathematical thinking preferred to use schematic representations more than analytic thinkers in their problem solving processes. The findings provided an insight about preservice teachers' preferences for a visual approach and the implications of these preferences for teacher education programs were discussed.

Keywords: Types of mathematical thinking, use of representations, visual-spatial abilities, word problems.

Introduction

Problem solving has an important role in mathematics and lies in the focus of almost every math curriculum (Van De Walle, Karp, & Bay-Williams, 2010). According to the American National Council of Teachers of Mathematics "one of the most significant aims of mathematics teaching and learning is to develop students' problem solving ability" (Deliyianni, Monoyiou, Elia, Georgiou, & Zannettou, 2009, p. 96). Understanding of problem solving process includes identifying, exploring, implementing, and using visual images, which is related with visualization (Deliyianni et al., 2009). With the rise of constructivism, the importance of the role of visualization in the learning process was emphasized more. The role of visualization in mathematical problem solving is investigated through three main constructs. These are mathematical thinking in terms of predisposition of visualization in problem solving, the use of visual-spatial representations and visual-spatial abilities.

Studies on visualization in mathematics often linked it with mathematical thinking. In a problem solving context, three types of mathematical thinking were suggested according to disposition of visualization (Krutetskii, 1976): the analytic type, the harmonic type, and the geometric type. Students who embrace the analytic thinking style do not feel the need to benefit from visual supports and also they do not have enough strength for the use of visual components. For geometric thinkers it is the contrary. They make use of visual-pictorial components and verbal-logical components have poor influence on their reasoning. The reasoning of harmonic thinker students includes both verbal-logical components and visual-pictorial components and their preferences can change according to the problems that they face.

Teachers' mathematical beliefs and learning experiences affect their mathematical thinking and visual approaches (Presmeg & Balderas-Cañas, 2001). Their thinking styles and use of visuality have an impact on their teaching (Presmeg, 1986b). Therefore preservice teachers' approaches towards different types of mathematical thinking and visual-spatial representations could be an important component of teacher education programs. This necessitates a careful study of the interrelations among teachers' visualization, mathematical thinking and problem solving performance before focusing on how these can be supported through teacher education programs.

The main purpose of the study was to investigate preservice teachers' preferences of problem solving strategies and how their mathematical thinking (analytic, harmonic or geometric types) might affect the visualization process in mathematical word problem solving. Considering the purpose, this study aimed to explore whether there is a significant difference in preservice teachers' mathematical word problem solving performance, use of visual-spatial representations (pictorial or schematic), and levels of visual-spatial abilities based on their types of mathematical thinking.

Method

Participants were selected by convenient sampling. The study was conducted with senior preservice teachers from one private and four state universities in İstanbul and Ankara, Turkey. 113 students were involved in the study and they were enrolled in Primary Mathematics Education (n = 91) and Secondary School Mathematics Education (n = 32) programs.

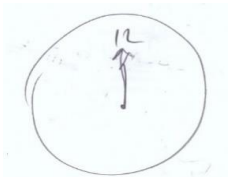
The data were collected with the implementation of two instruments; the Mathematical Processing Instrument (MPI) and the Spatial Ability Tests (SAT) during the second semester of the 2015-2016 academic year. The MPI, which was developed by Presmeg (1985) and adapted to Turkish by Taşova (2011), was used to measure participants' types of mathematical thinking, use of visual-spatial representations, and mathematical word problem solving performances. In order to measure preservice teachers' levels of visual-spatial abilities, SAT developed by Ekstrom and colleagues (1976) and adapted to Turkish by Delialioğlu (1996) were used.

The MPI was developed for the first time by Krutetskii (1976) to measure students' preferences of the use of visual methods. Then Suwarsano (1982) designed the instrument with the same name for elementary school students. According to Presmeg (1995), the instrument that was designed by Suwarsono (1982) was not convenient for teachers. Thus she arranged the instrument in three sections according to fieldwork in which both students and teachers participated. With the new arrangement, the instrument took its final form that consists of three sections. In this study, since participants were pre-service teachers, Section B and Section C of the MPI that were designed as appropriate for teachers was used.

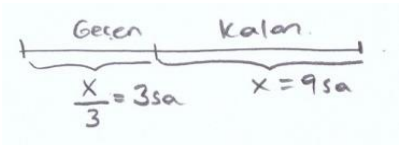
The MPI has two parts: a test that consists of 18 mathematical word problems and a questionnaire that includes a list of possible solutions for each problem. According to participants' responses on the test section of MPI, four different scores were generated. These were mathematical word problem solving performance, pictorial representation score, schematic representation score and visual-spatial representation score. The first score was the total number of problems solved correctly. The others were the total number of times that students reported using the specified type of representation. We used van Garderen and Montague (2003)'s coding for the classification of the representations. As shown in Figure 1, if preservice teachers "reported or drew an image of objects

or persons referred to in the problem" (van Garderen & Montague, 2003, p. 248), the representation was scored as primarily pictorial and if they "drew a diagram, showed the spatial relations between objects in a problem, or reported a spatial image of the relations expressed in the problem" (van Garderen & Montague, 2003, p. 248), the representation was scored as primarily schematic. Visual-spatial representation score was the summation of pictorial and schematic representations score.

C-2: If the elapsed time since noon (12:00) is accounted for 1 in 3 of the remaining time to midnight, what time is it now?



a pictorial representation



a schematic representation

Figure 1: Examples for preservice teachers' pictorial and schematic representations

According to participants' responses on the questionnaire of the MPI, visualizing mathematical scores were generated. In this score, without taking into consideration whether the students solved the problem correctly, if the participant chose only a visual problem solving strategy for a problem, 2 points were given. For the responses that did not include a visual problem solving strategy 0 points were given. For the responses including both visual and nonvisual strategies, 1 point was given. Therefore the possible minimum and maximum scores for preservice mathematical teachers' visualizing mathematical scores were 0 and 36 respectively. In order to group preservice teachers based on their mathematical thinking, participants' visualizing mathematical scores were used.

The SAT involves spatial orientation and spatial visualization tests. A person's SAT score was the summation of his or her spatial orientation test score and spatial visualization test score. A person's spatial orientation test score was obtained from the Card Rotation Test and the Cube Comparison Test. A person's spatial visualization test score was obtained from the Paper Folding Test and the Surface Development Test.

Results

In Table 1, descriptive statistics results of participants' scores for mathematical word problem solving performance, use of representations, and the SAT are presented.

Participants' Scores	Range	Mean	Std. Deviation
Mathematical Word Problem Solving Performance	7 - 18	14.89	2.28
Schematic Representations Score	2 - 19	8.07	3.23
Pictorial Representations Score	0 - 3	.95	.90
Visual-Spatial Representations Score	2 - 20	9.02	3.29
Visualizing Mathematical Score	5 - 28	13.97	4.73
The SAT Scores	81 - 260	172.80	48.22

Table 1: Descriptive statistics results for the variables

In particular, the participants used totally 1047 visual-spatial representations, of which 925 were schematic and 122 were pictorial. Although all participants used schematic representations in their problem solving processes, pictorial representations were rarely used by preservice teachers. The

results also showed that especially for five specific problems, participants did not prefer a visual method for the solutions.

Preservice teachers' types of mathematical thinking

In the literature, there were different methods suggested for the classification of analytic, harmonic and geometric thinking. For example Richardson (1977) determined the groups according to percentages as the first 15% segment of the distribution was analytic type, the last 15% segment of the distribution was geometric type, and others were harmonic type. Galindo-Morales (1994) determined the groups according to prearranged visualizing mathematical scores. Such as who had 22 points and above was a geometric thinker. In Taşova (2011)'s study, the range of visualizing mathematical scores was divided into three equal intervals. However, data from this study necessitated considerations upon the classification method. Results showed that for 5 specific problems preservice teachers did not tend to use any representations and also they did not select a visual solution in the questionnaire section. A participant with a visualizing mathematical score of 18, which is half of the maximum score, preferred a visual method in at least 9 of the remaining 13 problems. Under these circumstances, such a participant who preferred visual methods over nonvisual methods in approximately 70% of the remaining problems needed to be classified as a geometric thinker.

Due to the considerations mentioned in the previous paragraph, a new approach was adopted for the classification of types of mathematical thinking. The mean visualizing mathematical score was used while deciding on the center of the interval for the harmonic type and the intervals for all three types were found by taking the standard deviation of the scores into consideration. The minimum and maximum scores of the type of harmonic thinking were assigned by the half of the standard deviation of the preservice teachers' visualizing mathematical score around its mean. According to this classification, the number of people grouped for each type of mathematical thinking was 34 (30%) for the analytic type, 48 (43%) for the harmonic type and 31 (27%) for the geometric type.

Investigation of group differences

A Kruskal-Wallis H test was run to determine whether there were any differences in mathematical word problem solving performance between three groups of preservice teachers having different types of mathematical thinking. The results revealed that the distributions of mathematical word problem solving performance scores for each group with different types of mathematical thinking were similar. The medians of mathematical word problem solving scores were not significantly different among the analytic type (mean rank = 15.5), the harmonic type (mean rank = 15), and the geometric type (mean rank = 15), $\chi^2(2) = 14.468$, $p = .24$.

In order to determine whether there were any differences in the use of schematic, pictorial, and visual-spatial representations between three groups for types of mathematical thinking a Kruskal-Wallis H test was run. The results revealed that mean ranks of schematic representation scores and visual-spatial representation scores were significantly different between the groups. As a result of post hoc analysis, it was discovered there were statistically significant differences in preservice teachers' schematic representation scores and visual-spatial representation scores between the analytic type (mean rank = 6) and the harmonic type (mean rank = 8) ($p = .01$) and the analytic type and the geometric type (mean rank = 9) ($p = .01$). On the other hand, there were no significant

differences in schematic representation scores and visual-spatial representation scores between the geometric type and the harmonic type ($p > .05$). The results revealed that mean ranks of pictorial representation scores between the analytic type (mean rank = .5), the harmonic type (mean rank = 1), and the geometric type (mean rank = 1) were not significantly different, ($\chi 2(2) = 2.281$, $p = .32$).

One-way ANOVA test was run to investigate any differences between the SAT scores of groups of participants with different types of mathematical thinking. Shapiro-Wilk test was used to determine the normality of the distribution and the results showed participants' SAT scores were normally distributed ($p > .05$). Levene's Test of Homogeneity of Variance was used to investigate the homogeneity of the variances. A homogeneity of variances was discovered ($p > .05$). However, the SAT scores from the three groups, the analytic type ($M = 161.83$, $SD = 50.1$), geometric type ($M = 179.02$, $SD = 43.62$) and harmonic type ($M = 174.57$, $SD = 45.57$) did not differ significantly ($F (2, 97) = 1.233$, $p = .30$).

Discussion

In this section, a discussion of the group differences among types of mathematical thinking according to mathematical word problem solving performance, the use of visual-spatial representations, and levels of visual-spatial abilities is presented.

The structure of mathematical thinking adopted by preservice teachers

The findings showed that 30% of the preservice teachers were analytic type, 43% of preservice teachers were harmonic type, and 27% of preservice teachers were geometric type. The slightly high proportion of the trend for the harmonic type was consistent with the literature. Hacıömeroğlu and Hacıömeroğlu (2013) found that most of preservice teachers adopted the harmonic type of mathematical thinking. Taşova's findings (2011) supported that the harmonic type of thinking was the most commonly adopted by preservice teachers whereas the least percent of the preservice teachers were geometric thinkers. In the current study, these differences were not clearly seen and the classification method could be the reason for it. Hacıömeroğlu and Hacıömeroğlu (2013) found that senior preservice teachers used visual methods more than juniors. Therefore they related this difference with seniors' experiences through teaching mathematics and practicum courses. Since participants of the current study were also seniors and the data collection was done close to end of the second term, their final year experiences may have had an impact on their preferences.

Mathematical word problem solving performance according to types of mathematical thinking

Results showed that there was no significant difference among groups with analytic, harmonic, and geometric types of mathematical thinking in terms of problem solving performance. While the findings were supported by some studies (Kolloffel, 2012; Suwarsono, 1982) there were some conflicts in the literature. While Lean and Clements (1981) suggested that the preference had a significant effect on performance and students who preferred nonvisual strategies outperformed visualizers, Moses (1977) claimed that visual solution methods guide college students to more effective solutions. These controversial findings in the literature might be caused by sample selection. The studies applied the same instrument with some adjustments to different groups such as elementary school students, college students, and teachers. The participants' individual

differences like how they were taught, grade level, courses they were enrolled also could be factors influencing this relationship. In terms of performance, Presmeg (1986a, 1986b) suggested that there were internal and external factors, which could make a group superior compared to others. She discussed that textbooks and teachers' teaching styles emphasized nonvisual methods. Therefore this situation could favor for analytic thinkers. It could be also that school exams might constrain students' use of visual methods, which could take more time for solutions (Presmeg, 1986a). However with the educational developments the role of visualization and its importance in problem solving was recognized (Deliyianni et al., 2009). Visual approaches were included in both teacher education programs and curriculums. Therefore preservice teachers could be experienced with both visual and nonvisual approaches during their methods courses and school practices. These experiences through university life may reduce the influence of these internal or external factors on performance.

Use of visual-spatial representations according to types of mathematical thinking

The results showed significant differences in the use of schematic representations and visual-spatial representations among groups with different types of mathematical thinking while no difference was found in the use of pictorial representations. Preservice teachers did not tend to use pictorial representations as much as elementary or high school level students did as the previous studies suggested (van Garderen & Montegue, 2003). The frequency and the variance of preservice teachers' pictorial representations scores were very low. The rare use of pictorial representations by participants may be one reason for not observing significant differences between the groups.

In the current study harmonic thinkers and geometric thinkers had similar preferences for use of representations in problem solving whereas analytic thinkers were separated from the others by using fewer representations. These findings were different from Sevimli and Delice's study (2011). They found that analytic thinkers and harmonic thinkers had similar preferences for use of representations and their use of representations were significantly less frequent than geometric thinkers. There might be two reasons for the differences in these findings. One of them was the mathematical context of the studies. Sevimli and Delice (2011) carried out their study on a specific topic: definite integrals. They discussed that in calculus courses students were mainly taught nonvisual methods and algebraic expressions. The context of definite integral and how it is taught can lead the students to using algebraic solutions. On the other hand word problems that were used in this study might have promote preservice teachers to use representations in solutions.

The second reason could be that the participants did not express all problem solving procedures in their mind on the paper in the current study. For the context of working on definite integrals, although representations were not preferred by preservice teachers during the problem solving processes, when they used it might be a difficult procedure to operate representations in mind. The context requires specific graphical representations that include complex processes (Sevimli & Delice, 2011) and they could push the preservice teachers for operation on paper. However the representations that were used in solutions of word problems could be formed in mind. They did not have a complex structure as much as graphical representations that used in integral context. Further studies could be conducted for different mathematical contexts. Researchers might prefer interviews in data collection processes to detect representations that people construct in their mind.

Levels of visual-spatial abilities according to types of mathematical thinking

The findings of the current study did not show statistically significant differences in preservice teachers' visual-spatial abilities in terms of types of mathematical thinking. Taşova (2011) suggested that geometric thinkers were more successful in visual-spatial ability tests than analytic or harmonic thinkers. However he did not run a statistical analysis to compare the groups for types of mathematical thinking in terms of their levels of visual-spatial abilities. There are various other studies that documented no significant relationship between people's visual-spatial abilities and their preferences for visual or nonvisual methods (Hagarty & Kozhevnikov, 1999; Moses, 1977; Lean & Clements, 1981; Suwarsono, 1982). Krutetskii (1976) suggested that there were many other factors, which effects people's preferences like learning experiences. Therefore further studies could investigate such factors beyond focusing only on peoples' visual-spatial abilities.

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Teaching mathematics with multimedia-based representations – what about teachers' competencies?

Julia Ollesch, Fabian Grünig, Tobias Dörfler and Markus Vogel

Heidelberg, University of Education, Germany; olleschj@ph-heidelberg.de; gruenig@ph-heidelberg.de; doerfler@ph-heidelberg.de; vogel@ph-heidelberg.de;

Multimedia-based representations play a major role in mathematics and mathematics education. Consequently, they are important with regard to teaching purposes, as they are supposed to be useful to represent mathematical structures and processes in different ways. Within the presented project we developed an instrument by using video-vignettes in order to assess the competencies of mathematics teachers for multimedia use in mathematics lessons. For coping with complexity we reduced the instrument's focus on two facets: cognitive load and mutual supplement of multimedia representations. As the work is still in progress, we here focus on the project's theoretical background as well as on the development of the assessment instrument based on video-vignettes.

Keywords: Multimedia-based mathematical representations, technological pedagogical content knowledge, assessing teachers' competencies, video-vignettes, secondary school mathematics.

Introduction

Since mathematical objects are conceptual and invisible, the meaning of representations plays a major role in mathematics and mathematics education. The necessity of representations for the fundamental understanding of mathematical concepts has already been postulated by Duval (2006). Therefore, it is of importance that students work with multiple representations of the mathematical content early on. Doing so they can benefit from complementary expressions and viewpoints of the subject matter and are able to improve and deepen their understanding (Ainsworth, 1999). However, as a teacher it is insufficient to simply present multiple representations to the students. It is necessary that the students build and understand the connections between different representations and gain a coherent mental model (Seufert, 2003). Schnotz and Bannert (1999) illustrate the interaction between descriptional and depictional representations in their integrated model of information processing. According to their work, these two different kinds of representations complement each other in a synergetic way to form a mental model of the represented content. With the construction of a mental model through multiple representations, students also gain in cognitive flexibility (Spiro, Coulson, Feltovich, & Anderson, 1988). According to Mayer (1997) students can also achieve better results when learning with multiple representations, however, there is practical as well as empirical evidence that this is not always the case in classroom instruction. A teacher's knowledge about multiple mathematical representations and their kind of use in the classroom can obviously not be neglected.

In this paper we first establish the theoretical background for the use of multiple, dynamically linked representations in mathematics education and the related professional competencies required of teachers in this context. After that we describe the development of a test instrument to assess these competencies from a particular theoretical point of view.

Theoretical background

Especially multimedia can offer possibilities to develop and implement learning environments containing multiple mathematical representations. When working with multimedia-based representations, mathematics teachers should, among many other things, know the benefits and pitfalls of (dynamically) linking multiple representations while also being aware of the cognitive load generated by them. Therefore a variety of technological knowledge, skills and competencies must be combined with pedagogical knowledge and content knowledge of the subject matter.

Linking multiple mathematical representations

Computer-applets based on multimedia representations are not only suitable to illustrate both descriptional and depictorial representations (Schnotz & Bannert, 1999) at the same time, but they are also useful to establish a dynamic link between them. That way it is possible to present even more information about the mathematical content than the representations could provide without being linked to each other (Kaput, 1989). The dynamic linking and the mutual supplement of the different representations provide different approaches to the mathematical content, especially because of the automatic translation of effects when changing one representation. Providing different approaches could cause synergetic effects on the construction of coherent knowledge structures (Seufert, 2003). Moreover, the automatic translation between unrelated representations could decrease the cognitive load of the learner and leaves more capacities for the process of understanding (Ainsworth, 1999). Especially in the subject of mathematics, multimedia-based representations are appropriate to demonstrate the character of mathematical processes (Vogel, Girwidz, & Engel, 2007).

However, there are also disadvantages that come along with multimedia-based representations. As much as they can encourage a deeper understanding, they could also lead to misconceptions (Hadjidemetriou & Williams, 2002), if they are misleading with regard to their external arrangement. Likewise too many multiple dynamical representations could cause a heavy extraneous cognitive load, so that the students do not have any capacity for the intended germane load (Chandler & Sweller, 1991). If the extraneous cognitive load gets too heavy, students often tend to split their attention (split-attention-effect) and focus on one form of representation only (Brünken & Leutner, 2001). Hence reducing the extraneous cognitive load is of high importance when using multiple dynamic representations in mathematics teaching.

In his work, Mayer (2009) gives different principles that should be considered in constructing multiple dynamic representations: The coherence principle, for example, states that people learn better when irrelevant material is excluded. Particularly regarding the mutual supplement of multiple representations these principles are a good guideline for constructing effective multimedia-based mathematical learning environments.

Technological pedagogical content knowledge

Apparently, the profitable use of multimedia-based representations in mathematics lessons is not only a question of mathematics education, but concerns didactics of mathematics and psychology as an interdisciplinary field of multimedia learning. Certainly, teachers first have to decide from a mathematical point of view whether the mathematical content is adequate for the use of multimedia and which aspects of the content should be presented within this use of multimedia. In the second step it is important to implement the mathematical content into a computer-applet with regard to

available pedagogical and psychological insights of multimedia learning. The technological pedagogical content knowledge, that is needed for the profitable use of multimedia-based representations, is an "emerge of knowledge that goes beyond all three 'core' components (content knowledge, pedagogical knowledge and technological knowledge)" (Koehler & Mishra, 2009, p. 66) and requires extensive knowledge about all the aspects of multimedia learning. The TPACK-framework (Koehler & Mishra, 2009) extends the taxonomy of Shulman (1986) by adding technology knowledge which results in three new intersections: technological content knowledge, technological pedagogical knowledge and technological pedagogical content knowledge (TPACK).

Accordingly, the complexity of competencies needed to use multimedia-based representations in an effective way in mathematics lessons is high. Beyond their mathematical content knowledge teachers need an extensive knowledge about the media and technology they want to use as well as its chances and difficulties for multimedia learning. Consequently, teachers are confronted with new challenges (Koehler & Mishra, 2009) and need to develop the competencies to identify the chances and difficulties that go hand in hand with the use of multimedia-based representations (Spanhel, 1999). However, according to Koehler and Mishra (2009) many of the teachers do not feel prepared for the use of modern technologies to present these kinds of representations.

TPACK in context of multimedia learning in mathematics education

The internet provides many existing computer-applets which mathematics teachers could use in their lessons (for example see www.geogebra.org/materials). The question is if a chosen applet supports or prohibits the understanding of mathematical concepts and processes and how to determine its benefit. As far as we know there are no criteria given for evaluating an applet with regard to both mathematical and psychological aspects of multimedia learning. While several studies investigated the effect of multimedia-based representations on learning outcome in general, there is still little known on how to evaluate applets with interdisciplinary criteria of multimedia learning. Also it is little known about the competencies mathematics teachers need for an effective use of multimedia-based representations in their classrooms.

Hence it is the research goal of this study to develop a test instrument to assess competencies regarding the technological pedagogical content knowledge (TPACK, cf. Koehler & Mishra, 2009) and the interdisciplinary aspects of multimedia learning in mathematics education.

Assessing mathematics teachers' competencies in using multimedia-based mathematical representations by video-vignettes

As functional and geometrical thinking build an essential base for the understanding of mathematics and *elementary functions* as well as *geometry* also play a major role in the german curriculum of secondary schools (Kultusministerkonferenz, 2012) we decided to focus on these two mathematical contents when starting to develop the intended test instrument. These contents deeply involve the understanding of their dynamic aspects (for instance while studying covariance of functions, transformations of geometric figures or whole families of functions or geometric objects), so multimedia-based representations could be an appropriate tool in teaching functional and geometrical thinking. Functional thinking includes mainly three aspects of functions: aspect of assignment, aspect of covariance and view as a whole (cf. Vollrath, 1989). Especially for handling the aspect of covariance dynamic representations are an appropriate tool, because changes in one variable and their

effects can be directly visualized in other representations. Also for acquiring geometrical thinking, multimedia-based representations can be helpful: Geometrical thinking is based on the understanding of geometrical terms and conceptions (Ulfig & Neubrand, 2013). Young children already develop an understanding for geometrical terms, but mostly ignore the similarities (Heinze, 2002). For example they are not able to understand that a square is a special representative of rectangles. With dynamic representations it is possible to illustrate not just one example of a geometrical object, but to construct a whole class of objects by using the dynamic transformation (Kittel, 2009).

On base of our theoretical considerations it is necessary to investigate also psychological aspects beyond the mathematical ones. According to an intensive literature review, we determined eight facets of psychological aspects of multimedia learning as basis for the test instrument: *relation to the content* (e.g. Spanhel, 1999), *efficacy of the use of multimedia* (.e.g. Mandl, Gruber, & Renkl, 2002), *limitations of the representations* (e.g. Mandl et al., 2002), *misconceptions* (e.g. Mayer, 2009), *cognitive load* (e.g. Chandler & Sweller, 1991), *individual promotion of the learners* (Wauters, Desmet, & van den Noortgate, 2010), *mutual supplement of multiple representations* (e.g. Mayer, 2009) und *simplifying (mathematical) content* (e.g. Kittel, 2009). Within the development of the test instrument we conducted a multistage expert-rating in order to validate, but also to empirically support a selection of two of these eight facets for purposes of reducing complexity in this first approach. This process will be described more detailed later on.

Video-vignettes and their construction

Since video-vignettes are assumed to be an effective way of measuring teachers' competencies (Blomberg, Stürmer, & Seidel, 2011), we developed, for the time being, a pool of 36 video-vignettes, that show various situations during mathematics lessons using multimedia-based representations. Video-vignettes are short sequences of a classroom situation that show critical problems: to evaluate these situations the observing person needs special competencies (Rehm & Bölsterli, 2014). Figure 1 shows an example of a script for a video-vignette related to the psychological facet *mutual supplement of multiple representations*.

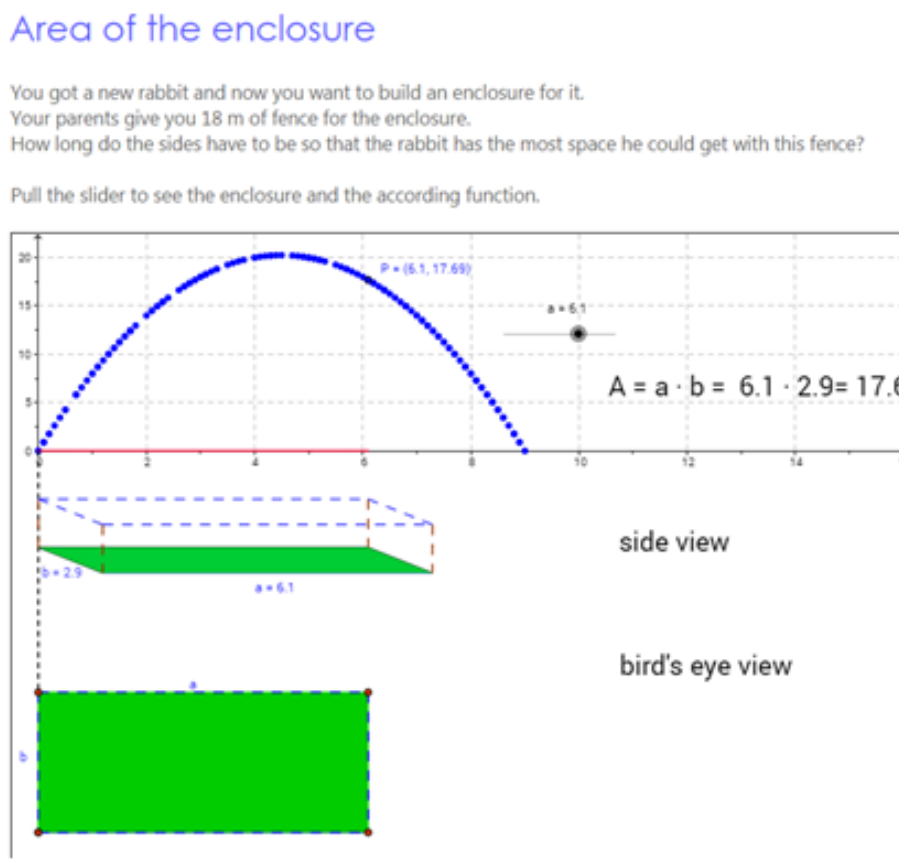
The vignettes are constructed with a closed-ended question type. Multiple statements have to be rated on a scale from one to six according to its appropriateness for the presented situation. An example of statements is shown in Figure 2.

After the development of 36 vignettes, they have been validated in a multistage expert-rating.

Validation of the constructed video-vignettes

First the constructed vignettes were evaluated by nine experts in a semi-standardized qualitative interview. The aim of these interviews was to assure the relevance and the clarity of the presented situations in the vignettes. Afterwards the vignettes were rated by 104 experts in a quantitative interview. The aim here was to reassure the evaluation of the qualitative interviews as well as to analyze the distribution of the answers on the scale from one to six of each statement. Moreover, the experts could give comments on each of the vignettes. The answers and the comments of the experts from the quantitative rating were analyzed regarding four criteria: *focus regarding the mathematical content of secondary school, distribution of the answers to the statements, relevance for school and clarity of the vignette* and *comments of the experts.*

In the 10th grade of the Paula-Fürst secondary school Miss Schäfer gives the following computer-applet to her students for working in pairs:



[In the background the following conversation can be heard:]

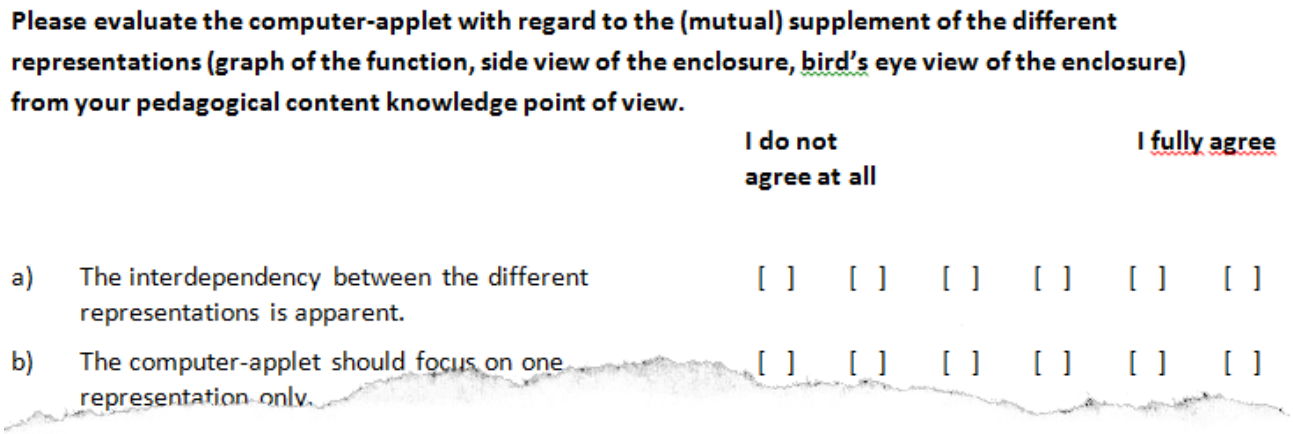
Miss Schäfer: *"Is everything alright?"*

Can: *"No, Timo does not understand that the function displays the area!"*

Timo: *"Yeah, I do not know why there is a round function while the area is a rectangle!"*

Figure 1: Example of a script for a video-vignette according to the psychological facet mutual supplement of multiple representations

Based on the multi-stage expert rating and psychometric properties of the instrument we chose the most appropriate vignettes and determined the two psychological facets of multimedia learning *cognitive load* and *mutual supplement of multimedia representations* as the focus for the final test instrument. Cognitive load refers to the trichotomy from Chandler and Sweller (1991): intrinsic cognitive load, extraneous cognitive load and germane cognitive load (cf. section theoretical background). So the aim is to explore if prospective teachers can estimate the cognitive load. The mutual supplement of multimedia representations refers to the interaction between two or more forms of representations of the same issue (cf. section theoretical background; Ainsworth, 1999; Kaput, 1989). Using these interactions between different forms of representations could involve many chances, but also risks. For example, it is important to link the different representations to gain an understanding of coherence (Seufert, 2003). However, as mentioned above different forms of representations could also cause the split-attention-effect (Brünken & Leutner, 2001).



Please evaluate the computer-applet with regard to the (mutual) supplement of the different representations (graph of the function, side view of the enclosure, bird's eye view of the enclosure) from your pedagogical content knowledge point of view.

		I do not agree at all					I fully agree
a)	The interdependency between the different representations is apparent.	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
b)	The computer-applet should focus on one representation only.	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]

Figure 2: Example of statements for the vignette from Figure 1

Assessment

After the reduction of the vignettes as well as the reduction of the psychological facets, we revised and adapted five vignettes for each of the psychological facets as well as six to seven items for each vignette. The formulation of the items was parallelized between the vignettes in order to assure that the important aspects of each of the psychological facets are tested.

These ten vignettes were again validated in a pilot study and as the results were promising, they were used in the final assessment. In this assessment we also included covariates to prove the discriminant validity of the test instrument: pedagogical knowledge and content knowledge. Preliminary results with 261 prospective teachers in Baden-Württemberg already show evidence for the discriminant validity of the test instrument: As expected from the TPACK-framework (Koehler & Mishra, 2009) the test score of the developed vignettes shows a weak correlation with the two constructs *pedagogical knowledge* ($r = .17$, $p = .01$) and *content knowledge* ($r = .29$, $p < .001$). Furthermore we could prove expected correlations with the educational progress of the prospective teachers ($r = .14$, $p = .03$) as well as the number of attended courses addressing the use of computers in mathematics lessons ($r = .17$, $p = .03$).

Discussion and outlook

The research goal was to develop a test instrument in order to assess the competencies mathematics secondary teachers need for an effective use of multimedia-based representations in mathematics lessons. Therefore, we considered both mathematical and psychological aspects of multimedia learning and developed a test instrument for the mathematical contents of *functions* and *geometry* as well as for the psychological facets *cognitive load* and *mutual supplement of multiple representations.* With the conducted multistage expert-rating and the preliminary results of the assessment we could confirm the validity of the test instrument.

At the moment, we conduct an assessment with the final test instrument in order to research the development of the previously described competencies during the practical phase of the studies of prospective teachers. Moreover, the test instrument will be complemented with further mathematical content: *algebra* and *stochastics*. At the current stage of the project, new vignettes are developed for these two subjects which will supplement the current test instrument. The new test instrument will

then be able to test a wide range of mathematical content knowledge combined with knowledge about the psychological aspects of multimedia learning.

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Children's drawings for word problems – design of a theory and an analysis tool

Barbara Ott

University of Education, St. Gallen, Switzerland; barbara.ott@phsg.ch

Representations are essential for mathematical understanding. In particular, graphic representations are taught as tools for solving word problems. However, not only do children often have difficulties in using them, but there is also a complete absence of research into whether or not children's own graphic productions actually represent the key mathematical elements of word problems. My project, therefore, focuses on this gap by developing a theoretical model and an analysis tool to categorise and map the drawings. This is based on primary school children's drawings, which have been identified as graphic representations of word problems.

Keywords: Graphic representations, visualisation, word problems, analysis tool, elementary school students.

Introduction

For doing mathematics, the use of external representations is essential. Without them it is virtually impossible to discuss or gain insight into mathematical problems (Dörfler, 2008; Hoffmann, 2005). In external representations mathematical patterns may be apparent and can be analysed. They are also important in the mathematical learning process. Bruner (1966) explains the power of external representations for learning "as its capacity, in the hands of a learner, to connect matters that, on the surface, seem quite separate" (p. 48). Accordingly, representation is quoted as a process standard in several standards and curricula (e.g. NCTM, 2000).

Within a range of external representations, graphic representations allow certain aspects of space to be mapped onto specific elements of content (Stern, Aprea & Ebner, 2003). They can be defined as a "data structure in which information is indexed by two-dimensional location" (Larkin & Simon, 1987, p. 68). Every element of a graphic representation contains information not only to do with its own location, but also nearby elements. Graphic representations, therefore, constitute a key tool for problem solving (Polya, 1967).

Nevertheless, it is often reported that children have difficulties in using graphic representations as a tool for problem solving (Fagnant & Vlassis, 2013; van Essen & Hamaker, 1990). In some studies, it is shown that the use of graphic representations can be trained (e.g. Fagnant & Vlassis, 2013; Diezmann, 2002). However, there is a lack of research into the actual extent to which children's individual drawings represent key mathematical aspects of a given word problem. This study (Ott, 2016) aims to make a contribution to this gap.

Research interest

This research focuses on self-generated children's drawings for word problems. It addresses the main question as to what extent these drawings represent the mathematical structure of the word problem. More specifically, two research questions are defined:

1) Which features are key in drawings for word problems?
2) How do they manifest themselves in children's drawings for word problems?

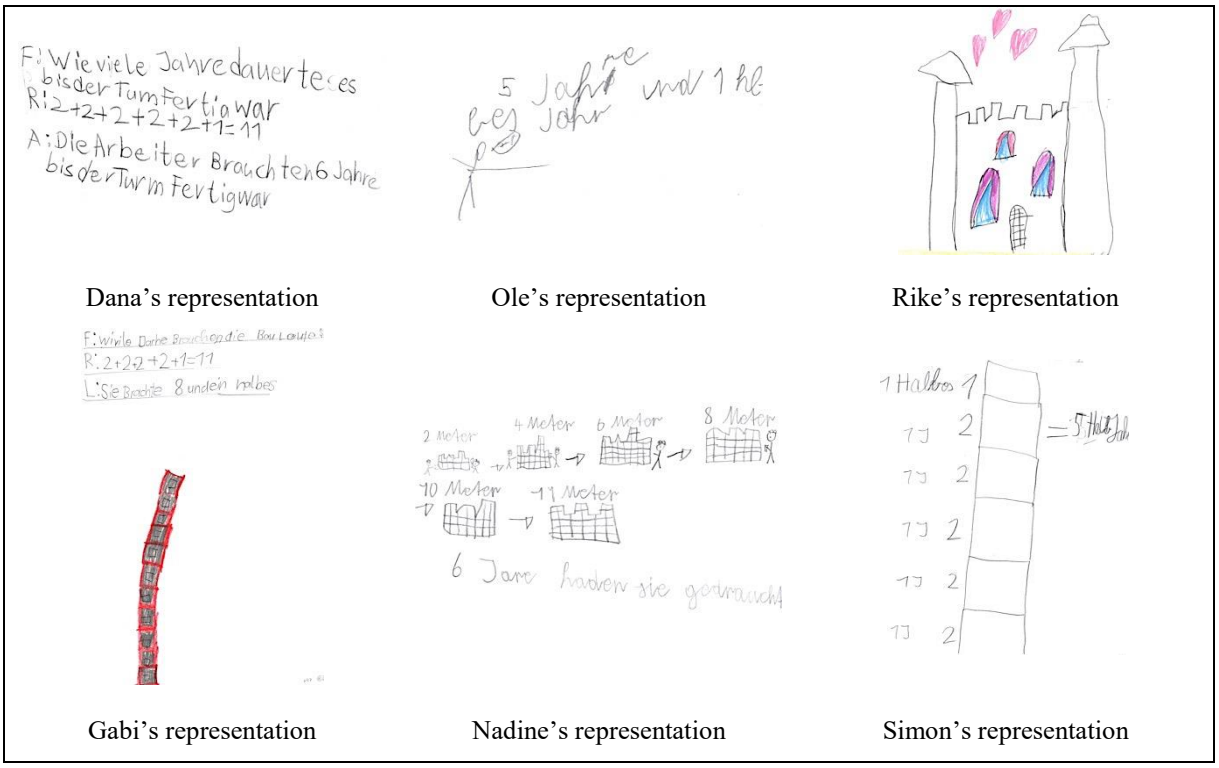
Finally, the study aims to design an analysis tool for children's word problem drawings, based on theory and split into various categories.

Methods

Data collection

A paper and pencil test was given to two first grade and two second grade classes in two German primary schools. 42 first graders and 35 second graders participated in the test. The average age of the first graders was 6 years and 4 months, whilst the average age of the second graders was 7 years. By pure chance, there were 35 girls and 42 boys.

The test was conducted at the end of the school year's first half. At that time, the first and second graders could solve addition and subtraction tasks in the number ranges up to 20 and 100 respectively. The first graders had experience in drawings for equations and finding equations to match drawings. Students of one second grade class (N=19) had additional experience in drawing sketches for word problems.



Dana's representation | Ole's representation | Rike's representation

Gabi's representation | Nadine's representation | Simon's representation

Figure 1: Children's drawings for the tower-item (Ott, 2016)

The test consisted of six word problems based on schoolbook tasks. The level of difficulty differed, in accordance with how much the verbal texts implicitly suggested objects and hence ways of drawing (Ott, 2015, 2016). Testing took place on two successive days, three test items a day. The instructor read the items out aloud to the students, who were then requested to draw their thoughts on a plain sheet of paper, so as to be understood afterwards by the instructor. Figure 1 shows six examples of children's drawings for the following word problem: *Once upon a time there was a king who wanted to have a tower of 11 meters in height. The tower was built over the course of several years. Every year, the workers built 2 meters. How long did it take for the tower to be built?*

Analysis

The children's 438 individual depictions from the test have been used both to develop a theory of drawings for word problems and also an analysis tool, based on various categories. To this end, the drawings have been analysed via a combination of qualitative content analysis (Mayring, 2010) and theoretical coding (Strauss & Corbin, 1996). Developing a theory and the design of an analysis tool took place simultaneously in an iterative process according to representation theories.

Results

The analysis of the children's drawings revealed as key either for a draft of a theory or an analysis tool the following three features: The *mathematical structure*, the *mathematical matching* and the *degree of abstraction*. In analysing the drawings, it is important to differentiate between these features, details of which are presented below.

Mathematical structure

Theory

A *mathematical structure* may be defined by set theory: An amorphous set is structured by defining relations and operations between its elements (Rinkens, 1973). This definition is used to identify both the mathematical structure of word problems and also of graphic representations.

Word problems are verbal descriptions of situations with a focus on mathematical relations (Veschaffel, Greer & de Corte, 2000). They can be characterised as "descriptional" (Schnotz, 2014, p. 47) representations. Information is presented sequentially with the quantities and nouns being related to each other by verbs and prepositions. Thus, structural information is integrated into the text and the word problem is thus given a mathematical structure. Accordingly, the verbs and prepositions serve as relational symbols (Schnotz, 2014, p. 47), without which the quantities and nouns would be unrelated – this applies equally to the elements of an amorphous set. Because of the verbal construction with verbs and prepositions as relational symbols, a relationship between the quantities will be defined. Consequently, a structure is given to the quantities and nouns. In the word problem presented here, the quantities and nouns "tower", "11 meters", "year(s)" and "2 meters" are related by the verbs and prepositions "of", "built" and the adverbial phrase "every year".

For a sound graphic representation, it is necessary to identify key objects in the text, e.g. quantities, objects or people mentioned, that are relevant and necessary for the mathematical structure. To achieve this, structurally relevant objects signs (Peirce, 1965) need to be invented, which can be regarded not only as physically analogous to the objects but also symbolic. Graphic representations can be characterised as "depictional" (Schnotz, 2014, p. 47). Compared with word problems, graphic representations do not include relational symbols. Indeed, a structure is given to the signs for structurally relevant objects by mapping them to certain aspects of space. To this end, the signs for structurally relevant objects have to be set out on the sheet in such a way that the arrangement represents the word problem's verbally described relationships. Such graphic representations have the character of diagrams (Dörfler, 2006). For instance, in Simon's graphic representation (Figure 1) the signs for structurally relevant objects "1J", "1 Halbes", "2", "1" and the rectangles are arranged in vertical columns side by side.

Analysis tool

Six categories of how the mathematical structure appears in children's drawings for word problems are identified: A representation is

- *non-graphic* if it consists only of calculations or texts;
- *off the text* if it possesses graphic elements, but there is no link to the text with regard to the content;
- *illustrative* if it possesses graphic elements with a link to the text but no structurally relevant objects are represented;
- *object-related* if it possesses graphic elements with a link to the text and structurally relevant objects are represented although relations between them are not identifiable in the arrangement;
- *implicitly diagrammatic* if it possesses graphic elements with a link to the text, structurally relevant objects are represented and relations between them are identifiable in the arrangement; the relations are not explicitly emphasised;
- *explicitly diagrammatic* if it possesses graphic elements with a link to the text, structurally relevant objects are represented and relations between them are identifiable in the arrangement; the relations are explicitly emphasised.

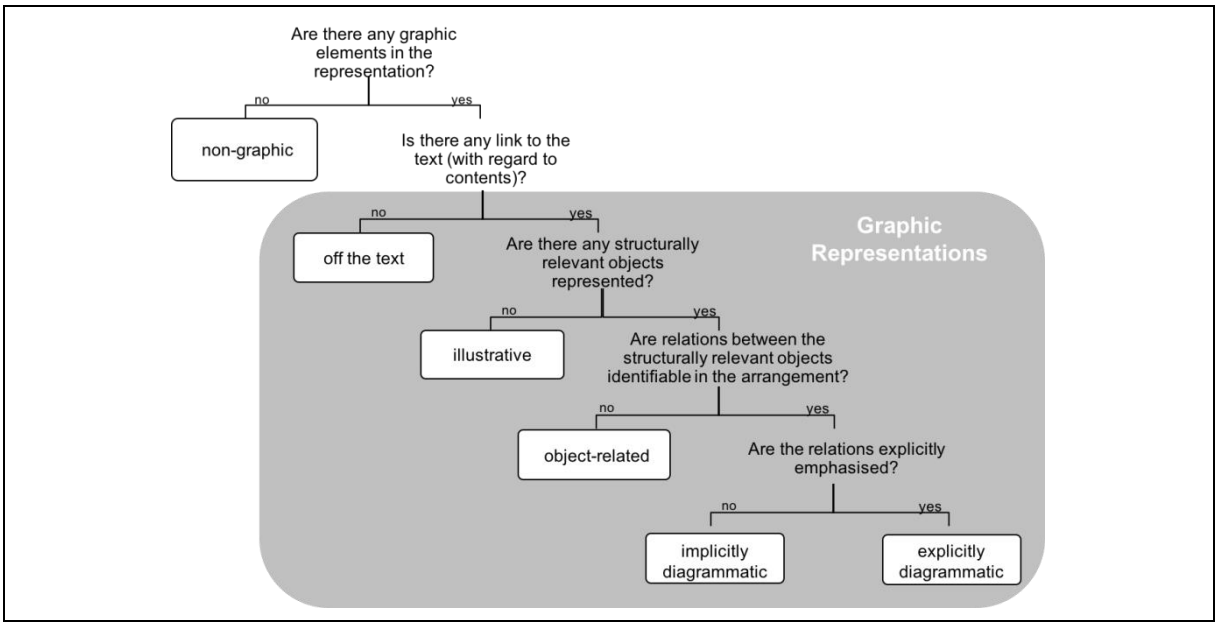


Figure 2: Analysis tool for the mathematical structure

In contrast to text analysis, such as the qualitative content analysis (Mayring 2010), it is impossible to analyse units step by step in a drawing. The analysis tool introduced here, therefore, arranges the categories in a decision tree guided by key questions, which lead to category definitions. The categorisation of a child's drawing takes place step by step in a strictly dichotomous procedure. Only if a question has to be answered with 'no', will the drawing be classified into the associated category. As long as questions can be answered with 'yes', the categorisation process is not yet finished. This way, each of the children's drawings can be clearly categorised. In Figure 2 we can see the decision tree.

We will now categorise the children's drawings shown in Figure 1. Dana's solution is an example of a *non-graphic* representation. It consists of text and a calculation but no graphic elements. Ole's

drawing is an example of a representation that is *off the text*. A stick figure with a speech bubble is drawn, which contains the answer. The drawing contains no link to the problem's content. In contrast, Rike's graphic representation with a castle and hearts contains a link to the text, because a tower as part of a castle is drawn. Gabi's representation of a tower also comprises graphic elements with a link to the text. The tower is subdivided into 11 rectangles, which could represent the 11 meters that are structurally relevant objects. Relations are not identifiable in this representation and hence it is an example of an *object-related* representation. Nadine and Simon's representations are both graphic with links to the text and structurally relevant objects, e.g. the tower. Nadine's representation is a sequence of pictures. In each picture, one can see the height of the tower in a given year. Thus, the relations between the meters and the years are identifiable and the representation is, therefore, diagrammatic. The relation is not explicitly emphasised and the representation is, therefore, an example of an *implicitly diagrammatic* representation. In Simon's drawing, the height of every tower section is shown in one tower only. Next to every section of the tower, one can find the years necessary for the work to be completed. Thus, the relation between meters and years is not only apparent but also emphasised and it is, therefore, an example of an *explicitly diagrammatic* representation.

Mathematical matching

Theory

In a representational system (Palmer, 1978) the correspondences between the represented and the representing "world" (p. 262) are important. In regard to the mathematical structure, this idea is used to define how a word problem is matched with a graphic representation. If there is a match, they are "informationally equivalent" (Palmer, 1978, p. 270).

We call a relation of text and drawing a *mathematical matching* if the word problem and the graphic representation are informationally equivalent on both object and relational levels. Accordingly, if there is a match between the quantities and nouns on the text side and the signs for the structurally relevant objects on the graphic side (object level), and if there is a match between the verbs and prepositions on the text side and the arrangement of the signs for the structurally relevant objects on the graphic side (relational level), then there exists a complete matching between the word problem and the graphic representation.

Analysis tool

The matching between word problems and graphic representations is analysed with regard to the mathematical structure. Only *object-related, implicitly* or *explicitly diagrammatic* representations are, therefore, analysed (see above). Representations of the other categories do not contain elements of a mathematical structure and accordingly there is no matching between them and the word problem.

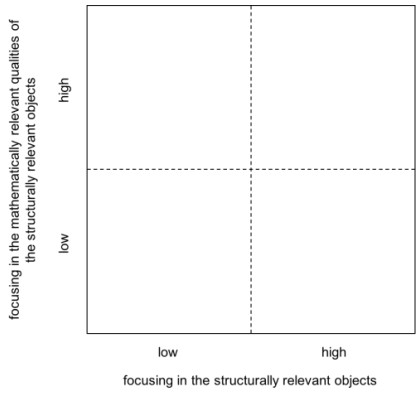
For analysing the match between the quantities and signs for structurally relevant objects, it is useful to distinguish between the measured value and the measuring unit. Numbers can also be considered as discrete quantities (Müller & Wittmann, 1984). For analysing the match between verbs and prepositions and the arrangement of signs for structurally relevant objects, we have to consider the operations of the word problems.

The variation of match between graphic representations and word problems with regard to the *measured value*, the *measuring unit* and the *operations* can be scaled (Mayring, 2010). The match can be *complete*, *partial* or *non-existent*. If it is non-existent, the quantities or operations graphically represented are other than those given in the word problem, e.g. an addition instead of a multiplication. It is also possible, that the quantities and operations are not apparent in the graphic representation. They are *without compliance*. In the analysis tool the possibilities are arranged in a 3x4 matrix, where each matching can be marked with a cross. Arithmetically, 64 combinations are possible (see Figure 3a).

In Simon's representation (see Figure 1), measured values (11 and 2) are apparent as well as measuring units (meters and years) and the operations (addition of meters and years, meters per year). In Nadine's drawing each measured value and operation are also apparent. With regard to the measuring units, only meters are identifiable. In Gabi's representation the operations are without compliance. With regard to the measured values, only the 11 is apparent. So far as measuring units are concerned, only the meters are identifiable in the squares.

	complete matching	partial matching	non-existent matching	without compliance
measured value	every measured value is apparent	some measured values are apparent	only other numbers are apparent	no measured value is apparent
measuring unit	every measuring unit is apparent	some measuring units are apparent	only other measuring units are apparent	no measuring unit is apparent
operations	every operation is apparent	some operations are apparent	only other operations are apparent	no operations are apparent (*object-related representations*)

a



b

Figure 3: Analysis tool for the a) mathematical matching, b) degree of abstraction

Degree of abstraction

Theory

Children's drawings for word problems are realistic to a greater or lesser extent. For analysing this, the idea of abstraction is used, which can be defined as a focusing of attention on certain aspects (Peschek, 1988). This idea is used to define the *degree of abstraction*.

We characterise the *degree of abstraction* in a graphic representation as being the degree of focusing on representation of the word problem's mathematical aspects (Ott, 2015). The foundations for the mathematical structure of a graphic representation are the structurally relevant objects (see above), that are important in defining the degree of abstraction, for which two indicators are identified:

1) Focus on the structurally relevant objects in the graphic representation - no objects other than the structurally relevant objects are drawn.
2) Focus on the mathematically relevant qualities of the structurally relevant objects - the signs for structurally relevant objects are not decorated and represent only the mathematically relevant qualities.

Analysis tool

Given that the degree of abstraction becomes apparent when focusing on the structurally relevant objects and their mathematically relevant qualities, only *object-related*, *implicitly* or *explicitly diagrammatic* representations are analysed (see above). Representations of the other categories do not contain structurally relevant objects and they are in themselves, therefore, not abstract.

Both of the indicators for the degree of abstraction can be pronounced 'low' or 'high'. Consequentially, it is possible to distinguish four categories for the degree of abstraction: *High-high* or *low-low* if each indicator is identified as high or low respectively, and *high-low* or *low-high* if one indicator is identified as low and the other as high. In the analysis tool the possibilities are arranged in a 2x2 matrix (Figure 3b). Both indicators have to be regarded separately in analysing a child's drawing, which is then classified in the category that fits both results.

In Simon's representation (Figure 1) the focus on both structurally relevant objects and their mathematically relevant qualities are high, contrary to Nadine's representation. Here, signs for other objects are drawn such as people and the tower decorated with battlements as a structurally relevant object. In Gabi's representation, the focus on the structurally relevant objects is high, because only the floors of the tower are drawn, which are decorated with colour and squares.

Conclusion

This study has focused on the main question of the extent to which children's drawings for word problems represent the actual mathematical structure. For this purpose, the *mathematical structure*, the *mathematical matching* and the *degree of abstraction* have been identified as substantial features of such graphic representations. Children's drawings for word problems vary greatly in the way they are depicted. In the analysis tool, the extent of the substantial features' occurrence in children's drawings has been modelled, so that each drawing can be clearly categorised. The analysis tool has been tested with an inter-rater reliability of K=0.81. It could, therefore, be applied in further research. In the second part of the research presented here it has been used in an intervention study looking at graphic representations of word problems (Ott, 2016).

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How the representations take on a key role in an inclusive educational sequence concerning fraction

Elisabetta Robotti

Università degli Studi di Torino, Dipartimento di Filosofia e Scienze dell'Educazione, Italy;
elisabetta.robotti@unito.it

The design analysis of an inclusive educational sequence concerning the teaching and learning of fractions (Robotti, et al., 2015) is the focus of this work. Referring to the principles of Universal Design for Learning, developed by the Centre for Applied Special Technology to reduce barriers in learning, we analyse the design of the educational sequence focussing on fractions and devoted to classes where students with certifications of mathematical learning disabilities (MLD) were present. From the point of view of mathematics education, we refer to the theory of semiotic mediation (Bartolini-Bussi, Mariotti, 1998) in order to clarify the cognitive role of artefacts taken into account in class activities. In particular, the use of artefacts to solve tasks produces representations that contribute to develop of mathematical meanings aimed in the teaching activity.

Keywords: Mathematical learning disabilities, Universal Design for Learning, fraction, number line, artefact.

Introduction and conceptual framework

The focus of this paper is the analysis of an inclusive educational sequence concerning the teaching and learning of fractions already described in detail in CERME9 (Robotti, *et al.,* 2015). In the paper presented in CERME9 we discussed how, in the context of Semiotic Mediation (Bartolini, Mariotti, 2008), the choice of particular artifacts and the design of tasks related to their use, allowed students to grasp different meanings of fraction. Now, I would like to discuss why that educational sequence was "inclusive" for learners with certifications of mathematical learning disabilities (MLD) or difficulties in math or with low achievement in math. To this aim, I will refer to the theoretical framework of Universal Design for Learning (UDL) and I will analyse how the design of the activities follows the guidelines and principles of UDL in order to reduce barriers in learning. Therefore, the rationale for referring both theoretical frameworks is to consider learning difficulties in a context of math education: if the Semiotic Mediation framework allowed us to design a structured educational sequence for teaching and learning different of meanings of fraction (for this, see CERME9 proceedings), the UDL framework, allowed us to design and analyse those tasks in order to be inclusive for MLD students, students with difficulties or with low achievements. Even if there isn't consensus on definition and identification of MLD students (Karagiannakis, *et al.,* 2014) and the inclusivity (Ianes, 2006) is not a construct used consistently across different fields (education, society...) or in different countries, in this research work, we considered as "inclusive educational activities" the educational activities, developed in the context of the class, which face to the special needs of MLD students including dyscalculic students, students with difficulties in math and students with low achievement in math. In other words, according to the UDL framework, we consider "inclusive educational activities" those that meet the needs of all students of the class. UDL is based on a set of principles and guidelines that have been elaborated to increase access to curriculum for all the students, including those with disabilities. These principles and guidelines have a general valence and they are devoted to various application contexts. In this paper we refer to the context of math

education. At the core of UDL is the belief that there are three networks in our brain that support learning: (i) The Knowledge/Recognition Networks, which are involved in identifying and interpreting sound, light, taste, smell, and touch. They are essential to learning because students are expected to comprehend a text, interpret formulas, identify cause/effect relationships, etc. UDL states that it is possible to support the knowledge/recognition networks providing Multiple Means of Representation (1° principle) to give learners various ways of acquiring information and knowledge. This principle has three supporting guidelines: provide options for Perception; provide options for Language, Mathematical expression, and symbols; provide options for Comprehension; (ii) The Strategic Networks, which are involved in planning, executing, and monitoring actions and skills. In learning, they occur, for example, to solve a problem, use an artefact, take notes and listen to a lecture. UDL states that it is possible to support the Strategic networks providing Multiple Means of Action and Expression (2° principle) to give learners alternatives for demonstrating what they know. This principle has three supporting guidelines: provide options for Physical action; provide options for Expressive skills and fluency; provide options for Executive function; (iii) The Affective Networks, which are responsible for establishing priorities and interests. UDL states that it is possible to support the Affective networks providing Multiple Means of Engagements (3° principle) to stimulate interest and motivation for learning. This principle has three supporting guidelines: provide options for Physical action Recruiting interest; provide options for Physical action Sustaining effort and persistence; provide options for Physical action Self-regulation. According to UDL frameworks, representations seem to be "the ways [to] perceive and comprehend information", produced by visual or auditory means, or contained in a printed text. Therefore, UDL considers representations as signs of different nature (visual, auditory, kinaesthetic…) produced by different means (or artefacts). In the Theory of Semiotic Mediation, Bartolini and Martiotti state that: "[…] any representation comes to life because of a human construction that makes it possible, in other words any representation is supported by an artefact" (Bartolini, Mariotti, 2008, p. 747). Even if the difficulty in articulating an accurate definition for the term "representation" is recognized in math education, we want to stress that, in his theoretical framework Duval states the importance of connections both within and amongst different representational registers as absolutely fundamental to deep understanding of mathematics (Duval, 1999). According to this, Arcavi (2003) stresses the essential role of visual representations in the learning of mathematics and he defines "visualization, as both the product and the process of creation, interpretation and reflection upon pictures and images, …" (Arcavi, 2003, p. 215). Visualization, visual representations and more general representations were taken into account in our analysis presented in CERME 9 discussion. As described in that analysis, the teaching of the notion of fraction is a quite delicate issue, which requires insightful ways of structuring didactical activities. In that occasion, we stressed how the importance of spatial processes, performed on the base of spatial skills, can be important in mathematical performances where explicit or implicit visualization is required, as in the case of learning of fractions. Therefore, the visual non-verbal, the kinaesthetic-tactile and the auditory channels were considered as preferential for designing inclusive educational activities concerning fractions. Finally, Goldin (1992) outlined a unified model for the psychology of mathematics learning, which incorporated cognitive and affective attributes of visualization as essential components in systems of representation in mathematical problem solving processes. From the above considerations, we can observe that, even in the domain of math education, the aspects related to multiple means of representations and their relations, multiple means of action and

expression and multiple means of engagement are taken into account. For this reason, we use the three principles of UDL to analyse the design of an educational sequence about fractions whose purpose was to be inclusive (in the UDL sense), that is to say, to meet the needs of all learners of the classes, even MLD students. Moreover, we will adopt the framework of Semiotic Mediation to analyse the role of the artefacts in the educational sequence in order to reach the aimed educational goals. More in detail, we will show how the design of the different tasks, requiring actions on artefacts, can be interpreted through the three principles of the UDL to analyse the efficacy both of construction of meanings for the notion of fraction and of inclusion for MLD students.

An example of inclusive educational sequence

I will briefly recall the main activities characterising the educational sequence concerning fractions, described in details in TWG13 of CERME 9 (Robotti & al., 2015), which was carried out in 22 classes (nine 5th grade classes, six 4th grade classes, and seven 3rd grade classes) involving around 400 students, of which 20 were certified MLD students (Robotti & al., 2016): (1) Partitioning of the A4 sheet of paper (named “placemat”). (2) Partitioning of a strip of squared paper: given a certain unit of measure (number of squares) on the strip, position a fraction on that strip (1/2 or 1/3 or 1/4, ….); given different units of measure on different strips, on each strip a same fraction is represented (1/2); chosen an appropriate unit of measure on the strip, different fractions are represented on that strip (e.g., 1/3 and 1/5). (3) Placing fractions on the number line. (4) Placing coloured tags, labelled with fractions, on a “string on the wall”.

The learning difficulties concerning fractions seem to be due, among other things, to the lack of connection between different meanings of fraction (Charalambous, Pitta-Pantazi, 2005) and to the fact that only some of them are fostered (e.g. the meaning of "part/whole"), (Pantziara, Philippou, 2011). Moreover, some visual representations are favoured (such that of the pizza) even if they could hinder the learning of the different meanings of fraction (Fandiño Pinilla, 2007). The main learning difficulties identified in Italian national assessment (INVALSI tests) are related to: managing the meaning of the “equal” sign (for instance, What does it means obtaining “equal parts” of the whole?); switching from a fraction to the unit that has generated it; managing equivalent fractions; ordering fractions on a straight-line even without transform them to decimal numbers. In the following, I will discuss how principles and guidelines of UDL can be effectively used to analyse the design of the tasks, which allow overcoming these difficulties. The analysis of actions required by tasks on the artefacts, will show how the educational aims and the aims of inclusion were achieved.

The artefact “placemat”

The “placemat” is a A4 sheet of paper that, at the beginning of the sequence will be in white colour but that, in the following, will be coloured. The mathematical meanings to be mediated by the “placemat” are: 1) construction of fractional units starting from a given unit of measure (the A4 sheet of paper) by folding and cutting out the A4 sheet of paper; 2) equivalence between fractional units, by folding or cutting the fractional units in order to show the equivalence between the surfaces. This allows to overcome the difficulty related to the interpretation of “equal parts of the whole” as “congruent fractional units” instead of “equivalent fractional units” (see above); 3) sum of fractional units in order to obtain the given unit of measure (in this case, the A4 sheet of paper). This can be realized covering the A4 sheet with different fractional units. This allows to come back to the unit of

measure starting from the fractional units. According to Guideline 1 of first Principle of UDL, learning is difficult if information is imperceptible to the learner, or when information is presented in formats that require extra effort for him/her (for example, decoding the text "one half" or decoding arithmetical expressions "1/2" for a dyslexic student). To reduce barriers to learning, it is important to ensure that key information are equally perceptible to all learners. For this reason we provided the same information through different modalities (e.g., through the possibility of touching and manipulating the pieces of A4 sheet, through vision of their drawing on the notebook, through hearing or reading an arithmetical expression). In other words, through different forms of representation (Guideline 2, Principle 1). Moreover, the colour, which characterises fractional units later on, plays the role of support to long-term memory for students with learning disabilities or simply students with math impairment. Once the information are made accessible, our educational sequence aims to help students transform them into useable knowledge. As pointed out by the Guideline 3 of the first Principle of UDL, this depends upon "information processing skills" like: selecting useful information, integrating new information with prior knowledge, strategic categorization and active memorization. Individuals differ greatly in their skills in information processing, but effective design of task and presentation of information in accessible ways can provide the scaffolds necessary to ensure that all learners are able to process information. Therefore, we designed educational activities in which students have to act on the artefact, in order to produce representations (for instance, by cutting or folding different pieces of paper referring to the same fractional unit in order to show the equivalence of their surfaces or choosing appropriate fractional units to cover the A4 sheet in order to show that the sum of appropriate fractional units gives the unit of measure 1), and they have to put the obtained representations into relation. Indeed, the actions, performed by students on the artefacts, produce situated signs (representations) through which students, with the help of teacher's mediation, construct the mathematical meanings aimed for (in that case, equivalent fractions or 2/8+3/4=1). In order to help students remember the mathematical meanings described, the teacher asks students to: reproduce on the note-book the operations performed with the artefact; write down the content of the class discussions; write texts in which the processes developed in the activity are explicitly linked to knowledge. This allows recalling and using knowledge in the future and it provides options for expression and communication. As a matter of fact, it is well known that there is no medium of expression that is equally suited for all learners and for all kinds of communication. This means that, according to the second Principle of UDL, students with learning disabilities may excel, for example, in interpretation of drawing data (e.g., the drawing of "placemat" covered by different coloured fractional units), but they may falter when asked to read data provided in a table or in arithmetical expression (e.g. 1= 2/8+3/4).

The artefact "strip of squared paper"

The strip of squared paper is a strip with squares of 1 cm, 10 cm high and approximately 1 m long. In the strip there are some integer numbers (0, 1, 2, 3, …) and fractions are constructed. The mathematical meanings mediated by the "strip of squared paper" are: 1) Fraction as operator on a given unit of measure: once a unit of measure is considered, the teacher asks students to place on the strip a given fractional unit and then a given fraction (equal o more than 1); Comparison, by means of a perceptive strategy, of fractional units that should be positioned on different strips of paper; 3) Relationship between a fractional unit and a chosen unit of measure: considering different units of measure on different strips, the teacher asks to place the same fractional unit (for instance, ½) on all

the strips; 4) Ordering fractional units: considering on a strip an appropriate unit of measure (given by the l.m.d. of the denominators), the teacher asks to place different fractional units. This allows students to compare them in a concrete way; 5) Equivalence between fractions: considering on a strip a given unit of measure, the teacher asks to place different fractions. Among them there are some equivalent fractions. We note that some mathematical meanings concerning fraction, such as equivalent fractions or comparison of fractions, are already presented in previous activities with the artefact "placemat". Once again, the first principle of the UDL is used in this educational sequence providing different means of representation (that is, different artefacts through which representations can be performed). In this activity, the students produce different representations: linguistic signs associated to the name of the fraction expressed in verbal language ("One half"), in verbal visual language (the writing "One half") and arithmetical language ("1/2"). The teacher institutionalizes the relationship between the different signs in terms of rational numbers. Thus, the construction of meaning related to the notion of rational number, is based on both the interplay between different types of semiotic representations, according to the first principle of UDL, and by different actions on artefacts producing those representations, according to the second principle of UDL. Moreover, we note that the colour, perceptive option already provided in the A4 sheet of paper, is used with the same aim also in the activities with strip of squared paper. Thus, the fractional units constructed on the strip are coloured with the same colour of the respective fractional units constructed by A4 sheet of paper. Therefore, the colour assumes the role of a support for memory and also of an artefact, which allows students to link the meaning of fractional unit constructed with the "placemat" (fraction as a *part of a whole*) with the meaning of fractional unit constructed by the strip of paper (fraction as *operator*). Moreover, we have taken into account different strips where different units of measure have been considered and where have been represented a given fractional units. The strips are put one beside the other (proving option for physical actions, as called for by the second principle of UDL) and the relationship between unit of measure and fractional units becomes perceptively evident, reifying Guideline 1 (provide options for perception) of the first principle of the UDL. In order to link the meaning of fraction as part of a whole and as an operator to the meaning of rational number on a number line, we need to switch from the artefact "strip of squared paper" to the artefact "number line". For this reason, the teacher asks the students to represent, on the *same* strip different fractional units: 1/8, 1/6, ¼, 1/3 and ½. Thus,, in order to represent all fractional units on the same strip, students need to find a suitable graphic strategy that maintains the colour without superposing different colours on the strip. They adopt coloured notches. Positioning on single strip different fractions makes the ordering of fractions exactly like that of the other perceptively evident numbers. Here the semiotic potential of the artefact "strip of squared paper" associated to the tasks proposed by the teacher becomes evident, playing a key role in fostering identification of fractions as rational number on the number line. I will show this in the following session.

The artefact "number line"

The number line is drawn by students on their note-book. It is a number line of positive numbers starting from the point 0 and it is presented as the natural "crushing" of the paper strip. Therefore, on the number line, students place fractions without transforming them into decimal numbers. The mathematical meanings supported by the "number line" and the tasks proposed are similar to those proposed in activities with the artefact "paper strip". This allows all students to find continuity both between the artifacts proposed and the construction of the aimed for mathematical meanings.

The artefact "string" on the wall

The artefact "string on the wall" consists of a string in nylon whose ends are fixed to two adjacent walls of the class (such as a wire for drying clothes). On it, students hang some tags labelled with fractions (in this case, tags are coloured accordingly with the colour used in previous activities) or natural numbers (in this case tags are white). The string simulates the number line starting from zero, which is placed at the left end and the position of the unit is made to vary dynamically sliding the corresponding labelled tag attached with a clothes' peg. In this case, however, the positions of the other tags do not vary dynamically at the same time or automatically as a consequence of the new placement of the unit: their motion requires a specific action in order for the numbers on the line to maintain the desired mathematical relationships. Tags are made so that they can be hung in "clusters" to ensure that tags corresponding to equivalent fractions have the same position on the string. The mathematical meanings supported by the "string" are: 1) Ordering of fractional units or fractions: once the unit of measure has been defined, by positioning the tag labelled with "1", the tags labelled with fractions need to be hung in the correct position; 2) Equivalent fractions: since their tags correspond to the same position on the string, they are hanging as "clusters" and they represent classes of equivalence. Students named them "caterpillars" whose first tag (corresponding to the irreducible fraction) was called "head of the class"; 3) Density of the numerical set Q: "enlarging" the unit of measure, that is to say increasing the distance between the tag corresponding to 0 and that corresponding to 1, it is possible to hang on the strip more and more fractions. This operation, repeated many times, allows constructing a mental image connected to the idea of infinity. We note that, when tags on the strip become numerous, one needs to "enlarge" their position to make room for other tags. This action on the string is a situated sign used by the teacher to introduce the idea of density in the set Q. Therefore, the representations of fraction on the string (tags labelled with fractions), their position on the string and the action performed on the tags are in line with the first principle of the UDL according to which different forms of representation are needed in order to capture information and transform them into knowledge. The colour of tags, the position of tags on the string, the dynamic position of tag labelled with "1" on the string, recall actions and representations performed in previously activities. Once again, this allows linking different representations and actions to the same mathematical meaning, by supporting long-term memory. According to the principles of the UDL, this contributes to making mathematical meanings accessible to the students.

Discussion and concluding remarks

The first principle of UDL (Providing Multiple Means of Representation) and the related guidelines provide useful references in order to choose artefacts and define tasks which allow overcoming obstacles and difficulties in understanding some of the different meanings of fractions (*part of a whole*, *measure* and *operator*). In order to provide different means of representation, we considered different artefacts: the "placemat", the strip of squared paper, the drawing number line and the "string" on the wall for "hanging out" fractions. Each of them allows students to produce different semiotic representations: pieces of paper as fractional units, coloured sections of the strip as fractions or fractional units, points on the number line, coloured tags labelled with fraction hanging on the string. We note that these representations are of different nature: physical, visual or symbolic. Thus, following the Semiotic Mediation Theory, each artefact allows students to produce situated signs (for example, the sign "enlarge" the tags' position on the string), which will be interpreted as mathematical

signs by the teacher's mediation (in the case above, the sign is used by the teacher to introduce the idea of density in the set Q) . The aim is to single out mathematical aspects relevant to the activity and make them accessible to all students through representations; this should allow them to use those mathematical aspects in future. To reach this aim, the design of educational activity (presentation of information, choice of the artefact on which students can act, task design in large sense…) is essential to ensure that all learners have access to knowledge. We note that in our educational sequence about fractions, learners construct mathematical meanings (for instance, the meaning of equivalent fractions) acting on different artefacts (A4 sheet, strip of paper, number line and "strip on the wall") and different representations carried out by those artefacts (fractional units of paper, coloured part of the strip, point on the line and coloured tags composing a "caterpillar"). According to the first principle of the UDL, we state that this is not yet sufficient to ensure the construction of the mathematical meaning, that is to say of usable knowledge (for example, the meaning of equivalent fractions). It is necessary that these different representations be put in relation with each other (for example, two equivalent fractions correspond to the same point on the number line or on the "strip" on the wall because they have the same surface extension as a sheet of A4 paper). Moreover, it is necessary that they be always available to learners during the whole teaching sequence. Thus, according to the second principle of the UDL (Providing Multiple Means of Action and Expression), the action performed by students on different artefacts allows to put into relation different representations associated to the same notion (such as, equivalent fraction) or associated to different meaning of fraction (such as "a part of the whole", in the case of placemat, or "fraction as operator", in the case of strip of paper). The third principle of the UDL (Providing multiple means for engagement) supports the engagement of students in the arithmetic activity concerning fractions. It suggests options to challenge all students, appropriately. Designing activity where a "real" situation requiring a solution was the starting point. As a matter of fact, acquiring new information must be received by the student as a necessity to deal with the challenge posed by the activity. Thus, since the narrative aspect is very important in primary school teaching, the work with the "placemat" started with a letter, sent by a pizzaiolo (pizza chef), where he asks students to realize coloured placemats for his restaurant. The need to make placemats according to the pizzaiolo's requests was, for primary school children, a stimulus to take in information and to process them in order to get new knowledge useful for the activity's aim. To create conditions for the students' self regulation and self assessment in the activity we asked students to work in pairs or in small groups and to compare, by the class discussion, the results of work with the other groups.

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Using manipulatives in upper secondary math education – a semiotic analysis

Marit Hvalsøe Schou

Laboratory for Coherent Education and Learning, University of Southern Denmark;
maritschou@imada.sdu.dk

The use of manipulatives in upper secondary education is not common, although they hold learning potential in contributing to students' conceptualizing and establishing of relations. In this case study it is investigated how the concept of recursive sequences as a description of growth can be seen in the written reflections from students having engaged in activities with manipulatives like matches and LEGO® and with only minor teacher interference. The analysis of students' answers is carried out by a combination of Radford's Theory of Objectification and Peirce's Theory of Signs. By interpreting the type of signs, the kind of arguing and the nature of the words used by students in various activities involving manipulatives, it is possible to acquire knowledge of the significance of the manipulatives in students' conceptualizing.

Keywords: Upper secondary mathematics education, manipulatives, semiotics, conceptualizing, objectification.

Introduction

For centuries manipulatives have been used in mathematics education as a pedagogical means for learning at all levels. A visit to the Institute of Mathematics at the University of Göttingen, Germany reveals an entire floor display of manipulatives used in university teaching (Stewart, Mühlhausen, & Miyazaki, 1993). Today the use of manipulatives is still common in primary and lower secondary school as a way of making the teaching concrete and meaningful. On higher levels where deduction and generalisation are important issues, manipulatives are rarely seen and research on their use is scarce (Bartolini & Martignone, 2014). Previous studies show that manipulatives can help students perform algebraic reflections and actions in pattern-generalizing tasks in primary and lower secondary school by joint student-teacher interaction (Radford, 2008). Here it will be investigated how the use of manipulatives can be seen in student reflections on the concept of growth in this upper secondary education.

Theoretical framework

Generalisation depends on symbolization (Otte, 2006) and many teachers use symbolic representations extensively without connecting them to other representations thereby preventing their students the chance to explore or reflect on these relations (Steinbring, 2005). The use of manipulatives is an alternative way of bringing in abstraction and generalisation in teaching, enhancing the development of mathematical knowledge (Mitchelmore, 2002). In his Theory of Objectification, Radford (2013, p. 26) states: "Learning […] is the noticing of something that is revealed in the emerging intention projected onto the signs […] in the course of practical concrete activity […] and is hence transformed into knowledge."

A semiotic analysis of the representations and the communication about these representations, i.e. the signs the students produce, is used to investigate the mathematical knowledge developed. Each

observation of student work is treated as *a semiotic bundle*, a system of signs produced by a student or a group of students while they solve a problem or discuss a mathematical question. (Arzarello, Paola, Robutti, & Sabena, 2009).

Presmeg et al. (2016) demonstrate the usefulness of Peirce's semiotics in analyzing students' work with representations. Peirce defines:

> A sign, or *representamen*, is something, which stands to somebody or something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. [Peirce 1965, 2.228, italics in original]

A sign to Peirce thus consists of a triad (representamen – object – sign). Each of these can be interpreted in terms of his categories (of which there are three: "possibility, existence and law"). Here I use in particular two trichotomies: *icon*, *index* and *symbol* referring to how the sign stands for its object, and the *token-type* distinction (omitting *quali-sign*).

An *icon* is a sign that shares a likeness with the object it represents either as an *image*, a *diagram* or a *metaphor*. An *image* has a resemblance or a simple quality in common with the object – a drawing of a real thing; a *diagram* is a sign representing relations, e.g., $x^2 + y^2 = r^2$ represents the points of a circle. A *metaphor* represents a parallelism where the character of one sign is expressed in a law-like manner in the other, like when a symptom, fever, of illness is seen as an increased temperature at a thermometer. An *index* is related to the object it represents by either a causal connection or by 'a purposeful act of connecting the signs' like using '*f*' as the name of a certain mathematical function. Finally a *symbol* is connected to its object by habit or law, e.g. designating π to refer to the ratio between the circumference and the diameter of a circle.

The *token-type* distinction applies to the sign alone. A *token* refers to the sign itself; consider the letter 'e', this letter appears many times – as a *token* – on this page. It is also possible to refer to 'e' as a general – as a *type*, stating that 'e' appears a particular number of times on this page.

A thorough explanation of Peirce's trichotomies of signs is offered in (Short, 2007) whereas mathematical examples can be found in (Otte, 2006) and (Sáenz-Ludlow & Kadunz, 2015).

The teaching design rests on the Theory of Didactical Situations, TDS (Brousseau, 2006). The epistemological foundation of TDS is that new knowledge is obtained in two steps: first as personal knowledge *connaissance* in working with a given problem and second as formal, scientific knowledge *savoir*, which can be share with and understood by others.

TDS distinguishes between *didactical situations* where the teacher participates, and *a-didactical situations* where students engage in activities on their own. *Devolution*, where the teacher introduces and hands over the problem, *validation* and *institutionalisation*, where the ideas and results are tested and then related to the scientific knowledge of the solution, are all didactical situations. The main focus of this case study is, however, the a-didactical situation where students interact with the *milieu*, which contains the tools that the students have at their disposal (symbolic or material as well as the other students in the group). An important issue in designing the milieu is to ensure that it provides feedback on how the students progress in order to solve the problem.

Within this framework the study will try to answer the question: *How does the use of manipulatives in teaching reflect in students' answers about growth, exemplified by recursive sequences?*

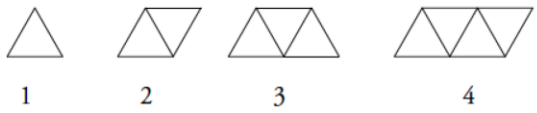
Method

The case study considers one lesson (100 min.) of a ten-week teaching sequence about differential equations in an upper secondary mathematics class with 25 students. The aim of the lesson was to enhance the students' understanding of the concept "growth" using recursive sequences. After a short devolution where the purpose of the lesson, describing the growth of certain patterns, and the milieu containing worksheets and manipulatives like matches and LEGO® were introduced, the students worked in groups of 4 for 60 min. During this period teacher intervention was very sparse. Figure 1 shows an example of an activity/worksheet. As preparation for the validation and institutionalisation the students were asked to share results, ideas and reflections with their fellow group members in a logbook prompted by questions given by the teacher. Part of the logbook writing was done in class and the rest at home using a Google Docs document shared by the group members. The empirical data consists of the logbooks and field notes from classroom observations.

The logbook entries were analysed through a qualitative content analysis (Hsieh & Shannon, 2005) for words indicating the concept of growth, like 'grow' and 'rise' (coded with underscore in the statements). In addition, words referring to the manipulatives (coded in **bold**) were noted.

To describe the patterns and how they evolved, the students introduced a 'sequence of signs'. They explained each sign and how they came on to it, and they used the sign as a starting point for the next sign they introduced. In order to examine the knowledge obtained by the students during the teaching sequence, data was analysed with respect to (1) the type of sign, applying Peirce's categories, (2) how the use of manipulatives show, according to the nature of the marked words, and (3) mathematical reasoning, according to Radford's definition of generalizing patterns (Radford, 2008).

The sequence of figures shown is made of matches. The first is made of 3 matches, the second of 5 matches etc. The sequence can be continued with figure no. 5, 6, 7, 8 …



Build and describe the next figures and tell how many matches you need to build them.

Figure no. 2 has been made by adding matches to figure no. 1 in a specific way. Figure no. 2 has been made from no. 3 etc. Explain how you move from one figure to the next in the sequence. Describe the pattern for adding new matches.

Explain how this pattern affects the *number* of matches, you need, to go from one figure to the next in the sequence. That is how can the number of matches in one figure be found from the number of matches in the previous figure?

If you want to know how many matches you need to build figure no. 37 it is rather time consuming to build and count the number. Use the pattern you have described above to predict how many matches you need for figure no. 37.

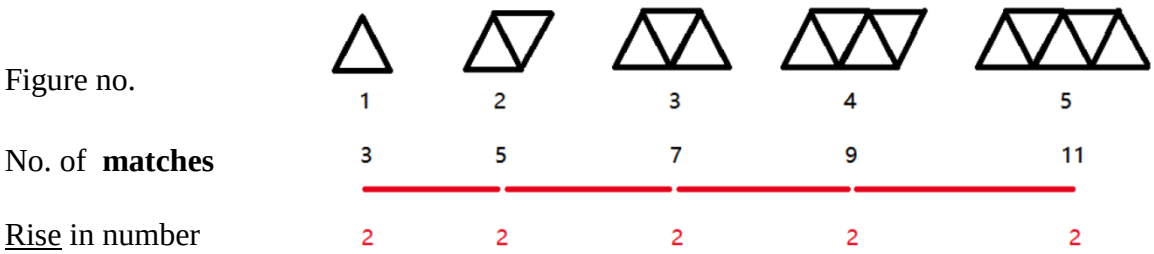
Figure 1: An example of an activity

Results

In this section, we will look at two logbook entries and one episode observed in class that show how the use of manipulatives contributes to obtaining knowledge of growth.

Figure 2 shows the first logbook entry on the question: *Choose the pattern from one of the activities and explain how the pattern developed. Use drawings/pictures/symbols describing the growth.*

For a start we observed how the number of **matches** grew. We found that there was a rise on two **matches** in every step. You can almost say that in each step a new triangle is **mirrored** on to the right side of the **figure**.



We decided to make a formula to calculate the numbers of **matches** in figure no. 37. As the number rises with 2 for each figure, 2 is a constant and has to be multiplied by the figure no. But then the number is one too little. Therefore we add another constant in the formula namely +1.

The formula can then be written: The number of **matches** in the figure = the number of **matches** the figure grows with * figure number + 1.

We also considered other ways of describing the growth after having discussed our results in class. Here we noticed the constant rise with 2. This was the constant change between the earlier and the next figure. Therefore we could write down the relations:

$T(1) = 3$

$T(2) = T(1) + 2$

$T(3) = T(2) + 2$

$T(4) = T(3) + 2$

Figure 2: A logbook entry with marked words

The semiotic bundle in figure 2 shows how the students, based on the pattern they have built of matches, introduce three signs: a table inclusive a drawing of the pattern, a 'formula' written in words and a list of the first four terms of the recursive sequence describing the pattern. On figure 3 one can see how any one sign evolves from a previous sign. The photo on the left shows the pattern build of matches.

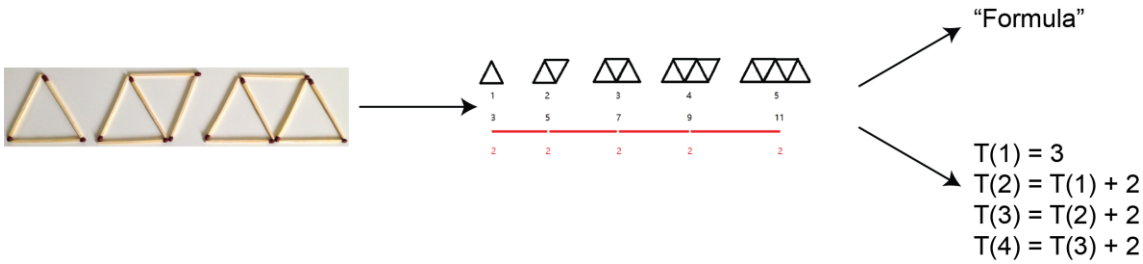


Figure 3: Signs introduced by the first group

The students begin by describing what they notice in the pattern they have built from matches, and they draw the pattern as the first line in a table. There is no explanation on the rest of the table but

my interpretation is that this is 'a conditioned reflex' as the students have made many tables like this with an independent variable (figure no.) and a dependent variable (no. of matches) when working with growth as a continuous function. The table is *iconic*: the previous text indicates that the students partly take it *diagrammatic* as it describes the relation between two subsequent figures. It also works in part as an *image* as the drawn pattern looks like the pattern made of matches.

In the table, the students also state the difference between two subsequent figures and this number is used for producing the next sign, a 'formula', from which they can calculate the number of matches in figure no. 37. The students notice that the increase is constant 2, which leads them to the first term of the expression. Using Radford's definition of pattern generalization (Radford, 2008) we see that they use naïve inductions for producing their formula. Instead of arguing for the *y*-intercept, they guess and test the value. The formula itself is a partial translation of the previous arguments to a semi-mathematical language. This sign is partly *diagrammatic* stating a relation between the figure number and the number of matches and containing the *symbols* '+' and '*'. It is interesting that the students do not finish the translation to a conventional formula (e.g. $f(x) = 2 \times x + 1$) introducing *indices f* and *x*. Most groups did at this stage. Instead they keep the original *iconical* terms 'figure no.' and 'no. of matches' closely related to the manipulatives.

Finally, after a validation and institutionalisation where the idea of explaining growth as a recursive sequence was discussed, the students decide to include this view as, not mentioning the important issue, though, that every new term is built on the previous term. They introduce an *index T* to stand for the number of matches but do not find a general expression using the symbol *n* for the figure number like $T(n) = T(n - 1) + 2$ seen in other logbook entries.

We see that the signs used for representing the object Growth include various kinds of icons as well as indices and symbols. In many of the icons there is a close link to the use of manipulatives i.e. the way the students obtain their 'data' and how they explain relations in the pattern. The manipulatives are not clearly visible in their reasoning, though. As a result of the analysis we may conclude that the understanding of growth for this group of students now includes some very concrete conceptions based on activities with manipulatives (matches), and these are pointing towards a more abstract understanding that has not yet been reached. The relations produced indicate that they take the pattern as a *token*. They have not yet reached the stage of considering the pattern as a *type* in order to produce general relations.

In some of the worksheets the use of manipulatives were not explicitly mentioned but replaced by drawings of squares, dots etc. Some groups, though, were using the manipulatives mentally, which in figure 4 is shown as a mixture of the terms 'squares'/'LEGO® blocks' in their explanations.

> […] In other words, the figure consists of a typical **tower** of **LEGO® blocks** where each level is 2 broader than the previous so that one **unit** is sticking out on each side. Therefore the number of extra **squares** in the figure rises for every figure because the bottom is 2 **squares** bigger every time. You start with $T(1) = 1$. That is the first **LEGO® block**. […]

Figure 4: Logbook entry showing the use of mentally evoked manipulatives

The group has not built this pattern physically, but in the context of other activities with manipulatives they use a mental picture made of LEGO®. Figure 5 shows the squares from the worksheet and the mentally evoked manipulatives. Like in the entry in figure 2, the text indicates that the *iconic* sign is partly *diagrammatic* as it describes the relation between two subsequent figures and at the same time works as an *image* as the drawn pattern looks like the pattern made of LEGO® blocks, which the students have evoked in their mind.

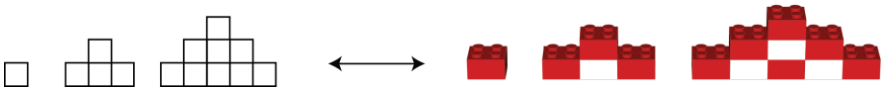


Figure 5: Alternating between to signs

As in figure 3, this group bring in various words describing growth; several terms refer to the manipulatives thereby showing the importance of the artefacts in the learning process.

The last example does not directly refer to the manipulatives from the activities but shows how a student seizes the idea of applying a concrete artefact in explaining a mathematical concept. The episode happened during the validation of the results from the activities described above. During the discussion it became obvious that many students had great difficulties with this new and different way of looking at the concept of growth. Their prior knowledge, connected to continuous functions, seemed very persistent. During the discussion one of the students reacted:

"No, no. We talk about the growth, the *change* – not the actual number. Look… [she jumps up from her chair, looks around and grabs a pile of **books** from the neighbouring table] …If we start with, say 1 **book**, [she puts one book on the table] and then in the first step we put 2 **books** on top of it, and then in the next step 2 more **books** and 2 more **books** and so on. [For every step she places 2 books on top of the stack on the table, see figure 6]. It doesn't matter if there are 1 or 3 or 5 **books** in the **pile**. The important thing is that it *grows* with two **books** every time. *The difference* is two."

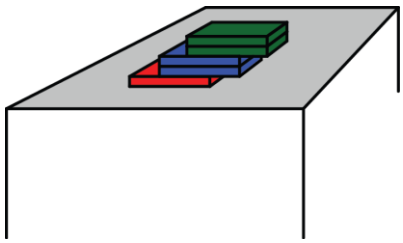


Figure 6: The pile of books holds many interpretations of a sign on Growth.

The student makes use of concrete artefacts in order to explain to her peers how to understand growth. She applies what is easily accessible in a classroom: books! As a representation of growth, the sign contain qualities of all three kinds of *icons*: the growing stable of books is an *image*, the relation of adding two more books in each step is a *diagram*, and by insisting on seeing the difference in each step (two more books every time) instead of the height of the stable, it becomes a *metaphor*. The student clearly states that this is just one way of visualizing how something grows. From the many different words used to indicate growth she demonstrates a wide understanding. It is an example of how manipulatives can be used as visualising a challenging concept rather than a starting point for general, symbolic expressions, which we saw in the logbook entries.

Discussion and conclusion

The case study demonstrates how the use of manipulatives shows in students' answers and reflections when describing a mathematical activity, e.g. building of patterns. By using concrete artefacts they are able to describe how patterns develop and translate this description to an expression of how the number of parts in each figure of the pattern grows. In the logbook entry (figure 2) the students use an intermediate stage mixing words and symbols, not reaching the final stage expressing the formula in formal mathematical language. Other groups, not considered here, argue for an algebraic expression.

Students use various words referring to different aspects of growth. Some are everyday language e.g. sticking out, broader, bigger, grow, rise and extra; whereas others are mathematical terms like add, multiply, change, growth etc. Often the everyday language is linked with the manipulatives, and the words might not have been evoked in a purely abstract context without the artefacts present. The semiotic analysis reveals that the signs introduced by students are predominantly iconic, many of these images. (Otte, 2006) argues "[…] iconic representations and perceptions are essential to introduce anything new into mathematical discourse", and he notices, "Mathematics teachers very often dislike iconic representations and perceptions, believing them to be confusing and not controllable with respect to their impact" (Otte, 1983). This case study shows how students benefit from the use of artefacts by bringing in numerous iconic representations.

While figure 3 shows that very simple patterns can be visualized as drawings without actual physical materials, classroom observations, not treated here, reveal that patterns in 3D demand the use of actual building blocks. In both cases students use the names of the manipulative i.e. "matches" or "LEGO®". Even when the physical artefact is not needed, students can apply a mental image of the artefact in their mathematical reasoning.

In former research results about the use of manipulatives in lower secondary education, Radford (2008) emphasizes the importance of joined-labour between students and teacher. In this study, the students were expected to be much more self-reliant as they were soon to end their upper secondary education, hence the use of a-didactical situations. The results, though, show that even students in the last year of upper secondary school need guidance in order to make their procedures and ideas converging with the mathematics curriculum as stated in the Theory of Objectification (Radford, 2008).

There are reasons to be somewhat cautious of the results. Using Peirce's categories of the relationship between object and sign is complicated as Presmeg, Radford, Roth, and Kadunz (2016) state: "the distinctions are subtle because they depend on the interpretation of the learner". They continue: "the distinctions may be useful to researchers or teachers for the purpose of identifying the subtlety of a learner's mathematical conceptions if differences in interpretation are taken into account." Thus, there are reasons to be careful in the interpretation. Earlier work of, among others, Radford (2000) shows comparable results indicating that the result of this case study is veridical. The semiotic analysis is based on students' statements, and it cannot always be taken for granted that students are conscious about the precise meaning of the words they use. The fact, though, that the analysis of most logbook entries share the features exemplified here indicates that the approach used in this study is rather robust with respect to the lack of carefulness in student language.

The results points towards several areas worth further investigation. In his study Radford (2008) shows how students can be led to algebraic generalisation, introducing mathematical symbols for unknown quantities. This can be taken further by studying how students make sense in already evolved algebraic expression, and whether manipulatives can be supportive. In upper secondary education students are presented with a large number of such algebraic expressions, which they are expected to manipulate and use but often fail to understand. Another interesting aspect is the significance of *prior knowledge*. In figure 2 we saw how the students were led to look at the relation between a figure number in the pattern and the total amount of matches in that figure resulting in a functional expression. This behaviour was even more apparent in other entries, not shown here, and could very well originate from their knowledge of continuous functions. Although prior knowledge often helps students in new contexts, it can also prevent them from going in the desired direction.

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The use of artifacts and different representations by producing mathematical audio-podcasts

Christof Schreiber and Rebecca Klose

Justus Liebig University, Department of Mathematics Education, Gießen, Germany

christof.schreiber@math.uni-giessen.de
rebecca.klose@math.uni-giessen.de

This paper focuses on the role of artifacts and different forms and modes of representation when learning mathematics at primary school level. This will be exemplified alongside the use of an interactive approach, in which mathematical audio-podcasts are produced. Audio-podcasts are a communication tool that generally focuses on oral means of representation. During the production process of mathematical audio-podcasts however, the students use different forms and modes of representation as well as produce and use different artifacts. With regard to this, field experiences will be shared.

Keywords: Mathematical audio-podcasts, artifacts, forms and modes of representation.

Artifacts and different representations

The role of artifacts for mathematical learning is of great importance. Artifacts generally encompass objects that are produced by human beings and are materially present and durable. A learning artifact in particular, refers to an object developed by students, which displays their knowledge (Kafai, 2006). Besides artifacts that are material in nature, there are virtual ones that comprise of tools of the information and communication technologies. Different artifacts support different forms of representation and have a great influence on cognitive development (Bartolini Bussi & Mariotti, 2008).

The education standards for teaching mathematics also emphasise the transfer of various modes of representations, since they enable students to communicate their mathematical ideas and understanding as well as support the modeling and interpretation of mathematical phenomena (NCTM, 2000). Kress (2010) defines 'mode' as

> a socially shaped and culturally given resource for making meaning. *Image, writing, layout, music, gesture, speech, moving image, soundtrack and 3D objects* are examples of modes used in representation and communication. [...] Modes offer different potentials for making meaning. These differing potentials have a fundamental effect on the choice(s) of mode in specific instances of communication. (p. 79)

Therefore, being able to use different modes of representation during a mathematics lesson is a central aspect and supporting competence across all school levels. The mathematical register in the classroom also comprises of the multiple semiotic modes:

> Language, mathematical expressions, and visual diagrams, as well as the gestures and actions of participants in the classroom, together construct meaning. (Schleppegrell, 2007, p. 142)

With regards to the role of written representation for mathematical learning, working with the products of students' writings has been common practice for some time now (Morgan, 1998). In the project 'Math-Chat' (Schreiber, 2013b), the use of written representations is the main means in the

problem solving process and focus of the analysis. In primary school, written-based communication is often used in the case of reflections and documentations of the students' learning processes, for example, in journals referred to as 'journal writing' (Borasi & Rose, 1989); and also after having solved math problems. Thus, writing is often considered "an activity that is in itself conductive to learning" (Morgan, 1998, p. 2). In this sense, it is assumed that writing might help students organise their thoughts by externalising them in a written form. This strategy of 'writing to learn' has to be distinguished from 'writing to communicate'. The tool 'writing to learn' enables students to become more active learners by questioning and reflecting on information, whereas 'writing to communicate' aims to demonstrate their learning in different forms of written assignments (Meiers, 2007).

It is oral communication that generally provides the participation in mathematical discussions and conversations, aiding with the understanding of mathematical signs (Meiers, 2010). Apart from speaking and explaining, mathematical learning also requires the acquisition of skills, such as listening, questioning, defining and proving (Ontario Ministry of Education, 2006). Pertaining to mathematical interaction and communication processes, research on mathematical didactics has attributed an increasing importance to the use of gestures (e.g. Goldin-Meadow, 2003; Huth, 2014). Kendon (2004) defines 'gesture' as "a name for visible action when it is used as utterance or as a part of an utterance" (p. 7). Gestures that are used in accompaniment with speech represent the same components of the expressions uttered. As such, gestures and spoken language can be seen as inseparable and interdependent modes of the same language system (Huth & Schreiber, 2017).

To further explain the interwoven use of oral and written communication in a medial and conceptual manner, the linguistic model of orality and writtenness by Koch and Oesterreicher (1985) and well explained by Fetzer as the "two dimensions of orality and writtenness" (2007, p. 79; translated by the authors: „Zwei Dimensionen von Mündlichkeit und Schriftlichkeit") is used. Koch and Oesterreicher (1985) have developed a model of communication that distinguishes between medial phonic and medial graphic communication and between communicative immediacy and communicative distance. The medial phonic and the medial graphic realisation of communication are dichotomous, whereas the conceptual realisation can be placed on a scale between the communicative immediacy and the communicative distance (Figure 4; Schreiber, 2013a).

Considering the example of a personal talk, it is not only medially phonic, as there is an emotional closeness to the dialogue partner and thus, also conceptually oral. Although writing in a diary is medially graphic, it is conceptually oral, informal and characterised by its 'closeness' to the reader. An administrative directive, in contrast, is medially graphic and also conceptually an example for communicative distance. In this case, the language is strictly formal.

This paper will refer to 'forms' of representation when pertaining to Koch and Oesterreicher's linguistic model of orality and writtenness and 'modes' of representation when used according to Kress and Schleppegrell's definitions as mentioned above.

Audio-podcasts for mathematics

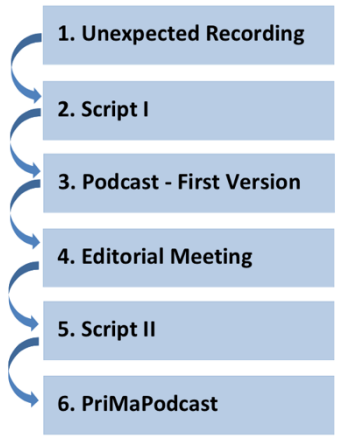

Figure 1: Process of production of PriMaPodcasts

Audio-Podcasts, which we use as learning artifacts in school, primarily focus on the aspect of orality, as the process of creating an audio-podcast produces an oral product. Schreiber (2013a) developed 'PriMaPodcasts' to serve as tool, in which oral and written forms of representation are alternated and interwoven permanently. 'PriMaPodcast' is the coined term for a mathematical audio-podcast produced by primary school students. The students undergo a process in which they develop specific mathematical content in stages up to a point where a version of the explanation is ready to be published on the internet (Figure 1). The different forms of representation, the medial and conceptual realisations as well as the use of artifacts will be presented by illustrating each production step below. For more information on the aims, the production steps, the audio files and the scripts, please visit our blog (www.uni-giessen.de/primapodcast-bili/). Before giving insights into the production process, three aims of the mathematical audio-podcasts will be described:

Learning: The exclusive use of oral representations can support learning. This can be brought forth when learners have to deal with one specific topic in depth and are required to refrain from using any written or graphical representations, thus, solely relying on oral means of communication.

Diagnosis: A teacher gains insight into a learner's mathematical concept and understanding by listening to the learner's recorded utterances. The spontaneous and planned recordings can be analysed during and after the production process.

Research: As mathematics is a written and graphically based science, focusing on oral representations is an interesting area for research. Therefore, using the recordings, the aim is to investigate to what extent learners use mathematical language to express their workings.

The focus of this paper will be on the last aim. This will be illustrated in the following. The production process will be depicted through excerpts from an empirical example. These consist of recordings and documentations that were produced at a bilingual school in Offenbach am Main, Germany. The selected study sample is from a dissertation project (Klose, 2015) and it consists of fourth graders that were taught mathematics bilingually (German/English) since first grade. Within the scope of the research project, the students' task was to produce a podcast in English. The empirical examples were transcribed and analysed by means of the interaction analysis (Krummheuer & Naujok, 1999). Excerpts of the transcribed utterances and scripts will be presented and commented thereafter. All citations of the transcripts are marked in angle brackets <like this>.

Production steps

In step 1 ('Unexpected Recording'), a small group of students is unexpectedly confronted with the mathematical question 'What is symmetry?'. The students record their responses with a voice recorder. Hence, in this step, the focus is on oral communication and representation. The medial realisation of the recording is phonic and conceptually oral as seen in the following.

In this example, the students answer the questions as follows:

ur	speaker:	statement:
01	Student 1:	what is symmetry (*4s*) o k a y . I think eh symmetry uhm there are symmetry lines/ . . uhm . and < miss brand had talk about it
02	Student 2:	< and
03	Student 2:	I think uhm in symmetry/ we have symmetry lines and uhm there are . uhm . . uhm . there are . .
04	Student 1:	(*whispers*) symmetrical% (*4s*) I think there are (*the thumb, index and middle finger of the right hand, which is open and facing upwards, are being spread apart and then joined to meet in the middle*) forms % . uhm #
05	Student 2:	# y yes
06	Student 1:	< (speaks softly) that are symmetrical and%
07	Student 2:	< there are forms that what you can do in three d and in two d (...)

In this excerpt, both students express their mathematical ways of thinking by speaking out aloud. The students associate the term ‘symmetry’ with ‘symmetry lines’ as seen in <ur01> and <ur03>. Student 1 remembers in <ur01> that her mathematics teacher had mentioned symmetry in class before. Furthermore, they relate ‘symmetry’ to “forms” <ur04> and “forms that what you can do in three d and in two d” <ur07>. As early as seen in this excerpt as well as later on, gestures are already going into action. This step not only prompts the students to think about the topic, but also supports them to clarify and organise their thoughts and reflect on specific content (Pimm, 1987).

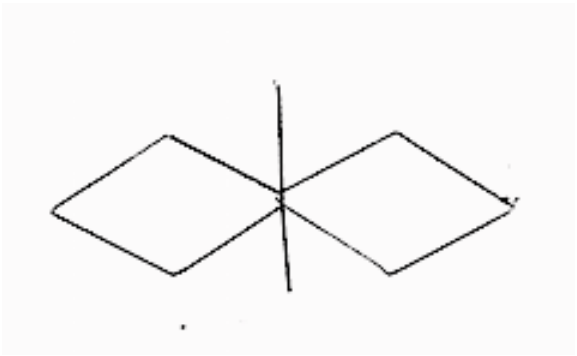

Figure 2: Script I, Student 2’s drawing as an approach to explain line symmetry

In step 2 (‘Script I’), the students research their topic and create a script. They are free to decide the format, structure and amount of detail they wish to include. In order to gather information, they are given different resources, such as the internet, textbooks and worksheets. They also have the possibility to use concrete material. Thus, different artifacts can serve as basis for oral communication and written-graphical representation.

The students of the empirical example were inspired by the work material and referred to various approaches of line symmetry. In particular, student 2 used the geoboard and drawings (Figure 2) to support her with the activities related to line symmetry. In their script, they explain ‘symmetry’ through the approach of folding (Figure 3).

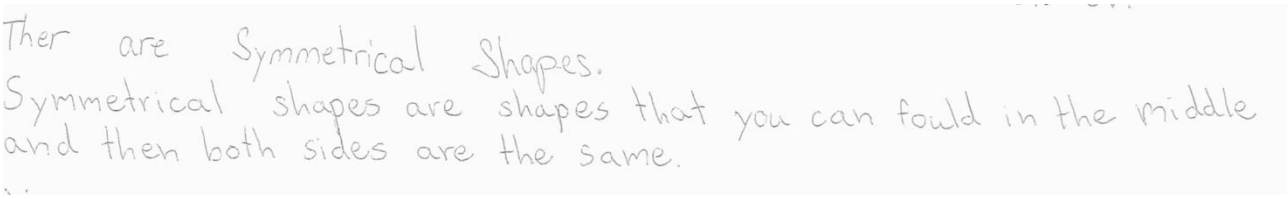

Figure 3: Script I, folding as an approach to explain line symmetry

The students continue by mentioning ‘drawing’ and ‘reflecting’. The writing activity itself may be considered a tool for ‘writing to learn’. Dealing with different artifacts and resources, this writing underlies the principle of connecting what has been read, viewed, heard and experienced with what

the students have understood so far (Meiers, 2007). The medial realisation of the script is graphic and its linguistic conception is - depending on the kind of script - a rather informal written type.

In step 3 (‘Podcast - First Version’), oral representation is essential, as the students are required to read the assigned parts of the script aloud for the recording of the first podcast version. Up to this stage, the team works independently: The students make their own decisions concerning the mathematical matter and their performances without any intervention from the instructor. The medial realisation of the recording is phonic and its linguistic conception is a more formal spoken type.

fv	speaker:	statement:
01	Student 2:	what is symmetry there are axis symmetry reflection symmetry and partition symmetry there are symmetrical shapes
02	Student 1:	symmetrical shapes are shapes that you can fold in the middle and then both sides are the same
03	Student 2:	you can draw symmetrical shapes with faces . vertices and edges
04	Student 1:	edges are the lines and vertices are the places where they come together faces are the places in between the edges and in German we call them flächen
06	Student 2:	you can make symmetrical shapes in two d and three d
07	Student 1:	examples for symmetrical shapes are a square a circle a triangle a hexagon a pentagon and a trapezoid
08	Student 2:	s some some shapes have got more than one m mirror line

In step 4 (‘Editorial Meeting’), the small group of students receives feedback from their peers and the instructor. They use this to reflect on their work in terms of content, style and language use. At this point, the instructor poses specific script-related questions and asks for more precise explanations. Again, different artifacts, such as the script, the audio files and objects and especially the students’ written products, support the discussion and serve as basis for better clarifications in oral communication.

In the presented empirical example, the different types of symmetry were correctly named and distinguished. For this purpose, various kinds of materials were used. By means of their drawing (Figure 2) the phrase ‘line of symmetry’ was introduced. Several locations to draw the line of symmetry were discussed. Thereby, the instructor’s focus shifted from 3D towards the symmetry of 2D shapes. Together with the other group, they reviewed and checked how many lines of symmetry a circle, quadrilaterals and different types of triangles could have.

In step 5 (‘Script II’), the students start to improve their first script in small groups. The students have the opportunity to use the same material and artifacts again. They may discuss and share their knowledge before writing down a second script. Thus, oral communication serves as basis for written representation. The writing activity of creating a second script may be linked to the strategy of ‘writing to communicate’ (Meiers, 2007). Since the second script serves as basis for the final podcast version, the learners decide on the content they wish to present to the audience. The script is medially graphic and conceptually rather an elaborated written type since the students clearly want to present their knowledge accurately and in a structured manner.

In the empirical example, the students took up most of the suggestions given during the editorial meeting as they tried to differentiate more distinctively between the different types of symmetry.

This, however, appeared to be challenging, as they focused on 3D shapes again and listed some of their properties instead of looking at 2D shapes.

In step 6 ('PriMaPodcast'), they record a final podcast version based on the second script. This will be published on the internet. Again, written representation serves as basis for the oral form of representation. Hence, the recording is medially phonic and conceptually more of an elaborated type.

pr	speaker:	statement:
01	Student 2:	what is symmetry. there are line symmetry reflection symmetry and rotation symmetry . there are symmetrical shapes
02	Student 1:	symmetrical shapes are shapes that you can fold in the middle and then both sides are the same . that's line symmetry
03	Student 2:	rotation symmetry is when you have a symmetrical shape and you rotate it a little bit and then the shape the shape looks the same as it was before
04	Student 1:	reflection symmetry is almost the same like line symmetry but you have to put a **mirror** in the middle of the shape\
05	Student 1:	you can draw symmetrical shapes with faces vertices and edges
06	Student 2:	examples for symmetrical shapes are a square a circle a triangle and in three d a sphere a pieramid and a cube
07	Student 1:	you can make symmetrical shapes in two d and three d
08	Student 2:	almost all shapes have got more than **one** line of sym**me**try

As described in the production process, the students' products can be classified into Koch and Oesterreicher's model as follows (Figure 4):

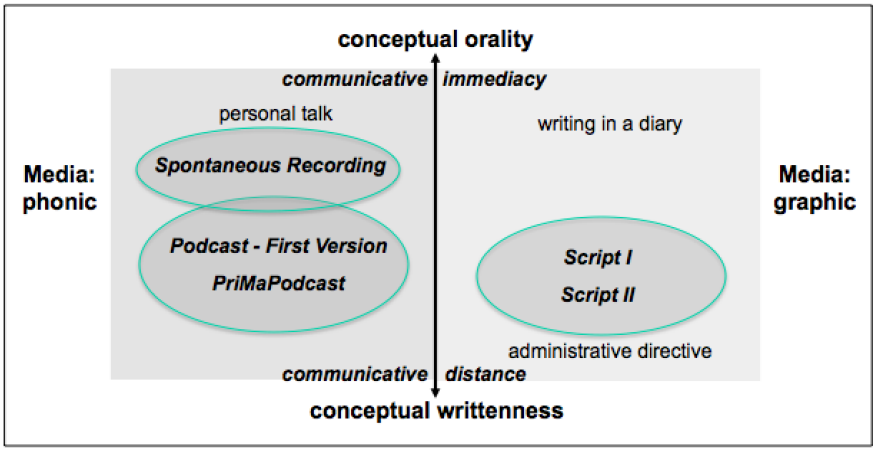

Figure 4: Model of writtenness and orality; (Fetzer, 2007, p. 79/ Schreiber, 2013a)

Conclusion

The use of artifacts and various representations in the mathematics classroom is of high relevance. The interactive approach of producing PriMaPodcasts follows this assumption by comprising various modes and forms of representation. The aim is to produce a learning artifact; a final product of oral nature. Publishing the final version on an online blog serves as an exceptional motivation for the students. In the process, they undergo various stages, in which they can expand their knowledge through discussions and learn different approaches to present them. Mathematical concepts and language competences can be strengthened as shown by the empirical example.

Field experience

So far, mathematical audio-podcasts have been used in different areas of teacher education. Firstly, we produced audio-podcasts for mathematics with students at primary school level. Besides German podcasts, some students also produced mathematical audio-podcasts in various other mother tongues (e.g. Russian and Turkish) as well as different languages of instruction (e.g. English and Spanish) in bilingual schools.

Secondly, students who are studying to become primary or secondary school teachers, created mathematical audio-podcasts. Moreover, they supervised the production of the mathematical podcasts of school students. It is important for the university students to have gone through the steps themselves prior, so as to understand the production procedure clearly. Hence, they gain a better understanding of how demanding it can be for younger learners to explain a concept without any preparation and solely using orality. The deepening of mathematical content and its reflection are another focus of the audio-podcasts. The university students become aware of the increase in knowledge and are able to internalise and reflect on already learnt content. Content knowledge ranges from secondary school mathematics to areas covered during university lectures.

Thirdly, the idea of mathematical audio-podcasts serves as research method. As part of a PhD project, the use of mathematical language in bilingual learning settings will be investigated further. The overall interest lies in the question of how bilingually taught learners use the language of mathematics in both their target languages (English and German) when asked to present mathematical content (Klose, 2015).

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The role of representations in promoting the quantitative reasoning

Lurdes Serrazina[1] and Margarida Rodrigues[1]

[1]Escola Superior de Educação de Lisboa, UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal; lurdess@eselx.ipl.pt, margaridar@eselx.ipl.pt

In this communication, we discuss the role of representations in the development of conceptual knowledge of 2nd grade students involved in additive quantitative reasoning through the analysis of the resolutions of two tasks that present transformation problems. Starting to discuss what is meant by additive quantitative reasoning and mathematical representation, we present after some empirical results in the context of a teaching experiment developed in a public school. The results show the difficulties with the inverse reasoning present in both situations proposed to students. Most students preferably use the symbolic representation, using also the written language as a way to express the meaning attributed to its resolution. The iconic representation was used only by a pair of students. Representations have assumed a dual role, that of being the means of understanding the students' thinking, and also supporting the development of their mathematical thinking.

Keywords: Mathematical representations, transformation problems, additive quantitative reasoning, inversion.

Introduction

This paper is part of the Project "Adaptive thinking and flexible computation: Critical issues" being developed by the Schools of Education of Lisboa, Setúbal and Portalegre. Its main goal is to discuss the role of representations in the development of 2nd grade students' conceptual knowledge that is present in different levels of understanding of numerical operations/relations when they solved two tasks. These were conceived with the aim to develop quantitative additive reasoning, as a means of understanding this same reasoning. The tasks present transformation problems included in the classes of search of initial state (Vergnaud, 2009). They are the last ones of a sequence of six tasks that was applied in the context of a teaching experiment developed in a public school in Lisbon. The empirical data analysis focused on representations aims to discuss the inferences we make in the reasoning of the students but also their role in the development of students' reasoning.

Mathematical representations and quantitative reasoning

Quantitative reasoning, within the additive structure, focuses mainly on relations between quantities (Thompson, 1993), being that the problems of transformation aimed at finding the initial state have increased cognitive complexity for 2nd grade students by requiring an inverse reasoning (Vergnaud, 2009). The representations are interconnected with the reasoning given the relevance of their role in the understanding of students' reasoning (NCTM, 2000). But, the representations also assume an important role in students' learning, constituting cognitive means with which they develop their mathematical thinking (NCTM, 2000; Ponte & Serrazina, 2000). In a broad sense, a representation is a setting that can represent something somehow (Goldin, 2008). The term "representation" refers both to the process of representing and the result of this process. In mathematics education, representations are privileged tools for students express their mathematical ideas, still working as helpers in the construction of new knowledge (NCTM, 2000). However, a mathematical representation cannot be

understood or interpreted in isolation, since only makes sense when part of a more comprehensive and structured system in which different representations are related (Goldin & Shteingold, 2001).

According to Stylianou (2010), the way as representations are used in classroom has impact in students learning and this largely depends on the role of the teacher, using "student-generated work as a launching point for discussions" (p. 339). This idea is reinforced by Ponte and Serrazina (2000) when they say that how mathematical ideas are represented influences profoundly the way they are understood and used. For example, according to Vergnaud (2009), the inverse transformation can be represented by two symbolic representations -- the algebraic one and the arrow diagram -- considering, however, that while the algebraic representation is not suitable for children in elementary school, using the diagram representation the teacher can help students connect, immediately, the different components of the relationship, namely the direct and inverse transformations, giving meaning to the temporal motion go forward and backward.

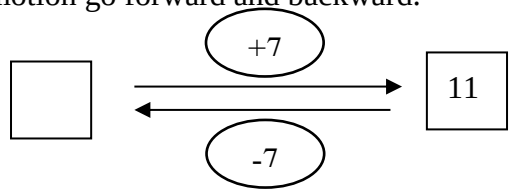


Figure 1: Arrow diagram (Vergnaud, 2009, p. 87)

Figure 1 presents a representative diagram of subtracting 7 to the final state in the situation "John has just won 7 marbles in playing with Meredith; now he has 11 marbles; how many marbles did he have before playing?" (Vergnaud, 2009, pp. 86-87). However, while recognizing the importance of this representation, the author states that children need several examples of the inverse transformation in order to be able to effectively understand it. "Several kinds of awareness are needed: you lose what you have just won; or you win what you have just lost; you go backwards as many steps as you have gone forwards and reciprocally" (p. 87).

For Ponte and Serrazina (2000), the main forms of representation used in the primary education are: (i) *the oral and written language*; (ii) *symbolic representations*, like numbers or the signs of the four operations and the equal sign; (iii) *iconic representations*, like figures or graphics; and (iv) *active representations*, like manipulative materials or other objects. It is through the analysis of the representations used by students that the teacher can become aware of their thinking and help them in the construction of own representations in mathematical language.

NCTM (2000) also emphasizes the role of idiosyncratic representations constructed by students when they are solving problems and investigating mathematical ideas, in that it can help them in understanding and solving problems and provide "meaningful ways to record a solution method and to describe the method to others" (p. 68). Observing these representations, teachers and researchers can understand the ways of interpreting and reasoning of students.

Methodology

This study follows a qualitative approach within an interpretive paradigm. Its methodology of *design research* is part of a perspective of learning design, in order to produce local theories of teaching and learning sequences that are resources and references available to inform the practices of teachers and researchers (Gravemeijer, 2015).

The data were collected in a second grade classroom (7-8 years old), with 26 students, of a public primary school in Lisboa. The Project team defined a sequence of tasks with the aim to develop the

calculation flexibility in addition and subtraction problems. The process of tasks elaboration included previous testing of some (namely the ones focused in this paper), through clinical interviews (Hunting, 1997) with students of the same grade. It is a technique that is directed by the researcher and seeks a description of the ways of thinking of respondents. The task sequence was previously discussed and analyzed with the classroom teacher having been made minor adjustments. Classroom teacher explored the task sequence with her students (a task every week). During their schooling the empty number line had been used regularly both by the teacher and students.

The data collection was made through participant observation of the authors of this paper, which drew up field notes and supported by video recording, subsequently transcribed. The written records of the students were also collected. All these data were analyzed and triangulated. By ethical reasons, the students' names were changed to ensure confidentiality.

In this communication we analyze two tasks (Figure 2), proposed to the students in the same class (given the similarity between them) and presented on the same sheet of paper, with space for the respective resolution.

Game of marbles I

Ana and Luís played a game of marbles together. At the beginning both had the same number of marbles.

Ana won 3 marbles from Luís and had 7 at the end of the game.

How many did Luís have at the end of the game, knowing that he did not win marbles?

Game of marbles II

Ana and Luís made a game of marbles.

Ana won 6 marbles from Luís and had 10 marbles at the end of the game.

Luís won nothing and had 3 marbles at the end of the game.

Compare the number of marbles of Ana and Luis before the game and at the end of the game.

Figure 2: Performed tasks

These were the last ones of the sequence solved by students. With the previous exploration of the other tasks, students had already worked the relationship between wins and losses over a marbles game, realizing that what a player wins, the other loses. All the tasks were first solved in pairs. In this class, after all pairs had solved the two tasks, the teacher promoted a collective discussion with whole class, from six pairs resolutions (three on each task) who presented their work on the blackboard.

Exploring the tasks

In this section we present and discuss some examples of tasks' resolutions. Their choice was made taking into account the diversity of representations presented by the different pairs and being representative of what happened in class.

Game of marbles I

The resolution of Alexandre and Rosa shows the inversion of reasoning as a critical aspect. Thus, they took first 10 marbles for both players from the sum of 3 ("Ana won 3 marbles") with 7, not mobilizing an inverse reasoning to determine the initial number of marbles.

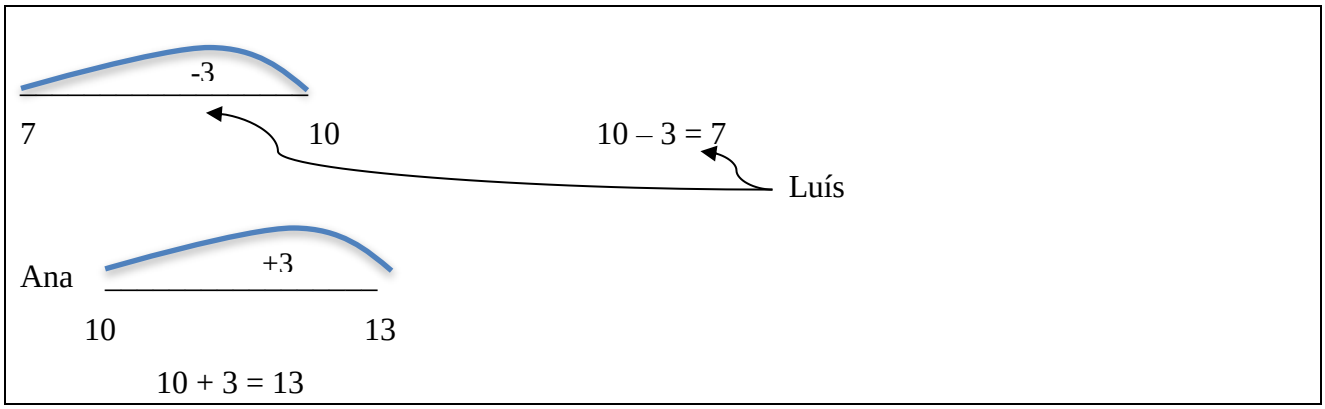


Figure 3: Alexandre's diagram for Game of marbles I

Alexandre's diagram (Figure 3) represents the wins and losses of marbles as they are thought by the pair of students. But, when asked by the teacher they erased all they did and made it again on their worksheets (this diagram was seen in the video records). The new representation (Figure 4) reveals the necessary inversion to find the initial number of marbles. They got it for Ana, but they seem to forget that both players had at the beginning the same number, as in Luis' allusive representation, they retired 3 marbles from 7.

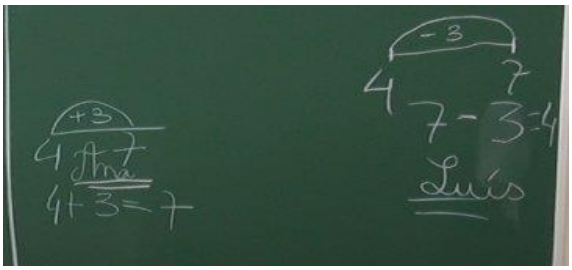


Figure 4: New Alexandre's diagram for Game of marbles I

Vítor and Joana can reverse the reasoning to the case of Ana but they show a lack of understanding of the situation alluding to Luís (Figure 5).

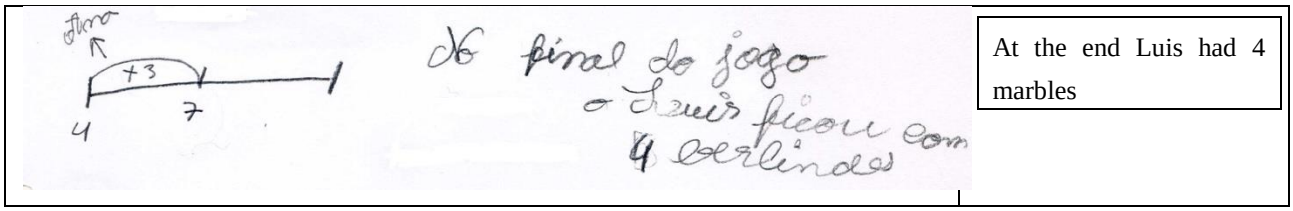


Figure 5: Vítor's diagram for Game of marbles I

This diagram helped the children's thinking in the sense that it respects the temporal order of the game: the initial marbles on the left and the final ones on the right side. However, they did not look again for the initial situation, that is, if Ana won, Luís has to have less at the end of the game.

Rui and António used symbolic representations presenting a number line with two curve lines that represent the inversion of addition and subtraction (Figure 6). Like the anterior pair, they reversed their reasoning to Ana but they assumed the same situation to the player Luís, stating that he won 7 marbles at the end of the game. Here the term "won" means the absolute number of marbles at the end of the game and not a quantitative difference. The distinction between the quantitative difference and the result of an arithmetical operation is a critical aspect that emerges from this resolution.

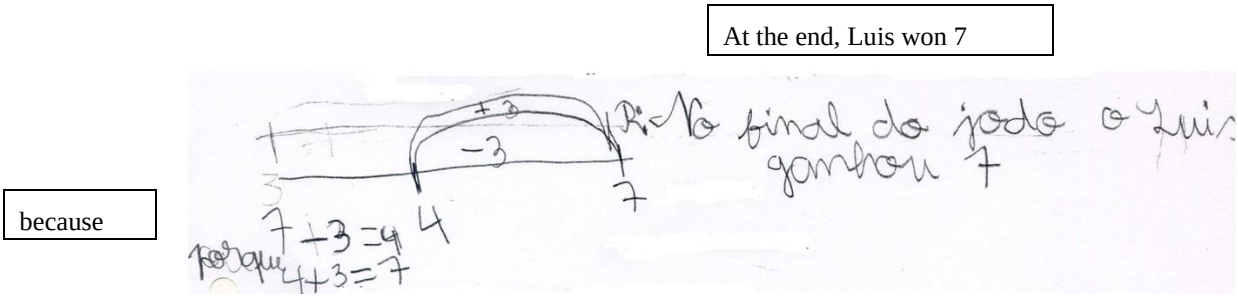


Figure 6: Rui's diagram for Game of marbles I

The word "because" explains the inversion used to determine the number of initial marbles, justifying it with the inverse relationship between addition and subtraction. The order in which they placed the operations shows the inverse reasoning process: first, they determined the initial number of marbles (the initial state), and after they confirmed the resultant final state with the inverse relationship.

Tiago and João used a table disposition (Figure 7), with columns for each of the two players and the lines for the different moments of the game, the top line to the beginning of the game and the bottom to the end.

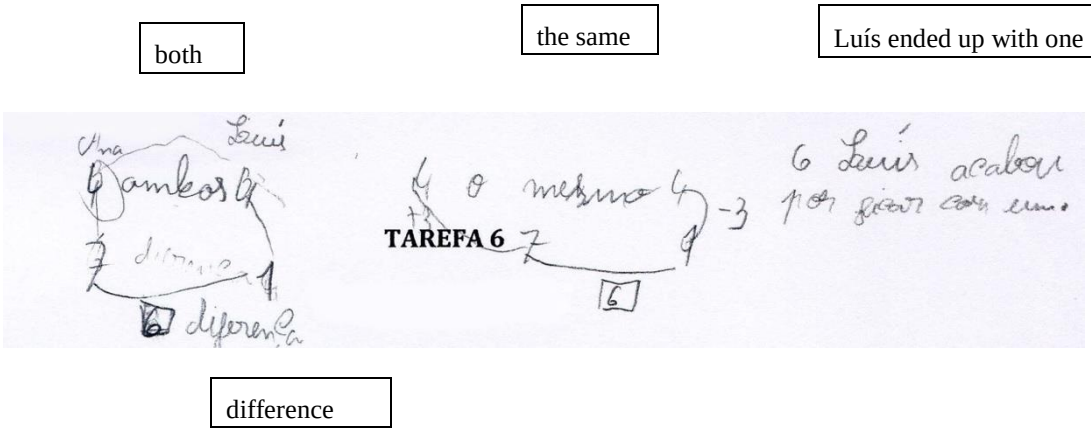


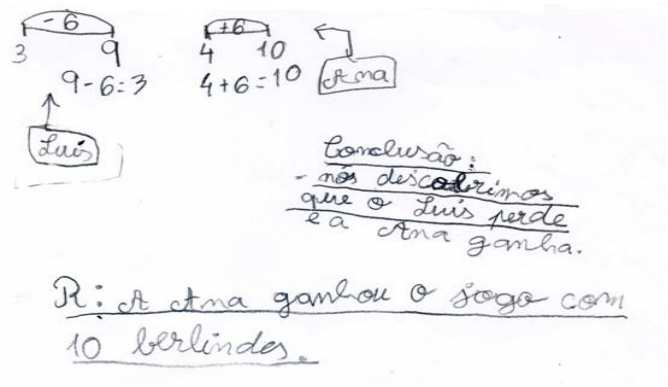
Figure 7: Tiago's diagram for Game of marbles I

It was the only pair in the class, which established the difference between the final numbers of marbles, although not required, focusing on the difference as an additive comparison of quantities. It seems that the representation used allowed them to manage two data at the same time, the quantity of marbles and the transformation after game.

Game of marbles II

Alexandre and Rosa read the problem. Immediately, Alexandre said: "Luís started with 9 and Ana started with 4." Once Alexandre had understood the previous problem when the teacher questioned their resolution, as mentioned before, here he already coped well with the unknown initial state, solving mentally the problem, through an inverse reasoning.

On the diagram's lines (Figure 8), the numbers are placed in ascending order, getting the initial number of marbles on the right for Luís and on the left for Ana. Their answer is focused on the absolute amount of marbles of Ana at the end of the game and not on the comparison.



Conclusion: we discover that Luís loses and Ana wins.

R: Ana won the game with 10 marbles.

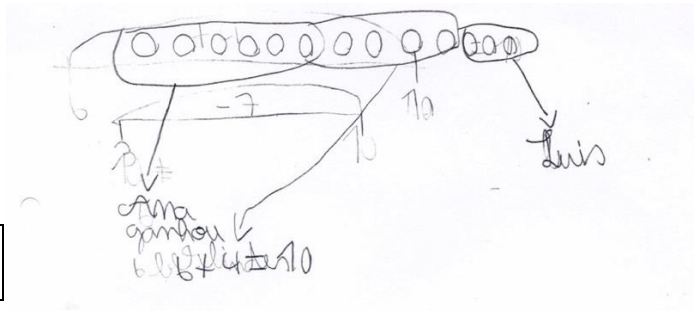
Figure 8: Alexandre's diagram for Game of marbles II

Vítor and Joana used the line representation (Figure 9), adopting the temporal criterion as they did in the previous task, putting the initial marbles on the left and the final ones on the right side and at this time they got the right solution.



Figure 9: Vítor's diagram for Game of marbles II

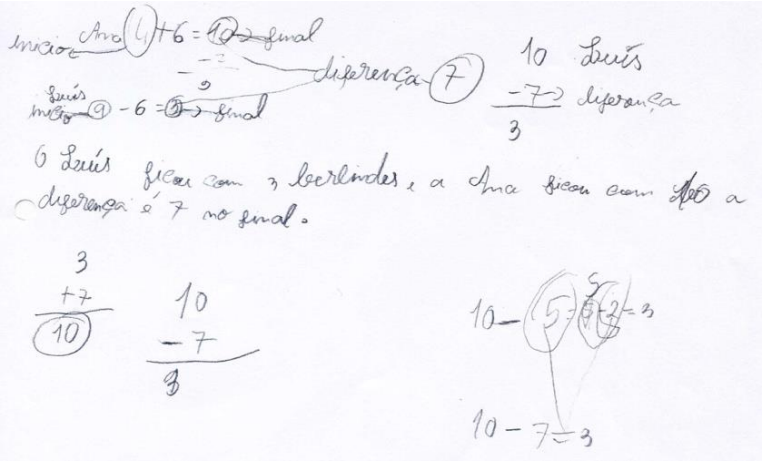
Rui used an iconic representation of Ana's marbles: first, the six marbles won from Luís; then the initial 4 marbles (probably counting them until the total 10); and finally the three final marbles of Luís. He was not able to reverse his thinking in order to determine the initial marbles of Luís.



Ana won

Figure 10: Rui's diagram for Game of marbles II

Tiago surrounded the numbers to assign their meaning, recording the player and the time of the game to which they relate (Figure 11) and respecting also the temporal order of the game.



Luis got 3 marbles and Ana got 10, the difference is 7 at the end.

Figure 11: Tiago's diagram for Game of marbles II

Tiago focused the additive comparison of the resultant final states, recording that "the difference is 7 at the end".

Final remarks

The inverse transformation is a critical aspect in solving the tasks (Vergnaud, 2009). Thus, the first representations used by Alexandre in *Game of marbles I* show a reasoning associated with prototypical situations of addition asking for final states and not initial ones. Students seem to cope more easily with this kind of transformation in the second task after having understood the inversion involved in *Game of marbles I*. For this understanding the teacher's questions seem to have been fundamental.

An example is the case of Alexandre and, although he did not fully solve the first problem, managed to overcome the obstacle of inversion in the second task solving it mentally very fast. Despite the inherent difficulty of the inverse transformation, the productions of the students in the class reveal a widespread understanding of an aspect of inverse thinking: what a player wins, the other loses.

Students seem to have essentially privileged two forms of representation (Ponte & Serrazina, 2000): the written language and symbolic representations. Just a pair of students used the iconic representation in support of symbolic representation (Rui and António). Among the symbolic representations used, there was a predominance of horizontal dispositions of the calculations, although the use of the empty number line also had had a significant expression, helping to think about the transformations involved in problems. We should stress that empty number line had been used by these students and her teacher since the first grade. Thus, the curved lines, which represented the transformation, supported the thought around the wins and losses, as well the temporal motion. The table disposition also seems to have helped the students (Tiago and João) to structure and relate the various elements of the problem: the two players and the two time points of the game.

Analyzing student productions, we can infer different levels of quantitative additive reasoning. While most learners focused on the absolute amounts of marbles, Tiago focused on quantitative difference as a quantitative result of comparing two quantities additively to find the relative change (Thompson, 1993). So, for the majority of students, the notion of difference as an additive comparison of quantities is a problematic aspect.

The representations used by the students had a dual role. On the one hand, they were windows to interpret their reasoning. On the other hand, they were scaffolds that helped to think mathematically demanding situations, taking into account their ages. It should be stressed that the teacher's role, making questions, not giving answers, while students were doing their work was also essential. As stated by Vergnaud (2009), the development of a conceptual field involves not only situations and schemes but also symbolic tools of representation.

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Maths Welcome Pack: A multi-sensory introduction to mathematics for high school students

Giulia Bini

LSS Leonardo da Vinci, Milano Italy; giulia.bini@lsdavincimilano.eu

Keywords: Secondary school mathematics, experiential learning, multisensory learning, mathematics activities, learning motivation.

Introduction and theoretical framework: Before the box

This poster is about the experience of an "on field" project started in 2014 with a group of 28 14-y.o. students attending to the first year (9th grade) of Liceo Scientifico and a 10cm cube cardboard box containing a collection of items that led to a two weeks journey through the focal points in the act of doing mathematics, which are the new entries of the high school approach to the subject, such as proofs, their necessity in mathematics and the related use of a proper symbolic language.

It has been conceived as a starter for the study of high school mathematics, but it also proved an effective strategy to boost students' learning motivation and long-term knowledge retention about the subject as a whole.

As Raymond Duval points out, "the goal of teaching mathematics, at primary and secondary level, is neither to train future mathematicians nor to give students tools, which can only possibly be useful to them many years later, but rather to contribute to the general development of their capacities of reasoning, analysis and visualization" (Duval, 2006, p. 105). Taking this perspective, I wanted to investigate the effectiveness of a multi-sensory approach, which is quite frequent for lower grades but isn't a common practice for secondary mathematics teaching and learning, through the means of Arzarello's *semiotic bundle* theory, widening the semiotic system horizon "to contain gestures, instruments, institutional and personal practices and, in general, extra-linguistic means of expression" (Arzarello, 2006).

The project had a double goal: from a research point of view, I aimed at focusing on the "cognitive functioning underlying the various mathematical processes" (Duval, 2006, p. 104) and exploring the efficacy of "a rich semiotic bundle with a variety of semiotic sets" (Arzarello, 2006) to foster students' understanding of some pivotal mathematical concepts. From a didactic point of view I wanted to verify that this approach, which took its strength from "a gradually growing and multimodal cognitive environment" (Arzarello, 2006) could help students in building up long lasting memories connected to the practice of mathematics through the act of giving "personal meanings to mathematical objects" (D'Amore, 2003, p. 19).

Methodology: Inside and Outside the box

The study explores how students answered to questions and problems prompted by the 12 items in the box, during the first two weeks of the school year (approximately 10 hours of teaching). Data has been collected by the teacher through classroom observation and assessment of the assigned homework.

The content of the box, described in detail in a multimedia presentation, varied from tactile objects like the piece of string which triggered the ice-breaker kinaesthetic activity **The Magic Knot** (Can you tie a knot in a string without letting go of the two ends?) to a paper cut with an extract from a 1742 letter from Christian Goldbach to Leonhard Euler which lead to the unveiling of **The Goldbach mystery (**What is the Goldbach conjecture? And why is it a conjecture and not a theorem?).

A good example to illustrate how the notion of semiotic bundle can be used to decode the activities of students solving a mathematical problem is **The mystery of the four triangles** (Can you build exactly 4 equilateral triangles using six toothpicks of equal length?). Searching for the answer to this problem students came up with the wrong construction seen in fig.1.



Figure 1

Discussing this solution, they brought into play an articulate semiotic bundle made up of:

- gesture: manipulating the toothpicks to build the solution
- speech: discussing why the construction in the figure is wrong
- written signs: drawing other possible solutions
- arithmetic representation: figuring out that the slanted toothpicks in Figure 1 weren't long enough to play as diagonals of the square
- geometric representation: figuring out that the triangles in Figure 1 couldn't be equilateral

Expanding the semiotic bundle from gesture to arithmetic representation, they became gradually aware that the construction couldn't be solved in two dimensional space and it eventually helped them to elaborate the correct solution.

Conclusions: Beyond the box

The described case shows how the use of a full range of different types of representation (verbal, gesture and iconic exemplification) has been a key point in the procedure of developing the meaning of the mathematical objects involved. In the following school years (14-15 and 15-16), this class group attested in two synergic ways how successful this opening experience has been: on one hand, they proved to remember vividly the content delivered through the box and on the other they become more metacognitively aware of the importance of creating a range of different representations in order to have access to a mathematical concept or problem.

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Representations of fractions in mathematics education: What do students learn?

Cecilia Sveider

Linköping University, Department of Behavioural Sciences and Learning, Linköping, Sweden
cecilia.sveider@liu.se

Keywords: Representation, fraction, primary school.

Introduction

During the past years the mathematics education community as well as the educational research of teaching and learning have seen representations as a useful tool for communicating information and understanding (e.g. Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; NCTM, 2000). NCTM (2000) advocates students to "create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems…" (NCTM 2000, p. 64). In the present study, 'representations' refers to Goldins (2002) definition that a representation can represent something else, for example, symbolic expressions, drawings, written words or diagrams. Different representations can be used to shed light onto different aspects of a complex mathematical concept or relationship (e.g. Cai, 2005; Cathcart, Pothier, Vance, & Bezuk, 2006). For example, when learning about parts of a whole, teachers and students can use a manipulative representation, as sectors of a circle or as pieces of a rectangle.

Purpose

The aim of the present study is twofold. Firstly, investigate how teachers and students use representations of fractions focusing on finding a fraction regarding either part- whole or part of set aspect. This would for instance entail examination of which type of representation, how they are used and in which order they are used. Secondly to examine what students learn in mathematics lessons, regarding representations of fractions.

Hypothesis

The hypothesis is that a more frequent use of different representations, by either teachers and or students, will enhance the students' knowledge of fraction as part of a set or part of a whole. This is in line with what have been suggested by Cai (2005); Cathcart, Pothier, Vance, and Bezuk, (2006) concerning different representations.

Method

The collection of data consists of two parts, (a) video recordings of lessons and (b) pre- and post-tests ("RB2 or RB3 Diamant" which is part of the Swedish National Agency for Education's test materials).Video recordings enable data to be analyzed repeatedly and seen by other researchers in order to contribute to the study. Data will be analyzed with respect to variation theory (see for instance Marton, 2015). The pre- and post-test are a measurement to document effects of using representations of fractions during the mathematics lessons. To examine group differences between pre- and posttest repeated measure ANOVA is used.

Participants

Ten primary school teachers and approximate 250 students in Sweden, New Zealand, and Singapore, respectively, will be part of the present study.

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